Unit # 1 Function and Limits

Function: A function $y = f(x)$ is a rule that assigns for each value of the independent variable (input) x a unique value of the dependent variable (output) y.

Domain and Range: Let $y = f(x)$ be a function of independent variable x. The set of real numbers than can be substituted for the independent variable x and give real numbers for the dependent variable y is called the domain of the function. The set of real numbers obtained for dependent variable y is called the range of the function.

Compound function: A function that defined by more than one equation is called a compound function. For example,

$$f(x) = \begin{cases} 3x+2 & \text{for } x \le 1 \\ x^2 - 1 & \text{for } x > 1 \end{cases}$$

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Graph of modulus function: Absolute value function $f(x) = |x|$ is defined as

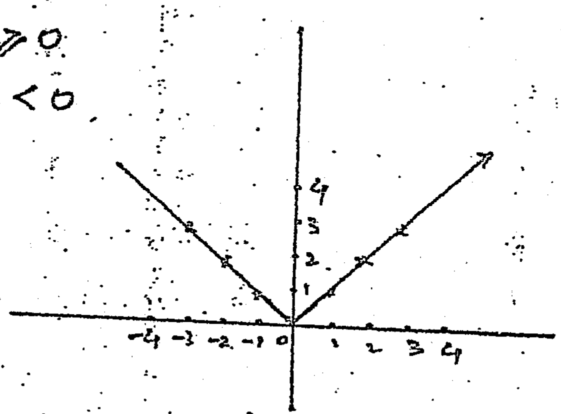
$$f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

$f(x) = x \quad , x \geq 0$

x	0	1	2	3
$f(x)$	0	1	2	3

$f(x) = -x$

x	0	-1	-2	-3
$f(x)$	0	1	2	3



Domain set is $(-\infty, \infty)$

Range set is $[0, \infty)$

One-to-One function: A function $f(x)$ is said (2) to be one-to-one if each range value corresponds to exactly one domain value, otherwise many-to-one function, if each range value corresponds to more than one domain values. For example,

$y = x^3$ is one-to-one function while
$y = x^2$ is many-to-one function.

Composition of functions: If $f(x)$ and $g(x)$ are two functions, then a composite function or composition of g and f is the function whose values are given by $g[f(x)]$ $\forall$ x in the domain of $f(x)$. For example

$f(x) = 4x+1$, $g(x) = x^2+2$

a) $g[f(x)] = g(4x+1)$
$= (4x+1)^2 + 2$
$= 16x^2 + 1 + 8x + 2$
$= 16x^2 + 8x + 3$

b) $f[g(x)] = f[x^2+2]$
$= 4(x^2+2) + 1$
$= 4x^2 + 8 + 1$
$= 4x^2 + 9$

Inverse functions: Let $y = f(x)$ be a function of x. This function takes y as dependent variable in response of x as independent variable. The function that takes x as dependent variable in response of y as independent variable is called the inverse function of $y = f(x)$ and is denoted by $x = f^{-1}(y)$

Note: ① The symbole $f^{-1}(y)$ means the inverse of f and does not mean $\frac{1}{f}$

② If $f(x)$ is not one-to-one function, then inverse of $f(x)$ is not a function.

EXERCISE # 1.1

1) Identify the independent and dependent variables for the following Problems:

a) $P = 64d$

independent variable : d ; dependent variable : P

b) $F(c) = \frac{9}{5}c + 32$

independent variable : c ; dependent variable : F

c) $C(F) = \frac{5}{9}(F-32)$

independent variable : F ; dependent variable : C

d) $S = f(r, \theta) = r \cdot \theta$

independent variable : r, θ ; dependent variable : S

e) $F = \theta(m, a) = ma$

independent variable : m, a ; dependent variable : F

f) $SA = \theta(l, w, h) = 2lw + 2lh + 2wh$

independent variable = l, w, h ; dependent variable : SA

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2) Evaluate the following functions for the indicated variables:

a) $f(x) = 3x^2 + 7x - 5 \longrightarrow (1)$ $\quad f(3);\ f(-4),\ f(a+h) = ?$

To find required results

put $x = 3$ in (1)

$f(3) = 3(3)^2 + 7(3) - 5$

$= 3(9) + 21 - 5 = 27 + 21 - 5 = 48 - 5 = 43$

$\Rightarrow \boxed{f(3) = 43}$

put $x = -4$ in (1)

$f(-4) = 3(-4)^2 + 7(-4) - 5$

$= 3(16) - 28 - 5 = 48 - 33 = 15$

$\Rightarrow \boxed{f(-4) = 15}$

Now put $x = a+h$ in (1)

$f(a+h) = 3(a+h)^2 + 7(a+h) - 5$

$= 3(a^2+h^2+2ah) + 7a + 7h - 5$

$f(a+h) = 3a^2+3h^2+6ah + 7a + 7h - 5$

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b) $f(t) = \frac{t+5}{t-3} \longrightarrow (1)$ $f(2)$, $f(7.4)$, $f(-3.7) = ?$

To find the required results

put $t=2$ in (1)

$f(2) = \frac{2+5}{2-3}$

$= \frac{7}{-1} = -7 \implies \boxed{f(2) = -7}$

Now put $t = 7.4$ in (1)

$f(7.4) = \frac{7.4+5}{7.4-3}$

$= \frac{12.4}{4.4} = 2.82 \implies \boxed{f(7.4) = 2.82}$

and now put $t = -3.7$ in (1)

$f(-3.7) = \frac{-3.7+5}{-3.7-3}$

$= \frac{1.3}{-6.7} = -0.19 \implies \boxed{f(-3.7) = -0.19}$

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c) $g(R) = \frac{R^2-R+6}{R-3} \longrightarrow (1)$ $g(2)$, $g(3)$, $g(3/8) = ?$

To find the required values

put $R=2$ in (1)

$g(2) = \frac{2^2-2+6}{2-3}$

$= \frac{4-2+6}{-1} = \frac{10-2}{-1} = -8 \implies \boxed{g(2) = -8}$

Now put $R = 3$ in (1)

$g(3) = \frac{3^2-3+6}{3-3}$

$= \frac{9-3+6}{0} = \frac{12}{0} = \infty \implies \boxed{g(3) = \infty}$

and now put $R = 3/8$ in (1)

$g(3/8) = \frac{(3/8)^2 - 3/8 + 6}{\frac{3}{8} - 3}$

$$= \frac{\frac{9}{64} - \frac{3}{8} + 6}{\frac{-21}{8}} = \frac{\frac{9-24+384}{64}}{\frac{-21}{8}} = \frac{\cancel{369}^{123}}{\cancel{64}_{8}} \times \frac{\cancel{-8}^{1}}{\cancel{21}_{7}} = -\frac{123}{8\times 7}$$

$g(3/8) = \frac{-123}{56}$ Ans

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d) $f(t) = 3t^2 + 2t - \sqrt{t} \longrightarrow (1)$ $f(3.217)$, $f(5.613)$, $f(\pi) = ?$

To find the req: values,

put $t = 3.217$ in (1)

$f(3.127) = 3(3.217)^2 + 2(3.217) - \sqrt{3.217}$

$= 3(3.217)^2 + 6.254 - 1.768$

$= 31.047 + 6.254 - 1.768$

$\boxed{f(3.127) = 35.533}$

Now put $t = 5.613$ in (1)

$f(5.613) = 3(5.613)^2 + 2(5.613) - \sqrt{5.613}$

$= 3(31.51) + 11.23 - 2.67$

$= 94.53 + 11.23 - 2.67$

$\boxed{f(5.613) = 103.09}$

put $t = \pi$ in (1)

$f(\pi) = 3\pi^2 + 2\pi - \sqrt{\pi}$ $\because \pi = 3.1416$ (App)

$= 3(3.1416)^2 + 2(3.1416) - \sqrt{3.1416}$

$= 3(9.87) + 6.28 - 1.77$

$= 29.61 + 6.28 - 1.77$

$\boxed{f(\pi) = 34.12}$ Ans

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3) The circumference of a circle is given by

$C(r) = 2\pi r \longrightarrow (1)$; where r is the length of the radius.

a) $C(2.34 \text{ in}) = ?$

So put $r = 2.34$ in (1)

$C(2.34) = 2\pi(2.34)$
$= 2 \times 3.1416 \times 2.34$
$= 14.7$ in

b) $C(6.41 \text{ in}) = ?$

put $r = 6.41$ in ①

$C(6.41) = 2\pi(6.41)$
$= 2 \times 3.1416 \times 6.41$
$= 40.28$ in

c) $C(\frac{5}{11} \text{ in}) = ?$

put $r = 5/11$ in ①

$C(\frac{5}{11}) = 2\pi(\frac{5}{11})$
$= \frac{10\pi}{11} = \frac{10 \times 3.1416}{11} = \frac{31.416}{11}$
$= 2.86$ in

4) The Area of a circle is given by

$A(r) = \pi r^2 \longrightarrow$ ① ; where r is the length of the radius.

a) $A(2.34 \text{ in}) = ?$

put $r = 2.34$ in ①

$A(2.34) = \pi(2.34)^2$
$= 3.1416 \times 5.48$
$= 17.22 \text{ in}^2$

b) $A(6.41 \text{ in}) = ?$

put $r = 6.41$ in ①

$A(6.41) = \pi(6.41)^2$
$= 3.1416 \times 41.09$
$= 129.09 \text{ in}^2$

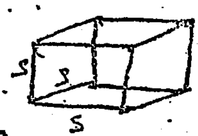
c) $A(\frac{5}{11} \text{ in}) = ?$

put $r = 5/11$ in ①

$A(\frac{5}{11}) = \pi(\frac{5}{11})^2$
$= 3.1416 \times \frac{25}{121} = \frac{78.54}{121} = 0.65 \text{ in}^2$

5) The total surface area of a cube is given by

$f(s) = 6s^2 \longrightarrow$ ① , where s is the length of the side of the cube.



a) $f(3.75 \text{ in}) = ?$

put $s = 3.75$ in ①

$f(3.75) = 6(3.75)^2$
$= 6 \times 14.06$
$= 84.38 \text{ in}^2$

b) $f(6.05 \text{ in}) = ?$

put $s = 6.05$ in ①

$f(6.05) = 6(6.05)^2$
$= 6 \times 36.6$
$= 219.62 \text{ in}^2$

c) $f(13.42) = ?$

put $s = 13.42$ in ①

$$f(13.42) = 6(13.42)^2$$
$$= 6 \times 180.1 = 1080$$

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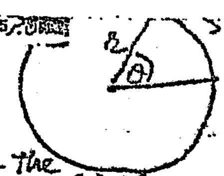
6) The measure of the angle θ is given by $\theta = f(s, r) = \frac{s}{r} \rightarrow$ ① where s is the length of the arc determined by θ and r is the length of the radius of the circle.



a) $f(4.71, 3) = ?$

put $s = 4.71$, $r = 3$ in ①

$$\theta = f(4.71, 3) = \frac{4.71}{3}$$
$$= 1.57 \text{ radians}$$

b) $f(15.71, 5) = ?$

put $s = 15.71$, $r = 5$ in ①

$$\theta = f(15.71, 5) = \frac{15.71}{5}$$
$$= 3.142 \text{ radians}$$

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7) If $f(x) = \begin{cases} x-3, & \text{for } x < 0 \\ 2x+5, & \text{for } x \geq 0 \end{cases}$ then find

a) $f(-1) = ?$

let $f(x) = x - 3$

put $x = -1$

$$f(-1) = -1 - 3$$
$$= -4$$

b) $f(0) = ?$

let $f(x) = 2x + 5$

put $x = 0$

$$f(0) = 2(0) + 5$$
$$= 0 + 5$$
$$= 5$$

c) $f(1) = ?$

let $f(x) = 2x + 5$

put $x = 1$

$$f(1) = 2(1) + 5$$
$$= 2 + 5$$
$$= 7$$

Ans

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8) If $f(x) = \begin{cases} \frac{x^2-4}{x-2}, & \text{for } x \neq 2 \\ 4, & \text{for } x = 2 \end{cases}$ then find

a) $f(0) = ?$

let $f(x) = \frac{x^2-4}{x-2}$

put $x = 0$

$$f(0) = \frac{0-4}{0-2} = \frac{-4}{-2}$$
$$= 2$$

b) $f(2) = ?$

let $f(x) = 4$

put $x = 2$

$$f(2) = 4$$

Ans

c) $f(4) = ?$

let $f(x) = \frac{x^2-4}{x-2}$

put $x = 4$

$$f(4) = \frac{4^2-4}{4-2}$$
$$= \frac{16-4}{2} = \frac{12}{2} = 6$$

Ans

9) Indicate whether each table specifies a function.

a) Domain → Range

3 → 0
5 → 1
7 → 2

It is a function because for each value in domain there is a unique value in range.

b) Domain → Range

−1 → 5
−2 → 7
−3 → 9

It is a function because for each value in domain there is a unique value in range.

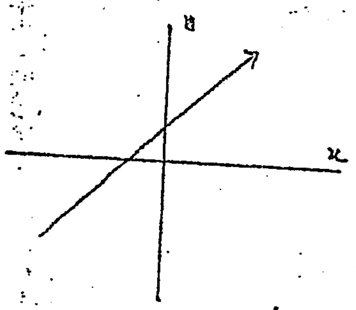
c) Domain → Range

3 → 5
3 → 6
4 → 6
4 → 7
5 → 8

It is not a function because for 3 & 4 images are not unique.

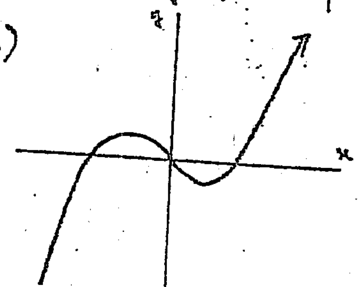
10) Indicate whether each graph specifies a function.

a)



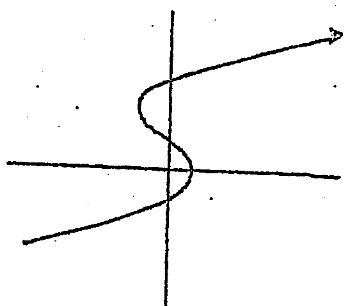
It is a function because any vertical line intersect it at exactly one Point.

b)



It is a function because any vertical line intersect it at exactly one Point.

c)



It can not be a function because a vertical line can intersect it at more than one Point.

Note

① Domain are the values for which function define and real.

② To find Domain use function as $y = f(x)$

③ To find range use function as $x = g(y)$

11) Determine the domain and range of the following functions

a) $y = 3x + 4$

$\because$ y is a real number $\forall x$, so Domain $= \{x \mid x \in R\}$

To find range $\because$ $y = 3x + 4$

$y - 4 = 3x$

$\frac{y-4}{3} = x$ or $\boxed{x = \frac{y-4}{3}}$

$\because$ x is real $\forall\ y$ so Range $= \{y \mid y \in R\}$

b) $f(t) = t^2 + 5$

$\because$ $f(t)$ is a real number $\forall\ t$.

so Domain $= \{t \mid t \in R\}$

To find Range

$\because\ f(t) = t^2 + 5$

$\Rightarrow\ f(t) - 5 = t^2$

or $t^2 = f(t) - 5$

taking sq: Root

$t = \pm\sqrt{f(t) - 5}$

$\because$ t is a real number $\forall\ f(t) \geq 5$

$\therefore$ Range $= \{f(t) \mid f(t) \geq 5\}$

Note Radius can not be -ive

c) $SA = f(r) = 4\pi r^2$

$\because$ $f(r)$ is a real number $\forall\ r$ & r can not be -ive

so Domain $= \{r \mid r \geq 0\}$

To find Range

$\because\ f(r) = 4\pi r^2$

$\Rightarrow\ \frac{f(r)}{4\pi} = r^2$ or $r^2 = \frac{f(r)}{4\pi}$

$\Rightarrow\ r = \sqrt{\frac{f(r)}{4\pi}}$

$\because$ r is a real number $\forall\ f(r) \geq 0$

$\therefore$ Range $= \{f(r) \mid f(r) \geq 0\}$

12) Find the composite functions $f[g(x)]$ and $g[f(x)]$ of the following functions

a) $f(x) = x^2 + 1$, $g(x) = 2x$

$f[g(x)] = f[2x]$
$= (2x)^2 + 1$
$= 4x^2 + 1$

Now
$g[f(x)] = g[x^2 + 1]$
$= 2(x^2 + 1)$
$= 2x^2 + 2$

b) $f(x) = \sin x$, $g(x) = 1 - x^2$

sol
$f[g(x)] = f(1 - x^2)$
$= \sin(1 - x^2)$

Now
$g[f(x)] = g[\sin x]$
$= 1 - (\sin x)^2$
$= 1 - \sin^2 x = \cos^2 x$

Note: $(\sin x)^2 = \sin^2 x$
but $(\sin x)^2 \neq \sin x^2$

c) $f(t) = \sqrt{t}$, $g(t) = t^2$

sol

$$f[g(t)] = f[t^2] = \sqrt{t^2} = t$$

Now

$$g[f(t)] = g[\sqrt{t}] = (\sqrt{t})^2 = t$$

d) $f(u) = \frac{u-1}{u+1}$, $g(u) = \frac{u+1}{1-u}$

1st Part

$$f[g(u)] = f\left[\frac{u+1}{1-u}\right] = \frac{\frac{u+1}{1-u} - 1}{\frac{u+1}{1-u} + 1}$$

L.C.M

$$= \frac{\frac{u+\cancel{1}-\cancel{1}+u}{\cancel{1-u}}}{\frac{\cancel{u}+1+1-\cancel{u}}{\cancel{1-u}}} = \frac{2u}{2} = u$$

2nd Part

$$g[f(u)] = g\left[\frac{u-1}{u+1}\right] = \frac{\frac{u-1}{u+1} + 1}{1 - \frac{u-1}{u+1}}$$

$$= \frac{\frac{u-\cancel{1}+u+\cancel{1}}{\cancel{u+1}}}{\frac{\cancel{u}+1-\cancel{u}+1}{\cancel{u+1}}} = \frac{2u}{2} = u$$

e) $f(x) = \sin x$, $g(x) = 2x+3$

1st Part $f[g(x)] = f(2x+3) = \sin(2x+3)$

2nd Part $g[f(x)] = g[\sin x] = 2\sin x + 3$

f) $f(x) = \frac{1}{x}$, $g(x) = \tan x$

$$f[g(x)] = f[\tan x] = \frac{1}{\tan x} \quad \text{Ans}$$

$$g[f(x)] = g\left[\frac{1}{x}\right] = \tan\frac{1}{x}$$

Note: To find inverse interchange dependent and independent variable.

13) Determine the inverse function of each of the following functions:

a) $f(x) = x+5$

let $f^{-1}(x+5) = x$ put $x+5 = t$, $x = t-5$

$\Rightarrow f^{-1}(t) = t-5$

Replace t by x

$f^{-1}(x) = x-5$ Ans

b) $f(x) = 2x+7$

let $f^{-1}(2x+7) = x$ put $2x+7 = t$, $2x = t-7$, $x = \frac{t-7}{2}$

$f^{-1}(t) = \frac{t-7}{2}$

Replace t by x

$f^{-1}(x) = \frac{x-7}{2}$

c) $f(x) = 2(x-4)$

let $f^{-1}(2(x-4)) = x$ put $2(x-4) = t$, $x-4 = \frac{t}{2}$ $\Rightarrow x = \frac{t}{2}+4$

$\Rightarrow f^{-1}(t) = \frac{t}{2}+4$

Replace t by x

$f^{-1}(x) = \frac{x}{2}+4$

d) $f(x) = \frac{x+4}{2}$

Let $f^{-1}\left(\frac{x+4}{2}\right) = x$ put $\frac{x+4}{2} = t$, $x+4 = 2t$, $x = 2t-4$

$\Rightarrow f^{-1}(t) = 2t-4$

Replace t by x

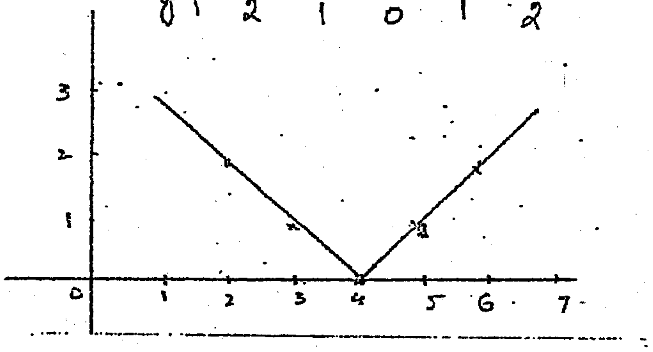
$f^{-1}(x) = 2x-4$

14) Graph each of the following absolute function

a) $y = |x-4|$

To sketch the graph we construct the table

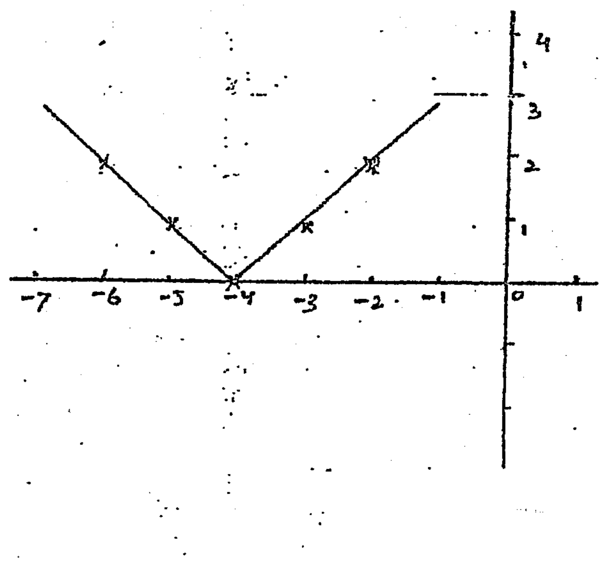
x	2	3	4	5	6
y	2	1	0	1	2



b) $y = |-4-x|$

To sketch the graph we construct the table

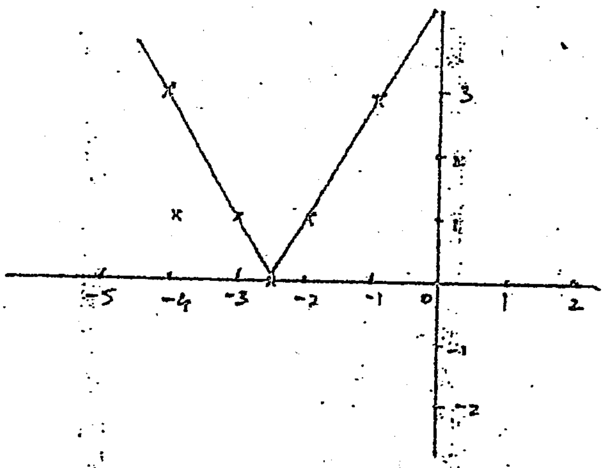
x	-6	-5	-4	-3	-2
y	2	1	0	1	2



c) $y = |2x+5|$

To sketch the graph we construct the table

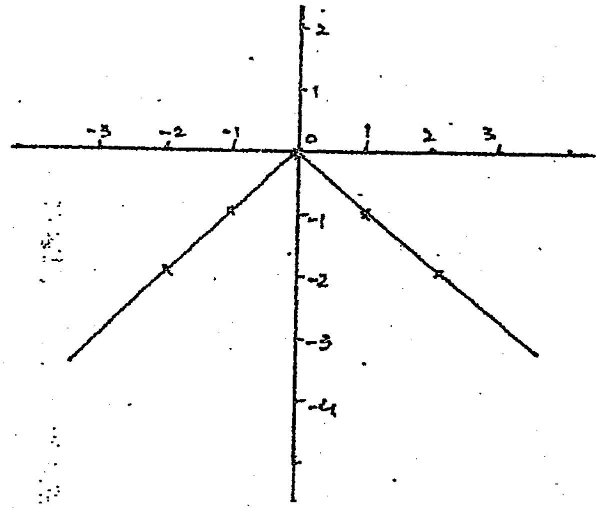
x	-4	-3	-2.5	-2	-1
y	3	1	0	1	3



d) $y = -|x|$

To sketch the graph we construct the table

x	-2	-1	0	1	2
y	-2	-1	0	-1	-2



Polynomial: An algebraic expression in which exponents are non-negative integers, highest exponent is called degree of the polynomial. For example

$x^2 + 3x - 4$ is Polynomial of degree 2

$\sqrt{3}x^3 + \frac{1}{2}x^2 + 1$ is Polynomial of degree 3

Algebraic function: Algebraic function is a function that can be defined as the root of a polynomial equation.

OR A function $f(x)$ is called algebraic if it can be constructed using algebraic operations starting with polynomials.

Any Rational function is an algebraic function.

Transcendental function: Functions that are not algebraic are called transcendental functions.

<u>Trigonometric function</u>: Trigonometric functions are the functions sine, cosine, tangent, secant, cosecant, cotangent.

<u>Exponential function</u>: An equation of the form

$f(x) = b^x$, $b > 0$, $b \neq 1$, b is positive constant.

<u>Exponent Laws</u>: ① $a^x a^y = a^{x+y}$ ② $\frac{a^x}{a^y} = a^{x-y}$

3) $(a^x)^y = a^{xy}$ 4) $(ab)^x = a^x b^x$ 5) $\left(\frac{a}{b}\right)^x = \frac{a^x}{b^x}$

<u>Note</u> ① $a^x = a^y$ iff $x = y$

② For $x \neq 0$; $a^x = b^x$ iff $a = b$

<u>Base e Exponential function</u>: $y = e^x$ is exponential function with base e. The base e is an irrational number like π it cannot be represented exactly by any finite decimal fraction. Approximated value of e is 2.718281.

<u>Logarithmic function</u>, The inverse of an exponential function is called a logarithmic function. For $b > 0$ and $b \neq 1$, the logarithmic function is $y = \log_b x$, which is equivalent to $x = b^y$

<u>common logarithm</u>: Common logarithms are logarithm with base 10. $y = \log_{10} x$ means $10^y = x$

<u>Natural logarithms</u>: Natural logarithms are logarithms with base e. $y = \ln x$ means $e^y = x$

<u>Logarithmic Notation</u>,

Common logarithmic: $\log_{10} x$

Natural logarithmic: $\ln x = \log_e x$

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Logarithmic - Exponential Relationships:

$\log_{10} x = y$ is equivalent to $x = 10^y$

$\ln x = y$ is equivalent to $x = e^y$

Properties of Logarithms:

If b, M, N are positive real numbers, $b \neq 1$ and p and x are also any positive numbers, then:

1) $\log_b 1 = 0$; 2) $\log_b b = 1$ 3) $\log_b b^x = x$

4) $b^{\log_b x} = x$ or $b^{\log_b M} = M$

5) $\log_b MN = \log_b M + \log_b N$ 6) $\log_b \frac{M}{N} = \log_b M - \log_b N$

7) $\log_b M^N = N \log_b M$ or $\log_b M^P = P \log_b M$

8) $\log_N M = \frac{1}{\log_M N}$ 9) $\frac{\log_b M}{\log_b N} = \log_N M$

10) $\log_b M = \log_b N \Rightarrow M = N$

Hyperbolic functions

$\sinh x = \frac{e^x - e^{-x}}{2} \Rightarrow \operatorname{cosech} x = \frac{2}{e^x - e^{-x}}$

$\cosh x = \frac{e^x + e^{-x}}{2} \Rightarrow \operatorname{sech} x = \frac{2}{e^x + e^{-x}}$

$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} \Rightarrow \coth x = \frac{e^x + e^{-x}}{e^x - e^{-x}}$

Note. ① "$\sinh x$" pronounced as "cinch"

② $\cosh x$ " " "kosh"

③ $\tanh x$ " " "tansh"

Note: ① $\cosh^2 x - \sinh^2 x = 1$

② $\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$

③ $\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$

④ $1 - \tanh^2 x = \operatorname{sech}^2 x$

Explicit and Implicit function:

If y is equated to an expression involving only x terms, that is $y = f(x)$, then we say that y expressed explicitly in terms of x. For example

$y = x^2+3$, $y = \sin x$, $y = e^{3x} - 2x$

Sometimes we have an equation connecting x and y but it is not easily transposed to the form $y = f(x)$. For example

$y = x^2 - y^3 + \sin x - \cos y$, $\sin(x+y) + e^x + e^{-y} = x^3 + y^3$

In these cases we say that y is expressed implicitly in terms of x.

Parametric Representation of Curves:

It is sometimes useful to define the variables x & y in the ordered Pair (x, y), so that they are each functions of some variable, say t:

$x = f(t)$ and $y = g(t)$

The Domain of these functions $f(t)$ and $g(t)$ is some interval D. The variable t is called the Parameter and $x = f(t)$ and $y = g(t)$ are called Parametric equation. For example

Consider the Pair (t, t^2), here $x = t$ & $y = t^2$ and define $\forall\ t \in R$, so $x = t$, $y = t^2$ are Parametric equations

---

EXERCISE 1.2

1) Simplify the following functions:

a) $(4^{3x})^{2y} = 4^{3x \times 2y} = 4^{6xy}$ $\quad (\because (a^m)^n = a^{mn})$

b) $10^{3x-1}\, 10^{4-x} = 10^{3x-1+4-x} = 10^{2x+3}$ $\quad (\because a^m a^n = a^{m+n})$

c) $\dfrac{e^{x-3}}{e^{x-4}} = e^{x-3-(x-4)} = e^{\cancel{x}-3-\cancel{x}+4} = e^1$ $\quad (\because \dfrac{a^m}{a^n} = a^{m-n})$

d) $\frac{e^x}{e^{3-x}} = e^{x-(3-x)} = e^{x-3+x} = e^{2x-3}$ $\left(\because \frac{a^m}{a^n} = a^{m-n}\right)$ (16

e) $(2e^{1.2t})^3 = 2^3 (e^{1.2t})^3 = 8\, e^{1.2t\times 3} = 8e^{3.6t}$ $\left((ab)^m = a^m b^m\right)$

f) $(3e^{-1.4x})^2 = 3^2 (e^{-1.4x})^2 = 9\, e^{-1.4x\times 2} = 9e^{-2.8x}$

Note (i) $a^x = a^y$ iff $x = y$ (ii) $a^x = b^x$ iff $a = b$

2) Solve the following equations:

a) $10^{2-3x} = 10^{5x-6}$

$\Rightarrow 2-3x = 5x-6$

$2+6 = 5x+3x$

$\cancel{8}^1 = \cancel{8}x$

$\Rightarrow \boxed{x=1}$

b) $5^{3x} = 5^{4x-2}$

$\Rightarrow 3x = 4x-2$

$\Rightarrow 2 = 4x-3x$

$\Rightarrow 2 = x$

$\Rightarrow \boxed{x=2}$

c) $4^{5x-x^2} = 4^{-6}$

$\Rightarrow 5x - x^2 = -6$

$\Rightarrow x^2-5x-6=0$

$x^2-6x+x-6=0$

$x(x-6)+1(x-6)=0$

$(x-6)(x+1)=0$

$x-6=0$ or $x+1=0$

$x=6$ $x=-1$

d) $7^{x^2} = 7^{2x+3}$

$\Rightarrow x^2 = 2x+3$

$\Rightarrow x^2-2x-3=0$

$x^2-3x+x-3=0$

$x(x-3)+1(x-3)=0$

$(x-3)(x+1)=0$

$x-3=0$, $x+1=0$

$x=3$ $x=-1$

e) $5^3 = (x+2)^3$

$\Rightarrow 5 = x+2$

$5-2 = x$

$\boxed{x=3}$

f) $(1-x)^5 = (2x-1)^5$

$\Rightarrow 1-x = 2x-1$

$1+1 = 2x+x$

$2 = 3x$

$\frac{2}{3} = x$

or $\boxed{x = 2/3}$

g) $(x-3)e^{x}=0$

$\because e^{x}\neq 0$ so $\div$ing by e^{x}

$x-3=0$

$\boxed{x=3}$

h) $2xe^{-x}=0$

$\because e^{-x}\neq 0$ so Dividing by e^{-x}

$2x=0$ $(\div$ing by 2

$\Rightarrow \boxed{x=0}$

i) $3xe^{-x}+x^2e^{-x}=0$

$\because e^{-x}\neq 0$ so Dividing by e^{-x}

$3x+x^2=0$

$\Rightarrow x(3+x)=0$

$\boxed{x=0}$, $3+x=0$

$\boxed{x=-3}$

j) $x^2e^{x}-5xe^{x}=0$

$\because e^{x}\neq 0$ so dividing by e^{x}

$x^2-5x=0$

$x(x-5)=0$

$\boxed{x=0}$, $x-5=0$

$\boxed{x=5}$

<u>Note</u> $\log_b x = y$ is equivalent to $x=b^y$

3) Rewrite in equivalent exponential form, the following logarithmic functions:

a) $\log_3 27 = 3 \Rightarrow 27 = 3^3$

b) $\log_2 32 = 5 \Rightarrow 32 = 2^5$

c) $\log_{10} 1 = 0 \Rightarrow 1 = 10^0$

d) $\log_e 1 = 0 \Rightarrow 1 = e^0$

e) $\log_4 8 = 3/2 \Rightarrow 8 = 4^{3/2}$

f) $\log_9 27 = 3/2 \Rightarrow 27 = 9^{3/2}$

4) Rewrite in equivalent logarithmic form, the following exponential function.

a) $49 = 7^2$

$\Rightarrow \log_7 49 = 2$

b) $36 = 6^2$

$\Rightarrow \log_6 36 = 2$

c) $8 = 4^{3/2}$

$\Rightarrow \log_4 8 = 3/2$

d) $9 = 27^{2/3}$

$\Rightarrow \log_{27} 9 = 2/3$

e) $A = b^4$

$\Rightarrow \log_b A = 4$

f) $M = b^x$

$\Rightarrow \log_b M = x$

5) Find x, y and b without a scientific calculator use: (18

a) $\log_3 x = 2$

$\Rightarrow x = 3^2$

$\boxed{x = 9}$

b) $\log_2 x = 2$

$\Rightarrow x = 2^2$

$\Rightarrow x = 4$

c) $\log_7 49 = y$

$\Rightarrow 49 = 7^y$

$\Rightarrow 7^2 = 7^y$

$\Rightarrow \boxed{y = 2}$

2nd method

$\log_7 49 = y$

$\log_7 7^2 = y$

$\boxed{2 = y}$

d) $\log_b 10^{-4} = -4$

$\Rightarrow 10^{-4} = b^{-4}$

$\Rightarrow 10 = b$

$\Rightarrow \boxed{b = 10}$

e) $\log_{1/3} 9 = y$

$\Rightarrow 9 = (\frac{1}{3})^y$

$\Rightarrow 3^2 = 3^{-y}$

$\Rightarrow 2 = -y \Rightarrow \boxed{y = -2}$

f) $\log_b 1000 = 3/2$

$\Rightarrow 1000 = b^{3/2}$

$10^3 = (b^{1/2})^3$

$\Rightarrow 10 = b^{1/2}$

Sq: both sides

$100 = b$

or $\boxed{b = 100}$

Note (1) $\log_b M^N \Leftrightarrow N \log_b M$

2) $\log_b MN \Leftrightarrow \log_b M + \log_b N$

3) $\log_b \frac{M}{N} \Leftrightarrow \log_b M - \log_b N$

4) $\log_b M = \log_b N \Leftrightarrow M = N$

6) Solve the following equations for unknown x

a) $\log_b x = \frac{2}{3} \log_b 8 + \frac{1}{2} \log_b 9 - \log_b 6$

$= \log_b 8^{2/3} + \log_b 9^{1/2} - \log_b 6$

$= \log_b (2^{\cancel{3}})^{2/\cancel{3}} + \log_b (3^{\cancel{2}})^{1/\cancel{2}} - \log_b 6$

$= \log_b 4 + \log_b 3 - \log_b 6$

$\log_b x = \log_b \frac{\overset{2}{\cancel{4}} \times \cancel{3}}{\cancel{6}}$

$\log_b x = \log_b 2$

$\Rightarrow \boxed{x = 2}$ Ans

b) $\log_b x = \frac{2}{3} \log_b 27 + 2 \log_b 2 - \log_b 3$

$= \log_b (27)^{2/3} + \log_b 2^2 - \log_b 3$

$= \log_b (3^{\cancel{3}})^{2/\cancel{3}} + \log_b 4 - \log_b 3$

$= \log_b 9 + \log_b 4 - \log_b 3$

$\log_b x = \log_b \frac{\overset{3}{\cancel{9}} \times 4}{\cancel{3}}$

$\log_b x = \log_b 12$

$\Rightarrow \boxed{x = 12}$ Ans

c) $\log_b x = \frac{3}{2}\log_b 4 - \frac{2}{3}\log_b 8 + 2\log_b 2$

$= \log_b 4^{3/2} - \log_b 8^{2/3} + \log_b 2^2$

$= \log_b (2^{\cancel{2}})^{3/\cancel{2}} - \log_b (2^{\cancel{3}})^{2/\cancel{3}} + \log_b 4$

$= \log_b 2^3 - \cancel{\log_b 4} + \cancel{\log_b 4}$

$\log_b x = \log_b 8$

$\Rightarrow \boxed{x = 8}$

d) $\log_b x + \log_b (x-4) = \log_b 21$

$\Rightarrow \log_b [x(x-4)] = \log_b 21$

$\Rightarrow x(x-4) = 21$

$x^2 - 4x - 21 = 0$

$x^2 - 7x + 3x - 21 = 0$

$x(x-7) + 3(x-7) = 0$

$(x-7)(x+3) = 0$

$x - 7 = 0$, $x + 3 = 0$

$\boxed{x = 7}$ Ans $\boxed{x = -3}$ rejected

---

e) $\log_{10}(x-1) - \log_{10}(x+1) = 1$

$\Rightarrow \log_{10} \frac{x-1}{x+1} = 1$

$\Rightarrow \frac{x-1}{x+1} = 10^1$

$\therefore x - 1 = 10(x+1)$

$x - 1 = 10x + 10$

$x - 10x = 10 + 1$

$-9x = 11 \Rightarrow \boxed{x = -\frac{11}{9}}$ rejected

so there is no solution

f) $\log_{10}(x+6) - \log_{10}(x-3) = 1$

$\Rightarrow \log_{10} \frac{x+6}{x-3} = 1$

$\Rightarrow \frac{x+6}{x-3} = 10^1$

$\Rightarrow x + 6 = 10(x-3)$

$x + 6 = 10x - 30$

$x - 10x = -30 - 6$

$\cancel{-9}x = \cancel{-36}\ 4 \Rightarrow \boxed{x = 4}$

---

7) Given equation is $S(t) = 125 + 83\log(5t+1) \longrightarrow (1)$

a) $S(0) = ?$

put $t = 0$ in (1)

$S(0) = 125 + 83\log(0+1)$

$S(0) = 125 + 83\log 1$

$= 125 + 83(0)$

$= 125 + 0 = 125$

Sale is about \$125000

b) $S(2) = ?$

put $t = 2$ in (1)

$S(2) = 125 + 83\log(5(2)+1)$

$= 125 + 83\log(11)$

$= 125 + 83(1.0414)$

$= 125 + 86.44 = 211.44$

Sale is about \$ 211000

c) $S(4) = ?$

put $t = 4$ in (1)

$S(4) = 125 + 83\log(5(4)+1)$

$= 125 + 83\log 21$

$= 125 + 83(1.3222)$

$= 125 + 109.74 = 234.74$

Sale is about \$235000

d) $S(31) = ?$ (20

put $t = 31$ in (1)

$S(31) = 125 + 83\log(5(31)+1)$

$= 125 + 83\log(156)$

$= 125 + 83(2.1931)$

$= 125 + 182.03 = 307.03$

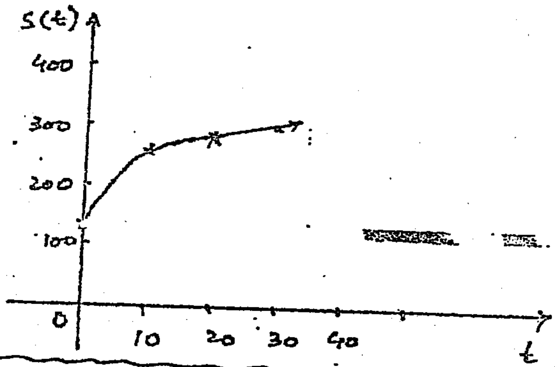
Sale is about \$ 307000

e) Draw and interpret the graph

$\because S(t) = 125 + 83\log(5t+1)$

To draw the graph construct the table

t	0	10	20	30
$S(t)$	125	266.7	291.4	305.9



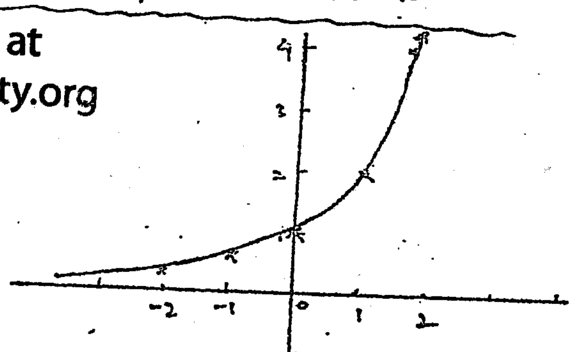
The sale of Product is increasing.

Note: Graph of 2^x

Let $y = 2^x$.

To draw the graph construct the table:

x	-2	-1	0	1	2
y	.25	.5	1	2	4

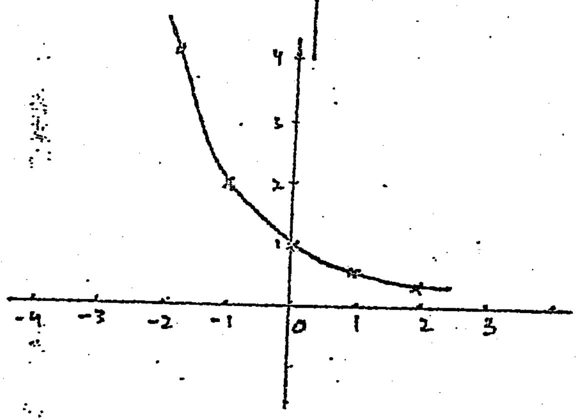


Graph of 2^{-x}

Let $y = 2^{-x}$ $\boxed{\left(\frac{1}{2}\right)^x}$

To draw the graph construct the table

x	-2	-1	0	1	2
y	4	2	1	.5	.25



Note: graph of $f(x) = b^x$, $b > 0$, $b \neq 1$

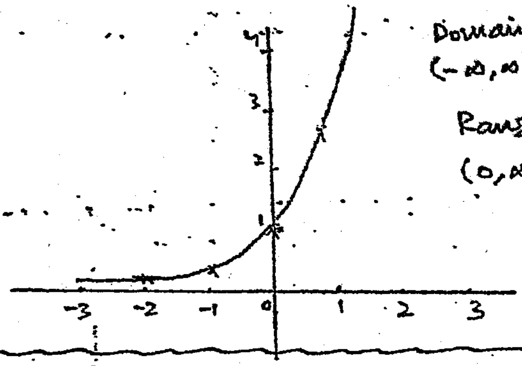
1) All graphs will pass through the Point $(0,1)$
2) If $b > 1$, then b^x increases as x increases.
3) If $0 < b < 1$, then b^x decreases as x increases.
4) All graphs are continuous curves, with no holes or jumps

Graph of e^x

Let $y = e^x$

To draw the graph we construct the table

x	-2	-1	0	1	2
y	0.14	0.37	1	2.7	7.38



Domain $(-\infty, \infty)$

Range $(0, \infty)$

Let $y = e^x$

Interchange x & y

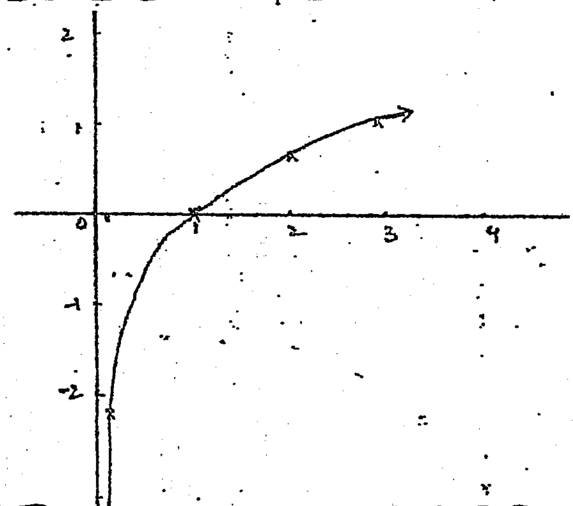
$x = e^y$

$\Rightarrow \ln x = y$.

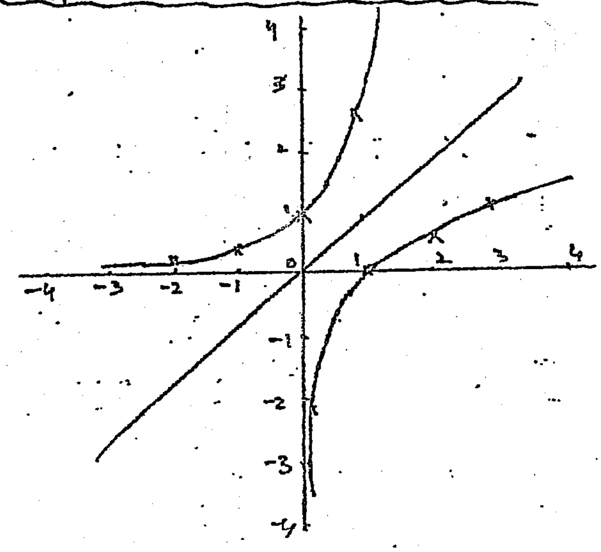
or $y = \ln x$

Now

x	0.1	1	2	3
y	-2.3	0	0.7	1.1



Note $y = e^x$ & $y = \ln x$ are inverse function and their graphs are symmetric about line $y = x$



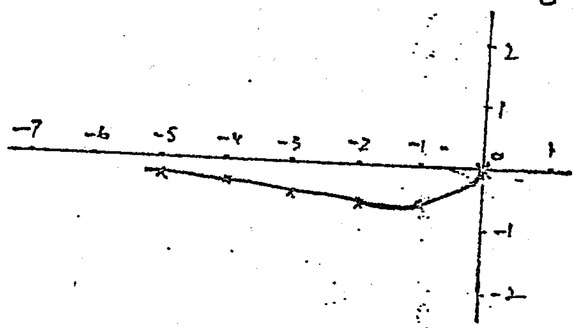
EXERCISE 1.3

1) Using a calculator and Point-by-point to plot the following exponential functions:

a) $h(x) = x(2^x)$; $[-5, 0]$

construct the table

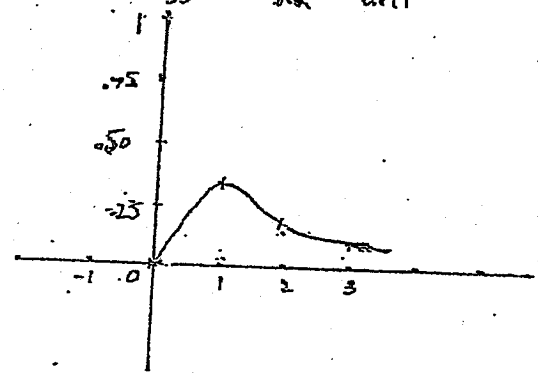
x	-5	-4	-3	-2	-1	0
y	-0.16	-.25	-.38	-.5	-.5	0



b) $m(x) = x(3^{-x})$; $[0, 3]$

construct the table

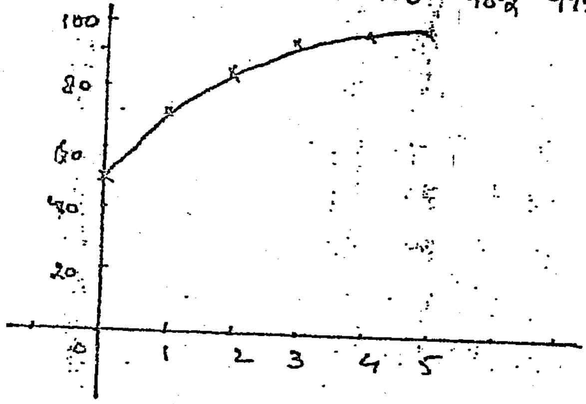
x	0	1	2	3
m	0	.33	.22	0.11



c) $N = \dfrac{100}{1+e^{-t}}$; $[0, 5]$

To sketch the graph we construct the table

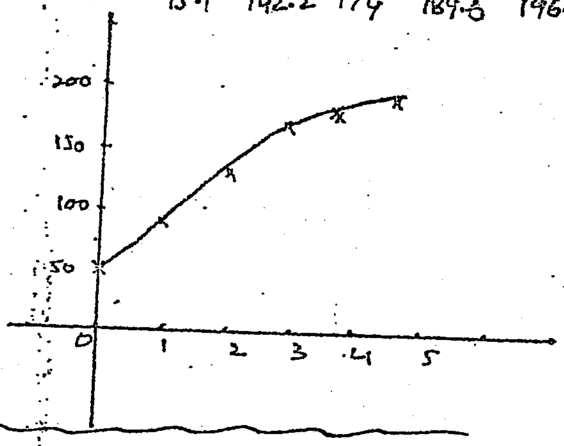
t	0	1	2	3	4	5
N	50	73.1	88.1	95.3	98.2	99.3



d) $N = \dfrac{200}{1+3e^{-t}}$ $[0, 5]$

construct the input output table

t	0	1	2	3	4	5
N	50	95.1	142.2	174	189.6	196.04

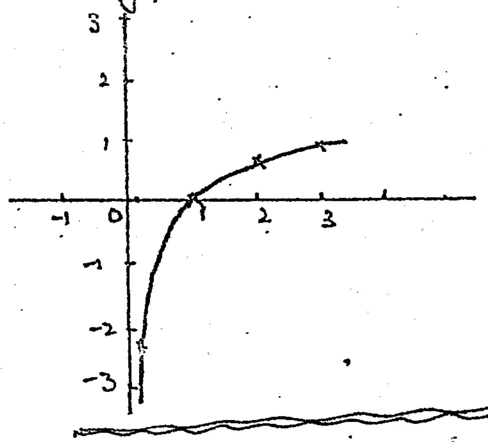


2) Using a calculator and Point-by-point to plot the following logarithmic functions:

See →

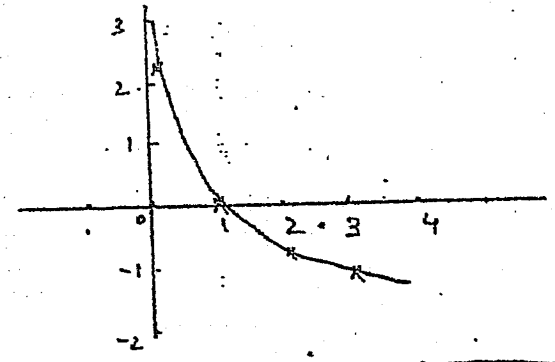
a) $y = \ln x$

x	0.1	1	2	3
y	-2.3	0	0.7	1.1



b) $u = -\ln x$

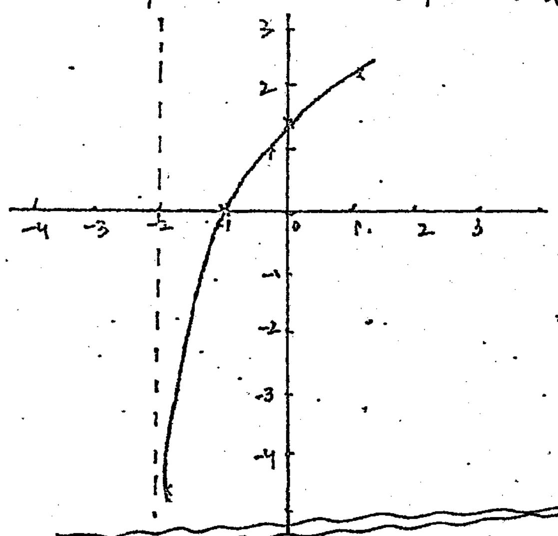
x	0.1	1	2	3
u	2.3	0	-0.7	-1.1



c) $y = 2\ln(x+2)$

Now

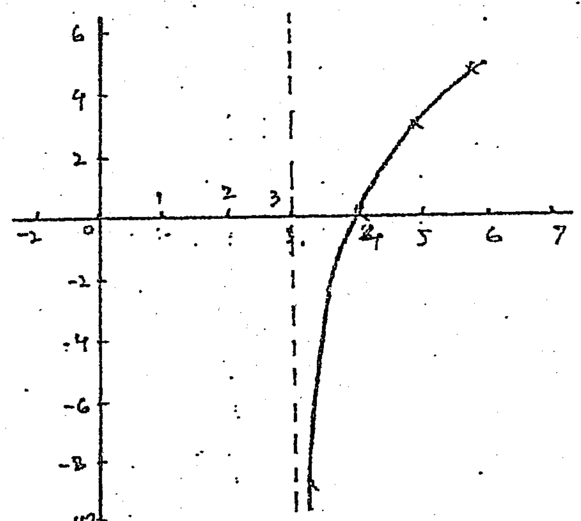
x	-1.9	-1	0	1
y	-4.6	0	1.4	2.2



d) $y = 4\ln(x-3)$

Now

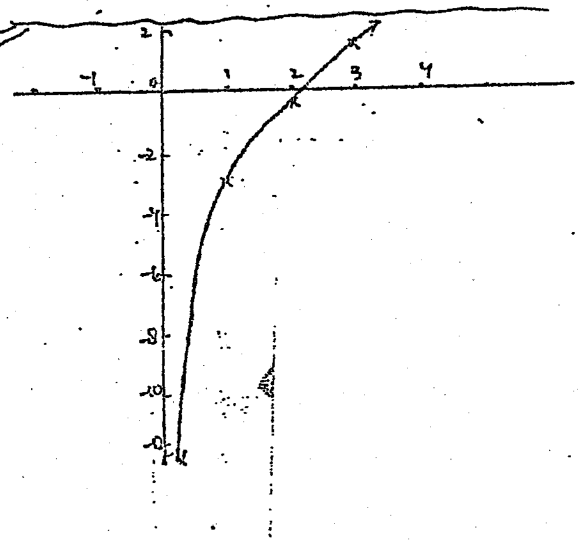
x	3.1	4	5	6
y	-9.2	0	2.8	4.4



e) $y = 4\ln x - 3$

Now

x	0.1	1	2	3
y	-12.2	-3	-0.2	1.4



Parametric equation:

Parametric equation of line is $x = x_0 + at$, $y = y_0 + bt$ which is passing through point (x_0, y_0) & parallel to position vector $\vec{u} = [a, b]$

---

3) Sketch the following parametric curves:

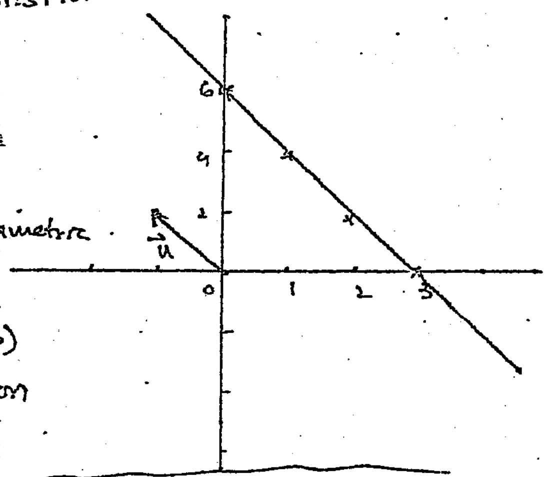
a) $(x(t), y(t)) = (3-t, 2t)$ t is a real number

By comparing corresponding coordinate we have

$x(t) = 3-t$, $y(t) = 2t$

To sketch the graph we construct the table

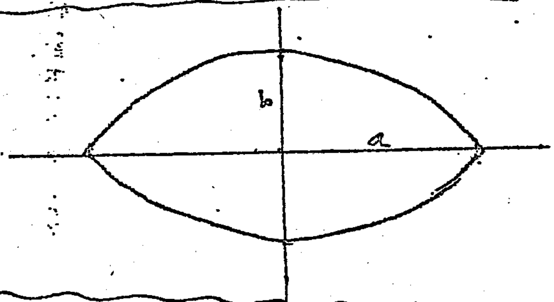
t	0	1	2	3
$x(t)$	3	2	1	0
$y(t)$	0	2	4	6



$\because$ $x = 3-t$, $y = 0+2t$ is parametric equation of straight line so it passing through $(3, 0)$ & parallel to the direction vector $\vec{u} = [-1, 2]$

Note Equation of ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$



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b) $(x(t), y(t)) = (4\cos t, -3\sin t)$

Comparing corresponding coordinates

$x = 4\cos t$, $y = -3\sin t$

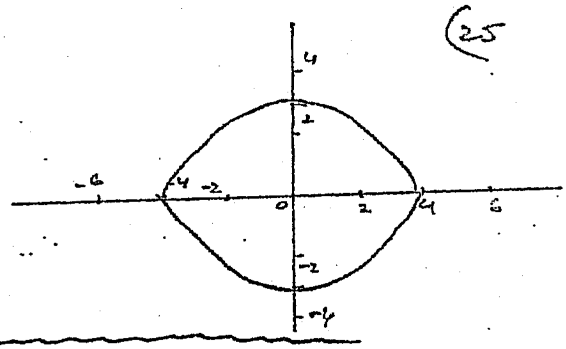
$\Rightarrow \frac{x}{4} = \cos t$ — (i) $\frac{y}{-3} = \sin t$ — (ii)

Squaring and add eq (i) & (ii)

$$\left(\frac{x}{4}\right)^2 + \left(\frac{y}{-3}\right)^2 = \cos^2 t + \sin^2 t$$

$$\frac{x^2}{4^2} + \frac{y^2}{3^2} = 1$$

which is an equation of ellipse.



c) $(x(t), y(t), z(t)) = (3+2t, 5-3t, 2-4t)$

compare corresponding coordinates

$x = 3+2t$, $y = 5-3t$, $z = 2-4t$

It is Parametric equation of straight line Passing through $(3,5,2)$ & Parallel to direction vector $\vec{u} = [2,-3,-4]$

Intervals: A set consist of all the real numbers between two Points is called an interval.

Inequality $a \le x \le b$ is called closed interval in which both a & b are included and denoted by $[a,b]$

Inequality $a < x < b$ is called open interval in which both a & b are not included & denoted by (a,b)

Inequality $a \le x < b$ is called half open and half closed in which a is included & b is not, denoted by $[a,b)$

Similarly $a < x \le b$ is also half open and half closed & denoted by $(a,b]$

Note: The set of all real number x such that $x \ge -2$

Its Interval notation is $[-2,\infty)$

Explanation of phrases $x \to 0$, $x \to a$, $x \to \infty$

1) $x \to 0$ (x approches zero)

Suppose a variable x assumes values as

$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots\ldots$

we notice that x is becoming smaller and smaller as no: of terms increases. This unending decrease of x is symbolically written as $x \to 0$ and is read as "x approches zero" or "x tends to zero"

Note: The symbol $x \to 0$ is quite different from $x = 0$

(1) $x \to 0$ means x is very very close to zero but not actually zero

(2) $x = 0$ means that x is actually zero

(ii) $x \to a$ (x approches a)

$x \to a$ means x is very very close to a but different from a, from both the left and right hand side of a. That is, $x-a$ becomes smaller and smaller as we please but $x-a \neq 0$

(iii) $x \to \infty$ (x approches infinity)

Suppose a variable x assumes values as

$1, 10, 100, 1000, 10000, \ldots \ldots$

It is clear that x is becoming larger and larger as no: of terms increases. This unending increase of x is symbolically written as $x \to \infty$ and is read as "x approches infinity" or "x tends to infinity"

Note ① Constant Rule $\lim_{x \to a} c = c$ i.e. $\lim_{x \to 2} 5 = 5$

② Limit Rule $\lim_{x \to a} f(x) = f(a)$ i.e. $\lim_{x \to 1} (x+5) = 1+5 = 6$

③ Multiple Rule $\lim_{x \to a} k\, f(x) = k \lim_{x \to a} f(x)$

Note If $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{0}{0}$ or $\frac{\infty}{\infty}$ then never use Limit Rule directly, first simplify then use limit Rule.

(27

Note: Limit of a function exist if left hand Limit is equal to Right hand limit.

Limit of a Sequence:

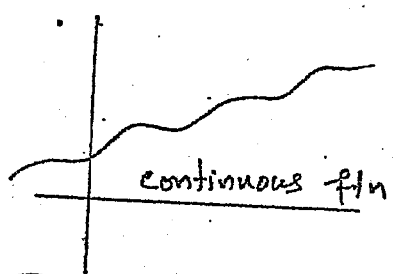
Let $\{a_n\}$ be a sequence, To find limit of sequence find $\lim\limits_{n\to\infty} a_n$, ...

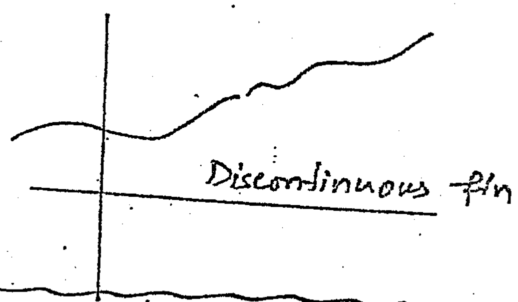
Note ① $\lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1$ ② $\lim\limits_{x\to\infty} \left(1+\dfrac{1}{x}\right)^x = e$

③ $\lim\limits_{x\to 0} (1+x)^{1/x} = e$ ④ $\lim\limits_{x\to\infty} (1+x)^{1/x} = 1$ or $\lim\limits_{n\to} x^{1/x} = 1$

Continuous & Discontinuous function

A continuous function has no "Jumps" or "holes" in the graph. For example the polynomial functions and some trigonometric functions like $\sin x$, $\cos x$ are continuous on the whole real set R, while $\tan x$ is continuous only in $(-\frac{\pi}{2}, \frac{\pi}{2})$.





Definition: A function $f(x)$ is continuous at a number "c" iff the following three conditions are satisfied.

i) $f(c)$ defined

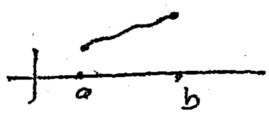
ii) $\lim\limits_{x\to c} f(x)$ exist

iii) $\lim\limits_{x\to c} f(x) = f(c)$

If one of the condition is fail then $f(x)$ is said to be discontinuous at "c"

Note ① In one-sided limit $f(x)$ is continuos at (28
$x=c$ if $\lim_{x\to c^+} f(x) = \lim_{x\to c^-} f(x) = f(c)$

② <u>One sided continuity</u>: If a & b are ends Points of the interval.



(i) If $\lim_{x\to a^+} f(x) = f(a)$ then $f(x)$ is continuous from right at a

(ii) If $\lim_{x\to b^-} f(x) = f(b)$ then $f(x)$ is continuous from left at b

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Properties of some specific functions

(i) A constant function $f(x)=k$, is always Continuous

(ii) A Polynomial function is always continuous $\forall x$.

(iii) A rational function is continuous $\forall x$ except those values that make a denominator 0.

(iv) $\sqrt[n]{f(x)}$ is continuous whenever $f(x)$ is continuous if n is an odd positive integer greater than 1

For example $\sqrt[3]{x^2}$ is cts $\forall x$

v) $\sqrt[n]{f(x)}$ is continuous whenever $f(x)$ is continuous & non-negative if n is an even positive integer.

For example $\sqrt[4]{x}$ is continuous on the interval $[0,\infty)$

<u>EXERCISE 1.4</u>

1) Evaluate the following limits

a) $\lim_{x\to -2} (x^2+3x-7) = (-2)^2+3(-2)-7$ (By using limit rule)
$= 4-6-7 = 4-13 = -9$

b) $\lim_{x\to 3}(x+5)(2x-7)$ Applying Limit Rule

$= (3+5)(2(3)-7)$

$= 8(6-7)$

$= 8(-1)$

$= -8$ Ans

c) $\lim_{z\to 1}\frac{z^2+z-3}{z+1} = \frac{1^2+1-3}{1+1}$

$= \frac{2-3}{2}$

$= \frac{-1}{2}$ Ans

d) $\lim_{x\to 4}\left(\frac{1}{x}+\frac{3}{x-5}\right)$ Applying Limit Rule

$= \frac{1}{4}+\frac{3}{4-5}$

$= \frac{1}{4}+\frac{3}{-1} = \frac{1}{4}-3 = \frac{-11}{4}$

e) $\lim_{x\to 1}\left(\frac{x^2+3x+2}{x^2+x+2}\right)^2$

$= \left[\lim_{x\to 1}\frac{x^2+3x+2}{x^2+x+2}\right]^2$ Applying Limit Rule

$= \left(\frac{1^2+3(1)+2}{1^2+1+2}\right)^2 = \left(\frac{6/3}{4/2}\right)^2 = \frac{9}{4}$

f) $\lim_{x\to 1}\frac{\sqrt{x}-1}{x-1}$ $\left(\frac{0}{0}\right)$ M&D

$= \lim_{x\to 1}\frac{\sqrt{x}-1}{x-1}\times\frac{\sqrt{x}+1}{\sqrt{x}+1}$

$= \lim_{x\to 1}\frac{(\sqrt{x})^2-1^2}{(x-1)(\sqrt{x}+1)}$

$= \lim_{x\to 1}\cancel{\frac{x-1}{(x-1)(\sqrt{x}+1)}}$

$= \lim_{x\to 1}\frac{1}{\sqrt{x}+1}$

$= \frac{1}{\sqrt{1}+1} = \frac{1}{1+1} = \frac{1}{2}$ Applying Limit Rule

g) $\lim_{x\to 0}\frac{\frac{1}{x+3}-\frac{1}{3}}{x}$ $\left(\frac{0}{0}\right)$

$= \lim_{x\to 0}\frac{\frac{3-(x+3)}{(x+3)\cdot 3}}{x}$

$= \lim_{x\to 0}\frac{\cancel{3}-x-\cancel{3}}{3(x+3)}\times\frac{1}{x}$

$= \lim_{x\to 0}\frac{-\cancel{x}}{3(x+3)\cancel{x}}$

$= \lim_{x\to 0}\frac{-1}{3(x+3)}$ Applying Limit Rule

$= \frac{-1}{3(0+3)} = \frac{-1}{9}$

h) $\lim_{x\to 1}\frac{\frac{1}{x}-1}{x-1}$ $\left(\frac{0}{0}\right)$

$= \lim_{x\to 1}\frac{\frac{1-x}{x}}{x-1} = \lim_{x\to 1}\frac{1-x}{x}\times\frac{1}{x-1} = \lim_{x\to 1}\frac{-\cancel{(x-1)}}{x\cancel{(x-1)}} = \lim_{x\to 1}\frac{-1}{x}$

$= -\frac{1}{1} = -1$ Ans

i) $\lim_{x\to 0} \dfrac{1-\sin x}{\cos^2 x}$

$= \dfrac{1-\sin(0)}{\cos^2 0} = \dfrac{1-0}{1}$

$= 1$ Ans

k) $\lim_{x\to 0} \dfrac{\sec x - 1}{x \sec x}$

$= \lim_{x\to 0} \dfrac{\frac{1}{\cos x} - 1}{x \cdot \frac{1}{\cos x}}$

$= \lim_{x\to 0} \dfrac{\frac{1-\cos x}{\cancel{\cos x}}}{\frac{x}{\cancel{\cos x}}}$

$= \lim_{x\to 0} \dfrac{1-\cos x}{x}$ $\left(\frac{0}{0}\right)$ MFD

$= \lim_{x\to 0} \dfrac{1-\cos x}{x} \times \dfrac{1+\cos x}{1+\cos x}$

$= \lim_{x\to 0} \dfrac{1^2-\cos^2 x}{x(1+\cos x)}$

$= \lim_{x\to 0} \dfrac{\sin^2 x}{x(1+\cos x)}$ $\left(\frac{0}{0}\right)$

$= \lim_{x\to 0} \dfrac{\sin x \cdot \sin x}{x(1+\cos x)}$

$= \lim_{x\to 0} \dfrac{\sin x}{x} \times \lim_{x\to 0} \dfrac{\sin x}{1+\cos x}$

$= 1 \times \dfrac{\sin 0}{1+\cos 0}$

$= 1 \times \dfrac{0}{1+1} = 0$

j) $\lim_{x\to 0} \dfrac{\tan x}{x}$ $\left(\frac{0}{0}\right)$

$= \lim_{x\to 0} \dfrac{\frac{\sin x}{\cos x}}{x} = \lim_{x\to 0} \dfrac{\sin x}{\cos x} \times \dfrac{1}{x}$

$= \lim_{x\to 0} \dfrac{\sin x}{x \cos x}$ $\left(\frac{0}{0}\right)$

Note

$\lim_{x\to 0} \dfrac{\sin x}{x} = 1$

$= \lim_{x\to 0} \dfrac{\sin x}{x} \times \lim_{x\to 0} \dfrac{1}{\cos x}$

$= 1 \times \dfrac{1}{\cos 0} = 1 \times \dfrac{1}{1} = 1$ Ans

l) $\lim_{x\to 0} \dfrac{\tan^2 x}{x}$ $\left(\frac{0}{0}\right)$

$= \lim_{x\to 0} \dfrac{\frac{\sin^2 x}{\cos^2 x}}{x}$

$= \lim_{x\to 0} \dfrac{\sin^2 x}{\cos^2 x} \times \dfrac{1}{x}$

$= \lim_{x\to 0} \dfrac{\sin^2 x}{x \cos^2 x}$ $\left(\frac{0}{0}\right)$

$= \lim_{x\to 0} \dfrac{\sin x \cdot \sin x}{x \cos^2 x}$

$= \lim_{x\to 0} \dfrac{\sin x}{x} \times \lim_{x\to 0} \dfrac{\sin x}{\cos^2 x}$

$= 1 \times \dfrac{\sin 0}{\cos^2 0}$

$= 1 \times \dfrac{0}{1}$

$= 0$ Ans

2) Use algebra and the rules of limits to evaluate the following limits.

a) $\lim_{x\to 4} \frac{-6}{(x-4)^2}$

Applying Limit Rule

$= \frac{-6}{(4-4)^2}$

$= \frac{-6}{0} = \infty$

b) Already done

See Part (g) Q 1

c) $\lim_{x\to 5} \frac{\sqrt{x}-\sqrt{5}}{x-5}$ $\left(\frac{0}{0}\right)$ M&D

$= \lim_{x\to 5} \frac{\sqrt{x}-\sqrt{5}}{x-5} \times \frac{\sqrt{x}+\sqrt{5}}{\sqrt{x}+\sqrt{5}}$

$= \lim_{x\to 5} \frac{(\sqrt{x})^2-(\sqrt{5})^2}{(x-5)(\sqrt{x}+\sqrt{5})}$

$= \lim_{x\to 5} \frac{\cancel{x-5}}{\cancel{(x-5)}(\sqrt{x}+\sqrt{5})}$

$= \lim_{x\to 5} \frac{1}{\sqrt{x}+\sqrt{5}}$ Applying Limit Rule

$= \frac{1}{\sqrt{5}+\sqrt{5}} = \frac{1}{2\sqrt{5}}$

d) $\lim_{x\to 25} \frac{\sqrt{x}-5}{x-25}$ $\left(\frac{0}{0}\right)$

$= \lim_{x\to 25} \frac{\sqrt{x}-5}{x-25} \times \frac{\sqrt{x}+5}{\sqrt{x}+5}$ M&D

$= \lim_{x\to 25} \frac{(\sqrt{x})^2-5^2}{(x-25)(\sqrt{x}+5)}$

$= \lim_{x\to 25} \frac{\cancel{x-25}}{\cancel{(x-25)}(\sqrt{x}+5)}$

$= \lim_{x\to 25} \frac{1}{\sqrt{x}+5}$ Applying Limit Rule

$= \frac{1}{\sqrt{25}+5}$

$= \frac{1}{5+5} = \frac{1}{10}$

Note: If limit of a sequence exist then it is called convergent otherwise it is called divergent

3) Find the limit of the convergent of the sequences

a) $\left\{\frac{5n}{n+7}\right\}$

Let $\lim_{n\to\infty} \left\{\frac{5n}{n+7}\right\} = \lim_{n\to\infty} \frac{5n}{n+7}$ $\left(\frac{\infty}{\infty}\right)$

$= \lim_{n\to\infty} \frac{5\cancel{n}}{\cancel{n}(1+\frac{7}{n})}$

Applying limit Rule

$= \frac{5}{1+\frac{7}{\infty}} = \frac{5}{1+0} = 5$

Ans

b) $\left\{\frac{4-7n}{8+n}\right\}$

Let $\lim_{n\to\infty} \left\{\frac{4-7n}{8+n}\right\} = \lim_{n\to\infty} \frac{4-7n}{8+n}$ $\left(\frac{\infty}{\infty}\right)$

$= \lim_{n\to\infty} \frac{\cancel{n}(\frac{4}{n}-7)}{\cancel{n}(\frac{8}{n}+1)}$

Applying limit Rule

$= \frac{\frac{4}{\infty}-7}{\frac{8}{\infty}+1} = \frac{0-7}{0+1} = -7$

Ans

c) $\lim_{n\to\infty}\left\{\frac{8n-500\sqrt{n}}{2n+800\sqrt{n}}\right\}$

$= \lim_{n\to\infty} \frac{8n-500\sqrt{n}}{2n+800\sqrt{n}} \quad \left(\frac{\infty}{\infty}\right)$

$= \lim_{n\to\infty} \frac{\cancel{n}\left(8-\frac{500\sqrt{n}}{n}\right)}{\cancel{n}\left(2+\frac{800\sqrt{n}}{n}\right)}$

$= \lim_{n\to\infty} \frac{8-\frac{500\cancel{\sqrt{n}}}{\sqrt{n}\cancel{\sqrt{n}}}}{2+\frac{800\cancel{\sqrt{n}}}{\sqrt{n}\cancel{\sqrt{n}}}}$ $\quad \because n=\sqrt{n}\sqrt{n}$

Applying limit Rule

$= \frac{8-\frac{500}{\sqrt{\infty}}}{2+\frac{800}{\sqrt{\infty}}} = \frac{8-0}{2+0} = 4$

Note ① $\lim_{n\to\infty} n^{1/n} = 1$

② $\frac{1}{\infty}=0$ or $\frac{1}{0}=\infty$

d) $\{n^{3/n}\}$

sol

$\lim_{n\to\infty}\{n^{3/n}\} \quad (\infty^0)$

$= \lim_{n\to\infty} n^{3/n}$

$= \lim_{n\to\infty} (n^{1/n})^3$

$= \left[\lim_{n\to\infty} n^{1/n}\right]^3 = 1^3 = 1$

4) $S(x) = 5000 + \frac{3600}{x+2}$ ——①

a) $S(2) = ?$

put $x=2$ in ①

$S(2) = 5000 + \frac{3600}{2+2}$

$= 5000 + \frac{\overset{900}{\cancel{3600}}}{\cancel{4}}$

$S(2) = 5900$

Sales in Rupees is Rs 5900

b) $\lim_{x\to 5} S(x)$

$= \lim_{x\to 5}\left(5000 + \frac{3600}{x+2}\right)$

Applying limit Rule

$= 5000 + \frac{3600}{5+2}$

$= 5000 + \frac{3600}{7} = 5000 + 514.29$

$= 5514.29$

Sales in Rupees is Rs 5514.29

c) $\lim_{x\to 16} S(x) = \lim_{x\to 16}\left(5000 + \frac{3600}{x+2}\right)$

Applying limit Rule

$= 5000 + \frac{3600}{16+2} = 5000 + \frac{\overset{200}{\cancel{3600}}}{\cancel{18}}$

$= 5000 + 200$

$= 5200$

Sales in Rupees is Rs 5200

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5) Use Properties of continuous function to test (33
the contiuity and discontinuity of the following f-ns.

a) $f(x) = 2x-3$

It is continuous $\forall x$ because it is a Polynomial function.

b) $g(x) = 3-5x$

It is continuous $\forall x$ because it is a Polynomial function.

c) $h(x) = \frac{2}{x-5}$

It is continious $\forall x$ exlcept $x=5$ (value that make denominator zero) because $h(x)$ is a Rational function

d) $k(x) = \frac{x}{x+3}$

since it is a Rational function so it is Continuous $\forall x$ except $x=-3$ (value that make denominator zero)

e) $g(x) = \frac{x-5}{(x-3)(x+2)}$

Since it is a Rational function so it is continuous $\forall x$ except $x=3, x=-2$ (values that make denominator zero)

f) $F(x) = \frac{1}{x(x+7)}$

since it is a Rational function so it is continuous $\forall x$ except $x=0, x=-7$ (values that make denominator zero)

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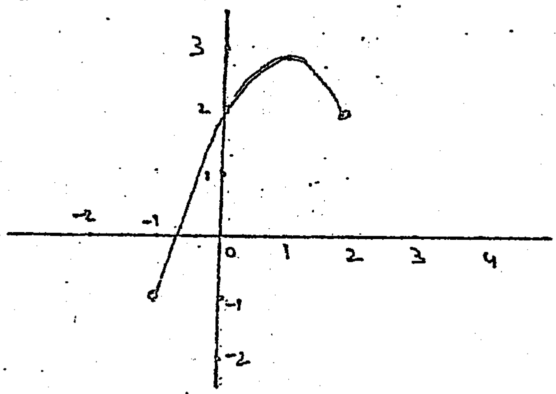
7) $g(x) = -x^2+2x+2$

a) Is $g(x)$ continuous on the open interval $(-1,2)$



Sol Yes $g(x)$ is continuous between -1 & 2 because there is no jump or hole in the graph.

b) Is $g(x)$ continuous from the right at $x=-1$?

Is $\lim_{x\to -1^+} g(x) = g(-1)$

Sol Yes it is continuous from right at $x=-1$

$\because g(x) = -x^2+2x+2$

put $x=-1$

$g(-1) = -(-1)^2+2(-1)+2$

$= -1 \cancel{-2} \cancel{+2} = -1 \Rightarrow \boxed{g(-1)=-1}$

from graph when x close to -1 from right then $g(x)$ closed to -1, $\therefore \boxed{\lim_{x\to -1^+} g(x) = -1}$

clearly $\lim_{x\to -1^+} g(x) = g(-1)$

c) yes it is continuous from left at $x=2$,

$\because g(x) = -x^2+2x+2$

put $x=2$

$g(2) = -(2)^2+2(2)+2 = \cancel{-4}\cancel{+4}+2 = 2$

$\Rightarrow \boxed{g(2)=2}$

from graph when x close to 2 from left then $g(x)$ close to 2 $\therefore \lim_{x\to 2^-} g(x) = 2$

Hence $\lim_{x\to 2^-} g(x) = g(2)$

d) Is $g(x)$ is cts on the closed interval $[-1,2]$

Sol Yes, because it is continuous between -1 & 2, it is continuous from right at $x=-1$ & it is continuous from left at $x=2$. Hence $g(x)$ is continuous everywhere in $[-1,2]$

END OF CH # 1