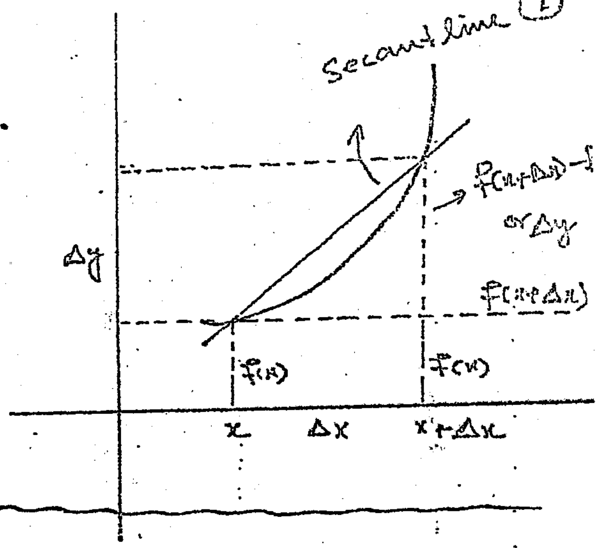
Unit # 2 Differentiation

slope of secant line
or
average rate of change of y per unit change in x is given by

$$\frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x}$$



Exercise 2.1

Find the average rate of change of the following functions over the indicated intervals

a) $y = x^2+4$ from $x=2$ to $x=3$

Sol Let $f(x) = x^2+4$

let $x = 2$
& $\Delta x = 3-2 = 1$

Replace x by $x+\Delta x$

$$f(x+\Delta x) = (x+\Delta x)^2+4$$
$$= x^2+\Delta x^2+2x\Delta x+4$$

$\because$ we know that

$$\text{Average rate of change} = \frac{f(x+\Delta x) - f(x)}{\Delta x}$$
$$= \frac{x^2+\Delta x^2+2x\Delta x+4-(x^2+4)}{\Delta x}$$
$$= \frac{\cancel{x^2}+\Delta x^2+2x\Delta x+\cancel{4}-\cancel{x^2}-\cancel{4}}{\Delta x}$$
$$= \frac{\cancel{\Delta x}(\Delta x+2x)}{\cancel{\Delta x}}$$
$$= \Delta x+2x$$
$$= 1+2(2) \qquad \because x=2, \Delta x = 1$$
$$= 1+4 = 5 \quad \text{Ans}$$

b) $y = x^2 + \frac{1}{3}x$ from $x=-3$ to $x=3$ (2)

Sol Let $f(x) = x^2 + \frac{1}{3}x$ Let $x = -3$

Replace x by Δx so $\Delta x = 3-(-3) = 3+3 = 6$

$f(x+\Delta x) = (x+\Delta x)^2 + \frac{1}{3}(x+\Delta x)$

$= x^2 + \Delta^2 x + 2x\Delta x + \frac{1}{3}x + \frac{1}{3}\Delta x$

$\because$ we know that

Average rate of change $= \dfrac{f(x+\Delta x) - f(x)}{\Delta x}$

$$= \frac{x^2 + \Delta^2 x + 2x\Delta x + \frac{1}{3}x + \frac{1}{3}\Delta x - (x^2 + \frac{1}{3}x)}{\Delta x}$$

$$= \frac{\cancel{x^2} + \Delta^2 x + 2x\Delta x + \cancel{\frac{1}{3}x} + \frac{1}{3}\Delta x - \cancel{x^2} - \cancel{\frac{1}{3}x}}{\Delta x}$$

$$= \frac{\cancel{\Delta x}(\Delta x + 2x + \frac{1}{3})}{\cancel{\Delta x}}$$

$= \Delta x + 2x + \frac{1}{3}$

$= 6 + 2(-3) + \frac{1}{3}$ $\because \Delta x = 6,\ x = -3$

$= \cancel{6} - \cancel{6} + \frac{1}{3} = \frac{1}{3}$ Ans

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c) $S = 2t^3 - 5t + 7$ from $t=1$ to $t=3$

Sol Let $f(t) = 2t^3 - 5t + 7$ Let $t = 1$

Replace t by $t+\Delta t$ Now $\Delta t = 3-1 = 2$

$f(t+\Delta t) = 2(t+\Delta t)^3 - 5(t+\Delta t) + 7$ $\left((a+b)^3 = a^3 + b^3 + 3a^2b + 3ab^2\right)$

$= 2[t^3 + \Delta^3 t + 3t^2\Delta t + 3t\Delta^2 t] - 5(t+\Delta t) + 7$

$= 2t^3 + 2\Delta^3 t + 6t^2\Delta t + 6t\Delta^2 t - 5t - 5\Delta t + 7$

$\because$ we know that

Average rate of change $= \dfrac{f(t+\Delta t) - f(t)}{\Delta t}$

$$= \frac{2t^3 + 2\Delta^3 t + 6t^2\Delta t + 6t\Delta^2 t - 5t - 5\Delta t + 7 - (2t^3 - 5t + 7)}{\Delta t}$$

$$= \frac{\cancel{2t^3} + 2\Delta^3 t + 6t^2\Delta t + 6t\Delta^2 t - \cancel{5t} - 5\Delta t + \cancel{7} - \cancel{2t^3} + \cancel{5t} - \cancel{7}}{\Delta t}$$

Average Rate of change $= \dfrac{\cancel{\Delta t}\left[2\Delta^2 t + 6t^2 + 6t\Delta t - 5\right]}{\cancel{\Delta t}}$

$= 2\Delta^2 t + 6t^2 + 6t\Delta t - 5$

$= 2(2)^2 + 6(1)^2 + 6(1)(2) - 5$ $\because t = 1,\ \Delta t = 2$

$= 8 + 6 + 12 - 5 = 21$

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d) $h = \sqrt{2t} - 7$ from $t = 8$ to $t = 8.5$

Sol Let $f(t) = \sqrt{2t} - 7$ let $t = 8$, so $\Delta t = 8.5 - 8 = 0.5$

Replace t by $t + \Delta t$

$f(t+\Delta t) = \sqrt{2(t+\Delta t)} - 7$

$= \sqrt{2t + 2\Delta t} - 7$

$\because$ we know that

Average Rate of change $= \dfrac{f(t+\Delta t) - f(t)}{\Delta t}$

$= \dfrac{\sqrt{2t+2\Delta t} - 7 - \{\sqrt{2t} - 7\}}{\Delta t}$

$= \dfrac{\sqrt{2t+2\Delta t} - \cancel{7} - \sqrt{2t} + \cancel{7}}{\Delta t}$

$= \dfrac{\sqrt{2t+2\Delta t} - \sqrt{2t}}{\Delta t}$

$= \dfrac{\sqrt{2(8)+2(.5)} - \sqrt{2\times 8}}{0.5}$ $\because t = 8,\ \Delta t = 0.5$

$= \dfrac{\sqrt{16+1} - \sqrt{16}}{0.5} = \dfrac{\sqrt{17} - 4}{0.5} = \dfrac{4.12 - 4}{0.5}$

$= \dfrac{0.12}{0.5} = 0.24$

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2) Use definition to find out the average rate of change over specified interval for the following functions:

a) $s = 2t - 3$ from $t = 2$ to $t = 5$

Sol Let $f(t) = 2t - 3$ let $t = 2$

Replace t by $t + \Delta t$ so $\Delta t = 5 - 2 = 3$

$f(t+\Delta t) = 2.(t+\Delta t) - 3$

$= 2t + 2\Delta t - 3$

$\because$ we know that

Average Rate of change $= \dfrac{f(t+\Delta t) - f(t)}{\Delta t}$

$= \dfrac{2t+2\Delta t - 3 - (2t-3)}{\Delta t}$

$= \dfrac{\cancel{2t}+2\Delta t - \cancel{3} - \cancel{2t} + \cancel{3}}{\Delta t} = \dfrac{2\cancel{\Delta t}}{\cancel{\Delta t}}$

$= 2$

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b) $y = x^2 - 6x + 8$ from $x = 3$ to $x = 3.1$

Sol let $f(x) = x^2 - 6x + 8$ let $\boxed{x = 3}$

Replace x by $x + \Delta x$ $\Delta x = 3.1 - 3 = .1$

$f(x+\Delta x) = (x+\Delta x)^2 - 6(x+\Delta x) + 8$

$= x^2 + \Delta x^2 + 2x\Delta x - 6x - 6\Delta x + 8$

$\because$ we know that

Average Rate of change $= \dfrac{f(x+\Delta x) - f(x)}{\Delta x}$

$= \dfrac{x^2 + \Delta x^2 + 2x\Delta x - 6x - 6\Delta x + 8 - (x^2 - 6x + 8)}{\Delta x}$

$= \dfrac{\cancel{x^2} + \Delta x^2 + 2x\Delta x - \cancel{6x} - 6\Delta x + \cancel{8} - \cancel{x^2} + \cancel{6x} - \cancel{8}}{\Delta x}$

$= \dfrac{\cancel{\Delta x}(\Delta x + 2x - 6)}{\cancel{\Delta x}}$

$= \Delta x + 2x - 6$ ✓ $\because x = 3$ & $\Delta x = .1$

$= .1 + 2(3) - 6$ ✓

$= .1 + \cancel{6} - \cancel{6} = .1$ Ans

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c) $A = \pi r^2$ from $r = 2$ to $r = 2.1$

Sol let $f(r) = \pi r^2$ let $r = 2$

Replace r by $r + \Delta r$ $\Delta r = 2.1 - 2 = .1$

$f(r+\Delta r) = \pi(r+\Delta r)^2$

$= \pi(r^2 + \Delta^2 r + 2r\Delta r) = \pi r^2 + \pi\Delta r^2 + 2\pi r\Delta r$

$\because$ we know that

Average rate of change $= \dfrac{f(r+\Delta r) - f(r)}{\Delta r}$

$= \dfrac{\cancel{\pi r^2} + \pi \Delta r^2 + 2\pi r \Delta r - \cancel{\pi r^2}}{\Delta r}$

$= \dfrac{\cancel{\Delta r}(\pi \Delta r + 2\pi r)}{\cancel{\Delta r}}$

$= \pi \Delta r + 2\pi r$

$= \pi(\Delta r + 2r)$

$= \pi(.1 + 2(2))$ $\quad \because r = 2, \Delta r = .1$

$= \pi(.1 + 4) = 4.1\pi$

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d) $h = \sqrt{t} - 9$ from $t = 9$ to $t = 16$

Sol let $f(t) = \sqrt{t} - 9$ let $t = 9$ & $\Delta t = 16 - 9 = 7$

Replace t by $t + \Delta t$

$f(t+\Delta t) = \sqrt{t+\Delta t} - 9$

$\because$ we know that

Average rate of change $= \dfrac{f(t+\Delta t) - f(t)}{\Delta t}$

$= \dfrac{\sqrt{t+\Delta t} - 9 - (\sqrt{t} - 9)}{\Delta t}$

$= \dfrac{\sqrt{t+\Delta t} - \cancel{9} - \sqrt{t} + \cancel{9}}{\Delta t}$

$= \dfrac{\sqrt{t+\Delta t} - \sqrt{t}}{\Delta t}$ $\quad \because t = 9, \Delta t = 7$

$= \dfrac{\sqrt{9+7} - \sqrt{9}}{7}$

$= \dfrac{\sqrt{16} - \sqrt{9}}{7} = \dfrac{4-3}{7} = 1/7$ Ans

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Note: (i) Average rate of change of $f(x) = \dfrac{f(x+\Delta x) - f(x)}{\Delta x} = \dfrac{\Delta f}{\Delta x}$

(2) Average rate of change of $h(t) = \dfrac{h(t+\Delta t) - h(t)}{\Delta t} = \dfrac{\Delta h}{\Delta t}$

3) A ball is thrown straight up. Its height after t seconds is given by the formula $h = -16t^2 + 80t$. Find the average velocity $\frac{\Delta h}{\Delta t}$ for the specified interval a) From $t=2$ to $t=2.1$ b) From $t=2$ to $t=2.01$ ⑥

Sol: Let $h(t) = -16t^2 + 80t$

Replace t by $t+\Delta t$

$$h(t+\Delta t) = -16(t+\Delta t)^2 + 80(t+\Delta t)$$
$$= -16(t^2 + \Delta t^2 + 2t\Delta t) + 80t + 80\Delta t$$
$$= -16t^2 - 16\Delta^2 t - 32t\Delta t + 80t + 80\Delta t$$

we know that average velocity is

$$\frac{\Delta h}{\Delta t} = \frac{h(t+\Delta t) - h(t)}{\Delta t}$$
$$= \frac{-16t^2 - 16\Delta_t^2 - 32t\Delta t + 80t + 80\Delta t - (-16t^2 + 80t)}{\Delta t}$$
$$= \frac{\cancel{-16t^2} - 16\Delta_t^2 - 32t\Delta_t + \cancel{80t} + 80\Delta t + \cancel{16t^2} - \cancel{80t}}{\Delta t}$$
$$= \frac{\cancel{\Delta t}(-16\Delta_t - 32t + 80)}{\cancel{\Delta t}}$$
$$\frac{\Delta h}{\Delta t} = -16\Delta_t - 32t + 80 \quad (1)$$

a) let $t=2$ & $\Delta t = 2.1 - 2 = .1$

put in RHS of (1)

$$\frac{\Delta h}{\Delta t} = -16(.1) - 32(2) + 80$$
$$= -1.6 - 64 + 80$$
$$= 14.4$$

b) Let $t = 2$, $\Delta t = 2.01 - 2 = .01$

put in RHS of (1)

$$\frac{\Delta h}{\Delta t} = -16(.01) - 32(2) + 80$$
$$= -.16 - 64 + 80$$
$$= 15.84$$

4) The rate of change of Price is called inflation. The Price P in rupees after t years is $P(t) = 3t^2 + t + 1$. Find the average rate of change of inflation from $t=3$ to $t=5$ years. What the rate of change of means? Explain

Sol Let $P(t) = 3t^2 + t + 1$ — Let $t = 3$ & $\Delta t = 5 - 3 = 2$

Replace t by $t + \Delta t$

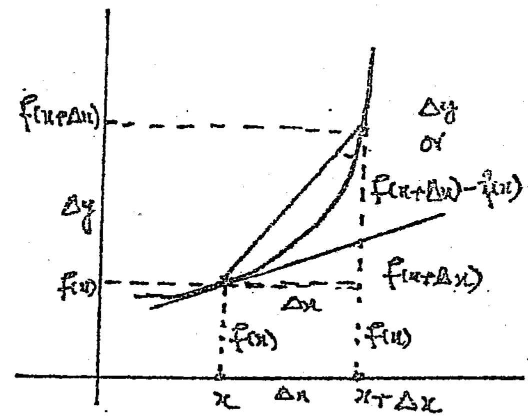
$$P(t+\Delta t) = 3(t+\Delta t)^2 + (t+\Delta t) + 1$$
$$= 3(t^2 + \Delta t^2 + 2t\Delta t) + t + \Delta t + 1$$
$$= 3t^2 + 3\Delta^2 t + 6t\Delta t + t + \Delta t + 1$$

$$\text{Average rate of change} = \frac{P(t+\Delta t) - P(t)}{\Delta t}$$
$$= \frac{3t^2 + 3\Delta^2 t + 6t\Delta t + t + \Delta t + 1 - (3t^2 + t + 1)}{\Delta t}$$
$$= \frac{\cancel{3t^2} + 3\Delta^2 t + 6t\Delta t + \cancel{t} + \Delta t + \cancel{1} - \cancel{3t^2} - \cancel{t} - \cancel{1}}{\Delta t}$$
$$= \frac{\cancel{\Delta t}(3\Delta t + 6t + 1)}{\cancel{\Delta t}}$$
$$= 3\Delta t + 6t + 1$$
$$= 3(2) + 6(3) + 1 \qquad \because t = 3, \Delta t = 2$$
$$= 6 + 18 + 1 = 25$$

The average rate of inflation is Rs 25 Per year

Derivative: slope of secant line or average rate of change of y per unit change in x is

$$\frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x}$$



Now slope of tangent line or Instantaneous rate of change is

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}, \text{ if limit exist}$$

slope of tangent line or Instantaneous rate of change of y w.r.t. x which we call the derivative of the function & denoted by $f'(x)$ or $\frac{dy}{dx}$, that is,

$$f'(x) \text{ or } \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}, \quad y = f(x)$$

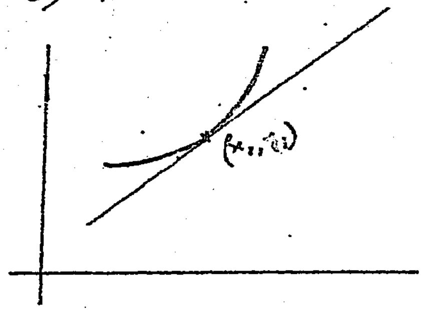
First Principle Rule: Differential coefficient or derivative of $f(x)$ by first Principle rule is $f'(x) = \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$

Note ① $f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$, if this limit does not exist, then there is no tangent (or no derivative) at the Point.

② Eq: of tangent line at (x_1, y_1)

$$y - y_1 = f'(x)(x - x_1)$$

where $f'(x)$ = slope of tangent line



EXERCISE 2.2

1) use first Principle Rule to determine the derivative of the following functions:

a) $f(x) = 3x$

Replace x by $x+\Delta x$

$f(x+\Delta x) = 3(x+\Delta x) = 3x + 3\Delta x$

Now $f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$

$= \lim_{\Delta x \to 0} \frac{\cancel{3x} + 3\Delta x - \cancel{3x}}{\Delta x}$

$= \lim_{\Delta x \to 0} 3\frac{\cancel{\Delta x}}{\cancel{\Delta x}} = \lim_{\Delta x \to 0}(3) = 3$ Ans

b) $f(x) = 5x + 6$

Replace x by $x + \Delta x$

$f(x+\Delta x) = 5(x+\Delta x) + 6$

$= 5x + 5\Delta x + 6$

Now $f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$

$= \lim_{\Delta x \to 0} \frac{5x + 5\Delta x + 6 - (5x+6)}{\Delta x}$

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$$f'(x) = \lim_{\Delta x \to 0} \frac{\cancel{5x} + 5\Delta x + \cancel{4} - \cancel{5x} - \cancel{4}}{\Delta x} = \lim_{\Delta x \to 0} \frac{5\cancel{\Delta x}}{\cancel{\Delta x}} = 5 \quad \text{Ans}$$

c) $f(x) = x^2 + 1$

Replace x by $x + \Delta x$

$$f(x+\Delta x) = (x+\Delta x)^2 + 1$$
$$= x^2 + \Delta^2 x + 2x\Delta x + 1$$

Now

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{x^2 + \Delta^2 x + 2x\Delta x + 1 - (x^2+1)}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{\cancel{x^2} + \Delta^2 x + 2x\Delta x + \cancel{1} - \cancel{x^2} - \cancel{1}}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{\cancel{\Delta x}(\Delta x + 2x)}{\cancel{\Delta x}}$$
$$= \lim_{\Delta x \to 0} (\Delta x + 2x) = 0 + 2x = 2x \quad \text{Ans}$$

d) $f(x) = 12 - x^2$

Replace x by $x + \Delta x$

$$f(x+\Delta x) = 12 - (x+\Delta x)^2$$
$$= 12 - (x^2 + \Delta x^2 + 2x\Delta x) = 12 - x^2 - \Delta x^2 - 2x\Delta x$$

Now

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{12 - x^2 - \Delta x^2 - 2x\Delta x - (12 - x^2)}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{\cancel{12} - \cancel{x^2} - \Delta^2 x - 2x\Delta x - \cancel{12} + \cancel{x^2}}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{\cancel{\Delta x}(-\Delta x - 2x)}{\cancel{\Delta x}}$$
$$= \lim_{\Delta x \to 0} (-\Delta x - 2x) = 0 - 2x = -2x \quad \text{Ans}$$

e) $f(x) = 16x^2 - 7x$

Replace x by $x+\Delta x$

$$f(x+\Delta x) = 16(x+\Delta x)^2 - 7(x+\Delta x)$$
$$= 16(x^2 + \Delta^2 x + 2x\Delta x) - 7(x+\Delta x)$$
$$= 16x^2 + 16\Delta^2 x + 32x\Delta x - 7x - 7\Delta x$$

Now

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{16x^2 + 16\Delta^2 x + 32x\Delta x - 7x - 7\Delta x - (16x^2 - 7x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\cancel{16x^2} + 16\Delta^2 x + 32x\Delta x - \cancel{7x} - 7\Delta x - \cancel{16x^2} + \cancel{7x}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\cancel{\Delta x}(16\Delta x + 32x - 7)}{\cancel{\Delta x}}$$

$$= \lim_{\Delta x \to 0} (16\Delta x + 32x - 7) = 16(0) + 32x - 7 = 32x - 7 \quad \text{Ans}$$

f) $f(x) = \frac{7}{x}$

Replace x by $x+\Delta x$

$$f(x+\Delta x) = \frac{7}{x+\Delta x}$$

Now

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\frac{7}{x+\Delta x} - \frac{7}{x}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\frac{7x - 7(x+\Delta x)}{(x+\Delta x)x}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\cancel{7x} - \cancel{7x} - 7\Delta x}{(x+\Delta x)x} \times \frac{1}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{-7\cancel{\Delta x}}{(x+\Delta x)x} \times \frac{1}{\cancel{\Delta x}}$$

$$= \frac{-7}{(x+0)x} = \frac{-7}{x^2} \quad \text{Ans}$$

g) $f(x) = \frac{3}{x+3}$

Replace x by $x+\Delta x$

$f(x+\Delta x) = \frac{3}{x+\Delta x+3}$

Now

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\frac{3}{x+\Delta x+3} - \frac{3}{x+3}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\frac{3(x+3) - 3(x+\Delta x+3)}{(x+\Delta x+3)(x+3)}}{\Delta x}$$

$$= \lim_{x \to 0} \frac{\cancel{3x}+\cancel{9} - \cancel{3x} - 3\Delta x - \cancel{9}}{(x+\Delta x+3)(x+3)} \times \frac{1}{\Delta x} = \lim_{\Delta x \to 0} \frac{-3\cancel{\Delta x}}{(x+\Delta x+3)(x+3)\cancel{\Delta x}}$$

$$= \lim_{\Delta x \to 0} \frac{-3}{(x+\Delta x+3)(x+3)} = \frac{-3}{(x+0+3)(x+3)} = \frac{-3}{(x+3)^2} \quad \text{Ans}$$

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h) Let $f(x) = \frac{5}{2x-4}$

Replace x by $x+\Delta x$

$$f(x+\Delta x) = \frac{5}{2(x+\Delta x)-4} = \frac{5}{2x+2\Delta x-4}$$

Now

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\frac{5}{2x+2\Delta x-4} - \frac{5}{2x-4}}{\Delta x} = \lim_{\Delta x \to 0} \frac{\frac{5(2x-4) - 5(2x+2\Delta x-4)}{(2x+2\Delta x-4)(2x-4)}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\cancel{10x} - \cancel{20} - \cancel{10x} - 10\Delta x + \cancel{20}}{(2x+2\Delta x-4)(2x-4)} \times \frac{1}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{-10\cancel{\Delta x}}{(2x+2\Delta x-4)(2x-4)} \times \frac{1}{\cancel{\Delta x}}$$

$$= \frac{-10}{(2x+0-4)(2x-4)} = \frac{-10}{(2x-4)^2} \quad \text{Ans}$$

i) $f(x) = 3x^2 + 4x - 9$

Replace x by $x+\Delta x$

$f(x+\Delta x) = 3(x+\Delta x)^2 + 4(x+\Delta x) - 9$

$= 3(x^2 + \Delta^2 x + 2x\Delta x) + 4(x+\Delta x) - 9$

$= 3x^2 + 3\Delta^2 x + 6x\Delta x + 4x + 4\Delta x - 9$

Now

$f'(x) = \lim\limits_{\Delta x \to 0} \dfrac{f(x+\Delta x) - f(x)}{\Delta x}$

$= \lim\limits_{\Delta x \to 0} \dfrac{3x^2 + 3\Delta^2 x + 6x\Delta x + 4x + 4\Delta x - 9 - (3x^2 + 4x - 9)}{\Delta x}$

$= \lim\limits_{\Delta x \to 0} \dfrac{\cancel{3x^2} + 3\Delta^2 x + 6x\Delta x + \cancel{4x} + 4\Delta x - \cancel{9} - \cancel{3x^2} - \cancel{4x} + \cancel{9}}{\Delta x}$

$= \lim\limits_{\Delta x \to 0} \dfrac{\cancel{\Delta x}(3\Delta x + 6x + 4)}{\cancel{\Delta x}}$

$= \lim\limits_{\Delta x \to 0} (3\Delta x + 6x + 4)$ Applying Limit Rule

$= 3(0) + 6x + 4 = 6x + 4$

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② Estimate the slope of the tangent line on a curve at a Point $P(x,y)$ for each of the following graphs:

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Equation of tangent line at Point $(\underset{x_1}{3},\underset{y_1}{12})$ is

$$y - y_1 = f'(x)(x - x_1)$$

$$\boxed{y - 12 = 1(x-3)}$$ Ans

$$y - 12 = x - 3$$

$$\Rightarrow \boxed{x - y + 9 = 0}$$ Ans

(b) Let $y = f(x) = 6x^2 - 11x - 10$, $x = 1$

put $x = 1$

$$y = 6(1)^2 - 11(1) - 10 = 6 - 11 - 10 = -15$$

Point of contact is $P(x,y) = P(1,-15)$

$\because$ $f(x) = 6x^2 - 11x - 10$

Replace x by $x + \Delta x$

$$f(x+\Delta x) = 6(x+\Delta x)^2 - 11(x+\Delta x) - 10$$

$$= 6(x^2 + \Delta^2 x + 2x\Delta x) - 11(x + \Delta x) - 10$$

$$= 6x^2 + 6\Delta^2 x + 12x\Delta x - 11x - 11\Delta x - 10$$

Now $f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$

$$= \lim_{\Delta x \to 0} \frac{6x^2 + 6\Delta^2 x + 12x\Delta x - 11x - 11\Delta x - 10 - (6x^2 - 11x - 10)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\cancel{6x^2} + 6\Delta^2 x + 12x\Delta x - \cancel{11x} - 11\Delta x - \cancel{10} - \cancel{6x^2} + \cancel{11x} + \cancel{10}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\cancel{\Delta x}(6\Delta x + 12x - 11)}{\cancel{\Delta x}}$$

$$= \lim_{\Delta x \to 0} (6\Delta x + 12x - 11) = 6(0) + 12x - 11$$

$$\boxed{f'(x) = 12x - 11}$$

At $x = 1$

$f'(1) = 12(1) - 11 = 1$, which is slope of tangent line at Point $(1,-15)$

$$\Rightarrow \boxed{f'(1) = 1}$$

3) slope of tangent line = ? Eq: of tangent line = ?

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a) Let

$y = f(x) = -x^2 + 7x$, $\boxed{x=3}$

Put $x = 3$

$y = -3^2 + 7(3) = -9 + 21$

$\boxed{y = 12}$

Point of contact is $(x, y) = (3, 12)$

$\because$ $f(x) = -x^2 + 7x$

Replace x by $x + \Delta x$

$f(x+\Delta x) = -(x+\Delta x)^2 + 7(x+\Delta x)$

$= -(x^2 + \Delta^2 x + 2x\Delta x) + 7x + 7\Delta x = -x^2 - \Delta x^2 - 2x\Delta x + 7x + 7\Delta x$

Now

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{-x^2 - \Delta^2 x - 2x\Delta x + 7x + 7\Delta x - (-x^2 + 7x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\cancel{-x^2} - \Delta^2 x - 2x\Delta x + \cancel{7x} + 7\Delta x + \cancel{x^2} - \cancel{7x}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\cancel{\Delta x}(-\Delta x - 2x + 7)}{\cancel{\Delta x}}$$

$$= \lim_{\Delta x \to 0} (-\Delta x - 2x + 7)$$

Applying limit rule

$= -0 - 2x + 7$

$\boxed{f'(x) = -2x + 7}$

At $x = 3$

$f'(3) = -2(3) + 7 = -6 + 7 = 1$, which is slope of tangent line at Point $(3, 12)$

$\boxed{f'(3) = 1}$

Eq: of tangent line at $(\underset{x_1}{1}, \underset{y_1}{-15})$ is

$$y - y_1 = f'(x)(x - x_1)$$

$$\boxed{y + 15 = 1(x - 1)} \quad \text{Ans}$$

$$y + 15 = x - 1$$

$$\Rightarrow \boxed{x - y - 16 = 0} \quad \text{Ans}$$

c) let $y = f(x) = 3x^2 - 6x - 10$, $\boxed{x = 0}$

put $x = 0$

$$y = 3(0) - 6(0) - 10 = -10$$

Point of contact is $(x, y) = (0, -10)$

$\because$ $f(x) = 3x^2 - 6x - 10$

Replace x by $x + \Delta x$

$$f(x+\Delta x) = 3(x+\Delta x)^2 - 6(x+\Delta x) - 10$$

$$= 3(x^2 + \Delta^2 x + 2x\Delta x) - 6(x + \Delta x) - 10$$

$$= 3x^2 + 3\Delta^2 x + 6x\Delta x - 6x - 6\Delta x - 10$$

Now

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{3x^2 + 3\Delta^2 x + 6x\Delta x - 6x - 6\Delta x - 10 - (3x^2 - 6x - 10)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\cancel{3x^2} + 3\Delta^2 x + 6x\Delta x - \cancel{6x} - 6\Delta x - \cancel{10} - \cancel{3x^2} + \cancel{6x} + \cancel{10}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\cancel{\Delta x}(3\Delta x + 6x - 6)}{\cancel{\Delta x}}$$

$$= \lim_{\Delta x \to 0} (3\Delta x + 6x - 6) = 3(0) + 6x - 6$$

$$\boxed{f'(x) = 6x - 6}$$

At $x = 0$

$f'(0) = 6(0) - 6 = -6$, which is slope of tangent line at $(0, -10)$

so

Equation of tangent line at $(0, -10)$ x_1 y_1

$y - y_1 = f'(x)(x - x_1)$

$\boxed{y + 10 = -6(x - 0)}$ Ans

$y + 10 = -6x \Rightarrow \boxed{6x + y + 10 = 0}$ Ans

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d) Let $y = f(x) = 2x^2 + 3x - 4$ $\boxed{x = 1}$

put $x = 1$

$y = 2(1)^2 + 3(1) - 4 = 2 + 3 - 4 = 1$

Point of contact is $(x, y) = (1, 1)$

$\because f(x) = 2x^2 + 3x - 4$

Replace x by $x + \Delta x$

$f(x+\Delta x) = 2(x+\Delta x)^2 + 3(x + \Delta x) - 4$

$= 2(x^2 + \Delta^2 x + 2x\Delta x) + 3x + 3\Delta x - 4$

$= 2x^2 + 2\Delta^2 x + 4x\Delta x + 3x + 3\Delta x - 4$

Now

$f'(x) = \lim_{\Delta x \to 0} \dfrac{f(x+\Delta x) - f(x)}{\Delta x}$

$= \lim_{\Delta x \to 0} \dfrac{2x^2 + 2\Delta^2 x + 4x\Delta x + 3x + 3\Delta x - 4 - (2x^2 + 3x - 4)}{\Delta x}$

$= \lim_{\Delta x \to 0} \dfrac{\cancel{2x^2} + 2\Delta^2 x + 4x\Delta x + \cancel{3x} + 3\Delta x \cancel{-4} \cancel{-2x^2} \cancel{-3x} \cancel{+4}}{\Delta x}$

$= \lim_{\Delta x \to 0} \dfrac{\cancel{\Delta x}(2\Delta x + 4x + 3)}{\cancel{\Delta x}}$

$= \lim_{\Delta x \to 0} (2\Delta x + 4x + 3) = 2(0) + 4x + 3 = 4x + 3$

$\boxed{f'(x) = 4x + 3}$

At $x = 1$

$f'(1) = 4(1) + 3 = 7$, which is slope of tangent line at $(1, 1)$

$\Rightarrow \boxed{f'(1) = 7}$

Eq: of tangent line at $\overset{x_1\,y_1}{(1,1)}$ is

$$y-y_1 = f'(x)(x-x_1)$$

$$\boxed{y-1 = 7(x-1)}$$ Ans

$$y-1 = 7x-7$$

$$\Rightarrow 7x-y-6=0$$ Ans

Note (1) $\frac{d}{dx}(c)=0$ (Constant Rule) $\left(\frac{d}{dx}(3)=0\right.$

2) $\frac{d}{dx}(cf) = c\frac{d}{dx}f$ (Constant multiple Rule)

3) $\frac{d}{dx}(f\pm g) = \frac{d}{dx}f \pm \frac{d}{dx}g$ (Sum/Difference Rule)

4) $\frac{d}{dx}(af+bg) = a\frac{d}{dx}f + b\frac{d}{dx}g$ (Linearity Rule)

Note: ① $\frac{d}{dx}(x)=1$ or $\frac{d}{dt}t=1$

② $\frac{d}{dx}x^n = nx^{n-1}$ (Power Rule)

3) $\frac{d}{dx}[f(x)]^n = n(f(x))^{n-1}\frac{d}{dx}f(x)$ (general Power Rule)

4) $\frac{d}{dx}(f\cdot g) = f\frac{d}{dx}g + g\frac{d}{dx}f$ (Product Rule)

(5) $\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{g\frac{d}{dx}f - f\frac{d}{dx}g}{g^2}$ (Quotient Rule)

Note $\frac{d}{dx}x^2 = 2x$, $\frac{d}{dx}x^3 = 3x^2$

{EXE 2.3} See next Page

Note

Product Rule

$$\frac{d}{dx}(f\cdot g) = f\frac{d}{dx}g + g\frac{d}{dx}f.$$

EXERCISE 2.3 b'2

1) Use Product rule to find out the derivative of the following functions

a) $y = (x^2-2)(3x+1)$

Diff: w.r.t. x

$$\frac{dy}{dx} = \frac{d}{dx}\left[(x^2-2)(3x+1)\right]$$

$$= (x^2-2)\frac{d}{dx}(3x+1) + (3x+1)\frac{d}{dx}(x^2-2)$$

$$= (x^2-2)(3\cdot1+0) + (3x+1)(2x-0)$$

$$= 3(x^2-2) + 2x(3x+1)$$

$$= 3x^2-6+6x^2+2x = 9x^2+2x-6 \quad \text{Ans}$$

b) $y = (6x^3+2)(5x-3)$

Diff: w.r.t. x

$$\frac{dy}{dx} = \frac{d}{dx}\left[(6x^3+2)(5x-3)\right]$$

$$= (6x^3+2)\frac{d}{dx}(5x-3) + (5x-3)\frac{d}{dx}(6x^3+2)$$

$$= (6x^3+2)[5(1)-0] + (5x-3)[6(3x^2)+0]$$

$$= 5(6x^3+2) + 18x^2(5x-3)$$

$$= 30x^3+10+90x^3-54x^2 = 120x^3-54x^2+10 \quad \text{Ans}$$

c) $y = (7x^4+2x)(x^2-4)$

Diff: w.r.t. x

$$\frac{dy}{dx} = \frac{d}{dx}\left[(7x^4+2x)(x^2-4)\right]$$

$$= (7x^4+2x)\frac{d}{dx}(x^2-4) + (x^2-4)\frac{d}{dx}(7x^4+2x)$$

$$= (7x^4+2x)[2x-0] + (x^2-4)[7(4x^3)+2(1)]$$

$$= 2x(7x^4+2x) + (x^2-4)(28x^3+2)$$

$$= 14x^5+4x^2+28x^5+2x^2-112x^3-8$$

$$= 42x^5-112x^3+6x^2-8 \quad \text{Ans}$$

d) $y=(2x^2+4x-3)(5x^3+x+2)$

Diff: w.r.t. x

$$\frac{dy}{dx}=\frac{d}{dx}\left[(2x^2+4x-3)(5x^3+x+2)\right]$$

$$=(2x^2+4x-3)\frac{d}{dx}(5x^3+x+2)+(5x^3+x+2)\frac{d}{dx}(2x^2+4x-3)$$

$$=(2x^2+4x-3)\left[5(3x^2)+1+0\right]+(5x^3+x+2)\left[2(2x)+4(1)-0\right]$$

$$=(2x^2+4x-3)(15x^2+1)+(5x^3+x+2)(4x+4)$$

$$=30x^4+2x^2+60x^3+4x-45x^2-3+20x^4+20x^3+4x^2+4x+8x+8$$

$$=50x^4+80x^3-39x^2+16x+5$$

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e) $y=(2x-3)(\sqrt{x}-1)$

Diff: w.r.t. x

$$\frac{dy}{dx}=\frac{d}{dx}\left[(2x-3)(\sqrt{x}-1)\right]$$

$$=(2x-3)\frac{d}{dx}[\sqrt{x}-1]+(\sqrt{x}-1)\frac{d}{dx}[2x-3]$$

$$=(2x-3)\left[\frac{1}{2}x^{\frac{1}{2}-1}-0\right]+(\sqrt{x}-1)\left[2(1)-0\right]$$

$$=(2x-3)\frac{1}{2}x^{-1/2}+(\sqrt{x}-1)(2)$$

$$=\frac{x^{-1/2}}{2}(2x-3)+2(\sqrt{x}-1)$$

$$=\frac{\cancel{2}x^{-\frac{1}{2}+1}}{\cancel{2}}-\frac{3\cdot x^{-1/2}}{2}+2\sqrt{x}-2$$

$$=x^{1/2}-\frac{3}{2}x^{-1/2}+2x^{1/2}-2$$

$$=3x^{1/2}-\frac{3}{2}x^{-1/2}-2 \qquad \text{Ans}$$

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f) $y=(-3\sqrt{x}+6)(4\sqrt{x}-2)$

Diff: w.r.t. x

$$\frac{dy}{dx}=\frac{d}{dx}\left[(-3\sqrt{x}+6)(4\sqrt{x}-2)\right]$$

$$=(-3\sqrt{x}+6)\frac{d}{dx}(4\sqrt{x}-2)+(4\sqrt{x}-2)\frac{d}{dx}(-3\sqrt{x}+6)$$

$$\frac{dy}{dx} = (-3\sqrt{x}+6)\left[4\left(\tfrac{1}{2}x^{\frac{1}{2}-1}\right)-0\right] + (4\sqrt{x}-2)\left[-3\left(\tfrac{1}{2}x^{\frac{1}{2}-1}\right)+0\right]$$

$$= (-3\sqrt{x}+6)\left[2x^{-1/2}\right] + (4\sqrt{x}-2)\left(-\tfrac{3}{2}x^{-1/2}\right)$$

$$= -6\,x^{\frac{1}{2}-1/2} + 12\,x^{-1/2} - 6\,x^{\frac{1}{2}-1/2} + 3x^{-1/2}$$

$$= -6x^{0} + 15x^{-1/2} - 6x^{0}$$

$$= -6 + 15\,x^{-1/2} - 6 = -12 + 15x^{-1/2} \quad \text{Ans}$$

Note Quotient Rule $\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{g\frac{d}{dx}f - f\frac{d}{dx}g}{g^2}$

2) Use the Quotient Rule to find out the derivative of the following functions

a) $y = \frac{3x-5}{x-4}$

Diff. w.r.t. x

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{3x-5}{x-4}\right)$$

$$= \frac{(x-4)\frac{d}{dx}(3x-5) - (3x-5)\frac{d}{dx}(x-4)}{(x-4)^2}$$

$$= \frac{(x-4)(3(1)-0) - (3x-5)(1-0)}{(x-4)^2}$$

$$= \frac{(x-4)(3) - (3x-5)(1)}{(x-4)^2} = \frac{\cancel{3x}-12-\cancel{3x}+5}{(x-4)^2} = \frac{-7}{(x-4)^2} \quad \text{Ans}$$

b) $y = \frac{2}{3x-5}$

Diff: w.r.t. x

$$\frac{dy}{dx} = \frac{d}{dx}\left[\frac{2}{3x-5}\right]$$

$$= \frac{(3x-5)\frac{d}{dx}(2) - 2\frac{d}{dx}(3x-5)}{(3x-5)^2}$$

$$\frac{dy}{dx} = \frac{(3x-5)(0) - 2[3(1)-0]}{(3x-5)^2}$$

$$= \frac{0 - 2(3)}{(3x-5)^2} = \frac{-6}{(3x-5)^2} \quad \text{Ans}$$

c) $f(t) = \dfrac{t^2+t}{t-1}$

Diff: w.r.t. t

$$\frac{d}{dt} f(t) = \frac{d}{dt}\left[\frac{t^2+t}{t-1}\right]$$

$$f'(x) = \frac{(t-1)\frac{d}{dt}(t^2+t) - (t^2+t)\frac{d}{dt}(t-1)}{(t-1)^2}$$

$$= \frac{(t-1)(2t+1) - (t^2+t)(1-0)}{(t-1)^2}$$

$$= \frac{2t^2 + \cancel{t} - 2t - 1 \quad - t^2 - \cancel{t}}{(t-1)^2} = \frac{t^2-2t-1}{(t-1)^2} \quad \text{Ans}$$

d) $y = \dfrac{-x^2+6x}{4x^3+1}$

Diff: w.r.t. x

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{-x^3+6x}{4x^3+1}\right)$$

$$= \frac{(4x^3+1)\frac{d}{dx}(-x^2+6x) - (-x^2+6x)\frac{d}{dx}(4x^3+1)}{(4x^3+1)^2}$$

$$= \frac{(4x^3+1)(-2x+6\cdot 1) - (-x^2+6x)(4(3x^2)+0)}{(4x^3+1)^2}$$

$$= \frac{(4x^3+1)(-2x+6) - (-x^2+6x)(12x^2)}{(4x^3+1)^2}$$

$$= \frac{-8x^4+24x^3-2x+6+12x^4-72x^3}{(4x^3+1)^2} = \frac{4x^4-48x^3-2x+6}{(4x^3+1)^2} \quad \text{Ans}$$

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e) $y = \dfrac{5x+6}{\sqrt{x}}$

Diff: w.r.t. x

$$\frac{dy}{dx} = \frac{d}{dx}\left[\frac{5x+6}{\sqrt{x}}\right]$$

$$= \frac{\sqrt{x}\,\frac{d}{dx}(5x+6) - (5x+6)\frac{d}{dx}\sqrt{x}}{(\sqrt{x})^2}$$

$$= \frac{\sqrt{x}[5(1)+0] - (5x+6)\frac{1}{2}x^{\frac{1}{2}-1}}{x}$$

$$= \frac{5\sqrt{x} - (5x+6)\frac{x^{-1/2}}{2}}{x}$$

$$= \frac{5\sqrt{x} - \frac{5}{2}x^{1-1/2} - 3x^{-1/2}}{x} = \frac{5\sqrt{x} - \frac{5}{2}x^{1/2} - \frac{3}{x^{1/2}}}{x}$$

$$= \frac{5\sqrt{x} - \frac{5}{2}\sqrt{x} - \frac{3}{\sqrt{x}}}{x} = \frac{(5-5/2)\sqrt{x} - \frac{3}{\sqrt{x}}}{x} = \frac{\frac{5}{2}\sqrt{x} - 3/\sqrt{x}}{x}$$ Ans

f) $y = \dfrac{x^2+7x-2}{x-2}$

Diff: w.r.t. x

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{x^2+7x-2}{(x-2)}\right)$$

$$= \frac{(x-2)\frac{d}{dx}(x^2+7x-2) - (x^2+7x-2)\frac{d}{dx}(x-2)}{(x-2)^2}$$

$$= \frac{(x-2)[2x+7(1)-0] - (x^2+7x-2)(1-0)}{(x-2)^2}$$

$$= \frac{(x-2)(2x+7) - (x^2+7x-2)}{(x-2)^2}$$

$$= \frac{2x^2+\cancel{7x}-4x-14-x^2-\cancel{7x}+2}{(x-2)^2} = \frac{x^2-4x-12}{(x-2)^2}$$ Ans

g) $f(p) = \dfrac{(2p+3)(4p-1)}{3p+2}$

Diff: w.r.t. p

$$\frac{d}{dp} f(p) = \frac{d}{dp}\left[\frac{(2p+3)(4p-1)}{3p+2}\right]$$

$$f'(p) = \frac{(3p+2)\frac{d}{dp}[(2p+3)(4p-1)] - (2p+3)(4p-1)\frac{d}{dp}(3p+2)}{(3p+2)^2}$$

$$= \frac{(3p+2)\left[(2p+3)\frac{d}{dp}(4p-1) + (4p-1)\frac{d}{dp}(2p+3)\right] - (8p^2-2p+12p-3)[3(1)+0]}{(3p+2)^2}$$

$$= \frac{(3p+2)[(2p+3)(4(1)-0) + (4p-1)(2(1)+0)] - (8p^2+10p-3)(3)}{(3p+2)^2}$$

$$= \frac{(3p+2)[(2p+3)(4) + (4p-1)(2)] - 24p^2 - 30p + 9}{(3p+2)^2}$$

$$= \frac{(3p+2)[8p+12+8p-2] - 24p^2 - 30p + 9}{(3p+2)^2}$$

$$= \frac{(3p+2)(16p+10) - 24p^2 - 30p + 9}{(3p+2)^2}$$

$$= \frac{48p^2 + \cancel{30p} + 32p + 20 - 24p^2 - \cancel{30p} + 9}{(3p+2)^2} = \frac{24p^2+32p+29}{(3p+2)^2} \quad \text{Ans}$$

2nd method

$$f(p) = \frac{(2p+3)(4p-1)}{(3p+2)} = \frac{8p^2-2p+12p-3}{3p+2}$$

$$f(p) = \frac{8p^2+10p-3}{(3p+2)}$$

Diff: w.r.t. p

$$\frac{d}{dp} f(p) = \frac{d}{dp}\left[\frac{8p^2+10p-3}{3p+2}\right]$$

$$f'(p) = \frac{(3p+2)\frac{d}{dp}(8p^2+10p-3) - (8p^2+10p-3)\frac{d}{dp}(3p+2)}{(3p+2)^2}$$

$$f'(p) = \frac{(3p+2)[8(2p)+10(1)-0] - (8p^2+10p-3)[3(1)+0]}{(3p+2)^2}$$

$$= \frac{(3p+2)(16p+10) - 3(8p^2+10p-3)}{(3p+2)^2}$$

$$= \frac{48p^2 + \cancel{30p} + 32p + 20 - 24p^2 - \cancel{30p} + 9}{(3p+2)^2}$$

$$= \frac{24p^2+32p+29}{(3p+2)^2}$$ Ans

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h) $$g(x) = \frac{x^3+1}{(2x+1)(5x+2)}$$

Diff: w.r.t. x.

$$\frac{d}{dx} g(x) = \frac{d}{dx}\left[\frac{x^3+1}{(2x+1)(5x+2)}\right]$$

$$g'(x) = \frac{(2x+1)(5x+2)\frac{d}{dx}(x^3+1) - (x^3+1)\frac{d}{dx}[(2x+1)(5x+2)]}{[(2x+1)(5x+2)]^2}$$

$$= \frac{(10x^2+4x+5x+2)[3x^2+0] - (x^3+1)\left[(2x+1)\frac{d}{dx}(5x+2)+(5x+2)\frac{d}{dx}(2x+1)\right]}{(2x+1)^2(5x+2)^2}$$

$$= \frac{(10x^2+9x+2)(3x^2) - (x^3+1)[(2x+1)(5)+(5x+2)(2)]}{(2x+1)^2(5x+2)^2}$$

$$= \frac{30x^4+27x^3+6x^2 - (x^3+1)[10x+5+10x+4)}{(2x+1)^2(5x+2)^2}$$

$$= \frac{30x^4+27x^3+6x^2 - (x^3+1)(20x+9)}{(2x+1)^2(5x+2)^2}$$

$$= \frac{30x^4+27x^3+6x^2-20x^4-9x^3-20x-9}{(2x+1)^2(5x+2)^2}$$

$$= \frac{10x^4+18x^3+6x^2-20x-9}{(2x+1)^2(5x+2)^2}$$ Ans

Note Eq: of tangent line $y - y_1 = f'(x)(x - x_1)$

3) Find an equation of a tangent line to the graph of the following function at the given point

a) $f(x) = 3x - 7$ at $(3, 2)$

Diff: w.r.t. x

$\frac{d}{dx} f(x) = \frac{d}{dx}(3x - 7)$

$f'(x) = 3(1) - 0$

$\boxed{f'(x) = 3}$

At Point $(\overset{x}{3}, \overset{y}{2})$ we have

$\boxed{f'(3) = 3}$

Eq: of tangent line at $(\overset{x_1}{3}, \overset{y_1}{2})$ is

$y - y_1 = f'(x)(x - x_1)$

$y - 2 = 3(x - 3)$

$y - 2 = 3x - 9$

$\Rightarrow 3x - y - 7 = 0$

b) Let $y = f(x) = x^3$ at $x = -\frac{1}{2}$

put $x = -1/2$

$y = (-\frac{1}{2})^3 = -\frac{1}{8}$

Point of contact is $(\overset{x}{-\frac{1}{2}}, \overset{y}{-\frac{1}{8}})$

$\because f(x) = x^3$

Diff w.r.t. x

$f'(x) = 3x^2$

At Point $(-\frac{1}{2}, -\frac{1}{8})$

$f'(-\frac{1}{2}) = 3(-\frac{1}{2})^2 = 3(\frac{1}{4}) = \frac{3}{4}$

Eq: of tangent line at point $(-\frac{1}{2}, -\frac{1}{8})$ is

$y - y_1 = f'(x)(x - x_1)$

$y + \frac{1}{8} = \frac{3}{4}(x + \frac{1}{2})$

$y + \frac{1}{8} = \frac{3}{4}x + \frac{3}{8}$

Xing by 8

$8y + 1 = 6x + 3$

$6x - 8y + 2 = 0$

÷ing by 2

$3x - 4y + 1 = 0$

c) Let $y = f(x) = \frac{1}{x+3}$ $\boxed{x = 2}$

put $x = 2$

$y = \frac{1}{2+3} = \frac{1}{5}$

Point of contact is $(x, y) = (2, 1/5)$

$\because \quad f(x) = \dfrac{1}{x+3}$

Diff: w.r.t. x

$$\frac{d}{dx} f(x) = \frac{d}{dx}\left(\frac{1}{x+3}\right)$$

$$f'(x) = \frac{(x+3)\frac{d}{dx}(1) - 1\frac{d}{dx}(x+3)}{(x+3)^2} = \frac{\cancel{(x+3)(0)} - 1(1+0)}{(x+3)^2}$$

$$f'(x) = \frac{-1}{(x+3)^2}$$

At Point $(\overset{x}{2}, \overset{y}{1/5})$

$$f'(2) = \frac{-1}{(2+3)^2} = -1/25$$

Eq: of tangent line at point $(\overset{x_1}{2}, \overset{y_1}{1/5})$ is

$$y - y_1 = f'(x)[x - x_1]$$

$$y - \frac{1}{5} = -\frac{1}{25}(x-2)$$

$$y - \frac{1}{5} = -\frac{1}{25}x + \frac{2}{25}$$

x'ing by 25

$$25y - 5 = -x + 2 \Rightarrow x + 25y - 7 = 0$$

---

d) $f(x) = \dfrac{x}{x-2}$ at $(3,3)$

Diff: w.r.t. x

$$f'(x) = \frac{d}{dx}\left(\frac{x}{x-2}\right)$$

$$= \frac{(x-2)\frac{d}{dx}x - x\frac{d}{dx}(x-2)}{(x-2)^2} = \frac{(x-2)1 - x(1-0)}{(x-2)^2}$$

$$= \frac{\cancel{x} - 2 - \cancel{x}}{(x-2)^2}$$

$$f'(x) = \frac{-2}{(x-2)^2}$$

At Point $(\overset{x}{3}, \overset{y}{3})$ we have

$$f'(3) = \frac{-2}{(3-2)^2} = \frac{-2}{1} = -2$$

Eq: of tangent line at Point $\underset{x_1,\ y_1}{(3,3)}$ is

$$y - y_1 = f'(x)(x - x_1)$$

$$y - 3 = -2(x-3)$$

$$y - 3 = -2x + 6 \Rightarrow 2x + y - 9 = 0 \quad \text{Ans}$$

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4) Given equation is

a) $$M(d) = \frac{200d}{3d+10}$$

Differentiat w.r.t. d (using Quotient Rule)

$$M'(d) = \frac{(3d+10)(200(1)) - 200d\,(3(1)+0)}{(3d+10)^2}$$

$$= \frac{(3d+10)(200) - (200d)3}{(3d+10)^2} = \frac{\cancel{600d} + 2000 - \cancel{600d}}{(3d+10)^2}$$

$$\boxed{M'(d) = \frac{2000}{(3d+10)^2}} \quad \text{—— (1)}$$

b) put $d = 2$ in (1)

$$M'(2) = \frac{2000}{(3(2)+10)^2}$$

$$= \frac{2000}{(16)^2} = \frac{\overset{125}{\cancel{2000}}}{\underset{16}{\cancel{256}}}$$

$$= \frac{125}{16} = 7.8125$$

Rate of change at the end of 2nd day.

put $d = 5$ in (1)

$$M'(5) = \frac{2000}{(3(5)+10)^2}$$

$$= \frac{2000}{(25)^2} = \frac{\overset{\overset{16}{\cancel{80}}}{\cancel{2000}}}{\underset{\underset{5}{\cancel{25}}}{\cancel{625}}}$$

$$= \frac{16}{5} = 3.2$$

Rate of change at the end of fifth day.

---

5) Given equation is

$$T = \frac{700}{t^2+4t+10} \quad \text{where } t \text{ is the time in hours}$$

Diff: w.r.t. t

$$\frac{dT}{dt} = \frac{d}{dt}\left(\frac{700}{t^2+4t+10}\right)$$

$$\frac{dT}{dt} = \frac{(t^2+4t+10)\frac{d}{dt}700 - 700\frac{d}{dt}(t^2+4t+10)}{(t^2+4t+10)^2}$$

$$= \frac{(t^2+4t+10)(0) - 700(2t+4(1)+10)}{(t^2+4t+10)^2} = \frac{0-700(2t+4)}{(t^2+4t+10)^2}$$

$$\frac{dT}{dt} = \frac{-700(2t+4)}{(t^2+4t+10)^2} \text{ ——— (1)}$$

a) $t = 1$ hour

put in (1)

$$\left(\frac{dT}{dt}\right)_{t=1} = \frac{-700(2(1)+4)}{(1^2+4(1)+10)^2}$$

$$= \frac{-700(6)}{(1+4+10)^2}$$

$$= -\frac{\cancel{4200}^{168}}{\cancel{225}_{9}} = \frac{\cancel{168}^{56}}{\cancel{9}_{3}}$$

$$= \frac{56}{3}$$

b) $t = 2$ hour

put in (1)

$$\left(\frac{dT}{dt}\right)_{t=2} = \frac{-700(2(2)+4)}{(2^2+4(2)+10)^2}$$

$$= \frac{-700(4+4)}{(4+8+10)^2}$$

$$= \frac{-700(8)}{(22)^2} = \frac{-5600}{484} = \frac{-1400}{121}$$

---

6) Given equation is

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{p}$$

a)

$$\frac{1}{y} = \frac{1}{p} - \frac{1}{x}$$

$$\frac{1}{y} = \frac{x-p}{px} \quad \text{(taking Reciprocal)}$$

$$\Rightarrow y = \frac{px}{x-p}$$

b) Diff: w.r.t. x.

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{px}{x-p}\right)$$

$$= \frac{(x-p)\frac{d}{dx}(px) - (px)\frac{d}{dx}(x-p)}{(x-p)^2}$$

$$\frac{dy}{dx} = \frac{(x-p)(p.1) - px(1-0)}{(x-p)^2} = \frac{(x-p)p - px}{(x-p)^2}$$

$$= \frac{\cancel{xp} - p^2 - \cancel{px}}{(x-p)^2} = \frac{-p^2}{(x-p)^2} \quad \text{Ans}$$

7) Given that

$$G(x) = \frac{1}{200}\left(\frac{800}{x} + x\right), \quad G'(x) = \frac{1}{200}\left(-\frac{800}{x^2} + 1\right) \longrightarrow (1)$$

a) put $x = 20$ in (1)

$$G'(20) = \frac{1}{200}\left(-\frac{800}{(20)^2} + 1\right)$$

$$= \frac{1}{200}\left(-\frac{\cancel{800}^{2}}{\cancel{400}} + 1\right)$$

$$= \frac{1}{200}(-2+1)$$

$$= \frac{-1}{200}$$

we will tell her go faster than that of 20 miles per hour.

b) put $x = 40$ in (1)

$$G'(40) = \frac{1}{200}\left(\frac{-800}{40^2} + 1\right)$$

$$= \frac{1}{200}\left(\frac{-800}{1600} + 1\right)$$

$$= \frac{1}{200}\left(\frac{-1}{2} + 1\right)$$

$$= \frac{1}{200}\left(\frac{1}{2}\right)$$

$$= \frac{1}{400}$$

In this case go slower than that of 40 miles per hour

Chain Rule: Consider Parametric equation

$x = f(t)$, $y = g(t)$

Find $\frac{dx}{dt}$ & $\frac{dy}{dt}$ To find $\frac{dy}{dx}$ use chain Rule,

that is, $\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$

General Power Rule

$$\frac{d}{dx}[f(x)]^n = n[f(x)]^{n-1}\frac{d}{dx}f(x)$$

Exe 2.4

1) Determine the indicated derivative in each case:

a) $s = 5(7-t)^4$

Diff: w.r.t. t

$$\frac{ds}{dt} = 5\frac{d}{dt}(7-t)^4$$
$$= 5\left\{4(7-t)^3\frac{d}{dt}(7-t)\right\}$$
$$= 20(7-t)^3(0-1)$$
$$= -20(7-t)^3$$

b) $w = 4(x^3-4x+2)^5$

Diff: w.r.t. x

$$\frac{dw}{dx} = 4\frac{d}{dx}(x^3-4x+2)^5$$
$$= 4\left\{5(x^3-4x+2)^4\frac{d}{dx}(x^3-4x+2)\right\}$$
$$= 20(x^3-4x+2)^4[3x^2-4(1)+0]$$
$$= 20(x^3-4x+2)^4(3x^2-4)$$

c) $x = -3(4-11s^2)^5$

Diff: w.r.t. s

$$\frac{dx}{ds} = -3\frac{d}{ds}(4-11s^2)^5$$
$$= -3\left\{5(4-11s^2)^4\frac{d}{ds}(4-11s^2)\right\}$$
$$= -15(4-11s^2)^4[0-11(2s)]$$
$$= -15(4-11s^2)^4(-22s)$$
$$= +330s(4-11s^2)^4$$

d) $y = \frac{(4x-x^3)^{11}}{5}$

Diff. w.r.t. x

$$\frac{dy}{dx} = \frac{1}{5}\frac{d}{dx}(4x-x^3)^{11}$$
$$= \frac{1}{5}\left[11(4x-x^3)^{10}\frac{d}{dx}(4x-x^3)\right]$$
$$= \frac{11}{5}(4x-x^3)^{10}[4(1)-3x^2]$$
$$= \frac{11}{5}(4x-x^3)^{10}(4-3x^2) \quad \text{Ans}$$

e) $u = \sqrt[3]{1-3t^2} = (1-3t^2)^{1/3}$

Diff: w.r.t. t

$$\frac{du}{dt} = \frac{d}{dt}(1-3t^2)^{1/3}$$
$$= \frac{1}{3}(1-3t^2)^{\frac{1}{3}-1}\frac{d}{dt}(1-3t^2)$$
$$= \frac{1}{3}(1-3t^2)^{-2/3}[0-3(2t)]$$
$$= \frac{1}{\cancel{3}}(1-3t^2)^{-2/3}(-\cancel{6}^{2}t)$$
$$= \frac{-2t}{(1-3t^2)^{2/3}} = \frac{-2t}{[(1-3t^2)^2]^{1/3}} = \frac{-2t}{\sqrt[3]{(1-3t^2)^2}} \quad \text{Ans}$$

f) $s = \frac{1}{(3t+1)^7} = (3t+1)^{-7}$

Diff: w.r.t. t

$$\frac{ds}{dt} = \frac{d}{dt}(3t+1)^{-7}$$
$$= -7(3t+1)^{-7-1}\frac{d}{dt}(3t+1)$$
$$= -7(3t+1)^{-8}[3(1)+0]$$
$$= -21(3t+1)^{-8} \quad \text{Ans}$$
$$= \frac{-21}{(3t+1)^{+8}}$$

g) $R = \dfrac{1}{(2x-1)^8} = (2x-1)^{-8}$

Diff: w.r.t. x

$$\frac{dR}{dx} = \frac{d}{dx}(2x-1)^{-8}$$

$$= -8(2x-1)^{-8-1}\frac{d}{dx}(2x-1)$$

$$= -8(2x-1)^{-9}(2(1)-0)$$

$$= -16(2x-1)^{-9}$$

$$= \frac{-16}{(2x-1)^9}$$

h) $R = \dfrac{1}{5(4x^2-7)^7} = \dfrac{(4x^2-7)^{-7}}{5}$

Diff: w.r.t. x

$$\frac{dR}{dx} = \frac{1}{5}\frac{d}{dx}(4x^2-7)^{-7}$$

$$= \frac{1}{5}\left[-7(4x^2-7)^{-7-1}\frac{d}{dx}(4x^2-7)\right]$$

$$= \frac{1}{5}\left[-7(4x^2-7)^{-8}(4\cdot 2x-0)\right]$$

$$= \frac{-7}{5}(4x^2-7)^{-8}(8x)$$

$$= \frac{-56x}{5(4x^2-7)^8}$$

2) Determine the derivative $f'(x)$ in each case

a) $f(x) = (2x-5)^3(5x-7)$

Diff: w.r.t. x

$$\frac{d}{dx}f(x) = \frac{d}{dx}\left[(2x-5)^3(5x-7)\right]$$

$$f'(x) = (2x-5)^3\frac{d}{dx}(5x-7) + (5x-7)\frac{d}{dx}(2x-5)^3$$

$$= (2x-5)^3[5(1)-0] + (5x-7)\left[3(2x-5)^2\frac{d}{dx}(2x-5)\right]$$

$$= 5(2x-5)^3 + 3(5x-7)(2x-5)^2[2(1)-0]$$

$$= 5(2x-5)^3 + 6(5x-7)(2x-5)^2 \quad \text{common}$$

$$= (2x-5)^2\left[5(2x-5)^1 + 6(5x-7)\right]$$

$$= (2x-5)^2[10x-25+30x-42]$$

$$= (2x-5)^2(40x-67) \quad \text{Ans}$$

b) $f(x) = \dfrac{(x+2)^2}{x-1}$

Diff w.r.t. x

$$f'(x) = \frac{d}{dx}\left[\frac{(x+2)^2}{x-1}\right]$$

$$= \frac{(x-1)\frac{d}{dx}(x+2)^2 - (x+2)^2\frac{d}{dx}(x-1)}{(x-1)^2}$$

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$$f'(x) = \frac{(x-1)\left[2(x+2)'\frac{d}{dx}(x+2)\right] - (x+2)^2[1-0]}{(x-1)^2}$$

$$= \frac{2(x-1)[(x+2)(1+0)] - (x+2)^2}{(x-1)^2}$$

$$= \frac{2(x-1)(x+2) - (x+2)^2}{(x-1)^2}$$

$$= \frac{(x+2)\left[2(x-1) - (x+2)\right]}{(x-1)^2} = \frac{(x+2)(2x-2-x-2)}{(x-1)^2}$$

$$= \frac{(x+2)(x-4)}{(x-1)^2}$$ Ans

c) $f(x) = \left(\frac{2x-5}{x-4}\right)^4$

Diff w.r.t. x

$$f'(x) = \frac{d}{dx}\left(\frac{2x-5}{x-4}\right)^4$$

$$= 4\left(\frac{2x-5}{x-4}\right)^3 \frac{d}{dx}\left(\frac{2x-5}{x-4}\right)$$

$$= 4\,\frac{(2x-5)^3}{(x-4)^3}\left[\frac{(x-4)\frac{d}{dx}(2x-5) - (2x-5)\frac{d}{dx}(x-4)}{(x-4)^2}\right]$$

$$= 4\,\frac{(2x-5)^3}{(x-4)^3}\left(\frac{(x-4)[2(1)-0] - (2x-5)[1-0]}{(x-4)^2}\right)$$

$$= \frac{4(2x-5)^3\left[\cancel{2x} - 8 - 2x + 5\right]}{(x-4)^5}$$

$$= \frac{4(2x-5)^3(-3)}{(x-4)^5} = \frac{-12(2x-5)^3}{(x-4)^5}$$ Ans

d) $f(x) = x\sqrt{2x^2+11}$

Diff: w.r.t. x

$$f'(x) = x\frac{d}{dx}\sqrt{2x^2+11} + \sqrt{2x^2+11}\frac{d}{dx}x$$

$$= x\left[\frac{1}{2}(2x^2+11)^{\frac{1}{2}-1}\frac{d}{dx}(2x^2+11)\right] + \sqrt{2x^2+11}\,(1)$$

$$= \frac{x}{\cancel{2}}(2x^2+11)^{-1/2}\left[\cancel{2}(2x)+0\right] + \sqrt{2x^2+11}$$

$$= \frac{2x^2}{(2x^2+11)^{1/2}} + \sqrt{2x^2+11}$$

$$= \frac{2x^2}{\sqrt{2x^2+11}} + \frac{\sqrt{2x^2+11}}{1}$$

$$= \frac{2x^2+\left(\sqrt{2x^2+11}\right)^2}{\sqrt{2x^2+11}} = \frac{2x^2+2x^2+11}{\sqrt{2x^2+11}} = \frac{4x^2+11}{\sqrt{2x^2+11}} \quad \text{Ans}$$

---

e) $f(x) = 3x\sqrt[3]{3x+7} = 3x(3x+7)^{1/3}$

Diff: w.r.t. x

$$f'(x) = 3\frac{d}{dx}\left[x(3x+7)^{1/3}\right]$$

$$= 3\left[x\frac{d}{dx}(3x+7)^{1/3} + (3x+7)^{1/3}\frac{d}{dx}x\right]$$

$$= 3\left[x\cdot\frac{1}{3}(3x+7)^{\frac{1}{3}-1}\frac{d}{dx}(3x+7) + (3x+7)^{1/3}\cdot 1\right]$$

$$= 3\left[\frac{x}{\cancel{3}}(3x+7)^{-2/3}(\cancel{3}\cdot1+0) + (3x+7)^{1/3}\right]$$

$$= 3\left[\frac{x}{(3x+7)^{2/3}} + (3x+7)^{1/3}\right]$$

$$= 3\left[\frac{x+(3x+7)^{\frac{1}{3}+\frac{2}{3}}}{(3x+7)^{2/3}}\right]$$

$$= 3\left[\frac{x+(3x+7)^{\cancel{3}/\cancel{3}}}{(3x+7)^{2/3}}\right]$$

$$= 3\left[\frac{x+3x+7}{(3x+7)^{2/3}}\right] = \frac{3(4x+7)}{(3x+7)^{2/3}} \quad \text{Ans}$$

f) $f(x) = \dfrac{\sqrt{2x+11}}{(3x-8)^2}$

Diff: w.r.t. x

$$\frac{d}{dx}f(x) = \frac{d}{dx}\left|\frac{\sqrt{2x+11}}{(3x-8)^2}\right|$$

$$f'(x) = \frac{(3x-8)^2\frac{d}{dx}\sqrt{2x+11} - \sqrt{2x+11}\,\frac{d}{dx}(3x-8)^2}{\left[(3x-8)^2\right]^2}$$

$$= \frac{(3x-8)^2\left[\frac{1}{2}(2x+11)^{\frac{1}{2}-1}\frac{d}{dx}(2x+11)\right] - \sqrt{2x+11}\left[2(3x-8)\frac{d}{dx}(3x-8)\right]}{(3x-8)^4}$$

$$= \frac{(3x-8)^2\left[\frac{1}{\cancel{2}}(2x+11)^{-1/2}(\cancel{2}\cdot 1+0)\right] - 2\sqrt{2x+11}\,(3x-8)(3\cdot 1-0)}{(3x-8)^4}$$

$$= \frac{1}{(3x-8)^4}\left[\frac{(3x-8)^2}{(2x+11)^{+1/2}} - 6\sqrt{2x+11}\,(3x-8)\right]$$

$$= \frac{\cancel{(3x-8)}^1}{(3x-8)^{\cancel{4}\,3}}\left[\frac{3x-8}{\sqrt{2x+11}} - 6\sqrt{2x+11}\right]$$

$$= \frac{1}{(3x-8)^3}\left[\frac{3x-8-6(\sqrt{2x+11})^2}{\sqrt{2x+11}}\right]$$

$$= \frac{3x-8-12x-66}{(3x-8)^3\sqrt{2x+11}} = \frac{-9x-74}{(3x-8)^3\sqrt{2x+11}} \quad \text{Ans}$$

---

g) $f(x) = \left(\dfrac{3x-8}{x+9}\right)^7$

Diff w.r.t. x

$$\frac{d}{dx}f(x) = \frac{d}{dx}\left(\frac{3x-8}{x+9}\right)^7$$

$$f'(x) = 7\left(\frac{3x-8}{x+9}\right)^6\frac{d}{dx}\left(\frac{3x-8}{x+9}\right)$$

$$= 7\,\frac{(3x-8)^6}{(x+9)^6}\cdot\left[\frac{(x+9)\frac{d}{dx}(3x-8) - (3x-8)\frac{d}{dx}(x+9)}{(x+9)^2}\right]$$

$$f'(x) = \frac{7(3x-6)^6}{(x+9)^6} \cdot \left[\frac{(x+9)[3(1)-0] - (3x-8)[1+0]}{(x+9)^2} \right]$$

$$= \frac{7(3x-6)^6 [\cancel{3x} + 27 - \cancel{3x} + 8]}{(x+9)^8}$$

$$= \frac{7(3x-6)^6 (35)}{(x+9)^8} = \frac{245(3x-6)^6}{(x+9)^8} \quad \text{Ans}$$

---

h) $f(x) = (2x-9)^2 \sqrt{3x+7}$

Diff: w.r.t. x

$$\frac{d}{dx} f(x) = \frac{d}{dx}\left[(2x-9)^2 \sqrt{3x+7}\right]$$

$$f'(x) = (2x-9)^2 \frac{d}{dx}\sqrt{3x+7} + \sqrt{3x+7}\,\frac{d}{dx}(2x-9)^2$$

$$= (2x-9)^2 \left[\frac{1}{2}(3x+7)^{\frac{1}{2}-1} \frac{d}{dx}(3x+7)\right] + \sqrt{3x+7}\left[2(2x-9)^1 \frac{d}{dx}(2x-9)\right]$$

$$= (2x-9)^2 \left[\frac{1}{2}(3x+7)^{-1/2}(3\cdot 1+0)\right] + 2\sqrt{3x+7}\,(2x-9)(2\cdot 1+0)$$

$$= \frac{3(2x-9)^2}{2(3x+7)^{+1/2}} + 4\sqrt{3x+7}\,(2x-9)$$

$$= (2x-9)\left[\frac{3(2x-9)}{2\sqrt{3x+7}} + 4\sqrt{3x+7}\right]$$

$$= (2x-9)\left[\frac{3(2x-9) + 8(\sqrt{3x+7})^2}{2\sqrt{3x+7}}\right]$$

$$= \frac{(2x-9)(6x-27+24x+56)}{2\sqrt{3x+7}} = \frac{(2x-9)(30x+29)}{2\sqrt{3x+7}} \quad \text{Ans}$$

Note Parametric equation

$x = f(t)$; $y = g(t)$

Diff w.r.t. t ; Diff w.r.t.

$\frac{dx}{dt} = \square$; $\frac{dy}{dt} = \Delta$

Chain Rule

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$

$$= \Delta \cdot \frac{1}{\square} \quad \text{Ans}$$

3) Find $\frac{dy}{dx}$ of the following functions in terms of Parameter t.

a) $x = 1+t^2$, $y = t^3+2t^2+1$ $\qquad \frac{dy}{dx} = ?$

Diff: w.r.t. t

$$\frac{dx}{dt} = \frac{d}{dt}(1+t^2)$$

$$= 0+2t$$

$$\boxed{\frac{dx}{dt} = 2t}$$

Diff: w.r.t. t

$$\frac{dy}{dt} = \frac{d}{dt}(t^3+2t^2+1)$$

$$= 3t^2+2(2t)+0$$

$$\boxed{\frac{dy}{dt} = 3t^2+4t}$$

By chain Rule

$$\frac{dy}{dx} = \frac{dy}{dt}\,\frac{dt}{dx}$$

$$= (3t^2+4t) \times \frac{1}{2t} = \frac{\cancel{t}(3t+4)}{2\cancel{t}} = \frac{3t+4}{2} \quad \text{Ans}$$

---

b) $x = 3at^2+2$, $y = 6t^4+9$

Diff: w.r.t. t

$$\frac{dx}{dt} = \frac{d}{dt}\left[3at^2+2\right]$$

$$= 3a(2t)+0$$

$$\boxed{\frac{dx}{dt} = 6at}$$

Diff: w.r.t. t

$$\frac{dy}{dt} = \frac{d}{dt}\left[6t^4+9\right]$$

$$= 6(4t^3)+0$$

$$\boxed{\frac{dy}{dt} = 24t^3}$$

By chain Rule

$$\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx}$$

$$= \cancel{24}^{\,4}\,t^{\cancel{3}\,2} \times \frac{1}{\cancel{6}a\cancel{t}} = \frac{4t^2}{a} \quad \text{Ans}$$

---

c) $x = \frac{a(1-t^2)}{1+t^2}$, $y = \frac{abt}{1+t^2}$

Diff w.r.t. t

$$\frac{dx}{dt} = a\,\frac{d}{dt}\left(\frac{1-t^2}{1+t^2}\right)$$

Diff: w.r.t. t

$$\frac{dy}{dt} = ab\,\frac{d}{dt}\left(\frac{t}{1+t^2}\right)$$

$$\frac{dx}{dt} = a\left[\frac{(1+t^2)\frac{d}{dt}(1-t^2) - (1-t^2)\frac{d}{dt}(1+t^2)}{(1+t^2)^2}\right]$$

$$= a\left[\frac{(1+t^2)(-2t) - (1-t^2)(2t)}{(1+t^2)^2}\right]$$

$$= \frac{a[-2t - \cancel{2t^3} - 2t + \cancel{2t^3}]}{(1+t^2)^2}$$

$$= \frac{a(-4t)}{(1+t^2)^2}$$

$$\boxed{\frac{dx}{dt} = -\frac{4at}{(1+t^2)^2}}$$

$$\frac{dy}{dt} = 2b\left[\frac{(1+t^2)\frac{d}{dt}t - t\frac{d}{dt}(1+t^2)}{(1+t^2)^2}\right]$$

$$= \frac{2b[(1+t^2)(1) - t(2t)]}{(1+t^2)^2}$$

$$= \frac{2b[1+t^2-2t^2]}{(1+t^2)^2}$$

$$\boxed{\frac{dy}{dt} = \frac{2b(1-t^2)}{(1+t^2)^2}}$$

By chain Rule.

$$\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx}$$

$$= \frac{\cancel{2}b(1-t^2)}{\cancel{(1+t^2)^2}} \times \frac{\cancel{(1+t^2)^2}}{-\cancel{4}_{2}at} = -\frac{b(1-t^2)}{2at} = \frac{b(t^2-1)}{2at} \quad \text{Ans}$$

(ii) $x = \dfrac{3at}{1+t^3}$

Diff: w.r.t. t

$$\frac{dx}{dt} = 3a\frac{d}{dt}\left(\frac{t}{1+t^3}\right)$$

$$= 3a\left[\frac{(1+t^3)\frac{d}{dt}t - t\frac{d}{dt}(1+t^3)}{(1+t^3)^2}\right]$$

$$= 3a\left[\frac{(1+t^3)(1) - t(0+3t^2)}{(1+t^3)^2}\right]$$

$$= 3a\frac{(1+t^3-3t^3)}{(1+t^3)^2}$$

$$\boxed{\frac{dx}{dt} = \frac{3a(1-2t^3)}{(1+t^3)^2}}$$

$y = \dfrac{3at^2}{1+t^3}$

Diff: w.r.t. t

$$\frac{dy}{dt} = 3a\frac{d}{dt}\left(\frac{t^2}{1+t^3}\right)$$

$$= 3a\left[\frac{(1+t^3)\frac{d}{dt}t^2 - t^2\frac{d}{dt}(1+t^3)}{(1+t^3)^2}\right]$$

$$= 3a\left[\frac{(1+t^3)2t - t^2(0+3t^2)}{(1+t^3)^2}\right]$$

$$= \frac{3a[2t+2t^4-3t^4]}{(1+t^3)^2}$$

$$\boxed{\frac{dy}{dt} = \frac{3a(2t-t^4)}{(1+t^3)^2}}$$

By chain Rule 72

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$

$$= \frac{\cancel{3a}(2t - t^4)}{\cancel{(1+t^3)^2}} \times \frac{\cancel{(1+t^3)^2}}{\cancel{3a}(1-2t^3)} = \frac{t(2-t^3)}{1-2t^3} \quad \text{Ans}$$

4) Given equation is

$$F(t) = 60 - \frac{150}{\sqrt{8+t^2}}$$

a) Diff: w.r.t. t

$$F'(t) = \frac{d}{dt}\left[60 - 150\,(8+t^2)^{-1/2}\right]$$

$$= 0 - 150\left[-\frac{1}{2}(8+t^2)^{-1/2-1}\frac{d}{dt}(8+t^2)\right]$$

$$= +75(8+t^2)^{-3/2}(0+2t)$$

$$F'(t) = \frac{150t}{(8+t^2)^{3/2}} \quad \text{—— (1)}$$

b) At what rate is the cashier speed increasing

i) After 5 hour

put $t = 5$ in (1)

$$F'(t) = \frac{150(5)}{(8+5^2)^{3/2}}$$

$$= \frac{750}{(33)^{3/2}} = 3.96$$

(ii) After 10 hours

put $t = 10$ in (1)

$$F'(10) = \frac{150(10)}{(8+10^2)^{3/2}}$$

$$= \frac{1500}{(108)^{3/2}} = 1.33$$

iii) After 20 hours

put $t = 20$ in (1)

$$F'(20) = \frac{150(20)}{(8+20^2)^{3/2}}$$

$$= \frac{3000}{(8+400)^{3/2}} = \frac{3000}{(408)^{3/2}} = 0.36$$

iv) After 40 hours

put $t = 40$ in (1)

$$F'(40) = \frac{150(40)}{(8+40^2)^{3/2}}$$

$$= \frac{6000}{(8+1600)^{3/2}} = \frac{6000}{(1608)^{3/2}} = 0.09$$

C = Total cost
q = no. of units
t = time in hours

5) Given equations are

$C(q) = 0.2q^2 + q + 900$

Diff: w.r.t. q

$\frac{dc}{dq} = \frac{d}{dq}(0.2q^2 + q + 900)$

$\frac{dc}{dq} = 0.2(2q) + 1 + 0$

$\boxed{\frac{dc}{dq} = 0.4q + 1}$

$q(t) = t^2 + 100t$ ——— ①

Diff : w.r.t. t

$\frac{dq}{dt} = \frac{d}{dt}(t^2 + 100t)$

$= 2t + 100 \cdot 1$

$\boxed{\frac{dq}{dt} = 2t + 100}$

Now $\frac{dc}{dt} = \frac{dc}{dq} \cdot \frac{dq}{dt}$ (chain Rule)

$\frac{dc}{dt} = (0.4q + 1)(2t + 100)$ ——— ②

After one hour
put $t = 1$ in ①
$q = 1^2 + 100(1) = 101$

Now put $t = 1$ & $q = 101$ in ②

$\frac{dc}{dt} = [0.4(101) + 1][2(1) + 100]$
$= (40.4 + 1)(2 + 100)$
$= (41.4)(102) = 4222.8/\text{hour}$

Derivative of Implicit function

If y is expressed explicitly in term of x; i.e. $y = f(x)$ then $\frac{dy}{dx}$ will also be expressed explicitly in term of x. If y is expressed implicitly in terms of x, then $\frac{dy}{dx}$ will be expressed in terms of x & y & it is known as Implicit differentiation. It consist differentiation of the both sides of the equation w.r.t x and then solve the resultant equation algebraically for $\frac{dy}{dx}$.

EXERCISE 2.5

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1) Use Implicit differentiation to Perform $\frac{dy}{dx}$

a) $x^2+y^2=25$

Diff: w.r.t. x

$$\frac{d}{dx}(x^2+y^2)=\frac{d}{dx}25$$

$$2x+2y\frac{dy}{dx}=0$$

$$\cancel{2}y\frac{dy}{dx}=-\cancel{2}x$$

$$\Rightarrow \frac{dy}{dx}=-\frac{x}{y} \quad \text{Ans}$$

b) $xy=25$

Diff: w.r.t. x

$$\frac{d}{dx}(xy)=\frac{d}{dx}(25)$$

$$x\frac{dy}{dx}+y\cdot\frac{d}{dx}x=0$$

$$x\frac{dy}{dx}+y\cdot 1=0$$

$$x\frac{dy}{dx}=-y$$

$$\Rightarrow \frac{dy}{dx}=-\frac{y}{x} \quad \text{Ans}$$

c) $xy(2x+3y)=2$

$$2x^2y+3xy^2=2$$

Diff: w.r.t. x

$$\frac{d}{dx}\left[2x^2y+3xy^2\right]=\frac{d}{dx}2$$

$$\Rightarrow 2\frac{d}{dx}x^2y+3\frac{d}{dx}xy^2=0$$

$$2\left[x^2\frac{dy}{dx}+y\frac{d}{dx}x^2\right]+3\left[x\frac{dy^2}{dx}+y^2\frac{d}{dx}x\right]=0$$

$$2\left[x^2\frac{dy}{dx}+y\cdot 2x\right]+3\left[x\cdot 2y\frac{dy}{dx}+y^2\cdot 1\right]=0$$

$$2x^2\frac{dy}{dx}+4xy+6xy\frac{dy}{dx}+3y^2=0$$

$$2x^2\frac{dy}{dx}+6xy\frac{dy}{dx}=-4xy-3y^2$$

$$(2x^2+6xy)\frac{dy}{dx}=-(4xy+3y^2)$$

$$\Rightarrow \frac{dy}{dx}=-\frac{(4xy+3y^2)}{2x^2+6xy} \quad \text{Ans}$$

d) $x^2+3xy+y^2=15$

Diff: w.r.t. x

$$\frac{d}{dx}\left[x^2+3xy+y^2\right]=\frac{d}{dx}15$$

$$2x+3\left[x\frac{dy}{dx}+y\frac{d}{dx}x\right]+2y\frac{dy}{dx}=0$$

$$2x+3\left[x\frac{dy}{dx}+y\cdot 1\right]+2y\frac{dy}{dx}=0$$

$$2x+3x\frac{dy}{dx}+3y+2y\frac{dy}{dx}=0$$

$$3x\frac{dy}{dx}+2y\frac{dy}{dx}=-2x-3y$$

$$(3x+2y)\frac{dy}{dx}=-(2x+3y)$$

$$\Rightarrow \frac{dy}{dx}=-\frac{(2x+3y)}{3x+2y} \quad \text{Ans}$$

e) $(x+y)^3+3y=3$

Diff: w.r.t. x

$$\frac{d}{dx}\left[(x+y)^3+3y\right]=\frac{d}{dx}(3)$$

$$\Rightarrow \frac{d}{dx}(x+y)^3+3\frac{dy}{dx}=0$$

$$3(x+y)^2\frac{d}{dx}(x+y)+3\frac{dy}{dx}=0$$

$$3(x+y)^2\left[1+\frac{dy}{dx}\right]+3\frac{dy}{dx}=0$$

$$3(x+y)^2+3(x+y)^2\frac{dy}{dx}+3\frac{dy}{dx}=0$$

$$\cancel{3}\left[(x+y)^2+1\right]\frac{dy}{dx}=-\cancel{3}(x+y)^2$$

$$\Rightarrow \frac{dy}{dx}=\frac{-(x+y)^2}{(x+y)^2+1}$$

f) $\frac{1}{y}+\frac{1}{x}=1$

Diff: w.r.t. x

$$\frac{d}{dx}\left[y^{-1}+x^{-1}\right]=\frac{d(1)}{dx}$$

$$-1\cdot y^{-2}\frac{dy}{dx}+(-1)x^{-2}=0$$

$$-\frac{1}{y^2}\frac{dy}{dx}-\frac{1}{x^{2}}=0$$

$$-\frac{1}{y^2}\frac{dy}{dx}=+\frac{1}{x^2}$$

$$\Rightarrow \frac{dy}{dx}=\frac{-y^2}{x^2}$$

Ans

2) Arrange the following functions explicity and implicity to perform $\frac{dy}{dx}$

a) <u>Explicitly</u>

$$x^2y^3+y^3=12$$

$$y^3(x^2+1)=12$$

$$y^3=\frac{12}{x^2+1} \Rightarrow y=\left(\frac{12}{x^2+1}\right)^{1/3}$$

$$y=\frac{(12)^{1/3}}{(x^2+1)^{1/3}}=\sqrt[3]{12}\,(x^2+1)^{-1/3}$$

Diff: w.r.t. x

$$\frac{dy}{dx}=\sqrt[3]{12}\,\frac{d}{dx}(x^2+1)^{-1/3}$$

$$=\sqrt[3]{12}\left\{-\frac{1}{3}(x^2+1)^{-\frac{1}{3}-1}\frac{d}{dx}(x^2+1)\right.$$

$$=-\frac{\sqrt[3]{12}}{3}(x^2+1)^{-4/3}(2x)$$

$$\boxed{\frac{dy}{dx}=-\frac{\sqrt[3]{12}\,(2x)}{3(x^2+1)^{+4/3}}}$$

<u>Implicitly</u>

$$\because x^2y^3+y^3=12$$

$$(x^2+1)y^3=12$$

Diff w.r.t. x

$$\frac{d}{dx}\left[(x^2+1)y^3\right]=\frac{d}{dx}12$$

$$(x^2+1)\frac{d}{dx}y^3+y^3\frac{d}{dx}(x^2+1)=0$$

$$(x^2+1)\,3y^2\frac{dy}{dx}+y^3(2x+0)=0$$

$$3y^2(x^2+1)\frac{dy}{dx}=-2xy^3$$

$$\Rightarrow \frac{dy}{dx}=\frac{-2xy^{\cancel{3}\,1}}{3\cancel{y^2}(x^2+1)}$$

$$\boxed{\frac{dy}{dx}=\frac{-2x\cdot y}{3(x^2+1)}}$$

Ans

b) Explicitly

$xy + 2y = x^2$

$(x+2)y = x^2$

$y = \frac{x^2}{x+2}$

Diff: w.r.t. x

$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{x^2}{x+2}\right)$

$= \frac{(x+2)\frac{d}{dx}x^2 - x^2\frac{d}{dx}(x+2)}{(x+2)^2}$

$= \frac{(x+2)2x - x^2(1+0)}{(x+2)^2}$

$= \frac{2x^2+4x-x^2}{(x+2)^2} = \frac{x^2+4x}{(x+2)^2}$

Implicitly

$xy + 2y = x^2$

$(x+2)y = x^2$

Diff w.r.t. x

$\frac{d}{dx}[(x+2)y] = \frac{d}{dx}x^2$

$(x+2)\frac{dy}{dx} + y\frac{d}{dx}(x+2) = 2x$

$(x+2)\frac{dy}{dx} + y(1+0) = 2x$

$(x+2)\frac{dy}{dx} + y = 2x$

$(x+2)\frac{dy}{dx} = 2x - y$

$\frac{dy}{dx} = \frac{2x-y}{x+2}$ Ans

c) Explicitly

$x + \frac{1}{y} = 5$

$\frac{1}{y} = 5 - x$

$\Rightarrow y = \frac{1}{5-x}$

Diff w.r.t. x

$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{1}{5-x}\right)$

$= \frac{(5-x)\frac{d}{dx}(1) - 1\frac{d}{dx}(5-x)}{(5-x)^2}$

$= \frac{(5-x)(0) - 1.(0-1)}{(5-x)^2}$

$= \frac{0+1}{(5-x)^2}$

$\frac{dy}{dx} = \frac{1}{(5-x)^2}$ Ans

Implicitly

$x + \frac{1}{y} = 5$

Diff: w.r.t. x

$\frac{d}{dx}(x+y^{-1}) = \frac{d}{dx}5$

$1 + (-1)y^{-2}\frac{dy}{dx} = 0$

$1 - y^{-2}\frac{dy}{dx}$

$-y^{-2}\frac{dy}{dx} = -1$

$\frac{1}{y^2}\frac{dy}{dx} = 1$

$\Rightarrow \boxed{\frac{dy}{dx} = y^2}$ Ans

d) <u>Explicitly</u>

$$xy - x = y + 2$$

$$xy - y = x + 2$$

$$(x-1)y = x+2$$

$$y = \frac{x+2}{x-1}$$

Diff w.r.t. x

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{x+2}{x-1}\right)$$

$$= \frac{(x-1)\frac{d}{dx}(x+2) - (x+2)\frac{d}{dx}(x-1)}{(x-1)^2}$$

$$= \frac{(x-1)(1+0) - (x+2)(1-0)}{(x-1)^2}$$

$$= \frac{\cancel{x}-1 - \cancel{x}-2}{(x-1)^2} = \frac{-3}{(x-1)^2}$$ Ans

<u>Implicitly</u>

$$xy - x = y + 2$$

Diff: w.r.t. x

$$\frac{d}{dx}[xy - x] = \frac{d}{dx}(y+2)$$

$$\frac{d}{dx}(xy) - \frac{d}{dx}(x) = \frac{dy}{dx} + 0$$

$$x\frac{dy}{dx} + y\frac{d}{dx}x - 1 = \frac{dy}{dx}$$

$$x\frac{dy}{dx} + y \cdot 1 - 1 = \frac{dy}{dx}$$

$$x\frac{dy}{dx} - \frac{dy}{dx} = 1 - y$$

$$(x-1)\frac{dy}{dx} = 1 - y$$

$$\frac{dy}{dx} = \frac{1-y}{x-1}$$ Ans

3) Let $\frac{u^2}{a^2} + \frac{v^2}{b^2} = 1$, where a & b are non-zero constants

a) $\frac{du}{dv} = ?$

$$\because \quad \frac{u^2}{a^2} + \frac{v^2}{b^2} = 1$$

Xing by a^2b^2

$$b^2u^2 + a^2v^2 = a^2b^2$$

Diff: w.r.t. v

$$\frac{d}{dv}[b^2u^2 + a^2v^2] = \frac{d}{dv}(a^2b^2)$$

$$b^2\frac{d}{dv}u^2 + a^2\frac{d}{dv}v^2 = 0$$

$$b^2\left\{2u\frac{du}{dv}\right\} + a^2 \cdot 2v = 0$$

$$\cancel{2}b^2u\frac{du}{dv} = -\cancel{2}a^2v$$

$$\Rightarrow \boxed{\frac{du}{dv} = -\frac{a^2v}{b^2u}}$$ Ans

b) $\frac{dv}{du} = ?$

$$\because \quad \frac{u^2}{a^2} + \frac{v^2}{b^2} = 1$$

Xing by a^2b^2

$$b^2u^2 + a^2v^2 = a^2b^2$$

Diff: w.r.t. u

$$\frac{d}{du}[b^2u^2 + a^2v^2] = \frac{d}{du}(a^2b^2)$$

$$b^2\frac{d}{du}u^2 + a^2\frac{d}{du}v^2 = 0$$

$$b^2\{2u\} + a^2\left\{2v\frac{dv}{du}\right\} = 0$$

$$\cancel{2}a^2v\frac{dv}{du} = -\cancel{2}b^2u$$

$$\Rightarrow \boxed{\frac{dv}{du} = -\frac{b^2u}{a^2v}}$$ Ans

4) Let $(a-b)u^3-(a+b)v^2=c$, where a, b & c are constants. Find

a) $\frac{du}{dv}=?$

$\because (a-b)u^3-(a+b)v^2=c$

Diff: w.r.t. v

$\frac{d}{dv}\left[(a-b)u^3-(a+b)v^2\right]=\frac{d}{dv}c$

$(a-b)\,3u^2\frac{du}{dv}-(a+b)\,2v=0$

$3(a-b)u^2\frac{du}{dv}=2(a+b)v$

$\Rightarrow \frac{du}{dv}=\frac{2(a+b)v}{3(a-b)u^2}$

b) $\frac{dv}{du}=?$

$\because (a-b)u^3-(a+b)v^2=c$

Diff: w.r.t. u

$\frac{d}{du}\left[(a-b)u^3-(a+b)v^2\right]=\frac{d}{du}(c)$

$(a-b)\,3u^2-(a+b)\,2v\frac{dv}{du}=0$

$\cancel{+}2(a+b)v\frac{dv}{du}=\cancel{+}3(a-b)u^2$

$\Rightarrow \frac{dv}{du}=\frac{3(a-b)u^2}{2(a+b)v}$

---

5) slope of tangent line at $P(-3,0)=?$

Sol Given equation is

$3x^2-7y^2+14y=27$

Diff: w.r.t. x

$\frac{d}{dx}(3x^2-7y^2+14y)=\frac{d}{dx}(27)$

$\frac{d}{dx}(3x^2)-\frac{d}{dx}(7y^2)+\frac{d}{dx}(14y)=0$

$3(2x)-7\left(2y\frac{dy}{dx}\right)+14\frac{dy}{dx}=0$

$6x-14y\frac{dy}{dx}+14\frac{dy}{dx}=0$

$-14y\frac{dy}{dx}+14\frac{dy}{dx}=-6x$

$\overset{7}{\cancel{14}}(-y+1)\frac{dy}{dx}=-\overset{3}{\cancel{6}}x$

$\Rightarrow \boxed{\frac{dy}{dx}=-\frac{3x}{7(1-y)}}$

At $P(\underset{x}{-3},\underset{y}{0})$

$\frac{dy}{dx}=\frac{-3(-3)}{7(1-0)}=\frac{+9}{7}$

slope of tangent line at $P(-3,0)=9/7$ Ans

6) slope at Point $P(2,1) = ?$

Given equation is

$$x^3y^3 + 4y = 3x^2$$

Diff: w.r.t. x

$$\frac{d}{dx}\left[x^3y^3 + 4y\right] = \frac{d}{dx}(3x^2)$$

$$\Rightarrow \frac{d}{dx}(x^3y^3) + \frac{d}{dx}(4y) = 3(2x)$$

$$x^3\frac{d}{dx}y^3 + y^3\frac{d}{dx}x^3 + 4\frac{dy}{dx} = 6x$$

$$x^3\left\{3y^2\frac{dy}{dx}\right\} + y^3 \cdot 3x^2 + 4\frac{dy}{dx} = 6x$$

$$3x^3y^2\frac{dy}{dx} + 4\frac{dy}{dx} = 6x - 3x^2y^3$$

$$(3x^3y^2 + 4)\frac{dy}{dx} = 6x - 3x^2y^3$$

$$\Rightarrow \frac{dy}{dx} = \frac{6x - 3x^2y^3}{3x^3y^2 + 4}$$

At Point $P(\overset{x}{2},\overset{y}{1})$

$$\frac{dy}{dx} = \frac{6(2) - 3(2)^2(1)^3}{3(2)^3 1^2 + 4} = \frac{12 - 3(4)}{3(8) + 4} = \frac{12-12}{28} = 0$$

slope of tangent line at $P(2,1) = 0$

---

7) Given equation is $(L+m)(V+n) = k$

m, n, k are constant

Diff: w.r.t. V

$$\frac{d}{dV}\left[(L+m)(V+n)\right] = \frac{d}{dV}k$$

$$(L+m)\frac{d}{dV}(V+n) + (V+n)\frac{d}{dV}(L+m) = 0$$

$$(L+m)\left[1+0\right] + (V+n)\left[\frac{dL}{dV} + 0\right] = 0$$

$$(V+n)\frac{dL}{dV} = -(L+m)$$

$$\Rightarrow \frac{dL}{dV} = -\frac{(L+m)}{V+n} \quad \text{Ans}$$

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Differentiation of Trigonometric functions

1) $\sin\alpha\cos\beta \pm \cos\alpha\sin\beta = \sin(\alpha\pm\beta)$

2) $\cos\alpha\cos\beta \mp \sin\alpha\sin\beta = \cos(\alpha\pm\beta)$

3) $\sin\alpha + \sin\beta = 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$

4) $\sin\alpha - \sin\beta = 2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$

5) $\cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$

6) $\cos\alpha - \cos\beta = -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$

7) $\lim\limits_{x\to 0}\frac{\sin x}{x} = 1$

$\sin(-\theta) = -\sin\theta$

$\cos(-\theta) = \cos\theta$

$\tan(-\theta) = -\tan\theta$

Derivative of $\sin x$

Let $f(x) = \sin x$

Replace x by $x+\Delta x$

$f(x+\Delta x) = \sin(x+\Delta x)$

Now $f'(x) = \lim\limits_{\Delta x\to 0}\dfrac{f(x+\Delta x) - f(x)}{\Delta x}$

$= \lim\limits_{\Delta x\to 0}\dfrac{\sin(x+\Delta x) - \sin x}{\Delta x}$

$= \lim\limits_{\Delta x\to 0}\dfrac{2\cos\frac{x+\Delta x+x}{2}\sin\frac{\cancel{x}+\Delta x-\cancel{x}}{2}}{\Delta x}$

$= \lim\limits_{\Delta x\to 0}\dfrac{2\cos\frac{2x+\Delta x}{2}\sin\frac{\Delta x}{2}}{\Delta x}$ $\quad\left(\frac{0}{0}\right)$

$= \lim\limits_{\Delta x\to 0} 2\cos\left(\frac{2x+\Delta x}{2}\right) \times \dfrac{\sin\frac{\Delta x}{2}}{\Delta x}$ $\quad\left(\because \lim\limits_{x\to 0}\frac{\sin x}{x} = 1\right)$

$= \cancel{2}\cos\left(\frac{\cancel{2}x+0}{\cancel{2}}\right)\cdot\lim\limits_{\Delta x\to 0}\dfrac{\sin\frac{\Delta x}{2}}{\cancel{2}\frac{\Delta x}{2}}$

M&D

$= \cos x \cdot 1 = \cos x$

Note $\frac{d}{dx}\sin x = \cos x$

$\frac{d}{dx}\cos x = ?$

let $f(x) = \cos x$

Replace x by $x+\Delta x$

$f(x+\Delta x) = \cos(x+\Delta x)$

Now

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\cos(x+\Delta x) - \cos x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{-2\sin\frac{x+\Delta x+x}{2} \cdot \sin\frac{\cancel{x}+\Delta x-\cancel{x}}{2}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{-2\sin\frac{2x+\Delta x}{2} \cdot \sin\frac{\Delta x}{2}}{\Delta x} \qquad \left(\frac{0}{0}\right)$$

$$= -2\sin\left(\frac{\cancel{2}x+0}{\cancel{2}}\right) \cdot \lim_{\Delta x \to 0} \frac{\sin\frac{\Delta x}{2}}{\Delta x}$$

$$= -\cancel{2}\sin x \cdot \lim_{\Delta x \to 0} \frac{\sin\frac{\Delta x}{2}}{\cancel{2}\frac{\Delta x}{2}}$$

ARBD

$$= -\sin x \times 1 = -\sin x$$

Note $\boxed{\frac{d}{dx}\cos x = -\sin x}$

$\frac{d}{dx}\tan x = ?$

Sol $\frac{d}{dx}\tan x = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right)$

$$= \frac{\cos x \frac{d}{dx}\sin x - \sin x \frac{d}{dx}\cos x}{\cos^2 x}$$

$$= \frac{\cos x(\cos x) - \sin x(-\sin x)}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

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$$\frac{d}{dx}\tan x = \frac{1}{\cos^2 x} \Rightarrow \boxed{\frac{d}{dx}\tan x = \sec^2 x}$$

Similarly we can show that

$\frac{d}{dx}\cot x = -\operatorname{cosec}^2 x$, $\frac{d}{dx}\sec x = \sec x \cdot \tan x$,

$\frac{d}{dx}\operatorname{cosec} x = -\operatorname{cosec} x \cdot \cot x$

Note ① $\sin^2 x + \cos^2 x = 1$

② $\sin^2 x = 1 - \cos^2 x$ or $\sin x = \sqrt{1-\cos^2 x}$

3) $\cos^2 x = 1 - \sin^2 x$ or $\cos x = \sqrt{1-\sin^2 x}$

4) $\sec^2 x = 1 + \tan^2 x$

5) $\operatorname{cosec}^2 x = 1 + \cot^2 x$

Derivative of Inverse trigonometric function

1) $\frac{d}{dx}\sin^{-1}x = ?$

let $y = \sin^{-1}x$

$\Rightarrow \boxed{\sin y = x}$

Diff: w.r.t. x

$\frac{d}{dx}\sin y = \frac{d}{dx}x$

$\cos y \frac{dy}{dx} = 1$

$\Rightarrow \frac{dy}{dx} = \frac{1}{\cos y}$

$\Rightarrow \frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-\sin^2 y}}$

$\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}$

2) $\frac{d}{dx}\cos^{-1}x = ?$

let $y = \cos^{-1}x$

$\Rightarrow \boxed{\cos y = x}$

Diff: w.r.t. x

$\frac{d}{dx}\cos y = \frac{d}{dx}x$

$-\sin y \frac{dy}{dx} = 1$

$\Rightarrow \frac{dy}{dx} = \frac{-1}{\sin y}$

$\Rightarrow \frac{d}{dx}(\cos^{-1}x) = \frac{-1}{\sqrt{1-\cos^2 y}}$

$\frac{d}{dx}\cos^{-1}x = \frac{-1}{\sqrt{1-x^2}}$

Similarly we can show that

$\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}$, $\frac{d}{dx}\cot^{-1}x = \frac{-1}{1+x^2}$

$\frac{d}{dx}\sec^{-1}x = \frac{1}{x\sqrt{x^2-1}}$, $\frac{d}{dx}\operatorname{cosec}^{-1}x = \frac{-1}{x\sqrt{x^2-1}}$

EXERCISE 2.6

Note ① $\frac{d}{dx}\sin u = \cos u \cdot \frac{du}{dx}$ & $\frac{d}{dx}\cos u = -\sin u \frac{du}{dx}$

② $\frac{d}{dx}\tan u = \sec^2 u \frac{du}{dx}$ & $\frac{d}{dx}\cot u = -\csc^2 u \frac{du}{dx}$

3) $\frac{d}{dx}\sec u = \sec u \cdot \tan u \frac{du}{dx}$ & $\frac{d}{dx}\csc u = -\csc u \cdot \cot u \frac{du}{dx}$

Note $\frac{d}{dx}\sin^n x = \frac{d}{dx}(\sin x)^n = n(\sin x)^{n-1}\frac{d}{dx}\sin x = n\sin^{n-1}x \cdot \cos x$

① Use any suitable rule of differentiation to Perform $\frac{dy}{dx}$ for the following functions:

a) $y = 4\sin e^x$

Diff: w.r.t. x

$$\frac{dy}{dx} = 4\frac{d}{dx}\sin e^x$$

$$= 4\cos e^x \frac{d}{dx}e^x$$

Note $\boxed{\frac{d}{dx}e^x = e^x}$

$$= 4\cos e^x \cdot e^x$$

$$= 4e^x \cos e^x \quad \text{Ans}$$

b) $y = \cos(x + \frac{\pi}{2})$

Diff w.r.t. x

$$\frac{dy}{dx} = \frac{d}{dx}\cos(x+\pi/2)$$

$$= -\sin(x+\frac{\pi}{2})\frac{d}{dx}(x+\pi/2)$$

$$= -\sin(x+\frac{\pi}{2})(1+0)$$

$$= -\sin(x+\pi/2) \quad \text{Ans}$$

c) $y = 3\sin(\ln x)$

Diff w.r.t. x.

Note $\boxed{\frac{d}{dx}\ln x = \frac{1}{x}}$

$$\frac{dy}{dx} = 3\frac{d}{dx}\sin(\ln x)$$

$$= 3\cos(\ln x)\frac{d}{dx}\ln x$$

$$= 3\cos(\ln x)\cdot\frac{1}{x}$$

$$= \frac{3\cos(\ln x)}{x} \quad \text{Ans}$$

d) $y = \sin x \cdot \cos x$

Diff: w.r.t. x.

$$\frac{dy}{dx} = \frac{d}{dx}(\sin x \cdot \cos x)$$

$$= \sin x \frac{d}{dx}\cos x + \cos x \frac{d}{dx}\sin x$$

$$= \sin x(-\sin x) + \cos x \cdot \cos x$$

$$= -\sin^2 x + \cos^2 x$$

$$= \cos^2 x - \sin^2 x \quad \text{Ans}$$

e) $y = \frac{\sin x}{\cos x}$

Diff w.r.t. x

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right)$$

$$= \frac{\cos x \frac{d}{dx}\sin x - \sin x \frac{d}{dx}\cos x}{\cos^2 x}$$

$$= \frac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$$ Ans

f) $y = \sin^3(\pi x^2)$

Diff: w.r.t. x

$$\frac{dy}{dx} = \frac{d}{dx}\sin^3(\pi x^2) \quad \text{(Power Rule)}$$

$$= 3\sin^2(\pi x^2)\frac{d}{dx}\sin(\pi x^2)$$

$$= 3\sin^2(\pi x^2).\cos \pi x^2 \frac{d}{dx}(\pi x^2)$$

$$= 3\sin^2(\pi x^2)\cos(\pi x^2).\pi.2x$$

$$= 6\pi x \sin^2 \pi x^2.\cos \pi x^2$$ Ans

2) a) $y = 2\cot 3x$

Diff: w.r.t. x

$$\frac{dy}{dx} = 2\frac{d}{dx}\cot 3x$$

$$= 2\left[-\operatorname{cosec}^2 3x \frac{d}{dx}3x\right]$$

$$= -2\operatorname{cosec}^2 3x.3.1$$

$$= -6\operatorname{cosec}^2 3x$$ Ans

b) $y = \sec \pi x$

Diff: w.r.t. x

$$\frac{dy}{dx} = \frac{d}{dx}\sec \pi x$$

$$= \sec \pi x.\tan \pi x \frac{d}{dx}(\pi x)$$

$$= \sec \pi x.\tan \pi x \ \pi.1$$

$$= \pi \sec \pi x \tan \pi x$$ Ans

c) $y = 4\operatorname{cosec} 2x$

Diff: w.r.t. x

$$\frac{dy}{dx} = 4\frac{d}{dx}\operatorname{cosec} 2x$$

$$= 4\left(-\operatorname{cosec} 2x.\cot 2x \frac{d}{dx}2x\right)$$

$$= -4\operatorname{cosec} 2x.\cot 2x.2.1$$

$$= -8\operatorname{cosec} 2x.\cot 2x$$ Ans

d) $y = 2\tan(x+3)^2$

Diff w.r.t. x

$$\frac{dy}{dx} = 2\frac{d}{dx}\tan(x+3)^2$$

$$= 2.\sec^2(x+3)^2 \frac{d}{dx}(x+3)^2$$

$$= 2\sec^2(x+3)^2 \left\{2(x+3)^1 \frac{d}{dx}(x+3)\right\}$$

$$= 4\sec^2(x+3)^2.(x+3)(1+0)$$

$$= 4(x+3)\sec^2(x+3)^2$$ Ans

e) $y = 4\cot\sqrt{x^2-1}$

Diff: w.r.t. x

$\frac{dy}{dx} = 4\frac{d}{dx}\cot\sqrt{x^2-1}$

$= 4\{-\operatorname{Cosec}^2\sqrt{x^2-1}\ \frac{d}{dx}\sqrt{x^2-1}\}$

$= -\cancel{4}^{2}\operatorname{Cosec}^2\sqrt{x^2-1}\cdot\frac{1}{\cancel{2}}(x^2-1)^{-1/2}\frac{d}{dx}(x^2-1)$

$= -\frac{2\operatorname{Cosec}^2\sqrt{x^2-1}}{(x^2-1)^{+1/2}}(2x-0)$

$= -\frac{4x\operatorname{Cosec}^2\sqrt{x^2-1}}{\sqrt{x^2-1}}$ Ans

f) $y = \sec^2 x^3$

Diff: w.r.t. x $\quad$ $/(\sec x^3)^2$

$\frac{dy}{dx} = \frac{d}{dx}\sec^2 x^3$

$= 2\sec^1 x^3\frac{d}{dx}\sec x^3$

$= 2\sec x^3\{\sec x^3.\tan x^3\frac{d}{dx}x^3\}$

$= 2\sec^2 x^3\tan x^3\cdot 3x^2$

$= 6x^2\sec^2 x^3.\tan x^3$

Ans

g) $y = 2\operatorname{cosec}^3(x+2)$

Diff: w.r.t. x

$\frac{dy}{dx} = 2\frac{d}{dx}\operatorname{cosec}^3(x+2)$

$= 2.\{3\operatorname{cosec}^2(x+2)\frac{d}{dx}\operatorname{Cosec}(x+2)\}$

$= 6\operatorname{cosec}^2(x+2)\{-\operatorname{Cosec}(x+2)\operatorname{Cot}(x+2)\frac{d}{dx}(x+2)$

$= -6\operatorname{Cosec}^3(x+2).\cot(x+2)\ (1+0)$

$= -6\operatorname{cosec}^3(x+2).\cot(x+2)$

h) $y = \frac{1+\tan 2x}{\operatorname{cosec}3x}$

Diff: w.r.t. x

$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{1+\tan 2x}{\operatorname{cosec}3x}\right)$

$= \frac{\operatorname{cosec}3x\ \frac{d}{dx}(1+\tan 2x) - (1+\tan 2x)\frac{d}{dx}\operatorname{cosec}3x}{\operatorname{cosec}^2 3x}$

$$\frac{dy}{dx} = \frac{\operatorname{cosec}3x\left[0+\sec^2 2x \frac{d}{dx}2x\right] - (1+\tan 2x)\left\{-\operatorname{cosec}3x.\cot 3x \frac{d}{dx}3x\right\}}{\operatorname{cosec}^2 3x}$$

$$= \frac{\operatorname{cosec}3x.\sec^2 2x.\ 2\cdot 1 + (1+\tan 2x)\operatorname{cosec}3x.\cot 3x\ 3\cdot 1}{\operatorname{cosec}^2 3x}$$

$$= \frac{2\operatorname{cosec}3x.\sec^2 2x + 3(1+\tan 2x).\operatorname{cosec}3x.\cot 3x}{\operatorname{cosec}^2 3x}$$

$$= \frac{\cancel{\operatorname{cosec}3x}\left[2\sec^2 2x + 3(1+\tan 2x).\cot 3x\right]}{\operatorname{cosec}^{\cancel{2}} 3x}$$

$$= \frac{2\sec^2 2x + 3(1+\tan 2x).\cot 3x}{\operatorname{cosec}3x} \quad \text{Ans}$$

Note ① $\frac{d}{dx}\sin^{-1}u = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$ & $\frac{d}{dx}\cos^{-1}u = \frac{-1}{\sqrt{1-u^2}}\frac{du}{dx}$

② $\frac{d}{dx}\tan^{-1}u = \frac{1}{1+u^2}\frac{du}{dx}$ & $\frac{d}{dx}\cot^{-1}u = \frac{-1}{1+u^2}\frac{du}{dx}$

3) $\frac{d}{dx}\sec^{-1}u = \frac{1}{u\sqrt{u^2-1}}\cdot\frac{du}{dx}$ & $\frac{d}{dx}\operatorname{cosec}^{-1}u = \frac{-1}{u\sqrt{u^2-1}}\frac{du}{dx}$

4) $\frac{d}{dx}e^u = e^u\frac{du}{dx}$ & $\frac{d}{dx}\ln u = \frac{1}{u}\frac{du}{dx}$

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③ Use any suitable rule of Differentiation

a) $y = \cos^{-1}(x+4)$

Diff : w.r.t.x

$$\frac{dy}{dx} = \frac{d}{dx}\cos^{-1}(x+4)$$

$$= \frac{-1}{\sqrt{1-(x+4)^2}}\frac{d}{dx}(x+4)$$

$$= \frac{-1}{\sqrt{1-(x+4)^2}}(1+0)$$

$$= \frac{-1}{\sqrt{1-(x+4)^2}} \quad \text{Ans}$$

b) $y = \tan^{-1}(11x)$

Diff : w.r.t.x

$$\frac{dy}{dx} = \frac{d}{dx}\tan^{-1}(11x)$$

$$= \frac{1}{1+(11x)^2}\frac{d}{dx}(11x)$$

$$= \frac{1}{1+(11x)^2}\cdot 11\cdot 1$$

$$= \frac{11}{1+(11x)^2} \quad \text{Ans}$$

c) $y = \cot^{-1}(e^x)$

Diff: w.r.t. x

$$\frac{dy}{dx} = \frac{d}{dx}\cot^{-1}(e^x)$$

$$= \frac{-1}{1+(e^x)^2}\frac{d}{dx}e^x$$

$$= \frac{-1}{1+e^{2x}}e^x = \frac{-e^x}{1+e^{2x}} \text{ Ans}$$

d) $y = \operatorname{cosec}^{-1}(e^{2x})$

Diff w.r.t. x

$$\frac{dy}{dx} = \frac{d}{dx}\operatorname{cosec}^{-1}(e^{2x})$$

$$= \frac{-1}{e^{2x}\sqrt{(e^{2x})^2-1}}\frac{d}{dx}e^{2x}$$

$$= \frac{-1}{\cancel{e^{2x}}\sqrt{e^{4x}-1}}\cancel{e^{2x}}\frac{d}{dx}(2x)$$

$$= \frac{-1}{\sqrt{e^{4x}-1}}\cdot 2\cdot 1 = \frac{-2}{\sqrt{e^{4x}-1}} \text{ Ans}$$

e) $y = \operatorname{cosec}^{-1}(x+3)$

Diff: w.r.t. x

$$\frac{dy}{dx} = \frac{d}{dx}\operatorname{cosec}^{-1}(x+3)$$

$$= \frac{-1}{(x+3)\sqrt{(x+3)^2-1}}\frac{d}{dx}(x+3)$$

$$= \frac{-1}{(x+3)\sqrt{(x+3)^2-1}}(1+0)$$

$$= \frac{-1}{(x+3)\sqrt{(x+3)^2-1}} \text{ Ans}$$

f) $y = \ln(\sin^{-1}x)$

Diff: w.r.t. x

$$\frac{dy}{dx} = \frac{d}{dx}\ln(\sin^{-1}x)$$

$$= \frac{1}{\sin^{-1}x}\frac{d}{dx}\sin^{-1}x$$

$$= \frac{1}{\sin^{-1}x}\cdot\frac{1}{\sqrt{1-x^2}}$$

$$= \frac{1}{\sin^{-1}x\cdot\sqrt{1-x^2}}$$

or

$$= \frac{1}{\sqrt{1-x^2}\cdot\sin^{-1}x}$$

4) Given Profit equation is

$$P(t) = 5 - 5\cos\frac{\pi t}{26} \qquad 0 \le t \le 104$$

a) Diff: w.r.t. t

$$P'(t) = \frac{d}{dt}\left[5 - 5\cos\frac{\pi t}{26}\right]$$

$$= 0 - 5\frac{d}{dx}\cos\frac{\pi t}{26}$$

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$$P'(t) = -5\left\{-\sin\frac{\pi t}{26}\ \frac{d}{dt}\frac{\pi t}{26}\right\}$$

$$= +5\sin\frac{\pi t}{26}\cdot\frac{\pi}{26}\cdot 1$$

$$\boxed{P'(t) = \frac{5\pi}{26}\sin\frac{\pi t}{26}} \longrightarrow (1)$$

$\boxed{\pi = 3.1416}$

Use calculator in radian mode

b)(i) Rate of change of profit 8 weeks after:

put $t=8$ in (1)

$$P'(8) = \frac{5\pi}{26}\sin\frac{\pi(8)}{26} = 0.497 \approx 0.5$$

Since Profit is in hundred of dollars for a week

so rate of change is 50 dollars per week

(ii) put $t=26$ in (1)

$$P'(26) = \frac{5\pi}{26}\sin\frac{\pi(\cancel{26})}{\cancel{26}}$$

$$= \frac{5\pi}{26}\sin\pi = 0$$

so rate of change is \$0/week

(iii) put $t=50$ in (1)

$$P'(50) = \frac{5\pi}{26}\sin\frac{\pi(50)}{26}$$

$$= -0.14$$

$\Rightarrow P'(50) = -14$ dollar/week

5) Given equation of Volume of air in lungs is

$$V(t) = 0.45 - 0.35\cos\frac{\pi t}{2} \qquad 0 \le t \le 8$$

a) Diff: w.r.t. t

$$V'(t) = \frac{d}{dt}\left[0.45 - 0.35\cos\frac{\pi t}{2}\right]$$

$$= 0 - 0.35\frac{d}{dx}\cos\frac{\pi t}{2}$$

$$= -0.35\left\{-\sin\frac{\pi t}{2}\ \frac{d}{dt}\frac{\pi t}{2}\right\}$$

$$= +0.35\sin\frac{\pi t}{2}\ \frac{\pi}{2}\cdot 1$$

(use calculator in radian mode)

$$V'(t) = 0.35\frac{\pi}{2}\sin\frac{\pi t}{2} \longrightarrow (1)$$

b) put $t=3$ in ①

$$v'(t) = 0.35\frac{\pi}{2}\sin\left(\frac{\pi(3)}{2}\right)$$

$$= 0.35\times\frac{3.1416}{2}\sin\frac{3\pi}{2}$$

$$= -0.55 \text{ liter/sec}$$

put $t=4$ in ①

$$v'(4) = 0.35\frac{\pi}{2}\sin\left(\frac{\pi(4)}{2}\right)$$

$$= 0.35\frac{\pi}{2}\sin 2\pi$$

$$= 0 \text{ liter/sec}$$

put $t=5$ in ①

$$v'(t) = 0.35\frac{\pi}{2}\sin\left(\frac{\pi(5)}{2}\right)$$

$$= \frac{0.35\times 3.1416}{2}\sin\frac{5\pi}{2}$$

$$= 0.55 \text{ liter/sec}$$

Differentiation of Exponential & logarithmic Function

Note $\lim_{x\to 0}\frac{a^x-1}{x} = \ln a$ & $\lim_{x\to 0}\frac{e^x-1}{x} = \ln e = 1$

Derivative of a^x

Let $f(x) = a^x$

Replace x by $x+\Delta x$

$$f(x+\Delta x) = a^{x+\Delta x} = a^x a^{\Delta x}$$

$$\because f'(x) = \lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$$

$$= \lim_{\Delta x\to 0}\frac{a^x a^{\Delta x}-a^x}{\Delta x} = \lim_{\Delta x\to 0}\frac{a^x(a^{\Delta x}-1)}{\Delta x}$$

$$= a^x\lim_{\Delta x\to 0}\frac{a^{\Delta x}-1}{\Delta x}$$

$$\left(\because \lim_{x\to 0}\frac{a^x-1}{x} = \ln a\right)$$

$$= a^x\ln a$$

Note $\boxed{\frac{d}{dx}a^x = a^x\ln a}$ Similarly $\frac{d}{dx}e^x = e^x\ln e$ ($\ln e = 1$)

$$\boxed{\frac{d}{dx}e^x = e^x}$$

Note $\frac{d}{dx}a^{f(x)} = a^{f(x)}\ln a\frac{d}{dx}f(x)$ & $\frac{d}{dx}e^{f(x)} = e^{f(x)}\frac{d}{dx}f(x)$

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Note ① $\frac{d}{dx}\log_a x = \frac{1}{x}\log_a e$ or $\frac{d}{dx}\log_a f(x) = \frac{1}{f(x)}\log_a e \, \frac{d}{dx}f(x)$

② $\frac{d}{dx}\ln x = \frac{1}{x}$ or $\frac{d}{dx}\ln f(x) = \frac{1}{f(x)} \cdot \frac{d}{dx}f(x)$

Differentiation of Hyperbolic functions

Note $\sinh x = \frac{e^x - e^{-x}}{2}$ & $\cosh x = \frac{e^x + e^{-x}}{2}$

$\operatorname{cosech} x = \frac{2}{e^x - e^{-x}}$ & $\operatorname{sech} x = \frac{2}{e^x + e^{-x}}$

$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$ & $\coth x = \frac{e^x + e^{-x}}{e^x - e^{-x}}$

Note ① $\cosh^2 x - \sinh^2 x = 1$

② $1 - \tanh^2 x = \operatorname{sech}^2 x$

③ $\coth^2 x - 1 = \operatorname{cosech}^2 x$

$\frac{d}{dx}\sinh x = ?$

Sol
$$\frac{d}{dx}\sinh x = \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right)$$
$$= \frac{\frac{d}{dx}e^x - \frac{d}{dx}e^{-x}}{2}$$
$$= \frac{e^x - e^{-x}\frac{d}{dx}(-x)}{2}$$
$$= \frac{e^x - e^{-x}(-1)}{2}$$
$$= \frac{e^x + e^{-x}}{2}$$
$$\boxed{\frac{d}{dx}\sinh x = \cosh x}$$

$\frac{d}{dx}\cosh x = ?$

Sol
$$\frac{d}{dx}\cosh x = \frac{d}{dx}\left(\frac{e^x + e^{-x}}{2}\right)$$
$$= \frac{\frac{d}{dx}e^x + \frac{d}{dx}e^{-x}}{2}$$
$$= \frac{e^x + e^{-x}\frac{d}{dx}(-x)}{2}$$
$$= \frac{e^x + e^{-x}(-1)}{2}$$
$$= \frac{e^x - e^{-x}}{2}$$
$$\boxed{\frac{d}{dx}\cosh x = \sinh x}$$

Now $\frac{d}{dx}\tanh x = \frac{d}{dx}\left(\frac{\sinh x}{\cosh x}\right)$

$$\frac{d}{dx}\tanh x = \frac{\cosh x \frac{d}{dx}\sinh x - \sinh x \frac{d}{dx}\cosh x}{\cosh^2 x}$$

$$= \frac{\cosh x . \cosh x - \sinh x . \sinh x}{\cosh^2 x} = \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} = \frac{1}{\cosh^2 x}$$

$$\boxed{\frac{d}{dx}\tanh x = \operatorname{sech}^2 x}$$

Similarly we can show that

$\frac{d}{dx}\coth x = -\operatorname{cosech}^2 x$, $\frac{d}{dx}\operatorname{sech} x = -\operatorname{sech} x . \tanh x$.

$\frac{d}{dx}\operatorname{cosech} x = -\operatorname{cosech} x . \coth x$

Derivative of Inverse hyperbolic functions

Note (1) $\cosh^2 x - \sinh^2 x = 1$

(2) $\cosh^2 x = 1 + \sinh^2 x$ or $\cosh x = \sqrt{1+\sinh^2 x}$

3) $\sinh^2 x = \cosh^2 x - 1$ or $\sinh x = \sqrt{\cosh^2 x - 1}$

4) $1 - \tanh^2 x = \operatorname{sech}^2 x$

5) $\coth^2 x - 1 = \operatorname{cosech}^2 x$

$\frac{d}{dx}\sinh^{-1}x = ?$

Sol Let $y = \sinh^{-1}x$

$\Rightarrow \boxed{\sinh y = x}$

Diff: w.r.t. x

$\cosh y \frac{dy}{dx} = 1$

$\frac{dy}{dx} = \frac{1}{\cosh y}$

$\Rightarrow \frac{d}{dx}\sinh^{-1}x = \frac{1}{\sqrt{1+\sinh^2 y}}$

$= \frac{1}{\sqrt{1+x^2}}$

$\frac{d}{dx}\cosh^{-1}x = ?$

Sol Let $y = \cosh^{-1}x$

$\Rightarrow \boxed{\cosh y = x}$

Diff: w.r.t. x

$\sinh y \frac{dy}{dx} = 1$

$\Rightarrow \frac{dy}{dx} = \frac{1}{\sinh y}$

$\Rightarrow \frac{d}{dx}\cosh^{-1}x = \frac{1}{\sqrt{\cosh^2 y - 1}}$

$= \frac{1}{\sqrt{x^2-1}}$

Similarly we can show that

$\frac{d}{dx}\tanh^{-1}x = \frac{1}{1-x^2}$, $\frac{d}{dx}\coth^{-1}x = \frac{-1}{x^2-1}$ or $\frac{1}{1-x^2}$

$\frac{d}{dx}\text{Sech}^{-1}x = \frac{-1}{x\sqrt{1-x^2}}$, $\frac{d}{dx}\text{cosech}^{-1}x = \frac{-1}{x\sqrt{x^2+1}}$

EXERCISE 2.7

Note ① $\frac{d}{dx}\ln f(x) = \frac{1}{f(x)}\frac{d}{dx}f(x)$ or $\frac{d}{dx}\ln x = \frac{1}{x}$

② $\ln a^b = b\ln a$ ③ $\ln ab = \ln a + \ln b$ ④ $\ln\frac{a}{b} = \ln a - \ln b$

Q1 a) $y = x\ln x^2$

Diff w.r.t. x

$\frac{dy}{dx} = \frac{d}{dx}(x\ln x^2)$

$= x\frac{d}{dx}\ln x^2 + \ln x^2\frac{d}{dx}x$

$= x\cdot\frac{1}{x^2}\frac{d}{dx}x^2 + \ln x^2\cdot 1$

$= \frac{1}{\cancel{x}}\cdot 2\cancel{x} + \ln x^2$

$= 2 + \ln x^2$ Ans

b) $y = \ln(x^2+3x+2)$

Diff w.r.t. x

$\frac{dy}{dx} = \frac{d}{dx}\ln(x^2+3x+2)$

$= \frac{1}{x^2+3x+2}\frac{d}{dx}(x^2+3x+2)$

$= \frac{1}{x^2+3x+2}(2x+3\cdot1+0)$

$= \frac{2x+3}{x^2+3x+2}$ Ans

c) $y = \frac{\ln 5x}{x^8}$

Diff w.r.t. x

$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\ln 5x}{x^8}\right)$

$= \frac{x^8\frac{d}{dx}\ln 5x - \ln 5x\frac{d}{dx}x^8}{(x^8)^2}$

$= \frac{x^{\cancel{8}7}\left\{\frac{1}{5\cancel{x}}\frac{d}{dx}5x\right\} - \ln 5x\cdot 8x^7}{x^{16}}$

$= \frac{x^7\{\frac{1}{5}5\cdot1\} - 8x^7\ln 5x}{x^{16}} = \frac{\cancel{x^7}(1-8\ln 5x)}{x^{\cancel{16}\,9}} = \frac{1-8\ln 5x}{x^9}$ Ans

d) $y = \ln(x^2+1)^{1/2}$ ($\ln a^b = b\ln a$)

$y = \frac{1}{2}\ln(x^2+1)$

Diff w.r.t. x

$\frac{dy}{dx} = \frac{1}{2}\frac{d}{dx}\ln(x^2+1)$

$= \frac{1}{2}\cdot\frac{1}{x^2+1}\frac{d}{dx}(x^2+1)$

$= \frac{1}{\cancel{2}}\cdot\frac{1}{x^2+1}\cdot(\cancel{2}x+0) = \frac{x}{x^2+1}$ Ans

e) $y = \dfrac{1}{\ln(x^2+1)}$

Diff w.r.t. x

$$\frac{dy}{dx} = \frac{d}{dx}\left[\ln(x^2+1)\right]^{-1}$$

$$= -1\left[\ln(x^2+1)\right]^{-2}\frac{d}{dx}\ln(x^2+1)$$

$$= \frac{-1}{(\ln(x^2+1))^2}\cdot\frac{1}{x^2+1}\frac{d}{dx}(x^2+1)$$

$$= \frac{-1}{(x^2+1)(\ln(x^2+1))^2}(2x+0)$$

$$= \frac{-2x}{(x^2+1)\left[\ln(x^2+1)\right]^2} \text{ Ans}$$

f) $y = \sqrt[3]{\ln(1-x^2)}$

Diff: w.r.t. x

$$\frac{dy}{dx} = \frac{d}{dx}\left[\ln(1-x^2)\right]^{1/3}$$

$$= \frac{1}{3}\left[\ln(1-x^2)\right]^{1/3-1}\frac{d}{dx}\ln(1-x^2)$$

$$= \frac{1}{3}\left[\ln(1-x^2)\right]^{-2/3}\frac{1}{1-x^2}\frac{d}{dx}(1-x^2)$$

$$= \frac{1}{3\left(\ln(1-x^2)\right)^{2/3}}\cdot\frac{1}{(1-x^2)}(0-2x)$$

$$= \frac{-2x}{3(1-x^2)\left[\ln(1-x^2)\right]^{2/3}} \text{ Ans.}$$

Note ① $\dfrac{d}{dx}e^u = e^u\dfrac{du}{dx}$ or $\dfrac{d}{dx}e^x = e^x$

② $\dfrac{d}{dx}a^u = a^u\ln a\dfrac{du}{dx}$ or $\dfrac{d}{dx}a^x = a^x\ln a$

Q2 a) $y = 5^{x+1}$

Diff: w.r.t. x

$$\frac{dy}{dx} = \frac{d}{dx}5^{x+1}$$

$$= 5^{x+1}\ln 5\frac{d}{dx}(x+1)$$

$$= 5^{x+1}\ln 5\,(1+0)$$

$$= 5^{x+1}\ln 5 \text{ Ans}$$

b) $y = e^{\sqrt{x}}$

Diff: w.r.t. x

$$\frac{dy}{dx} = \frac{d}{dx}e^{\sqrt{x}}$$

$$= e^{\sqrt{x}}\frac{d}{dx}\sqrt{x}$$

$$= e^{\sqrt{x}}\cdot\frac{1}{2}x^{-1/2}$$

$$= \frac{e^{\sqrt{x}}}{2x^{1/2}} = \frac{e^{\sqrt{x}}}{2\sqrt{x}} \text{ Ans}$$

c) $y = (e^{-x}+e^{x})^2$

Diff w.r.t. x

$$\frac{dy}{dx} = \frac{d}{dx}(e^{-x}+e^{x})^2$$

$$= 2(e^{-x}+e^{x})^1\frac{d}{dx}(e^{-x}+e^{x})$$

$$= 2(e^{-x}+e^{x})\left[e^{-x}\frac{d}{dx}(-x)+e^{x}\right]$$

$$= 2(e^{-x}+e^{x})\left[e^{-x}(-1)+e^{x}\right]$$

$$= 2(e^{-x}+e^{x})(-e^{-x}+e^{x})$$

$$= 2(e^{x}+e^{-x})(e^{x}-e^{-x})$$

$$= 2\left[(e^{x})^2-(e^{-x})^2\right]$$

$$= 2\left[e^{2x}-e^{-2x}\right] \text{ Ans}$$

c) 2nd method

$y = (e^{-x} + e^{x})^2 \rightarrow (a+b)^2 = a^2 + b^2 + 2ab$

$y = e^{-2x} + e^{2x} + 2\cancel{e^{-x}}\cancel{e^{x}}$

Diff: w.r.t. x

$\frac{dy}{dx} = \frac{d}{dx}[e^{-2x} + e^{2x} + 2]$

$= e^{-2x}\frac{d}{dx}(-2x) + e^{2x}\frac{d}{dx}2x + 0$

$= e^{-2x}(-2) + e^{2x}(2)$

$= 2(-e^{-2x} + e^{2x})$

$= 2(e^{2x} - e^{-2x})$ Ans

d) $y = (e^{3x} - 1)^4$

Diff: w.r.t. x

$\frac{dy}{dx} = \frac{d}{dx}(e^{3x} - 1)^4$

$= 4(e^{3x}-1)^3 \frac{d}{dx}(e^{3x}-1)$

$= 4(e^{3x}-1)^3 [e^{3x}\frac{d}{dx}3x - 0]$

$= 4(e^{3x}-1)^3 [e^{3x} 3 \cdot 1]$

$= 12e^{3x}(e^{3x}-1)^3$ Ans

e) $y = xe^{x\ln x}$

Diff: w.r.t. x

$\frac{dy}{dx} = \frac{d}{dx}(xe^{x\ln x})$

$= x\frac{d}{dx}e^{x\ln x} + e^{x\ln x}\frac{d}{dx}x$

$= x\{e^{x\ln x}\frac{d}{dx}x\ln x\} + e^{x\ln x} \cdot 1$

$= xe^{x\ln x}\{x\frac{d}{dx}\ln x + \ln x \cdot \frac{d}{dx}x\} + e^{x\ln x}$

$= xe^{x\ln x}\{\cancel{x} \cdot \frac{1}{\cancel{x}} + \ln x \cdot 1\} + e^{x\ln x}$

$= xe^{x\ln x}\{1 + \ln x\} + e^{x\ln x}$

$= e^{x\ln x}[x(1+\ln x) + 1]$

$= e^{x\ln x}[x + \ln x + 1]$ Ans

f) $y = 5^{x^2 - x}$

Diff: w.r.t. x

$\frac{dy}{dx} = \frac{d}{dx}5^{x^2-x}$

$= 5^{x^2-x}\ln 5 \frac{d}{dx}(x^2 - x)$

$= 5^{x^2-x}\ln 5\,(2x-1)$

$= (2x-1)\,5^{x^2-x}\ln 5$ Ans

Note: $\frac{d}{dx}\log_a x = \frac{1}{x}\log_a e$

or $\frac{d}{dx}\log_a u = \frac{1}{u}\log_a e \;\frac{d}{dx}u$

$\frac{d}{dx}\ln x = \frac{1}{x}$

or

$\frac{d}{dx}\ln u = \frac{1}{u}\frac{du}{dx}$

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3) a) $y = \log_{10}(3x^2+7)$

Diff w.r.t. x

$$\frac{dy}{dx} = \frac{d}{dx}\log_{10}(3x^2+7)$$

$$= \frac{1}{3x^2+7}\log_{10}e\,\frac{d}{dx}(3x^2+7)$$

$$= \frac{1}{3x^2+7}\log_{10}e\,(3\cdot 2x+0)$$

$$= \frac{6x\log_{10}e}{3x^2+7} \quad \text{Ans}$$

b) $y = \log_{10}(x^2+3x+2)$

Diff: w.r.t. x

$$\frac{dy}{dx} = \frac{d}{dx}\log_{10}(x^2+3x+2)$$

$$= \frac{1}{x^2+3x+2}\log_{10}e\,\frac{d}{dx}(x^2+3x+2)$$

$$= \frac{1}{x^2+3x+2}\log_{10}e\,(2x+3\cdot 1+0)$$

$$= \frac{(2x+3)\log_{10}e}{x^2+3x+2} \quad \text{Ans}$$

c) $y = \log_{10}\sqrt{x^2-7x} + x^3$ $\quad \log a^b = b\log a$

$$y = \frac{1}{2}\log_{10}(x^2-7x) + x^3$$

Diff w.r.t. x

$$\frac{dy}{dx} = \frac{d}{dx}\left[\frac{1}{2}\log_{10}(x^2-7x)+x^3\right]$$

$$= \frac{1}{2}\cdot\frac{1}{x^2-7x}\log_{10}e\,\frac{d}{dx}(x^2-7x) + 3x^2$$

$$= \frac{1}{2(x^2-7x)}\log_{10}e\,(2x-7\cdot 1) + 3x^2$$

$$= \frac{(2x-7)\log_{10}e}{2(x^2-7x)} + 3x^2 \quad \text{Ans}$$

d) $y = \ln[\sin(\ln x)]$

Diff w.r.t. x

$$\frac{dy}{dx} = \frac{d}{dx}\ln[\sin(\ln x)]$$

$$= \frac{1}{\sin(\ln x)}\frac{d}{dx}\sin(\ln x)$$

$$= \frac{1}{\sin(\ln x)}\cdot\cos(\ln x)\frac{d}{dx}\ln x$$

$$= \frac{\cos(\ln x)}{\sin(\ln x)}\cdot\frac{1}{x}$$

$$= \cot(\ln x)\cdot\frac{1}{x}$$

$$= \frac{\cot(\ln x)}{x} \quad \text{Ans}$$

e) $y = \log_{10}(\sin^{-1}x^2)$

Diff w.r.t. x

$$\frac{dy}{dx} = \frac{d}{dx}\log_{10}(\sin^{-1}x^2)$$

$$= \frac{1}{\sin^{-1}x^2}\log_{10}e\,\frac{d}{dx}\sin^{-1}x^2$$

$$= \frac{1}{\sin^{-1}x^2}\log_{10}e\cdot\frac{1}{\sqrt{1-(x^2)^2}}\frac{d}{dx}x^2 = \frac{\log_{10}e}{\sin^{-1}x^2\cdot\sqrt{1-x^4}}\cdot 2x = \frac{2x\log_{10}e}{\sin^{-1}x^2\cdot\sqrt{1-x^4}} \quad \text{Ans}$$

f) $y = \ln \tan\left[\frac{1}{2}x + \frac{1}{4}\pi\right]$

Diff: w.r.t. x

$$\frac{dy}{dx} = \frac{d}{dx}\ln\tan\left[\frac{x}{2}+\frac{\pi}{4}\right]$$

$$= \frac{1}{\tan\left(\frac{x}{2}+\frac{\pi}{4}\right)}\frac{d}{dx}\tan\left[\frac{x}{2}+\frac{\pi}{4}\right]$$

$$= \cot\left(\frac{x}{2}+\frac{\pi}{4}\right).\sec^2\left(\frac{x}{2}+\frac{\pi}{4}\right)\frac{d}{dx}\left(\frac{x}{2}+\pi/4\right)$$

$$= \frac{\cancel{\cos\left(\frac{x}{2}+\pi/4\right)}}{\sin\left(\frac{x}{2}+\pi/4\right)}\cdot\frac{1}{\cos^2\left(\frac{x}{2}+\frac{\pi}{4}\right)}\cdot\left(\frac{1}{2}+0\right)$$

$$= \frac{1}{2\sin\left(\frac{x}{2}+\frac{\pi}{4}\right)\cos\left(\frac{x}{2}+\frac{\pi}{4}\right)}$$

$\left(\because 2\sin\theta\cos\theta = \sin 2\theta\right.$

$$= \frac{1}{\sin 2\left(\frac{x}{2}+\frac{\pi}{4}\right)}$$

$$= \frac{1}{\sin\left(x+\frac{\pi}{2}\right)} = \frac{1}{\cos x}$$

$\sin\left(x+\frac{\pi}{2}\right) = \cos x$

or $\sin\left(x+\frac{\pi}{2}\right) = \sin x\cos\frac{\pi}{2} + \cos x\sin\frac{\pi}{2}$

$= \sin x \cdot 0 + \cos x \cdot 1$

$= \cos x$

$$\frac{dy}{dx} = \sec x \quad \text{Ans}$$

Note ① $\log a^b = b\log a$ ② $\log ab = \log a + \log b$ ③ $\log\frac{a}{b} = \log a - \log b$

Note to find Derivative of $[f(x)]^{g(x)}$ use log & $\log a^b = b\log a$

---

4) a) $y = \ln\sqrt{\frac{x+1}{x-1}}$

$y = \frac{1}{2}\ln\frac{x+1}{x-1}$

$y = \frac{1}{2}\left[\ln(x+1) - \ln(x-1)\right]$

Diff: w.r.t. x

$$\frac{dy}{dx} = \frac{1}{2}\frac{d}{dx}\left[\ln(x+1) - \ln(x-1)\right] = \frac{1}{2}\left[\frac{1}{x+1}\frac{d}{dx}(x+1) - \frac{1}{x-1}\cdot\frac{d}{dx}(x-1)\right]$$

$$= \frac{1}{2}\left[\frac{1}{x+1}(1) - \frac{1}{x-1}(1)\right] = \frac{1}{2}\left(\frac{x-1-(x+1)}{(x+1)(x-1)}\right) = \frac{1}{2}\left(\frac{\cancel{x}-1-\cancel{x}-1}{x^2-1^2}\right) = \frac{-2}{\cancel{2}(x^2-1)} = \frac{-1}{x^2-1}$$

Ans

Q7

b) $y = (\cos x)^{\ln x}$

taking ln of both sides

$\ln y = \ln(\cos x)^{\ln x}$ ($\ln a^b = b\ln a$)

$\ln y = \ln x \cdot \ln\cos x$

Diff: w.r.t. x.

$\frac{d}{dx}\ln y = \frac{d}{dx}(\ln x \cdot \ln\cos x)$

$\frac{1}{y}\frac{dy}{dx} = \ln x \cdot \frac{d}{dx}\ln\cos x + \ln\cos x \frac{d}{dx}\ln x$

$\frac{dy}{dx} = y\left[\ln x \cdot \frac{1}{\cos x}\frac{d}{dx}\cos x + \ln\cos x \cdot \frac{1}{x}\right]$

$= (\cos x)^{\ln x}\left[\frac{\ln x}{\cos x}(-\sin x) + \frac{\ln\cos x}{x}\right]$

$\frac{dy}{dx} = (\cos x)^{\ln x}\left[-\tan x \cdot \ln x + \frac{\ln\cos x}{x}\right]$

c) $y = (1+x^{-1})^{x}$

taking ln of B. sides

$\ln y = \ln(1+x^{-1})^{x}$ ($\ln a^b = b\ln a$)

$\ln y = x\ln(1+x^{-1})$

Diff w.r.t. x

$\frac{d}{dx}\ln y = \frac{d}{dx}[x\ln(1+x^{-1})]$

$\frac{1}{y}\frac{dy}{dx} = x\frac{d}{dx}\ln(1+x^{-1}) + \ln(1+x^{-1})\frac{d}{dx}x$

$\frac{dy}{dx} = y\left[x \cdot \frac{1}{1+x^{-1}}\frac{d}{dx}(1+x^{-1}) + \ln(1+x^{-1}) \cdot 1\right]$

$= y\left[x\frac{1}{1+\frac{1}{x}}(0-1\cdot x^{-2}) + \ln(1+x^{-1})\right]$

$= y\left[\frac{x}{\frac{x+1}{\cancel{x}}}\left(\frac{-1}{\cancel{x^2}}\right) + \ln(1+x^{-1})\right]$

$= (1+x^{-1})^{x}\left(\frac{-1}{1+x} + \ln(1+x^{-1})\right]$

d) $y = \frac{(1-x)^{1/2}(2-x^2)^{2/3}}{(3-x^2)^{3/4}(4-x^2)^{4/5}}$

taking ln of B. sides

$\ln y = \ln\frac{(1-x)^{1/2}(2-x^2)^{2/3}}{(3-x^2)^{3/4}(4-x^2)^{4/5}}$

$\ln y = \ln(1-x)^{1/2}(2-x^2)^{2/3} - \ln(3-x^2)^{3/4}(4-x^2)^{4/5}$ ($\ln\frac{a}{b} = \ln a - \ln b$)

$\ln y = \ln(1-x)^{1/2} + \ln(2-x^2)^{2/3} - \ln(3-x^2)^{3/4} - \ln(4-x^2)^{4/5}$ ($\ln ab = \ln a + \ln b$)

$\ln y = \frac{1}{2}\ln(1-x) + \frac{2}{3}\ln(2-x^2) - \frac{3}{4}\ln(3-x^2) - \frac{4}{5}\ln(4-x^2)$ ($\ln a^b = b\ln a$)

Diff: w.r.t. x.

$\frac{1}{y}\frac{dy}{dx} = \frac{1}{2}\cdot\frac{1}{1-x}\frac{d}{dx}(1-x) + \frac{2}{3}\cdot\frac{1}{2-x^2}\frac{d}{dx}(2-x^2) - \frac{3}{4}\frac{1}{3-x^2}\frac{d}{dx}(3-x^2) - \frac{4}{5}\cdot\frac{1}{4-x^2}\frac{d}{dx}(4-x^2)$

$\frac{dy}{dx} = y\left[\frac{1}{2(1-x)}(-1) + \frac{2}{3}\frac{1}{2-x^2}(0-2x) - \frac{3}{\cancel{4}_2(3-x^2)}(0-\cancel{2}x) - \frac{4}{5(4-x^2)}(0-2x)\right]$

$= \frac{(1-x)^{1/2}(2-x^2)^{2/3}}{(3-x^2)^{3/4}(4-x^2)^{4/5}}\left[\frac{-1}{2(1-x)} - \frac{4x}{3(2-x^2)} + \frac{3x}{2(3-x^2)} + \frac{8x}{5(4-x^2)}\right]$ Ans

e) $y = \dfrac{x\sqrt[3]{x^2+4}}{\sqrt{x^2+3}}$

taking ln of B. sides

$\ln y = \ln \dfrac{x(x^2+4)^{1/3}}{(x^2+3)^{1/2}}$ ($\ln\frac{a}{b} = \ln a - \ln b$

$\ln y = \ln x(x^2+4)^{1/3} - \ln(x^2+3)^{1/2}$ ($\ln ab = \ln a + \ln b$

$\ln y = \ln x + \ln(x^2+4)^{1/3} - \ln(x^2+3)^{1/2}$ ($\ln a^b = b\ln a$

$\ln y = \ln x + \frac{1}{3}\ln(x^2+4) - \frac{1}{2}\ln(x^2+3)$

Diff: w.r.t x

$$\frac{d}{dx}\ln y = \frac{d}{dx}\left(\ln x + \frac{1}{3}\ln(x^2+4) - \frac{1}{2}\ln(x^2+3)\right)$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x} + \frac{1}{3}\cdot\frac{1}{x^2+4}\frac{d}{dx}(x^2+4) - \frac{1}{2}\cdot\frac{1}{x^2+3}\frac{d}{dx}(x^2+3)$$

$$\frac{dy}{dx} = y\left[\frac{1}{x} + \frac{1}{3(x^2+4)}(2x+0) - \frac{1}{\cancel{2}(x^2+3)}(\cancel{2}x+0)\right]$$

$$= \frac{x\sqrt[3]{x^2+4}}{\sqrt{x^2+4}}\left[\frac{1}{x} + \frac{2x}{3(x^2+4)} - \frac{x}{x^2+3}\right] \quad \text{Ans}$$

---

f) $y = (\sin x)(\ln x)\,x^x$

taking ln of B. sides

$\ln y = \ln[(\sin x)(\ln x)\,x^x]$ ($\ln abc = \ln a + \ln b + \ln c$

$= \ln\sin x + \ln(\ln x) + \ln x^x$

$\ln y = \ln\sin x + \ln(\ln x) + x\ln x$ $\ln a^b = b\ln a$

Diff: w.r.t x

$\frac{d}{dx}\ln y = \frac{d}{dx}[\ln\sin x + \ln(\ln x) + x\ln x]$ (Product Rule)

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{\sin x}\frac{d}{dx}\sin x + \frac{1}{\ln x}\frac{d}{dx}\ln x + \left\{x\frac{d}{dx}\ln x + \ln x\frac{d}{dx}x\right\}$$

$$\frac{dy}{dx} = y\left[\frac{1}{\sin x}\cos x + \frac{1}{\ln x}\cdot\frac{1}{x} + \cancel{x}\cdot\frac{1}{\cancel{x}} + \ln x\cdot 1\right]$$

$$= (\sin x)(\ln x)\,x^x\left[\cot x + \frac{1}{x\ln x} + 1 + \ln x\right] \quad \text{Ans}$$

Note ① $\frac{d}{dx}\sinh u = \cosh u \frac{du}{dx}$ & $\frac{d}{dx}\cosh u = \sinh u \frac{du}{dx}$

② $\frac{d}{dx}\tanh u = \operatorname{sech}^2 u \frac{du}{dx}$ & $\frac{d}{dx}\coth u = -\operatorname{cosech}^2 u \frac{du}{dx}$

3) $\frac{d}{dx}\operatorname{sech} u = -\operatorname{sech} u \tanh u \frac{du}{dx}$ & $\frac{d}{dx}\operatorname{cosech} u = -\operatorname{cosech} u \coth u \frac{du}{dx}$

---

(5) a) $y = \cosh(2x^2+3x)$

Diff: w.r.t. x

$$\frac{dy}{dx} = \frac{d}{dx}\cosh(2x^2+3x)$$

$$= \sinh(2x^2+3x)\frac{d}{dx}(2x^2+3x)$$

$$= \sinh(2x^2+3x)[2\cdot 2x+3\cdot 1]$$

$$= (4x+3)\sinh(2x^2+3x)$$

b) $y = e^{\sinh^2 x}$

Diff: w.r.t. x

$$\frac{dy}{dx} = \frac{d}{dx}e^{\sinh^2 x}$$

$$= e^{\sinh^2 x}\frac{d}{dx}\sinh^2 x$$

$$= e^{\sinh^2 x}\{2\cdot\sinh x \frac{d}{dx}\sinh x\}$$

$$= e^{\sinh^2 x}\{2\sinh x\cdot\cosh x\}$$

$$= e^{\sinh^2 x}\sinh 2x \quad \text{Ans}$$

---

c) $y = \log(\cosh x)$

Diff: w.r.t. x

$$\frac{dy}{dx} = \frac{d}{dx}\log(\cosh x)$$

$$= \frac{1}{\cosh x}\frac{d}{dx}\cosh x$$

$$= \frac{1}{\cosh x}\cdot\sinh x$$

$$= \tanh x \quad \text{Ans}$$

d) $y = \operatorname{sech}(x^2+1) + \tanh(x^2+1)$

Diff w.r.t. x

$$\frac{dy}{dx} = \frac{d}{dx}\left(\operatorname{sech}(x^2+1) + \tanh(x^2+1)\right)$$

$$= -\operatorname{sech}(x^2+1)\tanh(x^2+1)\frac{d}{dx}(x^2+1) + \operatorname{sech}^2(x^2+1)\frac{d}{dx}(x^2+1)$$

$$= -\operatorname{sech}(x^2+1)\tanh(x^2+1)(2x+0) + \operatorname{sech}^2(x^2+1)(2x+0)$$

$$= 2x\left[-\operatorname{sech}(x^2+1)\cdot\tanh(x^2+1) + \operatorname{sech}^2(x^2+1)\right] \quad \text{Ans}$$

---

e) $y = \operatorname{cosech}(x^3+1)$

Diff w.r.t. x

$$\frac{dy}{dx} = \frac{d}{dx}\operatorname{cosech}(x^3+1)$$

$$= -\operatorname{cosech}(x^3+1)\coth(x^3+1)\frac{d}{dx}(x^3+1)$$

$$= -\operatorname{cosech}(x^3+1)\coth(x^3+1)(3x^2)$$

$$= -3x^2\operatorname{cosech}(x^3+1)\coth(x^3+1) \quad \text{Ans}$$

f) $x\cosh y = y\sinh x + 5$

Diff: w.r.t x

$$\frac{d}{dx}(x\cosh y) = \frac{d}{dx}(y\sinh x + 5)$$

$$x\frac{d}{dx}\cosh y + \cosh y\frac{d}{dx}x = y\frac{d}{dx}\sinh x + \sinh x\frac{d}{dx}y + 0$$

$$x\sinh y\frac{dy}{dx} + \cosh y \cdot 1 = y\cdot\cosh x + \sinh x\cdot\frac{dy}{dx}$$

$$x\sinh y\frac{dy}{dx} - \sinh x\frac{dy}{dx} = y\cosh x - \cosh y$$

$$(x\sinh y - \sinh x)\frac{dy}{dx} = y\cosh x - \cosh y$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y\cosh x - \cosh y}{x\sinh y - \sinh x}$$

Ans

Note ① $\frac{d}{dx}\sinh^{-1}u = \frac{1}{\sqrt{1+u^2}}\frac{du}{dx}$ & $\frac{d}{dx}\cosh^{-1}u = \frac{1}{\sqrt{u^2-1}}\frac{du}{dx}$

2) $\frac{d}{dx}\tanh^{-1}u = \frac{1}{1-u^2}\frac{du}{dx}$ & $\frac{d}{dx}\coth^{-1}u = \frac{-1}{u^2-1}\frac{du}{dx}$ or $\frac{1}{1-u^2}\frac{du}{dx}$

3) $\frac{d}{dx}\mathrm{sech}^{-1}u = \frac{-1}{u\sqrt{1-u^2}}\frac{du}{dx}$ & $\frac{d}{dx}\mathrm{cosech}^{-1}u = \frac{-1}{u\sqrt{1+u^2}}\frac{du}{dx}$

---

6 a) $y = \tanh^{-1}(\sin x)$

Diff: w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}\tanh^{-1}(\sin x)$$

$$= \frac{1}{1-\sin^2 x}\frac{d}{dx}\sin x$$

$$= \frac{1}{\cos^2 x}\cdot\cos x$$

$$= \frac{1}{\cos x} = \sec x$$

Ans

b) $y = \sinh^{-1}(\tan x)$

Diff w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}\sinh^{-1}(\tan x)$$

$$= \frac{1}{\sqrt{1+\tan^2 x}}\cdot\frac{d}{dx}\tan x$$

$$= \frac{1}{\sqrt{\sec^2 x}}\cdot\sec^2 x$$

$$= \frac{1}{\cancel{\sec x}}\cdot\sec^{\cancel{2}} x = \sec x$$

Ans

c) $y = \cosh^{-1}(\sec x)$

Diff: w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}\cosh^{-1}(\sec x)$$

$$= \frac{1}{\sqrt{\sec^2 x - 1}}\frac{d}{dx}\sec x.$$

$$= \frac{1}{\sqrt{\tan^2 x}}\cdot \sec x . \tan x$$

$$= \frac{1}{\cancel{\tan x}}\cdot \sec x\, \cancel{\tan x}$$

$$= \sec x . \quad \text{Ans}$$

d) $y = x\tanh^{-1}(3x)$

Diff: w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}\left[x\tanh^{-1}(3x)\right]$$

$$= x\frac{d}{dx}\tanh^{-1}3x + \tanh^{-1}3x\,\frac{d}{dx}x$$

$$= x\cdot\frac{1}{1-(3x)^2}\frac{d}{dx}(3x) + \tanh^{-1}3x\cdot 1$$

$$= \frac{x}{1-9x^2}(3\cdot 1) + \tanh^{-1}3x$$

$$= \frac{3x}{1-9x^2} + \tanh^{-1}(3x)$$

e) $y = x\cosh^{-1}x - \sqrt{x^2-1}$

Diff: w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}(x\cosh^{-1}x) - \frac{d}{dx}\sqrt{x^2-1}$$

$$= x\frac{d}{dx}\cosh^{-1}x + \cosh^{-1}x\frac{d}{dx}x - \frac{1}{2}(x^2-1)^{-1/2}\frac{d}{dx}(x^2-1)$$

$$= x\cdot\frac{1}{\sqrt{x^2-1}} + \cosh^{-1}x\cdot 1 - \frac{1}{\cancel{2}(x^2-1)^{1/2}}(\cancel{2}x-0)$$

$$= \cancel{\frac{x}{\sqrt{x^2-1}}} + \cosh^{-1}x - \cancel{\frac{x}{\sqrt{x^2-1}}} = \cosh^{-1}x \quad \text{Ans}$$

f) $\log(\cosh^{-1}x) + \sinh^{-1}y = 6$

Diff: w.r.t x

$$\frac{d}{dx}\log\cosh^{-1}x + \frac{d}{dx}\sinh^{-1}y = \frac{d}{dx}6$$

$$\frac{1}{\cosh^{-1}x}\frac{d}{dx}\cosh^{-1}x + \frac{1}{\sqrt{1+y^2}}\frac{dy}{dx} = 0$$

$$\frac{1}{\cosh^{-1}x}\cdot\frac{1}{\sqrt{x^2-1}} + \frac{1}{\sqrt{1+y^2}}\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{1}{\sqrt{1+y^2}}\frac{dy}{dx} = -\frac{1}{\sqrt{x^2-1}\cosh^{-1}x} \Rightarrow \frac{dy}{dx} = -\frac{\sqrt{1+y^2}}{\sqrt{x^2-1}\cosh^{-1}x} \quad \text{Ans}$$

7) Given equation is 102

$P(x) = 40 + 25\ln(x+1) \qquad 0 \le x \le 65$

Diff w.r.t. x

$P'(x) = \frac{d}{dx}[40 + 25\ln(x+1)]$

$= 0 + 25 \cdot \frac{1}{x+1} \frac{d}{dx}(x+1) = 25 \frac{1}{x+1}(1)$

$\boxed{P'(x) = \frac{25}{x+1}}$ —— (1)

$\boxed{\text{Rate of Pressure} = ?}$

At the end of 10 year

put $x = 10$ in (1)

$P'(10) = \frac{25}{10+1} = \frac{25}{11} = 2.27$

$P'(10) = 2.27$ mm of mercury/year

put $x = 30$ in (1)

$P'(30) = \frac{25}{30+1}$

$= 0.81$ mm of mercury/year

put $x = 60$ in (1)

$P'(60) = \frac{25}{60+1}$

$= \frac{25}{61}$

$= 0.41$ mm of mercury/year

---

8) Given equation is

$A(t) = 5000\, 2^{2t}$

Diff w.r.t. t

$A'(t) = 5000 \frac{d}{dt} 2^{2t}$

$= 5000 \{2^{2t} \ln 2 \frac{d}{dt} 2t\}$

$= 5000 \cdot 2^{2t} \cdot \ln 2 \cdot 2 \cdot 1$

$\boxed{A'(t) = 10000 \cdot \ln 2 \cdot 2^{2t}}$ —— (1)

put $t = 1$ in (1)

$A'(1) = 10000 \times \ln 2 \times 2^{2(1)}$

$= 27725$ bacteria/hour

It is the rate of change at the end of 1st hour

put $t = 5$ in (1)

$A'(5) = 10000 \cdot \ln 2 \cdot 2^{2(5)}$

$= 10000 \cdot \ln 2 \cdot 2^{10}$

$A'(5) = 7097827$ bacteria/hour

It is the rate of change of bacteria at the end of fifth hour.

END OF CH # 2