Unit # 4 Differentiation of Vector Functions

The relationship of calculus and vector methods forms what is called vector calculus.

Scalar function: A function $f(x)$ is a rule which operates on an input x (x is any scalar quantity) and produces always just a single scalar output y. This gives a proper notation of a scalar function:

$y = f(x)$

Vector function: If for each value of a scalar variable t, a vector $\vec{f}(t)$ is uniquely determined, $\vec{f}(t)$ is called a vector function of the scalar variable t.

A vector function $\vec{f}(t)$ is written as

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$$\vec{f}(t) = f_1(t)\,\vec{i} + f_2(t)\,\vec{j} + f_3(t)\,\vec{k}$$

Provided that $f_1(t)$, $f_2(t)$ & $f_3(t)$ are scalar functions of t, the functions $f_1(t)$, $f_2(t)$ & $f_3(t)$ are called components of $\vec{f}(t)$. $\vec{i}, \vec{j}$ & $\vec{k}$ are unit vectors associated with a rectangular coordinate system. We call such a vector a Position vector.

Note $\vec{f}(t) = f_1(t)\vec{i} + f_2(t)\vec{j}$ (2-space)
or $\vec{f}(t) = f_1(t)\vec{i} + f_2(t)\vec{j} + f_3(t)\vec{k}$ (3-space)

Domain and Range: The set of all t values used as input in $\vec{f}(t)$ is called the domain of a vector-valued function $\vec{f}(t)$ and the set of $\vec{f}(t)$ values that the vector function $\vec{f}(t)$ takes as t varies, is called the range of a vector function $\vec{f}(t)$

Vector Operations:

Vector functions

1) Addition: $\vec{f}(t) + \vec{g}(t) = (\vec{f}+\vec{g})t$

2) Subtraction: $\vec{f}(t) - \vec{g}(t) = (\vec{f}-\vec{g})t$

3) Scalar Product: $h(t)\,\vec{f}(t) = (h\vec{f})(t)$

4) Cross Product: $\vec{f}(t) \times \vec{g}(t) = (\vec{f}\times\vec{g})(t)$

Scalar Function:

5) Dot Product: $\vec{f}(t) \cdot \vec{g}(t) = (\vec{f}\cdot\vec{g})(t)$

Limit of a vector function: $\vec{f}(t) = f_1(t)\vec{i} + f_2(t)\vec{j} + f_3(t)\vec{k}$

$$\lim_{t\to t_0} \vec{f}(t) = \lim_{t\to t_0}\left[f_1(t)\vec{i} + f_2(t)\vec{j} + f_3(t)\vec{k}\right]$$
$$= \left[\lim_{t\to t_0} f_1(t)\right]\vec{i} + \left[\lim_{t\to t_0} f_2(t)\right]\vec{j} + \left[\lim_{t\to t_0} f_3(t)\right]\vec{k}$$

Continuity of a vector function.

A vector function $\vec{f}(t)$ is continuous at $t=t_0$
if 1) t_0 is in the domain of a vector function $\vec{f}(t)$
or $\vec{f}(t_0)$ is defined

2) $\lim_{t\to t_0} \vec{f}(t) = \vec{f}(t_0)$

EXERCISE 4.1

1) Find the domain for the following vector fns.

a) $\vec{F}(t) = 2t\vec{i} - 3t\vec{j} + t^{-1}k$

sol let $\vec{F}(t) = 2t\vec{i} - 3t\vec{j} + t^{-1}\vec{k} = f_1(t)\vec{i} + f_2(t)\vec{j} + f_3(t)\vec{k}$
The function $f_1(t) = 2t$ is defined $\forall t$,
$f_2(t) = -3t$ is defined $\forall t$, $f_3(t) = t^{-1} = 1/t$ is

defined $\forall$ values of t except $t=0$. Thus the domain of the function $\vec{F}(t)$ is $t \neq 0$.

b) let $\vec{F}(t) = (1-t)\vec{i} + \sqrt{t}\,\vec{j} - (t-2)^{-1}\vec{k} = f_1(t)\vec{i} + f_2(t)\vec{j} + f_3(t)\vec{k}$

The function $f_1(t) = 1-t$ is defined $\forall\, t$,

$f_2(t) = \sqrt{t}$ is defined & real for $t \geq 0$

& $f_3(t) = -(t-2)^{-1}$ is defined $\forall$ $t-2 \neq 0 \Rightarrow t \neq 2$

$= -\dfrac{1}{t-2}$

Hence Domain of the function $\vec{F}(t)$ is $t \geq 0$ but $t \neq 2$

c) $\vec{F}(t) = \sin t\,\vec{i} + \cos t\,\vec{j} + \tan t\,\vec{k}$

let $f_1(t) = \sin t$ is defined $\forall\, t$, $f_2(t) = \cos t$ is also defined $\forall\, t$, $f_3(t) = \tan t$ is defined $\forall\, t$ except $t = (2n+1)\frac{\pi}{2}$ $n \in \mathbb{Z}$

Hence domain of the function $\vec{F}(t)$ is $t \neq (2n+1)\frac{\pi}{2}, n \in \mathbb{Z}$

that is, $\vec{F}(t)$ is defined $\forall\, t$ except $t = (2n+1)\frac{\pi}{2}$ $n \in \mathbb{Z}$

d) $\vec{F}(t) = \cos t\,\vec{i} - \cot t\,\vec{j} + \operatorname{cosec} t\,\vec{j} = f_1(t)\vec{i} + f_2(t)\vec{j} + f_3(t)\vec{k}$

The function $f_1(t) = \cos t$ is defined $\forall\, t$,

function $f_2(t) = -\cot t = \dfrac{\cos t}{\sin t}$ is defind $\forall\, t$ except $t = n\pi, n \in \mathbb{Z}$

function $f_3(t) = \operatorname{cosec} t = \dfrac{1}{\sin t}$ is defined $\forall t$ except $t = n\pi, n \in \mathbb{Z}$

Hence Domain of the function $\vec{F}(t)$ is $t \neq n\pi$ $n \in \mathbb{Z}$

That is, $\vec{F}(t)$ is defined $\forall t$ except $t = n\pi$, $n \in \mathbb{Z}$

Note ① Domain of $\sin t$ or $\cos t$ is $\mathbb{R}$ or $-\infty < t < \infty$

② Domain of $\tan t$ or $\sec t$ is $-\infty < t < \infty$, $t \neq (2n+1)\pi/2$ $n \in \mathbb{Z}$

③ Domain of $\cot t$ or $\operatorname{cosec} t$ is $-\infty < t < \infty$, $t \neq n\pi$ $n \in \mathbb{Z}$

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e) $\vec{F}(t) = 3t\vec{j} + t^{-1}\vec{k}$, $\vec{G}(t) = 5t\vec{i} + \sqrt{10-t}\,\vec{j}$

Now $\vec{F}(t) + \vec{G}(t) = 3t\vec{j} + t^{-1}\vec{k} + 5t\vec{i} + \sqrt{10-t}\,\vec{j}$

$= 5t\vec{i} + (3t + \sqrt{10-t})\vec{j} + t^{-1}\vec{k}$

Let $f_1(t) = 5t$, $f_2(t) = 3t + \sqrt{10-t}$, $f_3(t) = t^{-1}$

The function $f_1(t) = 5t$ is defined $\forall\, t$, the function $f_2(t) = 3t + \sqrt{10-t}$ is defined & real $\forall\; 10 - t \geq 0$

$\Rightarrow 10 \geq t$ or $t \leq 10$, the function $f_3(t) = t^{-1} = 1/t$ is defined $\forall\, t$ except $t = 0$, Hence Domain of $\vec{F}(t) + \vec{G}(t)$ is $t \leq 10$ & $t \neq 0$

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f) $\vec{F}(t) = \ln t\,\vec{i} + 3t\vec{j} - t^2\vec{k}$, $\vec{G}(t) = \vec{i} + 5t\vec{j} - t^2\vec{k}$

Now $\vec{F}(t) - \vec{G}(t) = \ln t\,\vec{i} + 3t\vec{j} - t^2\vec{k} - (\vec{i} + 5t\vec{j} - t^2\vec{k})$

$= \ln t\,\vec{i} + 3t\vec{j} - \cancel{t^2\vec{k}} - \vec{i} - 5t\vec{j} + \cancel{t^2\vec{k}}$

$= (\ln t - 1)\vec{i} + (3t - 5t)\vec{j} = (\ln t - 1)\vec{i} - 2t\vec{j}$

Let $f_1(t) = \ln t - 1$, $f_2(t) = -2t$

The function $f_1(t) = \ln t - 1$ is defined $\forall\; t > 0$, the function $f_2(t) = -2t$ is defined $\forall\, t$

Hence Domain of $\vec{F}(t) - \vec{G}(t)$ is $t > 0$

Note: If $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$, $\vec{b} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$

then $\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

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g) $\vec{F}(t) = t^2\vec{i} - t\vec{j} + 2t\vec{k}$, $\vec{G}(t) = (t+2)^{-1}\vec{i} + (t+4)\vec{j} - \sqrt{-t}\,\vec{k}$

Now $\vec{F}(t) \times \vec{G}(t) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ t^2 & -t & 2t \\ (t+2)^{-1} & t+4 & -\sqrt{-t} \end{vmatrix}$

Expand by R_1

$$= \vec{i}\begin{vmatrix} -t & 2t \\ t+4 & -\sqrt{-t} \end{vmatrix} - \vec{j}\begin{vmatrix} t^2 & 2t \\ (t+2)^{-1} & -\sqrt{-t} \end{vmatrix} + \vec{k}\begin{vmatrix} t^2 & -t \\ (t+2)^{-1} & t+4 \end{vmatrix}$$

$$= \vec{i}\left[+t\sqrt{-t} - 2t(t+4)\right] - \vec{j}\left[-t^2\sqrt{-t} - 2t(t+2)^{-1}\right] + \vec{k}\left[t^2(t+4) + t(t+2)^{-1}\right]$$

$$= \vec{i}\left[t\sqrt{-t} - 2t^2 - 8t\right] + \vec{j}\left[t^2\sqrt{-t} + \frac{2t}{(t+2)^{+1}}\right] + \vec{k}\left[t^3 + 4t^2 + \frac{t}{(t+2)^{+1}}\right]$$

let $f_1(t) = t\sqrt{-t} - 2t^2 - 8t$, $f_2(t) = t^2\sqrt{-t} + \frac{2t}{t+2}$, $f_3(t) = t^3 + 4t^2 + \frac{t}{t+2}$

$\because$ function $f_1(t) = t\sqrt{-t} - 2t^2 - 8t$ is defined & real for $t \le 0$

& function $f_2(t) = t^2\sqrt{-t} + \frac{2t}{t+2}$ is defined & real for $t \le 0$ but $t \ne -2$

& function $f_3(t) = t^3 + 4t^2 + \frac{t}{t+2}$ is defined $\forall t$ except $t = -2$

Hence domain of $\vec{F}(t) \times \vec{G}(t)$ is $t \le 0$ but $t \ne -2$

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2) Sketch the following vector functions

a) $\vec{F}(t) = 2t\,\vec{i} + t^2\,\vec{j}$

Sol The vector function $\vec{F}(t)$ is used for different values of t to obtain the position vectors:

$t = -2 \Rightarrow \vec{F}(-2) = 2(-2)\vec{i} + (-2)^2\vec{j} = -4\vec{i} + 4\vec{j} = [-4, 4]$

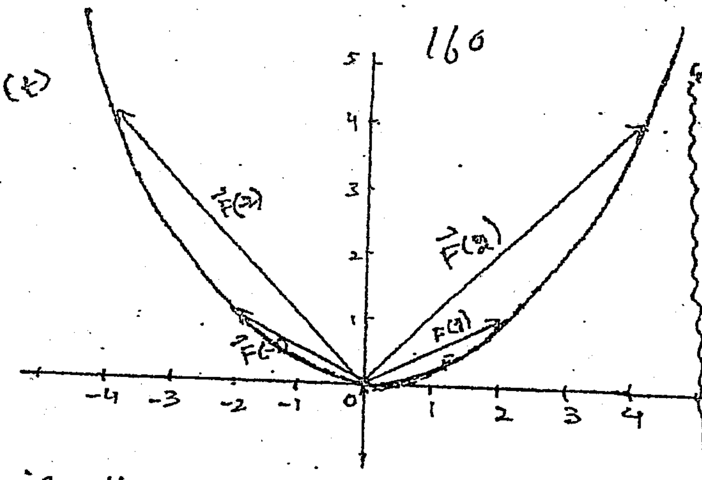
$t = -1 \Rightarrow \vec{F}(-1) = 2(-1)\vec{i} + (-1)^2\vec{j} = -2\vec{i} + \vec{j} = [-2, 1]$

$t = 0 \Rightarrow \vec{F}(0) = 2(0)\vec{i} + 0\,\vec{j} = 0\vec{i} + 0\vec{j} = [0, 0]$

$t = 1 \Rightarrow \vec{F}(1) = 2(1)\vec{i} + 1^2\vec{j} = 2\vec{i} + \vec{j} = [2, 1]$

$t = 2 \Rightarrow \vec{F}(2) = 2(2)\vec{i} + 2^2\vec{j} = 4\vec{i} + 4\vec{j} = [4, 4]$

we plot these position vectors. The terminal points of all the position vectors lie on the curve described parametrically by $x = 2t$, $y = t^2$ $\forall t \in R$



Note:

$\vec{F}(t) = 2t\vec{i} + t^2\vec{j}$

Here $x = 2t$, $y = t^2$

$\Rightarrow t = \frac{x}{2}$ | $y = \left(\frac{x}{2}\right)^2$

$y = \frac{x^2}{4}$

or

$x^2 = 4y$

which is the req: sketch

b) $\vec{G}(t) = \sin t\,\vec{i} - \cos t\,\vec{j}$ {Note use calculator in radian mode}

Sol The vector function $\vec{G}(t)$ is used for different values of t to obtain the Position vectors:

$t = -3 \Rightarrow \vec{G}(-3) = \sin(-3)\vec{i} - \cos(-3)\vec{j} = -0.1\,\vec{i} + 0.9\,\vec{j} = [-0.1, 0.9]$

$t = -2 \Rightarrow \vec{G}(-2) = \sin(-2)\vec{i} - \cos(-2)\vec{j} = -0.9\,\vec{i} + 0.4\,\vec{j} = [-0.9, 0.4]$

$t = -1 \Rightarrow \vec{G}(-1) = \sin(-1)\vec{i} - \cos(-1)\vec{j} = -0.8\,\vec{i} - 0.5\,\vec{j} = [-0.8, -0.5]$

$t = 0 \Rightarrow \vec{G}(0) = \sin(0)\vec{i} - \cos(0)\vec{j} = 0\vec{i} - \vec{j} = [0, -1]$

$t = 1 \Rightarrow \vec{G}(1) = \sin(1)\vec{i} - \cos(1)\vec{j} = 0.8\vec{i} - 0.5\vec{j} = [0.8, -0.5]$

$t = 2 \Rightarrow \vec{G}(2) = \sin 2\vec{i} - \cos 2\vec{j} = 0.9\vec{i} + 0.4\vec{j} = [0.9, 0.4]$

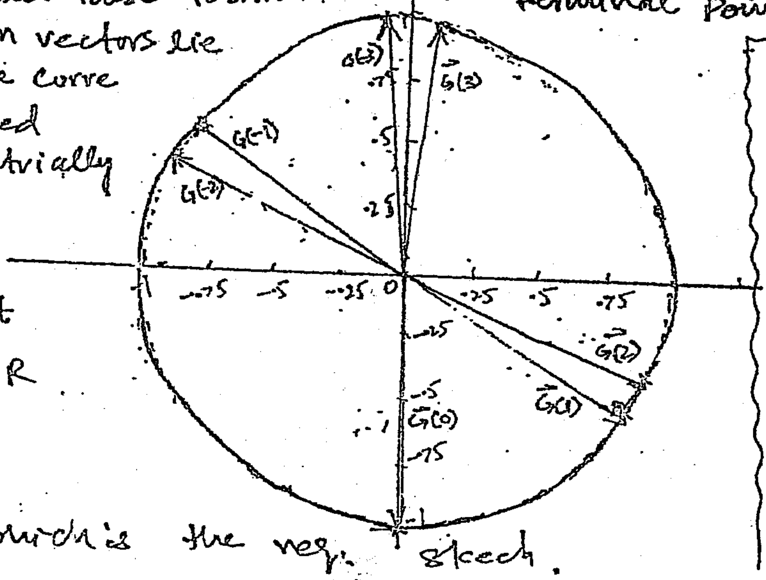
$t = 3 \Rightarrow \vec{G}(3) = \sin 3\vec{i} - \cos 3\vec{j} = 0.1\vec{i} + 0.9\vec{j} = [0.1, 0.9]$

we plot these Position vectors. The terminal points of all the Position vectors lie on the curve described Parametrically by

$x = \sin t$

$y = -\cos t$

$\forall\, t \in R$



Note $G(t) = \sin t\vec{i} - \cos t\vec{j}$

$\because x = \sin t,\ y = -\cos t$

Sq: & add

$x^2 + y^2 = \sin^2 t + \cos^2 t$

$x^2 + y^2 = 1$

It is eq: of Circle with Radius 1 & centre $(0, 0)$

which is the req. sketch.

(7)

3) Perform the operations of the following expressions with
$\vec{F}(t) = 2t\vec{i} - 5\vec{j} + t^2\vec{k}$, $\vec{G}(t) = (1-t)\vec{i} + \frac{1}{t}\vec{k}$, $H(t) = \sin t\,\vec{i} + e^t\vec{j}$

Sol a) $2\vec{F}(t) - 3\vec{G}(t) = 2(2t\vec{i} - 5\vec{j} + t^2\vec{k}) - 3[(1-t)\vec{i} + \frac{1}{t}\vec{k}]$

$= 4t\vec{i} - 10\vec{j} + 2t^2\vec{k} - 3(1-t)\vec{i} - 3\cdot\frac{1}{t}\vec{k}$

$= \{4t - 3(1-t)\}\vec{i} - 10\vec{j} + (2t^2 - \frac{3}{t})\vec{k}$

$= (4t - 3 + 3t)\vec{i} - 10\vec{j} + (2t^2 - \frac{3}{t})\vec{k}$

$= (7t-3)\vec{i} - 10\vec{j} + (2t^2 - 3/t)\vec{k}$

Note $\vec{a}\cdot\vec{b} = [a_1\vec{i} + a_2\vec{j} + a_3\vec{k}]\cdot[b_1\vec{i} + b_2\vec{j} + b_3\vec{k}] = a_1b_1 + a_2b_2 + a_3b_3$

b) $\vec{F}(t)\cdot\vec{G}(t) = (2t\vec{i} - 5\vec{j} + t^2\vec{k})\cdot[(1-t)\vec{i} + 0\vec{j} + \frac{1}{t}\vec{k}]$

$= 2t(1-t) - 5(0) + t^2(\frac{1}{t})$

$= 2t - 2t^2 - 0 + t = 3t - 2t^2$ Ans

c) $\vec{G}(t)\cdot\vec{H}(t) = ((1-t)\vec{i} + 0\vec{j} + \frac{1}{t}\vec{k})\cdot[\sin t\,\vec{i} + e^t\vec{j} + 0\vec{k}]$

$= (1-t)\cdot\sin t + 0\cdot e^t + \frac{1}{t}\cdot 0 = (1-t)\sin t + 0 + 0$

$= (1-t)\sin t$ Ans

Note $\vec{a}\times\vec{b} = (a_1\vec{i} + a_2\vec{j} + a_3\vec{k})\times(b_1\vec{i} + b_2\vec{j} + b_3\vec{k}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

d) $\vec{F}(t)\times\vec{H}(t) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2t & -5 & t^2 \\ \sin t & e^t & 0 \end{vmatrix}$

Expand by R_1

$= \vec{i}\begin{vmatrix} -5 & t^2 \\ e^t & 0 \end{vmatrix} - \vec{j}\begin{vmatrix} 2t & t^2 \\ \sin t & 0 \end{vmatrix} + \vec{k}\begin{vmatrix} 2t & -5 \\ \sin t & e^t \end{vmatrix}$

$= \vec{i}(0 - t^2e^t) - \vec{j}(0 - t^2\sin t) + \vec{k}(2te^t + 5\sin t)$

$= -t^2e^t\,\vec{i} + t^2\sin t\,\vec{j} + (2te^t + 5\sin t)\vec{k}$

e) $2e^t\vec{F}(t) + t\vec{G}(t) + 10\vec{H}(t)$

$= 2e^t[2t\vec{i} - 5\vec{j} + t^2\vec{k}] + t[(1-t)\vec{i} + \frac{1}{t}\vec{k}] + 10[\sin t\,\vec{i} + e^t\vec{j}]$

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$= \underline{\underline{4te^t\vec{i}}} - \cancel{10e^t\vec{j}} + 2t^2e^t\vec{k} + \underline{\underline{t(1-t)\vec{i}}} + 1\cdot\vec{k} + \underline{\underline{10\sin t\,\vec{i}}} + \cancel{10e^t\vec{j}}$

$= (4te^t + t(1-t) + 10\sin t)\vec{i} + (2t^2e^t + 1)\vec{k}$

$= (4te^t + t - t^2 + 10\sin t)\vec{i} + (2t^2e^t + 1)\vec{k}$ Ans

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4) Evaluate the following limits

Note: $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$

a) $\lim_{t\to1}(3t\vec{i} + e^{2t}\vec{j} + \sin\pi t\,\vec{k})$

$= \lim_{t\to1} 3t\,\vec{i} + \lim_{t\to1} e^{2t}\vec{j} + \lim_{t\to1}\sin\pi t\,\vec{k}$

Applying limit rule

$= 3(1)\vec{i} + e^{2(1)}\vec{j} + \sin\pi(1)\,\vec{k} = 3\vec{i} + e^2\vec{j} + 0\cdot\vec{k} = 3\vec{i} + e^2\vec{j}$

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b) $\lim_{t\to0}\left(\dfrac{\sin t\,\vec{i} - t\vec{k}}{t^2+t-1}\right) = \lim_{t\to0}\left[\dfrac{\sin t}{t^2+t-1}\vec{i} - \dfrac{t}{t^2+t-1}\vec{k}\right]$

$= \lim_{t\to0}\dfrac{\sin t}{t^2+t-1}\vec{i} - \lim_{t\to0}\dfrac{t}{t^2+t-1}\vec{k}$

Applying limit Rule

$= \dfrac{\sin 0}{0+0-1}\vec{i} - \dfrac{0}{0+0-1}\vec{k}$

$= \dfrac{0}{-1}\vec{i} - \dfrac{0}{-1}\vec{k} = 0\vec{i} + 0\vec{k}$

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c) $\lim_{t\to1}\left[\dfrac{t^3-1}{t-1}\vec{i} + \dfrac{t^2-3t+2}{t^2+t-2}\vec{j} + (t^2+1)e^{t-1}\vec{k}\right]$

The component of the vector function are

$f_1(t) = \dfrac{t^3-1}{t-1}$, $f_2(t) = \dfrac{t^2-3t+2}{t^2+t-2}$, $f_3(t) = (t^2+1)e^{t-1}$

$\Rightarrow \lim_{t\to1} f_1(t) = \lim_{t\to1}\dfrac{t^3-1}{t-1}\left(\dfrac{0}{0}\right)$

$= \lim_{t\to1}\dfrac{\cancel{(t-1)}(t^2+t(1)+1^2)}{\cancel{t-1}}$

$= \lim_{t\to1}(t^2+t+1)$

$\lim_{t\to1} f_2(t) = \lim_{t\to1}\dfrac{t^2-3t+2}{t^2+t-2}\left(\dfrac{0}{0}\right)$

$= \lim_{t\to1}\dfrac{t^2-2t-t+2}{t^2+2t-t-2}$

$\lim_{t\to1} f_3(t) = \lim_{t\to1}(t^2+1)e^{t-1}$

Applying limit Rule

$= (1+1)e^{1-1}$

$= 2e^0$

$= 2$

Applying limit rule

$= 1^2 + 1 + 1$

$\lim_{t \to 1} f_1(t) = 3$

$= \lim_{t \to 1} \frac{t(t-2) - 1(t-2)}{t(t+2) - 1(t+2)}$

$= \lim_{t \to 1} \frac{(t-2)\cancel{(t-1)}}{(t+2)\cancel{(t-1)}}$

$\lim_{t \to 1} f_2(t) = \lim_{t \to 1} \frac{t-2}{t+2} = \frac{1-2}{1+2} = -\frac{1}{3}$

Now

$$\lim_{t \to 1} \left[\frac{t^3 - 1}{t-1} \vec{i} + \frac{t^2 - 3t + 2}{t^2 + t - 2} \vec{j} + (t^2+1) e^{t-1} \vec{k} \right]$$

$$= \lim_{t \to 1} \left[f_1(t) \vec{i} + f_2(t) \vec{j} + f_3(t) \vec{k} \right]$$

$$= \lim_{t \to 1} f_1(t) \vec{i} + \lim_{t \to 1} f_2(t) \vec{j} + \lim_{t \to 1} f_3(t) \vec{k}$$

$$= 3\vec{i} - \frac{1}{3}\vec{j} + 2\vec{k} \qquad \text{Ans}$$

L'Hospital's Rule (L.H.R)

If $\lim_{x \to a} \frac{f(x)}{g(x)}$ is of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$

then use L'Hospital's Rule, that is,

$$\lim_{x \to a} \frac{\frac{d}{dx} f(x)}{\frac{d}{dx} g(x)}$$

we will continue this process untill the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ is finished.

Note $\frac{f(x)}{g(x)}$ is not to be differentiated by the quotient rule but $f(x)$ & $g(x)$ are to be differentiated separately.

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d) $\lim_{t \to 0} \left[\frac{te^t}{1-e^t} \vec{i} + \frac{e^{t-1}}{\cos t} \vec{j} \right]$

The component of the vector functions are

$f_1(t) = \frac{te^t}{1-e^t}$ & $f_2(t) = \frac{e^{t-1}}{\cos t}$

Note L.H.R can also be used for Part c

Now $\lim_{t\to 0} f_1(t) = \lim_{t\to 0}\left(\frac{te^t}{1-e^t}\right)$ $\left(\frac{0}{0}\right)$ & $\lim_{t\to 0} f_2(t) = \lim_{t\to 0} \frac{e^{t-1}}{\cos t}$

Applying L.H.R

$$\lim_{t\to 0} f_1(t) = \lim_{t\to 0} \frac{\frac{d}{dt}(te^t)}{\frac{d}{dt}(1-e^t)}$$

$$= \lim_{t\to 0} \frac{t\frac{d}{dt}e^t + e^t\frac{d}{dt}t}{0 - \frac{d}{dt}e^t}$$

$$= \lim_{t\to 0} \frac{t.e^t + e^t.1}{-e^t} = \frac{\cancel{e^t}(t+1)}{-\cancel{e^t}}$$

$$= \lim_{t\to 0}\left[-(t+1)\right] = -(0+1) = -1$$

Applying limit Rule

$$= \frac{e^{0-1}}{\cos 0}$$

$$= \frac{e^{-1}}{1} = e^{-1}$$

$$\lim_{t\to 0} f_2(t) = \frac{1}{e}$$

Now

$$\lim_{t\to 0}\left(\frac{te^t}{1-e^t}\vec{i} + \frac{e^{t-1}}{\cos t}\vec{j}\right) = \lim_{t\to 0}\left[f_1(t)\vec{i} + f_2(t)\vec{j}\right]$$

$$= \lim_{t\to 0} f_1(t)\vec{i} + \lim_{t\to 0} f_2(t)\vec{j}$$

$$= (-1)\vec{i} + \left(\frac{1}{e}\right)\vec{j} = -\vec{i} + \frac{1}{e}\vec{j} \quad \text{Ans}$$

---

e) $\lim_{t\to 0}\left[\frac{\sin t}{t}\vec{i} + \frac{1-\cos t}{t}\vec{j} + e^{1-t}\vec{k}\right]$

The component of the vetor function are

i... $f_1(t) = \frac{\sin t}{t}$, $f_2(t) = \frac{1-\cos t}{t}$, $f_3(t) = e^{1-t}$

Now

$\lim_{t\to 0} f_1(t) = \lim_{t\to 0}\frac{\sin t}{t}$ $\left(\frac{0}{0}\right)$

Applying LHR

$$= \lim_{t\to 0}\frac{\cos t}{1}$$

Applying limit Rule

$$= \cos 0 = 1$$

$\lim_{t\to 0} f_2(t) = \lim_{t\to 0}\frac{1-\cos t}{t}$ $\left(\frac{0}{0}\right)$

Applying LHR

$$= \lim_{t\to 0}\frac{0+\sin t}{1}$$

$$= \lim_{t\to 0}\sin t$$

$$= \sin 0 = 0$$

$\lim_{t\to 0} f_3(t) = \lim_{t\to 0} e^{1-t}$

Applying limit

$$= e^{1-0}$$

$$= e$$

Now

$$\lim_{t\to 0}\left[\frac{\sin t}{t}\vec{i} + \frac{1-\cos t}{t}\vec{j} + e^{1-t}\vec{k}\right] = \lim_{t\to 0}\left[f_1(t)\vec{i} + f_2(t)\vec{j} + f_3(t)\vec{k}\right]$$

$$= \lim_{t\to 0} f_1(t)\vec{i} + \lim_{t\to 0} f_2(t)\vec{j} + \lim_{t\to 0} f_3(t)\vec{k}$$

$$= (1)\vec{i} + (0)\vec{j} + (e)\vec{k}$$

$$= \vec{i} + 0\vec{j} + e\vec{k} = \vec{i} + e\vec{k} \quad \text{Ans}$$

f) $\lim_{t\to 0}\left[\frac{\sin 3t}{\sin 2t}\vec{i}+\frac{\ln(\sin t)}{\ln(\tan t)}\vec{j}+t^t\vec{k}\right]$

The components of the vector function are

$f_1(t)=\frac{\sin 3t}{\sin 2t}$, $f_2(t)=\frac{\ln\sin t}{\ln\tan t}$, $f_3(t)=t^t$ $\quad(\ln a^b=b\ln a)$

Now

$\lim_{t\to 0}f_1(t)=\lim_{t\to 0}\frac{\sin 3t}{\sin 2t}\ \left(\frac{0}{0}\right)$

Apply LHR

$=\lim_{t\to 0}\frac{\cos 3t\,(3)}{\cos 2t\,(2)}$

Applying limit rule

$=\frac{\cos 0\,(3)}{\cos 0\,(2)}=\frac{1(3)}{1(2)}=\frac{3}{2}$

$\lim_{t\to 0}f_2(t)=\lim_{t\to 0}\frac{\ln\sin t}{\ln\tan t}$

Apply LHR

$=\lim_{t\to 0}\frac{\frac{1}{\sin t}\frac{d}{dt}\sin t}{\frac{1}{\tan t}\frac{d}{dt}\tan t}$

$=\lim_{t\to 0}\frac{\frac{1}{\sin t}\cdot\cos t}{\cot t\cdot\sec^2 t}$

$=\lim_{t\to 0}\frac{\cancel{\cot t}}{\cancel{\cot t}\,\sec^2 t}$

$=\lim_{t\to 0}\cos^2 t=\cos^2 0=1$

$\lim_{t\to 0}f_3(t)=\lim_{t\to 0}t^t\ \left(0^0\right)$

taking ln of B-sides

$\lim_{t\to 0}\ln f_3(t)=\lim_{t\to 0}\ln t^t$

$=\lim_{t\to 0}t\cdot\ln t\ \ (0\cdot\infty)$

$=\lim_{t\to 0}\frac{\ln t}{1/t}\ \left(\frac{\infty}{\infty}\right)$

Apply LHR $\quad\boxed{\frac{d}{dt}\frac{1}{t}=-\frac{1}{t^2}}$

$=\lim_{t\to 0}\frac{\frac{1}{t}}{-\frac{1}{t^2}}$

$=\lim_{t\to 0}\left(-\frac{t}{1}\right)=-\frac{0}{1}$

$\lim_{t\to 0}\ln f_3(t)=0$ $\quad\boxed{\ln x=y\Rightarrow x=e^y}$

taking Antilog

$\lim_{t\to 0}f_3(t)=e^0=1$

Now

$\lim_{t\to 0}\left[\frac{\sin 3t}{\sin 2t}\vec{i}+\frac{\ln\sin t}{\ln\tan t}\vec{j}+t^t\vec{k}\right]$

$=\lim_{t\to 0}\left[f_1(t)\vec{i}+f_2(t)\vec{j}+f_3(t)\vec{k}\right]$

$=\lim_{t\to 0}f_1(t)\vec{i}+\lim_{t\to 0}f_2(t)\vec{j}+\lim_{t\to 0}f_3(t)\vec{k}$

$=\left(\frac{3}{2}\right)\vec{i}+(1)\vec{j}+(1)\vec{k}=\frac{3}{2}\vec{i}+\vec{j}+\vec{k}$ Ans

g) $\lim_{t\to 1}\left[2t\vec{i}-3\vec{j}+e^t\vec{k}\right]$ (there is no indeterminat form)

$=\lim_{t\to 1}(2t)\vec{i}-\left[\lim_{t\to 1}3\right]\vec{j}+\lim_{t\to 1}e^t\vec{k}$

Applying limit Rule

$=2(1)\vec{i}-3\vec{j}+e^1\vec{k}=2\vec{i}-3\vec{j}+e\vec{k}$ Ans

h) $\lim_{t\to 2}\left[\left(2\vec{i}-t\vec{j}+e^t\vec{k}\right)\times\left(t^2\vec{i}+4\sin t\,\vec{j}\right)\right]$

$=\lim_{t\to 2}\begin{vmatrix}\vec{i}&\vec{j}&\vec{k}\\2&-t&e^t\\t^2&4\sin t&0\end{vmatrix}$ Expand by R_1

$$= \lim_{t\to 2}\left[\vec{i}\begin{vmatrix}-t & e^t\\ 4\sin t & 0\end{vmatrix} - \vec{j}\begin{vmatrix}2 & e^t\\ t^2 & 0\end{vmatrix} + \vec{k}\begin{vmatrix}2 & -t\\ t^2 & 4\sin t\end{vmatrix}\right]$$

$$= \lim_{t\to 2}\left[\vec{i}(0-4e^t\sin t) - \vec{j}(0-t^2e^t) + \vec{k}(8\sin t + t^3)\right]$$

$$= \lim_{t\to 2}\left[-4e^t\sin t\,\vec{i} + t^2e^t\vec{j} + (8\sin t + t^3)\vec{k}\right]$$

$$= \lim_{t\to 2}(-4e^t\sin t)\vec{i} + \lim_{t\to 2}t^2e^t\,\vec{j} + \lim_{t\to 2}(8\sin t + t^3)\,\vec{k}$$

$$= -4e^2\sin(2)\vec{i} + 2^2e^2\vec{j} + (8\sin 2 + 2^3)\vec{k}$$

$$= -4e^2\sin(2)\,\vec{i} + 4e^2\vec{j} + (8\sin(2)+8)\,\vec{k}$$

$$= -4e^2\sin(2)\,\vec{i} + 4e^2\vec{j} + 8(\sin(2)+1)\vec{k} \qquad \text{Ans}$$

Theorem 4.2: A vector function $\vec{F}(t) = [f_1(t), f_2(t), f_3(t)]$ is continuous at $t = t_0$, iff.

1) $\vec{F}(t) = [f_1(t), f_2(t), f_3(t)]$ is defined at $t = t_0$

2) all the component functions $f_1(t)$, $f_2(t)$ & $f_3(t)$ are continuous at $t = t_0$, that is,

$$\lim_{t\to t_0} f_1(t) = f_1(t_0), \quad \lim_{t\to t_0} f_2(t) = f_2(t_0), \quad \lim_{t\to t_0} f_3(t) = f_3(t_0)$$

5) Test the continuity of the following expressions for all values of t,

a) $\vec{F}(t) = t\vec{i} + 3\vec{j} - (1-t)\vec{k}$

The components of a vector function are:

$f_1(t) = t, \quad f_2(t) = 3, \quad f_3(t) = -(1-t)$

The function $f_1(t) = t$ is continuous for all t, function $f_2(t) = 3$ is also continuous for all t, & the function $f_3(t)$ is continuous for all t. Thus $\vec{F}(t)$ is continuous for all values of t.

b) $\vec{G}(t) = t\vec{i} - t^{-1}\vec{k} = t\vec{i} - \frac{1}{t}\vec{k}$

$= t\vec{i} + 0\vec{j} - \frac{1}{t}\vec{k}$

The components of a vector function are

$f_1(t) = t$, $f_2(t) = 0$, $f_3(t) = \frac{-1}{t}$

The function $f_1(t) = t$ is continuous for all t;

function $f_2(t) = 0$ is continuous for all t

& function $f_3(t) = -\frac{1}{t}$ is continuous for all t except $t=0$

Thus $\vec{G}(t)$ is Continuous for all values of t except $t=0$

c) $\vec{G}(t) = \dfrac{\vec{i}+2\vec{j}}{t^2+t} = \dfrac{\vec{i}+2\vec{j}}{t(t+1)} = \dfrac{\vec{i}}{t(t+1)} + \dfrac{2\vec{j}}{t(t+1)}$

$= \dfrac{1}{t(t+1)}\vec{i} + \dfrac{2}{t(t+1)}\vec{j}$

The components of a vector function are

$f_1(t) = \dfrac{1}{t(t+1)}$ & $f_2(t) = \dfrac{2}{t(t+1)}$

The function $f_1(t) = \frac{1}{t(t+1)}$ is continuoses $\forall t$ except $t=0$ & $t=-$

& function $f_2(t) = \frac{2}{t(t+1)}$ is continuous $\forall t$ except $t=0$ & $t=-1$.

Thus $\vec{G}(t)$ is continuous for all values of t except $t=0$ & $t=-1$

d) $\vec{F}(t) = e^t \cdot \sin t\,\vec{i} + e^t \cos t\,\vec{k}$

The components of a vector function are

$f_1(t) = e^t \sin t$, $f_2(t) = 0$, $f_3(t) = e^t \cos t$

Since all the components functions are continuoses for all values of t, Thus $\vec{F}(t)$ is continuous for all values of t.

e) $\vec{F}(t) = e^t(t\vec{i} + t^{-1}\vec{j} + 3\vec{k}) = te^t\,\vec{i} + \frac{e^t}{t}\,\vec{j} + 3e^t\,\vec{k}$

The components of a vector function are:

$f_1(t) = te^t$, $f_2(t) = \frac{e^t}{t}$, $f_3(t) = 3e^t$

The function $f_1(t) = te^t$ is continuous $\forall\ t$,
The function $f_2(t) = \frac{e^t}{t}$ is continuous $\forall\ t$ except $t=0$
& function $f_3(t) = 3e^t$ is continuous for all t.
Thus, $\vec{F}(t)$ is continuous for all values of t except $t=0$

---

f) $\vec{G}(t) = \dfrac{t\vec{i} + \sqrt{t}\,\vec{j}}{\sqrt{t^2+t}} = \dfrac{t}{\sqrt{t^2+t}}\,\vec{i} + \dfrac{\sqrt{t}}{\sqrt{t^2+t}}\,\vec{j}$

The components of a vector function are:

$f_1(t) = \dfrac{t}{\sqrt{t^2+t}}$ & $f_2(t) = \dfrac{\sqrt{t}}{\sqrt{t^2+t}}$

The function $f_1(t) = \dfrac{t}{\sqrt{t^2+t}}$ is continuous for $t>0$
& function $f_2(t) = \dfrac{\sqrt{t}}{\sqrt{t^2+1}}$ is continuous for $t>0$.

Thus $\vec{G}(t)$ is continuous for all values of $t>0$

Derivative of a Vector function

The derivative of a vector function $\vec{F}(t)$ is the vector function $\vec{F}'(t)$ determined by taking the limit of a difference quotient $\frac{\Delta\vec{F}}{\Delta t}$, that is,

$$\vec{F}'(t) = \lim_{\Delta t \to 0} \frac{\Delta\vec{F}}{\Delta t} = \lim_{\Delta t \to 0} \frac{\vec{F}(t+\Delta t) - \vec{F}(t)}{\Delta t}$$, when this limit exists.

Note: In Leibnitz notation, the derivative of $\vec{F}(t)$ is denoted by $\frac{d\vec{F}}{dt}$.

Note: ① Derivative of a vector function

$\vec{F}(t) = f_1(t)\vec{i} + f_2(t)\vec{j} + f_3(t)\vec{k}$

Diff: w.r.t. t

$\vec{F}'(t) = \frac{d}{dt} f_1(t)\vec{i} + \frac{d}{dt} f_2(t)\vec{j} + \frac{d}{dt} f_3(t)\vec{k}$

2) Constant Rule: $\frac{d}{dt}(\text{constant vector}) = 0$

3) Linearity Rule: $\frac{d}{dt}(a\vec{F} + b\vec{G}) = a\frac{d}{dt}\vec{F} + b\frac{d}{dt}\vec{G}$

4) Scalar multiple Rule

$\frac{d}{dt}(h\vec{F}) = \left(\frac{d}{dt}h\right)\vec{F} + h\frac{d}{dt}\vec{F}$

5) Dot Product Rule: $\frac{d}{dt}(\vec{F}.\vec{G}) = \frac{d\vec{F}}{dt}.\vec{G} + \vec{F}.\frac{d}{dt}\vec{G}$

6) Cross Product Rule: $\frac{d}{dt}(\vec{F}\times\vec{G}) = \frac{d\vec{F}}{dt}\times\vec{G} + \vec{F}\times\frac{d}{dt}\vec{G}$

7) Chain Rule $\frac{d}{dt}F(h(t)) = \frac{dF(h(t))}{dt}.\frac{dh(t)}{dt}$

8) Quotient Rule: $\frac{d}{dt}\left(\frac{F}{h}\right)(t) = \frac{1}{h^2}\left[h\frac{d\vec{F}}{dt} - \frac{dh}{dt}F\right]$

EXERCISE 4.2

1) Find the vector derivative $F'(t)$

a) $\vec{F}(t) = t\vec{i} + t^2\vec{j} + (t+t^3)\vec{k}$.

Diff: w.r.t. t

$\vec{F}'(t) = \left[\frac{d}{dt}t\right]\vec{i} + \left[\frac{d}{dt}t^2\right]\vec{j} + \left[\frac{d}{dt}(t+t^3)\right]\vec{k}$

$= [1]\vec{i} + [2t]\vec{j} + [1+3t^2]\vec{k} = \vec{i} + 2t\vec{j} + (1+3t^2)\vec{k}$ A

b) $\vec{F}(s) = (s\vec{i} + s^2\vec{j} + s^2\vec{k}) + (2s^2\vec{i} - s\vec{j} + 3\vec{k})$

Diff: w.r.t. s

$\vec{F}(s) = \frac{d}{ds}\left[(s\vec{i} + s^2\vec{j} + s^2\vec{k}) + (2s^2\vec{i} - s\vec{j} + 3\vec{k})\right]$

$$\vec{F}'(s) = \frac{d}{ds}\left[s\vec{i} + s^2\vec{j} + s^2\vec{k}\right] + \frac{d}{ds}\left[2s^2\vec{i} - s\vec{j} + 3\vec{k}\right]$$
$$= \left(1\vec{i} + 2s\vec{j} + 2s\vec{k}\right) + \left[4s\vec{i} - 1\vec{j} + 0\vec{k}\right]$$
$$= (1+4s)\vec{i} + (2s-1)\vec{j} + 2s\vec{k}$$

2nd method

$$\vec{F}(s) = \left(s\vec{i} + s^2\vec{j} + s^2\vec{k}\right) + \left(2s^2\vec{i} - s\vec{j} + 3\vec{k}\right)$$
$$\Rightarrow \quad \vec{F}(s) = (s+2s^2)\vec{i} + (s^2-s)\vec{j} + (s^2+3)\vec{k}$$

Diff: w.r.t. s

$$\vec{F}'(s) = \frac{d}{ds}(s+2s^2)\vec{i} + \frac{d}{ds}(s^2-s)\vec{j} + \frac{d}{ds}(s^2+3)\vec{k}$$
$$= \left(1 + 2(2s)\right)\vec{i} + (2s-1)\vec{j} + (2s+0)\vec{k}$$
$$= (1+4s)\vec{i} + (2s-1)\vec{j} + 2s\vec{k} \quad \text{Ans}$$

c) $\vec{F}(\theta) = \cos\theta\left(\vec{i} + \tan\theta\,\vec{j} + 3\vec{k}\right)$

Diff w.r.t. θ

$$\vec{F}'(\theta) = \frac{d}{d\theta}\left[\cos\theta\left\{\vec{i} + \tan\theta\,\vec{j} + 3\vec{k}\right\}\right]$$
$$= \left(\frac{d}{d\theta}\cos\theta\right)\left\{\vec{i} + \tan\theta\,\vec{j} + 3\vec{k}\right\} + \cos\theta\,\frac{d}{dt}\left\{\vec{i} + \tan\theta\,\vec{j} + 3\vec{k}\right\}$$
$$= -\sin\theta\left\{\vec{i} + \frac{\sin\theta}{\cos\theta}\vec{j} + 3\vec{k}\right\} + \cos\theta\left\{0\vec{i} + \sec^2\theta\,\vec{j} + 0\vec{k}\right\}$$
$$= -\sin\theta\,\vec{i} - \frac{\sin^2\theta}{\cos\theta}\vec{j} - 3\sin\theta\,\vec{k} + \cos\theta\cdot\frac{1}{\cos^2\theta}\vec{j}$$
$$= -\sin\theta\,\vec{i} + \left(-\frac{\sin^2\theta}{\cos\theta} + \frac{1}{\cos\theta}\right)\vec{j} - 3\sin\theta\,\vec{k} = -\sin\theta\,\vec{i} + \frac{-\sin^2\theta + 1}{\cos\theta}\vec{j} - 3\sin\theta\,\vec{k}$$
$$= -\sin\theta\,\vec{i} + \frac{\cos^2\theta}{\cos\theta}\vec{j} - 3\sin\theta\,\vec{k} = -\sin\theta\,\vec{i} + \cos\theta\,\vec{j} - 3\sin\theta\,\vec{k} \quad \text{Ans}$$

2nd method $\vec{F}(\theta) = \cos\theta\left[\vec{i} + \tan\theta\cdot\vec{j} + 3\vec{k}\right]$

$$= \cos\theta\left[\vec{i} + \frac{\sin\theta}{\cos\theta}\vec{j} + 3\vec{k}\right]$$
$$\vec{F}(\theta) = \cos\theta\,\vec{i} + \sin\theta\,\vec{j} + 3\cos\theta\,\vec{k}$$

Diff: w.r.t. θ

$$\vec{F}'(\theta) = \frac{d}{d\theta}\cos\theta\,\vec{i} + \frac{d}{d\theta}\sin\theta\,\vec{j} + 3\frac{d}{d\theta}\cos\theta\,\vec{k}$$
$$= -\sin\theta\,\vec{i} + \cos\theta\,\vec{j} - 3\sin\theta\,\vec{k} \quad \text{Ans}$$

2) Find $\vec{F}'(t)$ & $\vec{F}''(t)$ of the following vector functions

a) $\vec{F}(t) = t^2\vec{i} + t^{-1}\vec{j} + e^{2t}\vec{k}$

Diff: w.r.t.t

$\vec{F}'(t) = \frac{d}{dt}t^2\vec{i} + \frac{d}{dt}t^{-1}\vec{j} + \frac{d}{dt}e^{2t}\vec{k}$

$= 2t\vec{i} + (-1)t^{-2}\vec{j} + e^{2t}\frac{d}{dt}(2t)\vec{k}$

$\boxed{\vec{F}'(t) = 2t\vec{i} - t^{-2}\vec{j} + 2e^{2t}\vec{k}}$

Diff w.r.t.t

$\vec{F}''(t) = \frac{d}{dt}(2t)\vec{i} - \frac{d}{dt}t^{-2}\vec{j} + 2\frac{d}{dt}e^{2t}\vec{k}$

$= 2\cdot1\,\vec{i} - (-2)t^{-3}\vec{j} + 2e^{2t}\frac{d}{dt}2t\,\vec{k}$

$= 2\vec{i} + 2t^{-3}\vec{j} + 2e^{2t}(2)\vec{k}$

$\boxed{\vec{F}''(t) = 2\vec{i} + 2t^{-3}\vec{j} + 4e^{2t}\vec{k}}$

---

b) $\vec{F}(s) = (1-2s^2)\vec{i} + s\cos(s)\vec{j} - s\vec{k}$

Diff w.r.t.s

$\vec{F}'(s) = \frac{d}{ds}[1-2s^2]\vec{i} + \frac{d}{ds}[s\cos s]\vec{j} - \frac{d}{ds}s\vec{k}$

$= (0-2(2s))\vec{i} + \{s\frac{d}{ds}\cos s + \cos s\frac{d}{ds}s\}\vec{j} - (1)\vec{k}$

$= -4s\vec{i} + \{-s\sin s + \cos s\,(1)\}\vec{j} - \vec{k}$

$\boxed{\vec{F}'(s) = -4s\vec{i} + \{\cos s - s\sin s\}\vec{j} - \vec{k}}$

Diff: w.r.t.s

$\vec{F}''(s) = \frac{d}{ds}(-4s)\vec{i} + \frac{d}{ds}[\cos s - s\sin s]\vec{j} - 0\vec{k}$

$= -4(1)\vec{i} + [-\sin s - \{s\frac{d}{ds}\sin s + \sin s\frac{d}{ds}s\}]\vec{j} - 0$

$= -4\vec{i} + [-\sin s - \{s\cos s + \sin s\cdot1\}]\vec{j} -$

$= -4\vec{i} + [-\sin s - s\cos s - \sin s]\vec{j}$

$= -4\vec{i} + [-2\sin s - s\cos s]\vec{j}$

$\boxed{\vec{F}''(s) = -4\vec{i} - (2\sin s + s\cos s)\vec{j}}$

Ans

c) $\vec{F}(s) = \sin s\,\vec{i} + \cos s\,\vec{j} + s^2\vec{k}$

Diff: w.r.t. s

$\vec{F}'(s) = \frac{d}{ds}[\sin s]\vec{i} + \left[\frac{d}{ds}\cos s\right]\vec{j} + \frac{d}{ds}s^2\vec{k}$

$\boxed{\vec{F}'(s) = \cos s\,\vec{i} - \sin s\,\vec{j} + 2s\vec{k}}$

Diff w.r.t. s

$\vec{F}''(s) = \frac{d}{ds}\cos s\,\vec{i} - \frac{d}{ds}\sin s\,\vec{j} + \frac{d}{ds}(2s)\vec{k}$

$\boxed{\vec{F}''(s) = -\sin s\,\vec{i} - \cos s\,\vec{j} + 2\vec{k}}$

d) $\vec{F}(\theta) = \sin^2\theta\,\vec{i} + \cos 2\theta\,\vec{j} + \theta^2\vec{k}$

Diff: w.r.t. θ

$\vec{F}'(\theta) = \frac{d}{d\theta}\sin^2\theta\,\vec{i} + \frac{d}{d\theta}\cos 2\theta\,\vec{j} + \frac{d}{d\theta}\theta^2\vec{k}$

$= 2\sin^1\theta\frac{d}{d\theta}\sin\theta\,\vec{i} + \{-\sin 2\theta\frac{d}{d\theta}2\theta\}\vec{j} + 2\theta\vec{k}$

$= 2\sin\theta\cos\theta\,\vec{i} - \sin 2\theta\cdot 2\cdot 1\,\vec{j} + 2\theta\vec{k}$

$\boxed{\vec{F}'(\theta) = \sin 2\theta\,\vec{i} - 2\sin 2\theta\,\vec{j} + 2\theta\vec{k}}$ $(\because \sin 2\theta = 2\sin\theta\cos\theta$

Diff: w.r.t. θ

$\vec{F}''(\theta) = \frac{d}{d\theta}\sin 2\theta\,\vec{i} - 2\frac{d}{d\theta}\sin 2\theta\,\vec{j} + \frac{d}{d\theta}(2\theta)\vec{k}$

$= \cos 2\theta\; 2\cdot 1\,\vec{i} - 2\cdot\cos 2\theta\cdot 2\cdot 1 + 2\cdot 1\,\vec{k}$

$\boxed{\vec{F}''(\theta) = 2\cos 2\theta\,\vec{i} - 4\cos 2\theta\,\vec{j} + 2\vec{k}}$

3) Differentiate the following scalar functions

a) $f(x) = [x\vec{i} + (x+1)\vec{j}]\cdot[2x\vec{i} - 3x^2\vec{j}]$

$= x(2x) + (x+1)(-3x^2) = 2x^2 - 3x^3 - 3x^2$

$f(x) = -3x^3 - x^2$

Diff w.r.t. x

$f'(x) = \frac{d}{dx}(-3x^3 - x^2) = -3(3x^2) - 2x = -9x^2 - 2x$ Ans

b) $f(x) = [\cos x\,\vec{i} + x\vec{j} - x\vec{k}] \cdot [\sec x\,\vec{i} - x^2\vec{j} + 2x\vec{k}]$

$= \cos x \cdot \sec x + x(-x^2) + (-x)(2x)$

$= \cancel{\cos x} \cdot \frac{1}{\cancel{\cos x}} - x^3 - 2x^2$

$f(x) = 1 - x^3 - 2x^2$

Diff w.r.t x

$f'(x) = \frac{d}{dx}(1 - x^3 - 2x^2)$

$= 0 - 3x^2 - 2(2x) = -3x^2 - 4x$

---

c) $g(x) = |\sin x\,\vec{i} - 2x\vec{j} + \cos x\,\vec{k}|$

$\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$

$|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$

$= \sqrt{\sin^2 x + (-2x)^2 + \cos^2 x}$

$= \sqrt{\sin^2 x + \cos^2 x + 4x^2}$

$g(x) = \sqrt{1 + 4x^2}$

Diff: w.r.t x

$g'(x) = \frac{d}{dx}\sqrt{1 + 4x^2}$

$= \frac{1}{2}(1 + 4x^2)^{-1/2} \frac{d}{dx}(1 + 4x^2)$

$= \frac{1}{2(1 + 4x^2)^{+1/2}}(0 + 4(2x)) = \frac{1}{\cancel{2}\sqrt{1 + 4x^2}}(\cancel{8x})^{4}$

$g'(x) = \frac{4x}{\sqrt{1 + 4x^2}}$ Ans

Position vector, Velocity and acceleration

Let Position vector or displacement vector is $\vec{R}(t)$

then velocity $\vec{V} = \frac{dR}{dt}$ or $R'(t)$

& acceleration $= \vec{A} = \frac{d\vec{V}}{dt}$ or $\vec{V}'(t)$

Speed is $|\vec{V}|$ (magnitude of the velocity)

Direction of motion is $\frac{\vec{V}}{|\vec{V}|}$ (unit vector)

4) Find the Particle's velocity, acceleration, speed and direction of motion for the indicated value of t, when the Position vector of a Particle's in space at time t is $\vec{R}(t)$: (20

a) $\vec{R}(t) = t\vec{i} + t^2\vec{j} + 2t\vec{k}$ at $\underline{t=1}$

Now velocity is

$$\vec{V}(t) = \frac{d}{dt}\vec{R}(t)$$
$$= \frac{d}{dt}(t\vec{i} + t^2\vec{j} + 2t\vec{k})$$
$$= 1\cdot\vec{i} + 2t\vec{j} + 2\vec{k}$$
$$\boxed{\vec{V}(t) = \vec{i} + 2t\vec{j} + 2\vec{k}}$$

Acceleration is

$$\vec{A}(t) = \frac{d\vec{V}}{dt}$$
$$= \frac{d}{dt}(\vec{i} + 2t\vec{j} + 2\vec{k})$$
$$= 0\vec{i} + 2\vec{j} + 0\vec{k}$$
$$\boxed{\vec{A}(t) = 2\vec{j}}$$

At $t=1$

$$\boxed{\vec{A}(1) = 2\vec{j}}$$

Now Velocity at $t=1$ is

$$\vec{V}(1) = \vec{i} + 2(1)\vec{j} + 2\vec{k}$$
$$\boxed{\vec{V}(1) = \vec{i} + 2\vec{j} + 2\vec{k}}$$

Now speed is

$$|\vec{V}(1)| = \sqrt{1^2 + 2^2 + 2^2}$$
$$= \sqrt{1+4+4} = \sqrt{9}$$
$$\boxed{|\vec{V}(1)| = 3}$$

Direction of motion at $t=1$ is

$$\frac{\vec{V}(1)}{|\vec{V}(1)|} = \frac{\vec{i} + 2\vec{j} + 2\vec{k}}{3} = \frac{\vec{i}}{3} + \frac{2\vec{j}}{3} + \frac{2}{3}\vec{k} = \frac{1}{3}\vec{i} + \frac{2}{3}\vec{j} + \frac{2}{3}\vec{k}$$ Ans

b) $\vec{R}(t) = (1-2t)\vec{i} - t^2\vec{j} + e^t\vec{k}$ at $t=0$

velocity

$$\vec{V}(t) = \frac{d\vec{R}(t)}{dt}$$
$$= \frac{d}{dt}\left((1-2t)\vec{i} - t^2\vec{j} + e^t\vec{k}\right)$$
$$\vec{V}(t) = (0-2)\vec{i} - 2t\vec{j} + e^t\vec{k}$$

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$$\boxed{\vec{V}(t) = -2\vec{i} - 2t\vec{j} + e^t\vec{k}} \Rightarrow$$

Velocity at $t=0$ is

$\vec{V}(0) = -2\vec{i} - 2(0)\vec{j} + e^0\vec{k}$

$\boxed{\vec{V}(0) = -2\vec{i} + 0\vec{j} + 1\cdot\vec{k}}$

speed at $t=0$

$|\vec{V}(0)| = \sqrt{(-2)^2 + 0^2 + 1^2}$

$= \sqrt{4+0+1}$

$\boxed{|\vec{V}(0)| = \sqrt{5}}$

Acceleration is

$\vec{A}(t) = \frac{d}{dt}\vec{V}(t)$

$= \frac{d}{dt}\left(-2\vec{i} - 2t\vec{j} + e^t\vec{k}\right)$

$= 0\vec{i} - 2(1)\vec{j} + e^t\vec{k}$

$\boxed{\vec{A}(t) = -2\vec{j} + e^t\vec{k}}$

acceleration at $t=0$

$\vec{A}(0) = -2\vec{j} + e^0\vec{k} \Rightarrow \boxed{\vec{A}(0) = -2\vec{j} + \vec{k}}$ $\quad e^0 = 1$

Direction of motion at $t=0$ is

$$\frac{\vec{V}(0)}{|\vec{V}(0)|} = \frac{-2\vec{i} + 0\vec{j} + 1\cdot\vec{k}}{\sqrt{5}}$$

$$= \frac{-2}{\sqrt{5}}\vec{i} + \frac{0}{\sqrt{5}}\vec{j} + \frac{1}{\sqrt{5}}\vec{k} \quad \text{or} \quad \frac{-2}{\sqrt{5}}\vec{i} + \frac{1}{\sqrt{5}}\vec{k}$$

---

c) $\vec{R}(t) = \cos t\,\vec{i} + \sin t\,\vec{j} + 3t\,\vec{k}$ at $t = \pi/4$

Velocity

$\vec{V}(t) = \frac{d}{dt}\vec{R}(t)$

$= \frac{d}{dt}\left[\cos t\,\vec{i} + \sin t\,\vec{j} + 3t\,\vec{k}\right]$

$\boxed{\vec{V}(t) = -\sin t\,\vec{i} + \cos t\,\vec{j} + 3\vec{k}}$

Velocity at $t = \pi/4$ is

$\vec{V}(\frac{\pi}{4}) = -\sin\frac{\pi}{4}\vec{i} + \cos\frac{\pi}{4}\vec{j} + 3\vec{k}$

(6) $\vec{V}(\frac{\pi}{4}) = -\frac{\sqrt{2}}{2}\vec{i} + \frac{\sqrt{2}}{2}\vec{j} + 3\vec{k}$

$\boxed{\vec{V}(\frac{\pi}{4}) = -\frac{1}{\sqrt{2}}\vec{i} + \frac{1}{\sqrt{2}}\vec{j} + 3\vec{k}}$

Note $\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$

$= \frac{\sqrt{2}}{\sqrt{4}} = \frac{\sqrt{2}}{2}$

$\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

speed at $t = \pi/4$ is

$|\vec{V}(\pi/4)| = \sqrt{\left(\frac{-1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + 3^2} = \sqrt{\frac{1}{2} + \frac{1}{2} + 9} = \sqrt{1+9} = \sqrt{10}$ ✓

Direction of motion is

$$\frac{\vec{V}(\pi/4)}{|\vec{V}(\frac{\pi}{4})|} = \frac{-\frac{1}{\sqrt{2}}\vec{i} + \frac{1}{\sqrt{2}}\vec{j} + 3\vec{k}}{\sqrt{10}} = \frac{1}{\sqrt{10}}\left[-\frac{1}{\sqrt{2}}\vec{i} + \frac{1}{\sqrt{2}}\vec{j} + 3\vec{k}\right]$$
$$= -\frac{1}{\sqrt{20}}\vec{i} + \frac{1}{\sqrt{20}}\vec{j} + \frac{3}{\sqrt{10}}\vec{k}$$
$$= -\frac{1}{2\sqrt{5}}\vec{i} + \frac{1}{2\sqrt{5}}\vec{j} + \frac{3}{\sqrt{10}}\vec{k}$$

Now Acceleration is

$$\vec{A}(t) = \frac{d}{dt}\vec{V}(t)$$
$$= \frac{d}{dt}\left(-\sin t\,\vec{i} + \cos t\,\vec{j} + 3\vec{k}\right] = -\cos t\,\vec{i} - \sin t\,\vec{j} + 0\,\vec{k}$$

$$\boxed{\vec{A}(t) = -\cos t\,\vec{i} - \sin t\,\vec{j}}$$

At $t = \pi/4$

$$\vec{A}(\pi/4) = -\cos\frac{\pi}{4}\vec{i} - \sin\frac{\pi}{4}\vec{j} = -\frac{1}{\sqrt{2}}\vec{i} - \frac{1}{\sqrt{2}}\vec{j} \quad \text{Ans}$$

---

(5) $\vec{V} = 2\vec{i} - \vec{j} + 5\vec{k}$, $\vec{w} = \vec{i} + 2\vec{j} - 3\vec{k}$

a) $\frac{d}{dt}(\vec{V} + t\vec{w})$

Sol
$$\vec{V} + t\vec{w} = (2\vec{i} - \vec{j} + 5\vec{k}) + t(\vec{i} + 2\vec{j} - 3\vec{k})$$
$$= 2\vec{i} - \vec{j} + 5\vec{k} + t\vec{i} + 2t\vec{j} - 3t\vec{k}$$
$$\vec{V} + t\vec{w} = (2+t)\vec{i} + (-1+2t)\vec{j} + (5-3t)\vec{k}$$

Diff: w.r.t t

$$\frac{d}{dt}(\vec{V} + t\vec{w}) = \frac{d}{dt}\left[(2+t)\vec{i} + (-1+2t)\vec{j} + (5-3t)\vec{k}\right]$$
$$= (0+1)\vec{i} + (0+2\cdot1)\vec{j} + (0-3\cdot1)\vec{k}$$
$$= \vec{i} + 2\vec{j} - 3\vec{k} \quad \text{Ans}$$

---

b) $\frac{d^2}{dt^2}(\vec{V}\cdot t^4\vec{w}) = ?$

Sol
$$\vec{V}\cdot t^4\vec{w} = (2\vec{i} - \vec{j} + 5\vec{k})\cdot t^4(\vec{i} + 2\vec{j} - 3\vec{k})$$
$$= (2\vec{i} - \vec{j} + 5\vec{k})\cdot(t^4\vec{i} + 2t^4\vec{j} - 3t^4\vec{k})$$
$$= 2(t^4) + (-1)(2t^4) + 5(-3t^4)$$

$$\vec{V}\cdot t^4\vec{\omega} = \cancel{2t^4} - \cancel{2t^4} - 15t^4$$

$$\vec{V}\cdot t^4\vec{\omega} = -15t^4$$

Diff: w.r.t. t

$$\frac{d}{dt}(\vec{V}\cdot t^4\vec{\omega}) = -15\frac{d}{dt}t^4$$

$$= -15(4t^3)$$

$$\frac{d}{dt}(\vec{V}\cdot t^4\vec{\omega}) = -60t^3$$

Diff: w.r.t. t

$$\frac{d}{dt}\left[\frac{d}{dt}(\vec{V}\cdot t^4\vec{\omega})\right] = -60\frac{d}{dt}t^3$$

$$\Rightarrow \frac{d^2}{dt^2}(\vec{V}\cdot t^4\vec{\omega}) = -60(3t^2) = -180t^2 \quad \text{Ans}$$

c) $\frac{d^2}{dt^2}\left(t|\vec{V}| + t^2|\vec{\omega}|\right)$

Sol

$$t|\vec{V}| + t^2|\vec{\omega}| = t|2\vec{i}-\vec{j}+5\vec{k}| + t^2|\vec{i}+2\vec{j}-3\vec{k}|$$

$$= t\sqrt{2^2+(-1)^2+5^2} + t^2\sqrt{1^2+2^2+(-3)^2}$$

$$= t\sqrt{4+1+25} + t^2\sqrt{1+4+9}$$

$$t|\vec{V}| + t^2|\vec{\omega}| = \sqrt{30}\,t + \sqrt{14}\,t^2$$

Diff: w.r.t. t

$$\frac{d}{dt}\left[t|\vec{V}| + t^2|\vec{\omega}|\right] = \frac{d}{dt}\left[\sqrt{30}\,t + \sqrt{14}\,t^2\right]$$

$$= \sqrt{30}\cdot 1 + \sqrt{14}\cdot 2t$$

$$\frac{d}{dt}\left[t|\vec{V}| + t^2|\vec{\omega}|\right] = \sqrt{30} + 2\sqrt{14}\,t$$

Diff: w.r.t. t

$$\frac{d}{dt}\left[\frac{d}{dt}\{t|\vec{V}| + t^2|\vec{\omega}|\}\right] = \frac{d}{dt}\left(\sqrt{30} + 2\sqrt{14}\,t\right)$$

$$\Rightarrow \frac{d^2}{dt^2}\left[t|\vec{V}| + t^2|\vec{\omega}|\right] = 0 + 2\sqrt{14}\cdot 1$$

$$= 2\sqrt{14} \quad \text{Ans}$$

d) $\frac{d}{dt}\left(t\vec{v} \times t^2\vec{w}\right) = ?$

Sol: $t\vec{v} \times t^2\vec{w} = t(2\vec{i} - \vec{j} + 5\vec{k}) \times t^2(\vec{i} + 2\vec{j} - 3\vec{k})$

$$= (2t\vec{i} - t\vec{j} + 5t\vec{k}) \times (t^2\vec{i} + 2t^2\vec{j} - 3t^2\vec{k})$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2t & -t & 5t \\ t^2 & 2t^2 & -3t^2 \end{vmatrix} \quad \text{Expand by } R_1$$

$$= \vec{i}\begin{vmatrix} -t & 5t \\ 2t^2 & -3t^2 \end{vmatrix} - \vec{j}\begin{vmatrix} 2t & 5t \\ t^2 & -3t^2 \end{vmatrix} + \vec{k}\begin{vmatrix} 2t & -t \\ t^2 & 2t^2 \end{vmatrix}$$

$$= \vec{i}(-3t^3 - 10t^3) - \vec{j}(-6t^3 - 5t^3) + \vec{k}(4t^3 + t^3)$$

$$t\vec{v} + t^2\vec{w} = -13t^3\vec{i} + 11t^3\vec{j} + 5t^3\vec{k}$$

Diff: w.r.t t

$$\frac{d}{dt}\left(t\vec{v} + t^2\vec{w}\right) = \frac{d}{dt}\left[-13t^3\vec{i} + 11t^3\vec{j} + 5t^3\vec{k}\right]$$

$$= -13(3t^2)\vec{i} + 11(3t^2)\vec{j} + 5(3t^2)\vec{k}$$

$$= -39t^2\vec{i} + 33t^2\vec{j} + 15t^2\vec{k} \quad \text{Ans}$$

---

6) $\boxed{\vec{F}(t) = (3+t^2)\vec{i} - \cos 3t\,\vec{j} + t^{-1}\vec{k}}$ $\boxed{\vec{G}(t) = \sin(2-t)\vec{i} - e^{2t}\vec{k}}$

a) verify that $(3F - 2G)'(t) = 3F'(t) - 2G'(t)$

LHS: $(3\vec{F} - 2\vec{G})'(t)$

$$= \frac{d}{dt}(3\vec{F} - 2\vec{G})(t) = \frac{d}{dt}\left[3\vec{F}(t) - 2\vec{G}(t)\right]$$

$$= \frac{d}{dt}\left[3\{(3+t^2)\vec{i} - \cos 3t\,\vec{j} + t^{-1}\vec{k}\} - 2\{\sin(2-t)\vec{i} - e^{2t}\vec{k}\}\right]$$

$$= \frac{d}{dt}\left[(9+3t^2)\vec{i} - 3\cos 3t\,\vec{j} + 3t^{-1}\vec{k} - 2\sin(2-t)\vec{i} + 2e^{2t}\vec{k}\right]$$

$$= \frac{d}{dt}\left[(9+3t^2 - 2\sin(2-t))\vec{i} - 3\cos 3t\,\vec{j} + (3t^{-1} + 2e^{2t})\vec{k}\right]$$

$$= \frac{d}{dt}(9+3t^2 - 2\sin(2-t))\vec{i} - 3\frac{d}{dt}\cos 3t\,\vec{j} + \frac{d}{dt}(3t^{-1} + 2e^{2t})\vec{k}$$

$= \{0 + 6t - 2\cos(2-t)\frac{d}{dt}(2-t)\}\vec{i} - 3\{-\sin 3t \frac{d}{dt}3t\}\vec{j} + \{3(-1)t^{-2} + 2e^{2t}\frac{d}{dt}2t\}\vec{k}$

$= \{6t - 2\cos(2-t)(0-1)\}\vec{i} + 3\sin 3t \; 3.1\,\vec{j} + \{-3t^{-2} + 2e^{2t}.2.1\}\vec{k}$

$(3F-2G)'(t) = \{6t + 2\cos(2-t)\}\vec{i} + 9\sin 3t\,\vec{j} + \{-3t^{-2} + 4e^{2t}\}\vec{k}$ — (1)

R.H.S: $3\vec{F}'(t) - 2\vec{G}'(t)$

$= 3\frac{d}{dt}\vec{F}(t) - 2\frac{d}{dt}\vec{G}(t)$

$= 3\frac{d}{dt}\left[(3+t^2)\vec{i} - \cos 3t\,\vec{j} + t^{-1}\vec{k}\right] - 2\frac{d}{dt}\left[\sin(2-t)\vec{i} - e^{2t}\vec{k}\right]$

$= 3\left[(0+2t)\vec{i} + \sin 3t\frac{d}{dt}3t\,\vec{j} + (-1)t^{-2}\vec{k}\right] - 2\left[\cos(2-t)\frac{d}{dt}(2-t)\vec{i} - e^{2t}\frac{d}{dt}2t\,\vec{k}\right]$

$= 3\left[2t\vec{i} + \sin 3t \; 3.1\,\vec{j} - t^{-2}\vec{k}\right] - 2\left[\cos(2-t)(0-1)\vec{i} - e^{2t}\,2.1\,\vec{k}\right]$

$= 6t\vec{i} + 9\sin 3t\,\vec{j} - 3t^{-2}\vec{k} + 2\cos(2-t)\vec{i} + 4e^{2t}\vec{k}$

$= \{6t + 2\cos(2-t)\}\vec{i} + 9\sin 3t\,\vec{j} + \{-3t^{-2} + 4e^{2t}\}\vec{k}$ — (2)

From (1) & (2) we have

$(3\vec{F} - 2\vec{G})'(t) = 3\vec{F}'(t) - 2\vec{G}'(t)$ verified.

---

b) $(F.G)'(t) = (F'.G)t + (F.G')(t)$

LHS: $(F.G)'(t) = \frac{d}{dt}(F.G)(t)$

$= \frac{d}{dt}(F(t).G(t))$

$= \frac{d}{dt}\left[\{(3+t^2)\vec{i} - \cos 3t\,\vec{j} + t^{-1}\vec{k}\}.\{\sin(2-t)\vec{i} + 0\vec{j} - e^{2t}\vec{k}\}\right]$

$= \frac{d}{dt}\left[(3+t^2)\sin(2-t) - (\cos 3t)(0) + t^{-1}(-e^{2t})\right]$

$= \frac{d}{dt}\left[(3+t^2)\sin(2-t) - 0 - t^{-1}e^{2t}\right]$

$= \frac{d}{dt}(3+t^2)\sin(2-t) - \frac{d}{dt}t^{-1}e^{2t}$

$$= (3+t^2)\frac{d}{dt}\sin(2-t) + \sin(2-t)\frac{d}{dt}(3+t^2) - \left\{ t^{-1}\frac{d}{dt}e^{2t} + e^{2t}\frac{d}{dt}t^{-1} \right\}$$

$$= (3+t^2)\cos(2-t)(0-1) + \sin(2-t)\cdot(0+2t) - \left\{ t^{-1}e^{2t}\cdot 2\cdot 1 + e^{2t}\{-1\cdot t^{-2}\} \right\}$$

$$(F\cdot G)'(t) = -(3+t^2)\cos(2-t) + 2t\sin(2-t) - 2t^{-1}e^{2t} + t^{-2}e^{2t} \quad \text{--- (1)}$$

RHS: $(\vec{F}'\cdot\vec{G})(t) + (\vec{F}\cdot\vec{G}')(t) = \vec{F}'(t)\cdot\vec{G}(t) + \vec{F}(t)\cdot\vec{G}'(t)$

$$= \frac{d}{dt}\vec{F}(t)\cdot\vec{G}(t) + \vec{F}(t)\cdot\frac{d}{dt}\vec{G}(t)$$

$$= \frac{d}{dt}\left[(3+t^2)\vec{i} - \cos 3t\,\vec{j} + t^{-1}\vec{k}\right]\cdot\left[\sin(2-t)\vec{i} + 0\vec{j} - e^{2t}\vec{k}\right]$$
$$+ \left\{(3+t^2)\vec{i} - \cos 3t\,\vec{j} + t^{-1}\vec{k}\right\}\cdot\frac{d}{dt}\left\{\sin(2-t)\vec{i} - e^{2t}\vec{k}\right\}$$

$$= \left[(0+2t)\vec{i} + \sin 3t\cdot 3\cdot 1\,\vec{j} - 1\cdot t^{-2}\vec{k}\right]\cdot\left[\sin(2-t)\vec{i} + 0\vec{j} - e^{2t}\vec{k}\right]$$
$$+ \left[(3+t^2)\vec{i} - \cos 3t\,\vec{j} + t^{-1}\vec{k}\right]\cdot\left[\cos(2-t)(-1)\vec{i} - e^{2t}\cdot 2\vec{k}\right]$$

$$= \left[2t\vec{i} + 3\sin 3t\,\vec{j} - t^{-2}\vec{k}\right]\cdot\left[\sin(2-t)\vec{i} + 0\vec{j} - e^{2t}\vec{k}\right]$$
$$+ \left[(3+t^2)\vec{i} - \cos 3t\,\vec{j} + t^{-1}\vec{k}\right]\cdot\left[-\cos(2-t)\vec{i} + 0\vec{j} - 2e^{2t}\vec{k}\right]$$

$$= 2t\sin(2-t) + \cancel{3\sin 3t\,(0)} - t^{-2}(-e^{2t})$$
$$+ (3+t^2)(-\cos(2-t)) - \cancel{\cos 3t\,(0)} + t^{-1}(2e^{2t})$$

$$= 2t\sin(2-t) + t^{-2}e^{2t} - (3+t^2)\cos(2-t) + 2t^{-1}e^{2t} \quad \text{--- (2)}$$

from (1) & (2), we have

$$(F\cdot G)'(t) = (F'\cdot G)(t) + (F\cdot G')(t) \qquad \text{verified}$$

---

7) If $\vec{F}(t)$ and $\vec{G}(t)$ are differentiable vector functions of t then Prove that

a) $(F\cdot G)'(t) = (F'\cdot G)(t) + (F\cdot G')(t)$

b) $(F\times G)'(t) = (F'\times G)(t) + (F\times G')(t)$

Sol Let.
$\vec{F}(t) = f_1(t)\vec{i} + f_2(t)\vec{j} + f_3(t)\vec{k}$, $\vec{G}(t) = g_1(t)\vec{i} + g_2(t)\vec{j} + g_3(t)\vec{k}$

$\Rightarrow \vec{F}'(t) = f_1'(t)\vec{i} + f_2'(t)\vec{j} + f_3'(t)\vec{k}$ & $\vec{G}'(t) = g_1'(t)\vec{i} + g_2'(t)\vec{j} + g_3'(t)\vec{k}$

a) prove $(\vec{F}.\vec{G})'(t) = (\vec{F}'.\vec{G})(t) + (\vec{F}.\vec{G}')(t)$

LHS: $(\vec{F}.\vec{G})'(t) = \frac{d}{dt}(\vec{F}.\vec{G})(t)$

$$= \frac{d}{dt}\left(\vec{F}(t).\vec{G}(t)\right)$$

$$= \frac{d}{dt}\left[f_1(t)g_1(t) + f_2(t)g_2(t) + f_3(t)g_3(t)\right]$$

$$= \frac{d}{dt}f_1(t)g_1(t) + \frac{d}{dt}f_2(t)g_2(t) + \frac{d}{dt}f_3(t)g_3(t)$$

$$= \{f_1'(t)g_1(t) + f_1(t)\underline{\underline{g_1'(t)}}\} + \{f_2'(t)g_2(t) + f_2(t)\underline{\underline{g_2'(t)}}\} + \{f_3'(t)g_3(t) + f_3(t)\underline{\underline{g_3'(t)}}\}$$

$$= \{f_1'(t)g_1(t) + f_2'(t)g_2(t) + f_3'(t)g_3(t)\} + \{f_1(t)g_1'(t) + f_2(t)g_2'(t) + f_3(t)g_3'(t)\}$$

$$= \vec{F}'(t).\vec{G}(t) + \vec{F}(t).\vec{G}'(t)$$

$$= (\vec{F}'.\vec{G})(t) + (\vec{F}.\vec{G}')(t) \quad \text{RHS}$$

---

b) Prove that $(\vec{F}\times\vec{G})'(t) = (\vec{F}'\times\vec{G})(t) + (\vec{F}\times\vec{G}')(t)$

Sol LHS: $(\vec{F}\times\vec{G})'(t)$

$$= \frac{d}{dt}(\vec{F}\times\vec{G})(t)$$

$$= \frac{d}{dt}\vec{F}(t)\times\vec{G}(t)$$

$$= \frac{d}{dt}\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ f_1(t) & f_2(t) & f_3(t) \\ g_1(t) & g_2(t) & g_3(t) \end{vmatrix} \quad \text{Expand by } R_1$$

$$= \frac{d}{dt}\left[\vec{i}\begin{vmatrix} f_2(t) & f_3(t) \\ g_2(t) & g_3(t)\end{vmatrix} - \vec{j}\begin{vmatrix} f_1(t) & f_3(t) \\ g_1(t) & g_3(t)\end{vmatrix} + \vec{k}\begin{vmatrix} f_1(t) & f_2(t) \\ g_1(t) & g_2(t)\end{vmatrix}\right]$$

$$= \frac{d}{dt}\left[\vec{i}\{f_2(t)g_3(t) - f_3(t)g_2(t)\} - \vec{j}\{f_1(t)g_3(t) - f_3(t)g_1(t)\} + \vec{k}\{f_1(t)g_2(t) - f_2(t)g_1(t)\}\right]$$

$$= \vec{i}\left[\frac{d}{dt} f_2(t)g_3(t) - \frac{d}{dt} f_3(t)g_2(t)\right] - \vec{j}\left[\frac{d}{dt} f_1(t)g_3(t) - \frac{d}{dt} f_3(t)g_1(t)\right] + \vec{k}\left[\frac{d}{dt} f_1(t)g_2(t) - \frac{d}{dt} f_2(t)g_1(t)\right]$$

$$= \vec{i}\left[f_2'(t)g_3(t) + f_2(t)g_3'(t) - \{f_3'(t)g_2(t) + f_3(t)g_2'(t)\}\right] - \vec{j}\left[f_1'(t)g_3(t) + f_1(t)g_3'(t) - \{f_3'(t)g_1(t) + f_3(t)g_1'(t)\}\right] + \vec{k}\left[f_1'(t)g_2(t) + f_1(t)g_2'(t) - \{f_2'(t)g_1(t) + f_2(t)g_1'(t)\}\right]$$

$$= \vec{i}\left[f_2'(t)g_3(t) - f_3'(t)g_2(t) + f_2(t)g_3'(t) - f_3(t)g_2'(t)\right] - \vec{j}\left[f_1'(t)g_3(t) - f_3'(t)g_1(t) + f_1(t)g_3'(t) - f_3(t)g_1'(t)\right] + \vec{k}\left[f_1'(t)g_2(t) - f_2'(t)g_1(t) + f_1(t)g_2'(t) - f_2(t)g_1'(t)\right]$$

$$= \vec{i}\left[f_2'(t)g_3(t) - f_3'(t)g_2(t)\right] - \vec{j}\left[f_1'(t)g_3(t) - f_3'(t)g_1(t)\right] + \vec{k}\left[f_1'(t)g_2(t) - f_2'(t)g_1(t)\right] + \vec{i}\left[f_2(t)g_3'(t) - f_3(t)g_2'(t)\right] - \vec{j}\left[f_1(t)g_3'(t) - f_3(t)g_1'(t)\right] + \vec{k}\left[f_1(t)g_2'(t) - f_2(t)g_1'(t)\right]$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ f_1'(t) & f_2'(t) & f_3'(t) \\ g_1(t) & g_2(t) & g_3(t) \end{vmatrix} + \begin{vmatrix} i & j & \vec{k} \\ f_1(t) & f_2(t) & f_3(t) \\ g_1'(t) & g_2'(t) & g_3'(t) \end{vmatrix}$$

$$= \vec{F}'(t) \times \vec{G}(t) + \vec{F}(t) \times \vec{G}'(t)$$

$$\therefore = (\vec{F}' \times \vec{G})(t) + (\vec{F} \times \vec{G}')(t) \quad RHS$$

Note: ① $|\vec{F}(t)| = \sqrt{|\vec{F}(t)|^2} = \sqrt{(\vec{F}(t))^2} = \sqrt{\vec{F}(t)\cdot\vec{F}(t)}$ $(\because \vec{a}\cdot\vec{a} = (\vec{a})^2$ or $|\vec{a}|^2$

② $\frac{d}{dt}|\vec{F}(t)| = \frac{d}{dt}\sqrt{\vec{F}(t)\cdot\vec{F}(t)} = \frac{1}{2}\left[\vec{F}(t)\cdot\vec{F}(t)\right]^{-1/2}\frac{d}{dt}\vec{F}(t)\cdot\vec{F}(t) = \frac{1}{2\left[\vec{F}(t)\cdot\vec{F}(t)\right]^{1/2}}\left[\vec{F}'(t)\cdot\vec{F}(t) + \vec{F}(t)\cdot\vec{F}'(t)\right]$

$$= \frac{\cancel{2}\,\vec{F}(t)\cdot\vec{F}'(t)}{\cancel{2}\sqrt{\vec{F}(t)\cdot\vec{F}(t)}} = \frac{\vec{F}(t)\cdot\vec{F}'(t)}{|\vec{F}(t)|}$$

$$\Rightarrow \therefore \frac{d}{dt}|\vec{F}(t)| = \frac{\vec{F}(t)\cdot\vec{F}'(t)}{|\vec{F}(t)|} \qquad \left(\frac{d}{dx}|x| = \frac{x}{|x|}\right.$$

8) Show that $\frac{d}{dt}\frac{\vec{F}(t)}{|\vec{F}(t)|} = \frac{F'(t)}{|F(t)|} - \frac{[F(t)\cdot F'(t)]\,F(t)}{|F(t)|^3}$

Sol LHS

$$\frac{d}{dt}\frac{\vec{F}(t)}{|\vec{F}(t)|} = \frac{|\vec{F}(t)|\frac{d}{dt}\vec{F}(t) - \left[\frac{d}{dt}|\vec{F}(t)|\right]\vec{F}(t)}{|\vec{F}(t)|^2}$$

$\left(\because |\vec{F}(t)| = \sqrt{|\vec{F}(t)|^2} = \sqrt{\vec{F}(t)\cdot\vec{F}(t)}\right.$

$$= \frac{\cancel{|\vec{F}(t)|}\frac{d}{dt}\vec{F}(t)}{|\vec{F}(t)|^{\cancel{2}}} - \frac{\left[\frac{d}{dt}\sqrt{\vec{F}(t)\cdot\vec{F}(t)}\right]\vec{F}(t)}{|\vec{F}(t)|^2}$$

$$= \frac{\vec{F}'(t)}{|\vec{F}(t)|} - \frac{\left[\frac{1}{2}(\vec{F}(t)\cdot\vec{F}(t))^{-1/2}\frac{d}{dt}(\vec{F}(t)\cdot\vec{F}(t))\right]\vec{F}(t)}{|\vec{F}(t)|^2}$$

Product Rule

$$= \frac{\vec{F}'(t)}{|\vec{F}(t)|} - \frac{1}{|\vec{F}(t)|^2}\left[\frac{1\{\vec{F}'(t)\cdot\vec{F}(t) + \vec{F}(t)\cdot\vec{F}'(t)\}}{2(\vec{F}(t)\cdot\vec{F}(t))^{+1/2}}\right]\vec{F}(t)$$

$$= \frac{\vec{F}'(t)}{|\vec{F}(t)|} - \frac{1}{|\vec{F}(t)|^2}\left[\frac{\cancel{2}\vec{F}(t)\cdot\vec{F}'(t)}{\cancel{2}\sqrt{\vec{F}(t)\cdot\vec{F}(t)}}\right]\vec{F}(t)$$

$$= \frac{\vec{F}'(t)}{|\vec{F}(t)|} - \frac{1}{|\vec{F}(t)|^2}\left[\frac{\vec{F}(t)\cdot\vec{F}'(t)}{|\vec{F}(t)|}\right]\vec{F}(t)$$

$$= \frac{\vec{F}'(t)}{|\vec{F}(t)|} - \frac{[\vec{F}(t)\cdot\vec{F}'(t)]\,\vec{F}(t)}{|\vec{F}(t)|^3}$$

Proved

RHS

END OF CH # 4