Integration

Introduction: In the Previous units we have learnt the different methods of finding the derivatives of the given functions, Now we are interested in inverse (reverse) Problem of finding a function whose derivative is given. If $F'(x) = f(x)$ then $F(x)$ is called antiderivative of $f(x)$.

The process of finding antiderivative is called antidifferentiation or integration.

MUHAMMAD ASHFAQ
Lecturer in Mathematics
Mobile: 0333-9000249

The concept of integration:

$\because \frac{d}{dx}x^2 = 2x$, $\frac{d}{dx}(x^2+3) = 2x$, $\frac{d}{dx}(x^2+\frac{1}{2}) = 2x$

Here antiderivative of $2x$ are $x^2, x^2+3, x^2+\frac{1}{2}$

Similarly we can find unlimited antiderivatives of $2x$ and these unlimited antiderivative of $2x$ are included in x^2+C, where c is an arbitrary constant, so we can say that Antiderivative of $2x$ is x^2+C & mathematically can be written as $\int 2x\,dx = x^2+C$

Note Antiderivatives $F_1(x)$ & $F_2(x)$ of a function $f(x)$ differ by a constant.

Note

$$\underset{\text{sign of integration}}{\int} \overset{\text{integrand}}{f(x)}\, \underset{\text{integrate w.r.t. x}}{dx} = \boxed{F(x)+C}$$

→ indefinite integral

Constant of integration

→ Particular integral

Ch-5

186 (2

Some standard integrals

① $\int 1\,dx = x + c$

② $\int k\,dx = kx + c$ (constant Rule)

3) $\int x^n\,dx = \frac{x^{n+1}}{n+1} + c$, $n \neq -1$ (Power Rule)

or $\int (ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c$, $n \neq -1$

4) $\int \frac{1}{x}\,dx = \ln|x| + c$, (logarithmic Rule)

or $\int \frac{1}{ax+b}\,dx = \frac{\ln(ax+b)}{a} + c$

5) $\int k\,f(x)\,dx = k\int f(x)\,dx$ (Constant multiple Rule)

6) $\int (f+g)\,dx = \int f\,dx + \int g\,dx$ (sum Rule)

7) $\int [f(x)]^n f'(x)\,dx = \frac{[f(x)]^{n+1}}{n+1} + c$, $n \neq -1$

8) $\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c$

9) $\int e^{f(x)} f'(x)\,dx = e^{f(x)} + c$

Integration by substitution

In above ⑦ ⑧ & ⑨ integrals if

$f(x) = u$ then differential is $f'(x)\,dx = du$

Note: ① $\int e^x\,dx = e^x + c$

② $\int e^{kx}\,dx = \frac{e^{kx}}{k} + c$ k is any real no: but $k \neq 0$

3) $\int a^x\,dx = \frac{a^x}{\ln a} + c$

4) $\int a^{kx}\,dx = \frac{a^{kx}}{k\ln a} + c$

Note: If all integrals exist, then:

① $\frac{d}{dx}\left[\int f(x)\,dx\right] = f(x)$ ② $\int\left[\frac{d}{dx}f(x)\right]dx = f(x)+c$

or

$\int f'(x)\,dx = f(x)+c$

EXERCISE 5.1

① Evaluate the following indefinite integrals and check the result through differentiation.

$\int x^n dx = \frac{x^{n+1}}{n+1}+c$

$\int k\,dx = kx+c$

a) $\int (x^2+3x^3-7)\,dx = \int x^2 dx + 3\int x^3 dx - \int 7\,dx$

$= \frac{x^{2+1}}{2+1} + 3\cdot\frac{x^{3+1}}{3+1} - 7x + c$

$\int (x^2+3x^3-7)\,dx = \frac{x^3}{3} + \frac{3x^4}{4} - 7x + C$ Ans

Check Differentiate both sides w.r.t. x

$\frac{d}{dx}\left[\int(x^2+3x^3-7)dx\right] = \frac{d}{dx}\left(\frac{x^3}{3}+\frac{3x^4}{4}-7x+c\right]$.

$\Rightarrow$ $x^2+3x^3-7 = \frac{1}{3}\frac{d}{dx}x^3 + \frac{3}{4}\frac{d}{dx}x^4 - 7\frac{d}{dx}x + \frac{d}{dx}C$

$= \frac{1}{\cancel{3}}(\cancel{3}x^2) + \frac{3}{\cancel{4}}(\cancel{4}x^3) - 7\cdot 1 + 0$

$x^2+3x^3-7 = x^2+3x^2-7$ Verified

b) $\int \frac{1}{x^3}dx = \int x^{-3}dx$

$= \frac{x^{-3+1}}{-3+1} = \frac{x^{-2}}{-2}+c = -\frac{1}{2x^{+2}}+c$ Ans

Check

$\frac{d}{dx}\int\frac{1}{x^3}dx = \frac{d}{dx}\left(-\frac{1}{2x^2}+c\right)$

$\frac{1}{x^3} = \frac{-1}{2}\frac{d}{dx}x^{-2} + \frac{d}{dx}c$

$\frac{1}{x^3} = \frac{\cancel{-1}}{\cancel{2}}(\cancel{-2}x^{-3}) + 0$

$\frac{1}{x^3} = x^{-3}$

$\frac{1}{x^3} = \frac{1}{x^3}$ (Verified)

188

c) $$\int (2x^4 + x^{-2/3} - x^{-5/3}]\,dx = 2\int x^4 dx + \int x^{-2/3} dx - \int x^{-5/3} dx$$

$$= 2\frac{x^{4+1}}{4+1} + \frac{x^{-2/3+1}}{-\frac{2}{3}+1} - \frac{x^{-5/3+1}}{-5/3+1} + C \quad \left\{ \int x^n dx = \frac{x^{n+1}}{n+1} + c \right.$$

$$= \frac{2x^5}{5} + \frac{x^{1/3}}{1/3} - \frac{x^{-2/3}}{-2/3} + c$$

$$= \frac{2}{5}x^5 + \frac{3}{1}x^{1/3} + \frac{3}{2}x^{-2/3} + c$$

$$\int (2x^4 + x^{-2/3} - x^{-5/3})\,dx = \frac{2}{5}x^5 + 3x^{1/3} + \frac{3}{2}x^{-2/3} + C \quad \text{Ans}$$

Check Diff: both sides w.r.t. x

$$\frac{d}{dx}\int (2x^4 + x^{-2/3} - x^{-5/3})\,dx = \frac{d}{dx}\left[\frac{2}{5}x^5 + 3x^{1/3} + \frac{3}{2}x^{-2/3} + C\right]$$

$$2x^4 + x^{-2/3} - x^{-5/3} = \frac{2}{5}\frac{d}{dx}x^5 + 3\frac{d}{dx}x^{1/3} + \frac{3}{2}\frac{d}{dx}x^{-2/3} + \frac{d}{dx}C$$

$$= \frac{2}{\cancel{5}}(\cancel{5}x^4) + \cancel{3}\left(\frac{1}{\cancel{3}}x^{\frac{1}{3}-1}\right) + \frac{\cancel{3}}{\cancel{2}}\left(\frac{-\cancel{2}}{\cancel{3}}x^{-\frac{2}{3}-1}\right) + 0$$

$$2x^4 + x^{-2/3} - x^{-5/3} = 2x^4 + x^{-2/3} - x^{-5/3} \quad \text{Verified}$$

d) $$\int (1+3t)\,t^3 dt = \int (t^3 + 3t^4)\,dt \quad \left(\int x^n dx = \frac{x^{n+1}}{n+1} + c \right.$$

$$= \frac{t^{3+1}}{3+1} + 3\frac{t^{4+1}}{4+1} + c$$

$$= \frac{t^4}{4} + 3\frac{t^5}{5} + c \quad \text{Ans}$$

Check

$$\frac{d}{dt}\left[\int (1+3t)\,t^3 dt\right] = \frac{d}{dt}\left[\frac{t^4}{4} + 3\frac{t^5}{5} + c\right]$$

$$\Rightarrow \quad (1+3t)\,t^3 = \frac{\cancel{4}t^3}{\cancel{4}} + \frac{3}{\cancel{5}}(\cancel{5}t^4) + 0$$

$$= t^3 + 3t^4$$

$$= t^3(1+3t) \quad \text{Verified.}$$

e) $\int (t^2-1)^2 dt = \int (t^4+1-2t^2)\,dt$ $\left((a-b)^2 = a^2+b^2-2ab\right.$

$= \int t^4 dt + \int 1 dt - 2\int t^2 dt$

$= \frac{t^5}{5} + t - 2\frac{t^3}{3} + C$ Ans

$\left(\because \int x^n dx = \frac{x^{n+1}}{n+1} + C\right.$ & $\int k\,dx = kx + C$

Check

$$\frac{d}{dt}\left[\int (t^2-1)^2 dt\right] = \frac{d}{dt}\left[\frac{t^5}{5} + t - 2\frac{t^3}{3} + C\right]$$

$$\Rightarrow (t^2-1)^2 = \frac{1}{5}\frac{d}{dt}t^5 + \frac{d}{dt}t - \frac{2}{3}\frac{d}{dt}(t^3) + \frac{d}{dt}C$$

$$= \frac{1}{\cancel{5}}(\cancel{5}t^4) + 1 - \frac{2}{\cancel{3}}(\cancel{3}t^2) + 0$$

$$= t^4 + 1 - 2t^2$$

$$(t^2-1)^2 = (t^2-1)^2$$ verified.

---

f) $\int \frac{x^3+1}{x^3}\,dx = \int\left(\frac{x^3}{x^3} + \frac{1}{x^3}\right)dx$

$= \int (1 + x^{-3})\,dx$

$= x + \frac{x^{-2}}{-2} + C$

$= x - \frac{1}{2x^{+2}} + C$

$= \frac{2x^3-1}{2x^2} + C$ Ans

$\left(\because \int 1\,dx = x + C\right.$ $\int x^n dx = \frac{x^{n+1}}{n+1} + C$

Check

$$\frac{d}{dx}\left[\int \frac{x^3+1}{x^3}dx\right] = \frac{d}{dx}\left[\frac{2x^3-1}{2x^2} + C\right]$$

$$\frac{x^3+1}{x^3} = \frac{2x^2\frac{d}{dx}(2x^3-1) - (2x^3-1)\frac{d}{dx}(2x^2)}{(2x^2)^2} + 0$$

$$= \frac{2x^2(6x^2-0) - (2x^3-1)(4x)}{4x^4}$$

$$\frac{x^3+1}{x^3} = \frac{12x^4 - 8x^4 + 4x}{4x^4}$$

$$\frac{x^3+1}{x^3} = \frac{4x^4+4x}{4x^4}$$

$$= \frac{\cancel{4x}(x^3+1)}{\cancel{4x^4}\,{}^3}$$

$$\therefore \frac{x^3+1}{x^3} = \frac{x^3+1}{x^3} \quad \text{(Verified)}$$

---

8) $\int z^2\sqrt{z}\,dz = \int z^2 z^{1/2} dz$

$$= \int z^{2+1/2} dz$$

$$= \int z^{5/2} dz$$

$$= \frac{z^{\frac{5}{2}+1}}{\frac{5}{2}+1} + C \qquad \left(\int z^n dz = \frac{z^{n+1}}{n+1} + C\right.$$

$$= \frac{z^{7/2}}{7/2} + C$$

$$= \frac{2}{7} z^{7/2} + C \quad \text{Ans}$$

Check

$$\frac{d}{dz}\int z^2\sqrt{z}\,dz = \frac{d}{dz}\left(\frac{2}{7} z^{7/2} + C\right)$$

$$z^2\sqrt{z} = \frac{\cancel{2}}{\cancel{7}}\left(\frac{\cancel{7}}{\cancel{2}} z^{\frac{7}{2}-1}\right) + 0$$

$$z^{2+1/2} = z^{5/2}$$

$$z^{5/2} = z^{5/2} \quad \text{verified.}$$

Integration by substitution

Sometimes it is Possible to convert an integral into a standard form or to an easy integral by a suitable change of a variable.

Note ① when integrand involve f(x) & f'(x) then put f(x)=u

② $\frac{d}{dx} f(x) = f'(x) \Rightarrow d\,f(x) = f'(x)\,dx$ it is called differential of f(x).

2) Evaluate the following indefinite integrals by method of substitution.

Ch-5

a) $\int(3x+4)^8 dx = \int u^8 \frac{du}{3}$

$= \frac{1}{3}\int u^8 du$

$= \frac{1}{3}\left(\frac{u^9}{9}\right)+c \quad \left(\because \int x^n dx = \frac{x^{n+1}}{n+1}+c\right)$

$= \frac{u^9}{27}+c$

$= \frac{(3x+4)^9}{27}+c \quad \because u = 3x+4$

put $3x+4 = u$

Diff w.r.t x

$\frac{d}{dx}(3x+4) = \frac{du}{dx}$

$3+0 = \frac{du}{dx}$

$\Rightarrow dx = \frac{du}{3}$

---

b) $\int 3x^2(x^3-4)dx = \int(x^3-4)\,3x^2 dx$

$= \int u^1 . du$

$= \frac{u^2}{2}+c$

$= \frac{(x^3-4)^2}{2}+c \quad \because x^3-4 = u$

put $x^3-4 = u$

Diff w.r.t. x

$3x^2 = \frac{du}{dx}$

$\Rightarrow 3x^2 dx = du$

---

c) $\int(3x^2+7)(x^3+7x)^8 dx = \int(x^3+7x)^8 . (3x^2+7)dx$

$= \int u^8 du$

$= \frac{u^9}{9}+c \quad \because \int x^n dx = \frac{x^{n+1}}{n+1}+c$

$= \frac{(x^3+7x)^9}{9}+c$

put $x^3+7x = u$

Diff: w.r.t. x

$3x^2+7 = \frac{du}{dx}$

$\Rightarrow (3x^2+7)dx = du$

---

d) $\int \frac{3x^2-7}{(x^3-7x)^4}dx = \int \frac{1}{(x^3-7x)^4}(3x^2-7)dx$

$\int \frac{1}{u^4}du$

$\int u^{-4}du$

$\frac{u^{-4+1}}{-4+1}+c = \frac{u^{-3}}{-3}+c = -\frac{u^{-3}}{3}+c$

$-\frac{1}{3u^{+3}}+c = \frac{-1}{3(x^3-7x)^3}+c$ Ans

put $x^3-7x = u$

Diff w.r.t. x

$3x^2-7 = \frac{du}{dx}$

$\Rightarrow (3x^2-7)dx = du$

192

⑥

e) $\int \frac{x+3x^2}{\sqrt{x}}dx = \int \frac{u^2+3(u^2)^2}{\cancel{u}}(2\cancel{u}du)$

$= 2\int (u^2+3u^4)\,du$

$= 2\left[\frac{u^3}{3}+3\cdot\frac{u^5}{5}\right]+C$

$= \frac{2u^3}{3}+\frac{6u^5}{5}+C$

$= \frac{2(\sqrt{x})^3}{3}+\frac{6(\sqrt{x})^5}{5}+C$

$= \frac{2x^{3/2}}{3}+\frac{6x^{5/2}}{5}+C$ Ans

put $\boxed{\sqrt{x}=u}$

Sq.

$\boxed{x=u^2}$

Diff: w.r.t. x

$1=2u\frac{du}{dx}$

$\Rightarrow dx = 2u\,du$

$\because \int x^n dx = \frac{x^{n+1}}{n+1}+C$

$(\because u=\sqrt{x}$

---

f) $\int\frac{x+1}{(x^2+2x+2)^2}dx = \int\frac{1}{(x^2+2x+2)^2}(x+1)dx$

$= \int \frac{1}{u^2}\,\frac{du}{2}$

$= \frac{1}{2}\int u^{-2}du$

$= \frac{1}{2}\cdot\frac{u^{-2+1}}{-2+1}+C = \frac{1}{2}\cdot\frac{u^{-1}}{-1}+C = -\frac{1}{2}u^{-1}+c$

$= -\frac{1}{2u}+C$

$= \frac{-1}{2(x^2+2x+2)}+C$ Ans

put $x^2+2x+2=u$

Diff w.r.t. x

$2x+2+0=\frac{du}{dx}$

$\Rightarrow 2(x+1)\,dx = du$

$\Rightarrow (x+1)dx = \frac{du}{2}$

Note ① $\int e^x dx = e^x+c$ ② $\int e^{kx}dx = \frac{e^{kx}}{k}+c$

③ $\int a^x dx = \frac{a^x}{\ln a}+c$ ④ $\int a^{kx}dx = \frac{a^{kx}}{k\ln a}+c$

---

③ Evaluate the following integrals by method of substitution

a) $\int 6e^{6t}dt = \int e^{6t}(6dt)$

$= \int e^u\,du$

$= e^u + c$

$= e^{6t}+C$ Ans $\because u=6t$

put $6t=u$

Diff w.r.t. t

$6\cdot1=\frac{du}{dt}$

$6dt = du$

b) $\int x e^{(5x^2+1)} dx = \int e^{5x^2+1} x dx$

$= \int e^u \frac{du}{10}$

$= \frac{1}{10} \int e^u du$

$= \frac{1}{10} e^u + c$

$= \frac{1}{10} e^{5x^2+1} + C \quad \because u = 5x^2+1$

$\left(\int e^x dx = e^x + c \right.$

put $5x^2+1 = u$
Diff: w. r.t x
$10x + 0 = \frac{du}{dx}$
$10x\, dx = du$
$\Rightarrow x\, dx = \frac{du}{10}$

---

c) $\int (x^2-2) e^{(x^3-6x+4)} dx = \int e^{(x^3-6x+4)} (x^2-2) dx$

$= \int e^u \frac{du}{3}$

$= \frac{1}{3} \int e^u du$

$= \frac{1}{3} e^u + c$

$= \frac{1}{3} e^{x^3-6x+4} + C$ Ans

put $x^3-6x+4 = u$
Diff w.r.t x
$3x^2 - 6 \cdot 1 + 0 = \frac{du}{dx}$
$3(x^2-2)\, dx = du$
$\Rightarrow (x^2-2) dx = \frac{du}{3}$

$\because \int e^x dx = e^x + c$

---

d) $\int 8^{(7-3x^2)} (-6x) dx = \int 8^u du$

$= \frac{8^u}{\ln 8} + c \quad (\because u = 7-3x^2$

$= \frac{8^{7-3x^2}}{\ln 8} + c \quad (\because \int a^x dx = \frac{a^x}{\ln a} + c$

put $7-3x^2 = u$
Diff w.r.t x
$0 - 6x = \frac{du}{dx}$
$(-6x) dx = du$

---

4) Find the equation of the Particular Curve that has a slope $4x^3 + 6x^2$ at a Point $(1,0)$

Sol $\because$ given that

slope $= 4x^3 + 6x^2$

$\Rightarrow \frac{dy}{dx} = 4x^3 + 6x^2$

integrate w.r.t x

$\int \left(\frac{dy}{dx}\right) dx = \int (4x^3 + 6x^2) dx$

$y = 4\int x^3 dx + 6 \int x^2 dx$

Note
① $\frac{d}{dx} \int f(x) dx = f(x)$
② $\int \frac{d}{dx} f(x) dx = f(x) + c$
or $\int \left(\frac{dy}{dx}\right) dx = y + c$

$$y = 4\cdot\frac{x^4}{4} + \overset{2}{\cancel{6}}\cdot\frac{x^3}{\cancel{3}} + C \qquad \left(\because \int x^n dx = \frac{x^{n+1}}{n+1} + C\right.$$

$$y = x^4 + 2x^3 + C \text{ —— } ①$$

∵ req: Curve Passes through Point (1, 0)

so put $x = 1,\ y = 0$ in ①

$$0 = 1^4 + 2(1)^3 + C$$

$$0 = 1 + 2 + C \Rightarrow \boxed{C = -3} \text{ put in } ①$$

①⇒ $y = x^4 + 2x^3 - 3$ is the required Curve.

---

5) ∵ given that the curve has a

$$\text{slope} = x(2x^2-1)^2$$

$$\Rightarrow \frac{dy}{dx} = x(2x^2-1)^2$$

integrate w.r.t. x

$$\int\left(\frac{dy}{dx}\right)dx = \int x(2x^2-1)^2 dx$$

$$y = \int (2x^2-1)^2\, x\,dx$$

$$y = \int u^2 \frac{du}{4}$$

$$= \frac{1}{4}\int u^2 du$$

$$y = \frac{1}{4}\left(\frac{u^3}{3}\right) + C$$

$$y = \frac{1}{12}(2x^2-1)^3 + C \text{ —— } ① \quad \left(\because u = 2x^2-1\right.$$

Note
$$\int f'(x)dx = f(x) + C$$
or
$$\int\left(\frac{dy}{dx}\right)dx = y + C$$

put
$$2x^2 - 1 = u$$
Diff: w.r.t. x
$$4x - 0 = \frac{du}{dx}$$
$$4x\,dx = du$$
$$\Rightarrow x\,dx = \frac{du}{4}$$

∵ Curve Passes through Point (3, 3)

so put $x = 3,\ y = 3$ in ①

$$3 = \frac{1}{12}\left[2(3)^2 - 1\right]^3 + C$$

$$3 = \frac{1}{12}(18-1)^3 + C$$

$$3 = \frac{1}{12}(17)^3 + C$$

$$3 = \frac{4913}{12} + C$$

$3 = 409.42 + C$

$3 - 409.42 = C \Rightarrow c = -406.42$ put in ①

① $\Rightarrow$ $y = \frac{1}{12}(2x^2-1)^2 - 406.42$ Ans

---

⑥ ∵ given that

$\text{slope} = x\sqrt{2x^2-1}$

$\Rightarrow \frac{dy}{dx} = x\sqrt{2x^2-1}$

integrate w.r.t. x

$\int\left(\frac{dy}{dx}\right)dx = \int x\sqrt{2x^2-1}\,dx$

$y = \int \sqrt{2x^2-1}\cdot x\,dx$

$y = \int \sqrt{u}\,\frac{du}{4}$

$y = \frac{1}{4}\int u^{1/2}du$

$y = \frac{1}{4}\cdot\frac{u^{1/2+1}}{\frac{1}{2}+1} + C$

$y = \frac{1}{\cancel{4}_2}\cdot\frac{u^{3/2}}{3/\cancel{2}} + C$

$y = \frac{1}{6}u^{3/2} + C$

$y = \frac{1}{6}(2x^2-1)^{3/2} + C$ —— ①

$\int f'(x)dx = f(x)$
or
$\int\left(\frac{d}{dx}y\right)dx = y$

put $2x^2-1 = u$
Diff: w.r.t. x
$4x - 0 = \frac{du}{dx}$
$4x\,dx = du$
$\Rightarrow x\,dx = \frac{du}{4}$

$\int x^n dx = \frac{x^{n+1}}{n+1} + C$.

∵ Curve passes through the Point (3,3)

so put $x=3$, $y=3$ in ①.

$3 = \frac{1}{6}[2(3)^2-1]^{3/2} + C$

$3 = \frac{1}{6}(18-1)^{3/2} + C \Rightarrow C = 3 - \frac{1}{6}(17)^{3/2} = 3 - \frac{70.1}{6}$

$C = 3 - 11.68$

$C = -8.68$ put in ①

① $\Rightarrow$ $y = \frac{1}{6}(2x^2-1)^{3/2} - 8.68$

which is the req: Curve.

196

(12

7) ∵ given that rate of change of weight W is

$\frac{dw}{dh} = 0.0015h^2$

integrate w.r.t. h

$\int \left(\frac{dw}{dh}\right) dh = \int .0015h^2\, dh$

$w = .0015 \frac{h^{2+1}}{2+1} + c$

$w = \overset{.0005}{\cancel{.0015}} \frac{h^3}{\cancel{3}} + c$

$w = .0005h^3 + C$ ——— ①

a) W = ? when w = 108 pounds & h = 60 inches

①⇒ ~~bo~~ $108 = 0.0005(60)^3 + C$

$108 = 0.0005 \times 216000 + C$

$\cancel{108} = \cancel{108} \cdot + c \Rightarrow c = 0$ put in ①

$w = .0005h^3 + 0$

$\boxed{w = .0005h^3.}$ ——— ②

b) W = ? when h = 5 feet 10 inches

h = 70 inches

put h = 70 in ②

$w = .0005(70)^3$

$= .0005 \times 343000$

$w = 171.5$ pounds

---

③ The rate of growth of the Population N(t) of newly incorporated city is

$\frac{dN}{dt} = 400 + 600\sqrt{t}$ $\quad 0 \le t \le 9$

integrate

(13

Ch-5

$$\int\left(\frac{dN}{dt}\right)dt = \int(400+600\sqrt{t})\,dt$$

$$N = 400t + 600\frac{t^{\frac{1}{2}+1}}{\frac{1}{2}+1} + c$$

$$N = 400t + 600\frac{t^{3/2}}{3/2} + C$$

$$N = 400t + \overset{200}{\cancel{600}} \times \frac{2}{\cancel{3}} \times t^{3/2} + C$$

$$N = 400t + 400t^{3/2} + C \quad \text{———} ①$$

∵ initially population was 15000

so put $t=0$ & $N = 5000$ in ①

$$5000 = 400(0) + 400(0)^{3/2} + C$$

$$5000 = 0+0+C \Rightarrow \boxed{C=5000}$$

put in ①

$$① \Rightarrow \boxed{N = 400t + 400t^{3/2} + 5000} \quad \text{———} ②$$

Now Population after 9 years

put $t=9$ in ②

$$N = 400(9) + 400(9)^{3/2} + 5000$$

$$N = 3600 + 400(3^{\cancel{2}})^{3/\cancel{2}} + 5000$$

$$= 3600 + 400\times 27 + 5000$$

$$= 3600 + 10800 + 5000 = 19400$$

Ans

Note ① $\int \sin x\,dx = -\cos x + c$, $\int \cos x\,dx = \sin x + c$

2) $\int \sec^2 x\,dx = \tan x + c$, $\int \operatorname{cosec}^2 x\,dx = -\cot x + c$

3) $\int \sec x \tan x\,dx = \sec x + c$, $\int \operatorname{cosec} x . \cot x\,dx = -\operatorname{cosec} x + c$

4) $\int \tan x\,dx = -\ln|\cos x| + c$ or $\ln \sec x + c$, $\int \cot x\,dx = \ln \sin x + c$

5) $\int \sec x\,dx = \ln(\sec x + \tan x) + c$, $\int \operatorname{cosec} x\,dx = \ln(\operatorname{cosec} x - \cot x) + c$

Note

1) $\int \frac{1}{a^2+x^2}dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$

2) $\int \frac{1}{x^2-a^2}dx = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C$

3) $\int \frac{1}{a^2-x^2}dx = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C$

4) $\int \frac{1}{\sqrt{a^2-x^2}}dx = \sin^{-1}\frac{x}{a} + C$

5) $\int \frac{1}{\sqrt{x^2+a^2}}dx = \ln|x+\sqrt{x^2+a^2}| + C$ or $\sinh^{-1}\frac{x}{a} + C$

6) $\int \frac{1}{\sqrt{x^2-a^2}}dx = \ln|x+\sqrt{x^2-a^2}| + C$ or $\cosh^{-1}\frac{x}{a} + C$

7) $\int \frac{1}{x\sqrt{x^2-a^2}}dx = \frac{1}{a}\sec^{-1}\frac{x}{a} + C$

Note ① when integrand involve $\sqrt{a^2-x^2}$ put $x = a\sin\theta$

② when integrand involve $\sqrt{a^2+x^2}$ put $x = a\tan\theta$

③ when integrand involve $\sqrt{x^2-a^2}$ put $x = a\sec\theta$

EXERCISE 5.2

① Evaluate the following indefinite integrals by method of substitution.

a) $\int \sin^4 x\cos x\,dx = \int u^4\,du$

$= \frac{u^{4+1}}{4+1} + C$

$= \frac{u^5}{5} + C$

$= \frac{\sin^5 x}{5} + C$ $\quad \because \sin x = u$

put $\sin x = u$
Diff w.r.t. x
$\cos x = \frac{du}{dx}$
$\Rightarrow \cos x\,dx = du$

---

b) $\int \frac{\cos x \ln\sin x}{\sin x}dx = \int \ln\sin x \cdot \frac{\cos x}{\sin x}dx$

$= \int u \cdot du$

$= \frac{u^2}{2} + C$

$= \frac{(\ln\sin x)^2}{2} + C$ Ans $\quad \because u = \ln\sin x$

put: $\ln\sin x = u$
Diff: w.r.t x
$\frac{1}{\sin x}\cos x = \frac{du}{dx}$
$\frac{\cos x}{\sin x}dx = du$

c) $\int e^x \sin e^x dx = \int \sin e^x . e^x dx$

$= \int \sin u . du$

$= -\cos u + c$

$= -\cos e^x + c$ Ans

put $e^x = u$
Diff: w.r.t. x
$e^x = \frac{du}{dx}$
$e^x dx = du$

$\because \int \sin x dx = -\cos x + C$

---

d). $\int (t+3) \cos(t+3)^2 dt = \int \cos(t+3)^2 (t+3) dt$

$= \int \cos u \frac{du}{2}$

$= \frac{1}{2} \int \cos u \, du$

$= \frac{1}{2}(+\sin u) + c$

$= \frac{1}{2} \sin(t+3)^2 + c$ Ans

put $(t+3)^2 = u$
Diff w.r.t. t
$2(t+3)^1(1) = \frac{du}{dt}$
$2(t+3)dt = du$
$(t+3)dt = \frac{du}{2}$

$\int \cos x dx = \sin x + C$

$\because u = (t+3)^2$

Note ① $\int \sin x dx = -\cos x + C$, $\int \cos x dx = \sin x + C$

② $\int \sec^2 x dx = \tan x + C$, $\int \csc^2 x dx = -\cot x + C$

3) $\int \sec x . \tan x dx = \sec x + C$, $\int \csc x . \cot x dx = -\csc x + C$

---

② Evaluate the following integrals by method of substitution.

a) $\int \tan 2x \sec 2x dx = \int \tan u . \sec u . \frac{du}{2}$

$= \frac{1}{2} \int \sec u . \tan u \, du$

$= \frac{1}{2} \sec u + C$

$= \frac{1}{2} \sec(2x) + C$ Ans

put $2x = u$
Diff: w.r.t. x
$2 = \frac{du}{dx}$
$\Rightarrow dx = \frac{du}{2}$

$\because u = 2x$

$\int \sec x . \tan x dx = \sec x + C$

200

b) $\int 4\sec^2 4x\,dx$

$= \int \cancel{4}\sec^2 u \frac{du}{\cancel{4}}$

$= \int \sec^2 u\,du$

$= \tan u + c$

$= \tan 4x + C$

put $4x = u$

Diff w.r.t. x

$4 = \frac{du}{dx}$

$dx = \frac{du}{4}$

$\int \sec^2 x\,dx = \tan x + C$

c) $\int \tan x \cdot \sec^2 x\,dx$

$= \int u \cdot du$

$= \frac{u^2}{2} + C$

$= \frac{\tan^2 x}{2} + C$

Ans

put $\tan x = u$

Diff w.r.t. x

$\sec^2 x = \frac{du}{dx}$

$\Rightarrow \sec^2 x\,dx = du$

$\int x^n dx = \frac{x^{n+1}}{n+1}$

$\because u = \tan x$

---

d) $\int (\tan 3x + \sec 3x)\,dx = \int (\tan u + \sec u)\frac{du}{3}$

$= \frac{1}{3}\int (\tan u + \sec u)\,du$

$= \frac{1}{3}\left[\ln \sec u + \ln(\sec u + \tan u)\right] + C$

$= \frac{1}{3}\left[\ln \sec 3x + \ln(\sec 3x + \tan 3x)\right] + C$

Ans

put $3x = u$

Diff: w.r.t. x

$3 = \frac{du}{dx}$

$\Rightarrow dx = \frac{du}{3}$

$\int \tan x\,dx = \ln \sec x + c$

$\int \sec x\,dx = \ln(\sec x + \tan x) + c$

---

e) $\int \frac{\cos^2 x}{\operatorname{cosec} x}\,dx = \int \cos^2 x \cdot \sin x\,dx$

$= \int u^2 (-du)$

$= -\int u^2 du$

$= -\frac{u^3}{3} + C$ $\quad \because u = \cos x$

$= -\frac{\cos^3 x}{3} + C$ Ans

put $\cos x = u$

Diff w.r.t. x

$-\sin x = \frac{du}{dx}$

$\Rightarrow \sin x\,dx = -du$

$\int x^n dx = \frac{x^{n+1}}{n+1} + C$

---

f) $\int \frac{\cot\sqrt{x}}{\sqrt{x}}\,dx = \int \cot\sqrt{x} \cdot \frac{1}{\sqrt{x}}\,dx$

$= \int \cot u \cdot 2du$

$= 2\int \cot u\,du$

$= 2(\ln \sin u) + C$

$= 2 \ln \sin\sqrt{x} + C$

Ans

put $\sqrt{x} = u$

Diff w.r.t. x

$\frac{1}{2}x^{-1/2} = \frac{du}{dx}$

$\frac{1}{2\sqrt{x}}dx = du$

$\Rightarrow \frac{1}{\sqrt{x}}dx = 2du$

$\because u = \sqrt{x}$

$\int \cot x\,dx = \ln \sin x + C$

g) $\int \frac{\sin x - \cos x}{\sin x + \cos x}\,dx = \int \frac{1}{\sin x + \cos x}(\sin x - \cos x)\,dx$

$= \int \frac{1}{u}(-du)$

$= -\int \frac{1}{u}\,du$

$= -\ln|u| + c$

$= -\ln|\sin x + \cos x| + c.$

put:
$\sin x + \cos x = u$
Diff w.r.t. x
$\cos x - \sin x = \frac{du}{dx}$
$-[\cos x + \sin x]\,dx = du$
$(\sin x - \cos x)\,dx = -du$

$\because u = \sin x + \cos x$

---

h) $\int \frac{\sin x}{3 + 2\cos x}\,dx = \int \frac{1}{3 + 2\cos x}\sin x\,dx$

$= \int \frac{1}{u}\left(-\frac{du}{2}\right)$

$= -\frac{1}{2}\int \frac{1}{u}\,du$

$= -\frac{1}{2}\ln u + c.$

$= -\frac{1}{2}\ln(3 + 2\cos x) + c.$ $\quad\because u = 3 + 2\cos x$

put $3 + 2\cos x = u$
Diff: w.r.t. x
$0 - 2\sin x = \frac{du}{dx}$
$\Rightarrow \sin x\,dx = -\frac{du}{2}$

$\int \frac{1}{x}\,dx = \ln x + c$

Note 1) $\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}x + c$ or $\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a} + c$

2) $\int \frac{1}{1+x^2}\,dx = \tan^{-1}x + c$ or $\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c$

3) $\int \frac{1}{x\sqrt{x^2-1}}\,dx = \sec^{-1}x + c$ or $\int \frac{1}{x\sqrt{x^2-a^2}}\,dx = \frac{1}{a}\sec^{-1}\frac{x}{a} + c$

or $\int \frac{1}{x\sqrt{x^2-1}}\,dx = -\operatorname{cosec}^{-1}x + c$ or $\int \frac{1}{x\sqrt{x^2-a^2}}\,dx = -\frac{1}{a}\operatorname{cosec}^{-1}\frac{x}{a} + c$

---

3) use suitable substitutions and tables to evaluate the following indefinite integrals

a) $\int \frac{1}{x^2+16}\,dx = \int \frac{1}{x^2+4^2}\,dx$

$= \frac{1}{4}\tan^{-1}\frac{x}{4} + c$ Ans

$\because \int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c$

2nd method →

2nd method $\int \frac{1}{x^2+16}dx = \int \frac{1}{x^2+4^2}dx$

$= \int \frac{1}{(4\tan\theta)^2+4^2}\cdot 4\sec^2\theta d\theta$

$= \int \frac{1}{16\tan^2\theta+16} \times 4\sec^2\theta d\theta$

$= \int \frac{\cancel{4}\sec^2\theta d\theta}{\cancel{16}_4(\tan^2\theta+1)}$

$= \frac{1}{4}\int \frac{\cancel{\sec^2\theta}}{\cancel{\sec^2\theta}} d\theta$

$= \frac{1}{4}\int 1 d\theta$

$= \frac{1}{4}\theta + C$

$= \frac{1}{4}\tan^{-1}\frac{x}{4} + C$

Ans

put
$x = 4\tan\theta$
Diff w.r.t. x.
$1 = 4\cdot\sec^2\theta\frac{d\theta}{dx}$
$\Rightarrow dx = 4\sec^2\theta d\theta$

$\because 1+\tan^2\theta = \sec^2\theta$

$\because x = 4\tan\theta$
$\frac{x}{4} = \tan\theta$
$\Rightarrow \tan^{-1}\frac{x}{4} = \theta$

b) $\int \frac{\sin x}{\cos^2x+1}dx = \int \frac{1}{\cos^2x+1}\sin x dx$

$= \int \frac{1}{u^2+1}(-du)$

$= -\int \frac{1}{u^2+1}du$

$= -\tan^{-1}u + C$

$= -\tan^{-1}(\cos x) + C$

put
$\cos x = u$
Diff: w.r.t. u
$-\sin x = \frac{du}{dx}$
$\Rightarrow \sin x dx = -du$

$\int \frac{1}{1+x^2}dx = \tan^{-1}x + C$

$\because u = \cos x$

c) $\int \frac{1}{\sqrt{5-2x^2}}dx = \int \frac{1}{\sqrt{5-(\sqrt{2}x)^2}}dx$ $(\sqrt{2})^2 = 2$

$= \int \frac{1}{\sqrt{5-u^2}}\frac{du}{\sqrt{2}}$

$= \frac{1}{\sqrt{2}}\int \frac{1}{\sqrt{(\sqrt{5})^2-u^2}}du$

put $\sqrt{2}x = u$
Diff: w.r.t. x
$\sqrt{2}\cdot 1 = \frac{du}{dx}$
$\Rightarrow dx = \frac{du}{\sqrt{2}}$

$5 = (\sqrt{5})^2$

$$= \frac{1}{\sqrt{2}} \sin^{-1}\frac{u}{\sqrt{5}} + c \qquad \left(\because \int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a} + c\right.$$

$$= \frac{1}{\sqrt{2}} \sin^{-1}\frac{\sqrt{2}x}{\sqrt{5}} + c \quad \underline{\underline{\text{Ans}}} \qquad \because u = \sqrt{2}x$$

or

$$= \frac{\sqrt{2}}{2} \sin^{-1}\frac{\sqrt{10}x}{5} + c \quad \text{Ans} \qquad \because \frac{1}{\sqrt{2}} = \sqrt{\frac{2}{4}} = \frac{\sqrt{2}}{2}$$

$$\& \quad \frac{\sqrt{2}}{\sqrt{5}} = \sqrt{\frac{2}{5}} = \sqrt{\frac{10}{25}} = \frac{\sqrt{10}}{5}$$

Note $\int \frac{1}{x\sqrt{x^2-a^2}}\,dx = \frac{1}{a}\sec^{-1}\frac{x}{a} + c$ or $-\frac{1}{a}\operatorname{cosec}^{-1}\frac{x}{a} + c$

---

d) $$\int \frac{1}{\sqrt{e^{2x}-4}}\,dx = \int \frac{1}{\sqrt{(e^x)^2-4}}\,dx$$

put $e^x = u$

Diff: w.r.t. x

$e^x = \frac{du}{dx}$

$\Rightarrow dx = \frac{du}{e^x}$

$\Rightarrow dx = \frac{du}{u}$ $\quad (u = e^x$

$$= \int \frac{1}{\sqrt{u^2-4}}\,\frac{du}{u}$$

$$= \int \frac{1}{u\sqrt{u^2-4}}\,du$$

$$= \int \frac{1}{u\sqrt{u^2-2^2}}\,du$$

$$= \frac{1}{2}\sec^{-1}\frac{u}{2}) + c$$

$$= \frac{1}{2}\sec^{-1}\left(\frac{e^x}{2}\right) + c \qquad \because u = e^x$$

Note ① $a^2 + 2ab + b^2 = (a+b)^2$ ② $\int \frac{1}{1+x^2}\,dx = \tan^{-1}x + c$

---

e) $$\int \frac{2x+5}{x^2+4x+5}\,dx = \int \frac{2x+5}{\underline{x^2+4x+4}+1}\,dx \qquad (5 = 4+1$$

$$= \int \frac{2x+5}{(x+2)^2+1}\,dx$$

$$= \int \frac{2x+4+1}{(x+2)^2+1}\,dx$$

$$= \int \frac{2(x+2)+1}{(x+2)^2+1}\,dx$$

$$= \int \left[\frac{2(x+2)}{(x+2)^2+1} + \frac{1}{(x+2)^2+1} \right] dx$$

$$= \int \frac{2(x+2)}{(x+2)^2+1} dx + \int \frac{1}{(x+2)^2+1} dx$$

$$= \int \frac{2u}{u^2+1} du + \int \frac{1}{u^2+1} du$$

$$= \int \frac{1}{t} dt + \tan^{-1} u + C$$

$$= \ln t + \tan^{-1} u + C$$

$$= \ln(u^2+1) + \tan^{-1} u + C$$

$$= \ln((x+2)^2+1) + \tan^{-1}(x+2) + C$$

$$= \ln[x^2+4+4x+1] + \tan^{-1}(x+2) + C$$

$$= \ln(x^2+4x+5) + \tan^{-1}(x+2) + C$$

put $x+2=u$
Diff w.r.t. x
$1+0 = \frac{du}{dx}$
$\Rightarrow dx = du$

In 1st integral
put $u^2+1 = t$
Diff w.r.t. u
$2u = \frac{dt}{du}$
$2u\,du = dt$

$t = u^2+1$
& $u = x+2$

---

f) $$\int \frac{2+x}{\sqrt{4-2x-x^2}} dx = \int \frac{2+x}{\sqrt{5-1-2x-x^2}} dx$$ ($4=5-1$

$$= \int \frac{2+x}{\sqrt{5-(1+2x+x^2)}} dx$$

$$= \int \frac{2+x}{\sqrt{5-(1+x)^2}} dx$$

$$= \int \frac{1+\boxed{1+x}}{\sqrt{5-(1+x)^2}} dx$$

$$= \int \left(\frac{1}{\sqrt{5-(1+x)^2}} + \frac{1+x}{\sqrt{5-(1+x)^2}} \right) dx$$

$$= \int \frac{1}{\sqrt{5-(1+x)^2}} dx + \int \frac{1+x}{\sqrt{5-(1+x)^2}} dx$$

$$= \int \frac{1}{\sqrt{5-u^2}} du + \int \frac{u}{\sqrt{5-u^2}} du$$

$$= \int \frac{1}{\sqrt{(\sqrt{5})^2-u^2}} du + \int \frac{1}{\sqrt{5-u^2}} u\,du$$

put $1+x = u$
Diff w.r.t. x
$1 = \frac{du}{dx}$
$\Rightarrow dx = du$

$$\int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1}\frac{x}{a} + C$$

$$= \sin^{-1}\frac{u}{\sqrt{5}}) + \int \frac{1}{\sqrt{5-u^2}}\,u\,du$$

$$= \sin^{-1}\frac{1+x}{\sqrt{5}} + \int \frac{1}{\sqrt{t}}\left(-\frac{dt}{2}\right)$$

$$= \sin^{-1}\frac{1+x}{\sqrt{5}} - \frac{1}{2}\int t^{-1/2}\,dt$$

$$= \sin^{-1}\frac{1+x}{\sqrt{5}} - \frac{1}{2}\,\frac{t^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c$$

$$= \sin^{-1}\frac{1+x}{\sqrt{5}} - \cancel{\frac{1}{2}}\cdot\frac{t^{1/2}}{\cancel{1/2}} + c$$

$$= \sin^{-1}\frac{1+x}{\sqrt{5}} - \sqrt{t} + c$$

$$= \sin^{-1}\frac{1+x}{\sqrt{5}} - \sqrt{4-2x-x^2} + c$$

Ans

put $5-u^2 = t$

Diff: w.r.t. u

$-2u = \frac{dt}{du}$

$-2u\,du = dt$

$\Rightarrow u\,du = -\frac{dt}{2}$

$\because u = 1+x$

$\because t = 5-u^2$

$= 5-(1+x)^2$

$= 5-(1+2x+x^2)$

$= 5-1-2x-x^2$

$t = 4-2x-x^2$

---

8) $$\int \frac{dx}{x\sqrt{7x^2-5}} = \int \frac{1}{x\sqrt{(\sqrt{7}x)^2-5}}\,dx$$

put $\sqrt{7}x = u$ or $x = \frac{u}{\sqrt{7}}$

Diff w.r.t. x

$\sqrt{7} = \frac{du}{dx}$

$\Rightarrow dx = \frac{du}{\sqrt{7}}$

$$= \int \frac{1}{\frac{u}{\cancel{\sqrt{7}}}\sqrt{u^2-5}}\,\frac{du}{\cancel{\sqrt{7}}}$$

$$= \int \frac{1}{u\sqrt{u^2-(\sqrt{5})^2}}\,du$$

$\int \frac{1}{x\sqrt{x^2-a^2}}\,dx = \frac{1}{a}\sec^{-1}\frac{x}{a} + c$

or

$= -\frac{1}{a}\,\text{cosec}^{-1}\frac{x}{a} + c$

$$= \frac{1}{\sqrt{5}}\sec^{-1}\frac{u}{\sqrt{5}} + c$$

$$= \frac{1}{\sqrt{5}}\sec^{-1}\frac{\sqrt{7}x}{\sqrt{5}} + c \qquad \because u = \sqrt{7}x$$

Ans

Note ① $\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + c$

② $\int \sqrt{x^2 \pm a^2}\,dx = \frac{x}{2}\sqrt{x^2 \pm a^2} + \frac{a^2}{2}\ln\left(x + \sqrt{x^2 \pm a^2}\right) + c$

③ $\int \frac{\sqrt{a^2-x^2}}{x}\,dx = \sqrt{a^2-x^2} - a\ln\left|\frac{a+\sqrt{a^2-x^2}}{x}\right| + c$

4) $\int \frac{1}{\sqrt{x^2 \pm a^2}} dx = \ln|x + \sqrt{x^2 \pm a^2}| + c$

Integration by Parts

$\because$ we know that

$$\frac{d}{dx}[f(x) \cdot g(x)] = f(x)\, g'(x) + g(x)\, f'(x)$$

$$\frac{d}{dx}[f(x)\, g(x)] - g(x)\, f'(x) = f(x)\, g'(x)$$

or $f(x)\, g'(x) = \frac{d}{dx}[f(x)\, g(x)] - g(x)\, f'(x)$

integrate B. sides w.r.t. x

$$\int f(x)\, g'(x)\, dx = \int \frac{d}{dx}[f(x)\, g(x)]\, dx - \int g(x)\, f'(x)\, dx$$

$$\boxed{\int f(x)\, g'(x) dx = f(x)\, g(x) - \int f'(x)\, g(x)\, dx} \longrightarrow ①$$

put $f(x) = u$ & $g'(x) = v$.

Diff w.r.t. x — integrate w.r.t. x

$f'(x) = \frac{du}{dx}$ — $g(x) = \int v dx$

$$① \Rightarrow \int u\, v dx = u \int v dx - \int \left(\frac{du}{dx}\right)\left(\int v dx\right) dx$$

or $$\boxed{\int \text{I}\,\text{II}\, dx = \text{I} \int \text{II}\, dx - \int \left(\frac{d}{dx}\text{I}\right)\left(\int \text{II}\, dx\right) dx}$$ ✓

Note

1) Use integration by Parts when integrand involve Product without $f(x)$ & $f'(x)$

2) If integrand is of the form $x^n f(x)$ then take x^n as the first function

3) If integrand involve logarithmic or inverse trig. function, then take such a function as 1st function

4) take 1 as the 2nd function in the following cases

i) $\int \log x \times 1 dx$ ii) $\int \sin^{-1}x \times 1 dx$ iii) $\int \sqrt{a^2 \pm x^2} \times 1\, dx$

Integration by Partial fraction

$\frac{P(x)}{Q(x)}$, $Q(x) \neq 0$, where $P(x)$ & $Q(x)$ are Polynomial & have no common factors, such a function is called rational function.

Rational functions are of two types

i) Proper ii) Improper

i) when degree of $P(x)$ is less than degree of $Q(x)$. then $\frac{P(x)}{Q(x)}$ is called Proper rational function, To use Partial fraction first factorize $Q(x)$

ii) when degree of $P(x)$ is greater or equal to degree of $Q(x)$ then $\frac{P(x)}{Q(x)}$ is called Improper rational function, To use Partial fraction first convert it into Proper rational function by using long division.

Partial fraction

Case-I $\frac{x}{(x+1)(x+3)} = \frac{A}{x+1} + \frac{B}{x+3}$

Case-II $\frac{x}{(x+1)(x+3)^3} = \frac{A}{x+1} + \frac{B}{x+3} + \frac{C}{(x+3)^2} + \frac{D}{(x+3)^3}$

Case-III $\frac{x}{(x+1)(x^2+3)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+3}$.

Case-IV $\frac{x}{(x+1)(x^2+3)^2} = \frac{A}{x+1} + \frac{Bx+C}{x^2+3} + \frac{Dx+E}{(x^2+3)^2}$

Conditional equation or equation: An equation which is true for some values. For example: $x^2-1=0$ is an equation

Identity: An equation which is true for all values.

For example, $x^2-1=(x-1)(x+1)$ is an identity.

Note: ① $\int \frac{1}{ax+b}\,dx = \frac{\ln(ax+b)}{a} + c$ ② $\int (ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C$

③ $\int \frac{f'(x)}{f(x)}\,dx = \ln f(x) + c$ ④ $\int [f(x)]^n f'(x)\,dx = \frac{[f(x)]^{n+1}}{n+1} + c$

EXERCISE 5.3

1) Evaluate the indefinite integrals after decomposing the following rational function into Partial fractions:

a) let $\frac{1}{x(x-3)} = \frac{A}{x} + \frac{B}{x-3} \longrightarrow ①$

x ing by $x(x-3)$

$1 = A(x-3) + Bx \longrightarrow ②$

put $x=0$ in ②

$1 = A(0-3) + B\cdot 0$

$1 = -3A + 0 \Rightarrow \boxed{A = -\frac{1}{3}}$

put $x-3=0 \Rightarrow x=3$ in ②

$1 = A(0) + B(3)$

$1 = 0 + 3B \Rightarrow \boxed{B = \frac{1}{3}}$

put values of A & B in ①

$\frac{1}{x(x-3)} = \frac{-\frac{1}{3}}{x} + \frac{\frac{1}{3}}{x-3}$

integrate w.r.t. x

$\int \frac{1}{x(x-3)}\,dx = -\frac{1}{3}\int \frac{1}{x}\,dx + \frac{1}{3}\int \frac{1}{x-3}\,dx$

$= -\frac{1}{3}\ln x + \frac{1}{3}\ln(x-3) + c$ Ans

(b ln a = ln a^b

$= \ln x^{-\frac{1}{3}} + \ln(x-3)^{\frac{1}{3}} + c$

$= \ln x^{-\frac{1}{3}}(x-3)^{\frac{1}{3}} + c.$

(ln a + ln b = ln ab

$= \ln\left[\frac{(x-3)^{\frac{1}{3}}}{x^{\frac{1}{3}}}\right] + c$ Ans

b) $\frac{3x^2+2x-1}{x(x+1)} = \frac{3x^2+2x-1}{x^2+x}$ (Improper)

$$x^2+x\,\overline{)\,3x^2+2x-1}\quad 3$$
$$\underline{3x^2+3x}$$
$$-x-1$$

$= 3 + \frac{-x-1}{x^2+x}$

$\frac{3x^2+2x-1}{x(x+1)} = 3 + \frac{-x-1}{x(x+1)}$

$\frac{3x^2+2x-1}{x(x+1)} = 3 + \frac{A}{x} + \frac{B}{x+1}$ —— ①

where $\frac{-x-1}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1}$

xing by $x(x+1)$

$-x-1 = A(x+1) + Bx \longrightarrow$ ②

put $x=0$ in ②

$0-1 = A(0+1) + B\cdot 0$

$-1 = A+0 \Rightarrow \boxed{A=-1}$

put $x+1=0 \Rightarrow x=-1$ in ②

$+1-1 = A(0) + B(-1)$

$0 = 0-B \Rightarrow \boxed{B=0}$

put $A=-1$ & $B=0$ in ①

$\frac{3x^2+2x-1}{x(x+1)} = 3 + \frac{-1}{x} + \frac{0}{x+1}$

$\frac{3x^2+2x-1}{x(x+1)} = 3 - \frac{1}{x}$

integrate

$\int \frac{3x^2+2x-1}{x(x+1)}\,dx = \int\left(3-\frac{1}{x}\right)dx$

$= 3x - \ln x + C$ Ans

---

c) Let $\frac{4x^3+4x^2+x-1}{x^2(x+1)^2} = \frac{A}{x^2} + \frac{B}{x^1} + \frac{C}{(x+1)^2} + \frac{D}{(x+1)^1}$ —— ①

xing by $x^2(x+1)^2$

$\boxed{4x^3+4x^2+x-1 = A(x+1)^2 + Bx(x+1)^2 + Cx^2 + Dx^2(x+1)}$ —— ②

$4x^3+4x^2+x-1 = A(x^2+2x+1) + Bx(x^2+2x+1) + Cx^2 + D(x^3+x^2)$

$\boxed{4x^3+4x^2+x-1 = A(x^2+2x+1) + B(x^3+2x^2+x) + Cx^2 + D(x^3+x^2)}$ —— ③

put $x=0$ in ②

$0+0+0-1 = A(0+1)^2 + 0 + 0 + 0$

$-1 = A \Rightarrow \boxed{A=-1}$

put $x+1=0 \Rightarrow x=-1$ in ②

$4(-1)^3 + 4(-1)^2 - 1 - 1 = A(0) + B(0) + C(-1)^2 + D(0)$

$\cancel{-4} + \cancel{4} - 2 = 0+0+C+0 \Rightarrow \boxed{C=-2}$

Now comparing coefficients in eq: ③

x^3: $\boxed{B+D=4}$ —— Ⅰ

x^2: $A+2B+C+D=4$

$-1+2B-2+D=4 \Rightarrow \boxed{2B+D=7}$ —— Ⅱ

Now (II) − (I)

$$\left.\begin{array}{r} 2B+D=7 \\ -B\pm D=4 \end{array}\right\}\text{ sub}$$

$\boxed{B=3}$ put in (I)

(I) ⇒ $3+D=4 \Rightarrow D=4-3 \Rightarrow \boxed{D=1}$

put values of A, B, C & D in ①

$$\frac{4x^3+4x^2+x-1}{x^2(x+1)^2}=\frac{-1}{x^2}+\frac{3}{x}+\frac{-2}{(x+1)^2}+\frac{1}{x+1}$$

$\int x^n dx = \frac{x^{n+1}}{n+1}+C$

integrate w.r.t. x

$$\int \frac{4x^3+4x^2+x-1}{x^2(x+1)^2}dx = \int\left[\frac{-1}{x^2}+\frac{3}{x}-\frac{2}{(x+1)^2}+\frac{1}{x+1}\right]dx$$

$$=-\int x^{-2}dx+3\int\frac{1}{x}dx-2\int(x+1)^{-2}dx+\int\frac{1}{x+1}dx$$

$$=-\frac{x^{-1}}{-1}+3\ln x-2\frac{(x+1)^{-1}}{-1}+\ln(x+1)+C$$

$$=+\frac{1}{x}+3\ln x+\frac{2}{x+1}+\ln(x+1)+C$$ Ans

OR

$$=\frac{1}{x}+\ln x^3+\frac{2}{x+1}+\ln(x+1)+C$$

$\ln a^b = b\ln a$

$\ln a+\ln b = \ln ab$

$$=\frac{1}{x}+\frac{2}{x+1}+\ln x^3(x+1)+C$$ Ans

---

d) $\frac{1}{x^3-1}=\frac{1}{(x-1)(x^2+x+1)}$

$\boxed{a^3-b^3=(a-b)(a^2+ab+b^2)}$

$$\Rightarrow \frac{1}{x^3-1}=\frac{A}{x-1}+\frac{Bx+C}{x^2+x+1} \longrightarrow ①$$

× ing by $(x-1)(x^2+x+1)$ or (x^3-1)

$$\boxed{1=A(x^2+x+1)+(Bx+C)(x-1)} \longrightarrow ②$$

$$1=A(x^2+x+1)+Bx(x-1)+C(x-1)$$

$$\boxed{1=A(x^2+x+1)+B(x^2-x)+C(x-1)} \longrightarrow ③$$

put $x-1=0 \Rightarrow x=1$ in ②

$\therefore 1 = A(1+1+1) + (Bx+C)(0)$

$1 = 3A + 0 \Rightarrow \boxed{A = 1/3}$

Now comparing coefficients in Eq. ③.

x^2: $A + B = 0$

$\frac{1}{3} + B = 0 \Rightarrow \boxed{B = -1/3}$

x: $A - B + C = 0$

$\frac{1}{3} + \frac{1}{3} + C = 0 \Rightarrow \frac{2}{3} + C = 0 \Rightarrow \boxed{C = -2/3}$

put values of A, B & C in ①

$$\frac{1}{x^3-1} = \frac{1/3}{x-1} + \frac{-1/3\,x - 2/3}{x^2+x+1}$$

integrate

$$\int \frac{1}{x^3-1}\,dx = \int\left[\frac{1/3}{x-1} + \frac{-1/3(x+2)}{x^2+x+1}\right]dx$$

$$= \frac{1}{3}\int\frac{1}{x-1}\,dx - \frac{1}{3}\int\frac{x+2}{x^2+x+1}\,dx$$

$$= \frac{1}{3}\ln(x-1) - \frac{1}{3}\int\frac{x+2}{x^2+x+(\frac{1}{2})^2+1-(\frac{1}{2})^2}\,dx$$ add & sub $(1/2)^2$

$$= \frac{1}{3}\ln(x-1) - \frac{1}{3}\int\frac{x+2}{(x+\frac{1}{2})^2+1-1/4}\,dx$$

$$= \frac{1}{3}\ln(x-1) - \frac{1}{3}\int\frac{x+2}{(x+\frac{1}{2})^2+3/4}\,dx$$ $2 = \frac{1}{2} + 3/2$

$$= \frac{1}{3}\ln(x-1) - \frac{1}{3}\int\frac{x+\frac{1}{2}+3/2}{(x+\frac{1}{2})^2+3/4}\,dx$$

$$= \frac{1}{3}\ln(x-1) - \frac{1}{3}\int\left[\frac{x+1/2}{(x+\frac{1}{2})^2+3/4} + \frac{3/2}{(x+\frac{1}{2})^2+3/4}\right]dx$$

$$= \frac{1}{3}\ln(x-1) - \frac{1}{3}\int\frac{x+1/2}{(x+\frac{1}{2})^2+3/4}\,dx - \frac{1}{3}\int\frac{3/2}{(x+\frac{1}{2})^2+3/4}\,dx$$

$$=\frac{1}{3}\ln(x-1)-\frac{1}{3}\int\frac{(x+\frac{1}{2})}{(x+\frac{1}{2})^2+3/4}\,dx-\frac{1}{\cancel{3}}\cdot\frac{\cancel{3}}{2}\int\frac{1}{(x+\frac{1}{2})^2+3/4}\,dx$$

$$=\frac{1}{3}\ln(x-1)-\frac{1}{3}\int\frac{u}{u^2+3/4}\,du-\frac{1}{2}\int\frac{1}{u^2+3/4}\,du$$

$$=\frac{1}{3}\ln(x-1)-\frac{1}{3}\int\frac{1}{t}\cdot\frac{dt}{2}-\frac{1}{2}\int\frac{1}{u^2+\left(\frac{\sqrt{3}}{2}\right)^2}\,du$$

$$=\frac{1}{3}\ln(x-1)-\frac{1}{6}\ln t-\frac{1}{\cancel{2}}\cdot\frac{1}{\frac{\sqrt{3}}{\cancel{2}}}\tan^{-1}\frac{u}{\frac{\sqrt{3}}{2}}+C$$

$$=\frac{1}{3}\ln(x-1)-\frac{1}{6}\ln(u^2+3/4)-\frac{1}{\sqrt{3}}\tan^{-1}\frac{x+1/2}{\sqrt{3}/2}+C$$

$$=\frac{1}{3}\ln(x-1)-\frac{1}{6}\ln\left[(x+\tfrac{1}{2})^2+3/4\right]-\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{\frac{2x+1}{\cancel{2}}}{\sqrt{3}/\cancel{2}}\right)+C$$

$$=\frac{1}{3}\ln(x-1)-\frac{1}{6}\ln\left[x^2+x+\tfrac{1}{4}+3/4\right]-\frac{1}{\sqrt{3}}\tan^{-1}\frac{2x+1}{\sqrt{3}}+C$$

$$=\frac{1}{3}\ln(x-1)-\frac{1}{6}\ln(x^2+x+1)-\frac{1}{\sqrt{3}}\tan^{-1}\frac{2x+1}{\sqrt{3}}+C$$ Ans

put
$x+\frac{1}{2}=u$
Diff. w.r.t. x
$1+0=\frac{du}{dx}$
$\Rightarrow dx=du$

on 1st integral
put $u^2+3/4=t$
Diff w.r.t. u
$2u+0=\frac{dt}{du}$
$\Rightarrow udu=\frac{dt}{2}$

$\int\frac{1}{a^2+x^2}dx=\frac{1}{a}\tan^{-1}x/a$

---

e) $\frac{x^4-x^2+2}{x^2(x-1)}=\frac{x^4-x^2+2}{x^3-x^2}$ (Improper)

$=x+1+\frac{2}{x^3-x^2}$ (By long division)

$$\begin{array}{r|l} & x+1 \\ \hline x^3-x^2 & \cancel{x^4}-x^2+2 \\ & \underset{-}{\cancel{x^4}}\ \mp x^3 \\ \hline & \cancel{x^3}-\cancel{x^2}+2 \\ & \underset{-}{\cancel{x^3}}\mp\cancel{x^2} \\ \hline & 2 \end{array}$$

$=x+1+\frac{2}{x^2(x-1)}$

$$\frac{x^4-x^2+2}{x^2(x-1)}=x+1+\frac{A}{x^2}+\frac{B}{x}+\frac{C}{x-1}\longrightarrow①$$

where $\frac{2}{x^2(x-1)}=\frac{A}{x^2}+\frac{B}{x}+\frac{C}{x-1}$

X'ing by $x^2(x-1)$

$2=A(x-1)+Bx(x-1)+Cx^2\longrightarrow②$

$2=A(x-1)+B(x^2-x)+Cx^2\longrightarrow③$

put $x=0$ in ②

$2=A(0-1)+0+0$

$2=-A \Rightarrow \boxed{A=-2}$

put $x-1=0 \Rightarrow x=1$ in ②

$$2 = A(0) + B(0) + C(1)^2$$

$$2 = 0 + 0 + C \Rightarrow \boxed{C=2}$$

Now comparing coefficients in eq: ③

x^2: $B+C=0$

$$B+2=0 \Rightarrow \boxed{B=-2}$$

put values of A, B & C in ①

$$\frac{x^4-x^2+2}{x^2(x-1)} = x+1+\frac{-2}{x^2}+\frac{-2}{x}+\frac{2}{x-1}$$

integrate

$$\int \frac{x^4-x^2+2}{x^2(x-1)}\,dx = \int\left(x+1-2x^{-2}-2\cdot\frac{1}{x}+2\cdot\frac{1}{x-1}\right)dx$$

$$= \frac{x^2}{2}+x-\frac{2x^{-1}}{-1}-2\ln x+2\ln(x-1)+c$$

$$= \frac{x^2}{2}+x+\frac{2}{x}-2\ln x+2\ln(x-1)+c$$ Ans

---

② Evaluate the following integrals through Partial fractions decomposition:

a) $\int \frac{dx}{x^2-1} = \int \frac{1}{(x-1)(x+1)}\,dx$

$$\int \frac{dx}{x^2-1} = \int\left(\frac{A}{x-1}+\frac{B}{x+1}\right)dx \longrightarrow ①$$

where $\frac{1}{(x-1)(x+1)} = \frac{A}{x-1}+\frac{B}{x+1}$

X'ing by $(x-1)(x+1)$

$$1 = A(x+1)+B(x-1) \longrightarrow ②$$

put $x-1=0 \Rightarrow x=1$ in ②

$$1 = A(1+1)+B\cdot 0$$

$$1 = 2A+0 \Rightarrow \boxed{A=\tfrac{1}{2}}$$

put $x+1=0 \Rightarrow x=-1$ in ②

$$1 = A(0)+B(-1-1)$$

$$1 = -2B \Rightarrow \boxed{B=-\tfrac{1}{2}}$$

put values of A & B in ①

$$\int \frac{dx}{x^2-1} = \int\left(\frac{1/2}{x-1} + \frac{-1/2}{x+1}\right) dx$$

$$= \frac{1}{2}\int \frac{1}{x-1}\,dx - \frac{1}{2}\int \frac{1}{x+1}\,dx$$

$$= \frac{1}{2}\ln(x-1) - \frac{1}{2}\ln(x+1) + c \quad \text{Ans}$$

OR

$$= \frac{1}{2}\left[\ln(x-1) - \ln(x+1)\right] + c$$

($\ln a - \ln b = \ln\frac{a}{b}$

$$= \frac{1}{2}\ln\frac{x-1}{x+1} + c \quad \text{Ans}$$

---

b) $\int \frac{3x+5}{x^2+2x-3}\,dx = \int \frac{3x+5}{(x-1)(x+3)}\,dx$

$$x^2+2x-3$$
$$= x^2+3x-x-3$$
$$= x(x+3)-1(x+3)$$
$$= (x+3)(x-1)$$

$$= \int\left(\frac{A}{x-1} + \frac{B}{x+3}\right) dx \longrightarrow ①$$

where $\frac{3x+5}{(x-1)(x+3)} = \frac{A}{x-1} + \frac{B}{x+3}$

Xing by $(x-1)(x+3)$

$$3x+5 = A(x+3) + B(x-1) \longrightarrow ②$$

put $x-1=0 \Rightarrow x=1$ in ②

$3(1)+5 = A(1+3) + 0$

$\overset{2}{\cancel{8}} = \cancel{4}A \Rightarrow \boxed{A=2}$

put $x+3=0 \Rightarrow x=-3$ in ②

$3(-3)+5 = A(0) + B(-3-1)$

$-9+5 = 0-4B$

$\cancel{-4} = \cancel{-4}B \Rightarrow \boxed{B=1}$

put values of A & B in ①

$$\int \frac{3x+5}{x^2+2x-3}\,dx = \int\left(\frac{2}{x-1} + \frac{1}{x+3}\right) dx$$

$$= 2\int \frac{1}{x-1}\,dx + \int \frac{1}{x+3}\,dx$$

$$= 2\ln(x-1) + \ln(x+3) + c \quad \text{Ans}$$

($\ln a^b = b\ln a$

$$= \ln(x-1)^2 + \ln(x+3) + c$$

$$= \ln\left[(x-1)^2(x+3)\right] + c$$

($\because \ln a + \ln b = \ln ab$

Ans

c) $\int \frac{-x-3}{2x^2-x-1}\,dx = \int \frac{-x-3}{(x-1)(2x+1)}\,dx$

$2x^2-x-1$
$= 2x^2-2x+x-1$
$= 2x(x-1)+1(x-1)$
$= (x-1)(2x+1)$

$= \int \left[\frac{A}{x-1} + \frac{B}{2x+1}\right] dx$ —①

where $\frac{-x-3}{(x-1)(2x+1)} = \frac{A}{x-1} + \frac{B}{2x+1}$

xing by $(x-1)(2x+1)$

$-x-3 = A(2x+1) + B(x-1) \longrightarrow ②$

put $x-1=0 \Rightarrow x=1$ in ②
$-1-3 = A(2(1)+1) + B(0)$
$-4 = 3A + 0 \Rightarrow \boxed{A = -4/3}$

put $2x+1=0 \Rightarrow x=-1/2$ in ②
$+\frac{1}{2}-3 = A(0) + B(-\frac{1}{2}-1)$
$\cancel{-}5/\cancel{2} = \cancel{-}\frac{3}{\cancel{2}}B \Rightarrow \boxed{B = 5/3}$

put values of A & B in ①

$\int \frac{-x-3}{2x^2-x-1}\,dx = \int \left(\frac{-4/3}{x-1} + \frac{5/3}{2x+1}\right) dx$

$= -\frac{4}{3}\int \frac{1}{x-1}\,dx + 5/3 \int \frac{1}{2x+1}\,dx$

$\left(\int \frac{1}{ax+b}\,dx = \frac{\ln(ax+b)}{a} + c\right.$

$= -\frac{4}{3}\ln(x-1) + 5/3\,\frac{\ln(2x+1)}{2} + c$

$= \frac{5}{6}\ln(2x+1) - \frac{4}{3}\ln(x-1) + c$ Ans $\left(\ln a^b = b\ln a\right.$

$= \ln(2x+1)^{5/6} - \ln(x-1)^{4/3} + c$

$= \ln\left[\frac{(2x+1)^{5/6}}{(x-1)^{4/3}}\right] + c$ $\quad \because \ln a - \ln b = \ln\frac{a}{b}$

---

d) $\int \frac{x^2-1}{x^2-2x-15}\,dx = \int \left[1 + \frac{2x+14}{x^2-2x-15}\right] dx$

$= \int \left[1 + \frac{2x+14}{(x-5)(x+3)}\right] dx$

$\int \frac{x^2-1}{x^2-2x-15}\,dx = \int \left(1 + \frac{A}{x-5} + \frac{B}{x+3}\right) dx$ ①

$$\begin{array}{r|l} x^2-2x-15 & \overline{\;x^2-1\;}\;\; 1 \\ & \underline{\pm x^2 \mp 15 \mp 2x} \\ & 2x+14 \end{array}$$

$x^2-2x-15$
$= x^2-5x+3x-15$
$= x(x-5)+3(x-5)$
$= (x-5)(x+3)$

where $\frac{2x+14}{(x-5)(x+3)} = \frac{A}{x-5} + \frac{B}{x+3}$

x'ing by $(x-5)(x+3)$

$2x+14 = A(x+3) + B(x-5)$ —— ②

put $x-5=0 \Rightarrow x=5$ in ②

$2(5)+14 = A(5+3) + 0$

$3\cancel{24} = \cancel{8}A \Rightarrow \boxed{A=3}$

put $x+3=0 \Rightarrow x=-3$ in ②

$2(-3)+14 = A(0) + B(-3-5)$

$-6+14 = 0 - 8B$

$\cancel{8} = -\cancel{8}B \Rightarrow \boxed{B=-1}$

put values of A & B in ①

$$\int \frac{x^2-1}{x^2-2x-15}\,dx = \int\left[1 + \frac{3}{x-5} + \frac{-1}{x+3}\right]dx$$
$$= x + 3\ln(x-5) - \ln(x+3) + c$$
$$= x + \ln(x-5)^3 - \ln(x+3) + c$$
$$= x + \ln\frac{(x-5)^3}{x+3} + c$$ Ans

① $\ln a^b = b\ln a$

⇒ $\ln a - \ln b = \ln\frac{a}{b}$

---

e) $\int \frac{x^2}{(x+1)^3}\,dx = \int\left(\frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3}\right)dx$ —— ①

where $\frac{x^2}{(x+1)^3} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3}$

x'ing by $(x+1)^3$

$x^2 = A(x+1)^2 + B(x+1) + C$ —— ②

$x^2 = A(x^2+2x+1) + B(x+1) + C$ —— ③

put $x+1=0 \Rightarrow x=-1$ in ②

$(-1)^2 = A(-1+1)^2 + B(-1+1) + C$

$1 = 0 + 0 + C \Rightarrow \boxed{C=1}$

Now comparing coefficients in eq: ③

x^2: $\boxed{A=1}$

x: $2\underset{\downarrow}{A} + B = 0$

$2(1) + B = 0 \Rightarrow \boxed{B=-2}$

put values of A, B & C in ①

$$\int \frac{x^2}{(x+1)^3}\,dx = \int\left[\frac{1}{x+1} + \frac{-2}{(x+1)^2} + \frac{1}{(x+1)^3}\right]dx$$

$$= \int \frac{1}{x+1}\,dx - 2\int (x+1)^{-2}\,dx + \int (x+1)^{-3}\,dx$$

$$= \ln(x+1) - 2\,\frac{(x+1)^{-1}}{-1} + \frac{(x+1)^{-2}}{-2} + c$$

$$= \ln(x+1) + \frac{2}{x+1} - \frac{1}{2(x+1)^2} + c \quad \text{Ans}$$

---

f) $\int \frac{x}{(x+1)(x^2+1)}\,dx = \int\left[\frac{A}{x+1} + \frac{Bx+C}{x^2+1}\right]dx$ —— ①

where $\frac{x}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1}$

×ing by $(x+1)(x^2+1)$

$$\boxed{x = A(x^2+1) + (Bx+C)(x+1)} \text{ —— ②}$$

$$= A(x^2+1) + Bx(x+1) + C(x+1)$$

$$\boxed{x = A(x^2+1) + B(x^2+x) + C(x+1)} \longrightarrow ③$$

put $x=-1$ in ②

$-1 = A((-1)^2+1) + (Bx+C)(0)$

$-1 = A(1+1) + 0$

$-1 = 2A \Rightarrow \boxed{A = -\frac{1}{2}}$

Now comparing coefficients in ③

x^2: $A + B = 0$

$-\frac{1}{2} + B = 0 \Rightarrow \boxed{B = \frac{1}{2}}$

x: $B + C = 1$

$\frac{1}{2} + C = 1 \Rightarrow C = 1 - \frac{1}{2} \Rightarrow \boxed{C = \frac{1}{2}}$

put values of A, B & C in ①

$$\int \frac{x^2}{(x+1)(x^2+1)}\,dx = \int\left[\frac{-\frac{1}{2}}{x+1} + \frac{\frac{1}{2}x + \frac{1}{2}}{x^2+1}\right]dx$$

$$= \int\left[\frac{-\frac{1}{2}}{x+1} + \frac{\frac{1}{2}x}{x^2+1} + \frac{\frac{1}{2}}{x^2+1}\right]dx$$

$$\int \frac{x}{(x+1)(x^2+1)}dx = -\frac{1}{2}\int \frac{1}{x+1}dx + \frac{1}{2}\int \frac{x}{x^2+1}dx + \frac{1}{2}\int \frac{1}{x^2+1}dx$$

(x & D by 2)

$$= -\frac{1}{2}\ln(x+1) + \frac{1}{2}\cdot\frac{1}{2}\int \frac{2x}{x^2+1}dx + \frac{1}{2}\tan^{-1}x + c$$

$$= -\frac{1}{2}\ln(x+1) + \frac{1}{4}\ln(x^2+1) + \frac{1}{2}\tan^{-1}x + C$$

$$= \frac{1}{4}\ln(x^2+1) - \frac{1}{2}\ln(x+1) + \frac{1}{2}\tan^{-1}x + C$$

$$= \ln(x^2+1)^{1/4} - \ln(x+1)^{1/2} + \frac{1}{2}\tan^{-1}x + c$$

$$= \ln\frac{(x^2+1)^{1/4}}{(x+1)^{1/2}} + \frac{1}{2}\tan^{-1}x + C \qquad \text{Ans}$$

$\int \frac{1}{1+x^2}dx = \tan^{-1}x + c$

$\int \frac{2x}{x^2+1}dx = \ln(x^2+1)+c$

---

7) $\int \frac{x^2+2}{(x^2+1)^2}dx = \int\left[\frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2}\right]dx \longrightarrow ①$

where $\frac{x^2+2}{(x^2+1)^2} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2}$

X'ing by $(x^2+1)^2$

$x^2+2 = (Ax+B)(x^2+1) + Cx+D$ —— ②

$x^2+2 = Ax(x^2+1) + B(x^2+1) + Cx+D$

$x^2+2 = A(x^3+x) + B(x^2+1) + Cx+D$ —— ③

Comparing coefficients in ③

x^3: $\boxed{A=0}$

x^2: $\boxed{B=1}$

x: $A+C=0$

$0+C=0 \Rightarrow \boxed{C=0}$

Constant: $B+D=2$

$1+D=2 \Rightarrow D=2-1 \Rightarrow \boxed{D=1}$

put values of A, B, C & D in ①

$$\int \frac{x^2+2}{(x^2+1)^2}dx = \int\left[\frac{0\cdot x+1}{x^2+1} + \frac{0x+1}{(x^2+1)^2}\right]dx$$

$$\int \frac{x^2+2}{(x^2+1)^2}dx = \int\left[\frac{1}{x^2+1} + \frac{1}{(x^2+1)^2}\right]dx$$

$$= \int \frac{1}{x^2+1}dx + \int \frac{1}{(x^2+1)^2}dx$$

$$= \tan^{-1}x + \int \frac{1}{(\tan^2\theta+1)^2}\sec^2\theta\, d\theta$$

$$= \tan^{-1}x + \int \frac{1}{(\sec^2\theta)^{\cancel{2}}}\cancel{\sec^2\theta}\, d\theta$$

$$= \tan^{-1}x + \int \cos^2\theta\, d\theta$$

$$= \tan^{-1}x + \int \frac{1+\cos 2\theta}{2}d\theta$$

$$= \tan^{-1}x + \frac{1}{2}\int(1+\cos 2\theta)d\theta$$

$$= \tan^{-1}x + \frac{1}{2}\left[\theta + \frac{\sin 2\theta}{2}\right] + C$$

$$= \tan^{-1}x + \frac{1}{2}\theta + \frac{1}{4}\cdot\sin 2\theta + C$$

$$= \tan^{-1}x + \frac{1}{2}\theta + \frac{1}{\cancel{4}_2}\cdot\frac{\cancel{2}\tan\theta}{1+\tan^2\theta} + C$$

$$= \underbrace{\tan^{-1}x + \frac{1}{2}\tan^{-1}x}_{\downarrow} + \frac{1}{2}\cdot\frac{x}{1+x^2} + C$$

$$= \frac{3}{2}\tan^{-1}x + \frac{x}{2(1+x^2)} + C \quad \text{Ans}$$

$\int \frac{1}{1+x^2}dx = \tan^{-1}x + C$

In 2nd Integral put

$x = \tan\theta$

Diff w.r.t. x

$1 = \sec^2\theta \frac{d\theta}{dx}$

$\Rightarrow dx = \sec^2\theta\, d\theta$

$1+\tan^2\theta = \sec^2\theta$

$\cos^2\theta = \frac{1+\cos 2\theta}{2}$

$\sin 2\theta = 2\sin\theta\cos\theta$

$\sin 2\theta = 2.\frac{\sin\theta}{\cos\theta}.\cos^2\theta$ M&D by $\cos\theta$

$= 2\tan\theta.\frac{1}{\sec^2\theta}$

$$\boxed{\sin 2\theta = \frac{2\tan\theta}{1+\tan^2\theta}}$$ Note

$\because x = \tan\theta$

$\tan^{-1}x = \theta$

$1+\frac{1}{2} = \frac{3}{2}$

---

h) $\int \frac{x^4+1}{x^4-1}dx = \int\left(1+\frac{2}{x^4-1}\right)dx$

$$= \int\left[1+\frac{2}{(x-1)(x+1)(x^2+1)}\right]dx$$

$$\int \frac{x^4+1}{x^4-1}dx = \int\left(1+\frac{A}{x-1}+\frac{B}{x+1}+\frac{Cx+D}{x^2+1}\right)dx \quad ①$$

$$x^4-1 \overline{\smash{)}\, x^4+1}$$ quotient 1

$\cancel{x^4}-1$ with signs $-\ +$

2

$x^4-1 = (x^2)^2 - 1^2$

$= (x^2-1)(x^2+1)$

$= (x-1)(x+1)(x^2+1)$

22^{0}

where

$$\frac{2}{(x-1)(x+1)(x^2+1)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{Cx+D}{x^2+1}$$

X'ing by $(x-1)(x+1)(x^2+1)$

$$2 = A(x+1)(x^2+1) + B(x-1)(x^2+1) + (Cx+D)(x-1)(x+1) \quad ②$$

$$2 = A(x^3+x+x^2+1) + B(x^3+x-x^2-1) + (Cx+D)(x^2-1)$$

$$2 = A(x^3+x^2+x+1) + B(x^3-x^2+x-1) + C(x^3-x) + D(x^2-1) \quad ③$$

put $x-1=0 \Rightarrow x=1$ in ②

$2 = A(1+1)(1+1) + B(0) + 0$

$\overset{1}{\cancel{2}} = \overset{2}{\cancel{4}}A \Rightarrow \boxed{A=\tfrac{1}{2}}$

put $x+1=0 \Rightarrow x=-1$ in ②

$2 = 0 + B(-1-1)((-1)^2+1) + 0$

$2 = B(-2)(2)$

$\cancel{2} = -\overset{2}{\cancel{4}}B \Rightarrow \boxed{B=-\tfrac{1}{2}}$

Now comparing coefficients in ③

x^3: $A + B + C = 0$

$\cancel{\tfrac{1}{2}} - \cancel{\tfrac{1}{2}} + C = 0 \Rightarrow \boxed{C=0}$

x^2: $A - B + D = 0$

$\tfrac{1}{2} + \tfrac{1}{2} + D = 0 \Rightarrow 1 + D = 0 \Rightarrow \boxed{D=-1}$

put these values in ①

$$\int \frac{x^4+1}{x^4-1}\,dx = \int \left[1 + \frac{\tfrac{1}{2}}{x-1} + \frac{-\tfrac{1}{2}}{x+1} + \frac{0x-1}{x^2+1}\right] dx$$

$$= \int 1\,dx + \frac{1}{2}\int \frac{1}{x-1}\,dx - \frac{1}{2}\int \frac{1}{x+1}\,dx - \int \frac{1}{x^2+1}\,dx$$

$$\int \frac{1}{1+x^2}\,dx = \tan^{-1}x + C$$

$$= x + \frac{1}{2}\ln(x-1) - \frac{1}{2}\ln(x+1) - \tan^{-1}x + C$$

Ans

$$= x + \frac{1}{2}\left[\ln(x-1) - \ln(x+1)\right] - \tan^{-1}x + C$$

$$= x + \frac{1}{2}\ln\frac{x-1}{x+1} - \tan^{-1}x + C$$

Ans

$$\ln a - \ln b = \ln\frac{a}{b}$$

Integration by Parts

$$\int I\,II\,dx = I\int II\,dx - \int\left(\frac{d}{dx}I\right)\left(\int II\,dx\right)dx$$

3) Use integration by Parts to evaluate the following integrals:

a) Let $I = \int \underset{I}{x}\,\underset{II}{e^x}\,dx$ Int: by Parts

$= x\int e^x dx - \int\left(\frac{d}{dx}x\right)\left(\int e^x dx\right)dx$

$= xe^x - \int 1\,(e^x)dx$

$= xe^x - \int e^x dx$

$= xe^x - e^x + c$ Ans

b) Let $I = \int \underset{I}{x}\,\underset{II}{\sin x}\,dx$ Int by Parts

$= x\int \sin x\,dx - \int\left(\frac{d}{dx}x\right)\left(\int \sin x\,dx\right)dx$

$= x(-\cos x) - \int(1)(-\cos x)\,dx$

$= -x\cos x + \int \cos x\,dx$

$= -x\cos x + \sin x + c$ Ans

c) Let $I = \int \tan^{-1}x\,dx$

$= \int \underset{I}{\tan^{-1}x} \times \underset{II}{1}\,dx$ Int: by Parts

$= \tan^{-1}x\int 1\,dx - \int\left(\frac{d}{dx}\tan^{-1}x\right)\left(\int 1\,dx\right)dx$

$= \tan^{-1}x \cdot x - \int \frac{1}{1+x^2}\cdot x\,dx$

put
$1+x^2 = u$
Diff: w.r.t. x
$2x = \frac{du}{dx}$
$2x\,dx = du$
$x\,dx = \frac{du}{2}$

$= x\tan^{-1}x - \int \frac{1}{u}\,\frac{du}{2}$

$= x\tan^{-1}x - \frac{1}{2}\int \frac{1}{u}\,du$

$= x\tan^{-1}x - \frac{1}{2}\ln u + c$

$= x\tan^{-1}x - \frac{1}{2}\ln(1+x^2) + c$ Ans

222

d) $I = \int \sin^{-1}x \, dx = \int \underset{I}{\sin^{-1}x} \times \underset{II}{1} \, dx$ Int: by Parts

$= \sin^{-1}x \int 1 dx - \int \left(\frac{d}{dx}\sin^{-1}x\right)\left(\int 1 dx\right) dx$

$= \sin^{-1}x \cdot x - \int \frac{1}{\sqrt{1-x^2}} \cdot x dx$

$= x\sin^{-1}x - \int \frac{1}{\sqrt{u}}\left(-\frac{du}{2}\right)$

$= x\sin^{-1}x + \frac{1}{2}\int u^{-1/2} du$

$= x\sin^{-1}x + \frac{1}{2} \frac{u^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c$

$= x\sin^{-1}x + \frac{\cancel{1}}{\cancel{2}} \frac{u^{1/2}}{\cancel{\frac{1}{2}}} + c$

$= x \cdot \sin^{-1}x + \sqrt{u} + c$

$= x\sin^{-1}x + \sqrt{1-x^2} + c$ $\quad \because u = 1-x^2$

} put
$1-x^2 = u$
Diff: w.r.t. x
$0-2x = \frac{du}{dx}$
$\Rightarrow x dx = -\frac{du}{2}$

---

e) $I = \int \underset{I}{x^2} \underset{II}{(x-3)^{11}} dx$ Int: by Parts

$= x^2 \int (x-3)^{11} dx - \int \left(\frac{d}{dx}x^2\right)\left(\int (x-3)^{11} dx\right) dx$

$= x^2 \frac{(x-3)^{12}}{12} - \int (\cancel{2}x)\left(\frac{(x-3)^{12}}{\cancel{12}_{6}}\right) dx$

$= \frac{x^2(x-3)^{12}}{12} - \frac{1}{6}\int \underset{I}{x} \underset{II}{(x-3)^{12}} dx$ Int: by Parts

$= \frac{x^2(x-3)^{12}}{12} - \frac{1}{6}\left[x \int (x-3)^{12} dx - \int \left(\frac{d}{dx}x\right)\left(\int (x-3)^{12} dx\right) dx\right]$

$= \frac{x^2(x-3)^{12}}{12} - \frac{1}{6}\left[x \frac{(x-3)^{13}}{13} - \int (1) \frac{(x-3)^{13}}{13} dx\right]$

$= \frac{x^2(x-3)^{12}}{12} - \frac{x(x-3)^{13}}{78} + \frac{1}{78}\int (x-3)^{13} dx$

$= \frac{x^2(x-3)^{12}}{12} - \frac{x(x-3)^{13}}{78} + \frac{1}{78}\frac{(x-3)^{14}}{14} + c$

$= \frac{x^2(x-3)^{12}}{12} - \frac{x(x-3)^{13}}{78} + \frac{(x-3)^{14}}{1092} + c$ Ans

f) $I = \int \underset{I}{e^x} \underset{II}{\cos x}\, dx$ Int. by Parts

$= e^x \int \cos x\, dx - \int \left(\frac{d}{dx} e^x\right)\left(\int \cos x\, dx\right) dx$

$= e^x \sin x - \int \underset{I}{e^x} \underset{II}{\sin x}\, dx$ Int by Parts

$= e^x \sin x - \left[e^x \int \sin x\, dx - \int \left(\frac{d}{dx} e^x\right)\left(\int \sin x\, dx\right) dx\right]$

$= e^x \sin x - \left[e^x(-\cos x) - \int e^x (-\cos x)\, dx\right]$

$= e^x \sin x - \left[-e^x \cos x + \int e^x \cos x\, dx\right]$

$I = e^x \sin x + e^x \cos x - I$

$2I = e^x(\sin x + \cos x)$

$\Rightarrow I = \frac{e^x}{2}(\sin x + \cos x) + C$ Ans

Note ① $\sin^2 x = \frac{1-\cos 2x}{2}$ ② $\cos^2 x = \frac{1+\cos 2x}{2}$ ③ $1+\tan^2 x = \sec^2 x$ ④ $1+\cot^2 x = \operatorname{cosec}^2 x$

g) $I = \int (x + \sin x)^2\, dx$

$= \int (x^2 + \sin^2 x + 2x \sin x)\, dx$

$= \int \left[x^2 + \frac{1-\cos 2x}{2} + 2x \sin x\right] dx$

$= \int \left[x^2 + \frac{1}{2} - \frac{\cos 2x}{2} + 2x \sin x\right] dx$

$= \int x^2 dx + \int \frac{1}{2} dx - \frac{1}{2}\int \cos 2x\, dx + 2\int \underset{I}{x} \underset{II}{\sin x}\, dx$

$= \frac{x^3}{3} + \frac{1}{2}x - \frac{1}{2}\left(\frac{\sin 2x}{2}\right) + 2\left[x \int \sin x\, dx - \int \left(\frac{d}{dx} x\right)\left(\int \sin x\, dx\right) dx\right]$

$= \frac{x^3}{3} + \frac{1}{2}x - \frac{1}{4}\sin 2x + 2\left[x(-\cos x) - \int 1(-\cos x)\, dx\right]$

$= \frac{x^3}{3} + \frac{1}{2}x - \frac{1}{4}\sin 2x + 2\left[-x \cos x + \int \cos x\, dx\right]$

$= \frac{x^3}{3} + \frac{1}{2}x - \frac{1}{4}\sin 2x - 2x \cos x + 2 \sin x + C$ Ans

---

Note: $\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C$, $n \neq -1$

h) $I = \int e^{2x}\sqrt{1-e^{x}}\,dx$

$= \int (e^{x})^{2}\sqrt{1-e^{x}}\,dx$

$= \int u^{\cancel{2}}\sqrt{1-u}\cdot\frac{du}{\cancel{u}}$

$= \int \underset{I}{u}\,\underset{II}{\sqrt{1-u}}\,du$ Int by Parts

Put $e^{x} = u$
Diff w.r.t. x
$e^{x} = \frac{du}{dx}$
$e^{x}dx = du$
$dx = \frac{du}{e^{x}}$
$dx = \frac{du}{u}$

$= u\int\sqrt{1-u}\,du - \int\left(\frac{d}{du}u\right)\left(\int\sqrt{1-u}\,du\right)du$

$= u\,\frac{(1-u)^{\frac{1}{2}+1}}{-1\left(\frac{1}{2}+1\right)} - \int 1\,\frac{(1-u)^{\frac{1}{2}+1}}{-1\left(\frac{1}{2}+1\right)}\,du$

$= u\,\frac{(1-u)^{3/2}}{-3/2} - \int\frac{(1-u)^{3/2}}{-3/2}\,du$

$= -\frac{2}{3}u(1-u)^{3/2} + \frac{2}{3}\int(1-u)^{3/2}du$

$= -\frac{2}{3}u(1-u)^{3/2} + \frac{2}{3}\,\frac{(1-u)^{3/2+1}}{-1\left(\frac{3}{2}+1\right)} + c$

$= -\frac{2}{3}u(1-u)^{3/2} + \frac{2}{3}\,\frac{(1-u)^{5/2}}{-5/2} + c$

$= -\frac{2}{3}e^{x}(1-e^{x})^{3/2} - \frac{4}{15}(1-e^{x})^{5/2} + C$ $\because u = e^{x}$

OR

$= -(1-e^{x})^{3/2}\left[+\frac{2}{3}e^{x} + \frac{4}{15}(1-e^{x})^{\frac{2}{2}=1}\right] + C$

$= -(1-e^{x})^{3/2}\left[\frac{10e^{x} + 4(1-e^{x})}{15}\right] + C$

$= -(1-e^{x})^{3/2}\frac{(10e^{x} + 4 - 4e^{x})}{15} + C$

$= -\frac{1}{15}(1-e^{x})^{3/2}(6e^{x}+4) + C$

$= -\frac{1}{15}(1-e^{x})^{3/2}\cdot 2(3e^{x}+2) + C$

$= -\frac{2}{15}(1-e^{x})^{3/2}(3e^{x}+2) + C$ Ans

Note $\sin 2x = 2\sin x\cos x \Rightarrow \frac{\sin 2x}{2} = \sin x\cos x$

---

i). $I = \int x\sin x\cos x\,dx$

$= \int x \cdot \frac{\sin 2x}{2}\,dx$

$= \frac{1}{2}\int \underset{I}{x} \cdot \underset{II}{\sin 2x}\,dx$ Int. by Parts

$= \frac{1}{2}\left[x\int \sin 2x\,dx - \int\left(\frac{d}{dx}x\right)\left(\int \sin 2x\,dx\right)dx\right]$

$= \frac{1}{2}\left[x\left(-\frac{\cos 2x}{2}\right) - \int (1)\left(-\frac{\cos 2x}{2}\right)dx\right]$

$= \frac{1}{2}\left[-\frac{x\cos 2x}{2} + \frac{1}{2}\int \cos 2x\,dx\right]$

$= -\frac{x\cos 2x}{4} + \frac{1}{4}\cdot\frac{\sin 2x}{2} + c$

$= -\frac{x\cos 2x}{4} + \frac{\sin 2x}{8} + c$ Ans

---

j) $I = \int x^2 \ln x\,dx$

$= \int \underset{I}{\ln x} \cdot \underset{II}{x^2}\,dx$ Int: by Parts

$= \ln x \cdot \frac{x^3}{3} - \int\left(\frac{1}{\cancel{x}}\right)\left(\frac{\overset{2}{\cancel{x^3}}}{3}\right)dx$

$= \frac{x^3\ln x}{3} - \frac{1}{3}\int x^2\,dx$

$= \frac{x^3\ln x}{3} - \frac{1}{3}\cdot\frac{x^3}{3} + c$

$= \frac{x^3\ln x}{3} - \frac{x^3}{9} + c$ Ans

---

4) Integrate the following integrals by Parts integration through appropriate substitution

See →

226

a) let $I = \int \frac{\ln x . \sin(\ln x)}{x} dx$

$= \int \ln x . \sin(\ln x) . \frac{1}{x} dx$

$= \int \underset{I}{u} . \underset{II}{\sin u} . du$

put $\ln x = u$
Diff: w.r.t. x
$\frac{1}{x} = \frac{du}{dx}$
$\frac{1}{x} dx = du$

Int: by Parts

$= u \int \sin u \, du - \int \left(\frac{d}{du} u\right)\left(\int \sin u \, du\right) du$

$= u(-\cos u) - \int (1)(-\cos u) \, du$

$= -u \cos u + \int \cos u \, du$

$= -u \cos u + \sin u + C$

$= -\ln x . \cos \ln x + \sin \ln x + C$ $\quad \because u = \ln x$

---

b) $I = \int \sin 2x \ln \cos x \, dx$

$= \int 2 \sin x \cos x \ln \cos x \, dx$

$= 2 \int \cos x \ln \cos x . \sin x \, dx$

$= 2 \int u \ln u \, (-du)$

$= -2 \int \underset{I}{\ln u} . \underset{II}{u} \, du$

put $\cos x = u$
Diff w.r.t. x
$-\sin x = \frac{du}{dx}$
$\Rightarrow \sin x \, dx = -du$

Int: by Parts

$= -2\left[\ln u . \frac{u^2}{2} - \int \left(\frac{1}{\cancel{u}}\right)\left(\frac{u^2}{2}\right) du\right]$

$= -2\left[\frac{u^2 \ln u}{2} - \frac{1}{2}\int u \, du\right]$

$= -u^2 \ln u + \int u \, du$

$= -u^2 \ln u + \frac{u^2}{2} + C$

$= -\cos^2 x . \ln \cos x + \frac{\cos^2 x}{2} + C$ $\quad$ Ans

---

c) $I = \int e^{2x} \sin e^x dx$

$= \int (e^x)^2 \sin e^x . dx$

$= \int u^2 \sin u . \frac{du}{\cancel{e^x}}$

put $e^x = u$
Diff w.r.t. x
$e^x = \frac{du}{dx}$
$\Rightarrow dx = \frac{du}{e^x} = \frac{du}{u}$

$I = \int \underset{I}{u}\, \underset{II}{\sin u}\, du$ Int by Parts

$= u(-\cos u) - \int (1)(-\cos u)\, du$

$= -u\cos u + \int \cos u\, du$

$= -u\cos u + \sin u + c$

$= -e^x \cos e^x + \sin e^x + c$ $\because u = e^x$

5) The rate at which the body eliminates a drug is given by

$$\frac{d}{dt}R(t) = \frac{60t}{(t+1)^2(t+2)}$$

integrate w.r.t. t

$$\int \left[\frac{d}{dt}R(t)\right] dt = \int \frac{60t}{(t+1)^2(t+2)}\, dt$$

$$R(t) = \int \left[\frac{A}{t+1} + \frac{B}{(t+1)^2} + \frac{c}{t+2}\right] dt \longrightarrow ①$$

where $\frac{60t}{(t+1)^2(t+2)} = \frac{A}{t+1} + \frac{B}{(t+1)^2} + \frac{c}{t+2}$

xing by $(t+1)(t+1)^2$

$60t = A(t+1)(t+2) + B(t+2) + c(t+1)^2 \longrightarrow ②$

$60t = A(t^2+2t+t+2) + B(t+2) + c(t^2+2t+1)$

$60t = A(t^2+3t+2) + B(t+2) + c(t^2+2t+1) \longrightarrow ③$

put $t+1=0 \Rightarrow t=-1$ in ②

$60(-1) = A(0) + B(-1+2) + 0$

$-60 = 0 + B + 0 \Rightarrow \boxed{B=-60}$

put $t+2=0 \Rightarrow t=-2$ in ②

$60(-2) = 0 + 0 + c(-2+1)^2$

$-120 = 0 + 0 + c(-1)^2$

$-120 = c \Rightarrow \boxed{c=-120}$

Now Comparing coefficients in ③

t^2: $A + c = 0$

$A - 120 = 0 \Rightarrow \boxed{A=120}$

put values of A, B & C in ①

$$R(t) = \int \left[\frac{120}{t+1} + \frac{-60}{(t+1)^2} + \frac{-120}{t+2} \right] dt$$

$$= 120 \int \frac{1}{t+1}\, du - 60 \int (t+1)^{-2} dt - 120 \int \frac{1}{t+2} dt$$

$$= 120 \ln(t+1) - 60 \frac{(t+1)^{-1}}{-1} - 120 \ln(t+2) + C$$

$$= 120 \ln(t+1) + \frac{60}{t+1} - 120 \ln(t+2) + C$$

$$= 120 [\ln(t+1) - \ln(t+2)] + \frac{60}{t+1} + C$$

$$\boxed{R(t) = 120 \ln\left(\frac{t+1}{t+2}\right) + \frac{60}{t+1} + C} \longrightarrow ④$$

∵ given that $R(0) = 0$

so put $t = 0$ in ④

$$R(0) = 120 \ln\left(\frac{0+1}{0+2}\right) + \frac{60}{0+1} + C$$

$$0 = 120 \ln\left(\tfrac{1}{2}\right) + 60 + C$$

$$0 = 120(-0.6931) + 60 + C$$

$$0 = -83.172 + 60 + C$$

$$0 = -23.172 + C \Rightarrow \boxed{C = +23.172}$$ put in ④

$$④ \Rightarrow \boxed{R(t) = 120 \ln\left(\frac{t+1}{t+2}\right) + \frac{60}{t+1} + 23.172} \longrightarrow ⑤$$

During the first hour

put $t = 1$ in ⑤

$$R(1) = 120 \ln\left(\frac{1+1}{1+2}\right) + \frac{60}{1+1} + 23.172$$

$$= 120 \ln\left(\tfrac{2}{3}\right) + \frac{60}{2} + 23.172$$

$$= 120(-0.4054) + 30 + 23.172$$

$$= -48.648 + 30 + 23.172$$

$$= 4.524 \text{ ml}$$

During the 4th hour = ?

put $t=3$ in ⑤

$$R(3) = 120\ln\left(\frac{3+1}{3+2}\right) + \frac{60}{3+1} + 23.172$$
$$= 120\ln\left(\frac{4}{5}\right) + \frac{60}{4} + 23.172$$
$$= 120(-0.2231) + 15 + 23.172$$
$$= -26.772 + 15 + 23.172$$
$$R(3) = 11.4 \quad \text{(In first 3-hours)}$$

put $t=4$ in 5

$$R(4) = 120\ln\left(\frac{4+1}{4+2}\right) + \frac{60}{4+1} + 23.172$$
$$= 120\ln\left(\frac{5}{6}\right) + \frac{60}{5} + 23.172$$
$$= 120(-0.1823) + 12 + 23.172$$
$$= -21.876 + 12 + 23.172$$
$$R(4) = 13.296 \quad \text{(In first four hours)}$$

Now During the 4th hour $= R(4) - R(3)$

$$= 13.296 - 11.4 = 1.896 \text{ ml},$$

---

6) The rate of change of the voting population of a city is estimated to be

$$\frac{dN}{dt} = \frac{100t}{(1+t^2)^2}$$

integrate w.r.t. t

$$\int\left(\frac{dN}{dt}\right)dt = \int\frac{100t}{(1+t^2)^2}\,dt$$

$$N = 100\int\frac{1}{(1+t^2)^2}\,t\,dt$$

put $1+t^2 = u$
Diff: w.r.t. t
$$2t = \frac{du}{dt}$$
$$\Rightarrow t\,dt = \frac{du}{2}$$

$$N = \overset{50}{\cancel{100}}\int\frac{1}{u^2}\,\frac{du}{\cancel{2}}$$
$$= 50\int u^{-2}\,du$$
$$= 50\,\frac{u^{-1}}{-1} + c$$

$$\left(\int x^n\,dx = \frac{x^{n+1}}{n+1} + c\right.$$

$$N(t) = -\frac{50}{u} + c$$

$$\boxed{N(t) = \frac{-50}{1+t^2} + c} \longrightarrow ① \quad \because u = 1+t^2$$

Let $N(0) = 0$ so put $t=0$ in ①

$$① \Rightarrow N(0) = \frac{-50}{1+0} + c$$
$$0 = \frac{-50}{1} + c \Rightarrow \boxed{c = 50} \text{ put in ①}$$

① ⇒ $N(t) = \frac{-50}{1+t^2} + 50$ ——— ②

put $t = 3$ in ②

$N(3) = \frac{-50}{1+3^2} + 50$

$= \frac{-50}{1+9} + 50 = -\frac{50}{10} + 50 = -5 + 50 = 45$

Hence the voting Population increases during the next 3-year = 45000

---

7) Given Rate of change is

$\frac{dR}{dt} = \frac{100}{\sqrt{t^2+9}}$, $t \geq 0$

Integrate w.r.t. t

$\int \left(\frac{dR}{dt}\right) dt = \int \frac{100}{\sqrt{t^2+9}} dt$

$\left(\int \frac{1}{\sqrt{x^2+a^2}} dx = \ln(x+\sqrt{x^2+a^2}) + c \right.$

$R = 100 \int \frac{1}{\sqrt{t^2+9}} dt$

$R = 100 \ln\left[t + \sqrt{t^2+9}\right] + c$ ——— ①

$\because R = 0$ when $t = 0$ so put in ①

$0 = 100 \ln(0 + \sqrt{0+9}) + c$

$0 = 100 \ln 3 + c$

$0 = 100(1.0986) + c$

$0 = 109.86 + c \Rightarrow \boxed{c = -109.86}$ put in ①

① ⇒ $\boxed{R = 100 \ln\left[t + \sqrt{t^2+9}\right] - 109.86}$ ——— ②

To find Radius of 1 Slick after 4 minutes put $t = 4$ in ②

$R = 100 \ln(4 + \sqrt{4^2+9}) - 109.86$

$= 100 \ln(4 + \sqrt{16+9}) - 109.86 = 100 \ln(4 + \sqrt{25}) - 109.86$

$= 100 \ln(4+5) - 109.86 = 100 \ln(9) - 109.86 = 219.72 - 109.86$

$= 109.86$ ft

8) Given that, the rate of change is :

$$\frac{dE}{dt} = te^{-0.2t} \quad , \quad \boxed{E(0)=0}$$

integrate w.r.t t

$$\int\left(\frac{dE}{dt}\right)dt = \int \underset{I}{t}\,\underset{II}{e^{-0.2t}}\,dt$$

Int: by Parts

$$E = t\int e^{-0.2t}dt - \int\left(\frac{d}{dt}t\right)\left(\int e^{-0.2t}dt\right)dt$$

$$= t\,\frac{e^{-0.2t}}{-0.2} - \int(1)\,\frac{e^{-0.2t}}{-0.2}\,dt$$

$$= -\frac{te^{-0.2t}}{0.2} + \frac{1}{0.2}\int e^{-0.2t}dt$$

$$= -\frac{te^{-0.2t}}{0.2} + \frac{1}{0.2}\left(\frac{e^{-0.2t}}{-0.2}\right) + c$$

$$\Rightarrow \quad E(t) = -\frac{te^{-0.2t}}{0.2} - \frac{e^{-0.2t}}{0.04} + c \quad \text{——} \quad ①$$

∵ given that $E(0)=0$ so put $t=0$

$$E(0) = \frac{-0.e^{0}}{0.2} - \frac{e^{0}}{0.04} + c$$

$$0 = 0 - \frac{1}{0.04} + c$$

$$0 = -25 + c \Rightarrow \boxed{c=25}$$

put in ①

$$① \Rightarrow E(t) = -\frac{te^{-0.2t}}{0.2} - \frac{e^{-0.2t}}{0.04} + 25 \longrightarrow ②$$

Now total amount of the drug that is assimilated into the bloodstream during the first 10 minutes after the Pill is taken, so put $t=10$ in ②

$$E(10) = \frac{-10e^{-0.2\times10}}{0.2} - \frac{e^{-0.2\times10}}{0.04} + 25 \qquad \boxed{e=2.718\ldots}$$

$$= \frac{-10e^{-2}}{0.2} - \frac{e^{-2}}{0.04} + 25 = \frac{-10}{0.2\times e^{2}} - \frac{1}{0.04\times e^{2}} + 25$$

$$= \frac{-10}{1.4778} - \frac{1}{0.2956} + 25 = -6.77 - 3.38 + 25 = 14.85$$

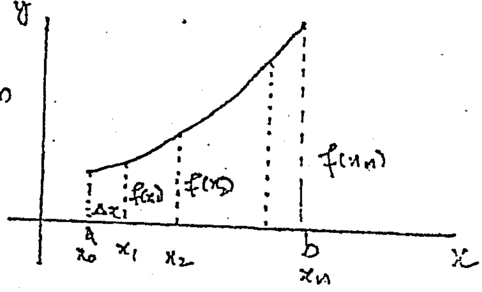
Ans

Definite Integrals

Before the definite integral, we need to develop the concept of the area and the area under a curve.

i) The area under the Curve

A technique for approximating the area of a region is to construct rectangles in the region and then take the sum of the areas of the rectangles.



The approximate area of a region bounded by the curve $y = f(x)$ (for $f(x) \geq 0$), the x-axis, $x = a$ and $x = b$ in the interval $[a, b]$ is the sum of the areas of the rectangles:

$$\text{Area} = A \approx f(x_1)\Delta x + f(x_2)\Delta x + \cdots\cdot + f(x_n)\Delta x$$

$$= \sum_{i=1}^{n} f(x_i)\Delta x \quad , \quad i = 1, 2, 3 \ldots, n \quad , \quad \Delta x = \frac{b-a}{n}$$

where interval $[a, b]$ is divided into n-equal subintervals & width of each interval is $\Delta x = \frac{b-a}{n}$

(ii) Definite integral as the limit of a sum

If $f(x)$ is continuous on the interval $[a, b]$ and interval $[a, b]$ is divided into n equal subintervals whose right hand points are $x_1, x_2, \ldots, x_n$, then the definite integral of $f(x)$ from $x = a$ to $x = b$ is

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

$$= \lim_{n\to\infty} \Delta x \left[f(x_1) + f(x_2) + f(x_3) + \cdots\cdot + f(x_n) \right)$$

$$= \lim_{n\to\infty} \frac{b-a}{n} \left[f(x_1) + f(x_2) + \cdots\cdots + f(x_n) \right]$$

In definite integral $\int_a^b f(x)\,dx$, a is called lower limit of integration & b is called upper limit of integration. It represents the actual area under the curve.

Ch-5

{EXERCISE 5.4}

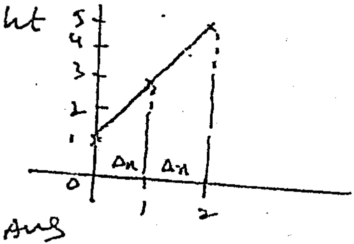
① In each case, determine the approximate area of the region bounded by $f(x)=2x+1$, $x=a$ and $x=b$ for n subintervals

a) $n=2,\ a=0,\ b=2,\ f(x)=2x+1$

width of each sub interval is: $\Delta x = \dfrac{b-a}{n} = \dfrac{2-0}{2} = 1$

	x	$f(x)=2x+1$
x_0	0	1 $f(x_0)$
x_1	1	3 $f(x_1)$
x_2	2	5 $f(x_2)$

The approximate area with the right end points of each subinterval is



$A \approx \Delta x f(x_1) + \Delta x f(x_2)$

$= 1(3) + 1(5) = 3+5 = 8$ Ans

---

b) $n=4,\ a=0,\ b=2,\ f(x)=2x+1$

The width of each subinterval is: $\Delta x = \dfrac{b-a}{n} = \dfrac{2-0}{4} = \dfrac{1}{2} = 0.5$

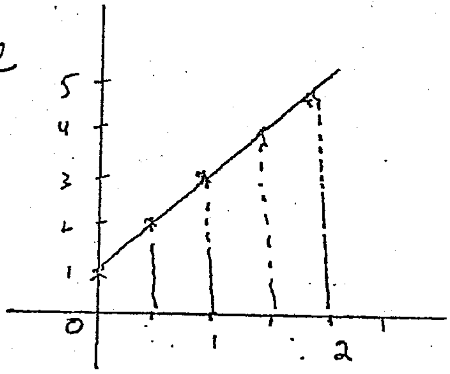
	x	$f(x)=2x+1$
x_0	0	1 $f(x_0)$
x_1	0.5	2 $f(x_1)$
x_2	1	3 $f(x_2)$
x_3	1.5	4 $f(x_3)$
x_4	2	5 $f(x_4)$

The approximate area with the right end points of each subinterval is

$A \approx \Delta x f(x_1) + \Delta x f(x_2) + \Delta x f(x_3) + \Delta x f(x_4)$
or
$= \Delta x\,[f(x_1) + f(x_2) + f(x_3) + f(x_4)]$

$= 0.5[2+3+4+5]$

$= 0.5[14] = 7$



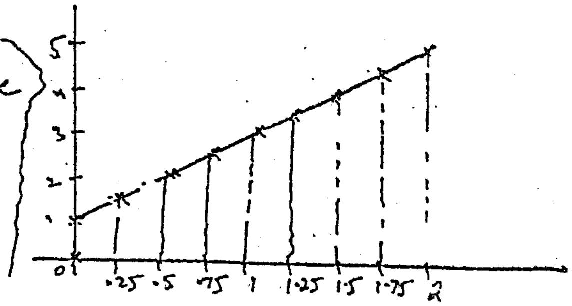
c) $n = 8, a = 0, b > 2$ $f(x) = 2x+1$

The width of each sub-interval is

$$\Delta x = \frac{b-a}{n} = \frac{2-0}{8} > \frac{1}{4} = 0.25$$

	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
x	0	.25	.5	.75	1	1.25	1.5	1.75	2
$f(x)$ $2x+1$	1	1.5	2	2.5	3	3.5	4	4.5	5
		$f(x_1)$	$f(x_2)$	$f(x_3)$	$f(x_4)$	$f(x_5)$	$f(x_6)$	$f(x_7)$	$f(x_8)$

The approximate area with the Right end Points of each Sub interval is



$A \approx \Delta x f(x_1) + \Delta x f(x_2) + \Delta x f(x_3) + \Delta x f(x_4) + \Delta x f(x_5) + \Delta x f(x_6) + \Delta x f(x_7) + \Delta x f(x_8)$

OR

$= \Delta x [f(x_1) + f(x_2) + f(x_3) + f(x_4) + f(x_5) + f(x_6) + f(x_7) + f(x_8)]$

$= 0.25 [1.5 + 2 + 2.5 + 3 + 3.5 + 4 + 4.5 + 5]$

$= 0.25 [26] = 6.5$ Ans

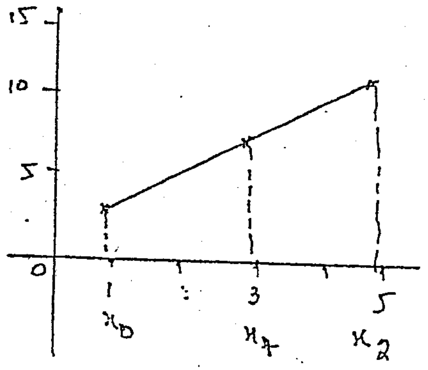
---

d) $n = 2$, $a = 1$, $b = 5$, $\boxed{f(x) = 2x+1}$

The width of each subinterval is $\Delta x = \frac{b-a}{n} = \frac{5-1}{2} = \frac{4}{2} = 2$

	x_0	x_1	x_2
x	1	3	5
$f(x) = 2x+1$	3	7	11
	$f(x_0)$	$f(x_1)$	$f(x_2)$

The approximate area with the right end Points of each subinterval is



$A \approx \Delta x f(x_1) + \Delta x f(x_2)$

or

$= \Delta x [f(x_1) + f(x_2)]$

$= 2 [7 + 11] = 2 (18) = 36$

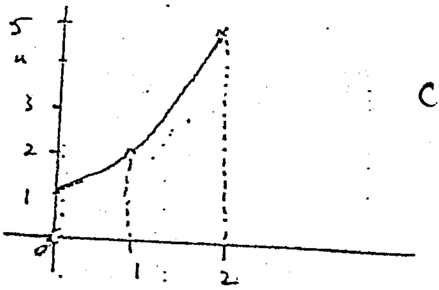
---

2) In each case, determine the approximate area of the Region bounded by $f(x) = x^2+1$, $x = a$, and $x = b$ for n subintervals.

a) $n=2,\ a=0,\ b=2,\ f(x)=x^2+1$

	x_0	x_1	x_1
x	0	1	2
$f(x)=x^2+1$	1	2	5
	$f(x_0)$	$f(x_1)$	$f(x_2)$

The width of each subinterval is $\Delta x = \frac{b-a}{n} = \frac{2-0}{2} = 1$



Ch-5

The approximate area with the Right end Points of each subinterval is

$$A \approx \Delta x\, f(x_1) + \Delta x\, f(x_2)$$

OR

$$A = \Delta x\,[f(x_1) + f(x_2)]$$

$$= 1\,[2+5] = 7$$

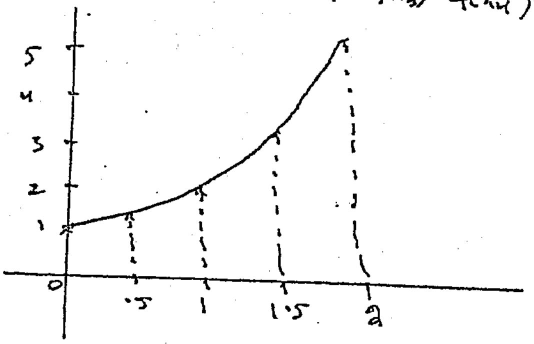
---

b) $n=4,\ a=0,\ b=2,\ f(x)=x^2+1$

	x_0	x_1	x_2	x_3	x_4
x	0	0.5	1	1.5	2
$f(x)=x^2+1$	1	1.25	2	3.25	5
	$f(x_0)$	$f(x_1)$	$f(x_2)$	$f(x_3)$	$f(x_4)$

The width of each subinterval is $\Delta x = \frac{b-a}{n} = \frac{2-0}{4} = \frac{1}{2} = 0.5$

The approximate area with the Right end points of each subinterval is



$$A \approx \Delta x\, f(x_1) + \Delta x\, f(x_2) + \Delta x\, f(x_3) + \Delta f(x_4)$$

$$A = \Delta x\,[f(x_1) + f(x_2) + f(x_3) + f(x_4)]$$

$$= 0.5\,[1.25 + 2 + 3.25 + 5]$$

$$= 0.5\,(11.5) = 5.75$$

---

c) $n=8,\ a=0,\ b=2$

	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
x	0	0.25	.5	.75	1	1.25	1.5	1.75	2
$f(x)=x^2+1$	1	1.0625	1.25	1.5625	2	2.5625	3.25	4.0625	5
	$f(x_0)$	$f(x_1)$	$f(x_2)$	$f(x_3)$	$f(x_4)$	$f(x_5)$	$f(x_6)$	$f(x_7)$	$f(x_8)$

The width of each subinterval is

$$\Delta x = \frac{b-a}{n} = \frac{2-0}{8} = \frac{1}{4} = 0.25$$

The approximate area with the Right end Points of each subinterval is $A \approx \Delta x\,[f(x_1) + f(x_2) + f(x_3) + f(x_4) + f(x_5) + f(x_6) + f(x_7) + f(x_8)]$

$$A = 0.25\,[1.0625 + 1.25 + 1.5625 + 2 + 2.5625 + 3.25 + 4.0625 + 5]$$

$$= 0.25\,[20.75] = 5.1875$$

d) $n=2,\ a=1,\ b=5,\ f(x)=x^2+1$

The width of each subinterval is $\Delta x = \frac{b-a}{n} = \frac{5-1}{2} = 2$

	x_0	x_1	x_2
x	1	3	5
$f(x)=x^2+1$	2	10	26
	$f(x_0)$	$f(x_1)$	$f(x_2)$

The approximate area with the right end points of each subinterval is

$$A \approx \Delta x f(x_1) + \Delta x f(x_2)$$
$$= \Delta x \left[f(x_1) + f(x_2) \right]$$
$$= 2\left[10 + 26 \right] = 2(36) = 72$$

---

3) In each case, determine the actual value of the integral using definition

a) $\int_{x=0}^{x=3} 3x\,dx$

Here $a=0,\ b=3,\ f(x)=3x$

For n subintervals, the width of each rectangle is

$\Delta x = \frac{b-a}{n} = \frac{3-0}{n} = \frac{3}{n}$

	x_0	x_1	x_2	x_3		x_n
x	0	$3/n$	$6/n$	$9/n$	$\cdots$	$\frac{3n}{n}$
$f(x)=3x$	0	$\frac{9}{n}$	$\frac{18}{n}$	$\frac{27}{n}$	$\cdots\cdots$	$\frac{9n}{n}$

The actual area with the right end points of each subinterval is

$$A = \int_0^3 3x\,dx$$
$$= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$
$$= \lim_{n\to\infty} \Delta x \left[f(x_1) + f(x_2) + f(x_3) + \cdots\cdots + f(x_n) \right]$$
$$= \lim_{n\to\infty} \frac{3}{n}\left[\frac{9}{n} + \frac{18}{n} + \frac{27}{n} \cdots\cdots + \frac{9n}{n} \right]$$
$$= \lim_{n\to\infty} \frac{3}{n}\cdot\frac{9}{n}\left[1 + 2 + 3 + \cdots\cdots + n \right]$$
$$= \lim_{n\to\infty} \frac{27}{\cancel{n^2}}\left[\frac{\cancel{n}(n+1)}{2} \right]$$

$$\boxed{\because 1+2+3+\cdots+n = \frac{n(n+1)}{2}}$$

$$A = \lim_{n\to\infty} \frac{27(n+1)}{2n} \qquad \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{n\to\infty} \frac{27\cdot \cancel{n}(1+\frac{1}{n})}{2\cancel{n}}$$

Applying limit rule

$$= \frac{27(1+\frac{1}{\infty})}{2} = \frac{27(1+0)}{2} = \frac{27}{2} = 13.5 \quad \text{Ans}$$

Note ① $1+2+3+ \cdots + n = \frac{n(n+1)}{2}$

② $1^2+2^2+3^2+ \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$

③ $\sum_{i=1}^{n} c = nc$

④ Actual area $A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$

$$= \lim_{n\to\infty} \left[\Delta x f(x_1) + \Delta x f(x_2) + \cdots + \Delta x f(x_n) \right]$$

---

b) $\int_0^3 (2x-4)\,dx$

Here $a=0$, $b=3$, $f(x)=2x-4$

For n subintervals, the width of each rectange is $\Delta x = \frac{b-a}{n} = \frac{3-0}{n} = 3/n$

The actual area with the right end points of each sub interval is

x		f(x) = 2x − 4	
x_0	0	-4	
x_1	$3/n$	$\frac{6}{n}-4$	$f(x_1)$
x_2	$6/n$	$\frac{12}{n}-4$	$f(x_2)$
x_3	$9/n$	$\frac{18}{n}-4$	$f(x_3)$
	⋮	⋮	⋮
x_n	$\frac{3n}{n}$	$\frac{6n}{n}-4$	$f(x_n)$

$$A = \int_0^3 (2x-4)\,dx$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$$

$$= \lim_{n\to\infty} \Delta x \left[f(x_1) + f(x_2) + f(x_3) + \cdots + f(x_n) \right]$$

$$= \lim_{n\to\infty} \frac{3}{n} \left[\left(\frac{6}{n}-4\right) + \left(\frac{12}{n}-4\right) + \left(\frac{18}{n}-4\right) + \cdots + \left(\frac{6n}{n}-4\right) \right]$$

$$= \lim_{n\to\infty} \frac{3}{n}\left[\frac{6}{n}+\frac{12}{n}+\frac{18}{n}\cdots+\frac{6n}{n}-4-4-4-\cdots-4\right]$$

$$= \lim_{n\to\infty} \frac{3}{n}\left[\frac{6}{n}(1+2+3+\cdots+n)-\underbrace{(4+4+4\cdots+4)}_{n-\text{terms}}\right]$$

$$= \lim_{n\to\infty} \frac{3}{n}\left[\frac{\cancel{6}^{3}}{\cancel{n}}\cdot\frac{\cancel{n}(n+1)}{\cancel{2}}-4n\right]$$

$$= \lim_{n\to\infty} \frac{3}{n}\left[3n+3-4n\right] = \lim_{n\to\infty} \frac{3}{n}\left[3-n\right] = \lim_{n\to\infty} \frac{3(3-n)}{n} \quad \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{n\to\infty} \frac{3\cdot\cancel{n}\left(\frac{3}{n}-1\right)}{\cancel{n}}$$

Applying limit rule

$$= 3\left(\frac{3}{\infty}-1\right) = 3(0-1) = -3 \quad \text{Ans}$$

---

c) $\int_0^2 x^2 dx = ?$

Here $a=0$, $b=2$, $f(x)=x^2$

For n subintervals, the width of each rectangle is

$$\Delta x = \frac{b-a}{n} = \frac{2-0}{n} = 2/n$$

The actual area with the right end points of each subinterval is

x	$f(x)=x^2$
0	0
$\frac{2}{n}$	$(\frac{2}{n})^2 = \frac{2^2}{n^2} = \frac{4}{n^2}$
$\frac{4}{n}$	$\frac{4^2}{n^2} = \frac{16}{n^2}$
$\frac{6}{n}$	$6^2/n^2 = \frac{36}{n^2}$
$\vdots$	
$\frac{2n}{n}$	$\frac{(2n)^2}{n^2} = \frac{4n^2}{n^2}$

$$A = \int_0^2 x^2 dx$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$$

$$= \lim_{n\to\infty} \Delta x\left[f(x_1)+f(x_2)+ \underline{\quad} + f(x_n)\right]$$

$$= \lim_{n\to\infty} \frac{2}{n}\left[\frac{4}{n^2}+\frac{16}{n^2}+\frac{36}{n^2}+ --- + \frac{4n^2}{n^2}\right]$$

$$= \lim_{n\to\infty} \frac{2}{n}\cdot\frac{4}{n^2}\left[1+4+9+ ---- + n^2\right]$$

$$= \lim_{n\to\infty} \frac{8}{n^3}\left(1^2+2^2+3^2+ - \text{——} + n^2\right)$$

$$= \lim_{n\to\infty} \frac{\cancel{8}^{4}}{n^{3}_{2}}\left[\frac{\cancel{n}(n+1)(2n+1)}{\cancel{6}_{3}}\right] = \lim_{n\to\infty} \frac{4(n+1)(2n+1)}{3n^2} \quad \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{n\to\infty} \frac{4\,\cancel{n}(1+1/n)\,\cancel{n}(2+1/n)}{3\,\cancel{n^2}}$$

$$= \lim_{n\to\infty} \frac{4(1+\frac{1}{n})(2+\frac{1}{n})}{3} = \frac{4(1+\frac{1}{\infty})(2+\frac{1}{\infty})}{3} = \frac{4(1+0)(2+0)}{3}$$

$$= \frac{4\times 2}{3} = \frac{8}{3} = 2.67$$

Ans

---

d) $\int_2^3 (x^2-4)\,dx$

Here $a=2$, $b=3$, $f(x)=x^2-4$

For n subintervals, the width of each rectangle is

$$\Delta x = \frac{b-a}{n} = \frac{3-2}{n} = \frac{1}{n}$$

The actual area with the right end points of each subinterval is

$$A = \int_2^3 (x^2-4)\,dx$$

	x	$f(x) = x^2-4$
x_0	2	0
x_1	$2+\frac{1}{n}$	$(2+\frac{1}{n})^2-4 = \cancel{4}+\frac{1}{n^2}+\frac{4}{n}-\cancel{4}$
x_2	$2+\frac{2}{n}$	$(2+\frac{2}{n})^2-4 = \cancel{4}+\frac{4}{n^2}+\frac{8}{n}-\cancel{4}$
x_3	$2+\frac{3}{n}$	$(2+\frac{3}{n})^2-4 = \cancel{4}+\frac{9}{n^2}+\frac{12}{n}-\cancel{4}$
	$\vdots$	$\vdots$
x_n	$2+\frac{n}{n}$	$(2+\frac{n}{n})^2-4 = \cancel{4}+\frac{n^2}{n^2}+\frac{4n}{n}-\cancel{4}$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$$

$$= \lim_{n\to\infty} \Delta x\left[f(x_1)+f(x_2)+f(x_3)+ \text{———} + f(x_n)\right]$$

$$= \lim_{n\to\infty} \frac{1}{n}\left[\left(\frac{1}{n^2}+\frac{4}{n}\right)+\left(\frac{4}{n^2}+\frac{8}{n}\right)+\left(\frac{9}{n^2}+\frac{12}{n}\right)+\cdots\cdots+\left(\frac{n^2}{n^2}+\frac{4n}{n}\right)\right]$$

$$= \lim_{n\to\infty} \frac{1}{n}\left[\underbrace{\frac{1}{n^2}+\frac{4}{n^2}+\frac{9}{n^2}+\cdots+\frac{n^2}{n^2}} + \underbrace{\frac{4}{n}+\frac{8}{n}+\frac{12}{n}+\text{—}\cdots+\frac{4n}{n}}\right]$$

$$= \lim_{n\to\infty} \frac{1}{n}\left[\frac{1}{n^2}(1+4+9+\cdots+n^2) + \frac{4}{n}(1+2+3+\cdots+n)\right]$$

$$= \lim_{n\to\infty} \frac{1}{n}\left[\frac{1}{n^2}(1+2^2+3^2+\cdots+n^2) + \cancel{\frac{4}{n}}\cdot\frac{\cancel{n}(n+1)}{\cancel{2}}\right]$$

$$= \lim_{n\to\infty} \frac{1}{n}\left[\frac{1}{\cancel{n^2}}\cdot\frac{\cancel{n}(n+1)(2n+1)}{6} + 2(\underline{n+1})\right]$$

$$= \lim_{n\to\infty} \frac{n+1}{n}\left[\frac{2n+1}{6n} + 2\right]$$

$$= \lim_{n\to\infty} \frac{n+1}{n}\left[\frac{2n+1+12n}{6n}\right] = \lim_{n\to\infty} \frac{(n+1)(14n+1)}{6n^2} \quad \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{n\to\infty} \frac{\cancel{n}(1+\frac{1}{n})\cancel{n}(14+\frac{1}{n})}{6\cancel{n^2}}$$

$$= \frac{(1+\frac{1}{\infty})(14+\frac{1}{\infty})}{6} = \frac{(1+0)(14+0)}{6} = \frac{14}{6} = \frac{7}{3} = 2.33$$

Ans

Fundamental theorem of Integral Calculus:

If a function $f(x)$ is continuous on the closed interval $[a,b]$, then the definite integral of a function $f(x)$ in the interval $[a,b]$ is

$$\int_{x=a}^{x=b} f(x)\,dx = \left|F(x)\right|_{x=a}^{x=b} = F(b) - F(a)$$

Here $F(x)$ is any function such that $F'(x) = f(x)\ \forall\ x \in [a,b]$

Note ① $\int_a^a f(x)\,dx = 0$

② $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ $\quad a<c<b$

③ $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

See Properties Book page 225

Definite integral as a Distance

The distance S travelled by an object moving with continuous positive rate function r(t) along a straight line from time t=a to t=b is

$$S = \int_a^b r(t)\,dt = F(b) - F(a)$$

EXERCISE 5.5

1) Evaluate the following definite integrals:

a) $\int_3^4 5dx = [5x]_3^4 = 5(4) - 5(3) = 20 - 15 = 5$

b) $\int_{12}^{20} dx = \int_{12}^{20} 1\,dx = [x]_{12}^{20} = 20 - 12 = 8$

c) $\int_1^2 (2x^{-2} - 3)dx = \left[2\frac{x^{-2+1}}{-2+1} - 3x\right]_1^2$

$$= \left[2\frac{x^{-1}}{-1} - 3x\right]_1^2 = \left[\frac{-2}{x} - 3x\right]_1^2$$

$$= \frac{-2}{2} - 3(2) - \left\{\frac{-2}{1} - 3(1)\right\}$$

$$= -1 - 6 - \{-2 - 3\}$$

$$= -7 - (-5) = -7 + 5 = -2$$ Ans

d) $\int_1^4 3\sqrt{x}\,dx = 3\int_1^4 \sqrt{x}\,dx$

$$= 3\left[\frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right]_1^4 = \cancel{3}\left[\frac{x^{3/2}}{\cancel{3}/2}\right]_1^4 = 2\left[x^{3/2}\right]_1^4$$

$$= 2\left[4^{3/2} - 1^{3/2}\right] = 2\left[(2^2)^{3/2} - 1\right]$$

$$= 2\left[2^3 - 1\right]$$

$$= 2(8-1) = 2(7) = 14$$ Ans

242

e) $\int_2^3 12(x^2-4)^5 x\,dx = 12\int_2^3 (x^2-4)^5\, x\,dx$

$= \overset{6}{\cancel{12}}\int_0^5 u^5 \frac{du}{\cancel{2}}$

$= 6\int_0^5 u^5\,du$

$= \cancel{6}\left[\frac{u^6}{\cancel{6}}\right]_0^5 = \left[u^6\right]_0^5$

$= 5^6 - 0^6 = 5^6 - 0 = 5^6 = 15625$ Ans

put $x^2-4=u$
Diff w.r.t. x
$2x-0=\frac{du}{dx}$
$\Rightarrow x\,dx=\frac{du}{2}$

when $x=2$ then $u=2^2-4$
$=0$

when $x=3$ then $u=3^2-4$
$=9-4=5$

2nd method $\int_2^3 12(x^2-4)^5 x\,dx = 12\int_2^3 (x^2-4)^5\, x\,dx$

M & D by 2

$= \frac{\overset{6}{\cancel{12}}}{\cancel{2}}\int_2^3 \underset{f}{(x^2-4)^5}\,\underset{f'}{2x\,dx}$

$\left(\int (f(x))^n \cdot f'(x) = \frac{(f(x))^{n+1}}{n+1}+c\right)$

$= \cancel{6}\left[\frac{(x^2-4)^6}{\cancel{6}}\right]_2^3$

$= \left[(x^2-4)^6\right]_2^3$

$= (3^2-4)^6 - (2^2-4)^6$

$= 5^6 - 0 = 15625$ Ans

3rd method $\int_2^3 12(x^2-4)^5 x\,dx = ?$

consider $\int 12(x^2-4)^5 x\,dx = 12\int (x^2-4)^5 x\,dx$

$= \overset{6}{\cancel{12}}\int u^5 \frac{du}{\cancel{2}}$

$= 6\int u^5\,du$

$= \cancel{6}\frac{u^6}{\cancel{6}} + c$

$= (x^2-4)^6 + c$

put $x^2-4=u$
Diff w.r.t-x
$2x=\frac{du}{dx}$
$\Rightarrow x\,dx=\frac{du}{2}$

Now $\int_2^3 12(x^2-4)^5 x\,dx = \left[(x^2-4)^6\right]_2^3$

$= (3^2-4)^6 - (2^2-4)^6 = 5^6 - 0 = 15625$ Ans

f) $\int_{-6}^{0}\sqrt{4-2x}\,dx = \int_{-6}^{0}(4-2x)^{1/2}dx$

$= \left|\frac{(4-2x)^{\frac12+1}}{-2(\frac12+1)}\right|_{-6}^{0}$

$= \left[\frac{(4-2x)^{3/2}}{-\cancel{2}(3/\cancel{2})}\right]_{-6}^{0} = \left[\frac{(4-2x)^{3/2}}{-3}\right]_{-6}^{0} = -\frac13\left[(4-2x)^{3/2}\right]_{-6}^{0}$

$= -\frac13\left[(4-2(0))^{3/2} - (4-2(-6))^{3/2}\right]$

$= -\frac13\left[4^{3/2} - (4+12)^{3/2}\right] = -\frac13\left[(2^{\cancel{2}})^{3/\cancel{2}} - (4^{\cancel{2}})^{3/\cancel{2}}\right]$

$= -\frac13\left[8-64\right] = -\frac13(-56) = +\frac{56}{3}$

$\int(ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)}+C$

$4+12=16=4^2$

2nd method $\int_{-6}^{0}\sqrt{4-2x}\,dx = \int_{4}^{2}u(-u\,du)$

$= -\int_{4}^{2}u^2du$

$= -\left[\frac{u^3}{3}\right]_{4}^{2}$

$= -\left[\frac{2^3}{3} - \frac{4^3}{3}\right]$

$= -\left[\frac{8}{3} - \frac{64}{3}\right]$

$= +\frac{56}{3}$ Ans

put $\sqrt{4-2x} = u$

sq

$4-2x = u^2$

Diff w.r.t.x

$-\cancel{2} = \cancel{2}u\frac{du}{dx}$

$\Rightarrow dx = -u\,du$

when $x=-6$ then $u=\sqrt{4-2(-6)}$

$=\sqrt{4+12}$

$=\sqrt{16}=4$ ✓

when $x=0$ then $u=\sqrt{4-0}$

$=\sqrt{4}=2$ ✓

---

g) $\int_{-1}^{7}\frac{x}{\sqrt{x+2}}dx = \int_{1}^{3}\frac{u^2-2}{\cancel{u}}(2\cancel{u}\,du)$

$= 2\int_{1}^{3}(u^2-2)\,du$

$= 2\left[\frac{u^3}{3}-2u\right]_{1}^{3}$

$= 2\left[\frac{3^3}{3}-2(3)-\left\{\frac{1^3}{3}-2(1)\right\}\right]$

$= 2\left[\frac{27}{3}-6-\frac{1}{3}+2\right]$

$= 2\left[\frac{26}{3}-4\right] = 2\left(\frac{14}{3}\right) = \frac{28}{3}$ Ans

put $\sqrt{x+2}=u$

sq

$x+2=u^2 \Rightarrow x=u^2-2$

Diff w.r.t x

$1=2u\frac{du}{dx}$

$\Rightarrow dx = 2u\,du$

when $x=-1$ then $u=\sqrt{-1+2}=1$

when $x=7$ then $u=\sqrt{7+2}=3$

h) $\int_0^1 (e^{2x}-2x)^2(e^{2x}-1)\,dx$

$= \int_1^{5.39} u^2 \frac{du}{2}$

$= \frac{1}{2}\int_1^{5.39} u^2\,du$

$= \frac{1}{2}\left[\frac{u^3}{3}\right]_1^{5.39} = \frac{1}{6}\left[u^3\right]_1^{5.39}$

$= \frac{1}{6}\left[(5.39)^3 - 1^3\right]$

$= \frac{1}{6}\left[156.591 - 1\right] = \frac{1}{6}(155.591) = 25.931$

put $e^{2x} - 2x = u$

Diff w.r.t. x

$e^{2x}\cdot 2 - 2\cdot 1 = \frac{du}{dx}$

$2(e^{2x}-1)\,dx = du$

$(e^{2x}-1)\,dx = \frac{du}{2}$

when $x=0$ then $u = e^0 - 2(0) = 1$

when $x=1$ then $u = e^2 - 2(1) = e^2 - 2$

$= (2.718)^2 - 2$

$= 7.39 - 2$

$= 5.39$

---

2) Evaluate the following definite integrals:

a) $\int_2^3 x\sqrt{2x^2-3}\,dx = \int_2^3 \sqrt{2x^2-3}\;x\,dx$

$= \int_{\sqrt{5}}^{\sqrt{15}} u\cdot\frac{u\,du}{2}$

$= \frac{1}{2}\int_{\sqrt{5}}^{\sqrt{15}} u^2\,du$

$= \frac{1}{2}\left[\frac{u^3}{3}\right]_{\sqrt{5}}^{\sqrt{15}}$

$= \frac{1}{6}\left[u^3\right]_{\sqrt{5}}^{\sqrt{15}}$

$= \frac{1}{6}\left[(\sqrt{15})^3 - (\sqrt{5})^3\right] = \frac{1}{6}\left[15^{3/2} - 5^{3/2}\right] = \frac{1}{6}\left[58.095 - 11.18\right]$

$= \frac{1}{6}(46.915) = 7.819$ Ans

put $\sqrt{2x^2-3} = u$

sq

$2x^2 - 3 = u^2$

Diff w.r.t. x

$\overset{2}{\cancel{4}}x = \cancel{2}u\frac{du}{dx}$

$x\,dx = \frac{u\,du}{2}$

when $x=2$ then $u = \sqrt{2(2)^2-3}$

$= \sqrt{8-3} = \sqrt{5}$

when $x=3$ then $u = \sqrt{2(3)^2-3}$

$= \sqrt{18-3} = \sqrt{15}$

2nd method $\int_2^3 x\sqrt{2x^2-3}\,dx = \int_2^3 \sqrt{2x^2-3}\;x\,dx$

$= \int_5^{15} \sqrt{u}\,\frac{du}{4}$

$= \frac{1}{4}\int_5^{15} \sqrt{u}\,du$

put $2x^2 - 3 = u$

Diff w.r.t. x

$4x - 0 = \frac{du}{dx}$

$x\,dx = \frac{du}{4}$

when $x=2$ then $u = 2(2)^2 - 3 = 5$

when $x=3$ then $u = 2(3)^2 - 3 = 15$

$$= \frac{1}{4}\left[\frac{u^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right]_5^{15} = \frac{1}{\cancel{4}_2}\left[\frac{u^{3/2}}{3/\cancel{2}}\right]_5^{15}$$

$$= \frac{1}{6}\left[u^{3/2}\right]_5^{15} = \frac{1}{6}\left[15^{3/2} - 5^{3/2}\right]$$

$$= \frac{1}{6}\left[58.095 - 11.18\right] = \frac{1}{6}(46.915) = 7.819$$ Ans

---

b) $$\int_0^1 x\sqrt{3x^2+2}\,dx = \int_0^1 \sqrt{3x^2+2}\;x\,dx$$

put $3x^2+2 = u$

Diff: w.r.t. x

$6x+0 = \frac{du}{dx}$

$\Rightarrow x\,dx = \frac{du}{6}$

when $x=0$ then $u = 3(0)+2 = 2$

when $x=1$ then $u = 3(1)^2+2 = 5$

$$= \int_2^5 \sqrt{u}\;\frac{du}{6}$$

$$= \frac{1}{6}\left[\frac{u^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right]_2^5$$

$$= \frac{1}{\cancel{6}3}\left[\frac{u^{3/2}}{3/\cancel{2}}\right]_2^5$$

$$= \frac{1}{9}\left[u^{3/2}\right]_2^5$$

$$= \frac{1}{9}\left[5^{3/2} - 2^{3/2}\right]$$

$$= \frac{1}{9}\left[11.18 - 2.828\right] = \frac{1}{9}(8.352) = 0.928$$

---

c) $$\int_0^1 \frac{x-1}{x^2-2x+3}\,dx = \int_0^1 \frac{x-1}{\underline{x^2-2x+1}+2}\,dx$$

put $x-1 = u$

Diff w.r.t. x

$1 = \frac{du}{dx}$

$\Rightarrow dx = du$

when $x=0$ then $u = 0-1 = -1$

when $x=1$ then $u = 1-1 = 0$

$$\int \frac{f'(x)}{f(x)}\,dx = \ln f(x) + c$$

$$= \int_0^1 \frac{x-1}{(x-1)^2+2}\,dx$$

$$= \int_{-1}^0 \frac{u}{u^2+2}\,du$$

$$= \frac{1}{2}\int_{-1}^0 \frac{2u}{u^2+2}\,du$$

$$= +\frac{1}{2}\left[\ln(u^2+2)\right]_{-1}^0$$

$$= \frac{1}{2}\left[\ln(0+2) - \ln((-1)^2+2)\right] = \frac{1}{2}\left[\ln 2 - \ln(1+2)\right]$$

$$= \frac{1}{2}\left[\ln 2 - \ln 3\right] = \frac{1}{2}\ln\frac{2}{3} = \frac{1}{2}(-0.4055) = -0.2028$$ Ans

d) $\int_{-1}^{1} \frac{(e^{-x}-e^{x})dx}{(e^{-x}+e^{x})^2} = \int_{-1}^{1} \frac{1}{(e^{-x}+e^{x})^2}(e^{-x}-e^{x})dx$

$$= \int_{e+e^{-1}}^{e+e^{-1}} \frac{1}{u^2}(-du)$$

$$= -\int_{e+e^{-1}}^{e+e^{-1}} u^{-2}\,du$$

$$= -\left[\frac{u^{-1}}{-1}\right]_{e+e^{-1}}^{e+e^{-1}}$$

$$= +\left[\frac{1}{u}\right]_{e+e^{-1}}^{e+e^{-1}}$$

$$= \frac{\cancel{1}}{\cancel{e}+e^{-1}} - \frac{1}{e\cancel{+e^{-1}}} = 0$$

put $e^{-x}+e^{x} = u$

Diff w.r.t.x

$e^{-x}(-1)+e^{x}.1 = \frac{du}{dx}$

$(-e^{-x}+e^{x})dx = du$

$-(e^{-x}-e^{x})dx = du$

$(e^{-x}-e^{x})dx = -du$

when $x=-1$ then $u = e^{+1}+e^{-1}$
$= e+e^{-1}$

when $x=1$ then $u = e^{-1}+e^{1}$
$= e+e^{-1}$

$\frac{3\pi}{2} = 270°$

$\frac{5\pi}{4} = 225°$

---

3) Evaluate the following definite integral

a) $\int_{\pi/2}^{\pi} \cos\left(\frac{x}{2}+\pi\right)dx = \int_{5\pi/4}^{3\pi/2} \cos u(2du)$

$$=2\left[\sin u\right]_{5\pi/4}^{3\pi/2}$$

$$=2\left[\sin\frac{3\pi}{2} - \sin\frac{5\pi}{4}\right]$$

$$=2\left[\sin(270°) - \sin 225°\right]$$

$$=2\left[-1-\left(\frac{-1}{\sqrt{2}}\right)\right]$$

$$=2\left[-1+\frac{1}{\sqrt{2}}\right]$$ Ans

$$=2\left[-1+\frac{\sqrt{2}}{2}\right] = \cancel{2}\left[\frac{-2+\sqrt{2}}{\cancel{2}}\right] = -2+\sqrt{2}$$ Ans

put $\frac{x}{2}+\pi = u$

Diff w.r.t x

$\frac{1}{2}+0 = \frac{du}{dx}$

$\Rightarrow dx = 2du$

when $x=\frac{\pi}{2}$ then $u = \frac{\pi/2}{2}+\pi$
$= \frac{\pi}{4}+\pi = \frac{5\pi}{4}$

when $x=\pi$ then $u = \frac{\pi}{2}+\pi = \frac{3\pi}{2}$

$\boxed{\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}}$

---

b) $\int_{0.75}^{2.5} x\cos x^2\,dx = \int_{0.75}^{2.5} \cos x^2 \cdot x\,dx$

$$= \int_{0.5625}^{6.25} \cos u \frac{du}{2}$$

put $x^2 = u$

Diff w.r.t. x

$2x = \frac{du}{dx} \Rightarrow x\,dx = \frac{du}{2}$

when $x = .75$ then $u = (.75)^2 = 0.5625$

when $x = 2.5$ then $u = (2.5)^2 = 6.25$

$$= \frac{1}{2}\int_{0.5625}^{6.25} \cos u \, du$$

$$= \frac{1}{2}\Big[\sin u\Big]_{0.5625}^{6.25}$$

$$= \frac{1}{2}\Big[\sin 6.25 - \sin 0.5625\Big]$$ (use calculator in radian mode

$$= \frac{1}{2}\Big[-0.0332 - 0.5333\Big]$$

$$= \frac{1}{2}\Big[-0.5665\Big] = -0.2833$$

---

c) $\int_0^{\pi/4} \sec^2\theta \, d\theta = \Big[\tan\theta\Big]_0^{\pi/4} = \tan\frac{\pi}{4} - \tan 0 = 1-0 = 1$ Ans

---

d) $\int_0^{\pi/4} \tan 2\pi x \, dx = \left[-\frac{\ln\cos 2\pi x}{2\pi}\right]_0^{\pi/4}$

$$= -\frac{1}{2\pi}\Big[\ln\cos 2\pi x\Big]_0^{\pi/4}$$

$$= -\frac{1}{2\pi}\Big[\ln\cos \cancel{2}\pi\left(\frac{\pi}{\cancel{4}_2}\right) - \ln\cos(2\pi(0))\Big]$$

$$= -\frac{1}{2\pi}\Big[\ln\cos\frac{\pi^2}{2} - \ln\cos 0\Big]$$

$$= -\frac{1}{2\pi}\Big[\ln(0.2206) - \ln 1\Big]$$

$$= -\frac{1}{2\pi}\Big[-1.5115 - 0\Big] = +0.241$$

($\int \tan kx \, dx = \frac{-\ln\cos kx}{k} + c$ or $\frac{\ln\sec kx}{k} + c$

2nd method $\int_0^{\pi/4} \tan 2\pi x \, dx$

put $2\pi x = u$

Diff w.r.t. x

$2\pi = \frac{du}{dx} \Rightarrow dx = \frac{du}{2\pi}$

when $x=0$ then $u = 2\pi(0) = 0$

when $x = \pi/4$ then $u = \cancel{2}\pi\left(\frac{\pi}{\cancel{4}_2}\right) = \frac{\pi^2}{2}$

$$= \int_0^{\pi^2/2} \tan u \, \frac{du}{2\pi}$$

$$= \frac{1}{2\pi}\int_0^{\pi^2/2} \tan u \, du$$

$$= \frac{1}{2\pi}\Big[-\ln\cos u\Big]_0^{\pi^2/2}$$

$$= -\frac{1}{2\pi}\Big[\ln\cos\pi^2/2 - \ln\cos 0\Big]$$

$$= -\frac{1}{2\pi}\Big[\ln(0.2206) - \ln 1\Big] = -\frac{1}{2\pi}\Big[-1.5115 - 0\Big] = 0.241$$

Ans

---

e) $\int_0^1 \frac{1}{x^2+1}\,dx = \left[\tan^{-1}x\right]_0^1 = \tan^{-1}1 - \tan^{-1}0 = \frac{\pi}{4} - 0 = \pi/4$

2nd method

$$\int_0^1 \frac{1}{x^2+1}\,dx = \int_0^{\pi/4} \frac{1}{\tan^2\theta+1}\cdot \sec^2\theta\, d\theta$$

$$= \int_0^{\pi/4} \frac{1}{\cancel{\sec^2\theta}}\cdot \cancel{\sec^2\theta}\, d\theta$$

$$= \int_0^{\pi/4} 1\cdot d\theta$$

$$= \left[\theta\right]_0^{\pi/4} = \frac{\pi}{4} - 0 = \pi/4 \quad \text{Ans}$$

put $x = 1\cdot\tan\theta \Rightarrow \underline{\underline{\theta = \tan^{-1}x}}$
Diff w.r.t x
$1 = \sec^2\theta \frac{d\theta}{dx}$
$\Rightarrow dx = \sec^2\theta\, d\theta$

when $x=0$ then $\theta = \tan^{-1}0 = 0$
when $x=1$ then $\theta = \tan^{-1}1 = \pi/4$

---

f) $\int_{\sqrt{2}}^{2} \frac{1}{x\sqrt{x^2-1}}\,dx = \left[\sec^{-1}x\right]_{\sqrt{2}}^{2}$ ↓

$$= \left[\cos^{-1}\frac{1}{x}\right]_{\sqrt{2}}^{2} = \cos^{-1}\frac{1}{2} - \cos^{-1}\frac{1}{\sqrt{2}}$$

$$= \frac{\pi}{3} - \frac{\pi}{4} = \frac{4\pi-3\pi}{12} = \frac{\pi}{12} \quad \text{Ans}$$

use calculator in radian mode

$\sec^{-1}x = \cos^{-1}\frac{1}{x}$

---

4) Evaluate the following definite integrals:

a) $\int_1^2 \frac{5t^2-3t+18}{t(9-t^2)}\,dt = \int_1^2 \frac{5t^2-3t+18}{t(3-t)(3+t)}\,dt$

$$\int_1^2 \frac{5t^2-3t+18}{t(9-t^2)} = \int_1^2 \left(\frac{A}{t} + \frac{B}{3-t} + \frac{C}{3+t}\right) dt \quad \text{——①}$$

where $\frac{5t^2-3t+18}{t(3-t)(3+t)} = \frac{A}{t} + \frac{B}{3-t} + \frac{C}{3+t}$

x ing by $t(3-t)(3+t)$

$$5t^2-3t+18 = A(3-t)(3+t) + Bt(3+t) + Ct(3-t) \quad \text{——②}$$

put $t=0$ in ②

$$0 - 0 + 18 = A(3-0)(3+0) + 0 + 0$$

$$\overset{2}{\cancel{18}} = \cancel{9}A \Rightarrow \boxed{A=2}$$

put $3-t=0 \Rightarrow t=3$ in ②

$5(3)^2-3(3)+18 = A(0) + B(3)(3+3) + 0$

$45-9+18 = 0 + B(3)(6)$

$3\ \cancel{54} = \cancel{18}B \Rightarrow \boxed{B=3}$

put $3+t=0 \Rightarrow t=-3$ in ②

$5(-3)^2-3(-3)+18 = A(0) + B(0) + C(-3)(3+3)$

$5(9)+9+18 = 0+0+C(-3)(6)$

$45+9+18 = -18C$

$4\ \cancel{72} = -\cancel{18}C \Rightarrow \boxed{C=-4}$

put values of A, B & C in ①

$$\int_1^2 \frac{5t^2-3t+18}{t(9-t^2)}\,dt = \int_1^2\left(\frac{2}{t}+\frac{3}{3-t}+\frac{-4}{3+t}\right)dt$$

$$=\left[2\ln t + \underbrace{3\ln(3-t)}_{-1} - 4\ln(3+t)\right]_1^2$$

$$=\left[2\ln t - 3\ln(3-t) - 4\ln(3+t)\right]_1^2$$

$$=\left[\ln t^2 + \ln(3-t)^{-3} + \ln(3+t)^{-4}\right]_1^2$$

$$=\left[\ln t^2(3-t)^{-3}(3+t)^{-4}\right]_1^2$$

$$=\left[\ln\left\{\frac{t^2}{(3-t)^{+3}(3+t)^4}\right\}\right]_1^2$$

$$=\ln\frac{2^2}{(3-2)^3(3+2)^4} - \ln\frac{1^2}{(3-1)^3(3+1)^4}$$

$$=\ln\left(\frac{4}{(1)625}\right) - \ln\frac{1}{8(256)}$$

$$=\ln\left(\frac{4}{625}\right) - \ln\frac{1}{2048}$$

$$=-5.0515+7.6246$$

$$=2.5731$$

25°

Let

b) $I = \int_2^3 \frac{4x^5-3x^4-6x^3+4x^2+6x-1}{(x-1)(x^2-1)}\,dx$ (Improper)

$$= \int_2^3 \frac{4x^5-3x^4-6x^3+4x^2+6x-1}{x^3-x^2-x+1}\,dx$$

$$\begin{array}{r|l} & 4x^2+x-1 \\ \hline x^3-x^2-x+1 & 4\cancel{x^5}-3x^4-6x^3+4\cancel{x^2}+6x-1 \\ & \underset{-}{\cancel{4x^5}}\ \underset{+}{-}4x^4\ \underset{+}{-}4x^3\ \underset{-}{+}\cancel{4x^2} \\ \hline & \cancel{x^4}-2x^3+0x^2+6x-1 \\ & \underset{-}{\cancel{x^4}}\ \underset{+}{-}x^3\ \underset{+}{-}x^2\ \underset{-}{+}x \\ \hline & -\cancel{x^3}+\cancel{x^2}+5x-\cancel{1} \\ & \underset{+}{-}\cancel{x^3}\ \underset{-}{+}\cancel{x^2}\ \underset{-}{+}x\ \underset{+}{-}\cancel{1} \\ \hline & 4x \end{array}$$

$$= \int_2^3 \left(4x^2+x-1 + \frac{4x}{x^3-x^2-x+1}\right) dx$$

$$= \int_2^3 \left(4x^2+x-1 + \frac{4x}{x^2(x-1)-1(x-1)}\right] dx$$

$$= \int_2^3 \left(4x^2+x-1 + \frac{4x}{(x-1)(x^2-1)}\right) dx$$

$$= \int_2^3 \left(4x^2+x-1 + \frac{4x}{(x-1)\{(x-1)(x+1)\}}\right\} dx$$

$$= \int_2^3 \left[4x^2+x-1 + \frac{4x}{(x-1)^2(x+1)}\right] dx$$

$$I = \int_2^3 \left[4x^2+x-1 + \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1}\right] dx \text{ ——— ①}$$

where $\frac{4x}{(x-1)^2(x+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1}$

×ing by $(x-1)^2(x+1)$

$$4x = A(x-1)(x+1) + B(x+1) + C(x-1)^2 \text{ ——— ②}$$

$$4x = A(x^2-1) + B(x+1) + C(x^2-2x+1) \text{ ——— ③}$$

put $x-1=0 \Rightarrow x=1$ in ②

$4(1) = 0 + B(1+1) + C(0)$

$4 = 2B \Rightarrow \boxed{B=2}$

put $x+1=0 \Rightarrow x=-1$ in ②

$4(-1) = A(0) + B(0) + C(-1-1)^2$

$-4 = C(-2)^2$

$-\cancel{4} = \cancel{4}C \Rightarrow \boxed{C=-1}$

Now comparing coefficients in eq ③

x^2: $A + C = 0$

$A + (-1) = 0 \Rightarrow \boxed{A=1}$

put values of A, B & c in ①

$$I = \int_2^3 \left[4x^2 + x - 1 + \frac{1}{x-1} + \frac{2}{(x-1)^2} + \frac{-1}{x+1}\right] dx$$

$$= \left[\frac{4x^3}{3} + \frac{x^2}{2} - x + \ln(x-1) + 2\,\frac{(x-1)^{-1}}{-1} - \ln(x+1)\right]_2^3$$

$$= \left[\frac{4x^3}{3} + \frac{x^2}{2} - x + \ln\frac{x-1}{x+1} - \frac{2}{x-1}\right]_2^3$$

$$= \frac{4(3)^3}{3} + \frac{3^2}{2} - 3 + \ln\frac{3-1}{3+1} - \frac{2}{3-1} - \left\{\frac{4(2)^3}{3} + \frac{2^2}{2} - 2 + \ln\frac{2-1}{2+1} - \frac{2}{2-1}\right\}$$

$$= \frac{4(\overset{9}{\cancel{27}})}{\cancel{3}} + \frac{9}{2} - 3 + \ln\left(\frac{2}{4}\right) - \frac{2}{2} - \frac{4(8)}{3} - \frac{\overset{2}{\cancel{4}}}{\cancel{2}} + 2 - \ln\frac{1}{3} + 2$$

$$= 36 + \frac{9}{2} - 3 + \ln\frac{1}{2} - 1 - \frac{32}{3} \cancel{-2} \cancel{+2} - \ln\frac{1}{3} + 2$$

$$= 34 + \frac{9}{2} - \frac{32}{3} + \ln\frac{1}{2} - \ln\frac{1}{3}$$

$$= 34 + 4.5 - 10.67 + (-0.6931) + 1.0986 = 28.2355$$ Ans

---

e) $$\int_3^5 \frac{x^2-2}{(x-2)^2}\,dx = \int_3^5 \frac{x^2-2}{x^2-4x+4}\,dx$$

$$\begin{array}{r|l} & 1 \\ x^2-4x+4 & \cancel{x^2}-2 \\ & \underline{-\cancel{x^2} \mp 4 \pm 4x} \\ & 4x-6 \end{array}$$

$$= \int_3^5 \left(1 + \frac{4x-6}{x^2-4x+4}\right) dx$$

$$= \int_3^5 \left(1 + \frac{4x-6}{(x-2)^2}\right) dx$$

$$= \int_3^5 \left[1 + \frac{A}{x-2} + \frac{B}{(x-2)^2}\right] dx \quad \text{———} ①$$

where $$\frac{4x-6}{(x-2)^2} = \frac{A}{x-2} + \frac{B}{(x-2)^2}$$

Xing by $(x-2)^2$

$$4x-6 = A(x-2) + B \quad \text{———} ②$$

put $x-2=0 \Rightarrow x=2$ in ②

$4(2)-6 = A(0) + B$

$2 = B \Rightarrow \boxed{B=2}$

comparing coefficient in ②

x: $\boxed{A=4}$

put values of A & B in ①

$$\int_3^5 \frac{x^2-2}{(x-2)^2}\,dx = \int_3^5 \left[1+\frac{4}{x-2}+\frac{2}{(x-2)^2}\right]dx$$

$$\int (x-2)^{-2}dx = \frac{(x-2)^{-1}}{-1}+c$$

$$= \left[x+4\ln(x-2)+2\,\frac{(x-2)^{-1}}{-1}\right]_3^5$$

$$= \left[x+\ln(x-2)^4-\frac{2}{x-2}\right]_3^5$$

$$= 5+\ln(5-2)^4-\frac{2}{5-2}-\left\{3+\ln(3-2)^4-\frac{2}{3-2}\right\}$$

$$= 5+\ln 81-\frac{2}{3}-3-\ln 1+\frac{2}{1}$$

$$= 5+4.39-0.67-3-0+2 = 7.72$$

---

d) $$\int_1^2 \frac{4}{t^3+4t}\,dt = \int_1^2 \frac{4}{t(t^2+4)}\,dt$$

$$= \int_1^2 \left[\frac{A}{t}+\frac{Bt+C}{t^2+4}\right]dt \quad \text{——} ①$$

where $$\frac{4}{t(t^2+4)} = \frac{A}{t}+\frac{Bt+C}{t^2+4}$$

X ing by $t(t^2+4)$

$$4 = A(t^2+4)+(Bt+C)t \quad \text{——} ②$$

$$4 = A(t^2+4)+Bt^2+Ct \quad \text{——} ③$$

put $t=0$ in ②

$$4 = A(0+4)+0$$

$$\overset{1}{\cancel{4}} = \cancel{4}A \Rightarrow \boxed{A=1}$$

Comparing coefficients in ③

t^2: $A+B=0$

$1+B=0 \Rightarrow \boxed{B=-1}$

t: $\boxed{C=0}$

put values of A, B & C in ①

$$\int_1^2 \frac{4}{t^3+4t}\,dt = \int_1^2 \left(\frac{1}{t} + \frac{-1\cdot t+0}{t^2+4}\right) dt$$

M&D by 2

$$\left(\int \frac{f'(x)}{f(x)}\,dx = \ln f(x) + C\right.$$

$$= \int_1^2 \left[\frac{1}{t} - \frac{2t}{2(t^2+4)}\right] dt$$

$$= \left[\ln t - \frac{1}{2}\overset{\curvearrowright}{\ln(t^2+4)}\right]_1^2$$

$$= \left[\ln t - \ln(t^2+4)^{1/2}\right]_1^2$$

$$= \left[\ln t - \ln\sqrt{t^2+4}\right]_1^2$$

$$= \left[\ln \frac{t}{\sqrt{t^2+4}}\right]_1^2$$

$$= \ln\frac{2}{\sqrt{2^2+4}} - \ln\frac{1}{\sqrt{1+4}}$$

$$= \ln\frac{2}{\sqrt{8}} - \ln\frac{1}{\sqrt{5}} = -0.3466 + 0.8047 = \underline{0.4581}$$

---

e) $\int_1^3 \ln(2x+1)\,dx = \int_1^3 \underset{I}{\ln(2x+1)} \times \underset{II}{1}\,dx$ Int: by Parts

consider $\int \underset{I}{\ln(2x+1)} \times \underset{II}{1}\,dx = \ln(2x+1)\int 1\,dx - \int \left[\frac{d}{dx}\ln(2x+1)\right]\left(\int 1\,dx\right) dx$

$$= \ln(2x+1)\cdot x - \int \frac{1}{2x+1}\frac{d}{dx}(2x+1)\cdot x\,dx$$

$$= x\ln(2x+1) - \int \frac{1}{2x+1}(2)\cdot x\,dx$$

$$= x\ln(2x+1) - \int \frac{2x}{2x+1}\,dx$$

$$\begin{array}{r} 1 \\ 2x+1\,\overline{)\,\cancel{2x}} \\ \underline{\cancel{2x}+1} \\ -1 \end{array}$$

$$= x\ln(2x+1) - \int\left(1 + \frac{-1}{2x+1}\right)dx$$

$$= x\ln(2x+1) - \left[x - \frac{\ln(2x+1)}{2}\right] + C$$

$$\int \frac{1}{ax+b}\,dx = \frac{\ln(ax+b)}{a}$$

$$= x\ln(2x+1) - x + \frac{\ln(2x+1)}{2} + C$$

$$\int \ln(2x+1)\,dx = \left(x + \frac{1}{2}\right)\ln(2x+1) - x + C$$

See →

Now $\int_1^3 \ln(2x+1)\,dx = \left[(x+\frac{1}{2})\ln(2x+1) - x\right]_1^3$

$= (3+\frac{1}{2})\ln(2(3)+1) - 3 - \left\{(1+\frac{1}{2})\ln(2(1)+1) - 1\right\}$

$= \frac{7}{2}\ln 7 - 3 - \frac{3}{2}\ln 3 + 1$

$= 6.81 - 2 - 1.65 = 3.16$ Ans

---

f) $\int_1^4 \frac{\ln x}{x^3}\,dx = \int_1^4 \underset{I}{\ln x}\,.\,\underset{II}{x^{-3}}\,dx$

Consider

$\int \frac{\ln x}{x^3}\,dx = \int \underset{I}{\ln x}\,.\,\underset{II}{x^{-3}}\,dx$ Int by Parts

$= \ln x\,.\int x^{-3}dx - \int\left(\frac{d}{dx}\ln x\right)\left(\int x^{-3}dx\right)dx$

$= \ln x\,.\,\frac{x^{-2}}{-2} - \int \frac{1}{x}\,.\,\frac{x^{-2}}{-2}\,dx$

$= \frac{x^{-2}\ln x}{-2} + \frac{1}{2}\int x^{-3}dx$

$= -\frac{\ln x}{2x^2} + \frac{1}{2}\,.\,\frac{x^{-2}}{-2} + c$

$= -\frac{\ln x}{2x^2} - \frac{1}{4x^2} + c = \frac{-2\ln x - 1}{4x^2} + c$

Now

$\int_1^4 \frac{\ln x}{x^3}\,dx = \left[\frac{-2\ln x - 1}{4x^2}\right]_1^4$

$= \frac{-2\ln 4 - 1}{4(4)^2} - \frac{-2\ln(1) - 1}{4(1)^2}$

$= \frac{-2\ln 4 - 1}{64} - \frac{0-1}{4}$

$= \frac{-2.773-1}{64} + \frac{1}{4} = \frac{-3.773}{64} + .25$

$= -0.059 + .25 = 0.191$

Ans

g) $\int_{1}^{2} \underset{I}{x}\,\underset{II}{(x-1)^6}\,dx$

Consider $\int \underset{I}{x}\,\underset{II}{(x-1)^6}\,dx$ Int. by Parts

$$= x\int(x-1)^6dx - \int\left(\frac{d}{dx}x\right)\left(\int(x-1)^6dx\right)dx$$

$$= x\,\frac{(x-1)^7}{7} - \int 1\,\frac{(x-1)^7}{7}\,dx$$

$$= \frac{x(x-1)^7}{7} - \frac{1}{7}\int(x-1)^7dx$$

$$= \frac{x(x-1)^7}{7} - \frac{1}{7}\cdot\frac{(x-1)^8}{8} + c$$

$$= \frac{x(x-1)^7}{7} - \frac{(x-1)^8}{56} + c$$

Now

$$\int_{1}^{2} x(x-1)^6dx = \left[\frac{x(x-1)^7}{7} - \frac{(x-1)^8}{56}\right]_{1}^{2}$$

$$= \frac{2(2-1)^7}{7} - \frac{(2-1)^8}{56} - \left\{\frac{1(1-1)^7}{7} - \frac{(1-1)^8}{56}\right\}$$

$$= \frac{2(1)^7}{7} - \frac{1^8}{56} - \{0 + 0\}$$

$$= \frac{2}{7} - \frac{1}{56} = \frac{16-1}{56} = \frac{15}{56}$$

---

h) $\int_{0}^{1} \underset{I}{(x-3)}\,\underset{II}{e^x}\,dx = ?$

Consider

$$\int \underset{I}{(x-3)}\,\underset{II}{e^x}\,dx = (x-3)\int e^xdx - \int\left(\frac{d}{dx}(x-3)\right)\left(\int e^xdx\right)dx$$

$$= (x-3)\,e^x - \int(1)\,e^x\,dx$$

$$= (x-3)\,e^x - \int e^xdx$$

$$= (x-3)e^x - e^x + c$$

$$= (x-3-1)\,e^x + c$$

$$= (x-4)\,e^x + c$$

Now $\int_0^1 (x-3)e^x \, dx = \left[(x-4)e^x\right]_0^1$

$= (1-4)e^1 - (0-4)e^0$

$= -3e + 4.1 = 4-3e$

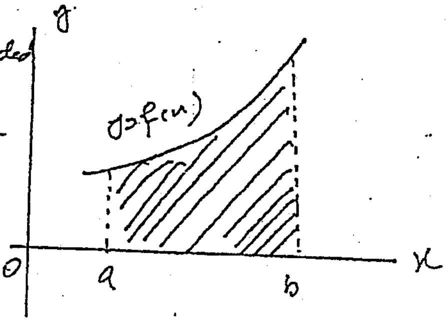
$= 4-3(2.718) = 4-8.154 = -4.154$

Ans

Area under the curve:

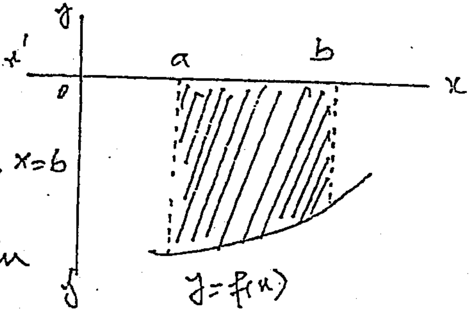
If A is the area which is bounded by curve $y = f(x)$, the x-axis and the lines $x=a$; $x=b$ then

$$A = \int_a^b f(x) \, dx$$



If A is the area which is bounded by curve $y=f(x)$, the x-axis and the lines $x=a, x=b$

then $A = \int_a^b [-f(x)] \, dx$ or $-\int_a^b f(x) dx$



5) Use the definite integral to find out the area between the curve f(x) and the x-axis over the indicated interval [a,b]

a) $f(x) = 4-x^2$, $[0,3]$

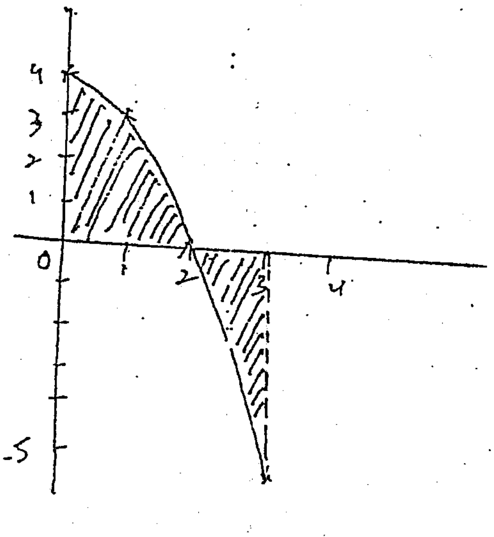
To draw the graph, we have

x	0	1	2	3
$f(x)=4-x^2$	4	3	0	-5

Required shaded Area is

$$A = \int_0^2 f(x)dx + \int_2^3 -f(x) \, dx$$

$$= \int_0^2 (4-x^2) + \int_2^2 -(4-x^2) \, dx$$



$$A = \int_0^2 (4-x^2)dx + \int_2^3 (x^2-4)dx$$

$$= \left[4x - \frac{x^3}{3}\right]_0^2 + \left[\frac{x^3}{3} - 4x\right]_2^3$$

$$= 4(2) - \frac{2^3}{3} - \{0-0\} + \frac{3^3}{3} - 4(3) - \left\{\frac{2^3}{3} - 4(2)\right\}$$

$$= 8 - \frac{8}{3} + \frac{\overset{9}{\cancel{27}}}{\cancel{3}} - 12 - \frac{8}{3} + 8$$

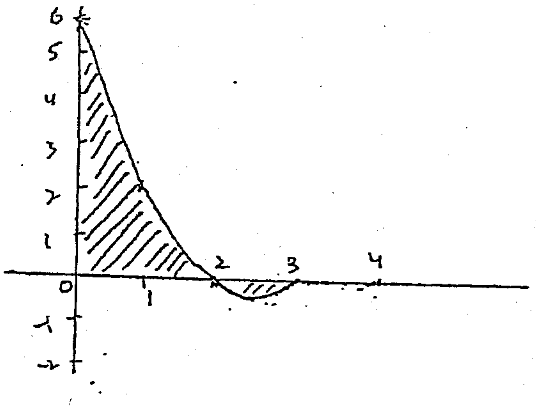
$$= 8 + 9 - 12 + 8 - \frac{8}{3} - \frac{8}{3} = 13 - \frac{16}{3} = \frac{23}{3} \text{ sq: unit}$$

---

b) $f(x) = x^2 - 5x + 6$ $[0,3]$

construct the table

x	0	1	2	2.5	3
$f(x) = x^2-5x+6$	6	2	0	-.25	0



Required area is

$$A = \int_0^2 (x^2-5x+6)dx + \int_2^3 -(x^2-5x+6)dx$$

$$= \int_0^2 (x^2-5x+6)dx - \int_2^3 (x^2-5x+6)dx$$

$$= \left[\frac{x^3}{3} - \frac{5x^2}{2} + 6x\right]_0^2 - \left[\frac{x^3}{3} - \frac{5x^2}{2} + 6x\right]_2^3$$

$$= \frac{2^3}{3} - \frac{5(2)^2}{2} + 6(2) - \left\{\frac{0}{3} - \frac{0}{2} + 0\right\} - \left[\frac{3^3}{3} - \frac{5(3)^2}{2} + 6(3) - \left\{\frac{2^3}{3} - \frac{5(2)^2}{2} + 6(2)\right\}\right]$$

$$= \frac{8}{3} - \frac{\overset{10}{\cancel{20}}}{\cancel{2}} + 12 - 0 - \left[\frac{\overset{9}{\cancel{27}}}{\cancel{3}} - \frac{45}{2} + 18 - \frac{8}{3} + \frac{\overset{10}{\cancel{20}}}{\cancel{2}} - 12\right]$$

$$= \frac{8}{3} - 10 + 12 - 9 + \frac{45}{2} - 18 + \frac{8}{3} - 10 + 12$$

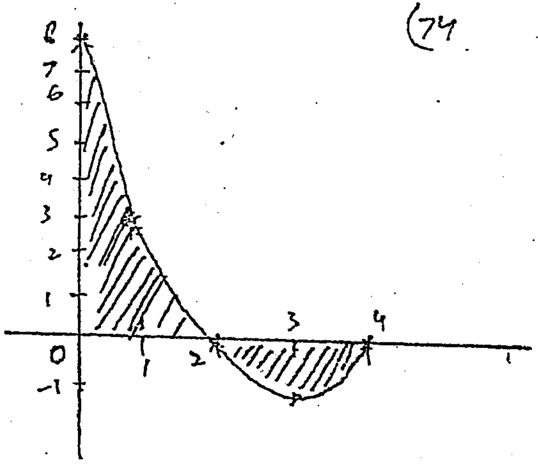
$$= \frac{16}{3} - 23 + \frac{45}{2}$$

$$= \frac{32 - 138 + 135}{6} = \frac{29}{6} \text{ sq: unit}$$

c) $f(x) = x^2 - 6x + 8$ [0, 4]

Construct the table

x	0	1	2	3	4
f(x)	8	3	0	-1	0



Required area is

$$A = \int_0^2 (x^2-6x+8)du + \int_2^4 -(x^2-6x+8)\,dx$$

$$= \int_0^2 (x^2-6x+8)du - \int_2^4 (x^2-6x+8)\,dx$$

$$= \left[\frac{x^3}{3} - \overset{3}{\cancel{6}}\frac{x^2}{\cancel{2}} + 8x\right]_0^2 - \left[\frac{x^3}{3} - \overset{3}{\cancel{6}}\frac{x^2}{\cancel{2}} + 8x\right]_2^4$$

$$= \frac{2^3}{3} - 3(2)^2 + 8(2) - \left\{\frac{0}{3} - 0 + 0\right\} - \left[\frac{4^3}{3} - 3(4)^2 + 8(4) - \left\{\frac{2^3}{3} - 3(2)^2 + 8(2)\right\}\right]$$

$$= \frac{8}{3} - 12 + 16 - 0 - \left[\frac{64}{3} - 48 + 32 - \frac{8}{3} + 12 - 16\right]$$

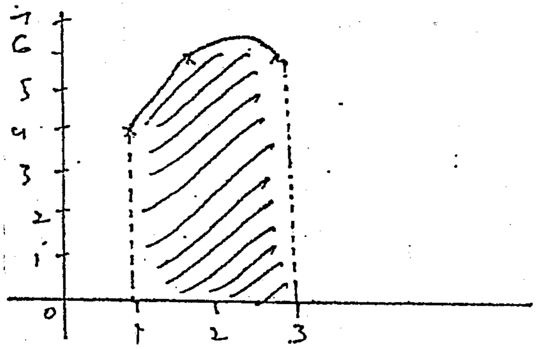
$$= \frac{8}{3} + 4 - \left(\frac{64}{3} - 20 - \frac{8}{3}\right)$$

$$= \frac{8}{3} + 4 - \frac{64}{3} + 20 + 8/3 = 24 - \frac{\overset{16}{\cancel{48}}}{\cancel{3}} = 24 - 16 = 8 \text{ sq. unit}$$

d) $f(x) = 5x - x^2$ [1, 3]

Construct the table

x	1	2	3
$f(x) = 5x - x^2$	4	6	6



Required area is

$$A = \int_1^3 (5x - x^2)\,du$$

$$= \left[5\frac{x^2}{2} - \frac{x^3}{3}\right]_1^3$$

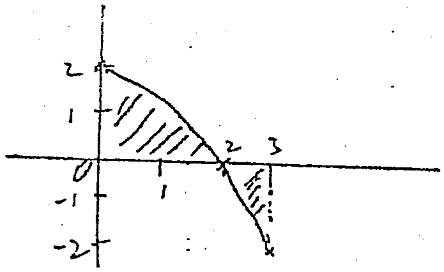
$$= \frac{5(3)^2}{2} - \frac{3^3}{3} - \left\{\frac{5(1)^2}{2} - \frac{1^3}{3}\right\}$$

$$= \frac{45}{2} - \frac{\overset{9}{\cancel{27}}}{\cancel{3}} - \frac{5}{2} + \frac{1}{3} = \frac{\overset{20}{\cancel{40}}}{\cancel{2}} - 9 + 1/3 = 11 + 1/3 = \frac{34}{3} = 11.33 \text{ sq. unit}$$

6) Set up definite integrals in Problems a to d that represents the indicated shaded area.

Ch-5

a) From the figure definite integral for the shaded area is



$$A = \int_0^2 (2-0.5x^2)\,dx + \int_2^3 -(2-0.5x^2)\,dx$$

$$= \int_0^2 \left(2 - \frac{1}{2}x^2\right) dx - \int_2^3 \left(2 - \frac{1}{2}x^2\right) dx$$

$$= \left[2x - \frac{1}{2}\frac{x^3}{3}\right]_0^2 - \left\{2x - \frac{1}{2}\cdot\frac{x^3}{3}\right]_2^3$$

$$= \left[2x - \frac{x^3}{6}\right]_0^2 - \left[2x - \frac{x^3}{6}\right]_2^3$$

$$= 2(2) - \frac{2^3}{6} - \{0-0\} - \left[2(3) - \frac{3^3}{6} - \left\{2(2) - \frac{2^3}{6}\right\}\right]$$

$$= 4 - \frac{\overset{4}{\cancel{8}}}{\cancel{6}_3} - 0 - \left[6 - \frac{\overset{9}{\cancel{27}}}{\cancel{6}_2} - 4 + \frac{\overset{4}{\cancel{8}}}{\cancel{6}_3}\right]$$

$$= 4 - \frac{4}{3} - \left[2 - \frac{9}{2} + \frac{4}{3}\right]$$

$$= 4 - \frac{4}{3} - 2 + \frac{9}{2} - \frac{4}{3}$$

$$= 2 - \frac{8}{3} + \frac{9}{2} = \frac{12-16+27}{6} = \frac{23}{6} \text{ sq. unit}$$

---

b) From the figure, definite integral for the shaded area is

$$A = \int_{-1}^{0} (x^2-2x)\,dx + \int_0^2 -(x^2-2x)\,dx$$

$$= \int_{-1}^{0} (x^2-2x)\,dx - \int_0^2 (x^2-2x)\,dx$$

$$= \left[\frac{x^3}{3} - \frac{\cancel{2}x^2}{\cancel{2}}\right]_{-1}^{0} - \left[\frac{x^3}{3} - \frac{\cancel{2}x^2}{\cancel{2}}\right]_0^2$$

260

$$A = \left[\frac{x^3}{3} - x^2\right]_{-1}^{0} - \left[\frac{x^3}{3} - x^2\right]_{0}^{2}$$

$$= \frac{0}{3} - 0 - \left\{\frac{(-1)^3}{3} - (-1)^2\right\} - \left[\frac{2^3}{3} - 2^2 - \left\{\frac{0}{3} - 0\right\}\right]$$

$$= 0 - 0 - \left\{-\frac{1}{3} - 1\right\} - \left[\frac{8}{3} - 4 - 0\right]$$

$$= +\frac{1}{3} + 1 - \frac{8}{3} + 4 = -\frac{7}{3} + 5 = \frac{8}{3} \text{ sq: unit}$$

---

c) From figure, definite integral for the shaded Area is

$$A = \int_1^2 \frac{e^2 - e^x}{2}\, du + \int_2^3 \left(\frac{e^2 - e^x}{2}\right) du$$

$$= \int_1^2 \frac{e^2 - e^x}{2}\, du - \int_2^3 \frac{e^2 - e^x}{2}\, du$$

$$= \frac{1}{2}\int_1^2 (e^2 - e^x)\, du - \frac{1}{2}\int_2^3 (e^2 - e^x)\, du$$

$$= \frac{1}{2}\left[xe^2 - e^x\right]_1^2 - \frac{1}{2}\left[xe^2 - e^x\right]_2^3$$

$$= \frac{1}{2}\left[\underline{2e^2 - e^2} - \{1.e^2 - e^1\}\right] - \frac{1}{2}\left[3e^2 - e^3 - \{\underline{2e^2 - e^2}\}\right]$$

$$= \frac{1}{2}\left[\cancel{e^2} - \cancel{e^2} + e\right] - \frac{1}{2}\left[3e^2 - e^3 - e^2\right]$$

$$= \frac{1}{2}\left[e\right] - \frac{1}{2}\left[2e^2 - e^3\right]$$

$$= \frac{1}{2}\left[e - 2e^2 + e^3\right]$$

$$= \frac{1}{2}\left[2.718 - 2(2.718)^2 + (2.718)^3\right]$$

$$= \frac{1}{2}\left[2.718 - 14.775 + 20.079\right] = \frac{1}{2}\left[8.022\right] = 4.011 \text{ sq unit}$$

---

n) From figure, definite integral for the shaded area is

$$A = \int_0^1 -(e^x - e)\,dx + \int_1^2 (e^x - e)\,dx$$

$$= \int_0^1 (e - e^x)\,dx + \int_1^2 (e^x - e)\,dx$$

$$= \left[xe - e^x\right]_0^1 + \left[e^x - ex\right]_1^2$$

$$= 1\cdot e - e^1 - \{0 - e^0\} + e^2 - e(2) - \{\cancel{e^1} - \cancel{e\cdot 1}\}$$

$$= \cancel{e} - \cancel{e} - \{0 - 1\} + e^2 - 2e - 0$$

$(e \approx 2.718\ldots$

$$= +1 + e^2 - 2e = 1 + 7.389 - 5.437 = 2.952 \text{ sq unit}$$

---

7) An oil tanker is leaking oil at a rate given in barrels per hour by

$$\frac{dL}{dt} = \frac{80\ln(t+1)}{t+1}$$

integrate w.r.t. t

$$\int\left(\frac{dL}{dt}\right)dt = \int 80\frac{\ln(t+1)}{t+1}\,dt$$

$$L = 80\int \ln(t+1)\cdot\frac{1}{t+1}\,dt$$

put $\ln(t+1) = u$

Diff: w.r.t. t

$$\frac{1}{t+1} = \frac{du}{dt}$$

$$\Rightarrow \frac{1}{t+1}dt = du$$

$$= 80\int u\,du$$

$$= 80\frac{u^2}{2} + c$$

$$= \overset{40}{\cancel{80}}\frac{[\ln(t+1)]^2}{\cancel{2}} + c$$

$$L = 40[\ln(t+1)]^2 + c \text{ ——— } ①$$

Initially $t = 0,\ L = 0$ put in ①

$$0 = 40[\ln(0+1)]^2 + c$$

$$0 = 40(0)^2 + c \implies \boxed{c = 0} \text{ put in } ①$$

$L = 40[\ln(t+1)]^2$. ——— ②

a) Leak on the first day = ?

put $t = 24$ in ②

$L = 40[\ln(24+1)]^2$

$L = 40(\ln 25)^2 = 40(3.2189)^2 = 40(10.3613)$

$\boxed{L = 414.45}$

The total no: of barrels that the ship will leak on the first day = About 414 barrels.

b) Leak on the second day = ?

put $t = 48$ in ②

$L = 40[\ln(48+1)]^2$

$= 40(\ln 49)^2 = 40(3.8918)^2 = 40(15.1461)$

$L = 605.84$

This leak is for two days

Now leak on the second day = $605.84 - 414.45$

$= 191.39$

Hence total no: of barrels that the ship will leak on the 2nd day = About 191 barrels

c) Over the long run the amount of oil leaked Per day decreasing to zero.

END OF CH # 5