ON STOLARSKY AND RELATED MEANS

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Abstract. We give a simple proof of the Stolarsky means inequality as well as some related inequalities for similar means of Stolarsky type.

1. Introduction and Preliminaries

Let us consider the following means

$$\begin{aligned} E(x,y;r,s) &= \left\{\tfrac{r(y^s-x^s)}{s(y^r-x^r)}\right\}^{\frac{1}{s-r}} \\ E(x,y;r,0) &= E(0,r) = \left\{\tfrac{y^r-x^r}{r(\ln y-\ln x)}\right\}^{1/r} \\ E(x,y;r,r) &= e^{-\frac{1}{r}}\left(\frac{x^{x^r}}{y^{y^r}}\right)^{1/(x^r-y^r)} \\ E(x,y;0,0) &= \sqrt{xy}, \end{aligned}$$

where x and y are positive real numbers $x \neq y$, r and s are any real numbers but 0.
These means, known in literature, are called Stolarsky means. Namely Stolarsky[1] in 1975 (see also [2, p.120]) introduced these means. Stolarsky proved that the function $E(r,s)$ is increasing in both r and s i.e. for $r \leq u$ and $s \leq v$, we have

$$E(x,y;r,s) \leq E(x,y;u,v). \tag{1}$$

In this paper, first we shall give a simple proof of inequality (1). Further we shall introduce two new classes of means of Stolarsky type.

2. A Simple Proof of Stolarsky Means Inequality

Note that $E(r,s)$ is continuous, this means it is enough to prove (1) in the case where $r,s,u,v \neq 0$, $r \neq s$ and $u \neq v$.

Key words and phrases. convex function, log-convex function, Stolarsky means.

We consider the following function

$$f(x) = p^2\varphi_r(x) + 2pq\varphi_t(x) + q^2\varphi_s(x) \qquad \text{where } t = \frac{r+s}{2} \text{ and } p, q \in \mathbb{R},$$

and

$$\varphi_r(x) = \begin{cases} x^r/r, & r \neq 0; \\ \ln x, & r = 0. \end{cases}$$

Now

$$\begin{aligned} f'(x) &= p^2x^{r-1} + 2pqx^{t-1} + q^2x^{s-1} \\ &= \left(px^{(r-1)/2} + qx^{(s-1)/2}\right)^2 \geq 0. \end{aligned}$$

This implies f is monotonically increasing. So for $x \neq y$

$$\frac{f(x) - f(y)}{x - y} \geq 0,$$

i.e.

$$p^2\frac{\varphi_r(x) - \varphi_r(y)}{x - y} + 2pq\frac{\varphi_t(x) - \varphi_t(y)}{x - y} + q^2\frac{\varphi_s(x) - \varphi_s(y)}{x - y} \geq 0.$$

Let

$$\phi(r) = \frac{\varphi_r(x) - \varphi_r(y)}{x - y},$$

then

$$p^2\phi(r) + 2pq\phi(t) + q^2\phi(s) \geq 0$$

i.e.

$$\phi^2(t) \leq \phi(r) \cdot \phi(s) \qquad \text{where} \quad t = \frac{r+s}{2}.$$

This implies ϕ is log-convex in Jensen sense.
Also $\lim_{r\to 0} \phi(r) = \phi(0)$, which implies ϕ is continuous for all $r \in \mathbb{R}$. And therefore log-convex.
We need following lemma which proof can be found in [2].

Lemma 2.1. *Let f be log-convex function and if, $x_1 \leq y_1$, $x_2 \leq y_2$, $x_1 \neq x_2$, $y_1 \neq y_2$, then the following inequality is valid:*

$$\left(\frac{f(x_2)}{f(x_1)}\right)^{1/(x_2-x_1)} \leq \left(\frac{f(y_2)}{f(y_1)}\right)^{1/(y_2-y_1)}. \tag{2}$$

Applying Lemma 2.1 for $f = \phi$, (let $r, s, u, v \neq 0$) we get an inequality

$$\left\{\frac{r\left(y^s - x^s\right)}{s\left(y^r - x^r\right)}\right\}^{1/(s-r)} \leq \left\{\frac{u\left(y^v - x^v\right)}{v\left(y^u - x^u\right)}\right\}^{1/(v-u)}.$$

Since $E(r, s)$ is continuous, we have (1).

Conclusion

In the literature, many researchers have published so many results on different major generalizations of convex function. Many authors today focus on interval-valued functions, which is known as the (h, m)-convex interval-valued function. Additionally, we give the rigorous proof of the famous Hermite-Hadamard type inequality for m-convex in interval-valued.

References

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