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Fluid mechanics:-

A branch of mechanics in which we deals with the study of fluid at rest or in motion is Called fluid mechanics.

Fluid mechanics

Fluid statics | Fluid kinematics | Fluid dynamics.

*) Fluid Statics deals with fluids at rest.

*) Fluid Kinematics deals with fluids in motion without discussing the Cause of motion.

*) Fluid dynamics deals with fluids in motion also discussing the forces acting on fluid.

why fluid mechanics ?

Knowledge and understanding of the basic principles of fluid mechanics are essential to analyse any system in which a fluid is the working medium.

*) we find fluid everywhere ; it is in our body ; in atmosphere ; in our rooms. A large portion of earth's surface and othe entire universe is in the fluid state

*) The designe of all types of fluid machinary including pumps ; fans; blowers and turbines clearly requires knowledge of the basic principles of fluid mechanics.

*) The circulatory system of our body is essentialy a fluid system.

*) Heating and ventilating system for our homes.

*) Movement of ships through water.

*) Airplanes fly in the air and air flows around wind machines.

so; the basic knowledge of fluid mechanics is necessary in every field of Science.

Fluid:-

Fluids are subtances that capable of flowing and conform to the shape of containing vessels.

or more precisely;

"A fluid is a substance that deformes Continuouly under the action of shear (tangential) stress; no matter how small the shear stress may be."

Fluids are usually divided into two groups liquids and gases. Liquids and gases behave in much the Same way; some specific differences are:

i) A liquid is difficult to compress and often regarded as being incompressible. A gas is easily to compress and usually treated as compressible.

ii) A given mass of liquid occupies a given volume and will form a free space. A gas has no fixed Volume it changes volume to expand to fill the containing vessel.

Pressure:- The magnitude of force per unit area exerted in a direction normal to that area.

$$P = F/A$$

Density:- Mass per unit volume is Called density or mass density i.e $\rho = \frac{m}{V}$

Specific weight:- weight per unit Volume is Called specific weight. $\gamma = \frac{W}{V} = \frac{mg}{V} = \rho g$

Specific volume:- The volume occupied by a unit mass of the fluid.

$$V_s = \frac{1}{\rho}$$

Specific gravity:- The specific gravity of a liquid (gas) is the ratio of the weight of the liquid (gas) to the the weight of an equal volume of water (air) at a standard temperature.

$$\text{specific gravity} = \frac{\text{weight of substance}}{\text{weight of equal volum of water}}$$

$$= \frac{\text{specific weight of substance}}{\text{specific weight of water}}$$

$$\text{specific gravity} = \frac{\text{density of substance}}{\text{density of water}}$$

Note:-
standard temperature of water is taken as 4°C while that of air is taken as 0°C.

Temperature:-
A measure of the intensity of heat is called temperature. It is a measure of average translational K.E associated with atoms and molecules of the fluid. Physical state of a substance changes with temperature.

Note that we can determine the state of of a moving fluid completely with the help of five quantities.

i) Three components of velocity $\vec{V}(x, y, z)$
ii) pressure P
iii) density ρ

Basic laws:-
The basic laws; which are applicable to any fluid are;

1) conservation of mass.
2) Newton's 2nd law of motion.
3) The principle of angular momentum.
4) The 1st law of thermodynamics.
5) The 2nd law of thermodynamics.

Note that all the basic laws are the same as those used in mechanics and thermodynamics; our task is to formulate these laws in suitable forms to solve fluid flow problems.

Methods of Analysis:-
The 1st step in solving a problem is to define the system that you are attempting to analyze.

→ In mechanics; we use free body diagrame.
→ In thermodynamics; we use closed or open system.
→ In fluid mechanics; we will use a system or a control volume.

System:- A system is defined as the fixed quantity of mass at rest or in motion; confined in a region of space and bounded by real or imaginary geometric boundries. The boundries may be fixed or movable but no mass crosses them.

Surroundings:- The region of physical space beyond the boundries of the system is called its surroundings.

Control Volume:- Control volume is an arbitrary volume in space through which fluid flows.

Control Surface:- Geometric boundries of the control volume is called a control surface. It may be real or imaginary; at rest or in motion.

Types of Control Volume:- In the analysis of fluid flow; there are two types of control volume:

i) Finite size control volume.
ii) Differential size control volume.

Finite size control volume is further divided into two types.

i) **Deformable control volume**:- In which the control surface is allowed to change its shape.

ii) **Non-deformable**:- In which the original shape of control surface remain unaltered.

Macroscopic system:- The word macroscopic refers to a quantity or a system large enough to be visible to the naked eye.

Microscopic system:- The word microscopic refers to a quantity or a system so small to be invisible with ou microscope.

Fluid as a Continuum:- Continuum means; a continuous distribution of matter with no empty spaces. Fluid can be treated as continuum.

SI System:-
In this system Mass [M], length [L] time [t], and temperature [T] are the primary dimensions.

Brithish System:-
In this system force [F], length [L] time [t], and temperature [T] are the primary dimensions.

English Engineering System:-
In this system; Force [F], mass [M] length [L], time [t] and temperature [T] are the primary dimensions.

Note:- force is a secondary dimension in SI system and its dimension is is;

$$[F] = \frac{[M][L]}{[t][t]} = [MLt^{-2}]$$

whereas in B.G system mass is a 2ndry dimension and;

$$[M] = \frac{[F][t^2]}{[L]}$$

Dimension	SI (unit)	B.G (unit)	Conversion
Mass [M]	kg	slug	1slug = 14.5939kg
length [L]	meter(m)	foot	1ft = 0.3048m
Time [t]	Second(s)	Second(s)	
Temperature [T]	Kelvin (K)	Rankine(°R)	1K = 1.8 °R

System of units:-
There are many ways available for selecting the units for each primary dimension.

MLtT:-
SI is an extension and refinement of the traditional metric system.
In the SI system of units
The unit of mass is kilogram (kg)
The unit of length is the meter (m)

Dimensions and units:-

units:- units are the arbitrary names (and mgnitudes) assigned to a quantity adopted as standards for measurement.

The quantitatively measurement of a fundamental quantity means to compare it with some standard quantity. The standard quantities in tems of which the fundamental quantities are measured are Called the fundamental units for those quantities.

Dimension:-

Dimension is used to refer any measurable quantity. A Dimension is the measure by which a physical variable is expressed quantatively.

In any particular system of dimensions; all measurable quantities can divided into two types:

Primary quantities:-

Primary quantities are those for which we set arbitrary scales of measure.

Generally; in fluid mechanics there are only four primary dimensions from which all other dimensions can be derived; mass, length, time and temperature.

Secondry quantities:-

On the other hand; secondry quantities are those whose dimensions are expressible in tems of the dimensions of the primary quantities e.g area; Volume; velocity; acceleration etc.

System of dimensions:-

Any valid eq that relates physical quantitres must be dimensionally homogeneous i.e each term in the eq must have same dimension.

we have three basic systems of dimensions corresponding to the different ways of specifying the primary dimensions:

The unit of time is second (S)
The unit of temperature is Kelvin (K)

Force as a 2ndry dimension has units newton (N) given by

$$1N = 1kg\,m/sec^2$$

In the absolute metric system of units;
The unit of mass is the gram.
The unit of length is the centrimeter.
The unit of time is the second.
The unit of temperature is the kelvin.

The unit of force in this system is; the dyne; given by;

$$1\,dyne = 1g\,cm/s^2$$

FLtT:-

In the British Gravitational system of units;
The unit of force is the pound (lbf)
Then unit of length is the foot (ft)
The unit of time is the second (s)
and the unit of temperature is the degree Rankine (°R)
Mass as a 2ndry dimension; has units Called slug; given as

$$1\,slug = 1\,lbf\cdot s^2/ft$$

FLMtT:-

In the English Engineering system of units;
unit of force is pound (lbf)
unit of mass is pound mass (lbm)
unit of length is foot (ft)
unit of time is second (s)
and unit of temperature is degree Rankine (°R)

Q:- A body weights 1000lbf when exposed to a standard earth gravity $g = 32.174\ ft/s^2$.

a) what is its mass in kg ?

b) what will the weight of this body be in N if it is exposed to the moon's standard acceleration $g_m = 1.62\ m/s^2$?

c) How fast will the body accelerates if a net force of 400lbf is applied to it on the moon or on the earth ?

Sol:-

a) $W = mg$

$1000\,lbf = m\,(32.174\ ft/s^2)$

$m = \dfrac{1000\ lbf}{32.174\ ft/s^2} = 31.08\ slugs.$

$m = 31.08 \times 14.5939\ kg = 554\ kg$

b) $W = mg_m = 454 \times 1.62 = 735\,N$

c) $F = 400\,lbf$

$ma = 400\,lbf$

$a = \dfrac{400\ lbf}{31.08\ slugs} = 12.87\ ft/s^2 = 3.92\ m/s^2$

Some Conversion factors:-

Length	$1\,in = 0.0254\,m$;	$1\,ft = 0.3048\,m$; $1\,mile = 5280\,ft$
Mass	$1\,lbm = 0.4536\,kg$;	$1\,slug = 14.59\,kg$
Force	$1\,lbf = 4.448\,N$	
Area	$1\,acre = 4047\,m^2$	
Volume	$1\,gal = 231\,in^3$;	$1\,gal = 3.785\,L$

Q:- Express mass and weight of 510g in SI, BG and EE units.

Sol:-

mass in SI unit

$$m = 510g = \frac{510}{1000} kg = 0.51 kg$$

mass in BG;

$$m = \frac{0.51}{14.59} slug = 0.0349 slug$$

mass in EE;

$$m = \frac{0.51}{0.4536} lbm = 1.12 lbm$$

Now; To find weight; we use

$$w = mg$$

In SI unit;

$$w = (0.51)(9.81) = 5N$$

In BG system;

$$w = (0.0349)(32.2) = 1.12 lbf$$

In EE units:-

$$w = mg/g_c$$

$$w = \frac{1.12 \times 32.2}{32.2} = 1.12 lbf$$

Q:- An early viscosity unit in the cgs system is the poise; or $g/cm \cdot s$ name after J.L.M. Poiseuille, a French physician who performed pioneering experiments in 1840 on water flow in pipes. The kinematic viscosity (ν) unit is the stokes, named after G.G stoke; a British physicist who in 1845 helped develop the basic differential eq/s of fluid m

1 stokes = $1 cm^2/s$.

water at 20°C has $\mu = 0.01$ poise and also $\nu = 0.01$ stokes. Express these results in a) SI and b) BG units.

Sol:- In SI units:-

$$\mu = 0.01\,P = 0.01\,\frac{g}{cm\cdot s}$$

$$\mu = 0.01 \times \frac{10^{-3}\,kg}{10^{-2}\,m\cdot s} = 0.001\,\frac{kg}{m\cdot s}$$

and

$$\nu = 0.01\,\text{stokes} = 0.01\,\frac{cm^2}{s}$$

$$\nu = 0.01\,\frac{10^{-4}\,m^2}{s} = 0.000001\,m^2/s$$

In BG units

$$\mu = 0.001\,\frac{kg}{m\cdot s}$$

$$\mu = 0.001 \times \frac{\frac{1}{14.59}\,slug}{\frac{1}{0.3048}\,ft\cdot s} = 0.00002089\,\frac{slug}{ft\cdot s}$$

and

$$\nu = 0.000001\,m^2/s = 0.000001\,\frac{\left(\frac{1}{0.3048}\right)^2 ft^2}{s}$$

$$\nu = 0.0000108\,ft^2/s$$

Q:- A useful theoretical eq for computing the relation b/w pressure; velocity and altitude in a steady flow of a nearly inviscid; nearly incompressible fluid with negligible heat transfor and shaft work is the Bernoulli relation; named after Daniel Bernoulli; who published a hydrodynamics textbook in 1738.

$$P_o = p + \frac{1}{2}\rho v^2 + \rho g z$$

where P_o = stagnation pressure

p = pressure in moving fluid

v = velocity

ρ = density

g = gravitational acceleration.

a) show that this eq satisfies the principle of dimensional homogeneity.

b) show that consistent units result without additional conversion factor in SI units.

c) rpeat (b) for B.G units.

Sol:-

a) $[P_o] = \frac{[F]}{[A]} = \frac{[M][LT^{-2}]}{[L^2]} = [ML^{-1}T^{-2}]$

Now;

$[ML^{-1}T^{-2}] = [ML^{-1}T^{-2}] + [ML^{-3}][L^2T^{-2}] + [ML^{-3}][LT^{-2}][L]$

$= [ML^{-1}T^{-2}] + [ML^{-1}T^{-2}] + [ML^{-1}T^{-2}]$

$= [ML^{-1}T^{-2}]$ for all terms.

b) Enter SI units for each quantity.

$N/m^2 = N/m^2 + \frac{Kg}{m^3} \cdot m^2/s^2 + \frac{Kg}{m^3} \cdot \frac{m}{s^2} \cdot m$

$= \frac{N}{m^2} + \frac{Kg}{m\,s^2}$

$= \frac{N}{m^2} + \frac{Kg\,m}{s^2} \cdot \frac{1}{m^2}$; $1N = kg\,m/s^2$

$= \frac{N}{m^2} + \frac{N}{m^2}$

$= \frac{N}{m^2}$

Thus all terms in Bernollis eq will have units of pascals ; Newton per squore meter ; when SI units are used, No conversion factors are needed ; which is true of all theoretical eq/s in fluid Mechanics.

c) Introducing BG units for each term ;

$\frac{lbf}{ft^2} = \frac{lbf}{ft^2} + \frac{slug}{ft^3} \cdot \frac{ft^2}{s^2} + \frac{slug}{ft^2} \cdot \frac{ft}{s^2}\,ft$

$= \frac{lbf}{ft^2} + \frac{slug}{ft \cdot s^2}$; $1\,slug = \frac{lbf\,s^2}{ft}$

$= \frac{lbf}{ft^2} + \frac{lbf\,s^2}{ft^2 \cdot s^2}$

$= \frac{lbf}{ft^2}$

All terms have the unit of pounds per square foot. No conversion factors are needed in B.G system

Compressibility and Bulk modulus:-

The compressibility of a fluid is a measure of the change of its volume under the action of external forces.

The compressibility of a fluid is expressed by its bulk modulus of elasticity. If the pressure p increased to $p+\Delta p$ then volume V decreased to $V-\Delta V$; since an increase in pressure always causes a decrease in volume.

Then the bulk modulus of elasticity is defined as;

$$K = -\frac{\Delta p}{\Delta V/V} = -\frac{\text{change in pressure}}{\text{volumetric strain}} \rightarrow (1)$$

in the limiting case $\Delta V \rightarrow 0$;

$$(1) \Rightarrow \quad K = -\frac{dp}{\frac{dV}{V}} = -V\frac{dp}{dV} \rightarrow (2)$$

in terms of density;

$$\rho = \frac{m}{V} \Rightarrow d\rho = -\frac{m}{V^2}dV = -\frac{\rho}{V}dV$$

$$\Rightarrow \frac{d\rho}{\rho} = -\frac{dV}{V}$$

so; eq (2); $\Rightarrow$

$$K = \frac{dp}{\frac{d\rho}{\rho}} = \rho\frac{dp}{d\rho}$$

Q:- When an increase in pressure of 30Mpa results in 1% decrease in volume of water; what is its bulk modulus of elasticity?

Sol:- Here $\Delta p = 30\,Mpa = 30\times10^6\,pa$

and $\Delta V = -1\%V = -\frac{V}{100} = -0.01V$

Now;

$$K = -\frac{\Delta p}{+\frac{\Delta V}{V}} = \frac{30\times10^6}{0.01} = 30\times10^8$$

$$K = 3\times10^9\,pa = 3\,Gpa.$$

Flow:- A material goes under deformation when different forces act upon it. If the deformation Continuously increases without limit; then the phenomenon is called flow.

There are many types of flow. Some of these are

1) **Uniform flow:-** A flow is said to be uniform when the velocity vector as well as other fluid properties do not change from pt to pt. Thus

$$\frac{\partial \vec{v}}{\partial s} = 0 \; ; \quad \frac{\partial \rho}{\partial s} = 0 ; \quad \frac{\partial P}{\partial s} = 0 \; --- \; \text{etc.}$$

i.e the partial derivative w.r.t "distance" of any quantity vanishes.

Example:- Flow of a liquid through a long straight pipe of constant diameter is a uniform flow.

2) **Non-uniform flow:-** A flow is said to be non-uniform if its velocity and other properties change from pt to pt in the fluid flow.

$$\text{i.e} \quad \frac{\partial \vec{v}}{\partial s} \neq 0$$

Example:- A liquid through a pipe of reducing section or through a curved pipe is a non-uniform flow.

3) **Laminar flow:-**

A flow in which each liquid particle has a definite path and the paths of individual particles do not cross each other is called the laminar flow.

Example:- The flow of high-viscosity fluids such as oils at low velocities is typically laminar.

4) **Turbulent flow:-**

A flow is said to be turbulent if it is not laminar. In other words; If the particles of the fluid move in an irregular fasion in all directions then the flow is said to turbulent.

Example:- The flow of low-viscosity fluid such as air at high velocities is typically turbulent.

5) **Steady flow:-**

A flow in which quantity of fluid flowing per second is constant. In other word; If the velocity vector and other fluid properties at every pt. in a fluid do not change with time; then the flow is said to be steady or stationary flow.

i.e $\frac{\partial \vec{V}}{\partial t} = 0$; $\frac{\partial \rho}{\partial t} = 0$; $\frac{\partial P}{\partial t} = 0$ -- etc.

Example:- The flow of water in a pipe of constent diameter at constent velocity is steady flow.

6) **Usteady flow:-**

A flow is said to be unsteady when fluid properties and conditions at any pt. in a fluid change with time. i.e $\frac{\partial \vec{V}}{\partial t} = 0$ etc.

Example:-

water being pumped through a fixed pipe at an increasing rate is an example of unsteady flow.

(7) **Compressible flow:-**

A flow in which the volume and thus the density of the flowing fluid changes during the flow. All the gases are considered to have compressible flow.

(8) **Incompressible flow:-** A flow in which the volume and thus the density of the flowing fluid does not change during the flow. Generally; all the liquids are considered to have incompressible flow.

(9) **Rotational flow:-** A flow in which the fluid partrcles rotate about their own axes during the flow.

so; the condition for rotational flow is;

$$\nabla \times \vec{V} \neq 0$$

(10) **Irrotational flow:-** A flow in which the fluid particles do not rotate about their own axes during the flow. Condition for this flow is;

$$\nabla \times \vec{V} = 0$$

(11) **1-Dimensional flow:-** A flow whose streamline may be represented by a straight line. It is because of the reason that a straight streamline; being a mathematical line; possesses one dimension only.
Example:- the flow in pipes and channels is 1-D flow.

(12) **2-Dimensional flow:-** A flow whose streamline may be represented by a curve; It is because of the reason that a curved streamline will be long two mutually ⊥ lines.
Example:- the flow b/w two non-parallel plates is 2-D flow.

(14) **3-D flow:-** A flow whose streamline may be represented in space.
Example:- The flow of water from a hole located in the bottom side of a tank is 3D-flow.

(15) **Baratropic flow:-** A flow is said to be baratropic when the pressure is a fn. of density alone.

Types of flow lines:-

Path lines:- The path or trajectory followed by a fluid in motion is called a pathline. Thus the pathline shows the direction of a particle; for a certain period of time or b/w two sections.

Streamlines:- The imaginary line drawn in the fluid in such a manner that the tangent to which at any point gives the direction of motion at that point is called streamline.
Thus the streamline shows the direction of motion of a number of particles at the same time.

Streamtube:-
An element of fluid; bounded by a number of streamlines; which confine the flow; is called stream tube. Since there is no movement of fluid across the streamline; therefore; no fluid can enter or leave the stream tube except at

the ends. It is thus obvious that the stream tube behaves as a solid tube.

Streaklines / filament lines:-

A streakline is a line consisting of all those fluid particles that have passed through a fixed pt in the flow field at some earlier instant. e.g the line formed by smoke particles ejected from a nozzle is a streakline.

Timelines:-

A time line is a set of fluid particles that form a line in a given flow field at a known instant of time. At later times both the shape and location of the timeline generally have changed. If a number of adjacent fluid particles in a flow field are marked at a given instant they form a line in the fluid at that instant and is called a time line.

Note:- In a steady flow all these lines are identical.

Force and its types:-

"An agent which brings or tends to bring a change in the state of a body is called force."

At a given instant of time there are many types of forces acting on the body. Forces are classified in a number of ways; but we here will focus on a very simple classification of forces. From the fluid mechanics pt of view; there are two types of forces:

i) surface force · ii) body force.

Surface force:-

Surface forces include all forces acting on the boundries of the medium through direct contact. These forces act only at the surface of the fluid. i.e pressure is an example of surface force.

Body forces:- Forces developed without physical contact and distributed over the volume of the fluid are termed as body forces e.g

Gravitational and electromagnetic forces are body forces.

Concept of field:- The term field refere to a scalar, vector or tensor quantity described by continuous fns. of time and space coordinates and is based on the concept of continuum.

Examples:- velocity field; temperature field; stress field air field; density field etc.

Stress:-

stress is defined as;

"Force per unit area is called stress."

i.e $\text{stress} = \dfrac{\text{force}}{\text{area}}$

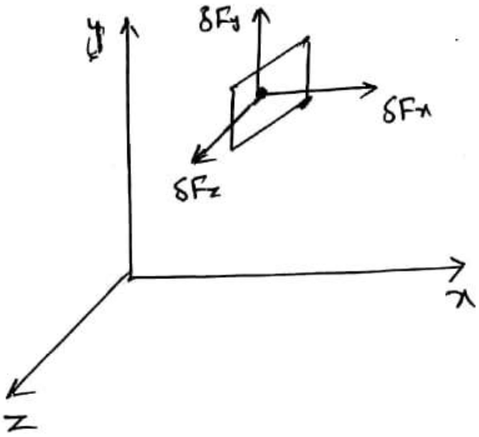
stress is a surface force and stress field has nine components and behaves as a 2nd order tensor. Thus stress field is a tensor field.

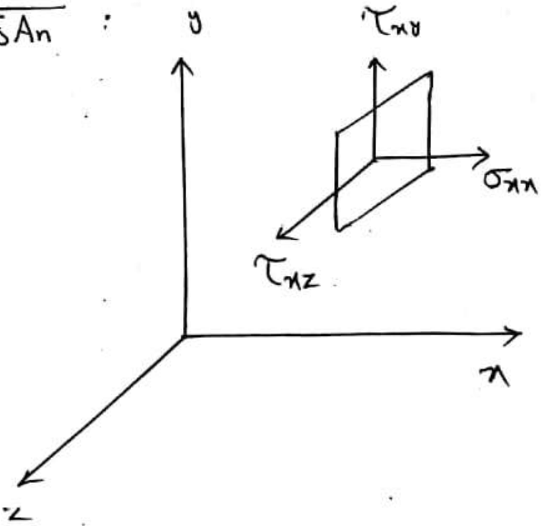
Normal stress:-

$$\sigma_n = \lim_{\delta A_n \to 0} \frac{\delta F_n}{\delta A_n}$$

shear (tangential) stress:-

$$\tau_n = \lim_{\delta A_n \to 0} \frac{\delta F_t}{\delta A_n}$$

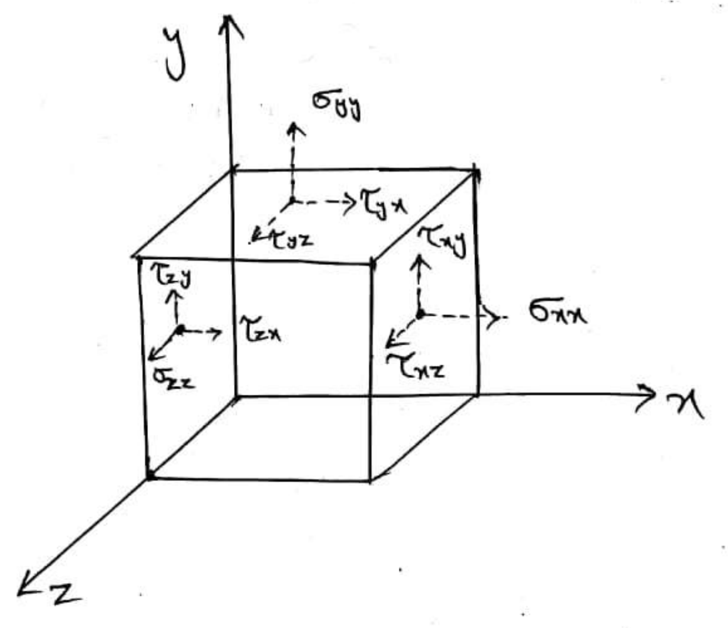




So; we have used a double subscript notation to label the stress. The 1st subscript indicates the plane|surface on which the stress act. The 2nd subscript indicates the direction in which the stress act.

The state of stress at a point can be described completely by specifying the stresses acting on three mutually ⊥ planes through the point;

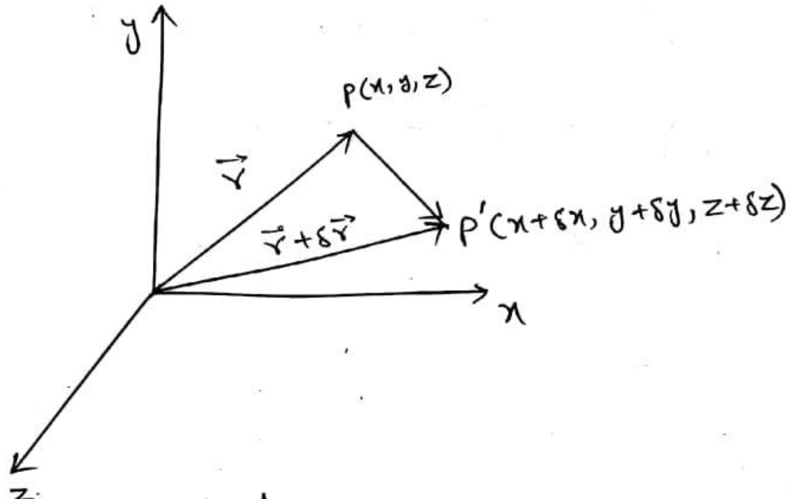
$$\tau = \begin{pmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{pmatrix}$$



Available at www.mathcity.org

Velocity of fluid at a point:-

Consider that at any time t a fluid particle is a pt $P(x,y,z)$ where $\vec{OP} = \vec{r}$ and after time δt the particle reaches a pt. P' such that $\vec{OP'} = \vec{r} + \delta\vec{r}$ at $t+\delta t$. Then in time δt particle is displaced through $\delta\vec{r}$;



Therefore; the avg velocity is given as;

$$\vec{V}_{avg} = \frac{\delta\vec{r}}{\delta t}$$

So that

$$\lim_{\delta t\to 0} \vec{V}_{avg} = \lim_{\delta t\to 0} \frac{\delta\vec{r}}{\delta t}$$

$$\vec{V} = \frac{d\vec{r}}{dt}$$

This expression gives the velocity of particle at point P; clearly; in general $\vec{V}$ depends on $\vec{r}$ as well as t;

$$\vec{V} = \vec{V}(\vec{r}, t)$$

If the pt P has coordinates (x,y,z) w.r.t a fixed frame of reference; then $\vec{V} = \vec{V}(x,y,z,t)$

Let us further assume that the Cartesian coordinates of $\vec{V}$ are u, v, w; Then

$$\vec{V} = [u, v, w]$$

$$\text{or } \vec{V} = u\hat{i} + v\hat{j} + w\hat{k}$$

Since $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

So; $\dfrac{d\vec{r}}{dt} = \dfrac{dx}{dt}\hat{i} + \dfrac{dy}{dt}\hat{j} + \dfrac{dz}{dt}\hat{k}$

so; in components form; $u = \dfrac{dx}{dt}$; $v = \dfrac{dy}{dt}$; $w = \dfrac{dz}{dt}$

Material derivative:-

Let $H = H(x,y,z,t)$ be any fluid property of the fluid; Now,

$$\frac{dH}{dt} = \frac{\partial H}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial H}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial H}{\partial z}\cdot\frac{dz}{dt} + \frac{\partial H}{\partial t}$$

$$\frac{dH}{dt} = \left(\frac{\partial H}{\partial x}\hat{i} + \frac{\partial H}{\partial y}\hat{j} + \frac{\partial H}{\partial z}\hat{k}\right)\cdot\left(\frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}\right) + \frac{\partial H}{\partial t}$$

$$\frac{dH}{dt} = \nabla H\cdot\vec{V} + \frac{\partial H}{\partial t}$$

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} + \vec{V}\cdot\nabla H$$

$$\frac{dH}{dt} = \left[\frac{\partial}{\partial t} + (\vec{V}\cdot\nabla)\right]H$$

$$\Rightarrow \frac{d}{dt} = \frac{\partial}{\partial t} + \vec{V}\cdot\nabla$$

$$\Rightarrow \frac{d}{dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$$

Here,

$\frac{d}{dt}$ = substantial derivative or stokes' derivative or total or material rate of change.

$\frac{\partial}{\partial t}$ = local rate of change

$\vec{V}\cdot\nabla$ = particular or conective rate of change.

The above result implies that the action of the operator $\frac{d}{dt}$ on a fn. is same as the action of the operator $\frac{\partial}{\partial t} + \vec{V}\cdot\nabla$.

Q:- Prove that material derivative in

i) cylindrical coordinates is

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + V_r\frac{\partial}{\partial r} + \frac{V_\theta}{r}\frac{\partial}{\partial \theta} + V_z\frac{\partial}{\partial z}$$

ii) spherical coordinates is

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + V_r\frac{\partial}{\partial r} + \frac{V_\theta}{r}\frac{\partial}{\partial \theta} + \frac{V_\phi}{r\sin\theta}\frac{\partial}{\partial \phi}$$

Q:- Given the velocity field $\vec{V}(x,y,z,t) = 3t\hat{i} + xz\hat{j} + ty^2\hat{k}$

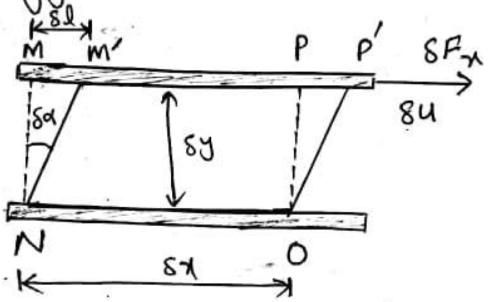
Find the expression for the acceleration of a fluid particle.

Viscosity:-

Viscosity of a fluid is the resistance of a fluid to its motion. or

The viscosity of a fluid is a measure of its resistance to shear or angular deformation. Viscosity of fluids is a physical property of fluids associated with shearing deformation of fluid particles subjected to the action of applied forces.

Consider the behavior of a fluid element b/w the two infinite plates; The rectangular fluid element is initially at rest at time t; Let us now suppose a Constant force δF_x is applied to the upper plate so that it is dragged across the fluid at Constant velocity δu;



The shear stress acting on the fluid element is given as;

$$\tau_{yx} = \lim_{\delta A_y \to 0} \frac{\delta F_x}{\delta A_y} = \frac{dF_x}{dA_y}$$

(The fluid directly in contact with the boundry has the same velocity as the boundry itself i.e there is no slip at the boundry. This is called the no slip condition.)

During the time interval δt the fluid is deformed from position MNOP to M'NOP'. The rate of deformation of fluid is given by

$$\text{deformation rate} = \lim_{\delta t \to 0} \frac{\delta \alpha}{\delta t} = \frac{d\alpha}{dt}$$

The distance δl b/w the pts. M and M' is given by;

$$\delta l = \delta u \, \delta t \qquad (s = vt)$$

for small angles;

$$\delta l = \delta y \, \delta \alpha \qquad (s = r\theta)$$

So;
$$\delta y\, \delta\alpha = \delta u\, \delta t$$
$$\Rightarrow \frac{\delta\alpha}{\delta t} = \frac{\delta u}{\delta y}$$
Taking limit on both sides; we have
$$\frac{d\alpha}{dt} = \frac{du}{dy}$$
So, deformation rate $= \frac{du}{dy}$

Thus the fluid element; when subjected to shear stress τ_{yx}; experiences a rate of deformation given by du/dy. So; we can say that any fluid that experiences a shear stress will flow.

Newton's law of viscosity:-

The rate of deformation (i.e velocity gradient) is directly proportional to the shear stress;
$$\tau \propto \frac{du}{dy}$$
$$\tau = \mu \frac{du}{dy}$$
Here μ is a constant of proportionality and is know as the absolute (dynamic) viscosity. This is known as Newton's law of viscosity.

Kinematic viscosity:-

The ratio of the absolute viscosity μ to the density ρ is called the kinematic viscosity of the fluid and is denoted by ν;

i.e $\nu = \frac{\mu}{\rho}$

Note:- i) In SI units; unit of dynamic viscosity μ is pa.s ($kg/m \cdot s$).

ii) unit of kinematic viscosity ν is m^2/s.

iii) For gases; viscosity increases with temperature while for liquids; viscosity decreases with temperature.

iv) In general;
$$\tau_{xy} = \mu \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)$$
for flows that are not 1-D.

Q:- A plate 0.5mm distant from a fixed plate moves at 0.25m/s and requires a force per unit area of 2pa. to maintain this velocity. Determine the viscosity of the fluid b/w the plates.

Sol:- Here

$$u = 0.25\,m/s \quad \text{and} \quad h = 0.5\,mm = \frac{0.5}{1000}\,m$$

and $\tau = 2\,pa$

Now;
$$\tau = \mu \frac{du}{dy} = \mu \frac{u}{h}$$

$$\frac{\tau h}{u} = \mu$$

So;
$$\mu = \frac{2 \times 0.5/1000}{0.25} = 0.004\,pa.s.$$

Q:- The density of a fluid is $1257.5\,kg/m^3$ and its absolute visocisity is 1.5 pa.s. Calculate its specific weight and kinematic viscosity.

Sol:- Here $\rho = 1257.5\,kg/m^3$

and $\mu = 1.5\,pa.s$

specific weight is given as;

$$\gamma = \rho g = 1257.5 \times 9.8 = 12323.5\,N/m^3$$

kinematic viscosity is;

$$\nu = \frac{\mu}{\rho} = \frac{1.5}{1257.5} = 1.193 \times 10^{-3}\,m^2/s$$

Q Carbon tetrachloride at 20°C has a viscosity of 0.000967 Pa s. what shear stress is required to deform this fluid at a strain rate of $5000\,s^{-1}$?

Sol:-

Classification of fluids:-

i) Real or viscous fluids:-

A real fluid is one which has finite viscosity and thus can exert a tangential stress on surface with which it is in contact.

i.e All fluids for which $\mu \neq 0$

ii) Ideal or Inviscid fluids:-

A fluid having zero viscosity i.e $\mu = 0$ is called an ideal fluid.

Note:- Actually no fluid is ever really ideal; but many flow problems are simplified by assuming that the fluid is ideal.

Real fluids are further subdivided into Newtonian and non-Newtonian fluids.

Newtonian fluids:-

Fluids in which the shear stress is directly proportional to the rate of deformation are called the Newtonian fluids. In other words; A fluid which obyes the Newton's law of viscosity is called Newtonian fluid.

$$\text{shear stress} \propto \frac{du}{dy}$$

$$\Rightarrow \tau = \mu \frac{du}{dy}$$

water and air are examples of Newtonian fluid.

Non-Newtonian fluids:-

A fluid which does not obey the Newton's law of viscosity is known as non-Newtonian fluid.

For such fluids; "the power-law modle" is;

$$\text{shear stress} \propto \left(\frac{du}{dy}\right)^n \quad ; \; n \neq 1$$

$$\tau = K \left(\frac{\partial u}{\partial y}\right)^n$$

where; n is the flow behaviour index and K is the consistency index

$$\tau = K \left(\frac{du}{\partial y}\right)^{n-1} \frac{\partial u}{\partial y}$$

$$\tau = \eta \frac{\partial u}{\partial y}$$

where $\eta = k\left(\frac{\partial u}{\partial y}\right)^{n-1}$ is refered to as the apparent viscosity.

Examples:- Milk, blood, butter, ketchup, honey, toothpaste shampoo, gets, greases etc. are the non-newtonian fluids.

Note:- For Newtonian fluids; the viscosity μ is independent of the rate of deformation. The graph b/w shear stress and rate of deformation is a straight line for a Newtonian fluid.

For Non-Newtonian fluids viscosity μ is not independent of the rate of defrmation. The graph b/w shear stress and rate of deformation will not be a straingt line.

Types of Non-newtonian fluids:-

Now-Newtonian fluids are divided into three groups.

i) Time independent fluids.
ii) Time dependent fluids.
iii) Viscoelastic fluids.

Time independent Non-Newtonian fluids:-

i) Pseudoplastic (shear thining) Fluids:- $(n < 1)$

Fluids in which the apparent viscosity decreases with increasing deformation rate i.e $n < 1$

Examples:- Polymer solution such as rubber; colloidal suspensions; blood; milk etc.

ii) Dilatant (or shear thickening) fluids:-

Fluids in which the apparent viscosity increases with increasing deformation rate i.e $n > 1$.

Examples:- Suspensions of starch and of sand; butter printing ink; suger in water etc.

iii) Ideal or Bingham plastic:-

Fluids that behave as solids until a miniumum yield stress; τ_y is exceeded and subsequently exhibits a linear relation b/w stress and rate of deformation

Mathematically;

$$\tau_{xy} = \tau_0 + \mu_p \frac{du}{dy}$$

Examples:- Drilling muds; toothpaste and clay suspensions; jellies etc.

Time-dependent Non-Newtonian fluids:-

1) Thixotropic fluids:-

Fluids that show a decrease in η with time under a constant applied shear stress.

Examples:- Lipstick; Some paints and enample etc.

2) Rheopectice fluids:-

Fluids that show an increase in η with time under a constant applied shear stress.

Examples:- gypsum suspension in water and bentonite solution etc.

Viscoelastic non-Newtonian fluids:-

Some fluids after deformation partially return to their original shape when the applied stress is released; such fluids are named as Viscoelastic.

Viscoelastic fluids have two major types:

i) linear viscoelastic fluids e.g The maxwell and jeffery's fluids; and

ii) non-linear viscoelastic fluids e.g walter's A and B, oldroyed A and B etc.

Q:- An infinite plate is moved over a 2nd plate on a layer of liquid. For a small gap width; $h = 0.3mm$; we assume a linear velocity distribution in the liquid; $u = 0.m/s$ The liquid viscosity is 0.65×10^{-3} kg/m.s and its specific gravity is 0.88. Find

i) The kinematic viscosity of the fluid.
ii) The shear stress on lower plate.
iii) Inidcate the direction of shear stress.

Sol:-

$$U = 0.3 \text{ m/s}$$
$$h = 0.3 \times 10^{-3} \text{m}$$
$$\mu = 0.65 \times 10^{-3}$$

specific gravity = 0.88

since ; specific gravity $= \dfrac{\rho_{sub}}{\rho_{water \text{ at } 4^\circ C}}$

So; $\rho_{sub} = 0.88 \times 1000$ kg/m³

Now;

i) $\nu = \dfrac{\mu}{\rho} = \dfrac{0.65 \times 10^{-3}}{0.88 \times 10^{3}} =$

ii) $\tau_{yx} = \tau_{lower} = \mu \dfrac{du}{dy} = \mu \dfrac{u}{h} = \dfrac{0.65 \times 10^{-3} \times 0.3}{0.3 \times 10^{-3}}$

$= 0.65$ kg/m.s²

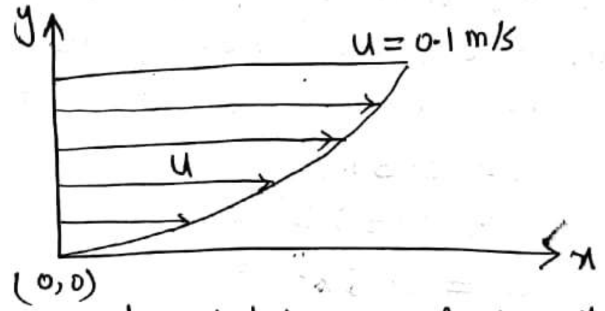
iii) Since τ_{yx} is +ive So the direction of shear stress is along +ive x-axis.

Q:- Suppose that the fluid being sheared b/w two plates is SEA 30 oil ($\mu = 0.29 \frac{kg}{ms}$) at 20°C. Compute the shear stress in the oil if $V = 3$ and $h = 2cm$.

Sol:- $\tau = \mu \dfrac{du}{dy} = \mu \dfrac{V}{h} = \dfrac{0.29 \times 3}{0.02}$

$\tau = 43$ pa.

Q:- Methyl iodide at a thickness of 10mm; and having a viscosity of 0.005 pa.s at a temperature of 20°C; is flowing over a flate plate. The velocity distribution of the thin film may be considered parabolic determine the shear stress at y = 0; 5 and 10mm. from the surface of the plate.



Sol:-
Since the velocity distribution of the thin film is considered to be parabolic.

So; $u = A + By + Cy^2 \longrightarrow (1)$

boundry conditions are;

a) $u = 0$; when $y = 0$ (no slip condition)

b) $u = 0.1 \, m/s$ at $y = 0.01 m$

c) $\frac{du}{dy} = 0$; when $y = 0.01 m$

using (a) in (1); we get $A = 0 \Rightarrow u = By + Cy^2 \rightarrow (2)$

using (b) in (2); we get

$$0.1 = 0.01B + 0.0001C$$

$$0.1B + 0.001C = 1 \rightarrow (3)$$

Now; using (c) in in (2);

$$\frac{du}{dy} = B + 2Cy$$

$$0 = B + 2C(0.01) \Rightarrow B = -0.02C$$

put value of B in (3);

$$(0.1)(-0.02C) + 0.001C = 1$$

$$-0.002C + 0.001C = 1$$

$$\Rightarrow -0.001C = 1$$

$$\Rightarrow C = -1000$$

So;

$$B = -0.02(-1000) = 20$$

So; eq ① $\Rightarrow$

$$u = 20y - 1000y^2 \Rightarrow \frac{du}{dy} = 20 - 2000y$$

i) for $y = 0$;

$$\tau_{yx} = \mu \left.\frac{du}{dy}\right|_{y=0} = (0.005)(20-0) = 0.1 \text{ pa.}$$

ii) for $y = 0.005\text{m}$

$$\tau_{yx} = \mu \left.\frac{du}{dy}\right|_{y=0.005} = (0.005)(20-10) = 0.05 \text{pa}$$

iii) for $y = 0.01\text{m}$;

$$\tau_{yx} = \mu \left.\frac{du}{dy}\right|_{y=0.01} = (0.005)(20-20) = 0$$

Q:- The viscous boundry layer velocity profile can be approximated by a cubic eq

$$u = a + b\left(\frac{y}{\delta}\right) + c\left(\frac{y}{\delta}\right)^3$$

The boundry condition is $u = V$ (the free stream velocity) at the boundry edge δ; (where the viscouse friction becomes zero.) Find the values of a, b and c.

Methods of description of fluid motion:-

A fluid consists of an inumerable number of particles; whose relative positions are never fix. whenever a fluid is in motion; these particles move along certain lines; depending upon the characteristic of the fluid and the shape of of the passage through which the fluid particles move.

For complete analysis of fluid motion; it is necessary to observe the motion of the fluid particles at various pts. and times. For the mathematical analysis of the fluid motion the following two methods are generally used:

i) Lagrangian method.

ii) Eulerian method.

1) Lagrangian method:-

It deals with the study of flow pattern of the individual particles. In this method we fix our attention on a particular fluid particle and follow its motion throughout its course.

Note:-

i) Lagrangian method is frequently used in solid mechanics and is rarely used in fluid mechanics.

ii) The merit of this method is that the motion and path of each fluid particle is know; so that at any time it is possible to trace the history of each fluid particle.

iii) This method has a serious drawback; the eq/s of motion in this method are non-linear in nature and are very difficult to solve.

In fact this method is used with an advantage only in 1-dimensional flow problems.

2) **Eulerian Method**:- It deals with the study of flow pattern of all the particles simultaneously at one section. This method based on the technique of selecting a fixed pt. in space occupied by the fluid and observing the changes in the properties of the fluid as it passes throu. that pt.

i) The drawback of Eulerian method is that the background information of individual particles is not known.

ii) The advantage of this method is that the eq/s of motion in this method can be easily linearized using acceptable approximations.

iii) The Eulerian method of specification is commonly used in fluid dynamics and is never used in solid mechanics.

Q:- The motion of a fluid particle in Lagrangin system is given by;

$$x = x_0 + y_0 t + z_0 t^2 \rightarrow (1)$$
$$y = y_0 + z_0 t + x_0 t^2 \rightarrow (2)$$
$$z = z_0 + x_0 t + y_0 t^2 \rightarrow (3)$$

Find the components of velocity in Eulerian system.

Sol:-

$$u = \frac{dx}{dt} = y_0 + 2z_0 t \rightarrow (4)$$
$$v = \frac{dy}{dt} = z_0 + 2x_0 t \rightarrow (5)$$
$$w = \frac{dz}{dt} = x_0 + 2y_0 t \rightarrow (6)$$

$(1) - t(2); \Rightarrow x - ty = x_0 - x_0 t^3 \Rightarrow x_0 = \frac{x - ty}{1 - t^3}$

$(2) - t(3); \Rightarrow y - tz = y_0 - y_0 t^3 \Rightarrow y_0 = \frac{y - tz}{1 - t^3}$

and

$(3) - t(1); \Rightarrow z - tx = z_0 - z_0 t^3 \Rightarrow z_0 = \frac{z - tx}{1 - t^3}$

So; velocity components in Eulerian form are;

$$u = \frac{y - tz}{1 - t^3} + 2\left(\frac{z - tx}{1 - t^3}\right)t = \frac{y + zt - 2xt^2}{1 - t^3}$$

Similarly; we get;

$$v = \frac{z + xt - 2yt^2}{1 - t^3} \text{ and } w = \frac{x + yt - 2zt^2}{1 - t^3}$$

Q:- For a 2D flow; the velocity components at a point in a fluid may be expressed as in Eulerian coordinates by $u = x+y+2t$ and $v = 2y+t$.

determine the Lagrange coordinates as a fn. of the initial positions x_0, y_0 and t.

Sol:-

$$u = \frac{dx}{dt} \quad \text{and}, \quad v = \frac{dy}{dt}$$

$$\frac{dx}{dt} = x+y+2t \quad \text{and} \quad \frac{dy}{dt} = 2y+t$$

Let $\frac{d}{dt} = D$ then

$$(D-1)x - y = 2t \longrightarrow (1) \quad \text{and} \quad (D-2)y = t \longrightarrow (2)$$

operating $(D-2)$ on (1) we get

$$(D-2)(D-1)x - (D-2)y = (D-2)(2t)$$

$$(D^2-3D+2)x - (D-2)y = 2-4t \rightarrow (3)$$

Adding (2) and (3) we get

$$(D^2-3D+2)x = 2-3t$$

So; $x_c = c_1e^t + c_2e^{2t}$

and

$$x_p = \frac{1}{(D^2-3D+2)}(2-3t)$$

$$x_p = \frac{1}{(D-2)(D-1)}(2-3t) = \frac{-1}{(D-2)(1-D)}(2-3t)$$

$$= \frac{-1}{D-2}(1-D)^{-1}(2-3t) = \frac{-1}{D-2}(1+D)(2-3t)$$

$$= \frac{-1}{D-2}(2-3t-3) = \frac{1}{2(1-\frac{D}{2})}(-3t-1)$$

$$= \frac{1}{2}(1-\frac{D}{2})^{-1}(-3t-1)$$

$$= \frac{1}{2}(1+\frac{D}{2})(-3t-1)$$

$$= \frac{1}{2}(-3t-1-\frac{3}{2}) = \frac{1}{2}(-3t-\frac{5}{2})$$

$$x_p = -\frac{1}{2}(3t+\frac{5}{2})$$

So; the general sol is;

$$x = x_c + x_p = c_1e^t + c_2e^{2t} - \frac{1}{2}(3t+\frac{5}{2})$$

Now; from eq; $(D-2)y = t$

$$\Rightarrow \frac{dy}{dt} - 2y = t \rightarrow A$$

which is linear eq in y; here $P(t) = -2$

So; $$I\cdot F = e^{\int -2dt} = e^{-2t}$$

$A \Rightarrow$ $$e^{-2t}\frac{dy}{dt} - 2ye^{-2t} = te^{-2t}$$

$$d(e^{-2t}y) = te^{-2t}$$

$$\int d(e^{-2t}y) = \int te^{-2t}dt + C_2$$

$$e^{-2t}y = t\frac{e^{-2t}}{-2} + \frac{1}{2}\int e^{-2t}dt + C_2$$

$$e^{-2t}y = \frac{te^{-2t}}{-2} + \frac{1}{2}\left(-\frac{1}{2}e^{-2t}\right) + C_2$$

$$e^{-2t}y = \frac{te^{-2t}}{-2} \cancel{+} - \frac{1}{4}e^{-2t} + C_2$$

$$\Rightarrow y = -\frac{t}{2} - \frac{1}{4} + C_2e^{2t}$$

$$\Rightarrow y = \frac{-1}{4}(2t+1) + C_2e^{2t}$$

To find C_1 and C_2 we use initial conditions;

i.e at $t = t_0 = 0$; $x = x_0$ and $y = y_0$

So; $x_0 = C_1 + C_2 - \frac{5}{4}$ and $y_0 = -\frac{1}{4} + C_2$

$C_2 = y_0 + \frac{1}{4}$

So; $x_0 = C_1 + y_0 + \frac{1}{4} - \frac{5}{4}$

$x_0 = C_1 + y_0 - 1$

$C_1 = x_0 - y_0 + 1$; so that

x and y are given as;

$$x = (x_0 - y_0 + 1)e^t + \left(y_0 + \frac{1}{4}\right)e^{2t} - \frac{1}{4}(6t+5)$$

and

$$y = \left(y_0 + \frac{1}{4}\right)e^{2t} - \frac{1}{4}(2t+1)$$

i) **The Lagrangian form of field representation:-**

In this form we study the fluid motion and associated properties for each fluid particle by following its position in space as a fn. of time.

Material description:- The description of motion with each fluid particle is called material description.

Material Coordinates:-

The set of space coordinates associated with each fluid particle are known as material coordinates.

Material Variables:- The space coordinates together with time are known as the material variables.

Material time derivative:-

Since the Lagranges form of representation studies motion behaviour by following each particle individually; the time derivative of each fn. is thus known as the material time derivative denoted by $\frac{d}{dt}$. It is also known as total derivative.

Note:- In the Lagrange's form; displacement is the base quantity and other properties e.g velocity and acceleration are derived quantities.

ii) **Euler form of field representation:-**

In this form no attention is paid to the motion of individual particles. Rather the state of motion of particles is studied at a fixed location as a fn. of time.

Spatial Position:- Each fixed location is called the spatial position and the state of motion is known as the spatial description.

Spatial Coordinates:- Each fixed location can be described by a set of space coordinates known as the spatial coordinates.

Spatial Variables:-

The space coordinates together with time are known as spatial variables.

Note:- In the Euler's form; velocity is the base quantity and other properties e.g displacement and acceleration are the derived quantities.

D'Alembert-Euler acceleration formula:-

Acceleration of a fluid particle is

$$\vec{a} = \frac{d\vec{V}}{dt}$$

$$\Rightarrow \quad \vec{a} = \frac{\partial \vec{V}}{\partial t} + (\vec{V}\cdot\nabla)\vec{V}$$

This is known as d'alembert-Euler acceleration formula;

In rectangular coordinates;

$$a_x = \frac{du}{dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$$

$$a_y = \frac{dv}{dt} = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}$$

$$a_z = \frac{dw}{dt} = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}$$

In cylindrical coordinates;

$$\vec{a} = \frac{\partial \vec{V}}{\partial t} + V_r\frac{\partial \vec{V}}{\partial r} + \frac{V_\theta}{r}\frac{\partial \vec{V}}{\partial \theta} + V_z\frac{\partial \vec{V}}{\partial z}$$

Example:- A velocity field $\vec{V} = u\hat{i} + v\hat{j} + w\hat{k}$ is given as;

$$u = x + 2y + 3z + 4t^2$$

$$v = xyz + t$$

$$w = (x+y)z^2 + 2t$$

Determine;

a) The local acceleration.

b) The convective acceleration

c) The total acceleration

At the point. $(1, 1, 1, 2)$.

Volumetric flow rate:-

The volume of fluid passing any normal cross section in unit time is called the volumetric flow rate or discharge. It is denoted by Q and its unit is m^3/s.

Mass flow rate:-

The mass of fluid passing any normal cross-section in unit time is called the mass flow rate it is denoted by $\dot{m}$ and its unit is kg/s.

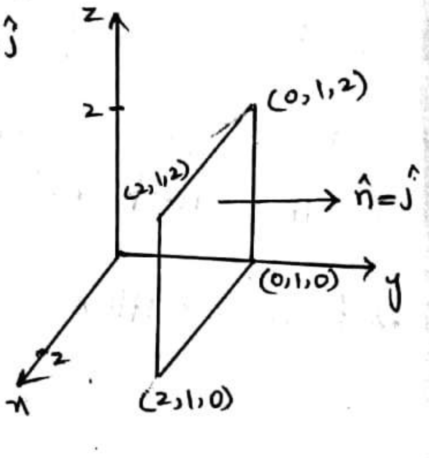
$$\text{mass flux through surface } S = \iint_S \rho \vec{V} \cdot \hat{n}\, ds$$

$$\text{Volume flux through surface } S = \iint_S \vec{V} \cdot \hat{n}\, ds$$

where $\vec{V}$ is the velocity and $\hat{n}$ be the outward drawn unit normal.

Example:- For the velocity vector $\vec{V} = 3t\hat{i} + xz\hat{j} + ty^2\hat{k}$ Evaluate the volumetric flow rate Q and the average velocity U_{av} through the square surface whose vertices are $(0,1,0)$, $(0,1,2)$, $(2,1,2)$, and $(2,1,0)$

Sol:-



Since $\vec{V} = 3t\hat{i} + xz\hat{j} + ty^2\hat{k}$ and $\hat{n} = \hat{j}$

So; $\vec{V} \cdot \hat{n} = xz$

So; volume flow rate is;

$$Q = \iint_S \vec{V} \cdot \hat{n}\, ds$$

$$= \iint_R \vec{V} \cdot \hat{n} \frac{dx\,dz}{|\hat{n} \cdot \hat{j}|}$$

$$= \int_0^2 \int_0^2 (xz)\, dx\,dz = 4\, m^3/s$$

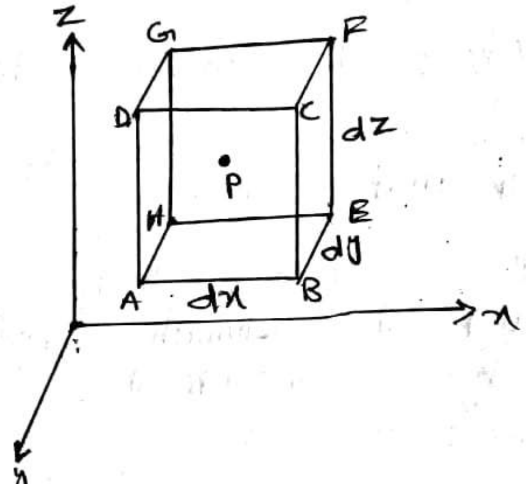
average velocity is;

$$V_{avg} = \frac{Q}{A} = \frac{4}{2 \times 2} = 1\, m/s.$$

Equation of Continuity:-

Eq of Continuity based on the principle of Conservation of mass; which states that the rate of increase of mass of fluid within the volume V must be equal to the rate of influx of mass of fluid across the surface S.

Consider the flow of fluid through a fixed element with centre at $P(x, y, z)$ having sides dx dy and dz. Let (u, v, w) be the components of velocity $\bar{V}$ at P;



x-component of velocity at the centre of face $BCFE = u + \frac{\partial u}{\partial x} \cdot \frac{dx}{2}$

density at the centre of face $BCFE = \rho + \frac{\partial \rho}{\partial x} \frac{dx}{2}$

Similarly;

x-component of velocity at the centre of face $ADHG = u - \frac{\partial u}{\partial x} \frac{dx}{2}$

and density at centre of face $ADHG = \rho - \frac{\partial \rho}{\partial x} \frac{dx}{2}$

So, the net mass efflux in x-direction is;

Net mass efflux = mass out flux — mass in flux

$$= \left(\rho + \frac{\partial \rho}{\partial x} \frac{dx}{2}\right)\left(u + \frac{\partial u}{\partial x} \frac{dx}{2}\right) dy\, dz$$

$$- \left(\rho - \frac{\partial \rho}{\partial x} \frac{dx}{2}\right)\left(u - \frac{\partial u}{\partial x} \frac{dx}{2}\right) dy\, dz$$

$$= \left(\rho \frac{\partial u}{\partial x} + u \frac{\partial \rho}{\partial x}\right) dx\, dy\, dz$$

Similarly;

Net mass efflux in y-direction $= \left(\rho \frac{\partial v}{\partial y} + v \frac{\partial \rho}{\partial y}\right) dx\, dy\, dz$

Net mass efflux in z-direction $= \left(\rho \frac{\partial w}{\partial z} + w \frac{\partial \rho}{\partial z}\right) dx\, dy\, dz$

total net mass of efflux $= \left[\rho\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)+u\frac{\partial \rho}{\partial x}+v\frac{\partial \rho}{\partial y}+w\frac{\partial \rho}{\partial w}\right]dxdydz$

since mass reduction in Control volume is;

$$-\frac{\partial \rho}{\partial t}\,dxdydz\ ;$$

So; total net mass efflux out of dV is equal to the reduction of mass in dV;

So;

$$\left[\rho\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)+u\frac{\partial \rho}{\partial x}+v\frac{\partial \rho}{\partial y}+w\frac{\partial \rho}{\partial z}\right]dxdydz = -\frac{\partial \rho}{\partial t}dxdydz$$

$$\rho\,\nabla\cdot\vec{V} + \vec{V}\cdot\nabla\rho = -\frac{\partial \rho}{\partial t}$$

$$\Rightarrow \nabla\cdot(\rho\vec{V}) + \frac{\partial \rho}{\partial t} = 0$$

$$\Rightarrow \frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\vec{V}) = 0$$

which is eq of Continuity.

1) If flow is steady; then density is independent of of time so; $\frac{\partial \rho}{\partial t}=0$;

and eq of Continuity becomes;

$$\nabla\cdot(\rho\vec{V})=0$$

2) If fluid is incompressible then density is constant; so

eq of Continuity becomes;

$$\nabla\cdot\vec{V}=0$$

Q:- Is the motion $u = \frac{kx}{x^2+y^2}$; $v = \frac{ky}{x^2+y^2}$; $w=0$ kinematically possible for an incompressible fluid flow ?

Sol:-

$$\frac{\partial u}{\partial x} = \frac{(x^2+y^2)k - kx(2x)}{(x^2+y^2)^2} = \frac{kx^2+ky^2-2kx^2}{(x^2+y^2)^2}$$

$$\frac{\partial u}{\partial x} = \frac{ky^2-kx^2}{(x^2+y^2)^2}$$

and

$$\frac{\partial v}{\partial y} = \frac{(x^2+y^2)k - ky(2y)}{(x^2+y^2)^2} = \frac{kx^2-ky^2}{(x^2+y^2)^2}$$

$$\frac{\partial w}{\partial z} = 0$$

Now;

$$\nabla\cdot\vec{V} = \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z} = \frac{ky^2-kx^2}{(x^2+y^2)^2}+\frac{kx^2-ky^2}{(x^2+y^2)^2}$$

$$\nabla\cdot\vec{V} = 0$$

Since u, v, w satisfy the eq of continuity for an incompressible flow ; so given velocity components represent an incompressible flow.

Q:- under what condition does the velocity field; $\vec{V} = (a_1x+b_1y+c_1z)\hat{i}+(a_2x+b_2y+c_2z)\hat{j}+(a_3x+b_3y+c_3z)\hat{k}$ reprent an incompressible flow?

Eq of Continuity in cylindrical polar Coordinates:-

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(\rho r V_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho V_\theta) + \frac{\partial}{\partial z}(\rho V_z) = 0$$

In spherical Coordinates:-

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\rho V_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\rho\sin\theta V_\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}(\rho V_\phi) = 0$$

Q:- show that the incompressible flow in cylindrical polar Coordinates given by;

$$V_r = c\left(\frac{1}{r^2} - 1\right)\cos\theta$$

$$V_\theta = c\left(\frac{1}{r^2} + 1\right)\sin\theta$$

$$V_z = 0$$

Satisfy the eq of Continuity.

Sol:- The eq of Continuity for incompressible flow in cylindrical coordinates is;

$$\frac{1}{r}\frac{\partial}{\partial r}(rV_r) + \frac{1}{r}\frac{\partial V_\theta}{\partial \theta} + \frac{\partial V_z}{\partial z} = 0$$

Now;

$$rV_r = c\left(\frac{1}{r} - r\right)\cos\theta$$

$$\frac{\partial}{\partial r}(rV_r) = c\left(-\frac{1}{r^2} - 1\right)\cos\theta$$

and $$\frac{\partial V_\theta}{\partial \theta} = \frac{\partial}{\partial \theta}\left[c\left(\frac{1}{r^2}+1\right)\sin\theta\right] = c\left(\frac{1}{r^2}+1\right)\cos\theta$$

$$\frac{\partial V_z}{\partial z} = 0;$$

So; $$\frac{1}{r}\frac{\partial}{\partial r}(rV_r) + \frac{1}{r}\frac{\partial V_\theta}{\partial \theta} + \frac{\partial V_z}{\partial z} = c\left(\frac{-1}{r^3} - \frac{1}{r}\right)\cos\theta + c\left(\frac{1}{r^3} + \frac{1}{r}\right)\cos\theta + 0$$

$$\frac{1}{r}\frac{\partial}{\partial r}(rV_r) + \frac{1}{r}\frac{\partial V_\theta}{\partial \theta} + \frac{\partial V_z}{\partial z} = 0$$

So; the eq of Continuity is satisfied.

Streamlines:- A streamline is a curve drawn in the fluid s.t the tangent to it at every pt. is in the direction of fluid velocity $\vec{V}$ at that pt. It is also called the line of flow.

Equation of the streamline:-

Since at each pt. of a streamline The velocity vector $\vec{V}$ is parallel to the unit tangent at that pt.

So;
$$\vec{V} \times \hat{t} = 0$$
$$\Rightarrow \vec{V} \times \frac{d\vec{r}}{ds} = 0$$
$$\Rightarrow \vec{V} \times d\vec{r} = 0$$
$$\Rightarrow \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u & v & w \\ dx & dy & dz \end{vmatrix} = 0$$

$$(v\,dz - w\,dy)\hat{i} + (w\,dx - u\,dz)\hat{j} + (u\,dy - v\,dx)\hat{k} = 0$$

$$\Rightarrow \quad v\,dz - w\,dy = 0 \Rightarrow \frac{dz}{w} = \frac{dy}{v}$$
$$w\,dx - u\,dz = 0 \Rightarrow \frac{dx}{u} = \frac{dz}{w}$$
and
$$u\,dy - v\,dx = 0 \Rightarrow \frac{dy}{v} = \frac{dx}{u}$$

So;
$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$$

is the differential eq for the streamlines.

Q:- Find the eqs of streamlines for the flow field; $u = \frac{kx}{x^2+y^2}$, $v = \frac{ky}{x^2+y^2}$

Sol:- eq of streamlines is ;
$$\frac{dx}{u} = \frac{dy}{v}$$

Q:- The velocity components for a certain three dimensional incompressible flow field are given by

$$u = ax \quad ; \quad v = ay \quad ; \quad w = -2az$$

Find the eq/s of the streamlines passing through the pt (1,1,1)

Sol:-

eq of streamline is

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$$

$$\frac{dx}{ax} = \frac{dy}{ay} = \frac{dz}{-2az}$$

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{-2z}$$

$$\frac{dx}{x} = \frac{dy}{y} \quad \text{and} \quad \frac{dy}{y} = \frac{dz}{-2z}$$

$$\Rightarrow \ln x = \ln y + c_1 \quad \text{and} \quad \ln y = -\frac{1}{2}\ln z + c$$

$$\Rightarrow y = c_1 x \;\longrightarrow (1) \qquad \ln y^2 + \ln z = c$$

$$zy^2 = c_2$$

at (1,1,1) eq (1) $\Rightarrow$ $c_1 = 1$

at (1,1,1) eq (2) $\Rightarrow$ $c_2 = 1$

So; required eq/s of streamline are

$$y = x \quad \text{and} \quad y^2 z = 1$$

Q:- Test whether the motion specified by

$$\vec{V} = \frac{K^2(x\hat{j} - y\hat{i})}{x^2 + y^2}$$

is a possible motion for an incompressible fluid. If so determine the eq/s of the streamlines.

Ans:- Incompressible ; $x^2 + y^2 = c_1$, $z = c_2$

Eq of streamline in cylindrical polar coordinates:-

$$\frac{dr}{V_r} = \frac{r\,d\theta}{V_\theta} = \frac{dz}{V_z}$$

In spherical coordinates:-

$$\frac{dr}{V_r} = \frac{r\,d\theta}{V_\theta} = \frac{r\sin\theta\,d\phi}{V_\phi}$$

Q:- The velocity components in a 2-D flow field are given by;

$$V_r = \frac{\cos\theta}{r^2} \; ; \; V_\theta = \frac{\sin\theta}{r^2}$$

Find the eq of streamline passing through the pt $r = 2$; $\theta = \pi/2$

Sol:-

For 2D flow field ; eq of streamline is

$$\frac{dr}{V_r} = \frac{r\,d\theta}{V_\theta}$$

$$\frac{dr}{\frac{\cos\theta}{r^2}} = \frac{r\,d\theta}{\frac{\sin\theta}{r^2}}$$

$$\frac{1}{r}\,dr = \frac{\cos\theta}{\sin\theta}\,d\theta$$

$$\ln r = \ln\sin\theta + C_1$$

$$\Rightarrow r = C\sin\theta$$

at $r = 2$ and $\theta = \pi/2$;

$$2 = C\sin\frac{\pi}{2} \Rightarrow C = 2$$

So; the eq of streamline is $r = 2\sin\theta$

Streamtube:- If we draw the streamlines through each pt. of a closed curve C lying in the fluid we obtain a tubular surface called the stream tube. The surface of streamtube is called a stream surface.

If the flow is unsteady ; the shape of the stream tube changes from instant to instant. If the flow is steady; the shape of streamtube remains the same at all times.

A streamtube of of infinitesimal cross-section is called a stream filament.

pathlines:- If we fix our attention on a particular fluid particle ; the curve which this particle describes during its motion is called a pathline.

when the motion is steady ; the pathlines coincide with the streamlines. Pathline is a lagrangian concept.

Differential eq for the pathlines:-

since a pathline describes the position of a particular fluid particle at each instant ; so the motion of particle is given as;

$$\frac{d\vec{r}}{dt} = \vec{V}$$

$$\Rightarrow \quad \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k} = u\hat{i} + v\hat{j} + w\hat{k}$$

$$\Rightarrow \quad \frac{dx}{dt} = u \; ; \quad \frac{dy}{dt} = v \; ; \quad \frac{dz}{dt} = w$$

Theses eq/s represent eq for the pathlines.

Q:- Find the eq of the pathlines for the following steady incompressible flow field

$$u = kx \; ; \quad v = -ky$$

Q:- The velocity components for an unsteady, 2D incompressible flow field are given by

$u = \frac{x}{t}$; $v = y$. Find the eq of pathline passing through the pt (1,1) at t=1.

Streakline:- A streakline is a line consisting of all those fluid particles that have passed through a fixed pt in the flow field at some earlier instant.

Note that if the flow is steady, streamlines pathlines, and streaklines are all same.

Example:- Find the eq of the streakline at any time t for the following steady incompressible flow field $u = kx$; $v = -ky$.

Sol:- we know that the paths of fluid particles are given by the eq's;

$$\frac{dx}{dt} = kx \quad ; \quad \frac{dy}{dt} = -ky$$

$$\frac{dx}{x} = k\,dt \quad ; \quad \frac{dy}{y} = -k\,dt$$

$$\ln x = kt + \ln c_1 \quad ; \quad \ln y = -kt + \ln c_2$$

$$x = c_1 e^{kt} \rightarrow (1) \quad ; \quad y = c_2 e^{-kt} \rightarrow (2)$$

to find values of c_1 and c_2; we assume that the fluid particles pass through the fixed pt. (x_1, y_1) at an earlier instant $t = s$;

Then from (1) and (2);

$$x_1 = c_1 e^{ks} \quad ; \quad y_1 = c_2 e^{-ks}$$

$$\Rightarrow c_1 = x_1 e^{-ks} \quad ; \quad c_2 = y_1 e^{ks}$$

so; (1) and (2); $\Rightarrow$

$$x = x_1 e^{k(t-s)} \quad ; \quad y = y_1 e^{-k(t-s)}$$

which are parametric eq's of pathlines; we can eliminate t; as;

$$xy = x_1 y_1$$

$$\Rightarrow xy = c$$

Example:- The velocity components for an unsteady 2D flow field are given by;

$u = \frac{x}{t}$; $v = y$ Then find the eq of the streakline passing through the pt. $(1,1)$; $t=1$

Sol:-

$$\frac{dx}{dt} = \frac{x}{t} \quad ; \quad \frac{dy}{dt} = y$$

$$\frac{dx}{x} = \frac{dt}{t} \quad ; \quad \frac{dy}{y} = dt$$

$$x = c_1 t \rightarrow (1) \quad \text{and} \quad y = c_2 e^{t} \rightarrow (2)$$

at $t = s$; (x_1, y_1)

$$x_1 = c_1 s \quad \text{and} \quad y_1 = c_2 e^{s}$$

$$c_1 = \frac{x_1}{s} \quad \text{and} \quad c_2 = y_1 e^{-s}$$

so; (1) and (2) becomes;

$$x = x_1 \frac{t}{s} \quad \text{and} \quad y = y_1 e^{t-s}$$

at $(x_1, y_1) = (1,1)$ and $t = 1$

$$x = \frac{1}{s} \quad ; \quad y = e^{1-s}$$

Eliminating s; we get

$$y = e^{1-\frac{1}{x}}$$

Stream function:- A fn. which describes the form of pattern of flow or in other words it is the discharge per unit thickness.

It is denoted by ψ and given as

$$\psi = \psi(x, y, t)$$

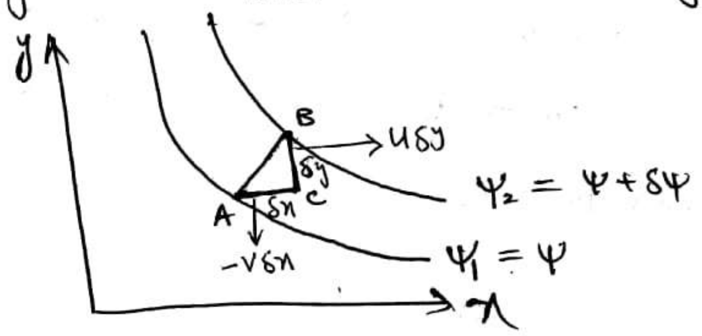
The stream fn. based on Continuity principle for stead-state flow

$$\psi = \psi(x, y)$$

Determination of velocity components from ψ:-

For the purpose of mass conservation; the control volume under consideration is choosen by ABC; with fluid flowing into the control volume through control surface AB. and leaving of through control surface AC and BC

Let a pt. along a streamline as shown in fig.



u = velocity component in x-direction at A

v = velocity component in y-direction at A

ψ = stream fn. at A;

Now let us consider another streamline. s.t pt. A is displaced through a small distance δy in y-direction and δx in x-direction

Let $\psi + \delta\psi$ = stream fn. of this new position

Now; The flow rate across δy will be;

$$\delta y = u\delta y \Rightarrow u = \frac{\delta\psi}{\delta y} \rightarrow (1)$$

Similarly; the flow rate across δx will be

$$\delta \psi = -v \, \delta x$$

$$v = -\frac{\delta \psi}{\delta x} \rightarrow (2)$$

-ive sign indicates that the velocity v acts downward.

In cylindrical coordinates;

$$V_\theta = -\frac{\partial \psi}{\partial r}$$

$$V_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta}$$

Example:- If for 2D-flow; the stream fn. is given by $\psi = 2xy$. Calculate the velocity at the pt. (3,6) **Ans:-** 13.42

Example:- The velocity components for a certain 2D incompressible fluid flow are

$$u = 2xy \quad ; \quad v = a^2 + x^2 - y^2$$

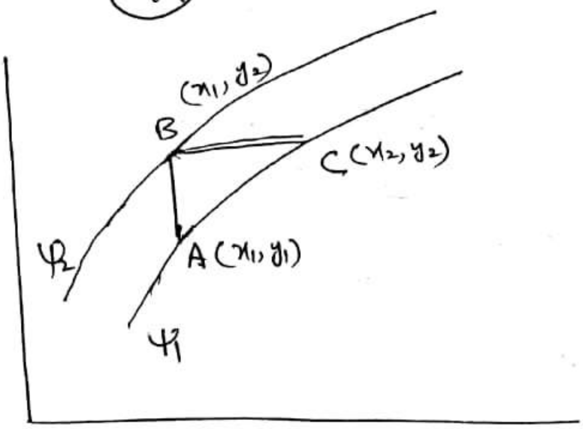
Determine the corresponding stream function.

Q:- show that the volume flow rate (per unit depth) b/w any two streamlines can be written as the difference b/w the constant values of ψ defining the two streamlines.

Sol:-

From the definition of a streamline; we recognize that there can be no flow across a streamline.

The volume flow rate, Q, b/w streamlines ψ_1 and ψ_2 can be evaluated by considering the flow across AB or across BC.



for unit depth the flow rate across AB is;

$$Q = \int_{y_1}^{y_2} u\,dy = \int_{y_1}^{y_2} \frac{\partial \psi}{\partial y}\,dy \rightarrow (1)$$

Since $\psi = \psi(x, y)$

$$d\psi = \frac{\partial \psi}{\partial x}dx + \frac{\partial \psi}{\partial y}dy \longrightarrow (A)$$

but along AB; $x = \text{Constant} \Rightarrow dx = 0$

$$\Rightarrow \quad d\psi = \frac{\partial \psi}{\partial y}dy$$

(1) $\Rightarrow$

$$Q = \int_{y_1}^{y_2} d\psi = \psi_2 - \psi_1 \longrightarrow (2)$$

Now; for a unit depth, the flow across BC;

$$Q = \int_{x_1}^{x_2} v\,dx = -\int_{x_1}^{x_2} \frac{\partial \psi}{\partial x}\,dx \rightarrow (3)$$

Since $y = \text{Constant}$ along BC;

$$\Rightarrow \quad dy = 0$$

So; (A) $\Rightarrow$ $d\psi = \frac{\partial \psi}{\partial x}dx$

So; (3) $\Rightarrow$ $Q = -\int_{x_1}^{x_2} d\psi = \psi_1 - \psi_2 \rightarrow (4)$

Hence from (2) and (4); desired result is complet.

Q:- Value of stream fn. is constant along a streamline.

Sol:- for a 2-D motion eq of streamline is

$$\frac{dx}{u} = \frac{dy}{v}$$

$$\Rightarrow \quad v\,dx - u\,dy = 0$$

$$\Rightarrow \quad u\,dy - v\,dx = 0 \rightarrow (1)$$

since $\psi = \psi(x, y)$

$$\Rightarrow d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy$$

$$\Rightarrow d\psi = -v\,dx + u\,dy$$

$$\Rightarrow d\psi = 0 \qquad \text{from (1)}$$

$$\psi = \text{Constant.}$$

This is the eq of streamline.

The Vorticity vector:-

The vorticity vector or rotation vector denoted by is defined as;

$$\vec{\xi} = \nabla \times \vec{V}$$

$$\Rightarrow \xi_x \hat{i} + \xi_y \hat{j} + \xi_z \hat{k} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ u & v & w \end{vmatrix}$$

Then

$$\xi_x = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \; ; \qquad \xi_y = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}$$

and $\xi_z = \dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial y}$

In 2D motion;

$$\vec{\xi} = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\hat{k}$$

In polar Coordinates (r, θ)

$$\xi_z = \frac{v_\theta}{r} + \frac{\partial v_\theta}{\partial r} - \frac{1}{r}\frac{\partial v_r}{\partial \theta}$$

In cylindrical Coordinates;

$$\xi_r = \frac{1}{r}\frac{\partial v_z}{\partial \theta} - \frac{\partial v_\theta}{\partial z} \quad ; \quad \xi_\theta = \frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}$$

$$\xi_z = \frac{v_\theta}{r} + \frac{\partial v_\theta}{\partial r} - \frac{1}{r}\frac{\partial v_r}{\partial \theta}$$

Q:- Determine the vorticity components

i) $u = 2xy$; $v = a^2 + x^2 - y^2$

ii) $v_r = r\sin\theta$; $v_\theta = 2r\cos\theta$

Vortex line:-

A vortex line is a curve drawn in the fluid s.t the tangent to it at every pt is in the direction of the vorticity vector.

Eq for a vortex line:-

Since $\vec{\xi}$ is parallel to the unit tangent at pt P, so

$$\vec{\xi} \times \frac{d\vec{r}}{ds} = 0$$

$$\Rightarrow \vec{\xi} \times d\vec{r} = 0$$

So; from here we get;

$$\frac{dx}{\xi_x} = \frac{dy}{\xi_y} = \frac{dz}{\xi_z}$$

is the eq for vortex line.

Irrotational flow:-

If $curl\,\vec{V}=0$ then the given flow field is irrotational.

Rotational flow:-

If $\nabla\times\vec{V}\neq 0$ Then the given flow field is rotatinal.

Conservative vector field:-

A vector field $\vec{F}$ is called conservative if there exist a differentiable fn. f s.t

$$\vec{F} = -\nabla f$$

The fn. f is called the potential fn. for $\vec{F}$.

Conservative force:-

a force $\vec{F}$ is conservative if

$$\nabla\times\vec{F}=0$$

and

$\nabla\times\vec{F}=0 \Rightarrow \vec{F}$ is gradient of some scalar fn. ϕ i-e

$$\vec{F} = -\nabla\phi$$

So; we can say that if a force is conservative then $\vec{F}$ can be expressed as;

$$\vec{F}=-\nabla\phi$$

Velocity potential:-

suppose that the motion is irrotational then $\nabla\times\vec{V}=0$; The necessary and sufficient condition for this eq to hold is $\vec{V}=-\nabla\phi$

where ϕ is a scalar. fn known as velocity fn. or velocity potential.

The velocity potential, ϕ, exists only for an irrotational flow.

Velocity components in terms of ϕ:-

$$\vec{V} = -\nabla\phi$$

$$u\hat{i} + v\hat{j} + w\hat{k} = -\frac{\partial\phi}{\partial x}\hat{i} - \frac{\partial\phi}{\partial y}\hat{j} - \frac{\partial\phi}{\partial z}\hat{k}$$

$$\Rightarrow u = -\frac{\partial\phi}{\partial x}\ ; \quad v = -\frac{\partial\phi}{\partial y}\ ; \quad w = -\frac{\partial\phi}{\partial z}$$

in cylindrical form;

$$V_r = -\frac{\partial\phi}{\partial r}\ ; \quad V_\theta = -\frac{1}{r}\frac{\partial\phi}{\partial\theta}\ ; \quad V_z = -\frac{\partial\phi}{\partial z}$$

Q:- For an incompressible fluid

$$\vec{V} = [-\omega y, \omega x, 0]$$

Discuss the nature of flow.

Sol:-

i) steady 2-D flow.

ii) $\nabla\cdot\vec{V} = 0 \Rightarrow$ flow is incompressible.

iii) $$\nabla\times\vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\omega y & \omega x & 0 \end{vmatrix} = 2\omega\hat{k} \neq 0$$

Thus the flow is not irrotation (or not of potential kind).

Q:- Determine whether the velocity potential for the velocity field $u = a(x^2-y^2)$; $v = -2axy$, $w=0$ exists? If it does, then find it.

Sol:-

$$\nabla\times\vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ a(x^2-y^2) & -2axy & 0 \end{vmatrix}$$

$$\nabla\times\vec{V} = 0\hat{i} + 0\hat{j} + (-2ay+2ay)\hat{k} = \vec{0}$$

Thus; the velocity potential for the given flow field exists.

Now; we find it

$$u = -\frac{\partial\phi}{\partial x}\ ; \quad v = -\frac{\partial\phi}{\partial y}$$

$$\frac{\partial\phi}{\partial x} = a(y^2-x^2) \longrightarrow (1)\ ; \quad \frac{\partial\phi}{\partial y} = 2axy \longrightarrow (2)$$

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Equipotential lines:-

The lines along which the value of the velocity potential ϕ does not change (i.e lines of constant are called the equipotential lines.

Thus $\phi(x, y, z) = \text{Constant}$

is the eq of the equipotential lines.

Note:- A velocity potential ϕ exists for an ideal and irrotational flow field only; where as a stream fn. exists for both ideal and real flow fields.

Eq for 2D, incompressible irrotational flow:-

Velocity components in terms ψ and ϕ are given as;

$$u = \frac{\partial \psi}{\partial y} \; ; \quad v = -\frac{\partial \psi}{\partial x} \rightarrow (1)$$

$$\text{and} \quad u = -\frac{\partial \phi}{\partial x} \; ; \quad v = -\frac{\partial \phi}{\partial y} \rightarrow (2)$$

from irrotationality condition; $(\nabla \times \vec{V} = 0)$

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 \rightarrow (3)$$

from (1) and (3); we get

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0 \rightarrow (A)$$

for incompressible fluid; $(\nabla \cdot \vec{V} = 0)$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \rightarrow (4)$$

from (2) and (4); we get

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \rightarrow (B)$$

eq (A) and (B) are Laplace's eq.

Q:- show that $\phi = x^3t + 2y^2t - 3txz^2 - 2z^2t$ is a possible velocity potential for a 3-D incompressible irrotational flow field.

Sol:-

$$\frac{\partial \phi}{\partial x} = 3x^2t - 3tz^2 \; ; \quad \frac{\partial \phi}{\partial y} = 4yt \; ; \quad \frac{\partial \phi}{\partial z} = -6txz - 4zt$$

$$\frac{\partial^2 \phi}{\partial x^2} = 6xt \; ; \quad \frac{\partial^2 \phi}{\partial y^2} = 4t \; ; \quad \frac{\partial^2 \phi}{\partial z^2} = -6tx - 4t$$

and

$$\nabla^2\phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$$

$$\nabla^2\phi = 6xt + 4t - 6tx - 4t$$

$$\nabla^2\phi = 0$$

So; ϕ is a possible velocity potential.

Q:- show that, lines of constant ψ and constant ϕ are orthogonal.

Sol:-

for constant ψ;

$$d\psi = 0$$

$$\frac{\partial \psi}{\partial x}dx + \frac{\partial \psi}{\partial y}dy = 0$$

Slope of a streamline is;

$$\left(\frac{dy}{dx}\right)_\psi = -\frac{\frac{\partial \psi}{\partial x}}{\frac{\partial \psi}{\partial y}} = -\frac{-v}{u} = \frac{v}{u} \rightarrow (1)$$

for constant ϕ;

$$d\phi = 0$$

$$\frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy = 0$$

slope of a potential line;

$$\left(\frac{dy}{dx}\right)_\phi = -\frac{\partial\phi/\partial x}{\partial\phi/\partial y} = -\frac{-u}{-v} = -\frac{u}{v} \rightarrow (2)$$

from (1) and (2);

$$\left(\frac{dy}{dx}\right)_\psi \left(\frac{dy}{dx}\right)_\phi = -1$$

Angular velocity vector:-

The angular velocity vector of a fluid element, denotedy by $\vec{\omega}$, is defined as;

$$\vec{\omega} = \frac{1}{2}\vec{\xi} = \frac{1}{2}\nabla \times \vec{V}$$

$$\omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)$$

$$\omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)$$

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

In cylindrical Coordinates;

$$\omega_r = \frac{1}{2}\left[\frac{1}{r}\frac{\partial V_z}{\partial \theta} - \frac{\partial V_\theta}{\partial z}\right]$$

$$\omega_\theta = \frac{1}{2}\left[\frac{\partial V_r}{\partial z} - \frac{\partial V_z}{\partial r}\right]$$

$$\omega_z = \frac{1}{2}\left[\frac{1}{r}\frac{\partial}{\partial r}(rV_\theta) - \frac{1}{r}\frac{\partial V_r}{\partial \theta}\right]$$

<—————————————>

Q:- For the velocity field

$$\vec{V} = 10x^2y\,\hat{i} + 20(yz+x)\,\hat{j} + 13\hat{k}$$

what is the total angular velocity of a fluid particle at $(1, 4, 3)$?

Ans: $-40\hat{i} + 5\hat{k}$

Q:- Find the components of angular velocity if $V_r = \frac{1}{r}$; $V_\theta = r^3$; $V_z = 2r\cos\theta$

Ans $(-\sin\theta, -\cos\theta, 2r^2)$

Flow along a curve:- The flow along any curve joining the pts. A and B is defined by either of the integrals;

$$\text{flow} = \int_A^B \vec{V}\cdot\hat{T}\,ds = \int_A^B V\cos\theta\,ds = \int_A^B \vec{V}\cdot d\vec{r}$$

Q:- Calculate the flow for the velocity field $u = x^2 y$; $v = x^2 - y^2$ along the paths

a) $y = 3x^2$

b) $y = 3x$

where $0 \le x \le 1$; $0 \le y \le 3$

Sol:-

$$\text{flow} = \int_A^B \vec{V}\cdot d\vec{r} = \int_A^B u\,dx + v\,dy = \int_A^B x^2 y\,dx + (x^2 - y^2)\,dy \longrightarrow (1)$$

a) Along the path $y = 3x^2$

$\Rightarrow dy = 6x\,dx$

So; (1) $\Rightarrow$

$$\text{flow} = \int_0^1 x^2(3x^2)\,dx + (x^2 - 9x^4)\,6x\,dx$$

$$= \int_0^1 (3x^4 + 6x^3 - 54x^5)\,dx$$

$$= \frac{3}{5}x^5 + \frac{6}{4}x^4 - \frac{54}{6}x^6 \Big|_0^1$$

$$= \frac{3}{5} + \frac{3}{2} - 9$$

$$= \frac{6 + 15 - 90}{10} = \frac{-69}{10}$$

b) Along path $y = 3x \Rightarrow dy = 3\,dx$

so; (1) $\Rightarrow$ $\text{flow} = \int_0^1 x^2(3x)\,dx + (x^2 - 9x^2)\,3\,dx$

$$= \int_0^1 (3x^3 - \text{[illegible]}\,x^2)\,dx$$

$$= \frac{3}{4}x^4 - \frac{24}{3}x^3 \Big|_0^1$$

$$= \frac{3}{4} - 8 = \frac{-29}{4}$$

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Circulation:-

The circulation, Γ, is defined as the line integral of the tangential component of the velocity vector arround a closed curve C fixed in the flow; thus

the circulation Γ around a curve C is given as;

$$\Gamma = \oint_C \vec{V} \cdot d\vec{r}$$

$$\Gamma = \oint_C u\,dx + v\,dy + w\,dz$$

In cylindrical polar Coordinates;

$$\Gamma = \oint_C V_r dr + V_\theta r\,d\theta + V_z dz$$

In spherical coordinates;

$$\Gamma = \oint_C V_r dr + V_\theta r\,d\theta + V_\phi r \sin\theta\, d\phi$$

Kelvin's theorem:-

In an ideal, homogeneous fluid, with conservative body forces, the circulation around a closed curve moving with the fluid remains constant with time

i.e $\dfrac{D\Gamma}{Dt} = 0$

Relationship b/w Circulation and Vorticity:-

Stoke's theorem:- The circulation around any closed curve C drawn in the fluid is the normal surface integral of the vorticity vector over any open two sided surface S lying entirely within the fluid. and having C as its boundry.

Proof:-

$$\Gamma = \oint_C \vec{V} \cdot d\vec{r}$$

$$\Gamma = \iint_S (\nabla \times \vec{V}) \cdot \hat{n}\, ds \quad \text{(By stoke's theorem)}$$

$$\Gamma = \iint_S \vec{\xi} \cdot d\vec{s} \quad \text{proved}$$

For 2D motion

$$\vec{\xi} = \xi_z \hat{k} = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\hat{k} \quad \text{and } \hat{n} = \hat{k}$$

So;

$$\Gamma = \iint_S \vec{\xi} \cdot \hat{n}\, ds$$

$$\Gamma = \iint_S \xi_z \hat{k} \cdot \hat{k}\, ds$$

$$= \iint_S \vec{\xi}_z\, ds$$

$$= \iint_S \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \frac{dx\,dy}{|\hat{k} \cdot \hat{k}|}$$

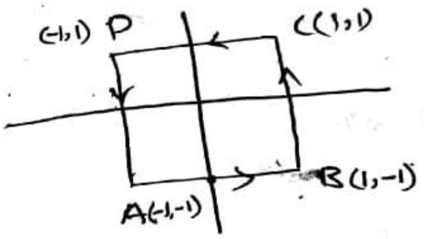
$$\Gamma = \iint_S \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy$$

This eq shows that circulation is the product of vorticity and the cross-sectional area bounded by the curve c.

Q:- The velocity components for a certain flow field are given by $u = x + y$; $v = x^2 - y$.

Calculate the circulation around the square enclosed by the lines $x = \pm 1$, $y = \pm 1$

Also verify the result by using stoke's Theorem.



$$\Gamma = \oint_{ABCDA} \vec{V}\cdot d\vec{r}$$

$$= \oint_{ABCD} u\,dx + v\,dy$$

$$\Gamma = \oint_{ABCD} (x+y)\,dx + (x^2-y)\,dy$$

$$\Gamma = \int_{AB} u\,dx+v\,dy + \int_{BC} u\,dx+v\,dy + \int_{CD} u\,dx+v\,dy + \int_{DA} u\,dx+v\,dy \longrightarrow (1)$$

Along AB ; $y=-1$; $\Rightarrow dy=0$ where x varies from -1 to 1.

So; $\int_{AB} u\,dx+v\,dy = \int_{AB} u\,dx = \int_{-1}^{1}(x+y)\,dx = \int_{-1}^{1}(x-1)\,dx = \frac{x^2}{2}\Big|_{-1}^{1} - x\Big|_{-1}^{1} = -2$

Along BC ; $x=1$; $\Rightarrow dx=0$; where y varies from -1 to 1.

$$\int_{BC} u\,dx+v\,dy = \int_{BC} v\,dy = \int_{-1}^{1}(x^2-y)\,dy = \int_{-1}^{1}(1-y)\,dy = y\Big|_{-1}^{1} - \frac{y^2}{2}\Big|_{-1}^{1} = 2$$

Along CD ; $y=1 \Rightarrow dy=0$ and x varies from 1 to -1;

$$\int_{CD} u\,dx+v\,dy = \int_{CD} u\,dx = \int_{1}^{-1}(x+y)\,dx = \int_{+1}^{-1}(x+1)\,dx = \frac{x^2}{2}\Big|_{+1}^{-1} + x\Big|_{1}^{-1} = -2$$

Along DA; $x=-1$; $\Rightarrow dx=0$ and y varies from 1 to -1;

$$\int_{DA} u\,dx+v\,dy = \int_{DA} v\,dy = \int_{1}^{-1}(x^2-y)\,dy = \int_{1}^{-1}(1-y)\,dy = y\Big|_{1}^{-1} - \frac{y^2}{2}\Big|_{1}^{-1} = -2$$

So; eq (1) becomes;

$$\Gamma = -2+2-2-2$$

$$\Gamma = -4$$

Now; by using stoke's theorem;

$$\Gamma = \iint_S \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy = \iint_S (2x-1)\,dx\,dy$$

$$\Gamma = \int_{-1}^{1}\int_{-1}^{1}(2x-1)\,dx\,dy = \int_{-1}^{1}(x^2-x)\Big|_{-1}^{1} dy = \int_{-1}^{1}(-2)\,dy$$

$$\Gamma = -2\,y\Big|_{-1}^{1} = -2(2) = -4$$

Q:- For the velocity components $u = 3x + y$; $v = 2x - 3y$ Calculate the circulation around the circle $(x-1)^2 + (y-6)^2 = 4$.

Sol:-

Given eq of circle is

$$(x-1)^2 + (y-6)^2 = 4$$

with centre $(1, 6)$ and radius $= 2$.

parametric eq/s of this circle are;

$$x = 1 + 2\cos\theta \quad \text{and} \quad y = 6 + 2\sin\theta$$

where $0 \le \theta \le 2\pi$; so

$$\Gamma = \oint_c (3x+y)\,dx + (2x-3y)\,dy$$

$$\Gamma = \int_0^{2\pi} \left[3(1+2\cos\theta) + 6 + 2\sin\theta\right](-2\sin\theta\,d\theta) + \left[2(1+2\cos\theta) - 3(6+2\sin\theta)\right](2\cos\theta)\,d\theta$$

$$\Gamma = \int_0^{2\pi} \left[-6\sin\theta - 12\sin\theta\cos\theta - 12\sin\theta - 4\sin^2\theta + 4\cos\theta + 8\cos^2\theta - 36\cos\theta - 12\sin\theta\cos\theta\right] d\theta$$

$$\Gamma = \int_0^{2\pi} \left[-18\sin\theta - 32\cos\theta - 24\sin\theta\cos\theta - 4\sin^2\theta + 8\cos^2\theta\right] d\theta$$

$$\Gamma = \int_0^{2\pi} \left[-18\sin\theta - 32\cos\theta - 12\sin 2\theta - 4\left[\frac{1-\cos 2\theta}{2}\right] + 8\left[\frac{1+\cos 2\theta}{2}\right]\right] d\theta$$

$$\Gamma = \int_0^{2\pi} \left[-18\sin\theta - 32\cos\theta - 12\sin 2\theta - 2 + 2\cos 2\theta + 4 + 4\cos 2\theta\right] d\theta$$

$$\Gamma = \int_0^{2\pi} \left[-18\sin\theta - 32\cos\theta - 12\sin 2\theta + 6\cos 2\theta + 2\right] d\theta$$

$$\Gamma = +18\cos\theta\Big|_0^{2\pi} - 32\sin\theta\Big|_0^{2\pi} + \frac{12}{2}\cos 2\theta\Big|_0^{2\pi} + \frac{6}{2}\sin 2\theta\Big|_0^{2\pi} + 2\theta\Big|_0^{2\pi}$$

$$\underset{0}{"} \qquad \underset{0}{"} \qquad \underset{0}{"} \qquad \underset{0}{"}$$

$$\Gamma = 2(2\pi - 0) = 4\pi$$

Q:- The circle $x^2 + y^2 - 2ax = 0$ is situated in a 2D flow field where $u = -by$; $v = bx$. Find the circulation in the circle.

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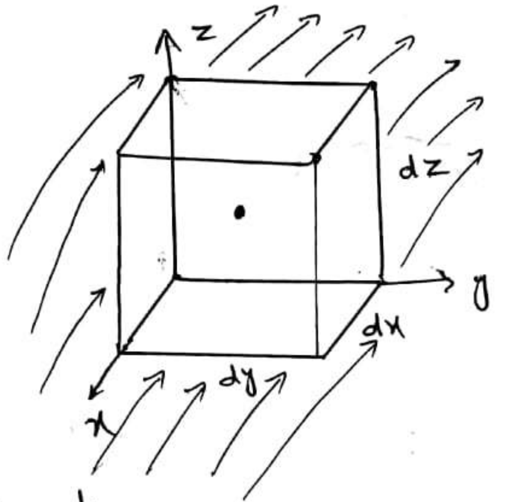
Euler's eq of motion:-

The eq/s of motion for frictionless flow are known as Euler's eq/s. These eqs are derived by applying Newton's law of motion to a fluid particle. The motion of a fluid particle under ideal conditions: i.e consider the forces; pressure, inertia, and gravity. All other forces such as surface tension and electro-magnetic forces are considered absent.

Let us consider a finite-size control volume through which an inviscid fluid is flowing; having sides dx, dy and dz. Also; let (u, v, w) be the components of the velocity $\vec{V}$ at the centre $P(x, y, z)$; and let the density of the fluid be ρ.



For x-direction;

$$\Sigma F_x = m a_x \rightarrow ①$$

$$\Sigma F_x = \text{surface forces} + \text{body forces}$$

$$\Sigma F_x = \left[\left(P - \frac{\partial P}{\partial x}\frac{dx}{2}\right) - \left(P + \frac{\partial P}{\partial x}\frac{dx}{2}\right)\right] dy\,dz + m g_x$$

$$m a_x = -\frac{\partial P}{\partial x} dx\,dy\,dz + m g_x$$

$$\rho a_x\,dx\,dy\,dz = -\frac{\partial P}{\partial x} dx\,dy\,dz + \rho g_x\,dx\,dy\,dz$$

$$\rho a_x\,dx\,dy\,dz = \left(\rho g_x - \frac{\partial P}{\partial x}\right) dx\,dy\,dz \rightarrow ②$$

Similarly; for y-direction;

$$\rho a_y\,dx\,dy\,dz = \left(\rho g_y - \frac{\partial P}{\partial y}\right) dx\,dy\,dz \rightarrow ③$$

and for z-direction;

$$\rho a_z\, dx dy dz = \left(\rho g_x - \frac{\partial p}{\partial z}\right) dx dy dz \rightarrow (4)$$

from (2), (3) and (4);

$$\rho(a_x\hat{i} + a_y\hat{j} + a_z\hat{k})\, dx dy dz = -\left[\frac{\partial p}{\partial x}\hat{i} + \frac{\partial p}{\partial y}\hat{j} + \frac{\partial p}{\partial z}\hat{k}\right] dx dy dz$$
$$+ \rho(g_x\hat{i} + g_y\hat{j} + g_z\hat{k})\, dx dy dz$$

dividing by $dV = dx dy dz$;

$$\Rightarrow \quad \rho\vec{a} = -\nabla p + \rho\vec{g}$$

$$\Rightarrow \quad \vec{a} = -\frac{1}{\rho}\nabla p + \vec{g}$$

$$\Rightarrow \quad \frac{d\vec{V}}{dt} = -\frac{1}{\rho}\nabla p + \vec{g}$$

$$\Rightarrow \quad \frac{\partial \vec{V}}{\partial t} + (\vec{V}\cdot\nabla)\vec{V} = -\frac{1}{\rho}\nabla p + \vec{g}$$

which is the vector form of Euler's eq.

Note:- If there is a body force $\rho\vec{F}$ other than gravity then Euler's eq becomes;

$$\frac{d\vec{V}}{dt} = \vec{F} - \frac{1}{\rho}\nabla p$$

i) tensor form:-

$$\frac{\partial u_i}{\partial t} + u_j\frac{\partial u_i}{\partial x_j} = F_i - \frac{1}{\rho}\frac{\partial p}{\partial x_i}$$

ii) Cylindrical form:-

$$\frac{\partial V_r}{\partial t} + V_r\frac{\partial V_r}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_r}{\partial \theta} + V_z\frac{\partial V_r}{\partial z} - \frac{V_\theta^2}{r} = F_r - \frac{1}{\rho}\frac{\partial p}{\partial r}$$

$$\frac{\partial V_\theta}{\partial t} + V_r\frac{\partial V_\theta}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_\theta}{\partial \theta} + V_z\frac{\partial V_\theta}{\partial z} + \frac{V_r V_\theta}{r} = F_\theta - \frac{1}{\rho r}\frac{\partial p}{\partial \theta}$$

$$\frac{\partial V_z}{\partial t} + V_r\frac{\partial V_z}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_z}{\partial \theta} + V_z\frac{\partial V_z}{\partial z} = F_z - \frac{1}{\rho}\frac{\partial p}{\partial z}$$

Q:- Given the following velocity field describes the motion of an incompressible fluid;

$$\tilde{V} = (x^2y+y^2)\hat{i} - xy^2\hat{j}$$

Find out a) pressure gradient in the x- and y-directions neglecting viscous (b) values of pressure gradient at (2,1); if the fluid is water.

Sol:-

$$\tilde{V} = (x^2y+y^2)\hat{i} - xy^2\hat{j}$$

$$u = x^2y+y^2 \quad ; \quad v = -xy^2$$

$$\frac{\partial u}{\partial x} = 2xy \quad ; \quad \frac{\partial v}{\partial x} = -y^2$$

$$\frac{\partial u}{\partial y} = x^2+2y \quad ; \quad \frac{\partial v}{\partial y} = -2xy$$

Euler's eqs of motion for 2-D flow neglecting viscous effects are

$$\cancel{\frac{\partial u}{\partial t}} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial x}$$

$$\cancel{\frac{\partial v}{\partial t}} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial y}$$

$$\frac{\partial P}{\partial x} = -\rho\left[(x^2y+y^2)(2xy) + (-xy^2)(x^2+2y)\right]$$

$$\frac{\partial P}{\partial x} = -\rho\left[2x^3y^2 + \cancel{2xy^3} - x^3y^2 - \cancel{2xy^3}\right]$$

$$\frac{\partial P}{\partial x} = -\rho x^3y^2 \rightarrow (1)$$

and

$$\frac{\partial P}{\partial y} = -\rho\left[(x^2y+y^2)(-y^2) + (-xy^2)(-2xy)\right]$$

$$\frac{\partial P}{\partial y} = -\rho\left[-x^2y^3 - y^4 + 2x^2y^3\right]$$

$$\frac{\partial P}{\partial y} = \rho\left[y^4 - x^2y^3\right] \rightarrow (2)$$

So; the pressure gradient is;

$$\nabla P = \frac{\partial P}{\partial x}\hat{i} + \frac{\partial P}{\partial y}\hat{j} = -\rho\left[x^3y^2\hat{i} + (x^2y^3 - y^4)\hat{j}\right]$$

at (2,1)

$$\nabla P = -1000\left[8\hat{i} + 3\hat{j}\right] \text{ N/m}^2$$

Bernolli's Equation:-

The Bernoulli's eq is an approximate relation b/w pressure, velocity and elevation; and is valid in regions of steady, incompressible flow where net frictional forces are negligible.

Statement:- Let the field of force be conservative flow is steady and density be the function of pressure alone Then

$$\int \frac{dp}{\rho} + \phi + \frac{1}{2}v^2 \text{ is constant}$$

along each streamline and each vortex.

Proof:- We know that the Euler's eq of motion is

$$\frac{\partial \vec{v}}{dt} + \vec{\Omega} \times \vec{v} + \frac{1}{2}\nabla v^2 = \vec{F} - \frac{1}{\rho}\nabla p \quad \rightarrow (1)$$

Now; since the flow is steady;

so; $\frac{\partial \vec{v}}{dt} = 0$; Also; the external (i.e body) force $\vec{F}$ is conservative so; $\vec{F} = -\nabla\phi$ where ϕ is the force potential. Also; $\rho = \rho(p)$.

so; (1) $\Rightarrow$

$$0 + \Omega \times \vec{v} + \frac{1}{2}\nabla v^2 = -\nabla\phi - \frac{1}{\rho}\nabla p$$

$$\nabla(\tfrac{1}{2}v^2) + \nabla\phi + \frac{1}{\rho}\nabla p = -\Omega \times \vec{v}$$

$$\nabla(\tfrac{1}{2}v^2) + \nabla\phi + \frac{1}{\rho}\nabla p = \vec{v} \times \Omega \rightarrow (2)$$

Taking dot product on both sides by $d\vec{r}$ along a streamline;

$$\nabla(\tfrac{1}{2}v^2)\cdot d\vec{r} + \nabla\phi\cdot d\vec{r} + \frac{1}{\rho}\nabla p\cdot d\vec{r} = -(\Omega \times \vec{v})\cdot d\vec{r}$$

$$\nabla(\tfrac{1}{2}v^2)\cdot d\vec{r} + \nabla\phi\cdot d\vec{r} + \frac{1}{\rho}\nabla p\cdot d\vec{r} = -(\Omega\cdot(\vec{v} \times d\vec{r})) \rightarrow (3)$$

Now; $d\vec{r}$ is parallel to $\vec{v}$ along a streamline

so; $\vec{v} \times d\vec{r} = 0$

Also; $\nabla\phi \cdot d\vec{r} = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz = d\phi$

similarly; $\nabla(\frac{1}{2}v^2) \cdot d\vec{r} = d(\frac{1}{2}v^2)$ and $\frac{\nabla P}{\rho} \cdot d\vec{r} = \frac{dP}{\rho}$

So; eq (3) $\Rightarrow$

$$d(\tfrac{1}{2}v^2) + d\phi + \frac{dP}{\rho} = 0$$

by integrating; we have;

$$\frac{1}{2}v^2 + \phi + \int \frac{dP}{\rho} = C \longrightarrow (4)$$

This is known as Bernoulli's eq for steady; inviscid flow. The constant of integration C called the Bernoulli's constant. In eq (4); C has same value along a given streamline but, in general, varies from streamline to streamline. Also; eq (4) is valid regardless of whether the flow is irrotational or rotational, and incompressible or compressible.

<u>Special Cases</u>:-

i) For an incompressible flow;

density is constant; so; eq (4) $\Rightarrow$

$$\frac{1}{2}v^2 + \phi + \frac{1}{\rho}\int dp = C$$

$$\frac{1}{2}v^2 + \phi + \frac{P}{\rho} = C \rightarrow (5)$$

eq (5) is restricted to steady; inviscid and incompersssible flow.

ii) In the absence of body forces;

eq takes of the form;

$$\frac{1}{2}v^2 + \frac{P}{\rho} = C$$

iii) when the body force is gravitational force;

Then $\vec{g} = +g\hat{k}$; $(\nabla z = \hat{k})$

$$\vec{g} = g\nabla z = \nabla(gz)$$

$$\Rightarrow \vec{F} = -\nabla(gz) \quad \Rightarrow \quad \phi = gz$$

So; Bernoulli's eq becomes;

$$\frac{1}{2}v^2 + \frac{P}{\rho} + gz = C$$

$$\frac{v^2}{2g} + \frac{P}{\rho g} + z = \frac{C}{g}$$

$$\frac{v^2}{2g} + \frac{P}{\rho g} + z = C^* \qquad \because C^* = \frac{C}{g}$$

This eq is applicable to ideal; rotational incompressible, barotropic and steady-state flow.

For unsteady; Irrotational, inviscid flow under Conservative forces:-

Euler's eq of motion is;

$$\frac{\partial \vec{V}}{\partial t} + \Omega \times \vec{V} + \frac{1}{2}\nabla V^2 = \vec{F} - \frac{1}{\rho}\nabla P \rightarrow (1)$$

Since flow is irrotational; so

$$\Omega = \nabla \times \vec{V} = 0 \quad \text{and} \quad \vec{V} = -\nabla\Phi_1$$

Also; $\vec{F}$ is conservative so;

$$\vec{F} = -\nabla\Phi$$

So; (1) $\Rightarrow$

$$\frac{\partial}{\partial t}(-\nabla\Phi_1) + 0 + \frac{1}{2}\nabla V^2 = -\nabla\Phi - \frac{1}{\rho}\nabla P$$

$$-\nabla\left(\frac{\partial\Phi_1}{\partial t}\right) + \nabla\left(\frac{1}{2}V^2\right) + \nabla\Phi + \frac{1}{\rho}\nabla P = 0$$

by taking dot product with $d\vec{r}$ along any line we have;

$$-d\left(\frac{\partial\Phi_1}{\partial t}\right) + d\left(\frac{1}{2}V^2\right) + d\Phi + \frac{dP}{\rho} = 0$$

by integrating we have;

$$-\frac{\partial\Phi_1}{\partial t} + \frac{1}{2}V^2 + \Phi + \int\frac{dP}{\rho} = f(t)$$

where $f(t)$ is any arbitrary fn. of time; Since t has been considered as constant

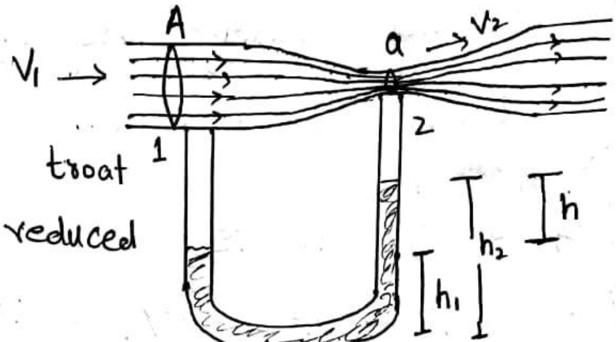
This eq hold for irrotational inviscid flow.

Applications of Bernoulli's eq:-

The venturi Meter:-

Venturi meter is a device used to measure the flow speed of a fluid in a pipe.

Let a fluid of density ρ_1 is flowing throug a pipe of cross-sectional are A. As show in fig.



At the troat area is reduced to a;

and a monometer tube is attached. Let the monometer liquid have a density ρ_2. Let V_1 and V_2 be th flow speed at pt 1, and 2. Now by applying Bernoulli's eq; we have;

$$P_1 + \frac{1}{2}\rho_1 V_1^2 + \rho_1 g h_1 = P_2 + \frac{1}{2}\rho_1 V_2^2 + \rho_1 g h_2$$

$$P_1 - P_2 = \frac{1}{2}\rho_1(V_2^2 - V_1^2) + \rho_1 g(h_2 - h_1) \rightarrow (1)$$

Now; $P_1 - P_2 = \rho_2 g h_2 - \rho_2 g h_1$

$$P_1 - P_2 = \rho_2 g(h_2 - h_1)$$

$$P_1 - P_2 = \rho_2 g h \quad \text{put in } (1)$$

$$\rho_2 g h = \frac{1}{2}\rho_1(V_2^2 - V_1^2) + \rho_1 g h$$

$$\Rightarrow (V_2^2 - V_1^2) = \frac{2(\rho_2 - \rho_1) g h}{\rho_1} \rightarrow (2)$$

By eq of continuity;

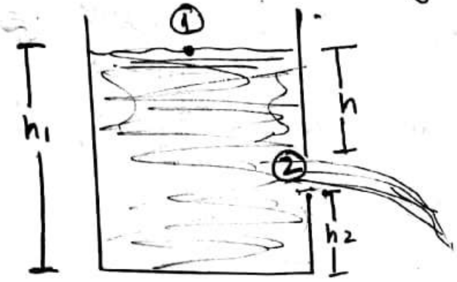
$$AV_1 = aV_2 \Rightarrow V_2 = \frac{AV_1}{a} \text{ put in } (2)$$

we have;

$$V_1 = a\sqrt{\frac{2(\rho_2 - \rho_1) g h}{\rho_1 (A^2 - a^2)}}$$

② Flow of a liquid from a large Tank:-

Let us consider a large tank;



through which a liquid is being discharged into the open atmosphere. Let v_1 be velocity at top surface and v_2 be velocity at orifice ②. Then by using Bernoulli's eq; we have

$$\frac{P_1}{\rho g} + \frac{v_1^2}{2g} + h_1 = \frac{P_2}{\rho g} + \frac{v_2^2}{2g} + h_2$$

$$\frac{v_2^2}{2g} = \frac{v_1^2}{2g} + \frac{P_1 - P_2}{\rho g} + h_1 - h_2$$

$$v_2^2 = v_1^2 + \frac{2(P_1 - P_2)}{\rho} + 2gh$$

③ Relation b/w speed and pressure:-

when a fluid is flowing horizontally with no significant change in height i.e $h_1 = h_2$

Then Benoulli's eq becomes;

$$P_1 + \frac{1}{2}\rho v_1^2 = P_2 + \frac{1}{2}\rho v_2^2$$

which tells us quantitatively that the speed is high where the pressure is low; and vice versa.

Head:- In fluid mechanics problems; it is convenient to work with energy expressed as a "head" i.e the amount of energy per unit weight of fluid. So; it has units of length.

In the eq $\frac{P}{\rho g} + \frac{v^2}{2g} + Z = C$

Each term on the left side has the dimensions of a length. So;

$\frac{P}{\rho g}$ is know as pressure head;

$\frac{V^2}{2g}$ is known as velocity head or kinetic head or dynamic head

and Z is known as gravitational or elevation head.

The Constant C on the R.h.s is known as total head; denoted by H.

So; $H = \frac{P}{\rho g} + \frac{V^2}{2g} + Z.$

Q:- water is flowing through a pipe of 70mm diameter under guage pressure of 3.5 kg/cm² and with a mean velocity of 1.5 m/sec. Neglecting friction; determine the total head if the pipe is 7 meters above the datum line.

Sol:-

diameter of pipe = 70mm = 7cm

pressure $= P = 3.5 kg/cm^2 = 35 \times 10^3 kg/m^2$

and $V = 1.5 m/s$

$Z = 7m$

So; $H = \frac{P}{\rho g} + \frac{V^2}{2g} + Z$

$$H = \frac{35 \times 10^3}{(1000)(9.8)} + \frac{(1.5)^2}{2(9.8)} + 7$$

$H =$

2-D Source:-

If the 2-D motion of a fluid is radially outward and symmetrical in all directions from a pt. in the reference plane; then the pt. is called a simple source in 2D.

So; A 2D source is a pt. at which fluid is continously created and distributed uniformly in all directions in the representative plane.

The strength m of a 2D source is defined to be the volume of fluid which emits in unit time i.e the strength is the total outward flux of fluid across any small closed curve surrounding it.

2-D Sink:-

If the two-Dim flow is such that the fluid is directed radially inwords to a pt. from all directions in the representative plane then the pt is called a Sink in 2D.

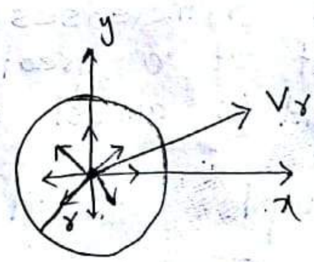
Thus; a sink is a pt. of inward radial flow at which fluid is continuously absorbeed or annihilated. So; a source of -ive strength is called a sink.

Velocity potential and stream fn. for A 2D Source:-

Let a source of strength m be placed at the origin. Since the flow is purely radial due to source

so; $\quad V_\theta = 0 \quad ; \quad V_r = V_r(r)$

Draw a circle C of radius r with centre at origin



we know that

$$m = \text{flux across } C$$

$$m = \oint_C \vec{V}\cdot\hat{n}\,ds$$

$$m = \int_0^{2\pi} V_r\, r\, d\theta = 2\pi r V_r$$

$$m = 2\pi r V_r$$

$$\Rightarrow V_r = \frac{m}{2\pi r}$$

The radial velocity V_r in terms of velocity potential ϕ is;

$$V_r = -\frac{\partial \phi}{\partial r}$$

$$\Rightarrow \frac{\partial \phi}{\partial r} = -\frac{m}{2\pi r}$$

$$\Rightarrow \partial \phi = -\frac{m}{2\pi r}\partial r$$

$$\Rightarrow \phi = -\frac{m}{2\pi}\ln r \rightarrow ①$$

Now; the radial velocity V_r in terms of stream fn. ψ; is

$$V_r = -\frac{1}{r}\frac{\partial \psi}{\partial \theta}$$

$$\Rightarrow \frac{\partial \psi}{\partial \theta} = -\frac{m}{2\pi}$$

$$\Rightarrow \psi = -\frac{m}{2\pi}\theta \rightarrow ②$$

eq ① shows that the equipotential lines are r = constant i.e concentric circles with centre at the source. Similarly; eq ② shows that the streamlines are θ = constant i.e straight lines radiating from the source at the origin.

Note:-

1) The pt. $r=0$; where V_r becomes infinite; is said to be a singularity of the solution.

2) From the eq $V_r = \frac{m}{2\pi r}$; it shows that as r increases; the speed decreases; so that at a great distance from the source the fluid is almost at rest.

$\longleftrightarrow$

Complex velocity potential for source and sink:-

The complex velocity potential $W(z)$ is given as;

$$W(z) = \phi + i\psi$$

$$W(z) = -\frac{m}{2\pi}\ln r - \frac{m}{2\pi}\theta i$$

$$= -\frac{m}{2\pi}(\ln r + i\theta)$$

$$W(z) = -\frac{m}{2\pi}\ln(re^{i\theta})$$

$$W(z) = -\frac{m}{2\pi} \ln z$$

which is the complex velocity potential due to a 2D source of strength m.

Now; the complex velocity potential due to a 2D sink of strength −m placed at origin is given by

$$W(z) = \frac{m}{2\pi} \ln z.$$

The complex velocity potentials due to a source and a sink of strengths m and −m placed at some pt. z_0 are given as;

$$W(z) = \frac{-m}{2\pi} \ln(z - z_0) \quad \text{and} \quad W(z) = \frac{m}{2\pi} \ln(z - z_0)$$

Two-Dimensional doublet or dipole:-

A combination of a source of strength m and a sink of strength −m at a small distance Δs apart; is said to form a doublet or dipole if in the limit as $\Delta s \to 0$ and $m \to \infty$ the product $m\Delta s$ remains finite and constant

i.e $\lim\limits_{\substack{\Delta s \to 0 \\ m \to \infty}} m\Delta s = \mu$ (say)

The constant μ is called the strength of dipole.

Complex velocity potential for doublet:-

Let there be a source of strength m at the pt. $ae^{i\alpha}$ and a sink of strength −m at the pt. $-ae^{i\alpha}$;

Then the complex velocity potential due to this doublet is;

$$W(z) = \frac{m}{2\pi} \ln(z + ae^{i\alpha}) - \frac{m}{2\pi} \ln(z - ae^{i\alpha})$$

$$= \frac{m}{2\pi} \left[\ln z\left(1 + \frac{ae^{i\alpha}}{z}\right) - \ln z\left(1 - \frac{ae^{i\alpha}}{z}\right) \right]$$

$$W(z) = \frac{m}{2\pi} \left[\cancel{\ln z} + \ln\left(1 + \frac{ae^{i\alpha}}{z}\right) - \cancel{\ln z} - \ln\left(1 - \frac{ae^{i\alpha}}{z}\right) \right]$$

$$W(z) = \frac{m}{2\pi} \left[\ln\left(1 + \frac{ae^{i\alpha}}{z}\right) - \ln\left(1 - \frac{ae^{i\alpha}}{z}\right) \right]$$

$$W(z) = \frac{m}{2\pi} \left[\frac{ae^{i\alpha}}{z} - \frac{a^2 e^{2i\alpha}}{2z^2} + \frac{a^3 e^{3i\alpha}}{3z^3} - \cdots - \left(-\frac{ae^{i\alpha}}{z} - \frac{a^2 e^{2i\alpha}}{2z^2} - \cdots \right) \right]$$

$$w(z) = \frac{m}{2\pi}\left[\frac{ae^{i\alpha}}{z} - \cancel{\frac{a^2e^{2i\alpha}}{2z^2}} + \frac{a^3e^{3i\alpha}}{3z^3} + \cdots \quad \frac{ae^{i\alpha}}{z} + \cancel{\frac{a^2e^{2i\alpha}}{2z^2}} + \frac{a^3e^{3i\alpha}}{3z^3} + \cdots\right.$$

$$w(z) = \frac{m}{2\pi}\left[\frac{2ae^{i\alpha}}{z} + \frac{2a^3e^{3i\alpha}}{3z^3} + \frac{2a^5e^{3i\alpha}}{5z^5} + \cdots\right] \rightarrow (1)$$

we know that for a doulet;

if $2a \to 0$ and $m \to \infty$

Then $2am \to \mu$

So; (1) $\Rightarrow$

$$w(z) = \lim_{\substack{2a\to 0\\ m\to\infty}} \frac{m}{2\pi}\left[\frac{2ae^{i\alpha}}{z} + \frac{2a^3e^{3i\alpha}}{3z^3} + \frac{2a^5}{5z^5} + \cdots\right]$$

$$w(z) = \frac{\mu}{2\pi}\frac{e^{i\alpha}}{z}$$

<u>Velocity potential and stream fn:-</u>

$$w(z) = \frac{\mu}{2\pi}\frac{e^{i\alpha}}{z}$$

$$\phi + i\psi = \frac{\mu}{2\pi}\frac{e^{i\alpha}}{re^{i\theta}}$$

$$= \frac{\mu}{2\pi r}e^{i(\alpha-\theta)}$$

$$\phi + i\psi = \frac{\mu}{2\pi r}\left[\cos(\alpha-\theta) + i\sin(\alpha-\theta)\right]$$

$$\Rightarrow \quad \phi = \frac{\mu}{2\pi r}\cos(\alpha-\theta)$$

and $\quad \psi = \dfrac{\mu}{2\pi r}\sin(\alpha-\theta)$

<u>Velocity components:-</u> since $w = \dfrac{\mu}{2\pi}\dfrac{e^{i\alpha}}{z}$

So; $\quad \dfrac{dw}{dz} = \dfrac{-\mu e^{i\alpha}}{2\pi z^2}$

$$(-V_r + iV_\theta)e^{-i\theta} = -\frac{\mu}{2\pi}\frac{e^{i\alpha}}{r^2e^{i2\theta}}$$

$$(-V_r + iV_\theta) = -\frac{\mu}{2\pi r^2}e^{i(\alpha-\theta)}$$

$$-V_r + iV_\theta = \frac{-\mu}{2\pi r^2}\left[\cos(\alpha-\theta) + i\sin(\alpha-\theta)\right]$$

So; $\quad V_r = \dfrac{\mu}{2\pi r^2}\cos(\alpha-\theta)$ and $V_\theta = \dfrac{-\mu}{2\pi r^2}\sin(\alpha-\theta)$

2D-Vortex:-

The fluid motion in which the stream lines are concentric circles is called a vortex.

Irrotational vortex:-

If the particles of fluid moving in a vortex do not rotate about their own centres. Then the vortex is called irrotational or free vortex or potential vortex.

Velocity field for an irrotational Vortex:-

Let an irrotational vortex be placed at the origin. Since the flow due to this vortex is purely circular; the radial and transverse components of velocity are given by;

$$V_r = 0 \; ; \quad V_\theta = V_\theta(r)$$

Since the flow is irrotational; so it must satisfy the vorticity eq $\xi_z = 0$

$$\Rightarrow \frac{\partial V_\theta}{\partial r} + \frac{V_\theta}{r} - \frac{1}{r}\frac{\partial V_r}{\partial \theta} = 0$$

$$\Rightarrow \frac{dV_\theta}{dr} + \frac{V_\theta}{r} = 0$$

$$\Rightarrow \frac{dV_\theta}{V_\theta} = -\frac{dr}{r}$$

$$\Rightarrow \ln V_\theta = -\ln r + c_1$$

$$\Rightarrow \ln V_\theta = \ln r^{-1} + c_1$$

$$\Rightarrow V_\theta = e^{\ln r^{-1} + c_1}$$

$$V_\theta = \frac{c}{r} \longrightarrow (1)$$

Here c is a constant to be determined.
Now; the circulation around a circle of radius r is given by;

$$\Gamma = \oint_c \vec{V}\cdot d\vec{r} = \oint_c \vec{V}\cdot\hat{T}\,ds = \oint_c V\,ds$$

$$\Gamma = \int_0^{2\pi} V_\theta\, r\, d\theta = \int_0^{2\pi} \frac{c}{r}\cdot r\, d\theta = c\,(2\pi - 0) = 2\pi c$$

$$\Rightarrow c = \frac{\Gamma}{2\pi}$$

So; (1) $\Rightarrow$ $$V_\theta = \frac{\Gamma}{2\pi r}$$

Here; Γ is known as the strength of the vortex

Velocity potential and stream fn:-

The vorticity is given by

$$\xi_z = \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}$$

$$= -\frac{\Gamma}{2\pi r^2} + \frac{\Gamma}{2\pi r^2}$$

$$= 0$$

So; the flow in this case irrotational; so the velocity potential exist.

Now;

$$v_\theta = -\frac{1}{r}\frac{\partial \phi}{\partial \theta}$$

$$\frac{\partial \phi}{\partial \theta} = -\frac{\Gamma}{2\pi} \Rightarrow \boxed{\phi = -\frac{\Gamma}{2\pi}\theta}$$

Now;

$$v_\theta = \frac{\partial \psi}{\partial r}$$

$$\Rightarrow \frac{\partial \psi}{\partial r} = \frac{\Gamma}{2\pi r} \Rightarrow \boxed{\psi = \frac{\Gamma}{2\pi}\ln r}$$

Now; eq of streamlines are;

$$\psi = \text{Const.}$$

$$\frac{\Gamma}{2\pi}\ln r = \text{Const} \Rightarrow \ln r = \text{Const}$$

$$\Rightarrow r = \text{Const.}$$

which are Concentric circles centre at the origin.

Now; equipotential lines are;

$$\phi = \text{Const}$$

$$\Rightarrow -\frac{\Gamma}{2\pi}\theta = \text{Const}$$

$$\Rightarrow \theta = \text{Const.}$$

which are straight lines starting from origin.

Complex Velocity potential:-

Complex velocity potential $w(z)$ is given as;

$$w(z) = \phi + i\psi$$

$$w(z) = -\frac{\Gamma}{2\pi}\theta + i\frac{\Gamma}{2\pi}\ln r$$

$$= \frac{i\Gamma}{2\pi}(\ln r + i\theta)$$

$$w = \frac{i\Gamma}{2\pi}(\ln re^{i\theta}) = \frac{i\Gamma}{2\pi}\ln z$$

Superposition of two equal Sources:-

Let two sources of equal strength m; placed at the pts. $(-a, 0)$ and $(a, 0)$;

complex velocity potential:-

The complex velocity potential for this combination is;

$$W(z) = \frac{-m}{2\pi} \ln(z+a) - \frac{m}{2\pi} \ln(z-a)$$

$$w = \frac{-m}{2\pi} \ln(z^2 - a^2)$$

$$\phi + i\psi = \frac{-m}{2\pi} \ln(x^2 - y^2 + 2xyi - a^2)$$

$$\phi + i\psi = \frac{-m}{2\pi} \left[\ln\sqrt{(x^2-y^2-a^2)^2 + 4x^2y^2} + i \tan^{-1} \frac{2xy}{x^2-y^2-a^2} \right]$$

So;

$$\phi = \frac{-m}{4\pi} \ln\left[(x^2-y^2-a^2)^2 + 4x^2y^2\right]$$

and $\psi = -\frac{m}{2\pi} \tan^{-1} \frac{2xy}{x^2-y^2-a^2}$

which are velocity potential and stream fn. for a combination of two sources of equal strength.

Velocity components:-

Since $w = \frac{-m}{2\pi} \ln(z^2 - a^2)$.

We know that

$$-u + iv = \frac{dw}{dz}$$

So;

$$-u + iv = \frac{-m}{\cancel{2}\pi} \frac{1}{z^2 - a^2} (\cancel{2}z)$$

$$-u + iv = \frac{-mz}{\pi(z^2 - a^2)} = \frac{-m}{\pi} \left[\frac{x + iy}{(x^2-y^2-a^2) + 2ixy} \right]$$

$$-u + iv = \frac{-m}{\pi} \left[\frac{(x+iy)((x^2-y^2-a^2) - 2ixy)}{(x^2-y^2-a^2)^2 + 4x^2y^2} \right]$$

$$-u + iv = \frac{-m}{\pi} \left[\frac{x(x^2-y^2-a^2) + 2xy^2 + i\left[y(x^2-y^2-a^2) - 2x^2y\right]}{(x^2-y^2-a^2)^2 + 4x^2y^2} \right]$$

So; $u = \frac{m}{\pi} \left[\frac{x(x^2-y^2-a^2) + 2xy^2}{(x^2-y^2-a^2)^2 + 4x^2y^2} \right]$

and $v = -\frac{m}{\pi} \left[\frac{y(x^2-y^2-a^2) - 2x^2y}{(x^2-y^2-a^2)^2 + 4x^2y^2} \right]$

Assignment:-

Q Find the velocity potential, stream fn. and velocity for the supperposition of equal strength.
i) A source and a sink of equal strength.
ii) A source and a vortex.

Ans:- i) $\phi = -\frac{m}{4\pi}\ln\frac{(x+a)^2+y^2}{(x-a)^2+y^2}$

$$\psi = \frac{m}{2\pi}\tan^{-1}\left[\frac{2ay}{x^2+y^2-a^2}\right]$$

and $V = \frac{ma}{\pi}\frac{1}{\sqrt{(x^2-y^2-a^2)^2+4x^2y^2}}$

ii) $\phi = -\frac{m}{4\pi}\ln(x^2+y^2) - \frac{\Gamma}{4\pi}\tan^{-1}\frac{y}{x}$

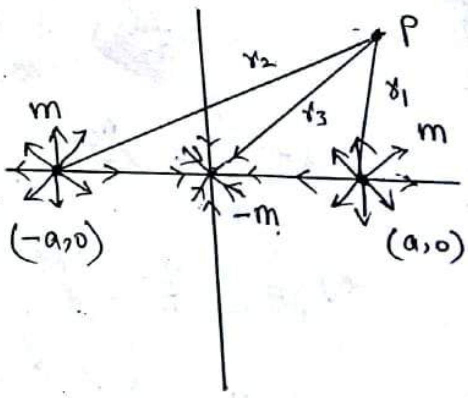
$$\psi = \frac{-m}{2\pi}\tan^{-1}\frac{y}{x} + \frac{\Gamma}{4\pi}\ln(x^2+y^2)$$

and $V = \frac{1}{2\pi}\sqrt{\frac{m^2+\Gamma^2}{x^2+y^2}}$

Q:- Find the expression for speed at a pt. due to two equal sources and an equal sink.

Sol:- Consider two sources each of strength m are placed at the pts. (a, o) and (-a, o) and a sink of strength -m at the origin;

The complex velocity potential at any pt P is given as;



$$w = -\frac{m}{2\pi}\ln(z-a) - \frac{m}{2\pi}\ln(z+a) + \frac{m}{2\pi}\ln z$$

$$w = \frac{-m}{2\pi}\left[\ln(z-a)+\ln(z+a)-\ln z\right]$$

Now;

$$\frac{dw}{dz} = \frac{-m}{2\pi}\left[\frac{1}{z-a}+\frac{1}{z+a}-\frac{1}{z}\right]$$

$$\frac{dw}{dz} = \frac{-m}{2\pi}\left[\frac{z^2+a^2}{(z-a)(z+a)z}\right]$$

Now; speed is given as;

$$V = \left|\frac{dw}{dz}\right|$$

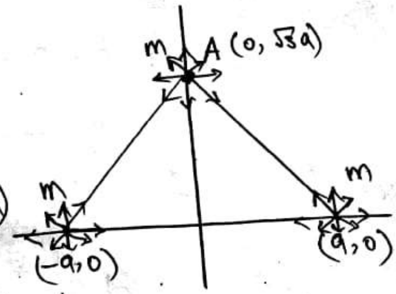
$$V = \frac{m|z^2+a^2|}{2\pi|z-a||z+a||z|} = \frac{m|z^2+a^2|}{2\pi r_1 r_2 r_3}$$

is required speed.

Stream fn. for equal sources 'm' placed at the corners of an equilateral triangle.

Let each side of the equilateral triangle ABC be 2a and let the coordinates of pts A, B, C be $(0, \sqrt{3}a)$ $(-a, 0)$ and $(a, 0)$.

Then the complex velocity potentia is given as;



$$w = -\frac{m}{2\pi}\ln(z-a) - \frac{m}{2\pi}\ln(z+a) - \frac{m}{2\pi}\ln(z - i\sqrt{3}a)$$

$$w = -\frac{m}{2\pi}\ln(z^2-a^2) - \frac{m}{2\pi}\ln(z - i\sqrt{3}a)$$

$$w = \frac{-m}{2\pi}\ln(x^2-y^2-a^2+2ixy) - \frac{m}{2\pi}\ln\left(x + i(y-\sqrt{3}a)\right)$$

$$\phi + i\psi = \frac{-m}{2\pi}\left[\frac{1}{2}\ln\left((x^2-y^2-a^2)^2 + i\tan^{-1}\frac{2xy}{x^2-y^2-a^2}\right)\right]$$

$$-\frac{m}{2\pi}\left[\frac{1}{2}\ln\left(x^2+(y-\sqrt{3}a)^2\right) + i\tan^{-1}\frac{y-\sqrt{3}a}{x}\right]$$

So; $\phi = \frac{-m}{4\pi}\left[\ln\left[(x^2-y^2-a^2)^2 + 4x^2y^2 + \ln(x^2+(y-\sqrt{3}a)^2\right)\right]$

and $\psi = \frac{-m}{2\pi}\left[\tan^{-1}\frac{2xy}{x^2-y^2-a^2} + \tan^{-1}\frac{y-\sqrt{3}a}{x}\right]$

$$\psi = -\frac{m}{2\pi}\tan^{-1}\left(\frac{\frac{2xy}{x^2-y^2-a^2} + \frac{y-\sqrt{3}a}{x}}{1 - \left(\frac{2xy}{x^2-y^2-a^2}\right)\left(\frac{y-\sqrt{3}a}{x}\right)}\right)$$

$$\psi = -\frac{m}{2\pi}\tan^{-1}\frac{2x^2y + (y-\sqrt{3}a)(x^2-y^2-a^2)}{x(x^2-y^2-a^2) - 2xy(y-\sqrt{3}a)}$$

Source in a uniform Stream:-

Let a source of strength m be placed at the origin. Let the uniform stream be flowing with velocity U in the +ive direction of the x-axis.

The complex velocity potential for this combination is given as;

$$w = -Uz - \frac{m}{2\pi} \ln z$$

$$\phi + i\psi = -Ure^{i\theta} - \frac{m}{2\pi} \ln re^{i\theta}$$

$$\phi + i\psi = -Ur(\cos\theta + i\sin\theta) - \frac{m}{2\pi}(\ln r + i\theta)$$

So;

$$\phi = -Ur\cos\theta - \frac{m}{2\pi} \ln r$$

$$\psi = -Ur\sin\theta - \frac{m}{2\pi}\theta$$

Velocity Components:-

$$\frac{dw}{dz} = -U - \frac{m}{2\pi z}$$

$$\Rightarrow (-V_r + iV_\theta)e^{-i\theta} = -U - \frac{m}{2\pi r}e^{-i\theta}$$

$$\Rightarrow -V_r + iV_\theta = -Ue^{i\theta} - \frac{m}{2\pi r}$$

$$\Rightarrow -V_r + iV_\theta = -U(\cos\theta + i\sin\theta) - \frac{m}{2\pi r}$$

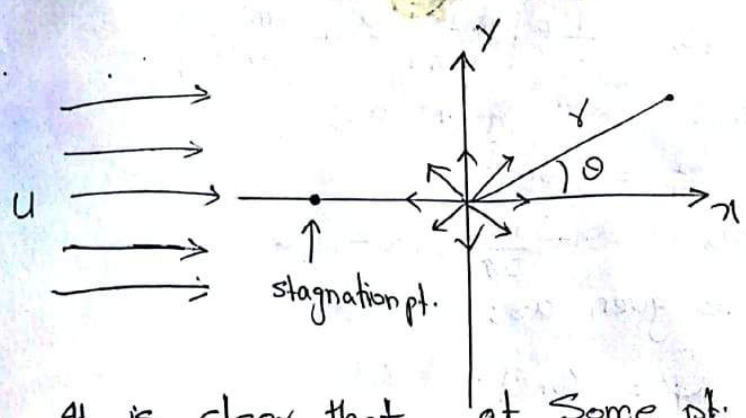
$$\Rightarrow V_r = U\cos\theta + \frac{m}{2\pi r}$$

and $V_\theta = -U\sin\theta$

and

$$V = \sqrt{V_r^2 + V_\theta^2}$$

$$V = \sqrt{U^2 + \frac{mU\cos\theta}{\pi r} + \frac{m^2}{4\pi^2 r^2}}$$



It is clear that at some pt. along the -ive x-axis the velocity due to the source will just cancel the velocity due to the uniform stream; and a stagnation pt. will be created.

To find stagnation pt

$$\frac{dw}{dz} = 0$$

$$-U - \frac{m}{2\pi z} = 0$$

$$\Rightarrow \frac{m}{2\pi z} = -U$$

$$\Rightarrow z = -\frac{m}{2\pi U}$$

$$\Rightarrow x = \frac{-m}{2\pi U} \quad \text{and} \quad y = 0$$

in polar coordinates; the stagnation pt is;

$$r = \frac{m}{2\pi U} \quad \text{and} \quad \theta = \pi$$

This the only stagnation pt There can not be a stagnation pt. on the right side of the origin since both velocities have the same sense.

The pressure distribution at any pt can be determined from the Bernoulli's eq. Thus applying the Bernoulli's eq b/w a pt far from the body; where the pressure is P_∞ and velocity is U; and some arbitrary pt. with pressure p and velocity V is;

$$p + \frac{1}{2}\rho V^2 = P_\infty + \frac{1}{\rho}U^2$$

$$\Rightarrow p = P_\infty + \frac{1}{2}\rho(U^2 - V^2)$$

$$\Rightarrow p = P_\infty + \frac{1}{2}\rho\left(\cancel{U^2} - \cancel{U^2} - \frac{mU\cos\theta}{\pi r} + \frac{m^2}{4\pi^2 r^2}\right)$$

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$$p = p_\infty - \frac{1}{2}\rho\left[\frac{mU\cos\theta}{\pi r} + \frac{m^2}{4\pi^2 r^2}\right]$$

equipotential lines are given as;

$$\phi = \text{Const}$$

$$Ur\cos\theta + \frac{m}{2\pi}\ln r = C_1$$

streamlines are given as;

$$\psi = \text{Const}$$

$$Ur\sin\theta + \frac{m}{2\pi}\theta = C_2$$

streamlines through stagnation pt are obtained as; by puting $r = \frac{m}{2\pi U}$ and $\theta = \pi$

Then $U\left[\frac{m}{2\pi U}\right]\underbrace{\sin\pi}_{0} + \frac{m}{2\pi}(\pi) = C_2$

$$\Rightarrow C_2 = \frac{m}{2}$$

So; streamline through stagnation pt is;

$$Ur\sin\theta + \frac{m\theta}{2\pi} = \frac{m}{2}$$

from which the radial distance to any pt on this streamline is

$$r = \frac{m(\pi-\theta)}{2\pi U\sin\theta}$$

Now;

as $x \to \infty$ then

$$r\cos\theta = \infty$$

$$\Rightarrow \frac{m(\pi-\theta)}{2\pi U\sin\theta}\cos\theta = \infty$$

$$\frac{2\pi U\tan\theta}{m(\pi-\theta)} = 0$$

$$\Rightarrow \tan\theta = 0$$

$$\Rightarrow \theta = 0$$

So; when $x \to \infty$ then this streamline becomes parallel to x-axis;

The ⊥ distance from x-axis to the streamline represents the maximum half-width of the body.

Now; at $x \to \infty$; θ becomes 0;

So;

$$y = r\sin\theta$$

$$y_{max} = \frac{m(\pi - \theta)}{2\pi U} = \frac{m\pi}{2\pi U}$$

$$y_{max} = \frac{m}{2U}$$

Total width $= 2\left(\frac{m}{2U}\right) = \frac{m}{U}$.

physically ; the combination of a uniform stream and a source can be used to describe the flow around a streamlined body placed in a uniform stream. The body is open at the down stream end ; and thus is called a half body or Rankine body or a semi-infinite body.

Method of images:-

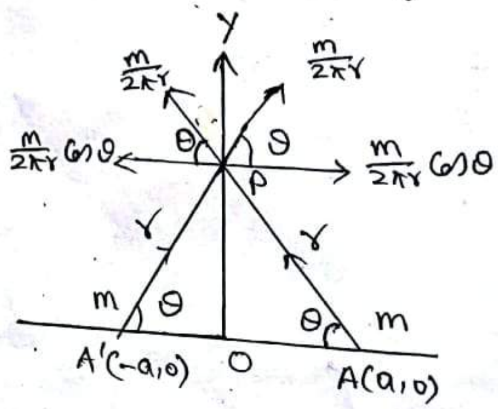
Method of images is used to determine the flow due to sources, sinks and vortices in the presence of rigid boundries.

suppose that a system S of sources; sinks; doublets and vortices is present in a region outside a known rigid boundry C. If it is possible to find another system S′ lying inside C so that the rigid boundry C is a streamline of the combined flow made up of the system S and S′; then S′ is said to be the image of system S w.r.t the rigid boundry C.

Image of a source w.r.t a plane:-

Let there be a 2-D source of strength placed at the pt A(a,o) and let the plane of a unit thickness by represented by y-axis.

We want to find the image of this source w.r.t y-axis. place an equal source of stength m at the pt. A′(-a,o).



Let P be any pt on the y-axis s.t

$AP = A'P = r$

Then velocity at P due to source A along $AP = \frac{m}{2\pi r}$

Similarly;

velocity at P due to A′ along $A'P = \frac{m}{2\pi r}$

Components of velocities ⊥ to y-axis at P are equal in magnitude but opposite in direction so;

resultant normal velocity at $P = -\frac{m}{2\pi r}\cos\theta + \frac{m}{2\pi r}\cos\theta$

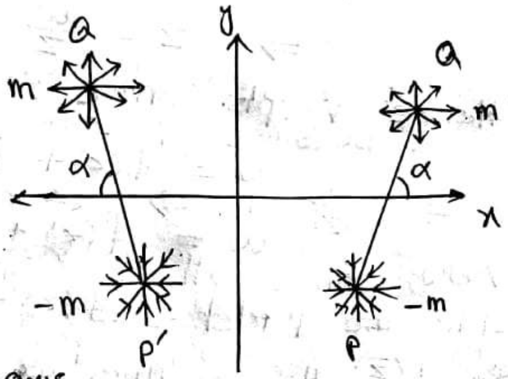
$= 0$

Hence the flow is entirely tangential to plane. Thus there will be no flow across y-axis.

Image of a doublet w.r.t a plane:-

Let PQ be a two dimensional doublet of strength μ with its axis making an angle α with the +ive direction of x-axis.

we can regard this doublet as a limiting case of the combination of a sink $-m$ at P and a source m at Q.



Let P′ and Q′ be the optical images of the pts. P and Q respectively; w.r.t y-axis regarded as representing the given plane. Then the image of the sink at P is an equal sink at P′ and the image of the source at Q is an equal source at Q′. proceeding to the limit as $P \to Q$; we have $P' \to Q'$ and the image of the doublet of strength μ making an angle α with the x-axis is therefore a doublet of equal strength symmetrically placed making an angle $\pi - \alpha$ with the +ive direction of x-axis.

Milne-Thomson Circle theorem:-

This theorem is used to calculate the flow outside the cylinder.

Statement:- Let there be 2D incompressible irrotational flow of an inviscid fluid in the z-plane. Let there be no rigid boundaries within the fluid and let the complex velocity potential of the flow be $w = f(z)$; where all the singularities of $f(z)$ are located at a distance greater than a from origin. Then if a solid circular cylinder $|z| = a$ is introduced into the flow the complex velocity potential of the resulting flow becomes

$$w = f(z) + \bar{f}\left(\frac{a^2}{z}\right) \quad \text{for } |z| \geq a$$

Proof:- To prove the theorem; we have to prove that

i) the circle $|z| = a$ represents the the streamline $\psi = 0$

ii) the singularities of $f(z)$ and $f(z) + \bar{f}\left(\frac{a^2}{z}\right)$ are the Same outside the circle $|z| = a$

Let c be the cross-section of the circular cylinder $|z| = a$; Then on the circle C; $|z| = a$

$$|z|^2 = a^2 \Rightarrow z\bar{z} = a^2$$

$$\Rightarrow \bar{z} = \frac{a^2}{z}$$ where $\bar{z}$ is image of z w.r.t circle.

If z is outside the circle then $\bar{z}$ is inside the circle. Because if z is outside then $|z| > a$

$$\Rightarrow \frac{a}{|z|} < 1 \Rightarrow \frac{a^2}{|z|} < a \Rightarrow \frac{a^2}{z}$$ is inside C.

Now; since all the singularities of $f(z)$ lie out side the circle $|z| = a$; and so the singularities of $f(\bar{z})$ and therefore those of $\bar{f}(\bar{z})$ lie inside. Therefore ; $\bar{f}(\bar{z})$ introduces no singularity outside the circle. Thus the fn. $f(z)$ and $f(z) + \bar{f}(\bar{z})$ both have the same singularities outside C. Therefore the conditions satisfied by $f(z)$ in the absence of the cylinder are satisfied by $f(z) + \bar{f}(\bar{z})$ in the presence of the cylinder.

So; the complex velocity potential after insertion of the cylinder $|z| = a$ is

$$w = f(z) + \bar{f}\left(\frac{a^2}{z}\right)$$

$$w = f(z) + \bar{f}(\bar{z})$$

$$\phi + i\psi = f(z) + \overline{f(z)}$$

$$= \text{a purely real quantity}$$

So; $\psi = 0$ on $|z| = a$

$\Rightarrow$ $|z| = a$ be a part of streamline $\psi = 0$ in the new flow.

Image System of a Source w.r.t a circular cylinder:-

consider a source of strength m placed at the pt. A(b,0). The complex velocity potential due to this Source in the absence of rigid boundries is $\frac{-m}{2\pi}\ln(z-b)$.

Let a circular cylinder of cross-section $|z|=a$ where $a<b$; be inserted into the flow ; then by the circle theorem; the com velocity potential is given by ;

$$w = f(z) + \bar{f}\left(\frac{a^2}{z}\right)$$

$$w = -\frac{m}{2\pi}\ln(z-b) - \frac{m}{2\pi}\ln\left(\frac{a^2}{z}-b\right)$$

$$\Rightarrow w = -\frac{m}{2\pi}\ln(z-b) - \frac{m}{2\pi}\ln\left(\frac{a^2-bz}{z}\right)$$

$$= \frac{-m}{2\pi}\ln(z-b) - \frac{m}{2\pi}\ln(a^2-bz) + \frac{m}{2\pi}\ln z$$

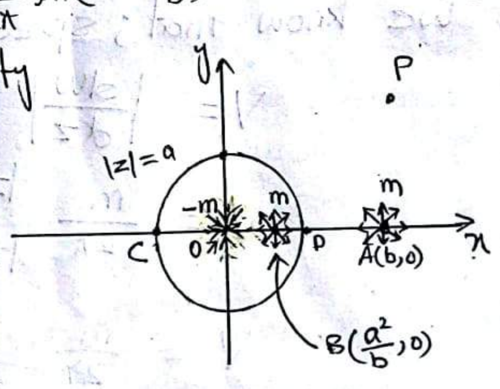
$$= \frac{-m}{2\pi}\ln(z-b) - \frac{m}{2\pi}\ln\left[(-b)\left(z-\frac{a^2}{b}\right)\right] + \frac{m}{2\pi}\ln z$$

$$w = \frac{-m}{2\pi}\ln(z-b) - \frac{m}{2\pi}\ln(-b) - \frac{m}{2\pi}\ln\left(z-\frac{a^2}{b}\right) + \frac{m}{2\pi}\ln z$$

neglecting the constant term; we have

$$w = -\frac{m}{2\pi}\ln(z-b) - \frac{m}{2\pi}\ln\left(z-\frac{a^2}{b}\right) + \frac{m}{2\pi}\ln z \rightarrow (1)$$

(1) represents the complex velocity potential due to



→ a source of strength m at $z=b$

→ a source of strength m at $z=\frac{a^2}{b}$

→ a sink of strength $-m$ at $z=0$

For this complex velocity potential ; the circle is a streamline becase $OA.OB = (b)\left(\frac{a^2}{b}\right) = a^2$; there for A and B are inverse pts. w.r.t the circle $|z|=a$. Also since $a<b$ therefore $a^2<ab \Rightarrow \frac{a^2}{b}<a$

hence B is inside the circle.

Thus the image system for a source of strength m outside a circular consists of a source of strength m at the inverse pt and a sink of strength −m at the centre of the circular cylinder.

Speed At any point:-

$$W = \frac{-m}{2m}\ln(z-b) - \frac{m}{2\pi}\ln\left(z-\frac{a^2}{b}\right) + \frac{m}{2\pi}\ln z$$

So;

$$\frac{dw}{dz} = \frac{-m}{2\pi}\left[\frac{1}{z-b} + \frac{1}{z-\frac{a^2}{b}} - \frac{1}{z}\right]$$

$$= \frac{-m}{2\pi}\left[\frac{z\left(z-\frac{a^2}{b}\right) + z(z-b) - (z-b)\left(z-\frac{a^2}{b}\right)}{z(z-b)\left(z-\frac{a^2}{b}\right)}\right]$$

$$= \frac{-m}{2\pi}\left[\frac{z^2 - \cancel{\frac{a^2}{b}z} + \cancel{z^2} - \cancel{bz} - \cancel{z^2} + \cancel{\frac{a^2}{b}z} + \cancel{bz} - a^2}{z(z-b)\left(z-\frac{a^2}{b}\right)}\right]$$

$$= \frac{-m}{2\pi}\left[\frac{z^2-a^2}{z(z-b)\left(z-\frac{a^2}{b}\right)}\right]$$

$$\frac{dw}{dz} = \frac{-m}{2\pi}\left[\frac{(z-a)(z+a)}{z(z-b)\left(z-\frac{a^2}{b}\right)}\right] \quad (1)$$

We know that; speed at any pt. is given as;

$$V = \left|\frac{dw}{dz}\right|$$

$$V = \frac{m}{2\pi}\,\frac{|z-a||z+a|}{|z|\,|z-b|\,\left|z-\frac{a^2}{b}\right|}$$

$$V = \frac{m}{2\pi}\,\frac{PD\cdot PC}{PO\cdot PA\cdot PB}$$

where C and D be pts. in which x-axis cuts the circle.

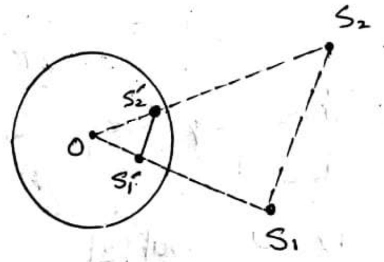
Corollary:- A source inside a circle and a sink at the centre has for image system an equal source at the inverse pt. of the given source.

Image of a doublet w.r.t a circular cylinder:-

Consider the combination of a sink of strength $-m$ at S_1 and a source of strength m at S_2 outside a circular cylinder of radius a with centre at the origin.

If S_1' and S_2' are the inverse pts. of S_1 and S_2; then the image of sink at S_1 is;

a sink of strength $-m$ at S_1' and a source of strength m at the centre O.



Similarly; the image of the source at S_2 is; a source of strength m at S_2' and a sink of strength $-m$ at O.

Combining these; we have a sink of strength $-m$ at S_1' and a source of strength m at S_2' since the source and sink at O cancel eachother.

Hence; the image of the given doublet S_1S_2 is an other doublet $S_1'S_2'$.

Let μ be the strength of doublet at the pt $z = b$; its axis being inclined at an angle α with the x-axis then;

the complex velocity potential in the absence of the cylinder is

$$f(z) = \frac{\mu}{2\pi}\frac{e^{i\alpha}}{z-b}$$

when the cylinder $|z| = a$ is inserted; then the complex velocity potential; by circle theorem; is given as;

$$w = f(z) + \bar{f}\left(\frac{a^2}{z}\right)$$

$$w = \frac{\mu}{2\pi}\frac{e^{i\alpha}}{z-b} + \frac{\mu}{2\pi}\frac{e^{-i\alpha}}{\frac{a^2}{z}-b}$$

$$w = \frac{\mu}{2\pi}\frac{e^{i\alpha}}{z-b} - \frac{\mu z e^{-i\alpha}}{2\pi b\left(z-\frac{a^2}{b}\right)}$$

$$w = \frac{\mu}{2\pi}\frac{e^{i\alpha}}{z-b} + \frac{\mu\left(z-\frac{a^2}{b}+\frac{a^2}{b}\right)e^{i(\pi-\alpha)}}{2\pi b\left(z-\frac{a^2}{b}\right)}$$

$$w = \frac{\mu}{2\pi}\frac{e^{i\alpha}}{z-b} + \frac{\mu e^{i(\pi-\alpha)}}{2\pi b} + \frac{\mu a^2}{2\pi b^2}\frac{e^{i(\pi-\alpha)}}{z-\frac{a^2}{b}}$$

neglecting constant term;

$$w = \frac{\mu}{2\pi}\frac{e^{i\alpha}}{z-b} + \frac{\mu\frac{a^2}{b^2}}{2\pi}\frac{e^{i(\pi-\alpha)}}{z-\frac{a^2}{b}}$$

This eq represents; the complex velocity potential due to;

i) a doublet of strength μ at $z=b$ inclined at an angle α with the x-axis.

ii) a doublet of strength $\mu\frac{a^2}{b^2}$ at the inverse pt. $z=\frac{a^2}{b}$ inclined at an angle $\pi-\alpha$ with the +ive x-axis.

Thus the image of a 2D doublet of strength μ outside a circular cylinder of radius a placed at a distance b from the centre of a cylinder is an anti-parallel doublet of strength $\mu\frac{a^2}{b^2}$ placed the inverse pt.

Q:- what arrangement of sources and sinks will give rise to the function $w = \log(z-\frac{a^2}{z})$? Also prove that two of the streamlines are a circle $r=a$ and $x=0$.

Sol:- Here

$$w = \log\left(z-\frac{a^2}{z}\right)$$

$$w = \log\left(\frac{z^2-a^2}{z}\right)$$

$$w = \log(z^2-a^2) - \log z$$

$$w = \log(z-a) + \log(z+a) - \log z$$

which represents

i). a sink at $z=a$; of strength -2π.

ii) a sink at $z=-a$; of strength -2π

iii) a source at $z=0$; of strength 2π.

Now; $\Phi + i\Psi = \log(x+iy-a) + \log(x+iy+a) - \log(x+iy)$

$$\Rightarrow \Phi + i\Psi = \log\sqrt{(x-a)^2+y^2} + i\tan^{-1}\frac{y}{x-a} + \sqrt{(x+a)^2+y^2} + i\tan^{-1}\frac{y}{x+a} - \log\sqrt{x^2+y^2} - i\tan^{-1}\frac{y}{x}$$

$$\Rightarrow \Psi = \tan^{-1}\frac{y}{x-a} + \tan^{-1}\frac{y}{x+a} - \tan^{-1}\frac{y}{x}$$

$$\Psi = \tan^{-1}\left[\frac{\frac{y}{x-a}+\frac{y}{x+a}}{1-\frac{y^2}{x^2-a^2}}\right] - \tan^{-1}\frac{y}{x}$$

$$\Psi = \tan^{-1}\frac{y(x+a)+y(x-a)}{x^2-a^2-y^2} - \tan^{-1}\frac{y}{x}$$

$$\Psi = \tan^{-1}\frac{2yx}{x^2-a^2-y^2} - \tan^{-1}\frac{y}{x}$$

$$\Psi = \tan^{-1}\left[\frac{\frac{2xy}{x^2-a^2-y^2} - \frac{y}{x}}{1+\left(\frac{2xy}{x^2-a^2-y^2}\right)\frac{y}{x}}\right]$$

$$\Psi = \tan^{-1}\left(\left[\frac{x^2+y^2+a^2}{x^2+y^2-a^2}\right]\frac{y}{x}\right)$$

eq of streamlines is given as;

$$\Psi = \text{Constant} = C \quad (\text{say})$$

Then $\quad (x^2+y^2+a^2)y = (x^2+y^2-a^2)x\tan C$

in particular if we take $C = \frac{\pi}{2}$ then;

$$(x^2+y^2-a^2)x = 0$$

$$x^2+y^2-a^2 = 0 \;;\; x = 0$$

$$x^2+y^2 = a^2 \;;\; x = 0$$

$$\Rightarrow \quad r = a \quad \text{and} \quad x = 0$$

proved

Stress Vector:-

Let S be the surface of a body which is subjected to a system of forces. Let $P(x_1, x_2, x_3)$ be a point on the surface element ΔS and $\hat{n}$ be the outward drawn unit normal to ΔS at P and let the orientation of ΔS be specified by $\hat{n}$ at P. Let $\Delta\vec{F}_n$ acted on ΔS Then the vector

$$\vec{T}_n = \lim_{\Delta S \to 0} \frac{\Delta \vec{F}_n}{\Delta S} = \frac{d\vec{F}_n}{dS} .$$

is called the stress vector on the surface element at the pt P.

The resultant vector $\vec{T}$ of all the strees vectors applied to the whole surface S is given by

$$\vec{T} = \iint_S \vec{T}_n \, dS$$

Stress Components:-

Let $\vec{T}_1$ be the stress vector acting upon the x_1 normal plane; then $\vec{T}_1$ can be resolved into 3 components τ_{11}, τ_{12} and τ_{13} in the directions of x_1, x_2, x_3 axis respectively.

Similarly; τ_{21}, τ_{22}, τ_{23} are components of $\vec{T}_2$ and $\tau_{31}, \tau_{32}, \tau_{33}$ are components of $\vec{T}_3$.

So; This nine components τ_{ij} ; $i, j = 1, 2, 3$ are called stress components.

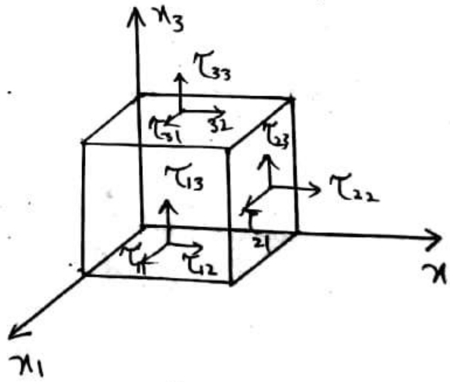
The components τ_{ii} ; $i = 1, 2, 3$ which act normally to the surface are called normal stresses. and the components τ_{ij} ; $i \neq j$ and $i, j = 1, 2, 3$ which act tangentially to the surface are called shearing stresses.

So;

$$\vec{T}_1 = \tau_{11}\hat{e}_1 + \tau_{12}\hat{e}_2 + \tau_{13}\hat{e}_3$$

$$\vec{T}_2 = \tau_{21}\hat{e}_1 + \tau_{22}\hat{e}_2 + \tau_{23}\hat{e}_3$$

$$\vec{T}_3 = \tau_{31}\hat{e}_1 + \tau_{32}\hat{e}_2 + \tau_{33}\hat{e}_3$$



In tensor notation;

$$\vec{T}_i = \tau_{ij}\,\hat{e}_j$$

or $$\tau_{ij} = \vec{T}_i \cdot \hat{e}_j$$

Note: ① In general; the stress vector depends on the orientation (direction) of the surface i.e $\vec{T}_n = \vec{T}_n(\hat{n})$ where $\hat{n}$ is the outwardly drawn unit normal to the surface.

we can prove that

$$\vec{T}_n = \tau_{ji}\, n_j\, \hat{e}_i$$

$$\Rightarrow \vec{T}_n \cdot \hat{e}_i = \tau_{ji}\, n_j$$

$$\Rightarrow (T_n)_i = \tau_{ji}\, n_j$$

② τ_{ij} is a 2nd order tensor.

Q:- The stress tensor at a point P is given by

$$\tau_{ij} = \begin{bmatrix} 7 & 0 & -2 \\ 0 & 5 & 0 \\ -2 & 0 & 4 \end{bmatrix}$$

Determine the stress vector at P on the plane whose unit normal is $\left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right)$.

Sol:- we know that

$$(T_n)_i = \tau_{ji}\, n_j$$

$$(T_n)_i = \tau_{ij}\, n_j \quad (\text{since } \tau_{ij} = \tau_{ji})$$

for $i = 1, 2, 3$ we get

$$(T_n)_1 = \tau_{1j}\, n_j = \tau_{11} n_1 + \tau_{12} n_2 + \tau_{13} n_3$$

$$(T_n)_2 = \tau_{2j}\, n_j = \tau_{21} n_1 + \tau_{22} n_2 + \tau_{23} n_3$$

$$(T_n)_3 = \tau_{3j}\, n_j = \tau_{31} n_1 + \tau_{32} n_2 + \tau_{33} n_3$$

In matrix form;

$$\begin{pmatrix} (T_n)_1 \\ (T_n)_2 \\ (T_n)_3 \end{pmatrix} = \begin{pmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}$$

$$\begin{bmatrix}(T_n)_1\\(T_n)_2\\(T_n)_3\end{bmatrix} = \begin{bmatrix}7 & 0 & -2\\0 & 5 & 0\\-2 & 0 & 4\end{bmatrix}\begin{bmatrix}2/3\\-2/3\\1/3\end{bmatrix}$$

$$= \begin{bmatrix}\frac{14}{3}-\frac{2}{3}\\-10/3\\-\frac{4}{3}+\frac{4}{3}\end{bmatrix} = \begin{bmatrix}4\\-10/3\\0\end{bmatrix}$$

So; stress vector is;

$$\vec{T_n} = (T_n)_i\,\hat{e}_i = (T_n)_1\hat{e}_1+(T_n)_2\hat{e}_2+(T_n)_3\hat{e}_3$$

$$= 4\hat{e}_1 - \frac{10}{3}\hat{e}_2 + 0\hat{e}_3$$

$$= 4\hat{e}_1 - \frac{10}{3}\hat{e}_2$$

Symmetry of stress tensor τ_{ij} :-

Let S be any arbitrary surface enclosing a volume V. Then for equilibrium; the two Conditions must be satisfied.

i) sum of all forces must be zero.
ii) sum of moments of all forces must be zero.

So; from (i) Condition;

total surface force + total body force = 0

$$\iint_S \vec{T_n}\,dS + \iiint_V \rho\vec{F}\,dV = 0$$

$$\Rightarrow \iint_S \tau_{ji}n_j\hat{e}_i\,dS + \iiint_V \rho F_i\hat{e}_i\,dV = 0$$

$$\Rightarrow \iint_S \tau_{ji}n_j\,dS + \iiint_V \rho F_i\,dV = 0$$

$$\Rightarrow \iiint_V \frac{\partial\tau_{ji}}{\partial x_j}\,dV + \iiint_V \rho F_i\,dV = 0 \quad \text{(By divergence theorem).}$$

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$$\iiint_V \left(\frac{\partial \tau_{ji}}{\partial x_j} + \rho F_i\right) dV = 0$$

$$\Rightarrow \quad \frac{\partial \tau_{ji}}{\partial x_j} + \rho F_i = 0$$

$$\Rightarrow \quad \frac{\partial \tau_{ji}}{\partial x_j} = -\rho F_i \rightarrow (1)$$

Now; we know that the moments of a force $\vec{F}$ at a pt whose position vector $\vec{r}$ is given by $\vec{r} \times \vec{F}$; ith component in tensor notation is $\epsilon_{ijk} x_j F_k$.

Now by using condition (ii)

moment of surface force + moment of body force = 0

$$\iint_S \epsilon_{ijk} x_j (T_n)_k ds + \iiint_V \epsilon_{ijk} x_j \rho F_k dV = 0$$

$$\iint_S \epsilon_{ijk} x_j (\tau_{lk} n_l) ds + \iiint_V \epsilon_{ijk} x_j \rho F_k dV = 0$$

$$\Rightarrow \iiint_V \epsilon_{ijk} \frac{\partial}{\partial x_l} (x_j \tau_{lk}) dV + \iiint_V \epsilon_{ijk} x_j \rho F_k dV = 0$$

$$\Rightarrow \iiint_V \epsilon_{ijk} \left(\frac{\partial x_j}{\partial x_l} \tau_{lk} + x_j \frac{\partial \tau_{lk}}{\partial x_l} \right) dV + \iiint_V \epsilon_{ijk} x_j \rho F_k dV = 0$$

$$\Rightarrow \iiint_V \epsilon_{ijk} \left(\delta_{jl} \tau_{lk} + x_j (-\rho F_k) \right) dV + \iiint_V \epsilon_{ijk} x_j \rho F_k dV = 0$$

$$\Rightarrow \iiint_V \epsilon_{ijk} (\tau_{jk} - x_j \rho F_k) dV + \iiint_V \epsilon_{ijk} x_j \rho F_k dV = 0$$

$$\Rightarrow \iiint_V \epsilon_{ijk} \tau_{jk} dV = 0$$

$$\Rightarrow \epsilon_{ijk} \tau_{jk} = 0$$

If $i=1$;

$$\epsilon_{123} \tau_{23} + \epsilon_{132} \tau_{32} = 0$$

$$\Rightarrow \tau_{23} - \tau_{32} = 0$$

$$\Rightarrow \tau_{23} = \tau_{32}$$

Similarly; we can prove that

$\tau_{12} = \tau_{21}$ and $\tau_{31} = \tau_{13}$

$\Rightarrow \tau_{ij}$ is symmetric.

So; the stress matrix becomes;

$$\tau_{ij} = \begin{bmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{12} & \tau_{22} & \tau_{23} \\ \tau_{13} & \tau_{23} & \tau_{33} \end{bmatrix}$$

which is also a symmetric matrix.

Rate of strain Tensor:-

When a continuous body of fluid is made to flow every element in it is displaced to a new position in the course of time. During this motion the elements of fluid become strained (deformed).

Let $P(x_1, x_2, x_3)$ and $Q(x_1+\Delta x_1, y_1+\Delta y_1, z_1+\Delta z_1)$ be two neighbouring pts at any time t;

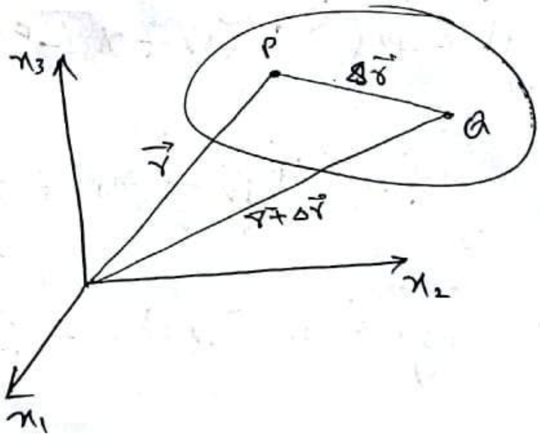
$\vec{OP} = \vec{r} = x_i \hat{e}_i$

$\vec{OQ} = \vec{r}+\Delta\vec{r} = (x_i+\Delta x_i)\hat{e}_i$

Let $\vec{V} = u_i \hat{e}_i$ and

$\vec{V}+\Delta\vec{V} = (u_i+\Delta u_i)\hat{e}_i$

be the velocities at P and Q;



Since $u_i = u_i(x_1, x_2, x_3, t)$; as t is not varying

So; $$\Delta u_i = \frac{\partial u_i}{\partial x_1}\Delta x_1 + \frac{\partial u_i}{\partial x_2}\Delta x_2 + \frac{\partial u_i}{\partial x_3}\Delta x_3$$

$$\Delta u_i = \frac{\partial u_i}{\partial x_j}\Delta x_j \longrightarrow (1)$$

in matrix form;

$$\begin{pmatrix} \Delta u_1 \\ \Delta u_2 \\ \Delta u_3 \end{pmatrix} = \begin{pmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{pmatrix} \begin{pmatrix} \Delta x_1 \\ \Delta x_2 \\ \Delta x_3 \end{pmatrix}$$

So; $\frac{\partial u_i}{\partial x_j}$ is a tensor of order 2 ; because because u_i be a tensor of order 1.

Note:- the partial differentiation of a tensor of order n gives a tensor of order n+1

Since any 2nd order tensor can be written as a sum of symmetric and anti-symmetric tensor; so

$$\frac{\partial u_i}{\partial x_j} = \frac{1}{2}\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right] + \frac{1}{2}\left[\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right]$$

where the symmetric part

$$e_{ij} = \frac{1}{2}\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right]$$

is called the strain rate tensor.

and the anti-symmetric part

$$\omega_{ij} = \frac{1}{2}\left[\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right]$$

is called the angular velocity tensor or spin tensor.

so; (1) $\Rightarrow$

$$\Delta u_i = (e_{ij} + \omega_{ij})\Delta x_j$$

$$= e_{ij}\Delta x_j + \omega_{ij}\Delta x_j$$

Cartesian form of strain rate tensor:-

$$e_{xx} = \frac{\partial u}{\partial x} \; ; \; e_{yy} = \frac{\partial v}{\partial y} \; ; \; e_{zz} = \frac{\partial w}{\partial z}$$

$$e_{xy} = e_{yx} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

$$e_{yz} = e_{zy} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)$$

$$e_{zx} = e_{xz} = \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)$$

Cylindrical form:-

$$e_{rr} = \frac{\partial V_r}{\partial r} \; ; \; e_{\theta\theta} = \frac{1}{r}\frac{\partial V_\theta}{\partial \theta} + \frac{V_r}{r} \; ; \; e_{zz} = \frac{\partial V_z}{\partial z}$$

$$e_{r\theta} = e_{\theta r} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial V_r}{\partial \theta} + \frac{\partial V_\theta}{\partial r} - \frac{V_\theta}{r}\right)$$

$$e_{\theta z} = e_{z\theta} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial V_z}{\partial \theta} + \frac{\partial V_\theta}{\partial z}\right)$$

$$e_{zr} = e_{rz} = \frac{1}{2}\left(\frac{\partial V_r}{\partial z} + \frac{\partial V_z}{\partial r}\right)$$

Spherical form:-

$$e_{rr} = \frac{\partial V_r}{\partial r} \; ; \; e_{\theta\theta} = \frac{1}{r}\frac{\partial V_\theta}{\partial \theta} + \frac{V_r}{r} \; ; \; e_{\phi\phi} = \frac{1}{r\sin\theta}\frac{\partial V_\phi}{\partial \phi} + \frac{V_r}{r} + \frac{V_\theta \cot\theta}{r}$$

$$e_{r\theta} = e_{\theta r} = \frac{1}{2}\left[r\frac{\partial}{\partial r}\left(\frac{V_\theta}{r}\right) + \frac{1}{r}\frac{\partial V_r}{\partial \theta}\right]$$

$$e_{\theta\phi} = e_{\phi\theta} = \frac{1}{2}\left[\frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\left(\frac{V_\phi}{\sin\theta}\right) + \frac{1}{r\sin\theta}\frac{\partial V_\theta}{\partial \phi}\right]$$

$$e_{\phi r} = e_{r\phi} = \frac{1}{2}\left[\frac{1}{r\sin\theta}\frac{\partial V_r}{\partial \phi} + r\frac{\partial}{\partial r}\left(\frac{V_\phi}{r}\right)\right]$$

Stress-Strain Rate Relationship for a Newtonian Fluid :-

when the viscous fluid is at rest (or when the inviscid fluid is moving), there are no tangential stresses. The only force acting on a matherial element of fluid is the normal stress (i.e pressure) which is same in all directions (i.e isotropic). This normal stress is independent of the direction of the normal to the surface element.

Therefore the stress tensor is given as;

$$\tau_{ij} = -p \quad \text{for } i=j \qquad \text{and} \qquad \tau_{ij} = 0 \quad \text{for } i \neq j$$

$$\Rightarrow \quad \tau_{ij} = -p\,\delta_{ij} \longrightarrow (1)$$

where p is the hydrostatic pressure; and δ_{ij} is the Kronecker delta.

Since the normal component of the stress acting across a surface element depends on the direction of the normal.

Therefore; the pressure at a pt. in a moving fluid is give as; minus the average of the three normal stresses.

i.e $p = -\frac{1}{3}\tau_{ii}$ from (1);

$$p = -\frac{1}{3}(\tau_{11} + \tau_{22} + \tau_{33}) \rightarrow (2)$$

we write the stress tensor as;

$$\tau_{ij} = -p\,\delta_{ij} + d_{ij} \longrightarrow (3)$$

where;

$-p\delta_{ij}$ = inviscid part of τ_{ij} due to fluid pressure p.

and d_{ij} = viscous part of τ_{ij} due to tangential stresses.

$-p\delta_{ij}$ is isotropic and d_{ij} is non-isotropic part of τ_{ij}.

The viscous or deviatoric stress tensor d_{ij} has zero trace;

So, ③ $\Rightarrow$

$$\begin{pmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{pmatrix} = \begin{bmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{bmatrix} + \begin{bmatrix} 0 & d_{12} & d_{13} \\ d_{21} & 0 & d_{23} \\ d_{31} & d_{32} & 0 \end{bmatrix}$$

So, eq ③ reduces to eq ①, when fluid is at rest i.e d_{ij} must be be zero for a stationary fluid.

It has been found experimentally that the deviatoric stress tensor for a Newtonian fluid is linearly related to a strain-rate tensor;

$$d_{ij} = A_{ijkl}\, e_{kl}$$

Now, from Cartesian tensor we know that the isotropic tensor of order 4; is given as;

$$A_{ijkl} = \lambda \delta_{ij}\delta_{kl} + \mu \delta_{ik}\delta_{jl} + \nu \delta_{il}\delta_{jk}$$

So; $d_{ij} = (\lambda \delta_{ij}\delta_{kl} + \mu_{ik}\delta_{jl} + \nu \delta_{il}\delta_{jk})\, e_{kl}$

$$d_{ij} = \lambda \delta_{ij} e_{kk} + \mu \delta_{ik} e_{kj} + \nu \delta_{il} e_{il}$$

$$d_{ij} = \lambda \delta_{ij} e_{kk} + \mu e_{ij} + \nu e_{ji} \rightarrow ④$$

since $e_{ij} = e_{ji}$; So d_{ij} is symmetric tensor, so $\mu = \nu$;

④ $\Rightarrow$

$$d_{ij} = \lambda \delta_{ij} e_{kk} + 2\mu e_{ij} \rightarrow ⑤$$

to find value of λ;

Since $d_{ii} = 0$, So

$$\lambda \delta_{ii} e_{kk} + 2\mu e_{ii} = 0$$

$$\Rightarrow 3\lambda e_{kk} + 2\mu e_{kk} = 0$$

$$\Rightarrow (3\lambda + 2\mu) e_{kk} = 0$$

$$\Rightarrow \lambda = -\frac{2\mu}{3}$$

so; (5) $\Rightarrow$

$$d_{ij} = 2\mu e_{ij} - \frac{2}{3}\mu \delta_{ij} e_{kk}$$

Now; (3) $\Rightarrow$

$$\tau_{ij} = -p\,\delta_{ij} + 2\mu e_{ij} - \frac{2}{3}\mu \delta_{ij} e_{kk} \rightarrow (6)$$

Since $e_{kk} = e_{11} + e_{22} + e_{33} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}$

$$e_{kk} = \frac{\partial u_k}{\partial x_k}$$

So; (6) $\Rightarrow$

$$\tau_{ij} = -p\delta_{ij} + 2\mu e_{ij} - \frac{2}{3}\mu \delta_{ij} \frac{\partial u_k}{\partial x_k}$$

This is known as deformation law for a Newtonian fluid.

This is also known as stokes' relationship for a viscous compressible fluid.

1) when fluid is at rest then

$$\tau_{ij} = -p\,\delta_{ij}$$

2) when fluid is incompressible then

$$\nabla \cdot \vec{V} = 0 \Rightarrow \frac{\partial u_k}{\partial x_k} = 0$$

So; $\tau_{ij} = -p\,\delta_{ij} + 2\mu e_{ij}$

$$\tau_{ij} = -p\,\delta_{ij} + 2\mu\, e_{ij} - \frac{2}{3}\mu\,\delta_{ij}\,\nabla\cdot\vec{V}$$

$$\tau_{ij} = -p\,\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) - \frac{2}{3}\mu\,\delta_{ij}\,\frac{\partial u_k}{\partial x_k}$$

Cartesian form:-

$$\tau_{xx} = -p + 2\mu\frac{\partial u}{\partial x} - \frac{2}{3}\mu\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial w}{\partial z}\right)$$

$$\tau_{yy} = -p + 2\mu\frac{\partial v}{\partial y} - \frac{2}{3}\mu\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)$$

$$\tau_{zz} = -p + 2\mu\frac{\partial v}{\partial y} - \frac{2}{3}\mu\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)$$

$$\tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

$$\tau_{yz} = \tau_{zy} = \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)$$

$$\tau_{zx} = \tau_{xz} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)$$

These are the constitutive equations.

Q:- An incompressible steady flow field has the velocity components

$$u = ax\,;\quad v = -ay\,;\quad w = 0$$

Evaluate the stress tensor τ_{ij}.

Sol:-

$$\tau_{xx} = -p + 2\mu\frac{\partial u}{\partial x} = -p + 2\mu a$$

$$\tau_{yy} = -p + 2\mu\frac{\partial v}{\partial y} = -p + 2\mu(-a)$$

$$\tau_{zz} = -p + 2\mu\frac{\partial w}{\partial z} = -p + 0 = -p$$

$$\tau_{xy} = \tau_{yx} = -p + \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = -p + \mu(0+0) = -p$$

$$\tau_{yz} = \tau_{zy} = -p + \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) = -p + \mu(0+0) = -p$$

$$\tau_{xz} = \tau_{zx} = -p + \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) = -p + 0 = -p$$

Navier-stokes' eqs of motion for a Compressible Viscous Fluid:-

The Navier-stokes' eqs are derived from Newton's 2nd law of motion; which states that "the rate of change of linear momentum of the body is equal to the sum of all external forces acting on the body.

Since the ~~surface~~ external forces acting on the body are the body forces and the surface forces. The surface forces are of two types: i) a normal force and ii) a tangential force

Let a fluid body of volume V be enclosed by any arbitrary surface S; let $\vec{V}$ be the velocity of body and ρ be density of fluid. Let δV be an element of volume. Then

mass of fluid element $= \rho\, \delta V$

momentum of fluid element $= \rho \vec{V}\, dV$

momentum of whole body $= \iiint_V \rho \vec{V}\, dV$

rate of change of momentum $= \iiint_V \rho \frac{d\vec{V}}{dt}\, dV$

$$= \iiint_V \rho \frac{du_i}{dt} \hat{e}_i\, dV$$

Let $\vec{F}$ = body force acting per unit mass

Then force acting on $\rho\, \delta V$ mass $= \vec{F} \rho\, \delta V$

total body force acting on body $= \iiint_V \rho \vec{F}\, dv$

$$= \iiint_V \rho F_i \hat{e}_i\, dv$$

Let the surface force acting per unit area = $\vec{T_n}$

Then the surface force acting on area $\delta S = \vec{T_n}\,\delta S$

and total surface force acting on body $= \iint_S \vec{T_n}\, dS$

$$= \iint_S \tau_{ji} n_j \hat{e}_i \, ds$$

$$= \iiint_V \frac{\partial \tau_{ji}}{\partial x_j} \hat{e}_i \, dV$$

Now;

rate of change of momentum = sum of forces

$$\iiint_V \rho \frac{du_i}{dt}\hat{e}_i dV = \iiint_V \rho F_i \hat{e}_i dV + \iiint_V \frac{\partial \tau_{ji}}{\partial x_j}\hat{e}_i dV$$

$$\Rightarrow \iiint_V \left(\rho \frac{du_i}{dt}\hat{e}_i - \rho F_i \hat{e}_i - \frac{\partial \tau_{ji}}{\partial x_j}\hat{e}_i\right) dV = 0$$

$$\Rightarrow \iiint_V \left[\rho \frac{du_i}{dt} - \rho F_i - \frac{\partial \tau_{ji}}{\partial x_j}\right]\hat{e}_i dV = 0$$

$$\Rightarrow \rho \frac{du_i}{dt} - \rho F_i - \frac{\partial \tau_{ji}}{\partial x_j} = 0$$

$$\Rightarrow \rho \frac{du_i}{dt} = \rho F_i + \frac{\partial \tau_{ji}}{\partial x_j} \rightarrow (1)$$

This eq^n is known as momentum eq^n;

In vector form;

$$\rho \frac{d\vec{V}}{dt} = \rho \vec{F_i} + \nabla \cdot \vec{T}$$

We know that $\tau_{ij} = \tau_{ji}$

Also; for viscous fluid; we know that

$$\tau_{ij} = -p\,\delta_{ij} + 2\mu e_{ij} - \frac{2}{3}\mu\,\delta_{ij}\frac{\partial u_k}{\partial x_k}$$

So; (1) $\Rightarrow$

$$\rho\frac{du_i}{dt} = \rho F_i + \frac{\partial}{\partial x_j}\left[-p\delta_{ij} + 2\mu e_{ij} - \frac{2}{3}\mu\delta_{ij}\frac{\partial u_k}{\partial x_k}\right]$$

$$= \rho F_i - \frac{\partial}{\partial x_j}(p\,\delta_{ij}) + \frac{\partial}{\partial x_j}(2\mu e_{ij}) - \frac{2}{3}\frac{\partial}{\partial x_j}\left(\mu\delta_{ij}\frac{\partial u_k}{\partial x_k}\right)$$

$$\rho\frac{du_i}{dt} = \rho F_i - \frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j}\left[\mu\left(2e_{ij} - \frac{2}{3}\delta_{ij}\frac{\partial u_k}{\partial x_k}\right)\right]$$

$$\rho\frac{du_i}{dt} = \rho F_i - \frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j}\left[\mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}\delta_{ij}\frac{\partial u_k}{\partial x_k}\right)\right]$$

This is known as Navier-stokes' eq/s of motion.

i) If viscosity is constant:-

Then $$\rho\frac{du_i}{dt} = \rho F_i - \frac{\partial p}{\partial x_i} + \mu\frac{\partial^2 u_i}{\partial x_j\partial x_j} + \mu\frac{\partial^2 u_j}{\partial x_j\partial x_i} - \frac{2\mu}{3}\frac{\partial}{\partial x_j}\left(\delta_{ij}\frac{\partial u_k}{\partial x_k}\right)$$

$$\rho\frac{du_i}{dt} = \rho F_i - \frac{\partial p}{\partial x_i} + \mu\frac{\partial^2 u_i}{\partial x_j\partial x_j} + \mu\frac{\partial}{\partial x_i}\left(\frac{\partial u_j}{\partial x_j}\right) - \frac{2}{3}\mu\frac{\partial}{\partial x_i}\left(\frac{\partial u_k}{\partial x_k}\right)$$

$$\rho\frac{du_i}{dt} = \rho F_i - \frac{\partial p}{\partial x_i} + \mu\frac{\partial^2 u_i}{\partial x_j\partial x_j} + \frac{1}{3}\mu\frac{\partial}{\partial x_i}\left(\frac{\partial u_j}{\partial x_j}\right)$$

ii) Incompressible fluid:-

$$\frac{\partial u_k}{\partial x_k} = \nabla\cdot\vec{v} = 0$$

Then $$\rho\frac{du_i}{dt} = \rho F_i - \frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j}\left[\mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\right]$$

iii) Incompressible with constant viscosity:-

$$\rho\frac{du_i}{dt} = \rho F_i - \frac{\partial p}{\partial x_i} + \mu\frac{\partial^2 u_i}{\partial x_j\partial x_j}$$

iv) Non-viscous or Inviscid Fluid:-

$\mu = 0$ for Inviscid fluid

So;
$$\rho \frac{du_i}{dt} = \rho F_i - \frac{\partial p}{\partial x_i}$$
which are Euler's eq/s of motion.

(Different forms with constant viscosity)

1) vector form:-

$$\rho \frac{d\vec{V}}{dt} = \rho \vec{F} - \nabla p + \mu \nabla^2 \vec{V} + \frac{1}{3}\mu \nabla(\nabla \cdot \vec{V})$$

$$\Rightarrow \frac{d\vec{V}}{dt} = \vec{F} - \frac{1}{\rho}\nabla p + \nu \nabla^2 \vec{V} + \frac{1}{3}\nu \nabla(\nabla \cdot \vec{V})$$

2) cartesian form:-

$$\rho\left[\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right] = F_x - \frac{\partial p}{\partial x} + \mu\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right] + \frac{1}{3}\mu \frac{\partial}{\partial x}\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right]$$

$$\rho\left[\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right] = F_y - \frac{\partial p}{\partial y} + \mu\left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right] + \frac{1}{3}\mu \frac{\partial}{\partial y}\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right]$$

$$\rho\left[\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right] = F_z - \frac{\partial p}{\partial z} + \mu\left[\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right] + \frac{1}{3}\mu \frac{\partial}{\partial z}\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right]$$

Q:- Consider a fluid flow with the velocity components
$$u(y) = U\frac{y}{h} + \frac{h^2}{2\mu}\left[-\frac{dp}{dx}\right]\frac{y}{h}\left[1 - \frac{y}{h}\right]$$
$$v = 0; \quad w = 0 \quad ; \quad p = p(x)$$
where μ, h, U and $\frac{dp}{dx}$ are constants.

Are the Navier-stoke's eq/s of motion for for an incompressible steady viscous flow with negligible body force satisfied?

Sol:- Navier stoke's eq/ of motion for viscous incompressible fluid with constant viscoustity is;

$$\rho \frac{d\vec{V}}{dt} = \rho \vec{F} - \nabla p + \mu \nabla^2 \vec{V}$$

$$\Rightarrow \frac{d\vec{V}}{dt} = \vec{F} - \frac{1}{\rho}\nabla p + \nu \nabla^2 \vec{V}$$

$$\Rightarrow \frac{\partial \vec{V}}{\partial t} + u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z} = \vec{F} - \frac{1}{\rho}\nabla p + \nu \nabla^2 \vec{V}$$

In Cartesian form;

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = F_x - \frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right]$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = F_y - \frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right]$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = F_z - \frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left[\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right]$$

since flow is steady so $\frac{\partial}{\partial t} = 0$ and body force is neglecteted, so $F_x = F_y = F_z = 0$; Also $v = w = 0$; so above eqs becomes;

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2}$$

$$\Rightarrow \frac{\partial p}{\partial x} = \mu\frac{\partial^2 u}{\partial y^2} \rightarrow (1)$$

$$R.H.S = \mu\frac{\partial^2 u}{\partial y^2} = \mu\frac{\partial^2}{\partial y^2}\left[\frac{y}{h} + \frac{h^2}{2\mu}\left(-\frac{dp}{dx}\right)\left(\frac{y}{h} - \frac{y^2}{h^2}\right)\right]$$

$$= \mu\frac{\partial}{\partial y}\left[\frac{1}{h} + \frac{h^2}{2\mu}\left(-\frac{dp}{dx}\right)\left(\frac{1}{h} - \frac{2y}{h^2}\right)\right]$$

$$= \mu\left[\frac{h^2}{2\mu}\left(-\frac{dp}{dx}\right)\left(-\frac{2}{h^2}\right)\right]$$

$$= \mu\left[\frac{1}{\mu}\frac{dp}{dx}\right]$$

$$= \frac{dp}{dx}$$

$$R.H.S = L.H.S$$

So; the given velocity field satisfies the Navier-stoke's eqs of motion.

Euler's eqs of motion:-

For an inviscid or Non-viscous fluid $\mu = 0$; so Navier-stoke's eq/s becomes;

$$\rho\left[\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right] = \rho F_x - \frac{\partial P}{\partial x}$$

$$\rho\left[\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right] = \rho F_y - \frac{\partial P}{\partial y}$$

$$\rho\left[\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right] = \rho F_z - \frac{\partial P}{\partial z}$$

These eqs are known as Euler's eq/s of motion; In vector form;

$$\rho\frac{d\vec{V}}{dt} = \rho\vec{F} - \nabla P$$

Note:- The Navier-stoke's eq/s are non-linear in general; Solving the eq/s is very difficult except for simple problems. In fact mathematicians are yet to prove that general sol's to these eq/s exist and is considered as the sixth most important unsolved problem in all of maths.

In addition the phenomenon of turbulence caused by the convective terms is considered to be the last unsolved problem of classical mechanics.

Three eq/s have four unknowns, P, u, v, w. They should be combined with the eq of Continuity to form four eq/s in these four unknowns.

Exact Solutions of Navier-Stoke's Eqs:-

Parallel flow:-

A flow is called parallel if there is only one velocity component i.e $v = w = 0$. The paractical application of this simple case is the flow b/w parallel flat plates; circular pipes and concentric rotating cylinders.

In such flows; The N-S eq/s simplify considerably; and infact permit an exact sol.

The eq/ of continuity becomes;

$$\frac{\partial u}{\partial x} = 0 \Rightarrow u = u(y, z, t)$$

Thus for parallel flow; velocity components are

$$u = u(y, z, t) \; ; \; v = 0 \; ; \; w = 0$$

and N-S eq/s are

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial y}$$

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial z}$$

last two eq/s indicate that $p = p(x, t)$.

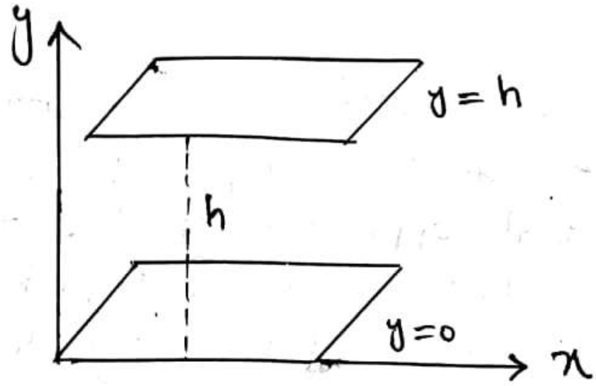
Now; we solve some problems analytically.

Steady leminar flow b/w Two parallel plates:-

Consider the steady leminar flow of an incompressible fluid with constant viscosity b/w two infinite parallel plates.

Let the direction of flow be x-axis and y-axis $\perp$ to the direction of flow. Also let the the distance b/w the plates be h and the width of plates in the z-direction be infinite.

So; $v = w = 0$; so the eq/ of continuity and N-S eq/s are; (with negligible body force)



$$\frac{\partial u}{\partial x} = 0 \quad \longrightarrow (1)$$

$$u\frac{\partial u}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\mu}{\rho}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) \longrightarrow (2)$$

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial y} \quad \longrightarrow (3)$$

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial z} \longrightarrow (4)$$

from (1) it is clear that $u = u(y,z)$ but we have already assumed that no flow in the z-direction; so $u = u(y)$ i.e u is only fn of y; Also; $\frac{\partial p}{\partial y} = 0$ and $\frac{\partial p}{\partial z} = 0$ shows that $p($ $p(x)$ i.e p is only fn. of x.

By using these information (2) $\Rightarrow$

$$\frac{dp}{dx} = \mu\frac{d^2u}{dy^2} \longrightarrow (5)$$

The partial derivatives of p and u are replaced with ordinary derivatives because u is only fn. of y and p is only fn. of x.

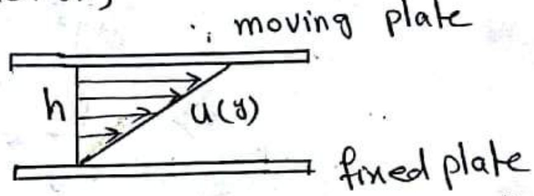
Now; we dissuss three different Cases.

i) Simple Couette flow:-

The simple couette flow or simple shear flow is the flow b/w two parallel plates one of which $y=0$ is at rest and the other $y=h$ moving with uniform velocity U parallel to itself in its own plane.

In this case; $p = Const \Rightarrow \frac{dp}{dx} = 0$

when the pressure is constant; the velocity is zero everywhere for the given flow field. To maintain a velocity field; it is necessary to set one of the plates in motion. so for this reason we set the upper plate into motion;



eq ⑤ $\Rightarrow \quad \frac{d^2u}{dy^2} = 0$

$$\Rightarrow \frac{du}{dy} = A$$

$$\Rightarrow \quad u = Ay + B \rightarrow ⑥$$

we use boundry conditions to find A and B;

$u = 0; \; at \; y = 0 \quad \Rightarrow \quad B = 0$

and $u = U$ at $y = h \quad \Rightarrow \quad A = \frac{U}{h}$

eq ⑥ $\Rightarrow \quad u = \frac{U}{h} y$

This eq shows that the velocity distribution across the gap of the parallel plates is linear. This type of flow is also called a plane couette flow

Average velocity:-

$$U_{av} = \frac{1}{h}\int_0^h u\,dy = \frac{1}{h}\int_0^h \frac{U}{h} y\,dy = \frac{U}{h^2} \frac{y^2}{2}\Big|_0^h = \frac{U}{2}$$

Also $u = 0$ is min and $u = U$ is maximum velocity

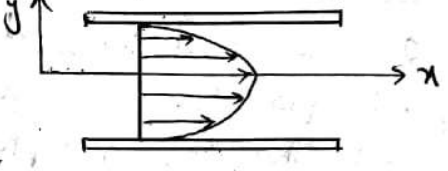
Shearing stress:-

$$\tau_{yx} = \mu \frac{du}{dy} = \mu \frac{d}{dy}\left(\frac{U}{h} y\right) = \mu \frac{U}{h}$$

ii) <u>Plane Poiseuille flow</u>:- (P varies linearly i.e $\frac{dP}{dx} = \text{Const}$)

If the two parallel plates are both stationary, the fully developed flow between the plates is generally refered to as plane poiseuille flow.

In this Case; flow is maintained by the pressure gradient. For a Const pressure gradient



eq (5) ⇒

$$\frac{d^2u}{dy^2} = \frac{1}{\mu}\frac{dP}{dx}$$

$$\Rightarrow \quad \frac{du}{dy} = \frac{1}{\mu}\frac{dP}{dx}y + A$$

$$\Rightarrow \quad u = \frac{1}{2\mu}\frac{dP}{dx}y^2 + Ay + B \rightarrow (A)$$

Let the plates are situated at $y = \pm h$

Then boundry conditions are $u = 0$ at $y = \pm h$

So; for $u = 0$ at $y = h$

(A) ⇒ $$0 = \frac{1}{2\mu}h^2\frac{dP}{dx} + Ah + B \rightarrow (7)$$

$$0 = \frac{1}{2\mu}h^2\frac{dP}{dx} - Ah + B \rightarrow (8)$$

(7) + (8) ⇒ $$B = -\frac{1}{2\mu}h^2\frac{dP}{dx} \quad \text{and} \quad A = 0$$

So; eq (A) ⇒

$$u = \frac{1}{2\mu}\frac{dP}{dx}y^2 - \frac{1}{2\mu}h^2\frac{dP}{dx}$$

$$u = \frac{-1}{2\mu}\frac{dP}{dx}(h^2 - y^2)$$

or $$u = \frac{-h^2}{2\mu}\frac{dP}{dx}\left[1 - \left(\frac{y}{h}\right)^2\right]$$

This eq shows that the velocity profile is parabolic.

Max velocity:-

$$\frac{du}{dy} = -\frac{1}{2\mu}\frac{dP}{dx}(-2y) = \frac{y}{\mu}\frac{dP}{dx}$$

put $\frac{du}{dy} = 0 \Rightarrow y = 0$

So; velocity is maximum at $y=0$

and $U_{max} = \frac{-h^2}{2\mu}\frac{dP}{dx}$

So; velocity distribution can be written as

$$u = U_{max}\left(1 - \left(\frac{y}{h}\right)^2\right]$$

Avg velocity:-

$$U_{avg} = \frac{1}{2h}\int_{-h}^{h} u\, dy$$

$$= \frac{U_{max}}{2h}\int_{-h}^{h}\left(1 - \frac{y^2}{h^2}\right)dy$$

$$= \frac{U_{max}}{2h}\left[y\Big|_{-h}^{h} - \frac{1}{3h^2}\, y^3\Big|_{-h}^{h}\right]$$

$$= \frac{U_{max}}{2h}\left(2h - \frac{2h^3}{3h^2}\right)$$

$$= \frac{U_{max}}{2h}\left(2h - \frac{2}{3}h\right)$$

$$U_{avg} = \frac{U_{max}}{2h}\left(\frac{4h}{3}\right) = \frac{2}{3}U_{max}$$

$$U_{avg} = -\frac{2}{3}\frac{h^2}{2\mu}\frac{dP}{dx}$$

$$U_{avg} = -\frac{h^2}{3\mu}\frac{dP}{dx}$$

iii) Generalised Couette flow:-

It is a simple couette flow with non-zero pressure gradient. So this is combination of 1st and 2nd cases.

In this case the velocity distribution is depends on both the motion of top plate and the existence of the pressure gradient.

For a constant pressure gradient

eq (5) $\Rightarrow$

$$u = \frac{1}{2\mu}\frac{dP}{dx}y^2 + Ay + B \rightarrow (B)$$

Then boundary conditions in this case are;

$u=0$ for $y=0$ and $u=U$ for $y=h$

So; (B) $\Rightarrow$

$B = 0$ and $U = \frac{1}{2\mu}\frac{dP}{dx}h^2 + Ah + B$

$$U = \frac{1}{2\mu}\frac{dP}{dx}h^2 + Ah$$

$$\Rightarrow A = \frac{U}{h} - \frac{h}{2\mu}\frac{dP}{dx}$$

eq (B) $\Rightarrow$

$$u = \frac{1}{2\mu}\frac{dP}{dx}y^2 + \left(\frac{U}{h} - \frac{h}{2\mu}\frac{dP}{dx}\right)y$$

$$u = \frac{U}{h}y - \frac{h^2}{2\mu}\frac{dP}{dx}\frac{y}{h}\left(1 - \frac{y}{h}\right)$$

$$u = \frac{U}{h}y - \frac{hy}{2\mu}\frac{dP}{dx}\left(1 - \frac{y}{h}\right)$$

which is the eq for the velocity distribution of the generalized coutte flow.

The pattern of the velocity distribution; can be investigated based on the value and direction of the pressure gradient;

i) when $\frac{dP}{dx} = 0$; then

$$u = \frac{Uy}{h}$$

$\Rightarrow$ The velocity distribution is a straight line.

ii) when $\frac{dP}{dx} < 0$; the pressure gradient is -ive then the fluid velocity is +ive in the direction of x-axis over the entire width b/w the plates

iii) For $\frac{dP}{dx} > 0$; In this case the velocity distribution may either be all +ive or a combination of +ive and -ive velocity distribution.

The +ive pressure gradient the separates these two kinds of velocity distribution is defined as the critical pressure gradient.

It can be evaluated at $y = 0$ after differentiating the velocity field;

$$\frac{du}{dy} = \frac{U}{h} - \frac{h}{2\mu}\frac{dP}{dx}\left(1 - \frac{2y}{h}\right)$$

$$\left(\frac{du}{dy}\right)_{y=0} = 0$$

$$\frac{U}{h} - \frac{h}{2\mu}\frac{dP}{dx} = 0$$

$$\left(\frac{dP}{dx}\right)_c = \frac{2\mu U}{h^2}$$

Average velocity :-

$$U_{av} = \frac{1}{h}\int_0^h u\,dy$$

$$= \frac{1}{h}\int_0^h \left[\frac{Uy}{h} - \frac{hy}{2\mu}\frac{dP}{dx}\left(1-\frac{y}{h}\right)\right]dy$$

$$= \frac{1}{h}\int_0^h \left[\frac{Uy}{h} - \frac{h\,dP}{2\mu\,dx}\left(y-\frac{y^2}{h}\right)\right]dy$$

$$= \frac{1}{h}\left[\frac{Uy^2}{2h}\Big|_0^h - \frac{h}{2\mu}\frac{dP}{dx}\left(\frac{y^2}{2}-\frac{y^3}{3h}\right)\Big|_0^h\right]$$

$$= \frac{1}{h}\left[\frac{Uh^2}{2h} - \frac{h}{2\mu}\frac{dP}{dx}\left(\frac{h^2}{2}-\frac{h^3}{3h}\right)\right]$$

$$= \frac{1}{h}\left[\frac{Uh}{2} - \frac{h}{2\mu}\frac{dP}{dx}\left(\frac{h^2}{2}-\frac{h^2}{3}\right)\right]$$

$$= \frac{U}{2} - \frac{1}{2\mu}\frac{dP}{dx}\left(\frac{h^2}{6}\right)$$

$$U_{av} = \frac{U}{2} - \frac{h^2}{12\mu}\frac{dP}{dx}$$

Maximum Velocity :-

To find maximum velocity, we put $\frac{du}{dy} = 0$;

$$u = \frac{Uy}{h} - \frac{h}{2\mu}\frac{dP}{dx}\left(y-\frac{y^2}{h}\right)$$

$$\frac{du}{dy} = \frac{U}{h} - \frac{h}{2\mu}\frac{dP}{dx}\left(1-\frac{2y}{h}\right)$$

put $\frac{du}{dy} = 0$

$$\frac{U}{h} - \frac{h}{2\mu}\frac{dP}{dx}\left(1-\frac{2y}{h}\right) = 0$$

$$\frac{h}{2\mu}\frac{dP}{dx}\left(1-\frac{2y}{h}\right) = \frac{U}{h}$$

$$1-\frac{2y}{h} = \frac{2U\mu}{h^2\frac{dP}{dx}}$$

$$\frac{2y}{h} = 1 - \frac{2\mu U}{h^2 \frac{dP}{dx}}$$

$$y = \frac{h}{2} + \frac{\mu U}{h \frac{dP}{dx}}$$

which is the position of the maximum velocity;

and

$$u_{max} = \frac{U}{h}\left(\frac{h}{2} - \frac{\mu U}{h\frac{dP}{dx}}\right) - \frac{h}{2\mu}\frac{dP}{dx}\left(\frac{h}{2} - \frac{\mu U}{h\frac{dP}{dx}}\right)\left[1 - \frac{1}{h}\left(\frac{h}{2} - \frac{\mu U}{h\frac{dP}{dx}}\right)\right]$$

Volume flow rate:-

The fluid discharge moving through the plates for the generalized coutte flow is given as;

$$Q = \int_0^y u\,dy$$

$$Q = \int_0^h \left[\frac{U}{h}y - \frac{h}{2\mu}\frac{dP}{dx}\left(y - \frac{y^2}{h}\right)\right]dy$$

$$Q = \frac{U}{h}\frac{y^2}{2}\Big|_0^h - \frac{h}{2\mu}\frac{dP}{dx}\left(\frac{y^2}{2} - \frac{y^3}{3h}\right)\Big|_0^h$$

$$Q = \frac{U}{h}\frac{h^2}{2} - \frac{h}{2\mu}\frac{dP}{dx}\left[\frac{h^2}{2} - \frac{h^3}{3h}\right]$$

$$Q = \frac{Uh}{2} - \frac{h}{2\mu}\frac{dP}{dx}\left[\frac{h^2}{2} - \frac{h^2}{3}\right]$$

$$Q = \frac{Uh}{2} - \frac{h}{2\mu}\frac{dP}{dx}\left[\frac{h^2}{6}\right]$$

$$Q = \frac{Uh}{2} - \frac{h^3}{12\mu}\frac{dP}{dx}$$

Shear Stress:-

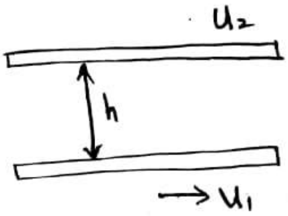
$$\tau_{yx} = \mu\frac{du}{dy} = \mu\left[\frac{U}{h} - \frac{h}{2\mu}\frac{dP}{dx}\left(1 - \frac{2y}{h}\right)\right]$$

$$\tau_{yx} = \frac{\mu U}{h} - \frac{h}{2}\frac{dP}{dx}\left(1 - \frac{2y}{h}\right)$$

$$= \frac{\mu U}{h} - \frac{dP}{dx}\left(\frac{h}{2} - y\right)$$

Flow b/w Two moving parallel plates:-

Consider the steady laminar flow of a viscous incompressible fluid b/w two infinite moving horizontal flat plates distance h aparts. Let the lower plate be moving with a velocity U_1 and the upper plate with a velocity U_2 in the direction parallel to to the direction of flow.



We know that the velocity distribution for the flow b/w parallel plates is given by;

$$u = \frac{1}{\mu}\cdot\frac{dP}{dx}\frac{y^2}{2} + Ay + B \rightarrow (1)$$

Boundry conditions in this case are;

$u = U_1$ at $y = 0$ and $u = U_2$ at $y = h$

1st B.C gives; $B = U_1$ and

2nd B.C gives;

$$U_2 = \frac{h^2}{2\mu}\frac{dP}{dx} + hA + U_1$$

$$\Rightarrow hA = U_2 - U_1 - \frac{h^2}{2\mu}\frac{dP}{dx}$$

$$\Rightarrow A = \frac{U_2 - U_1}{h} - \frac{h}{2\mu}\frac{dP}{dx}$$

So; eq (1) $\Rightarrow$

$$u = \frac{y^2}{2\mu}\frac{dP}{dx} + \left[\frac{U_2 - U_1}{h} - \frac{h}{2\mu}\frac{dP}{dx}\right]y + U_1$$

$$u = \frac{1}{2\mu}\frac{dP}{dx}\left[y^2 - yh\right] + \frac{U_2 - U_1}{h}y + U_1$$

Avg Velocity:-

$$U_{avg} = \frac{1}{h}\int_0^h u\,dy = \frac{1}{h}\int_0^h\left[\frac{1}{2\mu}\frac{dP}{dx}\left[y^2 - yh\right] + \frac{U_2 - U_1}{h}y + U_1\right]dy$$

$$U_{avg} = \frac{1}{h}\frac{1}{2\mu}\frac{dP}{dx}\left|\frac{y^3}{3} - \frac{y^2 h}{2}\right|_0^h + \frac{U_2 - U_1}{h^2}\frac{y^2}{2}\Big|_0^h + \frac{U_1}{h}y\Big|_0^h$$

$$= \frac{1}{2h\mu}\frac{dP}{dx}\left[\frac{h^3}{3} - \frac{h^3}{2}\right] + \frac{U_2 - U_1}{\cancel{h^2}}\frac{\cancel{h^2}}{2} + \frac{U_1\cancel{h}}{\cancel{h}}$$

$$= \frac{1}{2h\mu}\frac{dP}{dx}\frac{-h^3}{6} + \frac{U_2 - U_1}{2} + U_1$$

$$U_{avg} = \frac{-h^2}{12\mu}\frac{dP}{dx} + \frac{U_1 + U_2}{2}$$

Volumetric flow rate:-

$$Q = \int_0^y u\,dy$$

or

$$Q = h\,U_{avg}$$

$$Q = h\left[-\frac{h^2}{12\mu}\frac{dP}{dx} + \frac{U_1+U_2}{2}\right]$$

$$Q = -\frac{h^3}{12\mu}\frac{dP}{dx} + \frac{(U_1+U_2)h}{2}$$

Shear stress:-

$$\tau_{yx} = \mu\frac{du}{dy}$$

$$= \mu\left[\frac{1}{2\mu}\frac{dP}{dx}(2y-h) + \frac{U_2-U_1}{h}\right]$$

$$= \frac{1}{2}\frac{dP}{dx}(2y-h) + \frac{(U_2-U_1)\mu}{h}$$

shearing stress at lower plate is;

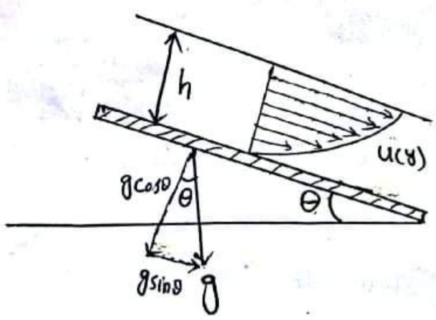
$$y=0\;;\quad \tau_{yx} = -\frac{h}{2}\frac{dP}{dx} + \frac{(U_2-U_1)\mu}{h}$$

shearing stress at upper plate;

$$y=h\;;\quad \tau_{yx} = \frac{h}{2}\frac{dP}{dx} + \frac{(U_2-U_1)\mu}{h}$$

Steady, Laminar flow over an inclined plane:-

Consider the steady flow of a viscous liquid over a wide flat plate inclined at an angle θ with the horizontal under the influence of gravity. There is no velocity $\perp$ to the plate and the pressure at free surface is constant.



Since the flow over the plate is parallel and occuring in the x-direction only; so $v = w = 0$

so; eq of continuity becomes

$$\frac{\partial u}{\partial x} = 0$$

since no flow is occurring in z-direction; u is not a fn. of z; Also; the flow is steady and pressure is constant; so $\frac{\partial}{\partial t} = 0$ and

$\frac{\partial p}{\partial x} = \frac{\partial p}{\partial y} = \frac{\partial p}{\partial z} = 0$;

Hence the N-S eqs for incompressible flow in this case including the body force force becomes;

$$0 = F_x + \frac{\mu}{\rho}\frac{\partial^2 u}{\partial y^2}$$

$$0 = g\sin\theta + \frac{\mu}{\rho}\frac{\partial^2 u}{\partial y^2}$$

$$\Rightarrow \frac{\partial^2 u}{\partial y^2} = -\frac{\rho g \sin\theta}{\mu}$$

$$\Rightarrow \frac{\partial u}{\partial y} = -\frac{\rho g \sin\theta}{\mu} y + A \longrightarrow (1)$$

Since flow is everywhere parallel to the plate;

so; $\frac{\partial u}{\partial y} = 0$ at $y = h$

$$\Rightarrow A = \frac{\rho g h \sin\theta}{\mu}$$

eq (1) $\Rightarrow$ $\frac{\partial u}{\partial y} = \frac{\rho g \sin\theta}{\mu}(h - y)$

again integrating; $u = \frac{\rho g \sin\theta}{\mu}\left(hy - \frac{y^2}{2}\right) + B \rightarrow (2)$

$u = 0$ at $y = 0$ so $B = 0$

(2) $\Rightarrow$

$$u = \frac{\rho g \sin\theta}{2\mu}\left[2hy - y^2\right]$$

Avg velocity:-

$$u_{avg} = \frac{1}{h}\int_0^h u\,dy$$

$$= \frac{1}{h}\int_0^h \frac{\rho g \sin\theta}{2\mu}\left[2hy - y^2\right] dy$$

$$= \frac{1}{h}\,\frac{\rho g \sin\theta}{2\mu}\left| hy^2 - \frac{y^3}{3}\right|_0^h$$

$$= \frac{\rho g \sin\theta}{2h\mu}\left[h^3 - \frac{h^3}{3}\right]$$

$$= \frac{\rho g \sin\theta}{h\mu} \times \frac{h^3}{3}$$

$$= \frac{\rho g h^2 \sin\theta}{3\mu}$$

maximum velocity:-

To find maximum velocity we put

$$\frac{\partial u}{\partial y} = 0$$

$$\frac{\rho g \sin\theta}{2\mu}\left[2h - 2y\right] = 0$$

$\Rightarrow$ $y = h$ which is the pt of max velocity; so at $y = h$

$$u_{max} = \frac{\rho g \sin\theta}{2\mu}\left[2h^2 - h^2\right]$$

$$u_{max} = \frac{\rho g h^2 \sin\theta}{2\mu}$$

Volumetric flow rate:-

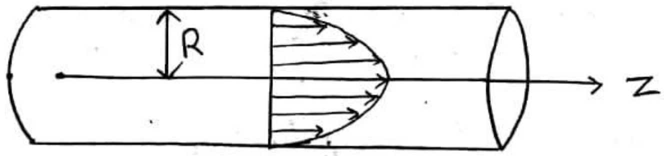
$$Q = \int_0^h u\,dy = h\,u_{avg}$$

$$Q = \frac{\rho g h^3 \sin\theta}{3\mu}$$

Flow through a circular pipe :- (The Hagen-Poiseuille flow)

Consider the steady laminar flow of a viscous incompressible fluid in an infinitely long straight horizontal circular pipe of radius R.

Let z-axis be along the axis of the pipe and r denote the radial direction measured outwards from the z-axis.



Let the direction of flow be along the axis of pipe i.e z-axis. This axially symmetric flow in a circular pipe is known as Hagen-poiseuille flow.

It is clear that flow is 1-D; so the radial and tangential velocity components are zero.

i.e $v_r = v_\theta = 0$;

The eq of continuity for steady flow is;

$$\frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} = 0$$

reduces to $\quad \dfrac{\partial v_z}{\partial z} = 0$

integrating $\quad v_z = v_z(r, \theta)$

which shows that v_z is independent of z, also due to axial symmetry of the flow; v_z will be independent of θ; so v_z is only fn. of r i.e $v_z = v_z(r)$

The N-S eqs without body forces in cylindrical Coordinates ;

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial r} \qquad (r\text{-Component})$$

$$0 = -\frac{1}{\rho r}\frac{\partial p}{\partial \theta} \qquad (\theta\text{-Component})$$

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \frac{\mu}{\rho}\left[\frac{\partial^2 v_z}{\partial r^2} + \frac{1}{r}\frac{\partial v_z}{\partial r}\right] \qquad (z\text{-Component})$$

1st two eqs. show p is independent of r and θ i.e $p = p(z)$.

and 3rd eq can be written as;

$$\Rightarrow \frac{dP}{dz} = \mu\left(\frac{\partial^2 V_z}{\partial r^2} + \frac{1}{r}\frac{\partial V_z}{\partial r}\right)$$

$$\Rightarrow \frac{r}{\mu}\frac{dP}{dz} = r\frac{d^2 V_z}{dr^2} + \frac{dV_z}{dr}$$

$$\Rightarrow \frac{r}{\mu}\frac{dP}{dz} = \frac{d}{dr}\left[r\frac{dV_z}{dr}\right]$$

integrating w.r.t r; we get;

$$r\frac{dV_z}{dr} = \frac{r^2}{2\mu}\frac{dP}{dz} + A$$

$$\Rightarrow \frac{dV_z}{dr} = \frac{r}{2\mu}\frac{dP}{dz} + \frac{A}{r}$$

again integrating

$$\Rightarrow V_z = \frac{r^2}{4\mu}\frac{dP}{dz} + A\ln r + B \longrightarrow (1)$$

The boundry conditions are

$V_z = 0$ at $r = R$ (no slip-condition)

$V_z =$ finite at $r = 0$ (velocity must be finite at the centre).

using 2nd condition $V_z(0) =$ finite; we must choose $A = 0$; otherwise V_z would become infinite at $r = 0$;

and using 1st condition; we get

$$B = -\frac{R^2}{4\mu}\frac{dP}{dz}$$

So; eq (1) $\Rightarrow$

$$V_z = \frac{1}{4\mu}\frac{dP}{dz}\left[r^2 - R^2\right]$$

$$V_z = -\frac{R^2}{4\mu}\frac{dP}{dz}\left[1 - \left(\frac{r}{R}\right)^2\right]$$

This velocity profile of the form of a paraboloid of revolution.

Maximum velocity:-

To find max ; we put $\frac{\partial V_z}{\partial r} = 0$

Then $$-\frac{R^2}{4\mu}\frac{dP}{dz}\left[0 - \frac{2r}{R^2}\right] = 0$$

$$\Rightarrow \quad r = 0$$

So ; the max velocity in this case occurs at the centre of the pipe where $r = 0$

So; $$V_{max} = -\frac{R^2}{4\mu}\frac{dP}{dz} \quad \text{where } \frac{dP}{dz} < 0$$

Avg velocity:-

$$V_{ag} = \frac{1}{\pi R^2}\int_0^{2\pi}\int_0^R V_z \, r \, dr \, d\theta$$

$$= \frac{1}{\pi R^2}\int_0^{2\pi}\int_0^R -\frac{R^2}{4\mu}\frac{dP}{dz}\left(1 - \frac{r^2}{R^2}\right) r \, dr \, d\theta$$

$$= \frac{-1}{4\pi\mu}\frac{dP}{dz}\int_0^{2\pi}\int_0^R \left(r - \frac{r^3}{R^2}\right) dr \, d\theta$$

$$= \frac{-1}{4\pi\mu}\frac{dP}{dz}\int_0^{2\pi}\left|\frac{r^2}{2} - \frac{r^4}{4R^2}\right|_0^R d\theta$$

$$= \frac{-1}{4\pi\mu}\frac{dP}{dz}\int_0^{2\pi}\left|\frac{R^2}{2} - \frac{R^4}{4R^2}\right| d\theta$$

$$= \frac{-1}{4\pi\mu}\frac{dP}{dz}\int_0^{2\pi}\frac{R^2}{4} d\theta = \frac{-R^2}{16\pi\mu}\frac{dP}{dz}\int_0^{2\pi} d\theta$$

$$= \frac{-R^2}{16\pi\mu}\frac{dP}{dz}(2\pi)$$

$$V_{avg} = \frac{-R^2}{8\mu}\frac{dP}{dz}$$

and $$\frac{V_{avg}}{V_{max}} = \frac{1}{2} = 0.5$$

Volumetric flow rate:-

$$Q = \int_0^{2\pi}\int_0^{R} V_z \, r\,dr\,d\theta$$

$$Q = \pi R^2 V_{avg}$$

$$Q = \pi R^2 \left(-\frac{R^2}{8\mu}\frac{dP}{dz} \right)$$

$$Q = -\frac{\pi R^4}{8\mu}\frac{dP}{dz}$$

Shearing Stress:-

$$\tau_{rz} = -\mu \frac{dV_z}{dr}$$

and
$$\frac{dV_z}{dr} = -\frac{R^2}{4\mu}\frac{dP}{dz}\left[0 - \frac{2r}{R^2}\right]$$

$$= \frac{r}{2\mu}\frac{dP}{dz}$$

So;
$$\tau_{rz} = -\frac{r}{2}\frac{dP}{dz}$$

shearing stress at the wall is given as;

$$(\tau_{rz})_{r=R} = -\frac{R}{2}\frac{dP}{dz}$$