Chapter 1 – Real Number System

Subject: *Real Analysis (Mathematics)* ***Level:*** *M.Sc.*
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The rational number system is inadequate for many purposes, both as a field and as an order set for many purpose. This leads to introduction of so called irrational numbers. We can prove in many ways that the rational number system has certain gaps and hence we fail to use it as an ordered set and as a field. ❑

✍ Theorem

There is no rational p such that $p^2 = 2$.

Proof

Let us suppose that there exists a rational p such that $p^2 = 2$.
This implies we can write

$$p = \frac{m}{n} \qquad \text{where } m, n \in \mathbb{Z} \text{ \& } m, n \text{ have no common factor.}$$

Then $p^2 = 2 \Rightarrow \frac{m^2}{n^2} = 2 \Rightarrow m^2 = 2n^2$

$\Rightarrow m^2$ is even

$\Rightarrow m$ is even

$\Rightarrow m$ is divisible by 2 and so m^2 is divisible by 4.

$\Rightarrow 2n^2$ is divisible by 4 and so n^2 is divisible by 2. $\qquad \because m^2 = 2n^2$

i.e. n^2 is even $\Rightarrow n$ is even

$\Rightarrow m$ and n both have common factor 2.

Which is contradiction. (because m and n have no common factor.)

Hence $p^2 = 2$ is impossible for rational p. ❑

✍ Theorem

Let A be the set of all positive rationals p such that $p^2 < 2$ and let B consist of all positive rationals p such that $p^2 > 2$ then A contain no largest member and B contains no smallest member.

Proof

We are to show that for every p in A there exists a rational $q \in A$ such that $p < q$ and for all $p \in B$ we can find rational $q \in B$ such that $q < p$.
Associate with each rational $p > 0$ the number

$$q = p - \frac{p^2 - 2}{p + 2} = \frac{2p + 2}{p + 2} \quad \ldots\ldots\ldots\ldots\ (i)$$

$$\text{Then } q^2 - 2 = \left(\frac{2p + 2}{p + 2}\right)^2 - 2 = \frac{2(p^2 - 2)}{(p + 2)^2} \quad \ldots\ldots\ldots\ldots\ (ii)$$

Now if $p \in A$ then $p^2 < 2 \Rightarrow p^2 - 2 < 0$

Since from (i) $\quad q = p - \frac{p^2 - 2}{p + 2} \Rightarrow q > p$

And $\frac{2(p^2 - 2)}{(p + 2)^2} < 0 \Rightarrow q^2 - 2 < 0 \Rightarrow q^2 < 2 \Rightarrow q \in A$

Now if $p \in B$ then $p^2 > 2 \Rightarrow p^2 - 2 > 0$

Since form (i) $q = p - \dfrac{p^2-2}{p+2} \Rightarrow q < p$

And $\dfrac{2(p^2-2)}{(p+2)^2} > 0 \Rightarrow q^2 - 2 > 0 \Rightarrow q^2 > 2 \Rightarrow q \in B$

The purpose of above discussion is simply to show that the rational number system has certain gaps, in spite of the fact that the set of rationals is dense i.e. we can always find a rational between any two given rational numbers. These gaps are filled by the irrational number. (e.g. if $r < s$ then $r < \dfrac{r+s}{2} < s$.) ❑

Order on a set

Let S be a non-empty set. An *order* on a set S is a relation denoted by "<" with the following two properties

(i) If $x \in S$ and $y \in S$,

then one and only one of the statement $x < y$, $x = y$, $y < x$ is true.

(ii) If $x, y, z \in S$ and if $x < y$, $y < z$ then $x < z$.

Ordered Set

A set S is said to be *ordered set* if an order is defined on S.

Bound

Let S be an ordered set and $E \subset S$. If there exists a $b \in S$ such that $x \le b \ \ \forall\ x \in E$, then we say that E is bounded above, and b is known as upper bound of E.

Lower bound can be define in the same manner with ≥ in place of ≤.

Least Upper Bound (Supremum)

Suppose S is an ordered set, $E \subset S$ and E is bounded above. Suppose there exists an $a \in S$ such that

(i) a is an upper bound of E.

(ii) If $g < a$ then g is not an upper bound of E.

Then a is called the *least upper bound* of E or *supremum* of E and is written as $\sup E = a$.

In other words a is the least member of the set of upper bound of E.

We can define the greatest lower bound or infimum of a set E, which is bounded below, in the same manner. ❑

Example

Consider the sets

$$A = \{p : p \in \mathbb{Q} \wedge p^2 < 2\}$$

$$B = \{p : p \in \mathbb{Q} \wedge p^2 > 2\}$$

where $\mathbb{Q}$ is set of rational numbers.

Then the set A is bounded above. The upper bound of A are the exactly the members of B. Since B contain no smallest member therefore A has no supremum in $\mathbb{Q}$.

Similarly B is bounded below. The set of all lower bounds of B consists of A and $r \in \mathbb{Q}$ with $r \le 0$. Since A has no largest member, therefore, B has no infimum in $\mathbb{Q}$.

Example

If a is supremum of E then a may or may not belong to E.

Let $E_1 = \{r : r \in \mathbb{Q} \wedge r < 0\}$

$E_2 = \{r : r \in \mathbb{Q} \wedge r \ge 0\}$

then $\sup E_1 = \inf E_2 = 0$ and $0 \notin E_1$ and $0 \in E_2$. ❑

✍ Example

Let E be the set of all numbers of the form $\dfrac{1}{n}$, where n is the natural numbers.

$$\text{i.e. } E=\left\{1,\frac{1}{2},\frac{1}{3},\frac{1}{4},\ldots\ldots\ldots\right\}$$

Then $\sup E=1$ which is in E, but $\inf E=0$ which is not in E. ❑

✍ Least Upper Bound Property

A set S is said to have the *least upper bound property* if the followings is true

(i) S is non-empty and ordered.

(ii) If $E\subset S$ and E is non-empty and bounded above then $\sup E$ exists in S.

Greatest lower bound property can be defined in a similar manner. ❑

✍ Example

Let S be set of rational numbers and

$$E=\left\{p : p\in\mathbb{Q}\ \wedge\ p^2<2\right\}$$

then $E\subset\mathbb{Q}$, E is non-empty and also bounded above but supremum of E is not in S, this implies that $\mathbb{Q}$ the set of rational numbers does not posses the least upper bound property. ❑

✍ Theorem

Suppose S is an ordered set with least upper bound property. $B\subset S$, B is non-empty and is bounded below. Let L be set of all lower bounds of B then $a=\sup L$ exists in S and also $a=\inf B$.

In particular infimum of B exists in S.

OR

An ordered set which has the least upper bound property has also the greatest lower bound property.

Proof

Since B is bounded below; therefore, L is non-empty.

Since L consists of exactly those $y\in S$ which satisfy the inequality.

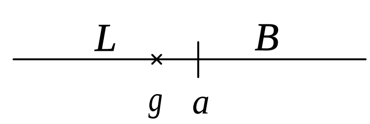
$$y\le x \qquad \forall\ x\in B$$

We see that every $x\in B$ is an upper bound of L.

$\Rightarrow$ L is bounded above.

Since S is ordered and non-empty therefore L has a supremum in S. Let us call it a.

If $g<a$, then g is not upper bound of L.



$\Rightarrow\ g\notin B$

$\Rightarrow\ a\le x \quad \forall\ x\in B \quad \Rightarrow a\in L$

Now if $a<b$ then $b\notin L$ because $a=\sup L$.

We have shown that $a\in L$ but $b\notin L$ if $b>a$. In other words, a is a lower bound of B, but b is not if $b>a$. This means that $a=\inf B$. ❑

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✎ Field

A set F with two operations called addition and multiplication satisfying the following axioms is known to be field.

Axioms for Addition:

(i) If $x, y \in F$ then $x+y \in F$. *Closure Law*

(ii) $x+y=y+x \quad \forall\ x, y \in F$. *Commutative Law*

(iii) $x+(y+z)=(x+y)+z \quad \forall\ x, y, z \in F$. *Associative Law*

(iv) For any $x \in F$, $\exists\ 0 \in F$ such that $x+0=0+x=x$ *Additive Identity*

(v) For any $x \in F$, $\exists\ -x \in F$ such that $x+(-x)=(-x)+x=0$ *+tive Inverse*

Axioms for Multiplication:

(i) If $x, y \in F$ then $xy \in F$. *Closure Law*

(ii) $xy=yx \quad \forall\ x, y \in F$ *Commutative Law*

(iii) $x(yz)=(xy)z \quad \forall\ x, y, z \in F$

(iv) For any $x \in F$, $\exists\ 1 \in F$ such that $x \cdot 1=1 \cdot x=x$ *Multiplicative Identity*

(v) For any $x \in F$, $x \neq 0$, $\exists\ \frac{1}{x} \in F$, such that $x\left(\frac{1}{x}\right)=\left(\frac{1}{x}\right)x=1$ $\times$*tive Inverse.*

Distributive Law

For any $x, y, z \in F$, (i) $x(y+z)=xy+xz$

(ii) $(x+y)z=xz+yz$ ❑

✎ Theorem

The axioms for addition imply the following:

(***a***) If $x+y=x+z$ then $y=z$

(***b***) If $x+y=x$ then $y=0$

(***c***) If $x+y=0$ then $y=-x$.

(***d***) $-(-x)=x$

Proof

(a) Suppose $x+y=x+z$.

Since $y=0+y$

$=(-x+x)+y$ $\quad \because\ -x+x=0$

$=-x+(x+y)$ by Associative law

$=-x+(x+z)$ by supposition

$=(-x+x)+z$ by Associative law

$=(0)+z$ $\quad \because\ -x+x=0$

$=z$

(b) Take $z=0$ in (a)

$x+y=x+0$

$\Rightarrow\ y=0$

(c) Take $z=-x$ in (a)

$x+y=x+(-x)$

$\Rightarrow\ y=-x$

(d) Since $(-x)+x=0$

then (c) gives $x=-(-x)$ ❑

Theorem

Axioms of multiplication imply the following.

(***a***) If $x \neq 0$ and $xy = xz$ then $y = z$.

(***b***) If $x \neq 0$ and $xy = x$ then $y = 1$.

(***c***) If $x \neq 0$ and $xy = 1$ then $y = \frac{1}{x}$.

(***d***) If $x \neq 0$, then $\frac{1}{1/x} = x$.

Proof

(a) Suppose $xy = xz$

$$\text{Since } y = 1\cdot y = \left(\frac{1}{x}\cdot x\right)y \qquad \because \frac{1}{x}\cdot x = 1$$

$$= \frac{1}{x}(xy) \qquad \text{by associative law}$$

$$= \frac{1}{x}(xz) \qquad \because xy = xz$$

$$= \left(\frac{1}{x}\cdot x\right)z \qquad \text{by associative law}$$

$$= 1\cdot z = z$$

(b) Take $z = 1$ in (a)

$$xy = x\cdot 1 \quad \Rightarrow \quad y = 1$$

(c) Take $z = \frac{1}{x}$ in (a)

$$xy = x\cdot\frac{1}{x} \quad \text{i.e. } xy = 1$$

$$\Rightarrow \quad y = \frac{1}{x}$$

(d) Since $\frac{1}{x}\cdot x = 1$

then (c) give

$$x = \frac{1}{1/x}$$

❑

Theorem

The field axioms imply the following.

(***i***) $0\cdot x = 0$

(***ii***) if $x \neq 0$, $y \neq 0$ then $xy \neq 0$.

(***iii***) $(-x)y = -(xy) = x(-y)$

(***iv***) $(-x)(-y) = xy$

Proof

(***i***) Since $0x + 0x = (0+0)x$

$$\Rightarrow \quad 0x + 0x = 0x$$

$$\Rightarrow \quad 0x = 0 \qquad \because x + y = x \Rightarrow y = 0$$

(***ii***) Suppose $x \neq 0$, $y \neq 0$ but $xy = 0$

$$\text{Since } 1 = \frac{1}{(x)(y)}\cdot xy$$

$$\Rightarrow 1 = \frac{1}{(x)(y)}(0) \qquad \because xy = 0,\ x \neq 0,\ y \neq 0$$

$$\Rightarrow 1 = 0 \qquad \text{from } (i) \quad \because x0 = 0$$

a contradiction, thus (*ii*) is true.

(***iii***) Since $(-x)y + xy = (-x + x)y = 0y = 0$ …….. (1)

Also $x(-y) + xy = x(-y + y) = x0 = 0$ ……… (2)

Also $-(xy) + xy = 0$ …………. (3)

Combining (1) and (2)

$$(-x)y + xy = x(-y) + xy$$

$$\Rightarrow (-x)y = x(-y) \text{ ………… (4)}$$

Combining (2) and (3)

$$x(-y) + xy = -(xy) + xy$$

$$\Rightarrow x(-y) = -xy \text{ …………. (5)}$$

From (4) and (5)

$$(-x)y = x(-y) = -xy$$

(***iv***) $(-x)(-y) = -[x(-y)] = -[-xy] = xy$ using (*iii*) ❑

Ordered Field

An ordered field is a field F which is also an ordered set such that

i) $x+y<x+z$ if $x,y,z\in F$ and $y<z$.

ii) $xy>0$ if $x,y\in F$, $x>0$ and $y>0$.

e.g. the set $\mathbb{Q}$ of rational number is an ordered field. ❑

Theorem

The following statements are true in every ordered field.

i) If $x>0$ then $-x<0$ and vice versa.

ii) If $x>0$ and $y<z$ then $xy<xz$.

iii) If $x<0$ and $y<z$ then $xy>xz$.

iv) If $x\neq 0$ then $x^2>0$ in particular $1>0$.

v) If $0<x<y$ then $0<\frac{1}{y}<\frac{1}{x}$.

Proof

i) If $x>0$ then $0=-x+x>-x+0$ so that $-x<0$.
If $x<0$ then $0=-x+x<-x+0$ so that $-x>0$.

ii) Since $z>y$ we have $z-y>y-y=0$
which means that $z-y>0$, Also $x>0$

$$\therefore\quad x(z-y)>0$$
$$\Rightarrow\quad xz-xy>0$$
$$\Rightarrow\quad xz-xy+xy>0+xy$$
$$\Rightarrow\quad xz+0>0+xy$$
$$\Rightarrow\quad xz>xy$$

iii) Since $y<z \Rightarrow -y+y<-y+z$
$\Rightarrow z-y>0$

Also $x<0 \Rightarrow -x>0$

Therefore $-x(z-y)>0$

$\Rightarrow -xz+xy>0 \qquad \Rightarrow -xz+xy+xz>0+xz$

$\Rightarrow xy>xz$

iv) If $x>0$ then $x\cdot x>0 \Rightarrow x^2>0$

If $x<0$ then $-x>0 \Rightarrow (-x)(-x)>0 \Rightarrow (-x)^2>0 \Rightarrow x^2>0$

i.e. if $x>0$ then $x^2>0$, since $1^2=1$ then $1>0$.

v) If $y>0$ and $v\le 0$ then $yv\le 0$, But $y\left(\frac{1}{y}\right)=1>0 \quad\Rightarrow \frac{1}{y}>0$

Likewise $\frac{1}{x}>0$ as $x>0$

If we multiply both sides of the inequality $x<y$ by the positive quantity $\left(\frac{1}{x}\right)\left(\frac{1}{y}\right)$

we obtain $\left(\frac{1}{x}\right)\left(\frac{1}{y}\right)x<\left(\frac{1}{x}\right)\left(\frac{1}{y}\right)y$

i.e. $\frac{1}{y}<\frac{1}{x}$

finally $0<\frac{1}{y}<\frac{1}{x}$ ❑

Existence of Real Field

There exists an ordered field $\mathbb{R}$ (set of reals) which has the least upper bound property and it contain $\mathbb{Q}$ (set of rationals) as a subfield. ❑

Theorem

a) If $x \in \mathbb{R}$, $y \in \mathbb{R}$ and $x > 0$ then there exists a positive integer *n* such that $nx > y$. (*Archimedean Property*)

b) If $x \in \mathbb{R}$, $y \in \mathbb{R}$ and $x < y$ then there exists $p \in \mathbb{Q}$ such that $x < p < y$.
i.e. between any two real numbers there is a rational number *or* $\mathbb{Q}$ is dense in $\mathbb{R}$.

Proof

a) Let $A = \{nx : n \in \mathbb{Z}^+ \wedge x > 0, x \in \mathbb{R}\}$

Suppose the given statement is false i.e. $nx \le y$.

$\Rightarrow$ y is an upper bound of *A*.

Since we are dealing with a set of reals, therefore, it has the least upper bound property.

Let $a = \sup A$

$\Rightarrow$ $a - x$ is not an upper bound of A.

$\Rightarrow$ $a - x < mx$ where $mx \in A$ for some positive integer *m*.

$\Rightarrow$ $a < (m+1)x$ where $m + 1$ is integer, therefore $(m+1)x \in A$

Which is impossible because a is least upper bound of A i.e. $a = \sup A$.

Hence we conclude that the given statement is true i.e. $nx > y$.

b) Since $x < y$, therefore $y - x > 0$

$\Rightarrow$ $\exists$ a +ive integer *n* such that

$n(y - x) > 1$ (*by Archimedean Property*)

$\Rightarrow$ $ny > 1 + nx$ …………… (*i*)

We apply (*a*) part of the theorem again to obtain two +ive integers m_1 and m_2 such that $m_1 \cdot 1 > nx$ and $m_2 \cdot 1 > -nx$

$\Rightarrow$ $-m_2 < nx < m_1$

then there exists an integers $m(-m_2 \le m \le m_1)$ such that

$m - 1 \le nx < m$

$\Rightarrow$ $nx < m$ and $m \le 1 + nx$

$\Rightarrow$ $nx < m < 1 + nx$

$\Rightarrow$ $nx < m < ny$ from (*i*)

$\Rightarrow$ $x < \frac{m}{n} < y$

$\Rightarrow$ $x < p < y$ where $p = m/n$ is a rational. ❑

Theorem

Given two real numbers *x* and *y*, $x < y$ there is an irrational number *u* such that

$x < u < y$

Proof

Take $x > 0$, $y > 0$

Then $\exists$ a rational number *q* such that

$0 < \frac{x}{a} < q < \frac{y}{a}$ where a is an irrational.

$\Rightarrow$ $x < aq < y$

$\Rightarrow$ $x < u < y$

Where $u = aq$ is an irrational as product of rational and irrational is irrational. ❑

Theorem
For every real number x there is a set E of rational number such that $x = \sup E$.

Proof

Take $E = \{q \in \mathbb{Q} : q < x\}$ where x is a real.

Then E is bounded above. Since $E \subset \mathbb{R}$ therefore supremum of E exists in $\mathbb{R}$.

Suppose $\sup E = l$.

It is clear that $l \leq x$.

If $l = x$ then there is nothing to prove.

If $l < x$ then $\exists$ $q \in \mathbb{Q}$ such that $l < q < x$

Which can not happen. Hence we conclude that real x is sup$E$. ❑

Theorem
For every real $x > 0$ and every integer $n > 0$, there is one and only one real y such that $y^n = x$.

This number y is written $\sqrt[n]{x}$ or $x^{1/n}$.

Proof

Take $y_1, y_2 \in \mathbb{R}$ such that $0 < y_1 < y_2$. Then $y_1^n < y_2^n$ i.e. there is at most one $y \in \mathbb{R}$ such that $y^n = x$. This shows the uniqueness of y.

Let us suppose E be the set of all positive real numbers t such that $t^n < x$.

$$\text{i.e. } E = \{t \ : \ t \in \mathbb{R} \wedge t^n < x\}$$

Take $t = \dfrac{x}{1+x}$ then $0 < t < 1$.

Hence $t^n < t$ and we have $t^n < x$

$\Rightarrow$ $t^n < t < x$

$\Rightarrow$ $t \in E$ and E is non-empty.

If $t > 1 + x$ then $t^n > t > x$ so that $t \notin E$.

Thus $1 + x$ is an upper bound of E.

Since E is non-empty and bounded above therefore $\sup E$ exists.

Take $y = \sup E$

To show that $y^n = x$ we will show that each of the inequality $y^n < x$ and $y^n > x$ leads to contradiction.

Consider

$$b^n - a^n = (b-a)(b^{n-1} + b^{n-2}a + b^{n-3}a^2 + \cdots\cdots\cdots + a^{n-1}) \qquad \text{where } n \in \mathbb{Z}^+.$$

Which yields the inequality (each a is replaced by b on R.H.S of above)

$$b^n - a^n < (b-a)(nb^{n-1}) \text{ ....}(i) \qquad \text{where } 0 < a < b.$$

Now assume $y^n < x$

Choose h so that $0 < h < 1$ and $h < \dfrac{x - y^n}{n(y+1)^{n-1}}$

Put $a = y$ and $b = y + h$ in (i)

Then $(y+h)^n - y^n < nh(y+h)^{n-1}$

$$< nh(y+1)^{n-1} \qquad \because \ h < 1$$

$$< x - y^n$$

$\Rightarrow$ $(y+h)^n < x$

$\Rightarrow$ $y + h \in E$

Since $y + h > y$ therefore it contradict the fact that y is $\sup E$.

Hence $y^n < x$ is impossible.

Now suppose $y^n > x$

Put $k = \dfrac{y^n - x}{ny^{n-1}}$, then $0 < k < y$

Now if $t \geq y - k$ we get

$$y^n - t^n < y^n - (y-k)^n < y^n - (y^n - nky^{n-1}) \qquad \textit{by binomial expansion}$$
$$< kny^{n-1} = y^n - x$$
$$\Rightarrow \; -t^n < -x \quad \Rightarrow \quad t^n > x \quad \text{and} \quad t \notin E$$

It follows that $y-k$ is an upper bound of E but $y - k < y$, which contradict the fact that y is $\sup E$.

Hence we conclude that $y^n = x$. ❑

✍ The Extended Real Numbers

The extended real number system consists of real field $\mathbb{R}$ and two symbols $+\infty$ and $-\infty$, We preserve the original order in $\mathbb{R}$ and define

$$-\infty < x < +\infty \quad \forall\, x \in \mathbb{R}.$$

The extended real number system does not form a field. Mostly we write $+\infty = \infty$.

We make following conventions

i) If x is real then $x + \infty = \infty$, $x - \infty = -\infty$, $\dfrac{x}{\infty} = \dfrac{x}{-\infty} = 0$

ii) If $x > 0$ then $x(\infty) = \infty$, $x(-\infty) = -\infty$.

iii) If $x < 0$ then $x(\infty) = -\infty$, $x(-\infty) = \infty$.

✍ Euclidean Space

For each positive integer *k*, let $\mathbb{R}^k$ be the set of all ordered *k*-tuples

$$\underline{x} = (x_1, x_2, \ldots\ldots\ldots\ldots, x_k)$$

where $x_1, x_2, \ldots\ldots\ldots\ldots, x_k$ are real numbers, called the *coordinates* of $\underline{x}$. The elements of $\mathbb{R}^k$ are called points, or vectors, especially when $k > 1$.

If $\underline{y} = (y_1, y_2, \ldots\ldots\ldots\ldots, y_n)$ and a is a real number, put

$$\underline{x} + \underline{y} = (x_1 + y_1, x_2 + y_2, \ldots\ldots\ldots\ldots, x_k + y_k)$$

and $$a\,\underline{x} = (a\,x_1, a\,x_2, \ldots\ldots\ldots\ldots, a\,x_k)$$

So that $\underline{x} + \underline{y} \in \mathbb{R}^k$ and $a\,\underline{x} \in \mathbb{R}^k$. These operations make $\mathbb{R}^k$ into a vector space over the real field.

The inner product or scalar product of $\underline{x}$ and $\underline{y}$ is defined as

$$\underline{x}.\underline{y} = \sum_{i=1}^{k} x_i y_i = (x_1 y_1 + x_2 y_2 + \ldots\ldots\ldots + x_k y_k)$$

And the norm of $\underline{x}$ is defined by

$$\|\,\underline{x}\,\| = (x \cdot x)^{1/2} = \left(\sum_{1}^{k} x_i^2 \right)^{1/2}$$

The vector space $\mathbb{R}^k$ with the above inner product and norm is called *Euclidean k-space*. ❑

✤..................................✤

Theorem

Let $\underline{x}$, $\underline{y} \in \mathbb{R}^n$ then

i) $\|\underline{x}\|^2 = x \cdot x$

ii) $\|\underline{x} \cdot \underline{y}\| \le \|\underline{x}\| \, \|\underline{y}\|$ (*Cauchy-Schwarz's inequality*)

Proof

i) Since $\|\underline{x}\| = (\underline{x} \cdot \underline{x})^{\frac{1}{2}}$ therefore $\|\underline{x}\|^2 = \underline{x} \cdot \underline{x}$

ii) For $l \in \mathbb{R}$ we have

$$0 \le \|\underline{x} - l\,\underline{y}\|^2 = (\underline{x} - l\,\underline{y}) \cdot (\underline{x} - l\,\underline{y})$$
$$= \underline{x} \cdot (\underline{x} - l\,\underline{y}) + (-l\,\underline{y}) \cdot (\underline{x} - l\,\underline{y})$$
$$= \underline{x} \cdot \underline{x} + \underline{x} \cdot (-l\,\underline{y}) + (-l\,\underline{y}) \cdot \underline{x} + (-l\,\underline{y}) \cdot (-l\,\underline{y})$$
$$= \|\underline{x}\|^2 - 2l\,(\underline{x} \cdot \underline{y}) + l^2 \|\underline{y}\|^2$$

Now put $l = \dfrac{\underline{x} \cdot \underline{y}}{\|\underline{y}\|^2}$ (*certain real number*)

$$\Rightarrow \; 0 \le \|\underline{x}\|^2 - 2\frac{(\underline{x} \cdot \underline{y})(\underline{x} \cdot \underline{y})}{\|\underline{y}\|^2} + \frac{(\underline{x} \cdot \underline{y})^2}{\|\underline{y}\|^4}\|\underline{y}\|^2 \quad \Rightarrow \; 0 \le \|\underline{x}\|^2 - \frac{(\underline{x} \cdot \underline{y})^2}{\|\underline{y}\|^2}$$

$$\Rightarrow \; 0 \le \|\underline{x}\|^2 \|\underline{y}\|^2 - \|\underline{x} \cdot \underline{y}\|^2$$

$$\Rightarrow \; 0 \le \left(\|\underline{x}\| \, \|\underline{y}\| + \|\underline{x} \cdot \underline{y}\|\right)\left(\|\underline{x}\| \, \|\underline{y}\| - \|\underline{x} \cdot \underline{y}\|\right)$$

Which hold if and only if

$$0 \le \|\underline{x}\| \, \|\underline{y}\| - \|\underline{x} \cdot \underline{y}\|$$

i.e. $\|\underline{x} \cdot \underline{y}\| \le \|\underline{x}\| \, \|\underline{y}\|$ ❑

Question

Suppose $\underline{x}, \underline{y}, \underline{z} \in \mathbb{R}^n$ the prove that

a) $\|\underline{x} + \underline{y}\| \le \|\underline{x}\| + \|\underline{y}\|$

b) $\|\underline{x} - \underline{z}\| \le \|\underline{x} - \underline{y}\| + \|\underline{y} - \underline{z}\|$

Proof

a) Consider $\|\underline{x} + \underline{y}\|^2 = (\underline{x} + \underline{y}) \cdot (\underline{x} + \underline{y})$

$$= \underline{x} \cdot \underline{x} + \underline{x} \cdot \underline{y} + \underline{y} \cdot \underline{x} + \underline{y} \cdot \underline{y}$$
$$= \|\underline{x}\|^2 + 2(\underline{x} \cdot \underline{y}) + \|\underline{y}\|^2$$
$$\le \|\underline{x}\|^2 + 2\|\underline{x}\| \, \|\underline{y}\| + \|\underline{y}\|^2 \qquad \because \; \|\underline{x}\| \, \|\underline{y}\| \ge \underline{x}.\underline{y}$$
$$= \left(\|\underline{x}\| + \|\underline{y}\|\right)^2$$

$$\Rightarrow \; \|\underline{x} + \underline{y}\| \le \|\underline{x}\| + \|\underline{y}\| \; \ldots\ldots\ldots\ldots \; (i)$$

b) We have $\|\underline{x} - \underline{z}\| = \|\underline{x} - \underline{y} + \underline{y} - \underline{z}\|$

$$\le \|\underline{x} - \underline{y}\| + \|\underline{y} - \underline{z}\| \qquad \text{from } (i)$$ ❑

Question

If r is rational and x is irrational then prove that $r+x$ and rx are irrational.

Proof

Let $r+x$ be rational.

$\Rightarrow\ r+x=\frac{a}{b}$ where $a,b\in\mathbb{Z}$, $b\neq 0$ such that $(a,b)=1$

$\Rightarrow\ x=\frac{a}{b}-r$

Since r is rational therefore $r=\frac{c}{d}$ where $c,d\in\mathbb{Z}$, $d\neq 0$ such that $(c,d)=1$

$\Rightarrow\ x=\frac{a}{b}-\frac{c}{d}\quad\Rightarrow\ x=\frac{ad-bc}{bd}$

Which is rational, which can not happened because x is given to be irrational.
Similarly let us suppose that rx is rational then

$rx=\frac{a}{b}$ for some $a,b\in\mathbb{Z}$, $b\neq 0$ such that $(a,b)=1$

$\Rightarrow\ x=\frac{a}{b}\cdot\frac{1}{r}$

Since r is rational therefore $r=\frac{c}{d}$ where $c,d\in\mathbb{Z}$, $d\neq 0$ such that $(c,d)=1$

$\Rightarrow\ x=\frac{a}{b}\cdot\frac{1}{c/d}=\frac{a}{b}\cdot\frac{d}{c}=\frac{ad}{bc}$

Which shows that x is rational, which is again contradiction; hence we conclude that $r+x$ and rx are irrational. ❑

Question

If n is a positive integer which is not perfect square then prove that $\sqrt{n}$ is irrational number.

Solution

There will be two cases

Case I. When n contain no square factor greater then 1.

Let us suppose that $\sqrt{n}$ is a rational number.

$\Rightarrow\ \sqrt{n}=\frac{p}{q}$ where $p,q\in\mathbb{Z}$, $q\neq 0$ and $(p,q)=1$

$\Rightarrow\ n=\frac{p^2}{q^2}\quad\Rightarrow\ p^2=nq^2$ ....(i)

$\Rightarrow\ q^2=\frac{p^2}{n}$

$\Rightarrow\ n\mid p^2\quad\Rightarrow\ n\mid p$ ....(ii) ($n\mid p$ means “ n divides p”)

Now suppose $\frac{p}{n}=c$ where $c\in\mathbb{Z}$

$\Rightarrow\ p=nc\quad\Rightarrow\ p^2=n^2c^2$

Putting this value of p^2 in equation (i)

$n^2c^2=nq^2$

$\Rightarrow\ nc^2=q^2\quad\Rightarrow\ c^2=\frac{q^2}{n}$

$\Rightarrow\ n\mid q^2\quad\Rightarrow\ n\mid q$ ....(iii)

From (ii) and (iii) we get p and q both have common factor n i.e. $(p,q)=n$
Which is a contradiction.
Hence our supposition is wrong.

Case II When n contain a square factor greater then 1.
Let us suppose $n=k^2m>1$

$$\Rightarrow \sqrt{n}=k\sqrt{m}$$

Where k is rational and $\sqrt{m}$ is irrational because m has no square factor greater than one, this implies $\sqrt{n}$, the product of rational and irrational, is irrational. ❑

✍ Question
Prove that $\sqrt{12}$ is irrational.

Proof
Suppose $\sqrt{12}$ is rational.

$\Rightarrow \sqrt{12}=\dfrac{p}{q}$ where $p,q\in\mathbb{Z}$, $q\neq 0$ and $(p,q)=1$

$\Rightarrow 12=\dfrac{p^2}{q^2}$ $\Rightarrow p^2=12q^2$ ………….. (i)

$\Rightarrow q^2=\dfrac{p^2}{12}$ $\Rightarrow q^2=\dfrac{p^2}{2^2\cdot 3}$

$\Rightarrow 2^2\,|\,p^2$ and $3\,|\,p^2$

$\Rightarrow 2\,|\,p$ and $3\,|\,p$

$\Rightarrow$ 2 and 3 are prime divisor of p.

$\Rightarrow 2\cdot 3\,|\,p$ i.e. $6\,|\,p$

$\Rightarrow \dfrac{p}{6}=c$, where c is an integer.

$\Rightarrow p=6c$

Put this value of p in equation (i) to get

$$36c^2=12q^2$$

$\Rightarrow 3c^2=q^2$ $\Rightarrow c^2=\dfrac{q^2}{3}$

$\Rightarrow 3\,|\,q^2$ $\Rightarrow 3\,|\,q$

$\Rightarrow (p,q)=3$, which is a contradiction.

Hence $\sqrt{12}$ is an irrational number. ❑

✍ Question
Let E be a non-empty subset of an ordered set, suppose a is a lower bound of E and b is an upper bound then prove that $a\leq b$.

Proof
Since E is a subset of an ordered set S i.e. $E\subseteq S$.
Also a is a lower bound of E therefore by definition of lower bound

$a\leq x \quad \forall \quad x\in E$ …………… (i)

Since b is an upper bound of E therefore by the definition of upper bound

$x\leq b \quad \forall \quad x\in E$ …………… (ii)

Combining (i) and (ii)

$$a\leq x\leq b$$

$\Rightarrow a\leq b$ as required. ❑

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Chapter 2 – Sequences and Series

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Sequence

A sequence is a function whose domain of definition is the set of natural numbers.

Or it can also be defined as an ordered set.

Notation:

An infinite sequence is denoted as

$\{S_n\}_{n=1}^{\infty}$ or $\{S_n : n\in \mathbb{N}\}$ or $\{S_1, S_2, S_3, ...\}$ or simply as $\{S_n\}$

e.g. i) $\{n\} = \{1, 2, 3, ...\}$

ii) $\left\{\frac{1}{n}\right\} = \left\{1, \frac{1}{2}, \frac{1}{3}, ...\right\}$

iii) $\left\{(-1)^{n+1}\right\} = \{1, -1, 1, -1, ...\}$

Subsequence

It is a sequence whose terms are contained in given sequence.

A subsequence of $\{S_n\}_{n=1}^{\infty}$ is usually written as $\{S_{n_k}\}^{\infty}$.

Increasing Sequence

A sequence $\{S_n\}$ is said to be an increasing sequence if $S_{n+1} \geq S_n \quad \forall \ n \geq 1$.

Decreasing Sequence

A sequence $\{S_n\}$ is said to be an decreasing sequence if $S_{n+1} \leq S_n \quad \forall \ n \geq 1$.

Monotonic Sequence

A sequence $\{S_n\}$ is said to be monotonic sequence if it is either increasing or decreasing.

$\{S_n\}$ is monotonically increasing if $S_{n+1} - S_n \geq 0$ or $\frac{S_{n+1}}{S_n} \geq 1, \ \forall \ n \geq 1$

$\{S_n\}$ is monotonically decreasing if $S_n - S_{n+1} \geq 0$ or $\frac{S_n}{S_{n+1}} \geq 1, \ \forall \ n \geq 1$

Strictly Increasing or Decreasing

$\{S_n\}$ is called strictly increasing or decreasing according as

$$S_{n+1} > S_n \quad \text{or} \quad S_{n+1} < S_n \qquad \forall \ n \geq 1.$$

Bernoulli's Inequality

Let $p\in \mathbb{R}$, $p \geq -1$ and $p \neq 0$ then for $n \geq 2$ we have

$$(1+p)^n > 1 + np$$

Proof:

We shall use mathematical induction to prove this inequality.

If $n = 2$

$L.H.S = (1+p)^2 = 1 + 2p + p^2$

$R.H.S = 1 + 2p$

$\Rightarrow \ L.H.S > R.H.S$

i.e. condition *I* of mathematical induction is satisfied.

Suppose $(1+p)^k > 1+kp$ .... (i) where $k \geq 2$

Now $(1+p)^{k+1} = (1+p)(1+p)^k$

$> (1+p)(1+kp)$ using (i)

$= 1+kp+p+kp^2$

$= 1+(k+1)p+kp^2$

$\geq 1+(k+1)p$ ignoring $kp^2 \geq 0$

$\Rightarrow (1+p)^{k+1} > 1+(k+1)p$

Since the truth for $n=k$ implies the truth for $n=k+1$ therefore condition *II* of mathematical induction is satisfied. Hence we conclude that $(1+p)^n > 1+np$.

Example

Let $S_n = \left(1+\frac{1}{n}\right)^n$ where $n \geq 1$

To prove that this sequence is an increasing sequence, we use $p = \frac{-1}{n^2}$, $n \geq 2$ in Bernoulli's inequality to have

$$\left(1-\frac{1}{n^2}\right)^n > 1-\frac{n}{n^2}$$

$$\Rightarrow \left(\left(1-\frac{1}{n}\right)\left(1+\frac{1}{n}\right)\right)^n > 1-\frac{1}{n}$$

$$\Rightarrow \left(1+\frac{1}{n}\right)^n > \left(1-\frac{1}{n}\right)^{1-n} = \left(\frac{n-1}{n}\right)^{1-n} = \left(\frac{n}{n-1}\right)^{n-1} = \left(1+\frac{1}{n-1}\right)^{n-1}$$

$$\Rightarrow S_n > S_{n-1} \qquad \forall \; n \geq 1$$

which shows that $\{S_n\}$ is increasing sequence.

Example

Let $t_n = \left(1+\frac{1}{n}\right)^{n+1}$; $n \geq 1$

then the sequence is decreasing sequence.

We use $p = \frac{1}{n^2-1}$ in Bernoulli's inequality.

$$\left(1+\frac{1}{n^2-1}\right)^n > 1+\frac{n}{n^2-1} \;\ldots\ldots\ldots\ldots (i)$$

where

$$1+\frac{1}{n^2-1} = \frac{n^2}{n^2-1} = \left(\frac{n}{n-1}\right)\left(\frac{n}{n+1}\right)$$

$$\Rightarrow \left(1+\frac{1}{n^2-1}\right)\left(\frac{n+1}{n}\right) = \left(\frac{n}{n-1}\right) \;\ldots\ldots\ldots\ldots\ldots (ii)$$

Now $t_{n-1} = \left(1+\frac{1}{n-1}\right)^n = \left(\frac{n}{n-1}\right)^n$

$$= \left(\left(1+\frac{1}{n^2-1}\right)\left(\frac{n+1}{n}\right)\right)^n \qquad \text{from } (ii)$$

$$=\left(1+\frac{1}{n^2-1}\right)^n\left(\frac{n+1}{n}\right)^n$$

$$>\left(1+\frac{n}{n^2-1}\right)\left(\frac{n+1}{n}\right)^n \qquad \text{from } (i)$$

$$>\left(1+\frac{1}{n}\right)\left(\frac{n+1}{n}\right)^n \qquad \because\ \frac{n}{n^2-1}>\frac{n}{n^2}=\frac{1}{n}$$

$$=\left(\frac{n+1}{n}\right)^{n+1}=t_n$$

i.e. $t_{n-1}>t_n$

Hence the given sequence is decreasing sequence.

Bounded Sequence

A sequence $\{S_n\}$ is said to be bounded if there exists a positive real number l such that $|S_n|<l \quad \forall\ n\in\mathbb{N}$

If S and s are the supremum and infimum of elements forming the bounded sequence $\{S_n\}$ we write $S=\sup S_n$ and $s=\inf S_n$

All the elements of the sequence S_n such that $|S_n|<l \quad \forall\ n\in\mathbb{N}$ lie with in the strip $\{y:-l<y<l\}$. But the elements of the unbounded sequence can not be contained in any strip of a finite width.

Examples

(i) $\{U_n\}=\left\{\frac{(-1)^n}{n}\right\}$ is a bounded sequence

(ii) $\{V_n\}=\{\sin nx\}$ is also bounded sequence. Its supremum is 1 and infimum is -1.

(iii) The geometric sequence $\{ar^{n-1}\}$, $r>1$ is an unbounded above sequence. It is bounded below by a.

(iv) $\left\{\tan\frac{np}{2}\right\}$ is an unbounded sequence.

Convergence of the Sequence

A sequence $\{S_n\}$ of real numbers is said to convergent to limit 's' as $n\to\infty$, if for every positive real number $e>0$, however small, there exists a positive integer n_0, depending upon e, such that $|S_n-s|<e \quad \forall\ n>n_0$.

Theorem

A convergent sequence of real number has one and only one limit (i.e. Limit of the sequence is unique.)

Proof:

Suppose $\{S_n\}$ converges to two limits s and t, where $s\neq t$.

Put $e=\frac{|s-t|}{2}$ then there exits two positive integers n_1 and n_2 such that

$$|S_n-s|<e \qquad \forall\ n>n_1$$

and $$|S_n-t|<e \qquad \forall\ n>n_2$$

$\Rightarrow$ $|S_n-s|<e$ and $|S_n-t|<e$ hold simultaneously $\forall\ n>\max(n_1,n_2)$.

Thus for all $n>\max(n_1,n_2)$ we have

$$|s-t|=|s-S_n+S_n-t|$$

$$\le |S_n - s| + |S_n - t|$$
$$< e + e = 2e$$
$$\Rightarrow |s-t| < 2\left(\frac{|s-t|}{2}\right)$$
$$\Rightarrow |s-t| < |s-t|$$

Which is impossible, therefore the limit of the sequence is unique.

Note: If $\{S_n\}$ converges to s then all of its infinite subsequence converge to s.

Cauchy Sequence

A sequence $\{x_n\}$ of real number is said to be a *Cauchy sequence* if for given positive real number e, $\exists$ a positive integer $n_0(e)$ such that

$$|x_n - x_m| < e \qquad \forall\ m,n > n_0$$

Theorem

A Cauchy sequence of real numbers is bounded.

Proof

Let $\{S_n\}$ be a Cauchy sequence.

Take $e = 1$, then there exits a positive integers n_0 such that

$$|S_n - S_m| < 1 \qquad \forall\ m,n > n_0.$$

Fix $m = n_0 + 1$ then

$$|S_n| = |S_n - S_{n_0+1} + S_{n_0+1}|$$
$$\le |S_n - S_{n_0+1}| + |S_{n_0+1}|$$
$$< 1 + |S_{n_0+1}| \qquad \forall\ n > n_0$$
$$< l \qquad \forall\ n > 1\text{, and } l = 1 + |S_{n_0+1}| \quad (n_0 \text{ changes as } e \text{ changes})$$

Hence we conclude that $\{S_n\}$ is a Cauchy sequence, which is bounded one.

Note:

(i) Convergent sequence is bounded.

(ii) The converse of the above theorem does not hold.

i.e. every bounded sequence is not Cauchy.

Consider the sequence $\{S_n\}$ where $S_n = (-1)^n$, $n \ge 1$. It is bounded sequence because

$$|(-1)^n| = 1 < 2 \qquad \forall\ n \ge 1$$

But it is not a Cauchy sequence if it is then for $e = 1$ we should be able to find a positive integer n_0 such that $|S_n - S_m| < 1$ for all $m,n > n_0$

But with $m = 2k+1$, $n = 2k+2$ when $2k+1 > n_0$, we arrive at

$$|S_n - S_m| = |(-1)^{2n+2} - (-1)^{2k+1}|$$
$$= |1+1| = 2 < 1 \quad \text{is absurd.}$$

Hence $\{S_n\}$ is not a Cauchy sequence. Also this sequence is not a convergent sequence. (it is an oscillatory sequence)

…………………………………

Divergent Sequence

A $\{S_n\}$ is said to be divergent if it is not convergent or it is unbounded.

e.g. $\{n^2\}$ is divergent, it is unbounded.

(ii) $\{(-1)^n\}$ tends to 1 or -1 according as n is even or odd. It oscillates finitely.

(iii) $\{(-1)^n n\}$ is a divergent sequence. It oscillates infinitely.

Note: If two subsequence of a sequence converges to two different limits then the sequence itself is a divergent.

Theorem

If $S_n < U_n < t_n \quad \forall \; n \geq n_0$ and if both the $\{S_n\}$ and $\{t_n\}$ converge to same limits as s, then the sequence $\{U_n\}$ also converges to s.

Proof

Since the sequence $\{S_n\}$ and $\{t_n\}$ converge to the same limit s, therefore, for given $e > 0$ there exists two positive integers $n_1, n_2 > n_0$ such that

$$|S_n - s| < e \qquad \forall \; n > n_1$$

$$|t_n - s| < e \qquad \forall \; n > n_2$$

i.e.
$$s - e < S_n < s + e \qquad \forall \; n > n_1$$

$$s - e < t_n < s + e \qquad \forall \; n > n_2$$

Since we have given

$$S_n < U_n < t_n \qquad \forall \; n > n_0$$

$$\therefore \; s - e < S_n < U_n < t_n < s + e \qquad \forall \; n > \max(n_0, n_1, n_2)$$

$$\Rightarrow \; s - e < U_n < s + e \qquad \forall \; n > \max(n_0, n_1, n_2)$$

i.e.
$$|U_n - s| < e \qquad \forall \; n > \max(n_0, n_1, n_2)$$

i.e.
$$\lim_{n\to\infty} U_n = s$$

Example

Show that $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$

Solution

Using Bernoulli's Inequality

$$\left(1 + \frac{1}{\sqrt{n}}\right)^n \geq 1 + \frac{n}{\sqrt{n}} \geq \sqrt{n} \geq 1 \qquad \forall \; n.$$

Also

$$\left(1 + \frac{1}{\sqrt{n}}\right)^2 = \left[\left(1 + \frac{1}{\sqrt{n}}\right)^n\right]^{\frac{2}{n}} > \left(\sqrt{n}\right)^{\frac{2}{n}} > n^{\frac{1}{n}} \geq 1$$

$$\Rightarrow \; 1 \leq n^{\frac{1}{n}} < \left(1 + \frac{1}{\sqrt{n}}\right)^2$$

$$\Rightarrow \; \lim_{n\to\infty} 1 \leq \lim_{n\to\infty} n^{\frac{1}{n}} < \lim_{n\to\infty}\left(1 + \frac{1}{\sqrt{n}}\right)^2$$

$$\Rightarrow \; 1 \leq \lim_{n\to\infty} n^{\frac{1}{n}} < 1$$

i.e. $\lim_{n\to\infty} n^{\frac{1}{n}} = 1.$

……………………..

Example

Show that $\lim_{n\to\infty}\left(\frac{1}{(n+1)^2}+\frac{1}{(n+2)^2}+\dots\dots\dots\dots+\frac{1}{(2n)^2}\right)=0$

Solution

We have

$$S_n=\left(\frac{1}{(n+1)^2}+\frac{1}{(n+2)^2}+\dots\dots\dots\dots+\frac{1}{(2n)^2}\right)$$

and

$$\frac{n}{(2n)^2} < S_n < \frac{n}{n^2}$$

$$\Rightarrow \frac{1}{4n} < S_n < \frac{1}{n}$$

$$\Rightarrow \lim_{n\to\infty}\frac{1}{4n} < \lim_{n\to\infty}S_n < \lim_{n\to\infty}\frac{1}{n}$$

$$\Rightarrow 0 < \lim_{n\to\infty}S_n < 0$$

$$\Rightarrow \lim_{n\to\infty}S_n = 0$$

Theorem

If the sequence $\{S_n\}$ converges to s then $\exists$ a positive integer n such that $|S_n|>\frac{1}{2}s$.

Proof

We fix $e=\frac{1}{2}|s|>0$

$\Rightarrow\ \exists$ a positive integer n_1 such that

$|S_n-s|<e \qquad$ for $n>n_1$

$$\Rightarrow |S_n-s|<\frac{1}{2}|s|$$

Now

$$\frac{1}{2}|s|=|s|-\frac{1}{2}|s|$$

$$<|s|-|S_n-s| \quad \le|s+(S_n-s)|$$

$$\Rightarrow \frac{1}{2}|s|<|S_n|$$

Theorem

Let a and b be fixed real numbers if $\{S_n\}$ and $\{t_n\}$ converge to s and t respectively, then

(*i*) $\{aS_n+bt_n\}$ converges to $as + bt$.

(*ii*) $\{S_n t_n\}$ converges to st.

(*iii*) $\left\{\frac{S_n}{t_n}\right\}$ converges to $\frac{s}{t}$, provided $t_n\neq 0\ \forall\ n$ and $t\neq 0$.

Proof

Since $\{S_n\}$ and $\{t_n\}$ converge to s and t respectively,

$$\therefore\ |S_n-s|<e \qquad \forall\ n>n_1\in\mathbb{N}$$

$$\left|t_n - t\right| < e \qquad \forall\ n > n_2 \in \mathbb{N}$$

Also $\exists\ l > 0$ such that $\left|S_n\right| < l \quad \forall\ n > 1 \qquad (\because \{S_n\}$ is bounded)

(*i*) We have

$$\left|(aS_n + bt_n) - (as + bt)\right| = \left|a(S_n - s) + b(t_n - t)\right|$$
$$\le \left|a(S_n - s)\right| + \left|b(t_n - t)\right|$$
$$< \left|a\right|e + \left|b\right|e \qquad \forall\ n > \max(n_1, n_2)$$
$$= e_1 \qquad \text{Where } e_1 = \left|a\right|e + \left|b\right|e \text{ a certain number.}$$

This implies $\{aS_n + bt_n\}$ converges to *as* + *bt*.

(*ii*) $\left|S_n t_n - st\right| = \left|S_n t_n - S_n t + S_n t - st\right|$

$$= \left|S_n(t_n - t) + t(S_n - s)\right| \le \left|S_n\right| \cdot \left|(t_n - t)\right| + \left|t\right| \cdot \left|(S_n - s)\right|$$
$$< l\,e + \left|t\right|e \qquad \forall\ n > \max(n_1, n_2)$$
$$= e_2 \qquad \text{where } e_2 = l\,e + \left|t\right|e \text{ a certain number.}$$

This implies $\{S_n t_n\}$ converges to *st.*

(*iii*) $\left|\dfrac{1}{t_n} - \dfrac{1}{t}\right| = \left|\dfrac{t - t_n}{t_n t}\right|$

$$= \frac{\left|t_n - t\right|}{\left|t_n\right|\left|t\right|} < \frac{e}{\frac{1}{2}\left|t\right|\left|t\right|} \qquad \forall\ n > \max(n_1, n_2) \qquad \because \left|t_n\right| > \frac{1}{2}t$$
$$= \frac{e}{\frac{1}{2}\left|t\right|^2} = e_3 \qquad \text{where } e_3 = \frac{e}{\frac{1}{2}\left|t\right|^2} \text{ a certain number.}$$

This implies $\left\{\dfrac{1}{t_n}\right\}$ converges to $\dfrac{1}{t}$.

Hence $\left\{\dfrac{S_n}{t_n}\right\} = \left\{S_n \cdot \dfrac{1}{t_n}\right\}$ converges to $s \cdot \dfrac{1}{t} = \dfrac{s}{t}$. (from (ii))

Theorem

For each irrational number *x*, there exists a sequence $\{r_n\}$ of distinct rational numbers such that $\lim\limits_{n\to\infty} r_n = x$.

Proof

Since *x* and *x* + 1 are two different real numbers

$\because\ \exists$ a rational number r_1 such that

$$x < r_1 < x + 1$$

Similarly $\exists$ a rational number $r_2 \ne r_1$ such that

$$x < r_2 < \min\left(r_1, x + \frac{1}{2}\right) < x + 1$$

Continuing in this manner we have

$$x < r_3 < \min\left(r_2, x + \frac{1}{3}\right) < x + 1$$
$$x < r_4 < \min\left(r_3, x + \frac{1}{4}\right) < x + 1$$

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$$x < r_n < \min\left(r_{n-1}, x + \frac{1}{n}\right) < x + 1$$

This implies that $\exists$ a sequence $\{r_n\}$ of the distinct rational number such that

$$x-\frac{1}{n}<x<r_n<x+\frac{1}{n}$$

Since

$$\lim_{n\to\infty}\left(x-\frac{1}{n}\right)=\lim_{n\to\infty}\left(x+\frac{1}{n}\right)=x$$

Therefore

$$\lim_{n\to\infty} r_n = x$$

Theorem

Let a sequence $\{S_n\}$ be a bounded sequence.

(i) If $\{S_n\}$ is monotonically increasing then it converges to its supremum.

(ii) If $\{S_n\}$ is monotonically decreasing then it converges to its infimum.

Proof

Let $S=\sup S_n$ and $s=\inf S_n$

Take $e>0$

(i) Since $S=\sup S_n$

$\therefore\ \exists\ S_{n_0}$ such that $S-e<S_{n_0}$

Since $\{S_n\}$ is $\uparrow$ ($\uparrow$ stands for monotonically increasing)

$\therefore\ S-e<S_{n_0}<S_n<S<S+e$ for $n>n_0$

$\Rightarrow\ S-e<S_n<S+e$ for $n>n_0$

$\Rightarrow\ |S_n-S|<e$ for $n>n_0$

$\Rightarrow\ \lim_{n\to\infty} S_n = S$

(ii) Since $s=\inf S_n$

$\therefore\ \exists\ S_{n_1}$ such that $S_{n_1}<s+e$

Since $\{S_n\}$ is $\downarrow$. ($\downarrow$ stands for monotonically decreasing)

$\therefore\ s-e<s<S_n<S_{n_1}<s+e$ for $n>n_1$

$\Rightarrow\ s-e<S_n<s+e$ for $n>n_1$

$\Rightarrow\ |S_n-s|<e$ for $n>n_1$

Thus $\lim_{n\to\infty} S_n = s$

Note

A monotonic sequence can not oscillate infinitely.

Example:

Consider $\{S_n\}=\left\{\left(1+\frac{1}{n}\right)^n\right\}$

As shown earlier it is an increasing sequence

Take $S_{2n}=\left(1+\frac{1}{2n}\right)^{2n}$

Then $\sqrt{S_{2n}}=\left(1+\frac{1}{2n}\right)^{n}$

$$\Rightarrow \frac{1}{\sqrt{S_{2n}}}=\left(\frac{2n}{2n+1}\right)^n \qquad \Rightarrow \frac{1}{\sqrt{S_{2n}}}=\left(1-\frac{1}{2n+1}\right)^n$$

Using Bernoulli’s Inequality we have

$$\Rightarrow \frac{1}{\sqrt{S_{2n}}} \geq 1 - \frac{n}{2n+1} > 1 - \frac{n}{2n} = \frac{1}{2} \qquad \because \left(1 - \frac{1}{2n+1}\right)^n \geq 1 - \frac{n}{2n+1}$$

$$\Rightarrow \sqrt{S_{2n}} < 2 \qquad \forall\ n = 1, 2, 3, \ldots\ldots\ldots$$

$$\Rightarrow S_{2n} < 4 \qquad \forall\ n = 1, 2, 3, \ldots\ldots\ldots$$

$$\Rightarrow S_n < S_{2n} < 4 \qquad \forall\ n = 1, 2, 3, \ldots\ldots\ldots$$

Which show that the sequence $\{S_n\}$ is bounded one.

Hence $\{S_n\}$ is a convergent sequence the number to which it converges is its supremum, which is denoted by 'e' and $2 < e < 3$.

Recurrence Relation

A sequence is said to be defined *recursively* or *by recurrence relation* if the general term is given as a relation of its preceding and succeeding terms in the sequence together with some initial condition.

Example

Let $t_1 > 0$ and let $\{t_n\}$ be defined by $t_{n+1} > 2 - \dfrac{1}{t_n}$; $n \geq 1$

$$\Rightarrow t_n > 0 \qquad \forall\ n \geq 1$$

Also $\quad t_n - t_{n+1} = t_n - 2 + \dfrac{1}{t_n}$

$$= \frac{t_n^2 - 2t_n + 1}{t_n} = \frac{(t_n - 1)^2}{t_n} > 0$$

$$\Rightarrow t_n > t_{n+1} \qquad \forall\ n \geq 1$$

This implies that t_n is monotonically decreasing.

Since $t_n > 1 \quad \forall\ n \geq 1$

$\Rightarrow t_n$ is bounded below $\quad \Rightarrow t_n$ is convergent.

Let us suppose $\lim\limits_{n\to\infty} t_n = t$

Then $\quad \lim\limits_{n\to\infty} t_{n+1} = \lim\limits_{n\to\infty} t_n$

$$\Rightarrow \lim_{n\to\infty}\left(2 - \frac{1}{t_n}\right) = \lim_{n\to\infty} t_n$$

$$\Rightarrow 2 - \frac{1}{t} = t \quad \Rightarrow \frac{2t-1}{t} = t \quad \Rightarrow 2t - 1 = t^2 \quad \Rightarrow t^2 - 2t + 1 = 0$$

$$\Rightarrow (t-1)^2 = 0 \quad \Rightarrow t = 1$$

Example

Let $\{S_n\}$ be defined by $S_{n+1} = \sqrt{S_n + b}$; $n \geq 1$ and $S_1 = a > b$.

It is clear that $S_n > 0\ \forall\ n \geq 1$ and $S_2 > S_1$ and

$$S_{n+1}^2 - S_n^2 = (S_n + b) - (S_{n-1} + b)$$

$$= S_n - S_{n-1}$$

$$\Rightarrow (S_{n+1} + S_n)(S_{n+1} - S_n) = S_n - S_{n-1}$$

$$\Rightarrow S_{n+1} - S_n = \frac{S_n - S_{n-1}}{S_{n+1} + S_n}$$

Since $S_{n+1} + S_n > 0 \quad \forall\ n \geq 1$

Therefore $S_{n+1} - S_n$ and $S_n - S_{n-1}$ have the same sign.

i.e. $S_{n+1} > S_n$ if and only if $S_n > S_{n-1}$ and

$S_{n+1} < S_n$ if and only if $S_n < S_{n-1}$.

But we know that $S_2 > S_1$ therefore $S_3 > S_2$, $S_4 > S_3$, and so on.
This implies the sequence is an increasing sequence.

Also $S_{n+1}^2 - S_n^2 = \left(\sqrt{S_n + b}\right)^2 - S_n^2 = S_n + b - S_n^2$

$$= -\left(S_n^2 - S_n - b\right)$$

Since $S_n > 0 \quad \forall\ n \ge 1$, therefore S_n is the root (+ive) of the

$$S_n^2 - S_n - b = 0$$

Take this value of S_n as a where $a = \dfrac{1+\sqrt{1+4b}}{2}$

> *For equation $ax^2 + bx + c = 0$*
> *The product of roots is $ab = {}^{c}\!/_{a}$*
> *i.e. the other root $b = \dfrac{c}{aa}$*

the other root of equation is therefore $\dfrac{-b}{a}$

Since $S_{n+1} > S_n \ \forall\ n \ge 1$

Also $-\left(S_n - a\right)\left(S_n + \dfrac{b}{a}\right) = S_{n+1}^2 - S_n^2 > 0$

$$\therefore\ S_n + \frac{b}{a} > 0 \quad \text{or} \quad -\left(S_n - a\right) \ge 0$$

$$\Rightarrow\ S_n < a \quad \forall\ n \ge 1$$

which shows that S_n is bounded and hence it is convergent.

Suppose $\lim\limits_{n\to\infty} S_n = s$

Then $\lim\limits_{n\to\infty} (S_{n+1})^2 = \lim\limits_{n\to\infty} (S_n + b)$

$$\Rightarrow\ s^2 = s + b \quad \Rightarrow\ s^2 - s - b = 0$$

Which shows that $a = \dfrac{1+\sqrt{1+4b}}{2}$ is the limit of the sequence.

Theorem

Every Cauchy sequence of real numbers has a convergent subsequence.

Proof

Suppose $\{S_n\}$ is a Cauchy sequence.

Let $e > 0$ then $\exists$ a positive integer $n_0 \ge 1$ such that

$$\left|S_{n_k} - S_{n_{k-1}}\right| < \frac{e}{2^k} \qquad \forall\ n_k, n_{k-1},\ k = 1, 2, 3, ...$$

Put $\quad b_k = \left(S_{n_1} - S_{n_0}\right) + \left(S_{n_2} - S_{n_1}\right) + ... + \left(S_{n_k} - S_{n_{k-1}}\right)$

$$\Rightarrow |b_k| = \left|\left(S_{n_1} - S_{n_0}\right) + \left(S_{n_2} - S_{n_1}\right) + ... + \left(S_{n_k} - S_{n_{k-1}}\right)\right|$$

$$\le \left|\left(S_{n_1} - S_{n_0}\right)\right| + \left|\left(S_{n_2} - S_{n_1}\right)\right| + ... + \left|\left(S_{n_k} - S_{n_{k-1}}\right)\right|$$

$$< \frac{e}{2} + \frac{e}{2^2} + ... + \frac{e}{2^k}$$

$$= e\left(\frac{1}{2} + \frac{1}{2^2} + ... + \frac{1}{2^k}\right) = e\left(\frac{\frac{1}{2}\left(1 - \frac{1}{2^k}\right)}{1 - \frac{1}{2}}\right) = e\left(1 - \frac{1}{2^k}\right)$$

$$\Rightarrow\ |b_k| < e \qquad \forall\ k \ge 1$$

$$\Rightarrow\ \{b_k\} \text{ is convergent}$$

$$\because\ b_k = S_{n_k} - S_{n_0} \quad \therefore\ S_{n_k} = b_k + S_{n_0}$$

Where S_{n_0} is a certain fix number therefore $\{S_{n_k}\}$ which is a subsequence of $\{S_n\}$ is convergent.

Theorem (Cauchy's General Principle for Convergence)

A sequence of real number is convergent if and only if it is a Cauchy sequence.

Proof

Necessary Condition

Let $\{S_n\}$ be a convergent sequence, which converges to s.

Then for given $e > 0$ $\exists$ a positive integer n_0, such that

$$|S_n - s| < \frac{e}{2} \quad \forall \; n > n_0$$

Now for $n > m > n_0$

$$\begin{aligned} |S_n - S_m| &= |S_n - s + S_m - s| \\ &\leq |S_n - s| + |S_m - s| \\ &< \frac{e}{2} + \frac{e}{2} = e \end{aligned}$$

Which shows that $\{S_n\}$ is a Cauchy sequence.

Sufficient Condition

Let us suppose that $\{S_n\}$ is a Cauchy sequence then for $e > 0$, $\exists$ a positive integer m_1 such that

$$|S_n - S_m| < \frac{e}{2} \qquad \forall \; n, m > m_1 \; \ldots\ldots\ldots.. (i)$$

Since $\{S_n\}$ is a Cauchy sequence

therefore it has a subsequence $\{S_{n_k}\}$ converging to s (say).

$\Rightarrow$ $\exists$ a positive integer m_2 such that

$$|S_{n_k} - s| < \frac{e}{2} \qquad \forall \; n > m_2 \; \ldots\ldots\ldots.. (ii)$$

Now

$$\begin{aligned} |S_n - s| &= |S_n - S_{n_k} + S_{n_k} - s| \\ &\leq |S_n - S_{n_k}| + |S_{n_k} - s| \\ &< \frac{e}{2} + \frac{e}{2} = e \qquad \forall \; n > \max(m_1, m_2) \end{aligned}$$

which shows that $\{S_n\}$ is a convergent sequence.

Example

Let $\{S_n\}$ be define by $0 < a < S_1 < S_2 < b$ and also

$$S_{n+1} = \sqrt{S_n \cdot S_{n-1}}, \quad n > 2 \; \ldots\ldots\ldots (i)$$

Here $S_n > 0$, $\forall \; n \geq 1$ and $a < S_1 < b$

Let for some $k > 2$

$$a < S_k < b$$

then $a^2 < a S_k < S_k S_{k-1} = (S_{k+1})^2 < b^2$ $\qquad \because \; S_{n+1} = \sqrt{S_n S_{n-1}}$

i.e. $a^2 < S_{k+1}^2 < b^2$

$\Rightarrow \; a < S_{k+1} < b$

$\Rightarrow \; a < S_n < b \quad \forall \; n \in \mathbb{N}$

$$\because \; \frac{S_n}{S_{n+1}} > \frac{a}{b}$$

$$\therefore \; \frac{S_n}{S_{n+1}} + 1 > \frac{a}{b} + 1$$

$$\Rightarrow \frac{S_n + S_{n+1}}{S_{n+1}} > \frac{a+b}{b}$$

$$\Rightarrow \frac{S_n + S_{n+1}}{S_n} > \frac{a+b}{b} \qquad S_{n+1} \text{ is replace by } S_n \quad \therefore\ S_n < S_{n+1}$$

And $\quad S_{n+1}^2 - S_n^2 = S_n \cdot S_{n-1} - S_n^2 \qquad \because\ S_{n+1} = \sqrt{S_n S_{n-1}}$

$$= S_n\left(S_{n-1} - S_n\right)$$

$$\Rightarrow \left|S_{n+1} - S_n\right| = \frac{S_n}{S_n + S_{n+1}}\left|S_{n-1} - S_n\right|$$

$$< \frac{b}{a+b}\left|S_{n-1} - S_n\right|$$

$$\Rightarrow \left|S_{n+1} - S_n\right| < \frac{b}{a+b}\left|S_n - S_{n-1}\right| \qquad \because\ \left|S_{n-1} - S_n\right| = \left|S_n - S_{n-1}\right|$$

$$< \left(\frac{b}{a+b}\right)^2 \left|S_{n-1} - S_{n-2}\right|$$

$$< \left(\frac{b}{a+b}\right)^3 \left|S_{n-2} - S_{n-3}\right|$$

$$\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots$$

$$< \left(\frac{b}{a+b}\right)^{n-1} (b-a)$$

Take $\quad r = \dfrac{b}{a+b} < 1$

Then for $n > m$ we have

$$\left|S_n - S_m\right| = \left|S_n - S_{n-1} + S_{n-1} - S_{n-2} + \ldots\ldots\ldots\ldots + S_{m+1} - S_m\right|$$

$$\leq \left|S_n - S_{n-1}\right| + \left|S_{n-1} - S_{n-2}\right| + \ldots\ldots\ldots\ldots + \left|S_{m+1} - S_m\right|$$

$$< \left(r^{n-2} + r^{n-3} + \ldots\ldots\ldots\ldots + r^{m-1}\right)(b-a)$$

$$= e$$

This implies that $\{S_n\}$ is a Cauchy sequence, therefore it is convergent.

Example

Let $\{t_n\}$ be defined by

$$t_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots\ldots\ldots\ldots + \frac{1}{n}$$

For $m, n \in \mathbb{N}$, $n > m$ we have

$$\left|t_n - t_m\right| = \frac{1}{m+1} + \frac{1}{m+2} + \ldots\ldots\ldots\ldots + \frac{1}{n}$$

$$> (n-m)\frac{1}{n} = 1 - \frac{m}{n}$$

In particular if $n = 2m$ then

$$\left|t_n - t_m\right| > \frac{1}{2}$$

This implies that $\{t_n\}$ is not a Cauchy sequence therefore it is divergent.

Theorem (nested intervals)

Suppose that $\{I_n\}$ is a sequence of the closed interval such that $I_n = [a_n, b_n]$, $I_{n+1} \subset I_n \ \forall \ n \geq 1$, and $(b_n - a_n) \to 0$ as $n \to \infty$ then $\bigcap I_n$ contains one and only one point.

Proof

Since $I_{n+1} \subset I_n$

$$\therefore \ a_1 < a_2 < a_3 < \ldots\ldots\ldots\ldots < a_{n-1} < a_n < b_n < b_{n-1} < \ldots\ldots\ldots < b_3 < b_2 < b_1$$

$\{a_n\}$ is increasing sequence, bounded above by b_1 and bounded below by a_1.

And $\{b_n\}$ is decreasing sequence bounded below by a_1 and bounded above by b_1.

$\Rightarrow$ $\{a_n\}$ and $\{b_n\}$ both are convergent.

Suppose $\{a_n\}$ converges to a and $\{b_n\}$ converges to b.

But $|a - b| = |a - a_n + a_n - b_n + b_n - b|$

$$\leq |a_n - a| + |a_n - b_n| + |b_n - b| \to 0 \quad \text{as} \quad n \to \infty.$$

$\Rightarrow$ $a = b$

and $\quad a_n < a < b_n \quad \forall \ n \geq 1$.

Theorem (Bolzano-Weierstrass theorem)

Every bounded sequence has a convergent subsequence.

Proof

Let $\{S_n\}$ be a bounded sequence.

Take $a_1 = \inf S_n$ and $b_1 = \sup S_n$

Then $a_1 < S_n < b_1 \quad \forall \ n \geq 1$.

Now bisect interval $[a_1, b_1]$ such that at least one of the two sub-intervals contains infinite numbers of terms of the sequence.

Denote this sub-interval by $[a_2, b_2]$.

If both the sub-intervals contain infinite number of terms of the sequence then choose the one on the right hand.

Then clearly $a_1 \leq a_2 < b_2 \leq b_1$.

Suppose there exist a subinterval $[a_k, b_k]$ such that

$$a_1 \leq a_2 \leq \ldots\ldots\ldots\ldots \leq a_k < b_k \leq \ldots\ldots\ldots\ldots \leq b_2 \leq b_1$$

$$\Rightarrow \ (b_k - a_k) = \frac{1}{2^k}(b_1 - a_1)$$

Bisect the interval $[a_k, b_k]$ in the same manner and choose $[a_{k+1}, b_{k+1}]$ to have

$$a_1 \leq a_2 \leq \ldots\ldots\ldots\ldots \leq a_k \leq a_{k+1} < b_{k+1} \leq b_k \leq \ldots\ldots\ldots\ldots \leq b_2 \leq b_1$$

and $$b_{k+1} - a_{k+1} = \frac{1}{2^{k+1}}(b_1 - a_1)$$

This implies that we obtain a sequence of interval $[a_n, b_n]$ such that

$$b_n - a_n = \frac{1}{2^n}(b_1 - a_1) \to 0 \quad \text{as} \quad n \to \infty.$$

$\Rightarrow$ we have a unique point s such that

$$s = \bigcap [a_n, b_n]$$

there are infinitely many terms of the sequence whose length is $e > 0$ that contain s.
For $e = 1$ there are infinitely many values of n such that

$$|S_n - s| < 1$$

Let n_1 be one of such value then

$$|S_{n_1} - s| < 1$$

Again choose $n_2 > n_1$ such that

$$\left| S_{n_2} - s \right| < \frac{1}{2}$$

Continuing in this manner we find a sequence $\{S_{n_k}\}$ for each positive integer k such that $n_k < n_{k+1}$ and

$$\left| S_{n_k} - s \right| < \frac{1}{k} \qquad \forall\ k = 1, 2, 3, \ldots\ldots\ldots\ldots$$

Hence there is a subsequence $\{S_{n_k}\}$ which converges to s.

Limit Inferior of the sequence

Suppose $\{S_n\}$ is bounded then we define limit inferior of $\{S_n\}$ as follow

$$\lim_{n\to\infty}(\inf S_n) = \lim_{n\to\infty} U_k \quad \text{where} \quad U_k = \inf\{S_n : n \geq k\}$$

If S_n is bounded below then

$$\lim_{n\to\infty}(\inf S_n) = -\infty$$

Limit Superior of the sequence

Suppose $\{S_n\}$ is bounded above then we define limit superior of $\{S_n\}$ as follow

$$\lim_{n\to\infty}(\sup S_n) = \lim_{n\to\infty} V_k \quad \text{where} \quad V_k = \inf\{S_n : n \geq k\}$$

If S_n is not bounded above then we have

$$\lim_{n\to\infty}(\sup S_n) = +\infty$$

Note:

(i) A bounded sequence has unique limit inferior and superior

(ii) Let $\{S_n\}$ contains all the rational numbers, then every real number is a subsequencial limit then limit superior of S_n is $+\infty$ and limit inferior of S_n is $-\infty$

(iii) Let $\{S_n\} = (-1)^n\left(1 + \frac{1}{n}\right)$

then limit superior of S_n is 1 and limit inferior of S_n is -1.

(iv) Let $U_k = \inf\{S_n : n \geq k\}$

$$= \inf\left\{\left(1+\frac{1}{k}\right)\cos kp\,, \left(1+\frac{1}{k+1}\right)\cos(k+1)p\,, \left(1+\frac{1}{k+2}\right)\cos(k+2)p\,, \ldots\ldots\ldots\ldots\ldots\right\}$$

$$= \begin{cases} \left(1+\frac{1}{k}\right)\cos kp & \text{if } k \text{ is odd} \\ \left(1+\frac{1}{k+1}\right)\cos(k+1)p & \text{if } k \text{ is even} \end{cases}$$

$$\Rightarrow \lim_{n\to\infty}(\inf S_n) = \lim_{n\to\infty} U_k = -1$$

Also $V_k = \sup\{S_n : n \geq k\}$

$$= \begin{cases} \left(1+\frac{1}{k+1}\right)\cos(k+1)p & \text{if } k \text{ is odd} \\ \left(1+\frac{1}{k}\right)\cos kp & \text{if } k \text{ is even} \end{cases}$$

$$\Rightarrow \lim_{n\to\infty}(\inf S_n) = \lim_{n\to\infty} V_k = 1$$

Theorem

If $\{S_n\}$ is a convergent sequence then

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}(\inf S_n) = \lim_{n\to\infty}(\sup S_n)$$

Proof

Let $\lim_{n\to\infty} S_n = s$ then for a real number $e > 0$, $\exists$ a positive integer n_0 such that

$$|S_n - s| < e \qquad \forall\ n \ge n_0 \ \ldots\ldots\ldots\ (i)$$

i.e. $\quad s - e < S_n < s + e \qquad \forall\ n \ge n_0$

If $\quad V_k = \sup\{S_n : n \ge k\}$

Then $\quad s - e < V_n < s + e \qquad \forall\ k \ge n_0$

$$\Rightarrow\ s - e < \lim_{k\to\infty} V_n < s + e \qquad \forall\ k \ge n_0 \ \ldots\ldots\ldots\ (ii)$$

from (*i*) and (*ii*) we have

$$s = \lim_{k\to\infty}\ \sup\{S_n\}$$

We can have the same result for limit inferior of $\{S_n\}$ by taking

$$U_k = \inf\{S_n : n \ge k\}$$

Infinite Series

Given a sequence $\{a_n\}$, we use the notation $\sum_{i=1}^{\infty} a_n$ or simply $\sum a_n$ to denotes the sum $a_1 + a_2 + a_3 + ...$ and called a infinite series or just series.

The numbers $S_n = \sum_{k=1}^{n} a_k$ are called the partial sum of the series.

If the sequence $\{S_n\}$ converges to *s*, we say that the series converges and write $\sum_{n=1}^{\infty} a_n = s$, the number *s* is called the sum of the series but it should be clearly understood that the '*s*' is the limit of the sequence of sums and is not obtained simply by addition.

If the sequence $\{S_n\}$ diverges then the series is said to be diverge.

Note:

The behaviors of the series remain unchanged by addition or deletion of the certain terms

Theorem

If $\sum_{n=1}^{\infty} a_n$ converges then $\lim_{n\to\infty} a_n = 0$.

Proof

Let $S_n = a_1 + a_2 + a_3 + ... + a_n$

Take $\lim_{n\to\infty} S_n = s = \sum a_n$

Since $a_n = S_n - S_{n-1}$

Therefore
$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}(S_n - S_{n-1})$$
$$= \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1}$$
$$= s - s = 0$$

Note:

The converse of the above theorem is false

Example

Consider the series $\sum_{n=1}^{\infty} \frac{1}{n}$.

We know that the sequence $\{S_n\}$ where $S_n = 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n}$ is divergent therefore $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent series, although $\lim_{n\to\infty} a_n = 0$.

This implies that if $\lim_{n\to\infty} a_n \neq 0$, then $\sum a_n$ is divergent.

It is know as basic divergent test.

Theorem (General Principle of Convergence)

A series $\sum a_n$ is convergent if and only if for any real number $e > 0$, there exists a positive integer n_0 such that

$$\left| \sum_{i=m+1}^{\infty} a_i \right| < e \qquad \forall \ n > m > n_0$$

Proof

Let $S_n = a_1 + a_2 + a_3 + ... + a_n$

then $\{S_n\}$ is convergent if and only if for $e > 0$ $\exists$ a positive integer n_0 such that

$$| S_n - S_m | < e \qquad \forall \ n > m > n_0$$

$$\Rightarrow \left| \sum_{i=m+1}^{\infty} a_i \right| = | S_n - S_m | < e$$

Example

If $|x| < 1$ then $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$

And if $|x| \geq 1$ then $\sum_{n=0}^{\infty} x^n$ is divergent.

Theorem

Let $\sum a_n$ be an infinite series of non-negative terms and let $\{S_n\}$ be a sequence of its partial sums then $\sum a_n$ is convergent if $\{S_n\}$ is bounded and it diverges if $\{S_n\}$ is unbounded.

Proof

Since $a_n \geq 0 \quad \forall \ n \geq 0$

$$S_n = S_{n-1} + a_n > S_{n-1} \qquad \forall \ n \geq 0$$

therefore the sequence $\{S_n\}$ is monotonic increasing and hence it is converges if $\{S_n\}$ is bounded and it will diverge if it is unbounded.

Hence we conclude that $\sum a_n$ is convergent if $\{S_n\}$ is bounded and it divergent if $\{S_n\}$ is unbounded.

Theorem (Comparison Test)

Suppose $\sum a_n$ and $\sum b_n$ are infinite series such that $a_n > 0$, $b_n > 0 \quad \forall \ n$. Also suppose that for a fixed positive number l and positive integer k, $a_n < l\, b_n \quad \forall\, n \geq k$

Then $\sum a_n$ converges if $\sum b_n$ is converges and $\sum b_n$ is diverges if $\sum a_n$ is diverges.

Proof

Suppose $\sum b_n$ is convergent and

$$a_n < l\, b_n \qquad \forall\, n \geq k \ \ldots\ldots\ldots\ldots \ (i)$$

then for any positive number $e > 0$ there exists n_0 such that

$$\sum_{i=m+1}^{n} b_i < \frac{e}{l} \qquad n > m > n_0$$

from (i)

$$\Rightarrow \sum_{i=m+1}^{n} a_i < l \sum_{i=m+1}^{n} b_i < e \qquad , \qquad n > m > n_0$$

$$\Rightarrow \sum a_n \text{ is convergent.}$$

Now suppose $\sum a_n$ is divergent then $\{S_n\}$ is unbounded.

$\Rightarrow \ \exists$ a real number $b > 0$ such that

$$\sum_{i=m+1}^{n} b_i > l\ b \qquad , \qquad n > m$$

from (i)

$$\Rightarrow \sum_{i=m+1}^{n} b_i > \frac{1}{l} \sum_{i=m+1}^{n} a_i > b \qquad , \qquad n > m$$

$$\Rightarrow \sum b_n \text{ is convergent.}$$

Example

We know that $\sum \frac{1}{n}$ is divergent and

$$n \geq \sqrt{n} \quad \forall \; n \geq 1$$

$$\Rightarrow \; \frac{1}{n} \leq \frac{1}{\sqrt{n}}$$

$\Rightarrow \; \sum \frac{1}{\sqrt{n}}$ is divergent as $\sum \frac{1}{n}$ is divergent.

Example

The series $\sum \frac{1}{n^a}$ is convergent if $a > 1$ and diverges if $a \leq 1$.

Let $S_n = 1 + \frac{1}{2^a} + \frac{1}{3^a} + ... + \frac{1}{n^a}$

If $a > 1$ then

$S_n < S_{2n}$ and $\frac{1}{n^a} < \frac{1}{(n-1)^a}$

Now $S_{2n} = \left[1 + \frac{1}{2^a} + \frac{1}{3^a} + \frac{1}{4^a} .... + \frac{1}{(2n)^a}\right]$

$$= \left[1 + \frac{1}{3^a} + \frac{1}{5^a} + ... + \frac{1}{(2n-1)^a}\right] + \left[\frac{1}{2^a} + \frac{1}{4^a} + \frac{1}{6^a} + ... + \frac{1}{(2n)^a}\right]$$

$$= \left[1 + \frac{1}{3^a} + \frac{1}{5^a} + ... + \frac{1}{(2n-1)^a}\right] + \frac{1}{2^a}\left[1 + \frac{1}{2^a} + \frac{1}{3^a} + ... + \frac{1}{(n)^a}\right]$$

$$< \left[1 + \frac{1}{2^a} + \frac{1}{4^a} + ... + \frac{1}{(2n-2)^a}\right] + \frac{1}{2^a} S_n$$

replacing 3 by 2, 5 by 4 and so on.

$$= 1 + \frac{1}{2^a}\left[1 + \frac{1}{2^a} + ... + \frac{1}{(n-1)^a}\right] + \frac{1}{2^a} S_n$$

$$= 1 + \frac{1}{2^a} S_{n-1} + \frac{1}{2^a} S_n \quad = 1 + \frac{1}{2^a} S_{2n} + \frac{1}{2^a} S_{2n} \qquad \because \; S_{n-1} < S_n < S_{2n}$$

$$= 1 + \frac{2}{2^a} S_{2n}$$

$$\Rightarrow \; S_{2n} < 1 + \frac{1}{2^{a-1}} S_{2n}$$

$$\Rightarrow \; \left(1 - \frac{1}{2^{a-1}}\right) S_{2n} < 1 \quad \Rightarrow \; \left(\frac{2^{a-1}-1}{2^{a-1}}\right) S_{2n} < 1 \quad \Rightarrow \; S_{2n} < \frac{2^{a-1}}{2^{a-1}-1}$$

i.e. $S_n < S_{2n} < \frac{2^{a-1}}{2^{a-1}-1}$

$\Rightarrow \; \{S_n\}$ is bounded and also monotonic. Hence we conclude that $\sum \frac{1}{n^a}$ is convergent when $a > 1$.

If $a \leq 1$ then

$$n^a \leq n \quad \forall \; n \geq 1$$

$$\Rightarrow \; \frac{1}{n^a} \geq \frac{1}{n} \quad \forall \; n \geq 1$$

$\because \; \sum \frac{1}{n}$ is divergent therefore $\sum \frac{1}{n^a}$ is divergent when $a \leq 1$.

Theorem

Let $a_n > 0$, $b_n > 0$ and $\lim\limits_{n\to\infty}\dfrac{a_n}{b_n} = l \neq 0$ then the series $\sum a_n$ and $\sum b_n$ behave alike.

Proof

Since $\lim\limits_{n\to\infty}\dfrac{a_n}{b_n} = l$

$$\Rightarrow \left|\frac{a_n}{b_n} - l\right| < e \quad \forall \; n \geq n_0.$$

Use $e = \dfrac{l}{2}$

$$\Rightarrow \left|\frac{a_n}{b_n} - l\right| < \frac{l}{2} \quad \forall \; n \geq n_0.$$

$$\Rightarrow l - \frac{l}{2} < \frac{a_n}{b_n} < l + \frac{l}{2}$$

$$\Rightarrow \frac{l}{2} < \frac{a_n}{b_n} < \frac{3l}{2}$$

then we got

$$a_n < \frac{3l}{2} b_n \quad \text{and} \quad b_n < \frac{2}{l} a_n$$

Hence by comparison test we conclude that $\sum a_n$ and $\sum b_n$ converge or diverge together.

Example

To check $\sum \dfrac{1}{n}\sin^2\dfrac{x}{n}$ diverges or converges consider

$$a_n = \frac{1}{n}\sin^2\frac{x}{n} \quad \text{and take} \quad b_n = \frac{1}{n^3}$$

then $\dfrac{a_n}{b_n} = n^2 \sin^2\dfrac{x}{n}$

$$= \frac{\sin^2 \frac{x}{n}}{\frac{1}{n^2}} = x^2\left(\frac{\sin\frac{x}{n}}{\frac{x}{n}}\right)^2$$

Applying limit as $n \to \infty$

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty} x^2\left(\frac{\sin\frac{x}{n}}{\frac{x}{n}}\right)^2 = x^2\left(\lim_{n\to\infty}\frac{\sin\frac{x}{n}}{\frac{x}{n}}\right)^2 = x^2(1) = x^2$$

$\Rightarrow \sum a_n$ and $\sum b_n$ have the similar behavior $\forall$ finite values of x except $x = 0$.

Since $\sum \dfrac{1}{n^3}$ is convergent series therefore the given series is also convergent for finite values of x except $x = 0$.

Theorem (Cauchy Condensation Test)

Let $a_n \geq 0$, $a_n > a_{n+1}$ $\forall$ $n \geq 1$, then the series $\sum a_n$ and $\sum 2^{n-1} a_{2^{n-1}}$ converges or diverges together.

Proof

Let us suppose

$$S_n = a_1 + a_2 + a_3 + \ldots\ldots\ldots\ldots + a_n$$

and $$t_n = a_1 + 2a_2 + 2^2 a_{2^2} + \ldots\ldots\ldots\ldots + 2^{n-1} a_{2^{n-1}}.$$

$\because$ $a_n \geq 0$ and $n < 2^{n-1} < 2^n - 1$

$\therefore$ $S_n < S_{2^{n-1}} < S_{2^n - 1}$ for $n > 2$

then

$$S_{2^n-1} = a_1 + a_2 + a_3 + \ldots\ldots + a_{2^n-1}$$

$$= a_1 + (a_2 + a_3) + (a_4 + a_5 + a_6 + a_7) + \ldots\ldots + \left(a_{2^{n-1}} + a_{2^{n-1}+1} + a_{2^{n-1}+2} + \ldots\ldots + a_{2^n-1}\right)$$

$$< a_1 + (a_2 + a_2) + (a_4 + a_4 + a_4 + a_4) + \ldots\ldots + \left(a_{2^{n-1}} + a_{2^{n-1}} + a_{2^{n-1}} + \ldots\ldots + a_{2^{n-1}}\right)$$

$$< a_1 + 2a_2 + 2^2 a_4 + \ldots\ldots + 2^{n-1} a_{2^{n-1}} = t_n$$

$\Rightarrow$ $S_n < t_n$

$\Rightarrow$ $S_n < t_n < 2S_{2^n}$ ………… (i)

Now consider

$$S_{2^n} = a_1 + a_2 + a_3 + \ldots\ldots\ldots\ldots + a_{2^n}$$

$$= a_1 + a_2 + (a_3 + a_4) + (a_5 + a_6 + a_7 + a_8) + \ldots\ldots + \left(a_{2^{n-1}+1} + a_{2^{n-1}+2} + a_{2^{n-1}+3} + \ldots + a_{2^n}\right)$$

$$> \frac{1}{2} a_1 + a_2 + (a_4 + a_4) + (a_8 + a_8 + a_8 + a_8) + \ldots\ldots + \left(a_{2^n} + a_{2^n} + a_{2^n} + \ldots\ldots + a_{2^n}\right)$$

$$= \frac{1}{2} a_1 + a_2 + 2a_4 + 2^2 a_8 + \ldots\ldots\ldots\ldots + 2^{n-1} a_{2^n}$$

$$= \frac{1}{2}\left(a_1 + 2a_2 + 2^3 a_4 + 2^3 a_8 + \ldots\ldots\ldots\ldots + 2^n a_{2^n}\right)$$

$\Rightarrow$ $S_{2n} > \frac{1}{2} t_n$ ………… (ii)

$\Rightarrow$ $2S_{2n} > t_n$

From (i) and (ii) we see that the sequence S_n and t_n are either both bounded or both unbounded, implies that $\sum a_n$ and $\sum 2^{n-1} a_{2^{n-1}}$ converges or diverges together.

Example

Consider the series $\sum \frac{1}{n^p}$

If $p \leq 0$ then $\lim\limits_{n\to\infty} \frac{1}{n^p} \neq 0$

therefore the series diverges when $p \leq 0$.

If $p > 0$ then the condensation test is applicable and we are lead to the series

$$\sum_{k=0}^{\infty} 2^k \frac{1}{(2^k)^p} = \sum_{k=0}^{\infty} \frac{1}{2^{kp-k}}$$

$$= \sum_{k=0}^{\infty} \frac{1}{2^{(p-1)k}} = \sum_{k=0}^{\infty} \left(\frac{1}{2^{(p-1)}}\right)^k$$

$$= \sum_{k=0}^{\infty} 2^{(1-p)k}$$

Now $2^{1-p} < 1$ iff $1 - p < 0$ i.e. when $p > 1$

And the result follows by comparing this series with the geometric series having common ratio less than one.

The series diverges when $2^{1-p}=1$ (i.e. when $p=1$)

The series is also divergent if $0<p<1$.

Example

If $p>1$, $\sum_{n=2}^{\infty}\frac{1}{n(\ln n)^p}$ converges and

If $p\le 1$ the series is divergent.

$\because$ $\{\ln n\}$ is increasing $\quad\therefore$ $\left\{\frac{1}{n\ln n}\right\}$ decreases

and we can use the condensation test to the above series.

We have $a_n=\frac{1}{n(\ln n)^p}$

$$\Rightarrow\ a_{2^n}=\frac{1}{2^n\left(\ln 2^n\right)^p}\qquad\Rightarrow\ 2^n a_{2^n}=\frac{1}{(n\ln 2)^p}$$

$\Rightarrow$ we have the series

$$\sum 2^n a_{2^n}=\sum\frac{1}{(n\ln 2)^p}=\frac{1}{(\ln 2)^p}\sum\frac{1}{n^p}$$

which converges when $p>1$ and diverges when $p\le 1$.

Example

Consider $\sum\frac{1}{\ln n}$

Since $\{\ln n\}$ is increasing there $\left\{\frac{1}{\ln n}\right\}$ decreases.

And we can apply the condensation test to check the behavior of the series

$$\because\ a_n=\frac{1}{\ln n}\qquad\therefore\ a_{2^n}=\frac{1}{\ln 2^n}$$

so $\quad 2^n a_{2^n}=\frac{2^n}{\ln 2^n}\qquad\Rightarrow\ 2^n a_{2^n}=\frac{2^n}{n\ln 2}$

since $\quad\frac{2^n}{n}>\frac{1}{n}\qquad\forall\ n\ge 1$

and $\sum\frac{1}{n}$ is diverges therefore the given series is also diverges.

Alternating Series

A series in which successive terms have opposite signs is called an alternating series.

e.g. $\sum \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + ...$ is an alternating series.

Theorem (Alternating Series Test or Leibniz Test)

Let $\{a_n\}$ be a decreasing sequence of positive numbers such that $\lim\limits_{n\to\infty} a_n = 0$ then the alternating series $\sum\limits_{n=1}^{\infty}(-1)^{n+1}a_n = a_1 - a_2 + a_3 - a_4 + ...$ converges.

Proof

Looking at the odd numbered partial sums of this series we find that

$$S_{2n+1} = (a_1 - a_2) + (a_3 - a_4) + (a_5 - a_6) + ... + (a_{2n-1} - a_{2n}) + a_{2n+1}$$

Since $\{a_n\}$ is decreasing therefore all the terms in the parenthesis are non-negative

$$\Rightarrow S_{2n+1} > 0 \quad \forall\ n$$

Moreover

$$S_{2n+3} = S_{2n+1} - a_{2n+2} + a_{2n+3}$$
$$= S_{2n+1} - \left(a_{2n+2} - a_{2n+3}\right)$$

Since $a_{2n+2} - a_{2n+3} \geq 0$ therefore $S_{2n+3} \leq S_{2n+1}$

Hence the sequence of odd numbered partial sum is decreasing and is bounded below by zero. (as it has +ive terms)

It is therefore convergent.

Thus S_{2n+1} converges to some limit l (say).

Now consider the even numbered partial sum. We find that

$$S_{2n+2} = S_{2n+1} - a_{2n+2}$$

and

$$\lim_{n\to\infty} S_{2n+2} = \lim_{n\to\infty}\left(S_{2n+1} - a_{2n+2}\right)$$
$$= \lim_{n\to\infty} S_{2n+1} - \lim_{n\to\infty} a_{2n+2}$$
$$= l - 0 = l \qquad \because \lim_{n\to\infty} a_n = 0$$

so that the even partial sum is also convergent to l.

$\Rightarrow$ both sequences of odd and even partial sums converge to the same limit.

Hence we conclude that the corresponding series is convergent.

Absolute Convergence

$\sum a_n$ is said to converge absolutely if $\sum |a_n|$ converges.

Theorem

An absolutely convergent series is convergent.

Proof:

If $\sum |a_n|$ is convergent then for a real number $e > 0$, $\exists$ a positive integer n_0 such that

$$\left|\sum_{i=m+1}^{n} a_i\right| < \sum_{i=m+1}^{n} |a_i| < e \qquad \forall\ n, m > n_0$$

$\Rightarrow$ the series $\sum a_n$ is convergent. (Cauchy Criterion has been used)

Note

The converse of the above theorem does not hold.

e.g. $\sum \frac{(-1)^{n+1}}{n}$ is convergent but $\sum \frac{1}{n}$ is divergent.

Theorem (The Root Test)

Let $\lim\limits_{n\to\infty} Sup|a_n|^{1/n} = p$

Then $\sum a_n$ converges absolutely if $p<1$ and it diverges if $p>1$.

Proof

Let $p<1$ then we can find the positive number $e>0$ such that $p+e<1$

$\Rightarrow |a_n|^{1/n} < p+e < 1 \quad \forall\ n>n_0$

$\Rightarrow |a_n| < (p+e)^n < 1$

$\because \sum (p+e)^n$ is convergent because it is a geometric series with $|r|<1$.

$\therefore \sum |a_n|$ is convergent

$\Rightarrow \sum a_n$ converges absolutely.

Now let $p>1$ then we can find a number $e_1>0$ such that $p-e_1>1$.

$\Rightarrow |a_n|^{1/n} > p+e > 1$

$\Rightarrow |a_n| > 1$ for infinitely many values of *n*.

$\Rightarrow \lim\limits_{n\to\infty} a_n \neq 0$

$\Rightarrow \sum a_n$ is divergent.

Note:

The above test give no information when $p=1$.

e.g. Consider the series $\sum \frac{1}{n}$ and $\sum \frac{1}{n^2}$.

For each of these series $p=1$, but $\sum \frac{1}{n}$ is divergent and $\sum \frac{1}{n^2}$ is convergent.

Theorem (Ratio Test)

The series $\sum a_n$

(i) Converges if $\lim\limits_{n\to\infty} Sup\left|\frac{a_{n+1}}{a_n}\right|<1$

(ii) Diverges if $\left|\frac{a_{n+1}}{a_n}\right|>1$ for $n\geq n_0$, where n_0 is some fixed integer.

Proof

If (i) holds we can find $b<1$ and integer *N* such that

$$\left|\frac{a_{n+1}}{a_n}\right| < b \quad \text{for } n\geq N$$

In particular

$$\left|\frac{a_{N+1}}{a_N}\right| < b$$

$\Rightarrow |a_{N+1}| < b|a_N|$

$\Rightarrow |a_{N+2}| < b|a_{N+1}| < b^2|a_N|$

$\Rightarrow |a_{N+3}| < b^3|a_N|$

……………………….

……………………….

……………………….

$\Rightarrow |a_{N+p}| < b^p|a_N|$

$\Rightarrow \ |a_n| < b^{n-N}|a_N|$ we put $N+p=n$.

i.e. $|a_n| < |a_N|\, b^{-N} b^n$ for $n \ge N$.

$\because \ \sum b^n$ is convergent because it is geometric series with common ration <1.

Therefore $\sum a_n$ is convergent (by comparison test)

Now if

$$|a_{n+1}| \ge |a_n| \quad \text{for } n \ge n_0$$

then $\lim\limits_{n\to\infty} a_n \ne 0$

$\Rightarrow \sum a_n$ is divergent.

Note

The knowledge $\left|\dfrac{a_{n+1}}{a_n}\right| = 1$ implies nothing about the convergent or divergent of series.

Example

Consider the series $\sum a_n$ with $a_n = \left[\dfrac{n}{n+1} - \left(\dfrac{n}{n+1}\right)^{n+1}\right]^{-n}$

$$\because \ \frac{n}{n+1} < 1 \qquad \therefore \ a_n > 0 \quad \forall \ n.$$

Also $(a_n)^{\frac{1}{n}} = \left[\dfrac{n}{n+1} - \left(\dfrac{n}{n+1}\right)^{n+1}\right]^{-1}$

$$= \left(\frac{n+1}{n}\right)\left[1 - \left(\frac{n}{n+1}\right)^{n}\right]^{-1} = \left(\frac{n+1}{n}\right)\left[1 - \left(\frac{n+1}{n}\right)^{-n}\right]^{-1}$$

$$= \left(1+\frac{1}{n}\right)\left[1 - \left(1+\frac{1}{n}\right)^{-n}\right]^{-1}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)\left[1 - \left(1+\frac{1}{n}\right)^{-n}\right]^{-1}$$

$$= \lim_{n\to\infty}\left(1+\frac{1}{n}\right) \lim_{n\to\infty}\left[1 - \left(1+\frac{1}{n}\right)^{-n}\right]^{-1}$$

$$= 1\cdot\left[1-e^{-1}\right]^{-1} = \left[1-\frac{1}{e}\right]^{-1} = \left[\frac{e-1}{e}\right]^{-1} = \left[\frac{e}{e-1}\right] > 1$$

$\Rightarrow$ the series is divergent.

Theorem (Dirichlet)

Suppose that $\{S_n\}$, $S_n = a_1 + a_2 + a_3 + \ldots\ldots\ldots\ldots + a_n$ is bounded. Let $\{b_n\}$ be positive term decreasing sequence such that $\lim\limits_{n\to\infty} b_n = 0$, then $\sum a_n b_n$ is convergent.

Proof

$\because \ \{S_n\}$ is bounded

$\therefore \ \exists$ a positive number l such that

$$|S_n| < l \qquad \forall \ n \ge 1.$$

Then $\quad a_i b_i = (S_i - S_{i-1}) b_i \qquad$ for $i \ge 2$

$$= S_i b_i - S_{i-1} b_i$$

$$= S_i b_i - S_{i-1} b_i + S_i b_{i+1} - S_i b_{i+1}$$

$$= S_i\left(b_i - b_{i+1}\right) - S_{i-1} b_i + S_i b_{i+1}$$

$$\Rightarrow \sum_{i=m+1}^{n} a_i b_i = \sum_{i=m+1}^{n} S_i\left(b_i - b_{i+1}\right) - \left(S_m b_{m+1} - S_n b_{n+1}\right)$$

$\because$ $\{b_n\}$ is decreasing

$$\therefore \left|\sum_{i=m+1}^{n} a_i b_i\right| = \left|\sum_{i=m+1}^{n} S_i\left(b_i - b_{i+1}\right) - S_m b_{m+1} + S_n b_{n+1}\right|$$

$$< \sum_{i=m+1}^{n} \left\{\left|S_i\right|\left(b_i - b_{i+1}\right)\right\} + \left|S_m\right| b_{m+1} + \left|S_n\right| b_{n+1}$$

$$< \sum_{i=m+1}^{n} \left\{l\left(b_i - b_{i+1}\right)\right\} + l\, b_{m+1} + l\, b_{n+1} \qquad \because \left|S_i\right| < l$$

$$= l\left(\sum_{i=m+1}^{n} \left(b_i - b_{i+1}\right) + b_{m+1} + b_{n+1}\right)$$

$$= l\left(\left(b_{m+1} - b_{n+1}\right) + b_{m+1} + b_{n+1}\right) = 2\,l\left(b_{m+1}\right)$$

$\Rightarrow \left|\sum_{i=m+1}^{n} a_i b_i\right| < e$ where $e = 2\,l\left(b_{m+1}\right)$ a certain number

$\Rightarrow$ The $\sum a_n b_n$ is convergent. (We have use Cauchy Criterion here.)

Theorem

Suppose that $\sum a_n$ is convergent and that $\{b_n\}$ is monotonic convergent sequence then $\sum a_n b_n$ is also convergent.

Proof

Suppose $\{b_n\}$ is decreasing and it converges to b.

Put $c_n = b_n - b$

$\Rightarrow$ $c_n \geq 0$ and $\lim\limits_{n\to\infty} c_n = 0$

$\because$ $\sum a_n$ is convergent

$\therefore$ $\{S_n\}$, $S_n = a_1 + a_2 + a_3 + \dots\dots\dots\dots + a_n$ is convergent

$\Rightarrow$ It is bounded

$\Rightarrow$ $\sum a_n c_n$ is bounded.

$\because$ $a_n b_n = a_n c_n + a_n b$ and $\sum a_n c_n$ and $\sum a_n b$ are convergent.

$\therefore$ $\sum a_n b_n$ is convergent.

Now if $\{b_n\}$ is increasing and converges to b then we shall put $c_n = b - b_n$.

Example

$\sum \dfrac{1}{(n\ln n)^a}$ is convergent if $a > 1$ and divergent if $a \leq 1$.

To see this we proceed as follows

$$a_n = \frac{1}{(n\ln n)^a}$$

Take $b_n = 2^n a_{2^n} = \dfrac{2^n}{\left(2^n \ln 2^n\right)^a} = \dfrac{2^n}{\left(2^n\; n \ln 2\right)^a}$

$$= \frac{2^n}{2^{na}\; n^a \left(\ln 2\right)^a} = \frac{1}{2^{na-n}\; n^a \left(\ln 2\right)^a}$$

$$= \frac{1}{(\ln 2)^a} \cdot \frac{\left(\frac{1}{2}\right)^{(a-1)n}}{n^a}$$

Since $\sum \frac{1}{n^a}$ is convergent when $a > 1$ and $\left(\frac{1}{2}\right)^{(a-1)n}$ is decreasing for $a > 1$ and it converges to 0. Therefore $\sum b_n$ is convergent

$\Rightarrow \sum a_n$ is also convergent.

Now $\sum b_n$ is divergent for $a \le 1$ therefore $\sum a_n$ diverges for $a \le 1$.

Example

To check $\sum \frac{1}{n^a \ln n}$ is convergent or divergent.

We have $a_n = \frac{1}{n^a \ln n}$

Take $b_n = 2^n a_{2^n} = \frac{2^n}{(2^n)^a (\ln 2^n)} = \frac{2^n}{2^{na} (n \ln 2)}$

$$= \frac{1}{\ln 2} \cdot \frac{2^{(1-a)n}}{n} = \frac{1}{\ln 2} \cdot \frac{\left(\frac{1}{2}\right)^{(a-1)n}}{n}$$

$\because \sum \frac{1}{n}$ is divergent although $\left\{\left(\frac{1}{2}\right)^{n(a-1)}\right\}$ is decreasing, tending to zero for $a > 1$ therefore $\sum b_n$ is divergent.

$\Rightarrow \sum a_n$ is divergent.

The series also divergent if $a \le 1$.
i.e. it is always divergent.

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Chapter 3 – Limit and Continuity

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❖ *Limit of the function*

Suppose

(i) (X, d_X) and (Y, d_Y) be two metric spaces

(ii) $E \subset X$

(iii) $f : E \to Y$ i.e. f maps E into Y.

(iv) p is the limit point of E.

We write $f(x) \to q$ as $x \to p$ or $\lim\limits_{x\to p} f(x) = q$, if there is a point q with the following property;

For every $e > 0$, there exists a $d > 0$ such that $d_Y\big(f(x), q\big) < e$ for all points $x \in E$ for which $d_X(x, p) < d$.

If X and Y are replaced by a real line, complex plane or by Euclidean space $\mathbb{R}^k$, then the distances d_X and d_Y are replaced by absolute values or by appropriate norms. ❑

Note: *i*) It is to be noted that $p \in X$ but that p need not a point of E in the above definition (p is a limit point of E which may or may not belong to E.)

ii) Even if $p \in E$, we may have $f(p) \neq \lim\limits_{x\to p} f(x)$. ❑

❖ *Example*

$$\lim_{x\to\infty} \frac{2x}{1+x} = 2$$

We have $\left|\dfrac{2x}{x-1} - 2\right| = \left|\dfrac{2x-2-2x}{1+x}\right| = \left|\dfrac{-2}{1+x}\right| < \dfrac{2}{x}$

Now if $e > 0$ is given we can find $d = \dfrac{2}{e}$ so that

$$\left|\frac{2x}{1+x} - 2\right| < e \quad \text{whenever} \quad x > d.$$ ❑

❖ *Example*

Consider the function $f(x) = \dfrac{x^2-1}{x-1}$.

It is to be noted that f is not defined at $x = 1$ but if $x \neq 1$ and is very close to 1 or less then $f(x)$ equals to 2. ❑

❖ *Definitions*

i) Let X and Y be subsets of $\mathbb{R}$, a function $f : X \to Y$ is said to tend to limit l as $x \to \infty$, if for a real number $e > 0$ however small, $\exists$ a positive number d which depends upon e such that distance

$$\left|f(x) - l\right| < e \quad \text{when } x > d \quad \text{and we write } \lim_{x\to\infty} f(x) = l.$$

ii) f is said to tend to a right limit l as $x \to c$ if for $e > 0$, $\exists$ $d > 0$ such that $\left|f(x) - l\right| < e$ whenever $x \in G$ and $0 < x < c + d$.

And we write $f(c+) = \lim\limits_{x\to c+} f(x) = l$

iii) f is said to tend to a left limit l as $x \to c$ if for $e > 0$, $\exists$ a $d > 0$ such that $\left|f(x) - l\right| < e$ whenever $x \in G$ and $0 < c - d < x < c$.

And we write $f(c-) = \lim\limits_{x\to c-} f(x) = l$. ❑

❖ *Theorem*

Suppose

(i) (X, d_x) and (Y, d_y) be two metric spaces

(ii) $E \subset X$

(iii) $f : E \to Y$ i.e. f maps E into Y.

(iv) p is the limit point of E.

Then $\lim\limits_{x\to p} f(x) = q$ iff $\lim\limits_{n\to\infty} f(p_n) = q$ for every sequence $\{p_n\}$ in E such that $p_n \neq p$, $\lim\limits_{n\to\infty} p_n = p$.

Proof

Suppose $\lim\limits_{x\to p} f(x) = q$ holds.

Choose $\{p_n\}$ in E such that $p_n \neq p$, $\lim\limits_{n\to\infty} p_n = p$, we are to show that $\lim\limits_{n\to\infty} f(p_n) = q$

Then there exists a $d > 0$ such that

$$d_y\left(f(x), q\right) < e \quad \text{if } x \in E \text{ and } 0 < d_x(x, p) < d \quad \ldots\ldots\ldots. (i)$$

Also $\exists$ a positive integer n_0 such that $n > n_0$

$$\Rightarrow d_x\left(p_n, p\right) < d \quad \ldots\ldots\ldots\ldots.. (ii)$$

from (*i*) and (*ii*), we have for $n > n_0$

$$d_y\left(f(p_n), q\right) < e$$

Which shows that limit of the sequence

$$\lim_{n\to\infty} f\left(p_n\right) = q$$

Conversely, suppose that $\lim\limits_{n\to\infty} f\left(p_n\right) = q$ is false.

Then $\exists$ some $e > 0$ such that for every $d > 0$, there is a point $x \in E$ for which $d_y\left(f(x), q\right) \geq e$ but $0 < d_x(x, p) < d$.

In particular, taking $d_n = \dfrac{1}{n}$, $n = 1, 2, 3, \ldots\ldots$

We find a sequence in E satisfied $p_n \neq p$, $\lim\limits_{n\to\infty} p_n = p$ for which $\lim\limits_{n\to\infty} f(p_n) = q$ is false. ❑

❖ *Example*

$$\lim_{x\to\infty} \sin\frac{1}{x} \quad \text{does not exist.}$$

Suppose that $\lim\limits_{x\to\infty} \sin\dfrac{1}{x}$ exists and take it to be l, then there exist a positive real number d such that

$$\left|\sin\frac{1}{x} - l\right| < 1 \quad \text{when} \quad 0 < \left|x - 0\right| < d \quad (\text{we take } e = 1 > 0 \text{ here})$$

We can find a positive integer n such that

$$\frac{2}{np} < d \quad \text{then} \quad \frac{2}{(4n+1)p} < d \quad \text{and} \quad \frac{2}{(4n+3)p} < d$$

It thus follows

$$\left|\sin\frac{(4n+1)p}{2} - l\right| < 1 \quad \Rightarrow \left|1 - l\right| < 1$$

$$\text{and} \quad \left|\sin\frac{(4n+3)p}{2} - l\right| < 1 \quad \Rightarrow \left|-1 - l\right| < 1 \quad \text{or} \quad \left|1 + l\right| < 1$$

So that

$$2 = \left|1+l+1-l\right| \le \left|1+l\right| + \left|1-l\right| < 1+1 \quad \Rightarrow \; 2 < 2$$

This is impossible; hence limit of the function does not exist. ❑

Alternative:

Consider $x_n = \dfrac{2}{(2n-1)p}$ then $\lim\limits_{x\to\infty} x_n = 0$

But $\{f(x_n)\}$ i.e. $\left\{\sin\dfrac{1}{x_n}\right\}$ is an oscillatory sequence

i.e. $\{1,-1,1,-1,..........\}$ therefore $\left\{\sin\dfrac{1}{x_n}\right\}$ diverges.

Hence we conclude that $\lim\limits_{x\to\infty} \sin\dfrac{1}{x}$ does not exit. ❑

❖ ***Example***

Consider the function

$$f(x) = \begin{cases} x & ; \quad x < 1 \\ 2+(x-1)^2 & ; \quad x \ge 1 \end{cases}$$

We show that $\lim\limits_{x\to 1} f(x)$ does not exist.

To prove this take $x_n = 1 - \dfrac{1}{n}$, then $\lim\limits_{x\to\infty} x_n = 1$ and $\lim\limits_{n\to\infty} f(x_n) = 1$

But if we take $x_n = 1 + \dfrac{1}{n}$ then $x_n \to 1$ as $n \to \infty$

and $\lim\limits_{x\to\infty} f(x_n) = \lim\limits_{x\to\infty} 2 + \left(1+\dfrac{1}{n}-1\right)^2 = 2$

This show that $\{f(x_n)\}$ does not tend to a same limit as for all sequences $\{S_n\}$ such that $x_n \to 1$.

Hence this limit does not exist. ❑

❖ ***Example***

Consider the function $f:[0,1] \to \mathbb{R}$ defined as

$$f(x) = \begin{cases} 0 & \textit{if } x \textit{ is rational} \\ 1 & \textit{if } x \textit{ is irratioanl} \end{cases}$$

Show that $\lim\limits_{x\to p} f(x)$ where $p \in [0,1]$ does not exist.

Solution

Let $\lim\limits_{x\to p} f(x) = q$, if given $e > 0$ we can find $d > 0$ such that

$$\left|f(x) - q\right| < e \quad \text{whenever} \quad \left|x - p\right| < d \;.$$

Consider the irrational $(r-s, r+s) \subset [0,1]$ such that r is rational and s is irrational.

Then $f(r) = 0$ & $f(s) = 1$

Suppose $\lim\limits_{x\to p} f(x) = q$ then

$$\begin{aligned} \left|f(s)\right| &= 1 \\ \Rightarrow 1 &= \left|f(s) - q + q\right| \\ &= \left|(f(s) - q + q - 0\right| \\ &= \left|f(s) - q + q - f(r)\right| \qquad \because\; 0 = f(r) \end{aligned}$$

$$\leq |f(s)-q| + |f(r)-q| < e + e$$

i.e. $1 < e + e$

$$\Rightarrow 1 < \frac{1}{4} + \frac{1}{4} \qquad \text{if } e = \frac{1}{4}$$

Which is absurd.

Hence the limit of the function does not exist. ❑

❖ ***Exercise***

$$\lim_{x\to 0} x\sin\frac{1}{x} = 0$$

We have

$$\left|x\sin\frac{1}{x} - 0\right| < e \quad \text{where } e > 0 \text{ is a pre-assigned positive number.}$$

$$\Rightarrow \left|x\sin\frac{1}{x}\right| < e$$

$$\Rightarrow |x|\left|\sin\frac{1}{x}\right| < e$$

$$\Rightarrow |x| < e \qquad \because \left|\sin\frac{1}{x}\right| \leq 1$$

$$\Rightarrow |x-0| < e = d$$

It shows that $\lim\limits_{x\to 0} x\sin\frac{1}{x} = 0$.

Same the case for function for $f(x) = x\cos\frac{1}{x}$

Also we can derived the result that $\lim\limits_{x\to 0} x^2\sin\frac{1}{x} = 0$. ❑

❖ ***Theorem***

If $\lim\limits_{x\to c} f(x)$ exists then it is unique.

Proof

Suppose $\lim\limits_{x\to c} f(x)$ is not unique.

Take $\lim\limits_{x\to c} f(x) = l_1$ and $\lim\limits_{x\to c} f(x) = l_2$ where $l_1 \neq l_2$.

$\Rightarrow \exists$ real numbers d_1 and d_2 such that

$$|f(x)-l_1| < e \quad \text{whenever} \quad |x-c| < d_1$$

$$\& \quad |f(x)-l_2| < e \quad \text{whenever} \quad |x-c| < d_2$$

Now $|l_1 - l_2| = |(f(x)-l_1) - (f(x)-l_2)|$

$$\leq |f(x)-l_1| + |f(x)-l_2|$$

$$< e + e \quad \text{whenever} \quad |x-c| < \min(d_1, d_2)$$

$$\Rightarrow l_1 = l_2$$ ❑

…………………

❖ Theorem

Suppose that a real valued function f is defined on an open interval G except possibly at $c \in G$. Then $\lim\limits_{x \to c} f(x) = l$ if and only if for every positive real number e, there is $d > 0$ such that $\left| f(t) - f(s) \right| < e$ whenever s & t are in $\left\{ x : \left| x - c \right| < d \right\}$.

Proof

Suppose $\lim\limits_{x \to c} f(x) = l$

$\therefore$ for every $e > 0$, $\exists$ $d > 0$ such that

$$\left| f(s) - l \right| < \frac{1}{2} e \quad \text{whenever} \quad 0 < \left| s - c \right| < d$$

$$\& \quad \left| f(t) - l \right| < \frac{1}{2} e \quad \text{whenever} \quad 0 < \left| t - c \right| < d$$

$$\Rightarrow \left| f(s) - f(t) \right| \le \left| f(s) - l \right| + \left| f(t) - l \right|$$

$$< \frac{e}{2} + \frac{e}{2} \qquad \text{whenever} \quad \left| s - c \right| < d \quad \& \quad \left| t - c \right| < d$$

$$\left| f(t) - f(s) \right| < e \quad \text{whenever} \quad s \ \& \ t \ \text{are in} \ \left\{ x : \left| x - c \right| < d \right\}.$$

Conversely, suppose that the given condition holds.

Let $\{ x_n \}$ be a sequence of distinct elements of G such that $x_n \to c$ as $n \to \infty$.

Then for $d > 0$ $\exists$ a natural number n_0 such that

$$\left| x_n - l \right| < d \quad \text{and} \quad \left| x_m - l \right| < d \qquad \forall \ m, n > n_0 .$$

And for $e > 0$

$$\left| f(x_n) - f(x_m) \right| < e \qquad \text{whenever} \quad m, n > n_0$$

$\Rightarrow \{ f(x_n) \}$ is a Cauchy sequence and therefore it is convergent. ❑

❖ Theorem (Sandwiching Theorem)

Suppose that f, g and h are functions defined on an open interval G except possibly at $c \in G$. Let $f \le h \le g$ on G.

If $\lim\limits_{x \to c} f(x) = \lim\limits_{x \to c} g(x) = l$, then $\lim\limits_{x \to c} h(x) = l$.

Proof

For $e > 0$ $\exists$ $d_1, d_2 > 0$ such that

$$\left| f(x) - l \right| < e \quad \text{whenever} \quad 0 < \left| x - c \right| < d_1$$

$$\& \ \left| g(x) - l \right| < e \quad \text{whenever} \quad 0 < \left| x - c \right| < d_2$$

$$\Rightarrow \ l - e < f(x) < l + e \quad \text{for} \quad 0 < \left| x - c \right| < d_1$$

$$\& \quad l - e < g(x) < l + e \quad \text{for} \quad 0 < \left| x - c \right| < d_2$$

$$\Rightarrow \ l - e < f(x) \le h(x) \le g(x) < l + e$$

$$\Rightarrow \ l - e < h(x) < l + e \quad \text{for} \quad 0 < \left| x - c \right| < \min(d_1, d_2)$$

$$\Rightarrow \ \lim_{x \to c} h(x) = l$$ ❑

…………………………

❖ *Theorem*

Let (i) (X,d) , (Y,d_y) be two metric spaces.

(ii) $E \subset X$

(iii) p is a limit point of E.

(iv) $f : E \to Y$.

(v) $g : E \to Y$

and $\lim_{x\to p} f(x) = A$ and $\lim_{x\to p} g(x) = B$ then

i- $\lim_{x\to p}\left(f(x) \pm g(x)\right) = A \pm B$

ii- $\lim_{x\to p}(fg)(x) = AB$

iii- $\lim_{x\to p}\left(\dfrac{f(x)}{g(x)}\right) = \dfrac{A}{B}$ provided $B \neq 0$.

Proof

Do yourself ❑

❖ *Continuity*

Suppose

i) $\left(X, d_X\right)$, $\left(Y, d_Y\right)$ are two metric spaces

ii) $E \subset X$

iii) $p \in E$

iv) $f : E \to Y$

Then f is said to be continuous at p if for every $e > 0$ $\exists$ a $d > 0$ such that $d_Y\left(f(x), f(p)\right) < e$ for all points $x \in E$ for which $d_X\left(x, p\right) < d$.

Note:

(i) If f is continuous at every point of E. Then f is said to be continuous on E.

(ii) It is to be noted that f has to be defined at p iff $\lim_{x\to p} f(x) = f(p)$. ❑

❖ *Examples*

$f(x) = x^2$ is continuous $\forall$ $x \in \mathbb{R}$.

Here $f(x) = x^2$, Take $p \in \mathbb{R}$

Then $\left| f(x) - f(p)\right| < e$

$\Rightarrow \left| x^2 - p^2 \right| < e$

$\Rightarrow \left|(x-p)(x+p)\right| < e$

$\Rightarrow \left| x - p\right| < e = d$

$\because$ p is arbitrary real number

$\therefore$ the function $f(x)$ is continuous $\forall$ real numbers. ❑

…………………………..

❖ *Theorem*

Let

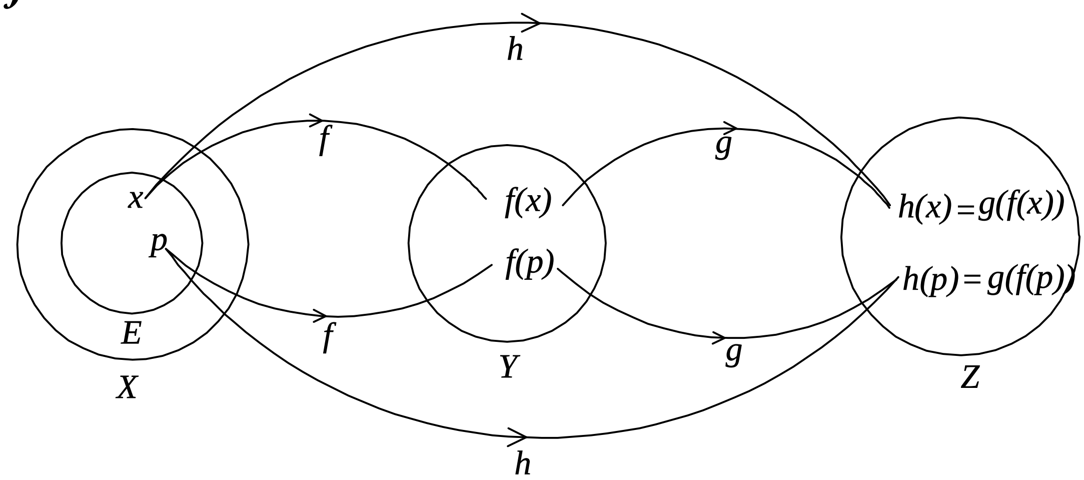
i) X, Y, Z be metric spaces

ii) $E \subset X$

iii) $f : E \to Y$, $g : f(E) \to Z$ and $h : E \to Z$ defined by $h(x) = g\left(f(x)\right)$

If f is continuous at $p \in E$ and if g is continuous at the point $f(p)$, then h is continuous at *p.*

Proof



$\because$ g is continuous at $f(p)$

$\therefore$ for every $e > 0$, $\exists$ a $d > 0$ such that

$$d_Z\left(g(y), g\left(f(p)\right)\right) < e \text{ whenever } d_Y\left(y, f(p)\right) < d_1 \text{ ……….. } (i)$$

$\because$ f is continuous at $p \in E$

$\therefore$ $\exists$ a $d > 0$ such that

$$d_Y\left(f(x), f(p)\right) < d_1 \text{ whenever } d_X\left(x, p\right) < d \text{ ………… } (ii)$$

Combining (i) and (ii), we have

$$d_Z\left(g(y), g\left(f(p)\right)\right) < e \quad \text{whenever} \quad d_X\left(x, p\right) < d$$

$$\Rightarrow d_Z\left(h(x), h(p)\right) < e \quad \text{whenever} \quad d_X(x, p) < d$$

which shows that the function h is continuous at p. ❑

❖ *Example*

(i) $f(x) = (1 - x^2)$ is continuous $\forall$ $x \in \mathbb{R}$ and $g(x) = \sqrt{x}$ is continuous $\forall$ $x \in [0, \infty]$, then $g\left(f(x)\right) = \sqrt{1 - x^2}$ is continuous $x \in \left(-1, 1\right)$.

(ii) Let $g(x) = \sin x$ and $f(x) = \begin{cases} x - p & , \ x \le 0 \\ x + p & , \ x > 0 \end{cases}$

Then $g\left(f(x)\right) = -\sin x$ $\forall$ x

Then the function $g\left(f(x)\right)$ is continuous at $x = 0$, although f is discontinuous at $x = 0$. ❑

❖ *Theorem*

Let f be defined on X. If f is continuous at $c \in X$ then $\exists$ a number $d > 0$ such that f is bounded on the open interval $(c - d, c + d)$.

Proof

Since f is continuous at $c \in X$.

Therefore for a real number $e > 0$, $\exists$ a real number $d > 0$ such that

$$\left|f(x) - f(c)\right| < e \quad \text{whenever} \ x \in X \ \text{and} \ \left|x - c\right| < d .$$

$$\Rightarrow \left|f(x)\right| = \left|f(x) - f(c) + f(c)\right|$$

$\leq \left| f(x) - f(c) \right| - \left| f(c) \right|$

$< e + \left| f(c) \right|$ whenever $\left| x - c \right| < d$.

It shows that f is bounded on the open interval $\left]c - d, c + d\right[$. ❑

❖ Theorem

Suppose f is continuous on $[a,b]$. If $f(c) > 0$ for some $c \in [a,b]$ then there exist an open interval $G \subset [a,b]$ such that $f(x) > 0 \;\; \forall \;\; x \in G$.

Proof

Take $e = \dfrac{1}{2} f(c)$

$\because$ f is continuous on $[a,b]$

$\therefore$ $\left| f(x) - f(c) \right| < e$ whenever $\left| x - c \right| < d$, $x \in [a,b]$

Take $G = \left\{ x \in [a,b] : \left| x - c \right| < d \right\}$

$\Rightarrow \left| f(x) \right| = \left| f(x) - f(c) + f(c) \right|$

$\leq \left| f(x) - f(c) \right| + \left| f(c) \right|$

$< e + \left| f(c) \right|$ whenever $\left| x - c \right| < d$

For $x \in G$, we have

$$f(x) = f(c) - \left(f(c) - f(x) \right) \geq f(c) - \left| f(c) - f(x) \right|$$

$$\geq f(c) - \left| f(x) - f(c) \right| > f(c) - \frac{1}{2} f(c)$$

$\Rightarrow f(x) > \dfrac{1}{2} f(c) > 0$ ❑

❖ Example

Define a function f by

$$f(x) = \begin{cases} x\cos x & ; \;\; x \neq o \\ 0 & ; \;\; x = 0 \end{cases}$$

This function is continuous at $x = 0$ because

$$\left| f(x) - f(0) \right| = \left| x\cos x \right| \leq \left| x \right| \qquad \left(\because \left| \cos x \right| \leq 1 \right)$$

Which shows that for $e > 0$, we can find $d > 0$ such that

$$\left| f(x) - f(0) \right| < e \quad \text{whenever} \quad 0 < \left| x - c \right| < d = e$$ ❑

❖ Example

$f(x) = \sqrt{x}$ is continuous on $\left[0, \infty\right[$.

Let c be an arbitrary point such that $0 < c < \infty$

For $e > 0$, we have

$$\left| f(x) - f(c) \right| = \left| \sqrt{x} - \sqrt{c} \right| = \frac{\left| x - c \right|}{\sqrt{x} + \sqrt{c}} < \frac{\left| x - c \right|}{\sqrt{c}}$$

$$\Rightarrow \left| f(x) - f(c) \right| < e \quad \text{whenever} \quad \frac{\left| x - c \right|}{\sqrt{c}} < e$$

i.e. $\left| x - c \right| < \sqrt{c}\, e = d$

$\Rightarrow$ f is continuous for $x = c$.

$\because$ c is an arbitrary point lying in $\left[0, \infty\right[$

$\therefore$ $f(x) = \sqrt{x}$ is continuous on $\left[0, \infty\right[$ ❑

…………………………..

❖ *Example*

Consider the function f defined on $\mathbb{R}$ such that

$$f(x)=\begin{cases}1 & , \ x \text{ is rational}\\ -1 & , \ x \text{ is irrational}\end{cases}$$

This function is discontinuous every where but $|f(x)|$ is continuous on $\mathbb{R}$. ❑

❖ *Theorem*

A mapping of a metric space X into a metric space Y is continuous on X iff $f^{-1}(V)$ is open in X for every open set V in Y.

Proof

Suppose f is continuous on X and V is open in Y.

We are to show that $f^{-1}(V)$ is open in X i.e. every point of $f^{-1}(V)$ is an interior point of $f^{-1}(V)$.

Let $p \in X$ and $f(p) \in V$

$\because$ V is open

$\therefore$ $\exists$ $e > 0$ such that $y \in V$ if $d_Y\left(y, f(p)\right) < e$ …….. (i)

$\because$ f is continuous at p

$\therefore$ $\exists$ $d > 0$ such that $d_Y\left(f(x), f(p)\right) < e$ when $d_X(x,p) < d$ ……… (ii)

From (i) and (ii), we conclude that

$$x \in f^{-1}(V) \quad \text{as soon as} \quad d_X\left(x,p\right) < d$$

Which shows that $f^{-1}(V)$ is open in X.

Conversely, suppose $f^{-1}(V)$ is open in X for every open set V in Y.

We are to prove that f is continuous for this.

Fix $p \in X$ and $e > 0$.

Let V be the set of all $y \in Y$ such that $d_Y\left(y, f(p)\right) < e$

V is open, $f^{-1}(V)$ is open

$\Rightarrow$ $\exists$ $d > 0$ such that $x \in f^{-1}(V)$ as soon as $d_X(x,p) < d$.

But if $x \in f^{-1}(V)$ then $f(x) \in V$ so that $d_Y\left(f(x), f(y)\right) < e$

Which proves that f is continuous. ❑

Note

The above theorem can also be stated as a mapping $f : X \to Y$ is continuous iff $f^{-1}(C)$ is closed in X for every closed set C in Y. ❑

❖ *Theorem*

Let $f_1, f_2, f_3, \ldots.., f_k$ be real valued functions on a metric space X and $\underline{f}$ be a mapping from X on to $\mathbb{R}^k$ defined by

$$\underline{f}(x) = \left(f_1(x), f_2(x), f_3(x), \ldots.., f_k(x)\right) \quad , \quad x \in X$$

then $\underline{f}$ is continuous on X if and only if $f_1, f_2, f_3, \ldots.., f_k$ are continuous on X.

Proof

Let us suppose that the function $\underline{f}$ is continuous on X, we are to show that $f_1, f_2, f_3, \ldots\ldots, f_k$ are continuous on X.

If $p \in X$, then $d_{\mathbb{R}^k}\left(\underline{f}(x), \underline{f}(p)\right) < e$ whenever $d_X(x,p) < d$

$\Rightarrow$ $\left\| \underline{f}(x) - \underline{f}(p) \right\| < e$ whenever $\left\| x - p \right\| < d$

$\Rightarrow \| f_1(x) - f_1(p), f_1(x) - f_1(p), \ldots\ldots f_k(x) - f_k(p) \| < e$ whenever $\| x - p \| < d$

$\Rightarrow \left[(f_1(x) - f_1(p))^2, (f_2(x) - f_2(p))^2, \ldots\ldots, (f_k(x) - f_k(p))^2 \right]^{1/2} < e$

whenever $\| x - p \| < d$

i.e. $\Rightarrow \left[\sum_{i=1}^{k} (f_i(x) - f_i(p))^2 \right]^{1/2} < e$ whenever $\| x - p \| < d$

$\Rightarrow \| f_1(x) - f_1(p) \| < e$ whenever $\| x - p \| < d$

$\| f_2(x) - f_2(p) \| < e$ whenever $\| x - p \| < d$

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$\| f_k(x) - f_k(x) \| < e$ whenever $\| x - p \| < d$

$\Rightarrow$ all the functions $f_1, f_2, f_3, \ldots\ldots, f_k$ are continuous at p.

$\because$ p is arbitrary point of x, therefore $f_1, f_2, f_3, \ldots\ldots, f_k$ are continuous on X.

Conversely, suppose that the function $f_1, f_2, f_3, \ldots\ldots, f_k$ are continuous on X, we are to show that $\underline{f}$ is continuous on X.

For $p \in X$ and given $e_i > 0$, $i = 1, 2, \ldots\ldots k$ $\exists$ $d_i > 0$, $i = 1, 2, \ldots\ldots, k$

Such that

$\| f_1(x) - f_1(p) \| < e_1$ whenever $\| x - p \| < d_1$

$\| f_2(x) - f_2(p) \| < e_2$ whenever $\| x - p \| < d_2$

....

....

....

$\| f_k(x) - f_k(x) \| < e_k$ whenever $\| x - p \| < d_k$

Take $d = \min(d_1, d_2, d_3, \ldots\ldots, d_k)$ then

$\| f_i(x) - f_i(p) \| < e_i$ whenever $\| x - p \| < d$

$\Rightarrow \left[(f_1(x) - f_1(p))^2 + (f_2(x) - f_2(p))^2 + \ldots. + (f_k(x) - f_k(p))^2 \right]^{1/2} < \left(e_1^2 + e_2^2 + \ldots. + e_k^2 \right)^{1/2}$

i.e. $\Rightarrow \left[(f_1(x) - f_1(p))^2 + (f_2(x) - f_2(p))^2 + \ldots\ldots + (f_k(x) - f_k(p))^2 \right]^{1/2} < e$

whenever $\| x - p \| < d$

where $\left(e_1^2 + e_2^2 + \ldots\ldots + e_k^2 \right)^{1/2} = e$

Then $d_{\mathbb{R}^k}\left(\underline{f}(x), \underline{f}(p) \right) < e$ whenever $d_X(x, p) < d$

$\Rightarrow$ $\underline{f}(x)$ is continuous at p.

$\because$ p is an arbitrary point therefore we conclude that $\underline{f}$ is continuous on X. ❑

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❖ Theorem

Suppose f is continuous on $[a,b]$

i) If $f(a)<0$ and $f(b)>0$ then there is a point c, $a<c<b$ such that $f(c)=0$.

ii) If $f(a)>0$ and $f(b)<0$, then there is a point c, $a<c<b$ such that $f(c)=0$.

Proof

i) Bisect $[a,b]$ then f must satisfy the given condition on at least one of the sub-interval so obtained. Denote this interval by $[a_2,b_2]$

If f satisfies the condition on both sub-interval then choose the right hand one $[a_2,b_2]$.

It is obvious that $a\le a_2\le b_2\le b$. By repeated bisection we can find nested intervals $\{I_n\}$, $I_{n+1}\subseteq I_n$, $I_n=[a_n,b_n]$ so that f satisfies the given condition on $[a_n,b_n]$, $n=1,2,.......$

And $a=a_1\le a_2\le a_3\le .....\le a_n\le b_n\le .....\le b_2\le b_1=b$

Where $b_n-a_n=\left(\frac{1}{2}\right)^n(b-a)$

Then $\bigcap\limits_{i=1}^{n} I_n$ contain one and only one point. Let that point be c such that $f(c)=0$

If $f(c)\neq 0$, let $f(c)>0$ then there is a subinterval $[a_m,b_m]$ such that $a_m<b_m<c$ Which can not happen. Hence $f(c)=0$

ii) *Do yourself as above* ❑

❖ Example

Show that $x^3-2x^2-3x+1=0$ has a solution $c\in[-1,1]$

Solution

Let $f(x)=x^3-2x^2-3x+1$

$\because$ $f(x)$ is polynomial

$\therefore$ it is continuous everywhere. (for being a polynomial continuous everywhere)

Now $f(-1)=(-1)^3-2(-1)^2-3(-1)+1$

$=-1-2+3+1=1>0$

$f(1)=(1)^3-2(1)^2-3(1)+1$

$=1-2-3+1=-3<0$

Therefore there is a point $c\in[-1,1]$ such that $f(c)=0$

i.e. c is the root of the equation. ❑

❖ Theorem (The intermediate value theorem)

Suppose f is continuous on $[a,b]$ and $f(a)\neq f(b)$, then given a number l that lies between $f(a)$ and $f(b)$, $\exists$ a point c, $a<c<b$ with $f(c)=l$.

Proof

Let $f(a)<f(b)$ and $f(a)<l<f(b)$.

Suppose $g(x)=f(x)-l$

Then $g(a)=f(a)-l<0$ and $g(b)=f(b)-l>0$

$\Rightarrow$ $\exists$ a point c between a and b such that $g(c)=0$

$\Rightarrow$ $f(c)-l=0$ $\Rightarrow$ $f(c)=l$

If $f(a)>f(b)$ then take $g(x)=l-f(x)$ to obtain the required result. ❑

❖ *Theorem*

Suppose f is continuous on $[a,b]$, then f is bounded on $[a,b]$
(Continuity implies boundedness)

Proof

Suppose that f is not bounded on $[a,b]$,
We can, therefore, find a sequence $\{x_n\}$ in the interval $[a,b]$ such that

$f(x_n) > n$ for all $n \geq 1$.

$\Rightarrow \{f(x_n)\}$ diverges.

But $a \leq x_n \leq b$; $n \geq 1$

$\Rightarrow \exists$ a subsequence $\{x_{n_k}\}$ such that $\{x_{n_k}\}$ converges to l.

$\Rightarrow \{f(x_{n_k})\}$ also converges to l.

$\Rightarrow \{f(x_n)\}$ converges to l.

Which is contradiction
Hence our supposition is wrong. ❑

❖ *Uniform continuity*

Let f be a mapping of a metric space X into a metric space Y. We say that f is uniformly continuous on X if for every $e > 0$ there exists $d > 0$ such that

$$d_Y(f(p), f(q)) < e \quad \forall \quad p,q \in X \quad \text{for which} \quad d_x(p,q) < d$$

The uniform continuity is a property of a function on a set i.e. it is a global property but continuity can be defined at a single point i.e. it is a local property. Uniform continuity of a function at a point has no meaning.

If f is continuous on X then it is possible to find for each $e > 0$ and for each point p of X, a number $d > 0$ such that $d_Y(f(x), f(p)) < e$ whenever $d_X(x,p) < d$. Then number d depends upon e and on p in this case but if f is uniformly continuous on X then it is possible for each $e > 0$ to find one number $d > 0$ which will do for all point p of X.

It is evident that every uniformly continuous function is continuous.

To emphasize a difference between continuity and uniform continuity on set S, we consider the following examples. ❑

❖ *Example*

Let S be a half open interval $0 < x \leq 1$ and let f be defined for each x in S by the formula $f(x) = x^2$. It is uniformly continuous on S. To prove this observe that we have

$$\begin{aligned}|f(x) - f(y)| &= |x^2 - y^2| \\ &= |x-y||x+y| \\ &< 2|x-y|\end{aligned}$$

If $|x-y| < d$ then $|f(x) - f(y)| < 2d = e$

Hence if e is given we need only to take $d = \frac{e}{2}$ to guarantee that

$$|f(x) - f(y)| < e \text{ for every pair } x, y \text{ with } |x-y| < d$$

Thus f is uniformly continuous on the set S. ❑

❖ ***Example***

$f(x)=x^n$, $n \geq 0$ is uniformly continuous of $[0,1]$

Solution

For any two values x_1, x_2 in $[0,1]$ we have

$$\left|x_1^n - x_2^n\right| = \left|(x_1 - x_2)\left(x_1^{n-1} + x_1^{n-2}x_2 + x_1^{n-3}x_2^2 + \dots + x_2^{n-1}\right)\right|$$
$$\leq n\left|x_1 - x_2\right|$$

Given $e>0$, we can find $d=\dfrac{e}{n}$ independent of x_1 and x_2 such that

$$\left|x_1^2 - x_2^2\right| < n\left|x_1 - x_2\right| < e \quad \text{whenever} \quad x_1, x_2 \in [0,1] \text{ and } \left|x_1 - x_2\right| < d = \frac{e}{n}$$

Hence the function f is uniformly continuous on $[0,1]$. ❑

❖ ***Example***

Let S be the half open interval $0<x\leq 1$ and let a function f be defined for each x in S by the formula $f(x)=\dfrac{1}{x}$. This function is continuous on the set S, however we shall prove that this function is not uniformly continuous on S.

Solution

Let suppose $e=10$ and suppose we can find a d , $0<d<1$, to satisfy the condition of the definition.

Taking $x=d$, $y=\dfrac{d}{11}$, we obtain

$$\left|x-y\right| = \frac{10d}{11} < d$$

and

$$\left|f(x)-f(y)\right| = \left|\frac{1}{d}-\frac{11}{d}\right| = \frac{10}{d} > 10$$

Hence for these two points we have $\left|f(x)-f(y)\right| > 10$ (always)

Which contradict the definition of uniform continuity.

Hence the given function being continuous on a set S is not uniformly continuous on S. ❑

❖ ***Example***

$f(x)=\sin\dfrac{1}{x}$; $x \neq 0$. is not uniformly continuous on $0<x\leq 1$ i.e $(0,1]$.

Proof

Suppose that f is uniformly continuous on the given interval then for $e=1$, there is $d>0$ such that

$$\left|f(x_1)-f(x_2)\right| < 1 \quad \text{whenever} \quad \left|x_1-x_2\right| < d$$

Take $x_1=\dfrac{1}{(n-\frac{1}{2})p}$ and $x_2=\dfrac{1}{3(n-\frac{1}{2})p}$, $n\geq 1$.

So that $\left|x_1-x_2\right| < d = \dfrac{2}{3(n-\frac{1}{2})p}$

But $\left|f(x_1)-f(x_2)\right| = \left|\sin(n-\frac{1}{2})p - \sin 3(n-\frac{1}{2})p\right| = 2 > 1$

Which contradict the assumption.

Hence f is not uniformly continuous on the interval. ❑

❖ Example

Prove that $f(x)=\sqrt{x}$ is uniformly continuous on $[0,1]$.

Solution

Suppose $e=1$ and suppose we can find d, $0<d<1$ to satisfy the condition of the definition.

Taking $x=d^2$, $y=\frac{d^2}{4}$

Then $|x-y| = d^2-\frac{d^2}{4} = \frac{3d^2}{4} < d$

And $|f(x)-f(y)| = \left|\sqrt{d^2}-\sqrt{\frac{d^2}{4}}\right|$

$$= \left|d-\frac{d}{2}\right| = \left|\frac{d}{2}\right| < 1=e$$

Hence f is uniformly continuous on $[0,1]$. ❑

❖ Theorem

If f is continuous on a closed and bounded interval $[a,b]$, then f is uniformly continuous on $[a,b]$.

Proof

Suppose that f is not uniformly continuous on $[a,b]$ then $\exists$ a real number $e>0$ such that for every real number $d>0$.

We can find a pair u, v satisfying

$$|u-v|<d \quad \text{but} \quad |f(u)-f(v)| \geq e > 0$$

If $d=\frac{1}{n}$, $n=1,2,3,....$

We can determine two sequence $\{u_n\}$ and $\{v_n\}$ such that

$$|u_n-v_n| < \frac{1}{n} \quad \text{but} \quad |f(u_n)-f(v_n)| \geq e$$

$\because$ $a \leq u_n \leq b$ $\forall$ $n=1,2,3.......$

$\therefore$ there is a subsequence $\{u_{n_k}\}$ which converges to some number u_0 in $[a,b]$

$\Rightarrow$ for some $l>0$, we can find an integer n_0 such that

$$|u_{n_k}-u_0|< l \quad \forall \quad n \geq n_0$$

$$\Rightarrow |v_{n_k}-u_0| \leq |v_{n_k}-u_{n_k}|+|u_{n_k}-u_0| < \frac{1}{n}+l$$

$\Rightarrow$ $\{v_{n_k}\}$ also converges to u_0.

$\Rightarrow$ $\{f(u_{n_k})\}$ and $\{f(v_{n_k})\}$ converge to $f(u_0)$.

Consequently, $\left|f(u_{n_k})-f(v_{n_k})\right| < e$ whenever $|u_{n_k}-v_{n_k}| < e$

Which contradict our supposition.

Hence we conclude that f is uniformly continuous on $[a,b]$. ❑

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❖ Theorem

Let $\underline{f}$ and $\underline{g}$ be two continuous mappings from a metric space X into $\mathbb{R}^k$, then the mappings $\underline{f}+\underline{g}$ and $\underline{f}\cdot\underline{g}$ are also continuous on X.

i.e. the sum and product of two continuous vector valued function are also continuous.

Proof

$i)$ $\because$ $\underline{f}$ & $\underline{g}$ are continuous on X.

$\therefore$ by the definition of continuity, we have for a point $p\in X$.

$$\left\|\underline{f}(x)-\underline{f}(p)\right\| < \frac{e}{2} \quad \text{whenever} \quad \left\|x-p\right\| < d_1$$

and $$\left\|\underline{g}(x)-\underline{g}(p)\right\| < \frac{e}{2} \quad \text{whenever} \quad \left\|x-p\right\| < d_2$$

Now consider

$$\begin{aligned}
&\left\|\underline{f}(x)+\underline{g}(x)-\underline{f}(x)-\underline{g}(p)\right\| \\
&=\left\|\underline{f}(x)-\underline{f}(p)+\underline{g}(x)-\underline{g}(p)\right\| \\
&\le \left\|\underline{f}(x)-\underline{f}(p)\right\|+\left\|\underline{g}(x)-\underline{g}(p)\right\| \\
&< \frac{e}{2}+\frac{e}{2}=e \quad \text{whenever} \quad \left\|x-p\right\|<d \quad \text{where } d=\min\left(d_1,d_2\right)
\end{aligned}$$

which shows that the vector valued function $\underline{f}+\underline{g}$ is continuous at $x=p$ and hence on X.

$ii)$ $$\underline{f}\cdot\underline{g} = \sum_{i=1}^{k} f_i\cdot g_i$$

$$= f_1g_1+f_2g_2+f_3g_3+.....+f_kg_k$$

$\because$ the function $\underline{f}$ and $\underline{g}$ are continuous on X

$\therefore$ their components f_i and g_i are continuous on X. ❑

❖ Question

Suppose f is a real valued function define on $\mathbb{R}$ which satisfies

$$\lim_{h\to 0}\left[f(x+h)-f(x-h)\right] = 0 \quad \forall \ x\in\mathbb{R}$$

Does this imply that the function f is continuous on $\mathbb{R}$.

Solution

$$\because \lim_{h\to 0}\left[f(x+h)-f(x-h)\right]=0 \quad \forall \ x\in\mathbb{R}$$

$$\Rightarrow \lim_{h\to 0} f(x+h) = \lim_{h\to 0} f(x-h)$$

$$\Rightarrow f(x+0) = f(x-0) \ \forall \ x\in\mathbb{R}$$

Also it is given that $f(x) = f(x+0) = f(x-0)$

It means f is continuous on $x\in\mathbb{R}$. ❑

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❖ Discontinuities

If x is a point in the domain of definition of the function f at which f is not continuous, we say that f is discontinuous at x or that f has a discontinuity at x.

If the function f is defined on an interval, the discontinuity is divided into two types

1. Let f be defined on (a,b). If f is discontinuous at a point x and if $f(x+)$ and $f(x-)$ exist then f is said to have a discontinuity of first kind or a simple discontinuity at x.
2. Otherwise the discontinuity is said to be second kind.

For simple discontinuity

i. either $f(x+) \neq f(x-)$ [$f(x)$ is immaterial]

ii. or $f(x+) = f(x-) \neq f(x)$ ❑

❖ Example

i) Define $f(x) = \begin{cases} 1 & , \ x \text{ is rational} \\ 0 & , \ x \text{ is irrational} \end{cases}$

The function f has discontinuity of second kind on every point x because neither $f(x+)$ nor $f(x-)$ exists. ❑

ii) Define $f(x) = \begin{cases} x & , \ x \text{ is rational} \\ 0 & , \ x \text{ is irrational} \end{cases}$

Then f is continuous at $x = 0$ and has a discontinuity of the second kind at every other point. ❑

iii) Define $f(x) = \begin{cases} x+2 & (-3 < x < -2) \\ -x-2 & (-2 < x < 0) \\ x+2 & (0 < x < 1) \end{cases}$

The function has simple discontinuity at $x = 0$ and it is continuous at every other point of the interval $(-3,1)$ ❑

iv) Define $f(x) = \begin{cases} \sin\dfrac{1}{x} & , \ x \neq 0 \\ 0 & , \ x = 0 \end{cases}$

$\because$ neither $f(0+)$ nor $f(0-)$ exists, therefore the function f has discontinuity of second kind.

f is continuous at every point except $x = 0$. ❑

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Chapter 4 – Differentiation

Subject: *Real Analysis* **Level:** *M.Sc.*
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❖ *Derivative of a function:*

Let f be defined and real valued on $[a,b]$. For any point $c\in[a,b]$, form the quotient

$$\frac{f(x)-f(c)}{x-c}$$

and define

$$f'(c)=\lim_{x\to c}\frac{f(x)-f(c)}{x-c}$$

provided this limit exits.

We thus associate a function f' with the function f, where domain of f' is the set of points at which the above limit exists.

The function f' is so defined is called the derivative of f.

(i) If f' is defined at point x, we say that f is differentiable at x.

(ii) $f'(c)$ exists if and only if for a real number $\varepsilon>0$, $\exists$ a real number $\delta>0$ such that

$$\left|\frac{f(x)-f(c)}{x-c}-f'(c)\right|<\varepsilon \qquad \text{whenever} \qquad |x-c|<\delta$$

(iii) If $x-c=h$ then we have

$$f'(c)=\lim_{h\to 0}\frac{f(c+h)-f(c)}{h}$$

(iv) f is differentiable at c if and only if c is a removable discontinuity of the function $\varphi(x)=\dfrac{f(x)-f(c)}{x-c}$.

❖ *Example*

(i) A function $f:\mathbb{R}\to\mathbb{R}$ defined by

$$f(x)=\begin{cases} x^2\sin\frac{1}{x} & ;\ x\neq 0 \\ 0 & ;\ x=0\end{cases}$$

This function is differentiable at $x=0$ because

$$\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0}\frac{x^2\sin\frac{1}{x}-0}{x-0}$$
$$=\lim_{x\to 0}\frac{x^2\sin\frac{1}{x}}{x}=\lim_{x\to 0}x\sin\tfrac{1}{x}=0$$

(ii) Let $f(x)=x^n$; $n\geq 0$ (n is integer), $x\in\mathbb{R}$.

Then

$$\lim_{x\to c}\frac{f(x)-f(c)}{x-c}=\lim_{x\to c}\frac{x^n-c^n}{x-c}$$
$$=\lim_{x\to c}\frac{(x-c)(x^{n-1}+cx^{n-2}+\dots\dots\dots+c^{n-2}x+c^{n-1})}{x-c}$$
$$=\lim_{x\to c}(x^{n-1}+cx^{n-2}+\dots\dots\dots+c^{n-2}x+c^{n-1})$$
$$=nc^{n-1}$$

implies that f is differentiable every where and $f'(x)=nx^{n-1}$.

❖ *Theorem*

Let f be defined on $[a,b]$, if f is differentiable at a point $x \in [a,b]$, then f is continuous at x. (Differentiability implies continuity)

Proof

We know that

$$\lim_{t\to x} \frac{f(t)-f(x)}{t-x} = f'(x) \qquad \text{where } t \neq x \quad \text{and} \quad a<t<b$$

Now

$$\lim_{t\to x}\big(f(t)-f(x)\big) = \lim_{t\to x}\left(\frac{f(t)-f(x)}{t-x}\right)\lim_{t\to x}(t-x)$$
$$= f'(x)\cdot 0$$
$$= 0$$
$$\Rightarrow \lim_{t\to x} f(t) = f(x).$$

Which show that f is continuous at x.

NOTE

(i) The converse of the above theorem does not hold.

Consider $f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x<0 \end{cases}$

$f'(0)$ does not exists but $f(x)$ is continuous at $x=0$

(ii) If f is discontinuous at $c \in \mathbf{D}_f$ then $f'(c)$ does not exists.

e.g.

$$f(x) = \begin{cases} 1 & \text{if } x>0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

is discontinuous at $x=0$ therefore it is not differentiable at $x=0$.

(iii) f is differentiable at a point c if and only if $D_+ f(c)$ (right derivative) and $D_- f(c)$ (left derivative) exists and equal.

i.e. $D_+ f(c) = D_- f(c) = Df(c)$

❖ *Example*

Let $f:\mathbb{R}\to\mathbb{R}$ be defined by

$$f(x) = \begin{cases} x^2 & \text{if } x>1 \\ x^3 & \text{if } x \leq 1 \end{cases}$$

then

$$D_+ f(1) = \lim_{\substack{x\to 1+h \\ h\to 0}} \frac{f(x)-f(1)}{x-1}$$
$$= \lim_{h\to 0}\frac{f(1+h)-f(1)}{1+h-1} = \lim_{h\to 0}\frac{(1+h)^2-1}{h}$$
$$= \lim_{h\to 0}\frac{1+2h+h^2-1}{h} = \lim_{h\to 0}(2+h) = 2$$

and

$$D_- f(1) = \lim_{\substack{x\to 1-h \\ h\to 0}} \frac{f(x)-f(1)}{x-1}$$
$$= \lim_{h\to 0}\frac{f(1-h)-f(1)}{1-h-1} = \lim_{h\to 0}\frac{(1-h)^3-1}{-h}$$
$$= \lim_{h\to 0}\frac{1-3h+3h^2-h^3-1}{-h} = \lim_{h\to 0}\left(3-3h+h^2\right) = 3$$

Since $D_+ f(1) \neq D_- f(1) \Rightarrow f'(1)$ does not exist even though f is continuous at $x=1$. $f'(x)$ exist for all other values of x.

❖ *Theorem*

Suppose f and g are defined on $[a,b]$ and are differentiable at a point $x \in [a,b]$, then $f+g$, fg and $\dfrac{f}{g}$ are differentiable at x and

(*i*) $(f+g)'(x) = f'(x) + g'(x)$

(*ii*) $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$

(*iii*) $\left(\dfrac{f}{g}\right)'(x) = \dfrac{g(x)f'(x) - f(x)g'(x)}{g^2(x)}$

The proof of this theorem can be get from any F.Sc or B.Sc text book.

NOTE

The derivative of any constant is zero.

And if f is defined by $f(x) = x$ then $f'(x) = 1$

And for $f(x) = x^n$ then $f'(x) = nx^{n-1}$ where n is positive integer, if $n < 0$ we have to restrict ourselves to $x = 0$.

Thus every polynomial $P_n(x) = a_0 + a_1x + a_2x^2 + \dots\dots\dots + a_nx^n$ is differentiable every where and so every rational function except at the point where denominator is zero.

❖ *Theorem (Chain Rule)*

Suppose f is continuous on $[a,b]$, $f'(x)$ exists at some point $x \in [a,b]$. A function g is defined on an interval I which contains the range of f, and g is differentiable at the point $f(x)$.

If $h(t) = g\big(f(t)\big)$; $a \leq t \leq b$

Then h is differentiable at x and $h'(x) = g'\big(f(x)\big) \cdot f'(x)$.

Proof

Let $y = f(x)$

By the definition of the derivative we have

$$f(t) - f(x) = (t-x)\big[f'(x) + u(t)\big] \quad \dots\dots\dots\dots (i)$$

and $$g(s) - g(y) = (s-y)\big[g'(y) + v(s)\big] \quad \dots\dots\dots\dots (ii)$$

where $t \in [a,b]$, $s \in I$ and $u(t) \to 0$ as $t \to x$ and $v(s) \to 0$ as $s \to y$.

Let us suppose $s = f(t)$ then

$$\begin{aligned} h(t) - h(x) &= g\big(f(t)\big) - g\big(f(x)\big) \\ &= \big[f(t) - f(x)\big]\big[g'(y) + v(s)\big] && \text{by } (ii) \\ &= (t-x)\big[f'(x) + u(t)\big]\big[g'(y) + v(s)\big] && \text{by } (i) \end{aligned}$$

or if $t \neq x$

$$\frac{h(t) - h(x)}{t - x} = \big[f'(x) + u(t)\big]\big[g'(y) + v(s)\big]$$

taking the limit as $t \to x$ we have

$$\begin{aligned} h'(x) &= \big[f'(x) + 0\big]\big[g'(y) + 0\big] \\ &= g'\big(f(x)\big) \cdot f'(x) && \because \ y = f(x) \end{aligned}$$

which is the required result.

It is known as *chain rule*.

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❖ *Example*

Let f be defined by

$$f(x)=\begin{cases} x\sin\frac{1}{x} & ; \quad x\neq 0 \\ 0 & ; \quad x=0 \end{cases}$$

$$\Rightarrow \quad f'(x)=\sin\frac{1}{x}-\frac{1}{x}\cos\frac{1}{x} \qquad \text{where } x\neq 0.$$

$\because$ at $x=0$, $\frac{1}{x}$ is not defined.

$\therefore$ Applying the definition of the derivative we have

$$f'(0)=\lim_{t\to 0}\frac{f(t)-f(0)}{t-0}=\lim_{t\to 0}\frac{t\sin\frac{1}{t}}{t}=\lim_{t\to 0}\sin\frac{1}{t}$$

which does not exit.

The derivative of the function $f(x)$ does not exist at $x=0$ but it is continuous at $x=0$ (i.e. it is not differentiable although it is continuous at $x=0$)

Same the case with absolute value function.

❖ *Example*

Let f be defined by

$$f(x)=\begin{cases} x^2\sin\frac{1}{x} & ; \quad x\neq 0 \\ 0 & ; \quad x=0 \end{cases}$$

We have $f'(x)=2x\sin\frac{1}{x}-\cos\frac{1}{x}$ where $x\neq 0$.

$\because$ at $x=0$, $\frac{1}{x}$ is not defined.

$\therefore$ Applying the definition of the derivative we have

$$\left|\frac{f(t)-f(0)}{t-0}\right|=\left|t\sin\frac{1}{t}\right|\leq t \qquad , (t\neq 0)$$

Taking limit as $t\to 0$ we see that $f'(0)=0$

Thus f is differentiable at points x but f' is not a continuous function, since $\cos\frac{1}{x}$ does not tend to a limit as $x\to 0$.

❖ *Local Maximum*

Let f be a real valued function defined on a metric space X , we say that f has a local maximum at a point $p\in X$ if there exist $\delta>0$ such that $f(q)\leq f(p)$ $\forall$ $q\in X$ with $d(p,q)<\delta$.

Local minimum is defined likewise.

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❖ *Theorem*

Let f be defined on $[a,b]$, if f has a local maximum at a point $x\in[a,b]$ and if $f'(x)$ exist then $f'(x)=0$.

(The analogous for local minimum is of course also true)

Proof

Choose δ such that

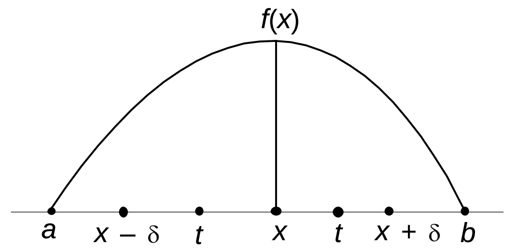
$$a<x-\delta<x<x+\delta<b$$

Now if $x-\delta<t<x$ then

$$\frac{f(t)-f(x)}{t-x}\geq 0$$

Taking limit as $t\to x$ we get

$$f'(x)\geq 0 \;\ldots\ldots\ldots\ldots (i)$$



If $x<t<x+\delta$

Then

$$\frac{f(t)-f(x)}{t-x}\leq 0$$

Again taking limit when $t\to x$ we get

$$f'(x)\leq 0 \;\ldots\ldots\ldots\ldots\ldots (ii)$$

Combining (i) and (ii) we have

$$f'(x)=0$$

❖ *Generalized Mean Value Theorem*

If f and g are continuous real valued functions on closed interval $[a,b]$, then there is a point $x\in(a,b)$ at which

$$[f(b)-f(a)]g'(x)=[g(b)-g(a)]f'(x)$$

The differentiability is not required at the end point.

Proof

Let

$$h(t)=[f(b)-f(a)]g(t)-[g(b)-g(a)]f(t) \qquad (a\leq t\leq b)$$

$\because$ h involves f and g therefore h is

i) Continuous on close interval $[a,b]$.

ii) Differentiable on open interval (a,b).

iii) and $h(a)=h(b)$.

To prove the theorem we have to show that $h'(x)=0$ for some $x\in(a,b)$

There are two cases to be discussed

(i) h is constant function.

$\Rightarrow$ $h'(x)=0 \quad \forall \quad x\in(a,b)$

(ii) If h is not constant.

then $h(t)>h(a)$ for some $t\in(a,b)$

Let x be the point in the interval (a,b) at which h attain its maximum,

then $h'(x)=0$

Similarly,

if $h(t)<h(a)$ for some $t\in(a,b)$ then $\exists$ a point $x\in(a,b)$ at which the function h attain its minimum and since the derivative at a local minimum is zero therefore we get $h'(x)=0$

Hence

$$h'(x)=[f(b)-f(a)]g'(x)-[g(b)-g(a)]f'(x)=0$$

This gives the desire result.

❖ Geometric Interpretation of M.V.T.

Consider a plane curve C represented by $x = f(t)$, $y = g(t)$ then theorem states that there is a point S on C between two points $P\big(f(a), g(a)\big)$ and $Q\big(f(b), g(b)\big)$ of C such that the tangent at S to the curve C is parallel to the chord PQ.

❖ Lagrange's M.V.T.

Let f be

i) continuous on $[a,b]$
ii) differentiable on (a,b)

then $\exists$ a point $x \in (a,b)$ such that $\dfrac{f(b)-f(a)}{b-a} = f'(x)$.

Proof

Let us design a new function

$$h(t) = \big[f(b)-f(a)\big]t - (b-a)f(t) \qquad ,(a \le t \le b)$$

then clearly $h(a) = h(b)$

Since $h(t)$ depends upon t and $f(t)$ therefore it possess all the properties of f.

Now there are two cases

i) h is a constant.
implies that $h'(x) = 0 \qquad \forall \ x \in (a,b)$

ii) h is not a constant, then
if $h(t) > h(a)$ for some $t \in (a,b)$
then $\exists$ a point $x \in (a,b)$ at which h attains its maximum
implies that $h'(x) = 0$
and if $h(t) < h(a)$
then $\exists$ a point $x \in (a,b)$ at which h attain its minimum
implies that $h'(x) = 0$
$\because \ h(t) = \big[f(b)-f(a)\big]t - (b-a)f(t)$
$\therefore \ h'(x) = \big[f(b)-f(a)\big] - (b-a)f'(x)$

Which gives

$$\frac{f(b)-f(a)}{b-a} = f'(x) \qquad \text{as desired.}$$

❖ Theorem (Intermediate Value Theorem or Darboux,s Theorem)

Suppose f is a real differentiable function on some interval $[a,b]$ and suppose $f'(a) < \lambda < f'(b)$ then there exist a point $x \in (a,b)$ such that $f'(x) = \lambda$.

A similar result holds if $f'(a) > f'(b)$.

Proof

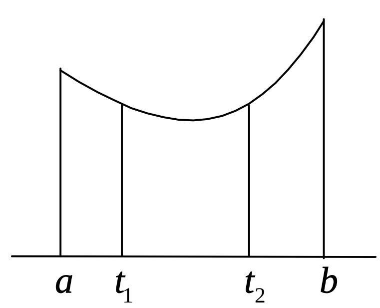
Put $g(t) = f(t) - \lambda t$

Then $g'(t) = f'(t) - \lambda$

If $t = a$ we have

$$g'(a) = f'(a) - \lambda$$

$\because \ f'(a) - \lambda < 0 \qquad \therefore \ g'(a) < 0$



implies that g is monotonically decreasing at a.

$\Rightarrow \ \exists$ a point $t_1 \in (a,b)$ such that $g(a) > g(t_1)$.

Similarly,

$$g'(b) = f'(b) - \lambda$$

$\because \ f'(b) - \lambda > 0 \qquad \therefore \ g'(b) > 0$

implies that g is monotonically increasing at b.

$\Rightarrow$ $\exists$ a point $t_2 \in (a,b)$ such that $g(t_2) < g(b)$

$\Rightarrow$ the function attain its minimum on (a,b) at a point x (say)

such that $g'(x) = 0 \quad \Rightarrow \quad f'(x) - \lambda = 0$

$$\Rightarrow \quad f'(x) = \lambda .$$

Note

We know that a function f may have a derivative f' which exist at every point but is discontinuous at some point however not every function is a derivative. In particular derivatives which exist at every point on the interval have one important property in common with function which are continuous on an interval is that intermediate value are assumed.

The above theorem relates to this fact.

❖ ***Question***

If a and c are real numbers, $c > 0$ and f is defined on $[-1,1]$ by

$$f(x) = \begin{cases} x^a \sin x^{-c} & ; \quad x \neq 0 \\ 0 & ; \quad x = 0 \end{cases}$$

then discuss the differentiability as well as continuity at $x = 0$.

Solution

$$f'(x) = \lim_{t\to x} \frac{f(t) - f(x)}{t - x}$$

$$= \lim_{t\to x} \frac{t^a \sin t^{-c} - x^a \sin x^{-c}}{t - x}$$

$$\Rightarrow \quad f'(0) = \lim_{t\to 0} \frac{t^a \sin t^{-c}}{t}$$

$$= \lim_{t\to 0} t^{a-1} \sin t^{-c}$$

If $a - 1 > 0$, then $\lim\limits_{t\to 0} t^{a-1} \sin t^{-c} = 0 \quad \Rightarrow \quad f'(0) = 0$ when $a > 0$.

If $a - 1 < 0$ i.e. when $a < 1$ we have $t^{a-1} = t^{-b}$ where $b > 0$

And $\lim\limits_{t\to 0} t^{a-1} \sin t^{-c} = \lim\limits_{t\to 0} t^{-b} \sin t^{-c}$

Which does not exist.

If $a - 1 = 0$, we get $\lim\limits_{t\to\infty} \sin t^{-c}$

Which also does not exist.

Hence $f'(0)$ exists if and only if $a > 1$.

Also $\lim\limits_{x\to 0} x^a \sin x^{-c}$ exist and zero when $a > 0$, which equals the actual value of the function $f(x)$ at zero.

Hence the function is continuous at $x = 0$.

❖ ***Question***

Let f be defined for all real x and suppose that

$\left| f(x) - f(y) \right| \leq (x - y)^2 \quad \forall$ real x & y. Prove that f is constant.

Solution

Since $\quad \left| f(x) - f(y) \right| \leq (x - y)^2$

Therefore

$$-(x - y)^2 \leq f(x) - f(y) \leq (x - y)^2$$

Dividing throughout by $x - y$, we get

$$-(x-y)\le \frac{f(x)-f(y)}{x-y}\le (x-y) \quad \text{when} \quad x>y$$

and

$$-(x-y)\ge \frac{f(x)-f(y)}{x-y}\ge (x-y) \quad \text{when} \quad x<y$$

Taking limit as $x\to y$, we get

$$\left.\begin{array}{l} 0\le f'(y)\le 0 \\ 0\ge f'(y)\ge 0 \end{array}\right] \Rightarrow f'(y)=0$$

which shows that function is constant.

❖ *Question*

If $f'(x)>0$ in (a,b) then prove that f is strictly increasing in (a,b) and let g be its inverse function, prove that the function g is differentiable and that

$$g'\left(f(x)\right)=\frac{1}{f(x)} \quad ; \quad a<x<b$$

Solution

Let $y\in\left(f(a),f(b)\right)$

$\Rightarrow y=f(x)$ for some $x\in(a,b)$

$$\Rightarrow g'(y)=\lim_{z\to y}\frac{g(z)-g(y)}{z-y}$$

$$=\lim_{x_z\to x} g\left(f(x_z)\right)=\frac{g\left(f(x_z)\right)-g\left(f(x)\right)}{f(x_z)-f(x)}$$

$$=\lim_{x_z\to x}\frac{f^{-1}\left(f(x_z)\right)-f^{-1}\left(f(x)\right)}{f(x_z)-f(x)}$$

$$=\lim_{x_z\to x}\frac{x_z-x}{f(x_z)-f(x)}=\frac{1}{\lim\limits_{x_z\to x}\dfrac{f(x_z)-f(x)}{x_z-x}}=\frac{1}{f'(x)}$$

❖ *Question*

Suppose f is defined and differentiable for every $x>0$ and $f'(x)\to 0$ as $x\to+\infty$ put $g(x)=f(x+1)-f(x)$. Prove that $g(x)\to 0$ as $x\to+\infty$.

Solution

Since f is defined and differentiable for $x>0$ therefore we can apply the Lagrange's M.V. T. to have

$$f(x+1)-f(x)=(x+1-x)f'(x_1) \quad \text{where} \quad x<x_1.$$

$\because f'(x)\to 0$ as $x\to\infty$

$\therefore f'(x_1)\to 0$ as $x\to\infty$

$\Rightarrow f(x+1)-f(x)\to 0$ as $x\to 0$

$\Rightarrow g(x)\to 0$ as $x\to 0$

....

❖ Question (L Hospital Rule)

Suppose $f'(x), g'(x)$ exist, $g'(x) \neq 0$ and $f(x) = g(x) = 0$.

Prove that $\lim\limits_{t\to x} \dfrac{f(t)}{g(t)} = \dfrac{f'(x)}{g'(x)}$

Proof

$$\lim_{t\to x} \frac{f(t)}{g(t)} = \lim_{t\to x} \frac{f(t)-0}{g(t)-0} = \lim_{t\to x} \frac{f(t)-f(x)}{g(t)-(x)} \qquad \because\ f(x)=g(x)=0$$

$$= \lim_{t\to x} \frac{f(t)-f(x)}{t-x} \cdot \frac{t-x}{g(t)-(x)}$$

$$= \lim_{t\to x} \frac{f(t)-f(x)}{t-x} \cdot \lim_{t\to x} \frac{1}{\dfrac{g(t)-(x)}{t-x}}$$

$$= \lim_{t\to x} \frac{f(t)-f(x)}{t-x} \cdot \frac{1}{\lim\limits_{t\to x} \dfrac{g(t)-(x)}{t-x}} = f'(x) \cdot \frac{1}{g'(x)} = \frac{f'(x)}{g'(x)}$$

Q.E.D.

❖ Question

Suppose f is defined in the neighborhood of a point x and $f''(x)$ exists.

Show that $\lim\limits_{h\to 0} \dfrac{f(x+h)+f(x-h)-2f(x)}{h^2} = f''(x)$

Solution

By use of Lagrange's Mean Value Theorem

$$f(x+h)+f(x) = hf'(x_1) \quad \text{where} \quad x < x_1 < x+h \ \ldots\ldots\ldots\ldots\ldots (i)$$

and

$$-[f(x-h)-f(x)] = hf'(x_2) \quad \text{where} \quad x-h < x_2 < x \ \ldots\ldots\ldots\ldots\ldots (ii)$$

Subtract (ii) from (i) to get

$$f(x+h)+f(x-h)-2f(x) = h[f'(x_1)-f'(x_2)]$$

$$\Rightarrow \frac{f(x+h)+f(x-h)-2f(x)}{h^2} = \frac{f'(x_1)-f'(x_2)}{h}$$

$\because\ x_2 - x_1 \to 0$ as $h \to 0$

therefore

$$\therefore \lim_{h\to 0} \frac{f(x+h)+f(x-h)-2f(x)}{h^2} = \lim_{x_1\to x_2} \frac{f'(x_1)-f'(x_2)}{x_1-x_2}$$

$$= f''(x_2)$$

❖ Question

If $c_0 + \dfrac{c_1}{2} + \dfrac{c_2}{3} + \ldots\ldots\ldots + \dfrac{c_{n-1}}{n} + \dfrac{c_n}{n+1} = 0$

Where $c_0, c_1, c_2, \ldots\ldots\ldots, c_n$ are real constants.

Prove that $c_0 + c_1 x + c_2 x^2 + \ldots\ldots\ldots + c_n x^n = 0$ has at least one real root between 0 and 1.

Solution

Suppose $f(x) = c_0 x + \dfrac{c_1}{2} x^2 + \ldots\ldots\ldots\ldots + \dfrac{c_n}{n+1} x^{n+1}$

Then $f(0) = 0$ and $f(1) = c_0 + \dfrac{c_1}{2} + \dfrac{c_2}{3} + \ldots\ldots\ldots\ldots + \dfrac{c_n}{n+1} = 0$

$\Rightarrow\ f(0) = f(1) = 0$

$\because\ f(x)$ is a polynomial therefore we have

i) It is continuous on $[0,1]$
ii) It is differentiable on $(0,1)$
iii) And $f(a)=0=f(b)$

$\Rightarrow$ the function f has local maximum or a local minimum at some point $x\in(0,1)$

$\Rightarrow$ $f'(x)=c_0+c_1x+c_2x^2+\dots\dots\dots+c_nx^n=0$ for some $x\in(0,1)$

$\Rightarrow$ the given equation has real root between 0 and 1.

❖ *Riemann Differentiation of Vector valued function*

If $f(t)=f_1(t)+i\,f_2(t)$

$f'(t)=f_1'(t)+i\,f_2'(t)$

where $f_1(t)$ and $f_2(t)$ are the real and imaginary part of $f(t)$.

The Rule of differentiation of real valued functions are valid in case of vector valued function but the situation changes in the case of Mean Value Theorem.

❖ *Example*

Take $f(x)=e^{ix}=\cos x+i\sin x$ in $(0,2\pi)$.

Then $f(2\pi)=\cos 2\pi+i\sin 2\pi=1$

$f(0)=\cos(0)+i\sin(0)=1$

$\Rightarrow$ $f(2\pi)-f(0)=0$ but $f'(x)=ie^{ix}$

$\Rightarrow$ $\dfrac{f(2\pi)-f(0)}{2\pi-0}\neq ie^{ix}$ (there is no such x)

$\Rightarrow$ the M.V.T. fails.

In case of vector valued functions, the M.V.T. is not of the form as in the case of real valued function.

❖ *Theorem*

Let f be a continuous mapping of the interval $[a,b]$ into a space $\mathbb{R}^k$ and $\underline{f}$ be differentiable in (a,b) then $\exists$ $x\in(a,b)$ such that $\left|\underline{f}(b)-\underline{f}(a)\right|\le(b-a)\left|\underline{f}'(x)\right|$.

Proof

Put $\underline{z}=\underline{f}(b)-\underline{f}(a)$

And suppose $\varphi(t)=\underline{z}\cdot\underline{f}(t)$ $(a\le t\le b)$

$\varphi(t)$ so defined is a real valued function and it possess the properties of $f(t)$.

$\Rightarrow$ M.V.T. is applicable to $\varphi(t)$.

We have $\varphi(b)-\varphi(a)=(b-a)\varphi'(x)$

i.e. $\varphi(b)-\varphi(a)=(b-a)\underline{z}\cdot\underline{f}'(x)$ for some $x\in(a,b)$ ……….. (i)

Also $\varphi(b)=\underline{z}\cdot\underline{f}(b)$ and $\varphi(a)=\underline{z}\cdot\underline{f}(a)$

$\Rightarrow$ $\varphi(b)-\varphi(a)=\underline{z}\cdot\left(\underline{f}(b)-\underline{f}(a)\right)$ …………. (ii)

from (i) and (ii)

$$\underline{z}\cdot\underline{z}=(b-a)\underline{z}\cdot f'(x)$$
$$\le(b-a)\left|\underline{z}\right|\left|\underline{f}'(x)\right|$$

$\Rightarrow$ $\left|\underline{z}\right|^2\le(b-a)\left|\underline{z}\right|\left|\underline{f}'(x)\right|$

$\Rightarrow$ $\left|\underline{z}\right|\le(b-a)\left|\underline{f}'(x)\right|$

i.e. $\left|\underline{f}(b)-\underline{f}(a)\right|\le(b-a)\left|\underline{f}'(x)\right|$ $\because$ $\underline{z}=\underline{f}(b)-\underline{f}(a)$

which is the required result.

❖ *Question*

If $f(x)=\left|x^3\right|$, then compute $f'(x),\ f''(x)$ and $f'''(x)$, and show that $f'''(0)$ does not exist.

Solution

$$f(x)=\left|x^3\right|=\begin{cases} x^3 & if \quad x\geq 0 \\ -x^3 & if \quad x<0 \end{cases}$$

Now $D_+f(0)=\lim\limits_{x\to 0+0}\dfrac{f(x)-f(0)}{x-0}=\lim\limits_{x\to 0+0}\dfrac{x^3-0}{x-0}=\lim\limits_{x\to 0+0}x^2=0$

& $D_-f(0)=\lim\limits_{x\to 0-0}\dfrac{f(x)-f(0)}{x-0}=\lim\limits_{x\to 0-0}\dfrac{-x^3-0}{x-0}=\lim\limits_{x\to 0-0}(-x^2)=0$

$\because \quad D_+f(x)=D_-f(x)$

$\therefore \quad f'(x)$ exists at $x=0$ & $f'(0)=0$.

Now if $x\neq 0$ and $x>0$ then

$f(x)=x^3 \quad \Rightarrow \quad f'(x)=3x^2$

and if $x\neq 0$ and $x<0$ then

$f(x)=-x^3 \quad \Rightarrow \quad f'(x)=-3x^2$

i.e. $f'(x)=\begin{cases} 3x^2 & if \quad x>0 \\ 0 & if \quad x=0 \\ -3x^2 & if \quad x<0 \end{cases}$

Now $D_+f'(0)=\lim\limits_{x\to 0+0}\dfrac{f'(x)-f'(0)}{x-0}=\lim\limits_{x\to 0+0}\dfrac{3x^2-0}{x-0}$

$=\lim\limits_{x\to 0+0}3x=0$

And Now $D_-f'(0)=\lim\limits_{x\to 0-0}\dfrac{f'(x)-f'(0)}{x-0}=\lim\limits_{x\to 0-0}\dfrac{-3x^2-0}{x-0}$

$=\lim\limits_{x\to 0+0}(-3x)=0$

$\because \quad D_+f'(x)=D_-f'(x)$

$\therefore \quad f''(x)$ exists at $x=0$ & $f''(0)=0$.

Now if $x\neq 0$ and $x>0$ then

$f'(x)=3x^2 \quad \Rightarrow \quad f''(x)=6x$

and if $x\neq 0$ and $x<0$ then

$f'(x)=-3x^2 \quad \Rightarrow \quad f''(x)=-6x$

i.e. $f''(x)=\begin{cases} 6x & if \quad x>0 \\ 0 & if \quad x=0 \\ -6x & if \quad x<0 \end{cases}$

Now $D_+f''(0)=\lim\limits_{x\to 0+0}\dfrac{f''(x)-f''(0)}{x-0}=\lim\limits_{x\to 0+0}\dfrac{6x-0}{x-0}=6$

And $D_-f''(0)=\lim\limits_{x\to 0-0}\dfrac{f''(x)-f''(0)}{x-0}=\lim\limits_{x\to 0-0}\dfrac{-6x-0}{x-0}=-6$

$\because \quad D_+f''(0)\neq D_-f''(0)$

$\therefore \quad f'''(0)$ doest not exist.

But $f'''(0)$ exist if $x\neq 0$, and equal to 6 if $x>0$ and equal to -6 if $x<0$.

…………………………………

ENOUGH

Chapter 5 – Function of Several Variables

Subject: *Real Analysis* ***Level:*** *M.Sc.*
Source: *Syed Gul Shah (Chairman, Department of Mathematics, US Sargodha)*

❖ *Introduction*

There is the basic difference between the calculus of functions of one variable and the calculus of functions of two variables. But there is a slight difference between the calculus of two variable and the calculus of functions of three, four or of many variables. Therefore we shall emphasise mainly on the study of functions of two variables.

❖ *Function of two variables*

If to each point (x, y) of a certain part of $xy-$plane, there is assigned a real number z, then z is known to be a function of two variable x and y.

e.g. $z = x^2 - y^2$, $z = x^2 + y^2$, $z = xy$ etc.

❖ *Neighbourhood (nhood)*

A neighbourhood of radius δ of a point (x_0, y_0) of the $xy-$plane is the set of points which lies inside a circle with centre at (x_0, y_0) and has radius δ.

$$N_\delta(x_0, y_0) = (x - x_0)^2 + (y - y_0)^2 < \delta^2$$

Similarly, a nhood of a radius δ of a point (x_0, y_0, z_0) of a space is a sphere with centre at (x_0, y_0, z_0) and radius δ.

$$N_\delta(x_0, y_0, z_0) = (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 < \delta^2$$

This definition can be extended to the definition of a nhood of a point of a space of any dimension.

❖ *Open Set*

A set is known to be open set if each point (x_0, y_0) of the set has a nhood which totally lies inside the set.

❖ *Domain*

A set D which is not empty and open is known to be a domain, if any two points of the set can be joined by a broken line which lies completely with in D.

❖ *Region*

A domain D is known to be a region if some or all of the boundary points are contained in D.

❖ *Closed Region*

A region is known to be closed if it contains all the boundary points.

e.g. i) $x^2 + y^2 < 1$ (Domain)
$x^2 + y^2 = 1$ (Boundary)
$x^2 + y^2 \le 1$ (Closed region)

ii) $xy < 1$ (Domain)
$xy = 2$ (Boundary)
$xy \le 1$ (Closed Region)

❖ *Limit & Continuity*

Let $z = f(x, y)$ be a function of two variables defined in a domain D. Suppose there is a point $(x_0, y_0) \in D$ or is a boundary point then

$$\lim_{\substack{x \to x_0 \\ y \to y_0}} f(x, y) = c$$

It means that given $\varepsilon > 0$ $\exists$ a $\delta > 0$ such that

$$|f(x, y) - c| < \varepsilon \quad \text{whenever} \quad |(x, y) - (x_0, y_0)| < \delta \quad \forall \ (x, y) \in N_\delta(x_0, y_0)$$

If limit of a function is equal to actual value of function then f is said to be continuous at the point (x_0, y_0)

$$\lim_{\substack{x\to x_0\\ y\to y_0}} f(x,y) = f(x_0, y_0)$$

If f is continuous at every point of D, then f is said to be continuous on D.

❖ *Theorem*

Let $f(x,y)$ & $g(x,y)$ be defined in a domain D and suppose that

$$\lim_{\substack{x\to x_0\\ y\to y_0}} f(x,y) = u_1 \quad \& \quad \lim_{\substack{x\to x_0\\ y\to y_0}} g(x,y) = v_1$$

$a)$ then (i) $\lim_{\substack{x\to x_0\\ y\to y_0}} \left[f(x,y)+g(x,y)\right] = u_1 + v_1$

(ii) $\lim_{\substack{x\to x_0\\ y\to y_0}} \left[f(x,y)\cdot g(x,y)\right] = u_1 v_1$

(iii) $\lim_{\substack{x\to x_0\\ y\to y_0}} \dfrac{f(x,y)}{g(x,y)} = \dfrac{u_1}{v_1}$

$b)$ If $f(x,y)$ & $g(x,y)$ are defined in D, then

$$\lim_{\substack{x\to x_0\\ y\to y_0}} f(x,y) = f(x_0, y_0) \quad \& \quad \lim_{\substack{x\to x_0\\ y\to y_0}} g(x,y) = g(x_0, y_0)$$

i.e. $f(x,y)$, $g(x,y)$ are continuous at (x_0, y_0) then so are the functions

$$f(x,y)+g(x,y), \quad f(x,y)g(x,y) \text{ and } \frac{f(x,y)}{g(x,y)}, \quad \text{provided } g(x,y)\neq 0.$$

Proof

$a)$ (i) $\because \lim_{\substack{x\to x_0\\ y\to y_0}} f(x,y) = u_1$, $\lim_{\substack{x\to x_0\\ y\to y_0}} g(x,y) = v_1$

$\therefore$ given $\dfrac{\varepsilon}{2} > 0$ $\exists$ a $\delta_1, \delta_2 > 0$ such that

$$\left| f(x,y) - u_1 \right| < \frac{\varepsilon}{2} \quad \forall \ (x,y) \in N_{\delta_1}(x_0, y_0)$$

$$\& \quad \left| g(x,y) - v_1 \right| < \frac{\varepsilon}{2} \quad \forall \ (x,y) \in N_{\delta_2}(x_0, y_0)$$

then
$$\begin{aligned}\left|\left[f(x,y)+g(x,y)\right]-\left[u_1+v_1\right]\right| &= \left|\left[f(x,y)-u_1\right]+\left[g(x,y)-v_1\right]\right| \\ &\le \left|f(x,y)-u_1\right| + \left|g(x,y)-v_1\right| \\ &< \frac{\varepsilon}{2}+\frac{\varepsilon}{2} \quad \forall \ (x,y)\in N_\delta(x_0,y_0) \\ &\text{where } \delta = \min(\delta_1, \delta_2)\end{aligned}$$

Which show that

$$\lim_{\substack{x\to x_0\\ y\to y_0}} \left[f(x,y)+g(x,y)\right] = u_1 + v_1$$

(ii)
$$\begin{aligned}\left| f(x,y)\cdot g(x,y) - u_1 v_1 \right| &= \left| f(x,y)\cdot g(x,y) - u_1 g(x,y) + u_1 g(x,y) - u_1 v_1 \right| \\ &= \left| g(x,y)\left[f(x,y)-u_1\right] + u_1\left[g(x,y)-v_1\right] \right| \\ &\le \left| g(x,y)\left[f(x,y)-u_1\right] \right| + \left| u_1\left[g(x,y)-v_1\right] \right| \\ &< \left| g(x,y) \right| \frac{\varepsilon}{2} + u_1 \frac{\varepsilon}{2} = \varepsilon_1 \quad \forall \ (x,y)\in N_\delta(x_0,y_0)\end{aligned}$$

$$\Rightarrow \lim_{\substack{x\to x_0\\ y\to y_0}} f(x,y)\cdot g(x,y) = u_1 v_1$$

iii) We prove that $\lim_{\substack{x\to x_0\\ y\to y_0}} \frac{1}{g(x,y)} = \frac{1}{v_1}$

$$\left|\frac{1}{g(x,y)} - \frac{1}{v_1}\right| = \left|\frac{v_1 - g(x,y)}{v_1\, g(x,y)}\right|$$

$$= \frac{|g(x,y)-v_1|}{|v_1||g(x,y)|} < \frac{\varepsilon/2}{|v_1||g(x,y)|}$$

$$< \frac{\varepsilon/2}{|v_1||g(x,y)-v_1+v_1|} < \frac{\varepsilon/2}{|v_1|\left(|g(x,y)-v_1|+|v_1|\right)}$$

$$< \frac{\varepsilon/2}{|v_1|\left(\varepsilon/2+|v_1|\right)} = \varepsilon_1 \quad \forall\ (x,y)\in N_{\delta_2}(x_0,y_0)$$

$$\Rightarrow \lim_{\substack{x\to x_0\\ y\to y_0}} \frac{1}{g(x,y)} = \frac{1}{v_1}$$

$$\because \Rightarrow \lim_{\substack{x\to x_0\\ y\to y_0}} f(x,y) = u_1 \quad \& \quad \lim_{\substack{x\to x_0\\ y\to y_0}} g(x,y) = v_1$$

By (*ii*) of theorem

$$\lim_{\substack{x\to x_0\\ y\to y_0}} f(x,y)\cdot\frac{1}{g(x,y)} = \lim_{\substack{x\to x_0\\ y\to y_0}} \frac{f(x,y)}{g(x,y)} = \frac{u_1}{v_1}$$

b) Since it is given that the limiting values are the same as the actual values of the functions $f(x,y)+g(x,y)$, $f(x,y)\cdot g(x,y)$ and $\frac{f(x,y)}{g(x,y)}$ at the point (x_0,y_0) therefore these function are continuous on (x_0,y_0).

Note

It is to be noted that there is a difference between $\lim_{(x,y)\to(a,b)} f(x,y)$ and $\lim_{\substack{x\to a\\ y\to b}} f(x,y)$

i.e. $\lim_{\substack{x\to a\\ y\to b}} f(x,y) = \lim_{y\to b}\left(\lim_{x\to a} f(x,y)\right)$ or $\lim_{\substack{x\to a\\ y\to b}} f(x,y) = \lim_{x\to a}\left(\lim_{y\to b} f(x,y)\right)$

Obviously in the two cases limits are taken first w.r.t one variable and then w.r.t other variable. These limits are called the repeated limits. Since these are taken along the special path, therefore repeated limits are the special cases of limits.

$\lim_{(x,y)\to(a,b)} f(x,y)$ exists if and only if limiting vales are not depend upon any path along which $(x,y)\to(a,b)$.

❖ *Example*

Consider $f:\mathbb{R}^2\to\mathbb{R}$ given by

$$f(x,y) = \begin{cases} \dfrac{x^2y^2}{x^4+y^4} & , (x,y)\neq(0,0) \\ 0 & , (x,y)=(0,0) \end{cases}$$

Now $\lim_{\substack{x\to 0\\ y\to 0}} f(x,y) = \lim_{\substack{y\to 0\\ x\to 0}} f(x,y) = 0$

However along the straight line $y = mx$, we have

$$\lim_{(x,y)\to(0,0)} f(x,y) = \frac{m^4}{1+m^4}$$

which is different for different values of m. Hence $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

❖ *Example*

Consider $f:\mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x,y) = \begin{cases} \dfrac{x^2\cos x - y^2\cos y}{x^2+y^2} & , \ (x,y) \neq 0 \\ 0 & , \ (x,y) = 0 \end{cases}$$

then $\lim_{x\to 0}\left[\lim_{y\to 0} f(x,y)\right] = \lim_{x\to 0}\cos x = 1$

and $\lim_{y\to 0}\left[\lim_{x\to 0} f(x,y)\right] = \lim_{y\to 0}(-\cos y) = -1$

$\Rightarrow \lim_{\substack{x\to 0\\ y\to 0}} f(x,y)$ does not exist.

❖ *Example*

Consider $f:\mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x,y) = \begin{cases} (x+y)\dfrac{\sin(x^2+y^2)}{x^2+y^2} & , \ (x,y) \neq (0,0) \\ 0 & , \ (x,y) = (0,0) \end{cases}$$

Use $\dfrac{\sin x}{x} < 1$ to get

$$\| f(x,y) - 0 \| \leq |x+y| < |x| + |y|$$

Thus $\| f(x,y) - 0 \| < \varepsilon$ whenever $|x| < \dfrac{\varepsilon}{2}$, $|y| < \dfrac{\varepsilon}{2}$

Take $\delta = \dfrac{\varepsilon}{2}$,

It follows that for given $\varepsilon > 0$, we can find $\delta > 0$ such that

$$\| f(x,y) - f(0,0) \| < \varepsilon \quad \text{whenever } \sqrt{(x-0)^2 + (y-0)^2} < \delta$$

i.e. $\forall \ (x,y) \in N_\delta(0,0)$

Limit of the function at $(0,0)$ is equal to actual value of function at $(0,0)$.

Hence f is continuous at $(0,0)$.

❖ *Partial Derivative*

Let $z = f(x,y)$ be defined in a domain D of xy-plane and take $(x_0, y_0) \in D$, then $f(x, y_0)$ is a function of x alone and its derivative may exist. If it exists then its value at (x_0, y_0) is known to be the partial derivative of $f(x,y)$ at (x_0, y_0) and is denoted as $\left.\dfrac{\partial f}{\partial x}\right|_{(x_0,y_0)}$ or $\left.\dfrac{\partial z}{\partial x}\right|_{(x_0,y_0)}$

The other notations are z_x , f_x , f_1.

$$\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} = \lim_{\Delta x\to 0}\frac{f(x+\Delta x, y_0) - f(x, y_0)}{\Delta x}$$

We can define $\dfrac{\partial f}{\partial y}$ in the same manner.

❖ *Geometrical Interpretation*

$z = f(x,y)$ represents a surface in space. $y = y_0$ is a plane. $z = (x, y_0)$ is the curve which arises when $y = y_0$ cuts the surface $z = f(x,y)$. Thus $\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)}$ denotes the slope of tangent to the curve $z = f(x, y_0)$ at $x = x_0$. Similarly $\left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)}$ denotes the slope of the tangent to the curve $z = f(x_0, y)$ at $y = y_0$.

If the point (x_0, y_0) varies, then f_x & f_y are themselves functions of x & y.
In the case of functions of more than three variables it is necessary to indicate the variable held constant during the process of differentiation as a suffix to avoid the confusion.

For example, $z = f(x,y,u,v)$, then partial derivatives are written as $\left(\frac{\partial z}{\partial u}\right)_x$, $\left(\frac{\partial z}{\partial y}\right)_v$ and so on. We take an example: $x = u + v$, $y = u - v$

$\left(\frac{\partial x}{\partial u}\right)_v = 1$, $\left(\frac{\partial x}{\partial v}\right)_u = 1$, $\left(\frac{\partial y}{\partial u}\right)_v = 1$, $\left(\frac{\partial y}{\partial v}\right)_u = -1$

Also $x + y = 2u$ and $x = 2u - y$, then

$\left(\frac{\partial x}{\partial u}\right)_y = 2$ & $\left(\frac{\partial x}{\partial y}\right)_u = -1$ and so on.

❖ *Total Differential*

In the case of partial derivative we have considered increments Δx & Δy separately.

Now take (x,y) & $(x+\Delta x, y+\Delta y)$ two points in the domain of definition of z then if $(z+\Delta z)$ correspond to the point $(x+\Delta x, y+\Delta y)$ we have

$$\Delta z = f(x+\Delta x, y+\Delta y) - f(x,y)$$

If the increment Δz can be expressed as

$$\Delta z = a\Delta x + b\Delta y + \varepsilon_1\Delta x + \varepsilon_2\Delta y$$

and $\varepsilon_1, \varepsilon_2 \to 0$ as $\Delta x, \Delta y \to 0$, then $a\Delta x + b\Delta y$ is known to be the total differential of z denoted by dz, and we write

$$\Delta z = dz + \varepsilon_1\Delta x + \varepsilon_2\Delta y$$

In case when z is differentiable function dz gives very close approximation of Δz.

❖ *Theorem*

If $z = f(x,y)$ has a total differential at a point $(x,y) \in D$, then

$$a = \frac{\partial z}{\partial x} \quad \& \quad b = \frac{\partial z}{\partial y}.$$

Proof

We have

$$\Delta z = dz + \varepsilon_1\Delta x + \varepsilon_2\Delta y \quad \text{where } \varepsilon_1, \varepsilon_2 \to 0 \text{ as } \Delta x, \Delta y \to 0$$

Let us suppose that $\Delta y = 0$

then $\Delta z = a\Delta x + \varepsilon_1\Delta x$

Taking the limit as $\Delta x \to 0$

$$\frac{\partial z}{\partial x} = a$$

Similarly we can get $\frac{\partial z}{\partial y} = b$.

❖ *Theorem (Fundamental Lemma)*

If $z = f(x,y)$ has a continuous first order partial derivative in D then z has total differential $dz = \dfrac{\partial z}{\partial x}\Delta x + \dfrac{\partial z}{\partial y}\Delta y$ at every point $(x,y) \in D$.

Proof

Take a point (x,y) as a fixed point in the domain D. Suppose x changes alone. Then we have

$$\Delta z = f(x+\Delta x, y) - f(x,y)$$
$$= f_x(x_1, y)\Delta x \qquad \left(x < x_1 < x+\Delta x\right) \text{ (It is by M. V. Theorem)}$$

$\because$ f_x is continuous

$\therefore$ $\varepsilon_1 = f_x(x_1, y) - f_x(x,y) \to 0$ as $\Delta x \to 0$

$\Rightarrow$ $f(x+\Delta x, y) - f(x,y) = f_x(x,y)\Delta x + \varepsilon_1 \Delta x$ ………. (i)

Now if both x, y changes, we obtain a change Δz in z as

$$\Delta z = f(x+\Delta x, y+\Delta y) - f(x,y)$$
$$= \left[f(x+\Delta x, y) - f(x,y)\right] + \left[f(x+\Delta x, y+\Delta y) - f(x+\Delta x, y)\right]$$

that is we have expressed Δz as the sum of terms representing the effect of a change in x alone and subsequent change in y alone.

Now $f(x+\Delta x, y+\Delta y) - f(x+\Delta x, y) = f_y(x+\Delta x, y_1)\Delta y \quad \left(y < y_1 < y+\Delta y\right)$

(It is by use of M.V. theorem)

$\because$ f_y is given to be continuous

$\therefore$ $\varepsilon_2 = f_y(x+\Delta x, y_1) - f_y(x,y) \to 0$ as $\Delta x, \Delta y \to 0$

$\Rightarrow$ $f(x+\Delta x, y+\Delta y) - f(x+\Delta x, y) = f_y(x,y)\Delta y + \varepsilon_2 \Delta y$ ……….. (ii)

Using (i) & (ii), we have

$$\Delta z = f_x(x,y)\Delta x + f_y(x,y)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y \quad \text{where } \varepsilon_1, \varepsilon_2 \to 0 \text{ as } \Delta x, \Delta y \to 0$$

which shows that the total differential dz of z exist & is given by

$$dz = f_x(x,y)\Delta x + f_y(x,y)\Delta y \quad \ldots\ldots\ldots\ldots.. (iii)$$

Note

(a) For reasons to be explained later; Δx & Δy can be replaced by dx & dy in (iii).

Thus we have $dz = \dfrac{\partial z}{\partial x}dx + \dfrac{\partial z}{\partial y}dy$

Which is the customary way of writing the differential. The preceding analysis extends at once to functions of three or more variables. For example, if

$$w = f(x,y,u,v), \quad \text{then} \quad dw = \frac{\partial w}{\partial x}dx + \frac{\partial w}{\partial y}dy + \frac{\partial w}{\partial u}du + \frac{\partial w}{\partial v}dv.$$

(b) In the following discussion, the function and their Ist order partial derivatives will be considered to be continuous in their respective domain of definition.

Example

If $z = x^2 - y^2$, then $dz = 2xdx - 2ydy$.

Example:

If $w = \dfrac{xy}{z}$, then $dw = \dfrac{y}{x}dx + \dfrac{x}{z}dy - \dfrac{xy}{z^2}dz$

PROBLEMS

1) Evaluate $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if

a) $z = \frac{x}{x^2 + y^2}$ Ans: $\frac{\partial z}{\partial x} = \frac{y^2 - x^2}{\left(x^2 + y^2\right)^2}$, $\frac{\partial z}{\partial x} = \frac{-2xy}{\left(x^2 + y^2\right)^2}$

b) $z = x \sin xy$ Ans: $\frac{\partial z}{\partial x} = \sin xy + xy \cos xy$, $\frac{\partial z}{\partial y} = x^2 \cos xy$

c) $x^3 + xy^2 - x^2 z + z^3 - 2 = 0$ Ans: $\frac{\partial z}{\partial x} = \frac{3x^2 + y^2 - 2xz}{x^2 - 3z^2}$, $\frac{\partial z}{\partial y} = \frac{e^{x+2y} - y}{\sqrt{e^{x+2y} - y^2}}$

2) Evaluate the indicated partial derivatives:

a) $\left(\frac{\partial u}{\partial x}\right)_y$ and $\left(\frac{\partial v}{\partial y}\right)_x$ if $u = x^2 - y^2$, $v = x + 2y$

b) $\left(\frac{\partial x}{\partial u}\right)_y$ and $\left(\frac{\partial y}{\partial v}\right)_u$ if $u = x - 2y$, $v = u + 2y$ Ans: $\left(\frac{\partial x}{\partial u}\right)_y = 1$, $\left(\frac{\partial y}{\partial v}\right)_u = \frac{1}{2}$

3) Find the differentials of the following functions

a) $z = \frac{x}{y}$ Ans: $\frac{ydx - xdy}{y^2}$

b) $z = \log\sqrt{x^2 + y^2}$ Ans: $\frac{xdx + ydy}{x^2 + y^2}$

c) $z = \tan^{-1}\left(\frac{y}{x}\right)$ Ans: $\frac{-ydx + ydy}{x^2 + y^2}$

d) $u = \frac{1}{\sqrt{x^2 + y^2 + z^2}}$ Ans: $\frac{-(xdx + ydy + zdz)}{(x^2 + y^2 + z^2)^{3/2}}$

4) If $z = x^2 + 2xy$, find Δz in terms of Δx, Δy for $x = 1$, $y = 1$.

Ans: $\Delta z = 4\Delta x + 2\Delta y + \overline{\Delta x^2} + 2\Delta x \Delta y$, $dz = 4\Delta x + 2\Delta y$, $dz = 4\Delta x + 2\Delta y$.

……………………………………..

❖ *Derivative and Differential of functions of functions*

In the following discussion, the function and their first order partial derivatives will be considered to be continuous in their respective domain of definitions.

❖ *Theorem (Chain Rule I)*

Let $z = f(x,y)$, $x = g(t)$ & $y = h(t)$ be defined in a domain D, then

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$$

Proof

$\because$ $z = f(x,y)$, $x = g(t)$, $y = h(t)$ are defined in D, are continuous and have Ist order partial derivatives.

$\therefore$ By using the fundamental lemma we have

$$\Delta z = \frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial y}\Delta y + \varepsilon_1\Delta x + \varepsilon_2\Delta y \;\ldots\ldots\ldots\ldots\; (i)$$

where $\varepsilon_1, \varepsilon_2 \to 0$ as $\Delta x, \Delta y \to 0$

Also $\Delta x = g(t+\Delta t) - g(t)$

$\Delta y = h(t+\Delta t) - h(t)$

Dividing (i) by Δt, we get

$$\frac{\Delta z}{\Delta t} = \frac{\partial z}{\partial x}\cdot\frac{\Delta x}{\Delta t} + \frac{\partial z}{\partial y}\cdot\frac{\Delta y}{\Delta t} + \varepsilon_1\frac{\Delta x}{\Delta t} + \varepsilon_2\frac{\Delta y}{\Delta t}$$

Take the limit as $\Delta t \to 0$, we get

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt} \quad \text{as desired.}$$

❖ *Theorem (Chain Rule II)*

Let $z = f(x,y)$, $x = g(u,v)$, $y = h(u,v)$ be defined in a domain D and have continuous first order partial derivative in D, then

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}$$

$$\text{and}\quad \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}$$

Proof

$\because$ the functions are continuous having first order partial derivatives in D, therefore by the fundamental lemma, we have

$$\Delta z = \frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial y}\Delta y + \varepsilon_1\Delta x + \varepsilon_2\Delta y \;\ldots\ldots\ldots\ldots\; (i)$$

where $\Delta x = g(u+\Delta u, v) - g(u,v)$, $\Delta y = h(u+\Delta u, v) - h(u,v)$

and $\varepsilon_1, \varepsilon_2 \to 0$ as $\Delta x, \Delta y \to 0$ i.e. $\Delta u \to 0$

Dividing (i) by Δu throughout to have

$$\frac{\Delta z}{\Delta u} = \frac{\partial z}{\partial x}\cdot\frac{\Delta x}{\Delta u} + \frac{\partial z}{\partial y}\cdot\frac{\Delta y}{\Delta u} + \varepsilon_1\frac{\Delta x}{\Delta u} + \varepsilon_2\frac{\Delta y}{\Delta u}$$

Taking the limit as $\Delta u \to 0$ i.e. $\Delta x, \Delta y \to 0$, we have

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}$$

Similarly if $\Delta x = g(u, v+\Delta v) - g(u,v)$

$\Delta y = h(u, v+\Delta v) - h(u,v)$

Then dividing (i) by Δv throughout, we obtain

$$\frac{\Delta z}{\Delta v}=\frac{\partial z}{\partial x}\cdot\frac{\Delta x}{\Delta v}+\frac{\partial z}{\partial y}\cdot\frac{\Delta y}{\Delta v}+\varepsilon_1\frac{\Delta x}{\Delta v}+\varepsilon_2\frac{\Delta y}{\Delta v}$$

Taking the limit as $\Delta v\to 0$, we have

$$\frac{\partial z}{\partial v}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}$$

❖ *Note*

We have proved in chain rule I, that if $z=f(x,y)$, $x=g(t)$, $y=h(t)$, then

$$\frac{dz}{dt}=\frac{\partial z}{\partial x}\cdot\frac{dx}{dt}+\frac{\partial z}{\partial y}\cdot\frac{dy}{dt} \quad \ldots\ldots\ldots\ (i)$$

The three functions of t considered here: $x=g(t)$, $y=h(t)$, $z=f\left(g(t),h(t)\right)$ have differentials $dx=\frac{dx}{dt}\Delta t$, $dy=\frac{dy}{dt}\Delta t$, $dz=\frac{dz}{dt}\Delta t$.

From (*i*) we conclude that

$$\frac{dz}{dt}\Delta t=\frac{\partial z}{\partial x}\left(\frac{dx}{dt}\Delta t\right)+\frac{\partial z}{\partial y}\left(\frac{dy}{dt}\Delta t\right)$$

$$\Rightarrow\ dz=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy \quad \ldots\ldots\ldots\ (ii)$$

Similarly,

$$dx=\frac{\partial x}{\partial u}\Delta u+\frac{\partial x}{\partial v}\Delta v$$

$$dy=\frac{\partial y}{\partial u}\Delta u+\frac{\partial y}{\partial v}\Delta v$$

$$dz=\frac{\partial z}{\partial u}\Delta u+\frac{\partial z}{\partial v}\Delta v$$

are the corresponding differentials when $z=f(x,y)$, $x=g(u,v)$, $y=h(u,v)$

$$\Rightarrow\ dz=\left(\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}\right)\Delta u+\left(\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}\right)\Delta v$$

$$=\frac{\partial z}{\partial x}\left(\frac{\partial x}{\partial u}\Delta u+\frac{\partial x}{\partial v}\Delta v\right)+\frac{\partial z}{\partial y}\left(\frac{\partial y}{\partial u}\Delta u+\frac{\partial y}{\partial v}\Delta v\right)$$

$$=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy$$

which is again (*ii*)

The generalization of this permits to conclude that:

The differential formula

$$dz=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy+\frac{\partial z}{\partial t}dt+\ldots\ldots$$

which holds when $z=f(x,y,t,\ldots.)$ and $dx=\Delta x$, $dy=\Delta y$, $dt=\Delta t$,……, remain the true when $x,y,t,\ldots\ldots$, and hence z, are all functions of other independent variables and $dx,dy,dt,\ldots\ldots,dz$ are the corresponding differentials.

As a consequence we can conclude:

Any equation in differentials which is correct for one choice of independent variables remains true for any other choice. Another way of saying this is that any equation in differentials treats all variables on an equal basis.

Thus, if $dz=2dx-3dy$ at a given point, then $dx=\frac{1}{2}dz+\frac{3}{2}dy$ is the corresponding differentials of x in terms of y and z.

❖ ***Example***

If $z = \dfrac{x^2 - 1}{y}$, then $dz = \dfrac{2xy\,dx - (x^2 - 1)dy}{y^2}$

Hence $\dfrac{\partial z}{\partial x} = \dfrac{2x}{y}$, $\dfrac{\partial z}{\partial y} = \dfrac{1 - x^2}{y^2}$

❖ ***Example***

If $r^2 = x^2 + y^2$, then $rdr = xdx + ydy$

and $\left(\dfrac{\partial r}{\partial x}\right)_y = \dfrac{x}{r}$, $\left(\dfrac{\partial r}{\partial y}\right)_x = \dfrac{y}{r}$, $\left(\dfrac{\partial x}{\partial r}\right)_y = \dfrac{r}{x}$, etc.

❖ ***Example***

If $z = \tan^{-1}\left(\dfrac{y}{x}\right)$ $(x \neq 0)$, then

$$dz = \frac{1}{1 + \left(\dfrac{y}{x}\right)^2}.d\left(\frac{y}{x}\right) = \frac{xdy - ydx}{x^2 + y^2}$$

and hence

$$\frac{\partial z}{\partial x} = -\frac{y}{x^2 + y^2} \quad , \quad \frac{\partial z}{\partial y} = \frac{x}{x^2 + y^2}$$

…………………………………..

❖ *Implicit Function*

If $F(x,y,z)$ is a given function of x,y & z, then the equation $F(x,y,z)=0$ is a relation which may describe one or several functions z of x & y.

Thus if $x^2+y^2+z^2-1=0$, then

$$z=\sqrt{1-x^2-y^2} \quad \text{or} \quad z=-\sqrt{1-x^2-y^2}$$

Where both functions being defined for $x^2+y^2\le 1$. Either function is said to be implicitly defined by the equation $x^2+y^2+z^2-1=0$.

Similarly, an equation $F(x,y,z,w)=0$ may define one or more implicit functions w of x,y,z. If two such equations are given;

$$F(x,y,z,w)=0 \quad , \quad G(x,y,z,w)=0 \quad ,$$

It is in general possible (at least in theory) to reduce the equations by elimination to the form

$$w=f(x,y) \quad , \quad z=g(x,y)$$

i.e. to obtain two functions of two variables. In general, if m equations in n unknown are given $(m<n)$, it is possible to solve for m of the variables in terms of the remaining $n-m$ variables; the number of dependent variables equals the number of equations

❖ *Example*

If $3x+2y+z+2w=0$

$2x+3y-z-w=0$

then $w=f(x,y)=-5x-5y$ & $z=g(x,y)=7x+8y$

❖ *Example*

Suppose that the functions $w=f(x,y)$ & $z=g(x,y)$ are implicitly defined by

$$2x^2+y^2+z^2-zw=0$$

$$x^2+y^2+2z^2-8+zw=0$$

Then taking the differentials, we obtain

$$4xdx+2ydy+2zdz-wdz-zdw=0 \quad \ldots\ldots\ldots (i)$$

$$wdz+zdw+2xdx+2ydy+4zdz=0 \quad \ldots\ldots\ldots (ii)$$

Eliminate dw between (i) and (ii) to have

$$6xdx+4ydy+6zdz=0$$

$$\Rightarrow \; dz=-\frac{x}{z}dx-\frac{2y}{3x}dy$$

$$\Rightarrow \; \frac{\partial z}{\partial x}=-\frac{x}{z} \quad , \quad \frac{\partial z}{\partial y}=-\frac{2y}{3z}$$

Eliminating of dz from (i) and (ii) gives

$$6x(2z+w)dx+4y(z+w)dy-6z^2dw = 0$$

$$\Rightarrow \; dw = \frac{x(2z+w)}{z^2}dx+\frac{2y(z+w)}{3z^2}dy$$

$$\frac{\partial w}{\partial x}=\frac{x(2x+w)}{x^2} \quad , \quad \frac{\partial w}{\partial y}=\frac{2y(z+w)}{z^2}$$

❖ Examples

Suppose that the functions $w = f(x,y)$ & $z = g(x,y)$ are implicitly define by

$$F(x,y,z,w)=0 \quad \text{and} \quad G(x,y,z)=0\text{, then}$$

$$F_x dx + F_y dy + F_z dz + F_w dw = 0$$

and $$G_x dx + G_y dy + G_z dz + G_w dw = 0$$

$$\Rightarrow F_z dz + F_w dw = -\left[F_x dx + F_y dy\right]$$

and $$G_z dz + G_w dw = -\left[G_x dx + G_y dy\right]$$

Then by crammer rule, we have

$$dz = \frac{-\begin{vmatrix} F_x dx + F_y dy & F_w \\ G_x dx + G_y dy & G_w \end{vmatrix}}{\begin{vmatrix} F_z & F_w \\ G_z & G_w \end{vmatrix}} = -\frac{\begin{vmatrix} F_x & F_w \\ G_x & G_w \end{vmatrix}}{\begin{vmatrix} F_z & F_w \\ G_z & G_w \end{vmatrix}} dx - \frac{\begin{vmatrix} F_y & F_w \\ G_y & G_w \end{vmatrix}}{\begin{vmatrix} F_z & F_w \\ G_z & G_w \end{vmatrix}} dy$$

$$\Rightarrow \frac{\partial z}{\partial x} = -\frac{\begin{vmatrix} F_x & F_w \\ G_x & G_w \end{vmatrix}}{\begin{vmatrix} F_z & F_w \\ G_z & G_w \end{vmatrix}} \quad , \quad \frac{\partial z}{\partial y} = -\frac{\begin{vmatrix} F_y & F_w \\ G_y & G_w \end{vmatrix}}{\begin{vmatrix} F_z & F_w \\ G_z & G_w \end{vmatrix}}$$

$$\Rightarrow \frac{\partial z}{\partial x} = -\frac{\frac{\partial(F,G)}{\partial(x,w)}}{\frac{\partial(F,G)}{\partial(z,w)}} \quad , \quad \frac{\partial z}{\partial y} = -\frac{\frac{\partial(F,G)}{\partial(y,w)}}{\frac{\partial(F,G)}{\partial(z,w)}} \qquad \text{provided } \frac{\partial(F,G)}{\partial(z,w)} \neq 0$$

Similarly, we have

$$dw = -\frac{\begin{vmatrix} F_z & F_x dx + F_y dy \\ G_z & G_x dx + G_y dy \end{vmatrix}}{\begin{vmatrix} F_z & F_w \\ G_z & G_w \end{vmatrix}}$$

and we can find $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$ in the same manner.

❖ Particular Cases

i) One equation in 2 unknowns i.e. $F(x,y)=0$

$$\Rightarrow F_x dx + F_y dy = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{F_x}{F_y} \quad \left(F_y \neq 0\right)$$

ii) One equation in 3 unknowns i.e. $F(x,y,z)=0$

$$F_x dx + F_y dy + F_z dz = 0$$

$$\Rightarrow \frac{\partial z}{\partial x} = -\frac{F_x}{F_z} \quad , \quad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z} \quad (F_z \neq 0)$$

iii) 2 equations in 3 unknown

$$F(x,y,z)=0 \quad , \quad G(x,y,z)=0$$

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial(F,G)}{\partial(y,x)}}{\frac{\partial(F,G)}{\partial(y,z)}} \quad , \quad \frac{\partial z}{\partial y} = -\frac{\frac{\partial(F,G)}{\partial(w,y)}}{\frac{\partial(F,G)}{\partial(z,w)}}$$

…………………………………

❖ ***Example***

Find the partial derivatives w.r.t x & y, when

$$u+2v-x^2+y^2=0$$
$$2u-v-2xy=0$$

Solution

Take the differentials

$$du+2dv-2x\,dx+2y\,dy=0 \;\ldots\ldots\ldots\;(i)$$
$$2du-dv-2x\,dy-2y\,dx=0 \;\ldots\ldots\ldots\;(ii)$$

Eliminating dv between (i) and (ii), we have

$$5du-(2x+4y)dx+(2y-4x)dy=0$$
$$\Rightarrow\; du=\frac{1}{5}(2x+4y)dx-\frac{1}{5}(2y-4x)dy$$
$$\Rightarrow\; \frac{\partial u}{\partial x}=\frac{1}{5}(2x+4y) \quad \& \quad \frac{\partial u}{\partial y}=-\frac{1}{5}(2y-4x)$$

Eliminating du between (i) and (ii), we get

$$5dv-(4x-2y)dx+(4y+2x)dy=0$$
$$\Rightarrow\; dv=\frac{1}{5}(4x-2y)dx-\frac{1}{5}(4y+2x)dy$$
$$\Rightarrow\; \frac{\partial v}{\partial x}=\frac{1}{5}(4x-2y) \quad \& \quad \frac{\partial v}{\partial y}=-\frac{1}{5}(4y+2x)$$

❖ ***Question***

Give that

$$2x+y-3z-2u=0$$
$$\& \quad x+2y+z+u=0$$

Find $\left(\frac{\partial x}{\partial y}\right)_z$, $\left(\frac{\partial y}{\partial x}\right)_u$, $\left(\frac{\partial z}{\partial u}\right)_x$, $\left(\frac{\partial y}{\partial z}\right)_x$

Solution

Take the differentials

$$2dx+dy-3dz-2du=0 \;\ldots\ldots\ldots\;(i)$$
$$dx+2dy+dz+du=0 \;\ldots\ldots\ldots\;(ii)$$

Eliminating du between (i) and (ii), we have

$$4dx+5dy-dz=0 \;\ldots\ldots\;(iii)$$
$$\Rightarrow\; dx=-\frac{5}{4}dy+\frac{1}{4}dz$$
$$\Rightarrow\; \left(\frac{\partial x}{\partial y}\right)_z=-\frac{5}{4}$$

From (iii), we have

$$5dy=dz-4dx$$
$$\Rightarrow\; dy=\frac{1}{5}dz-\frac{4}{5}dx$$
$$\Rightarrow\; \left(\frac{\partial y}{\partial z}\right)_x=\frac{1}{5}$$

Eliminating dz between (i) & (ii), we get

$$5dx+7dy+du=0$$

$$\Rightarrow\ dy = -\frac{5}{7}dx - \frac{1}{7}du$$

$$\Rightarrow\ \left(\frac{\partial y}{\partial x}\right)_u = -\frac{5}{7}$$

Now eliminating dy between (i) & (ii), we get

$$-3dx - 5dz - 3du = 0$$

$$\Rightarrow\ dz = -\frac{3}{5}dx - \frac{3}{5}du$$

$$\Rightarrow\ \left(\frac{\partial z}{\partial u}\right)_x = -\frac{3}{5}$$

❖ *Question*

Given that

$$x^2 + y^2 + z^2 - u^2 + v^2 = 1 \ \ldots\ldots\ldots\ldots\ldots (i)$$

$$x^2 - y^2 + z^2 + u^2 + 2v^2 = 2 \ \ldots\ldots\ldots\ldots (ii)$$

a) Find du & dv in terms of dx, dy & dz at the point

$$x = 1\ ,\ y = 1\ ,\ z = 2\ ,\ u = 3\ \ \&\ \ v = 2\,.$$

b) Find $\left(\frac{\partial u}{\partial x}\right)_{(y,z)}$, $\left(\frac{\partial v}{\partial y}\right)_{(x,z)}$ at the point given above.

c) Find approximately the values of u & v for $x = 1\cdot 1$, $y = 1.2$, $z = 1.8$

Solutions

Differential gives

$$2xdx + 2ydy + 2zdz - 2udu + 2vdv = 0 \ \ldots\ldots\ldots (iii)$$

$$2xdx - 2ydy + 2zdz + 2udu + 2vdv = 0 \ \ldots\ldots\ldots (iv)$$

a) Putting $x = 1,\ y = 1,\ z = 2,\ u = 3$ & $v = 2$ in (iii) & (iv), we obtain

$$2dx + 2dy + 4dz - 6du + 4dv = 0 \ \ldots\ldots\ldots\ldots (v)$$

$$\&\quad 2dx - 2dy + 4dz + 6du + 8dv = 0 \ \ldots\ldots\ldots\ldots (vi)$$

Adding gives

$$12dv = -(4dx + 8dz)$$

$$\Rightarrow\ dv = -\frac{1}{3}(dx + 0\cdot dy + 2dz)$$

Similarly eliminating dv between (v) and (vi), we get

$$du = \frac{1}{9}(dx + 3dy + 2dz)$$

b) $\because\ du = \frac{1}{9}(dx + 3dy + 2dz)$

$$\therefore\ \left(\frac{\partial u}{\partial x}\right)_{y,z} = \frac{1}{9}$$

& $\because\ dv = -\frac{1}{3}(dx + 0\cdot dy + 2dz)$

$$\therefore\ \left(\frac{\partial v}{\partial y}\right)_{x,z} = 0$$

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❖ ***Question***

Find the transformation of $x = r\cos\theta$, $y = r\sin\theta$ from rectangular to polar coordinates. Verify the relations

a) $dx = \cos\theta dr - r\sin\theta d\theta$

$dy = \sin\theta dr + r\cos\theta d\theta$

b) $dr = \cos\theta dx + \sin\theta dy$

$d\theta = -\frac{\sin\theta}{r}dx + \frac{\cos\theta}{r}dy$

c) $\left(\frac{\partial x}{\partial r}\right)_\theta = \cos\theta$, $\left(\frac{\partial x}{\partial r}\right)_y = \sec\theta$, $\frac{\partial(r,\theta)}{\partial(x,y)} = \frac{1}{r}$

Solutions

Given that $x = r\cos\theta$ & $y = r\sin\theta$

a) Differential gives

$$dx = \cos\theta dr - r\sin\theta d\theta \quad \ldots\ldots\ldots (i)$$
$$dy = \sin\theta dr + r\cos\theta d\theta \quad \ldots\ldots\ldots (ii)$$

b) Multiplying (*i*) by $\cos\theta$ & (*ii*) by $\sin\theta$ and adding, we get

$$dr = \cos\theta dx + \sin\theta dy$$

Now multiply (*i*) by $\sin\theta$ & (*ii*) by $\cos\theta$ and subtract to obtain

$$d\theta = -\frac{\sin\theta}{r}dx + \frac{\cos\theta}{r}dy$$

c) Given $x = r\cos\theta$

$$\Rightarrow \left(\frac{\partial x}{\partial r}\right)_\theta = \cos\theta$$

We have already shown that $dr = \cos\theta dx + \sin\theta dy$

Which can be written as $dx = \frac{dr}{\cos\theta} - \tan\theta dy$

$$\Rightarrow \left(\frac{\partial x}{\partial r}\right)_y = \sec\theta$$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta \;= r$$

and $$\frac{\partial(r,\theta)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{vmatrix} = \begin{vmatrix} \cos\theta & \sin\theta \\ -\frac{\sin\theta}{r} & \frac{\cos\theta}{r} \end{vmatrix} = \frac{1}{r}\cos^2\theta + \frac{1}{r}\sin^2\theta \;= \frac{1}{r}$$

❖ ***Question***

Given that $x^2 - y^2\cos uv + z^2 = 0$

$x^2 + y^2 - \sin uv + 2z^2 = 2$

and $xy - \sin u\cos v + z = 0$

Find $\left(\frac{\partial x}{\partial u}\right)_v$, $\left(\frac{\partial x}{\partial v}\right)_u$ at $x = 1,\ y = 1,\ u = \frac{\pi}{2},\ v = 0,\ z = 0$

Solution

Differential gives

$$2x\,dx - 2y\cos uv\,dy + y^2\sin uv \cdot udv + y^2\sin uv \cdot vdu + 2zdz = 0 \quad \ldots\ldots (i)$$

$$2x\,dx + 2y\,dy - \cos uv \cdot u dv - \cos uv \cdot v du + 4z dz = 0 \;\ldots\ldots\ldots\; (ii)$$

& $$x\,dy + y\,dx - \cos u \cdot \cos v\,du + \sin u \cdot \sin v\,dv + dz = 0 \;\ldots\ldots\ldots\; (iii)$$

At the given point, these equations reduce to

$$2dx - 2dy = 0 \;\ldots\ldots\ldots\ldots\ldots\; (iv)$$

$$2dx + 2dy - \frac{\pi}{2}dv = 0 \;\ldots\ldots\ldots\; (v)$$

& $$dx + dy + dz = 0 \;\ldots\ldots\ldots\ldots\ldots\; (vi)$$

Adding (iv) & (v), we have

$$4dx - \frac{\pi}{2}dv = 0$$

$$\Rightarrow\; dx = \frac{\pi}{8}dv + 0 \cdot du \quad \Rightarrow \left(\frac{\partial x}{\partial u}\right)_v = 0 \;,\; \left(\frac{\partial x}{\partial v}\right)_u = \frac{\pi}{8}$$

❖ *Question*

Find $\left(\frac{\partial u}{\partial x}\right)_y$ if $x^2 - y^2 + u^2 + 2v^2 = 1$

$$x^2 + y^2 - u^2 - v^2 = 2$$

Solution

Taking the differentials, we have

$$2x\,dx - 2y\,dy + 2u\,du + 4v\,dv = 0$$
$$2x\,dx + 2y\,dy - 2u\,du - 2v\,dv = 0$$

Eliminating dv, we get

$$6x\,dx + 2y\,dy - 2u\,du = 0$$

$$\Rightarrow\; du = \frac{3x}{u}dx + \frac{y}{u}dy$$

$$\Rightarrow \left(\frac{\partial u}{\partial x}\right)_y = \frac{3x}{u}$$

❖ *Question*

Given the transformation

$$x = u - 2v$$
$$y = 2u + v$$

a) Write the equations of the inverse transformation

b) Evaluate the Jacobian of the transformation and that of the inverse transformation.

Solution

a) From the equations, we have

$$u = \frac{1}{5}x + \frac{2}{5}y$$

$$v = -\frac{2}{5}x + \frac{1}{5}y$$

which are the equations of the inverse transformation.

b) Jacobian of the given transformation $= \dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$

$$= \begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix} = 5$$

Jacobian of the inverse transformation $=\dfrac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix}$

$$= \begin{vmatrix} \frac{1}{5} & \frac{2}{5} \\ -\frac{2}{5} & \frac{1}{5} \end{vmatrix} = \frac{1}{5}$$

❖ *Question*

Given the transformation $x = f(u,v)$, $y = g(u,v)$ with Jacobian $J = \dfrac{\partial(x,y)}{\partial(u,v)}$, show that for the inverse transformation one has

$$\frac{\partial u}{\partial x} = \frac{1}{J}\frac{\partial y}{\partial v} \quad , \quad \frac{\partial u}{\partial y} = -\frac{1}{J}\frac{\partial x}{\partial v} \quad , \quad \frac{\partial v}{\partial x} = -\frac{1}{J}\frac{\partial y}{\partial u} \quad , \quad \frac{\partial u}{\partial y} = \frac{1}{J}\frac{\partial x}{\partial u}$$

Solution

The given equations are

$$f(u,v) - x = 0 \;\ldots\ldots\ldots\; (i)$$
$$g(u,v) - y = 0 \;\ldots\ldots\ldots\; (ii)$$

Differentiating w.r.t. x, we get

$$f_u \frac{\partial u}{\partial x} + f_v \frac{\partial v}{\partial x} - 1 = 0$$
$$g_u \frac{\partial u}{\partial x} + g_v \frac{\partial v}{\partial x} - 0 = 0$$

Solving these equations by Crammer's rule, we have

$$\frac{\partial u}{\partial x} = -\frac{\begin{vmatrix} -1 & f_v \\ 0 & g_v \end{vmatrix}}{\begin{vmatrix} f_u & f_v \\ g_u & g_v \end{vmatrix}} = \frac{g_v}{J} = \frac{1}{J}\frac{\partial y}{\partial v} \qquad \left(\because \frac{\partial y}{\partial v} = g_v\right)$$

$$\frac{\partial v}{\partial x} = -\frac{\begin{vmatrix} f_u & -1 \\ g_u & 0 \end{vmatrix}}{J} = -\frac{g_u}{J} = -\frac{1}{J}\frac{\partial y}{\partial u}$$

Differentiating (i) & (ii) w.r.t. y, we have

$$f_u \frac{\partial u}{\partial y} + f_v \frac{\partial v}{\partial y} - 0 = 0$$
$$g_u \frac{\partial u}{\partial y} + g_v \frac{\partial v}{\partial y} - 1 = 0$$

Solving these equations by Crammer's rule, we get

$$\frac{\partial u}{\partial y} = -\frac{\begin{vmatrix} 0 & f_v \\ -1 & g_v \end{vmatrix}}{J} = -\frac{f_v}{J} = -\frac{1}{J}\frac{\partial x}{\partial v}$$

$$\frac{\partial v}{\partial y} = -\frac{\begin{vmatrix} f_u & 0 \\ g_u & -1 \end{vmatrix}}{J} = \frac{f_u}{J} = \frac{1}{J}\frac{\partial x}{\partial u}$$

❖ Question

Given the transformation

$$x = u^2 - v^2$$
$$y = 2uv$$

a) Compute its Jacobian.

b) Evaluate $\left(\frac{\partial u}{\partial x}\right)_y$ & $\left(\frac{\partial v}{\partial x}\right)_y$

Solution

The given equations can be written as

$$u^2 - v^2 - x = 0 \;\ldots\ldots\ldots\; (i)$$
$$2uv - y = 0 \;\ldots\ldots\ldots\; (ii)$$

Differentiating (*i*) & (*ii*) partially w.r.t. x, we have

$$2u\frac{\partial u}{\partial x} - 2v\frac{\partial v}{\partial x} - 1 = 0 \;\ldots\ldots\ldots\; (iii)$$

$$2v\frac{\partial u}{\partial x} + 2u\frac{\partial v}{\partial x} - 0 = 0 \;\ldots\ldots\ldots\; (iv)$$

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2u & -2v \\ 2v & 2u \end{vmatrix} = 4(u^2 + v^2)$$

Solving (*iii*) & (*iv*) by Crammer's rule, we have

$$\left(\frac{\partial u}{\partial x}\right)_y = -\frac{\begin{vmatrix} -1 & -2v \\ 0 & 2u \end{vmatrix}}{J} = \frac{2u}{4(u^2+v^2)} = \frac{u}{2(u^2+v^2)}$$

$$\left(\frac{\partial v}{\partial x}\right)_y = -\frac{\begin{vmatrix} 2u & -1 \\ 2v & 0 \end{vmatrix}}{J} = \frac{-2v}{4(u^2+v^2)} = \frac{-v}{2(u^2+v^2)}$$

Note

$\left(\frac{\partial u}{\partial y}\right)_x$ & $\left(\frac{\partial v}{\partial y}\right)_x$ can be determined in the same manner.

❖ Question

Prove that if $F(x,y,z) = 0$, then

$$\left(\frac{\partial z}{\partial x}\right)_y \cdot \left(\frac{\partial x}{\partial y}\right)_z \cdot \left(\frac{\partial y}{\partial z}\right)_x = -1$$

Solution

$$F(x,y,z) = 0$$
$$\Rightarrow\; F_x\,dx + F_y\,dy + F_z\,dz = 0$$
$$\Rightarrow\; dx = -\frac{F_y}{F_x}dy - \frac{F_z}{F_x}dz \qquad \Rightarrow\; \left(\frac{\partial x}{\partial y}\right)_z = -\frac{F_y}{F_x}$$
$$\&\;\; dy = -\frac{F_x}{F_y}dx - \frac{F_z}{F_y}dz \qquad \Rightarrow\; \left(\frac{\partial y}{\partial z}\right)_x = -\frac{F_z}{F_y}$$
$$dz = -\frac{F_x}{F_z}dx - \frac{F_y}{F_z}dy \qquad \Rightarrow\; \left(\frac{\partial z}{\partial x}\right)_y = -\frac{F_x}{F_z}$$

Hence

$$\left(\frac{\partial z}{\partial x}\right)_y \cdot \left(\frac{\partial x}{\partial y}\right)_z \cdot \left(\frac{\partial y}{\partial z}\right)_x = \left(-\frac{F_x}{F_z}\right)\cdot\left(-\frac{F_y}{F_x}\right)\cdot\left(-\frac{F_z}{F_y}\right) = -1$$

❖ *Question*

Prove that, if $x = f(u,v)$, $y = g(u,v)$, then

$$\left(\frac{\partial x}{\partial u}\right)_v \left(\frac{\partial u}{\partial x}\right)_y = \left(\frac{\partial y}{\partial v}\right)_u \left(\frac{\partial v}{\partial y}\right)_x$$

and $$\left(\frac{\partial x}{\partial v}\right)_u \left(\frac{\partial v}{\partial x}\right)_y = \left(\frac{\partial u}{\partial y}\right)_x \left(\frac{\partial y}{\partial u}\right)_v$$

also that $$\left(\frac{\partial x}{\partial y}\right)_u \left(\frac{\partial y}{\partial x}\right)_u = 1$$

Solution

$$\because \ f(u,v) - x = 0$$
$$g(u,v) - y = 0$$

$$\therefore \left(\frac{\partial u}{\partial x}\right)_y = \frac{g_v}{J} \quad , \quad \left(\frac{\partial v}{\partial x}\right)_y = -\frac{g_u}{J}$$

$$\left(\frac{\partial u}{\partial y}\right)_x = -\frac{f_v}{J} \quad , \quad \left(\frac{\partial v}{\partial y}\right)_x = \frac{f_u}{J}$$ as already shown

Taking differentials of the given equations, we have

$$f_u\,du + f_v\,dv - dx = 0$$
$$g_u\,du + g_v\,dv - dy = 0$$
$$\Rightarrow \ dx = f_u\,du + f_v\,dv \ \ldots\ldots\ldots\ldots (i)$$
$$dy = g_u\,du + g_v\,dv \ \ldots\ldots\ldots\ldots (ii)$$

$$\Rightarrow \left(\frac{\partial x}{\partial u}\right)_v = f_u \quad , \quad \left(\frac{\partial x}{\partial v}\right)_u = f_v$$

$$\left(\frac{\partial y}{\partial u}\right)_v = g_u \quad , \quad \left(\frac{\partial y}{\partial v}\right)_u = g_v$$

Now $$\left(\frac{\partial x}{\partial u}\right)_v \cdot \left(\frac{\partial u}{\partial x}\right)_y = \left(\frac{\partial y}{\partial v}\right)_u \cdot \left(\frac{\partial v}{\partial y}\right)_x$$

$$\Rightarrow \ f_u \cdot \frac{g_v}{J} = g_v \cdot \frac{f_u}{J},$$ which is true

Similarly, we have the second relation.

Eliminating dv between (i) & (ii), we get

$$\left(f_u \cdot g_v - f_v \cdot g_u\right)du - g_v\,dx + f_v\,dy = 0$$

$$\Rightarrow \ dx = \frac{f_u\,g_v - f_v\,g_u}{g_v}\cdot du + \frac{f_v}{g_v}\,dy$$

and $$dy = \frac{g_v}{f_v}dx - \frac{f_u\,g_v - f_v\,g_u}{f_v}\,du$$

$$\Rightarrow \left(\frac{\partial x}{\partial y}\right)_u = \frac{f_v}{g_v} \quad \& \quad \left(\frac{\partial y}{\partial x}\right)_u = \frac{g_v}{f_v}$$

$$\Rightarrow \left(\frac{\partial x}{\partial y}\right)_u \cdot \left(\frac{\partial y}{\partial x}\right)_u = \frac{f_v}{g_v}\cdot\frac{g_v}{f_v} = 1$$

❖ Question

Given that $x = f(u,v,w)$, $y = g(u,v,w)$, $z = h(u,v,w)$ with the Jacobian $J = \frac{\partial(x,y,z)}{\partial(u,v,w)}$, show that for the inverse transformation one has

i) $\frac{\partial u}{\partial x} = \frac{1}{J}\frac{\partial(y,z)}{\partial(v,w)}$, $\frac{\partial u}{\partial y} = \frac{1}{J}\frac{\partial(z,x)}{\partial(v,w)}$, $\frac{\partial u}{\partial z} = \frac{1}{J}\frac{\partial(x,y)}{\partial(v,w)}$

ii) $\frac{\partial v}{\partial x} = \frac{1}{J}\frac{\partial(y,z)}{\partial(w,u)}$, $\frac{\partial v}{\partial y} = \frac{1}{J}\frac{\partial(z,x)}{\partial(w,u)}$, $\frac{\partial v}{\partial z} = \frac{1}{J}\frac{\partial(x,y)}{\partial(w,u)}$

iii) $\frac{\partial w}{\partial x} = \frac{1}{J}\frac{\partial(y,z)}{\partial(u,v)}$, $\frac{\partial w}{\partial y} = \frac{1}{J}\frac{\partial(z,x)}{\partial(u,v)}$, $\frac{\partial w}{\partial z} = \frac{1}{J}\frac{\partial(x,y)}{\partial(u,v)}$

Solution

We have
$$f(u,v,w) - x = 0$$
$$g(u,v,w) - y = 0$$
$$h(u,v,w) - z = 0$$

Differentiating w.r.t. to x, we get
$$f_u\frac{\partial u}{\partial x} + f_v\frac{\partial v}{\partial x} + f_w\frac{\partial w}{\partial x} - 1 = 0$$
$$g_u\frac{\partial u}{\partial x} + g_v\frac{\partial v}{\partial x} + g_w\frac{\partial w}{\partial x} - 0 = 0$$
$$h_u\frac{\partial u}{\partial x} + h_v\frac{\partial v}{\partial x} + h_w\frac{\partial w}{\partial x} - 0 = 0$$

By Crammer's rule, we have
$$\frac{\partial u}{\partial x} = -\frac{\begin{vmatrix} -1 & f_v & f_w \\ 0 & g_v & g_w \\ 0 & h_v & h_w \end{vmatrix}}{J} = \frac{\begin{vmatrix} g_v & g_w \\ h_v & h_w \end{vmatrix}}{J} = \frac{1}{J}\frac{\partial(g,h)}{\partial(v,w)} = \frac{1}{J}\frac{\partial(y,z)}{\partial(v,w)}$$
$$\frac{\partial v}{\partial x} = -\frac{\begin{vmatrix} f_u & -1 & f_w \\ g_u & 0 & g_w \\ h_u & 0 & h_w \end{vmatrix}}{J} = -\frac{\begin{vmatrix} g_u & g_w \\ h_u & h_w \end{vmatrix}}{J} = \frac{1}{J}\frac{\partial(g,h)}{\partial(w,u)} = \frac{1}{J}\frac{\partial(y,z)}{\partial(w,u)}$$
$$\frac{\partial w}{\partial x} = -\frac{\begin{vmatrix} f_u & f_v & -1 \\ g_u & g_v & 0 \\ h_u & h_v & 0 \end{vmatrix}}{J} = \frac{\begin{vmatrix} g_u & g_v \\ h_u & h_v \end{vmatrix}}{J} = \frac{1}{J}\frac{\partial(g,h)}{\partial(u,v)} = \frac{1}{J}\frac{\partial(y,z)}{\partial(u,v)}$$

We can find the other relations in the same way by differentiating given relation w.r.t. y and w.r.t. z respectively.

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❖ *Partial Derivative of Higher Order*

Let a function $z = f(x,y)$ be given. Then its two partial derivatives $\frac{\partial z}{\partial x}$ & $\frac{\partial z}{\partial y}$ are themselves functions of x & y.

$$\text{i.e.}\quad \frac{\partial z}{\partial x} = f_x(x,y) \quad , \quad \frac{\partial z}{\partial y} = f_y(x,y)$$

Hence each can be differentiable w.r.t. x & y.

Thus, we obtain four partial derivatives

$$\frac{\partial^2 z}{\partial x^2} = f_{xx}(x,y) \quad , \quad \frac{\partial^2 z}{\partial x \partial y} = f_{xy}(x,y)$$

$$\frac{\partial^2 z}{\partial y \partial x} = f_{yx}(x,y) \quad , \quad \frac{\partial^2 z}{\partial y^2} = f_{yy}(x,y)$$

$\frac{\partial^2 z}{\partial x^2}$ is the result of differentiating $\frac{\partial z}{\partial x}$ w.r.t. x, where $\frac{\partial^2 z}{\partial y \partial x}$ is the result of differentiating $\frac{\partial z}{\partial x}$ w.r.t. y. If all the derivatives concerned are continuous in the domain considered, then $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$ i.e. order of differentiation is immaterial.

Third and higher order partial derivatives are defined in the same manner and under appropriate assumptions of continuity the order of differentiation does not matter.

❖ *Laplacian of z*

If $z = f(x,y)$, then the Laplacian of z is denoted by $\nabla^2 z$ is the expression

$$\nabla^2 z = \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}$$

if $w = f(x,y,z)$, the Laplacian of w is the expression

$$\nabla^2 w = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}$$

The symbol "∇" is a vector differential operator define as

$$\nabla = \frac{\partial}{\partial x}\underline{\hat{i}} + \frac{\partial}{\partial y}\underline{\hat{j}} + \frac{\partial}{\partial x}\underline{\hat{k}}$$

We then have symbolically

$$\nabla^2 = \nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

❖ *Harmonic Function*

If $z = f(x,y)$ has continuous second order derivatives in a domain D and $\nabla^2 z = 0$ in D, then z is said to be Harmonic in D. The same term is used for the function of three variables which has continuous 2nd derivatives in a domain D in space and whose Laplacian is zero in D. The two equations for harmonic functions

$$\nabla^2 z = \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$$

$$\nabla^2 w = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} = 0$$

are known as the Laplace equations in two and three dimensions respectively.

❖ Bi-Harmonic Equations

Another important combination of derivatives occurs in the equation

$$\frac{\partial^4 z}{\partial x^4}+2\frac{\partial^4 z}{\partial x^2\partial y^2}+\frac{\partial^4 z}{\partial y^4}=0$$

which is known to be the Bi-harmonic equation. This combination can be expressed in terms of Laplacian as

$$\nabla^2\left(\nabla^2 z\right) = \nabla^4 z = 0$$

The solutions of $\nabla^4 z=0$ are termed as Pri-harmonic functions.

❖ Higher Derivatives of Functions of Functions

(1) Let $z=f(x,y)$ and $x=g(t)$, $y=h(t)$ so that z can be expressed in terms of t alone. Then

$$\frac{dz}{dt}=\frac{\partial z}{\partial x}\frac{dx}{dt}+\frac{\partial z}{\partial y}\frac{dy}{dt} \quad \ldots\ldots\ldots.. (i)$$

$$\frac{d^2z}{dt^2}=\frac{d}{dt}\left(\frac{dz}{dt}\right)=\frac{\partial z}{\partial x}\frac{d^2x}{dt^2}+\frac{dx}{dt}\frac{d}{dt}\left(\frac{\partial z}{\partial x}\right)+\frac{\partial z}{\partial y}\frac{d^2y}{dt^2}+\frac{dy}{dt}\frac{d}{dt}\left(\frac{\partial z}{\partial y}\right) \quad \ldots\ldots\ldots (ii)$$

Using (i), we have

$$\frac{d}{dt}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial^2 z}{\partial x^2}\frac{dx}{dt}+\frac{\partial^2 z}{\partial y\,\partial x}\frac{dy}{dt}$$

& $$\frac{d}{dt}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial^2 z}{\partial x\partial z}\frac{dx}{dt}+\frac{\partial^2 z}{\partial y^2}\frac{dy}{dt}$$

Putting these values in (ii), we have

$$\frac{d^2z}{dt^2}=\frac{\partial z}{\partial x}\frac{d^2x}{dt^2}+\frac{\partial^2 z}{\partial x^2}\left(\frac{dx}{dt}\right)^2+2\frac{\partial^2 z}{\partial x\partial y}\frac{dx}{dt}\cdot\frac{dy}{dt}+\frac{\partial^2 z}{\partial y^2}\left(\frac{dy}{dt}\right)^2+\frac{\partial z}{\partial y}\frac{d^2y}{dt^2}$$

(2) If $z=f(x,y)$ and $x=g(u,v)$, $y=h(u,v)$, then

$$\frac{\partial z}{\partial u}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u} \quad \ldots\ldots\ldots.. (iii)$$

$$\frac{\partial z}{\partial v}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v} \quad \ldots\ldots\ldots.. (iv)$$

$$\frac{\partial^2 z}{\partial u^2}=\frac{\partial z}{\partial x}\cdot\frac{\partial^2 x}{\partial u^2}+\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial x}\right)\cdot\frac{\partial x}{\partial u}+\frac{\partial z}{\partial y}\left(\frac{\partial^2 y}{\partial u^2}\right)+\frac{\partial y}{\partial u}\cdot\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial y}\right) \quad \ldots\ldots\ldots\ldots (iv)$$

Using (iii), we have

$$\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial^2 z}{\partial x^2}\cdot\frac{\partial x}{\partial u}+\frac{\partial^2 z}{\partial y\,\partial x}\cdot\frac{\partial y}{\partial u}$$

and $$\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial^2 z}{\partial x\partial y}\frac{\partial x}{\partial u}+\frac{\partial^2 z}{\partial y^2}\frac{\partial y}{\partial u}$$

Putting these values in (iv), we get

$$\frac{\partial^2 z}{\partial u^2}=\frac{\partial z}{\partial x}\frac{\partial^2 x}{\partial u^2}+\frac{\partial^2 z}{\partial x^2}\left(\frac{\partial x}{\partial u}\right)+2\frac{\partial^2 z}{\partial x\partial y}\frac{\partial x}{\partial u}\cdot\frac{\partial y}{\partial u}+\frac{\partial^2 z}{\partial y^2}\left(\frac{\partial y}{\partial u}\right)^2+\frac{\partial z}{\partial y}\cdot\frac{\partial^2 y}{\partial u^2}$$

We can find the values of $\frac{\partial^2 z}{\partial u\,\partial v}$ & $\frac{\partial^2 z}{\partial v^2}$ in the same manner.

…………………………………………..

❖ *The Laplacian in Polar, Cylindrical and Spherical Co-ordinate*

We consider first the two-dimensional Laplacian

$$\nabla^2 w = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}$$

and its expression in terms of polar co-ordinates r & θ.

Thus we are given $w = f(x, y)$ and $x = r\cos\theta$, $y = r\sin\theta$ and we wish to express $\nabla^2 w$ in terms of r, θ and derivatives of w with respect to r and θ. The solution is as follows. One has

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial w}{\partial \theta}\cdot\frac{\partial \theta}{\partial x}$$

$$\frac{\partial w}{\partial y} = \frac{\partial w}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial w}{\partial \theta}\cdot\frac{\partial \theta}{\partial y} \quad \text{by chain rule}$$

To evaluate $\frac{\partial r}{\partial x}$, $\frac{\partial \theta}{\partial x}$, $\frac{\partial r}{\partial y}$, $\frac{\partial \theta}{\partial y}$, we use the equations

$$dx = \cos\theta\, dr - r\sin\theta\, d\theta$$

$$dy = \sin\theta\, dr + r\cos\theta\, d\theta$$

These can be solved for dr and $d\theta$ by determinants or by elimination to give

$$dr = \cos\theta\, dx + \sin\theta\, dy$$

$$d\theta = -\frac{\sin\theta}{r}dx + \frac{\cos\theta}{r}dy$$

Hence $\frac{\partial r}{\partial x} = \cos\theta$, $\frac{\partial r}{\partial y} = \sin\theta$, $\frac{\partial \theta}{\partial x} = -\frac{\sin\theta}{r}$ and $\frac{\partial \theta}{\partial y} = \frac{\cos\theta}{r}$

Putting these values above in expressions of $\frac{\partial w}{\partial x}$ & $\frac{\partial w}{\partial y}$, we have

$$\left.\begin{aligned}\frac{\partial w}{\partial x} &= \cos\theta\frac{\partial w}{\partial r} - \frac{\sin\theta}{r}\frac{\partial w}{\partial \theta}\\ \frac{\partial w}{\partial y} &= \sin\theta\frac{\partial w}{\partial r} + \frac{\cos\theta}{r}\frac{\partial w}{\partial \theta}\end{aligned}\right\}\text{..............} (i)$$

These equations provide general rules for expressing derivatives w.r.t. x or y in terms of derivatives w.r.t. r and θ. By applying the first equation to the function $\frac{\partial w}{\partial x}$, one finds that

$$\frac{\partial^2 w}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x}\right) = \cos\theta\frac{\partial}{\partial r}\left(\frac{\partial w}{\partial x}\right) - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\left(\frac{\partial w}{\partial x}\right)$$

By (i) this can be written as follows:

$$\frac{\partial^2 w}{\partial x^2} = \cos\theta\frac{\partial}{\partial r}\left(\cos\theta\frac{\partial w}{\partial r} - \frac{\sin\theta}{r}\frac{\partial w}{\partial \theta}\right) - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\left(\cos\theta\frac{\partial w}{\partial r} - \frac{\sin\theta}{r}\frac{\partial w}{\partial \theta}\right)$$

The rule for differentiation of a product gives finally

$$\frac{\partial^2 w}{\partial x^2} = \cos^2\theta\cdot\frac{\partial^2 w}{\partial r^2} - \frac{2\sin\theta\cos\theta}{r}\cdot\frac{\partial^2 w}{\partial r\,\partial\theta} + \frac{\sin^2\theta}{r^2}\cdot\frac{\partial^2 w}{\partial \theta^2}$$

$$+\frac{\sin^2\theta}{r}\cdot\frac{\partial w}{\partial \theta} + \frac{2\sin\theta\cos\theta}{r^2}\cdot\frac{\partial w}{\partial \theta} \text{ .... } (ii)$$

In the same manner one finds

$$\frac{\partial^2 w}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial w}{\partial y}\right) = \sin\theta\frac{\partial}{\partial r}\left(\sin\theta\frac{\partial w}{\partial r} + \frac{\cos\theta}{r}\frac{\partial w}{\partial \theta}\right) + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial w}{\partial r} + \frac{\cos\theta}{r}\frac{\partial w}{\partial \theta}\right)$$

$$= \sin^2\theta \cdot \frac{\partial^2 w}{\partial r^2} + \frac{2\sin\theta\cos\theta}{r} \cdot \frac{\partial^2 w}{\partial r\,\partial\theta} + \frac{\cos^2\theta}{r^2} \cdot \frac{\partial^2 w}{\partial\theta^2}$$

$$+ \frac{\cos^2\theta}{r} \cdot \frac{\partial w}{\partial r} - \frac{2\sin\theta\cos\theta}{r^2} \cdot \frac{\partial w}{\partial\theta} \quad \ldots\ldots (iii)$$

Adding (*ii*) & (*iii*), we conclude

$$\nabla^2 w = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = \frac{\partial^2 w}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 w}{\partial\theta^2} + \frac{1}{r}\frac{\partial w}{\partial r} \quad \ldots\ldots\ldots (iv)$$

This is the desired result.

Equation (*iv*) at once permits one to write the expression for the 3-demensional Laplacian in cylindrical co-ordinates for the transformation of coordinates

$$x = r\cos\theta \quad , \quad y = r\sin\theta \quad , \quad z = z$$

involves only x & y. In the same way as above, we have

$$\nabla^2 w = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}$$

$$= \frac{\partial^2 w}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 w}{\partial\theta^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{\partial^2 w}{\partial z^2}$$

❖ *Laplacian in Spherical Polar Coordinates*

The transformation form rectangular to spherical polar coordinates is

$$x = \rho\sin\varphi\cos\theta \ , \ y = \rho\sin\varphi\sin\theta \ , \ z = \rho\cos\varphi$$

Writing $r = \rho\sin\varphi$, we have

$$x = r\cos\theta \ , \ y = r\sin\theta \ , \ z = z$$

Which can be considered as a transformation from rectangular to cylindrical coordinates (r,θ,z)

We have

$$\nabla^2 w = \frac{\partial^2 w}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 w}{\partial\theta^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{\partial^2 w}{\partial z^2} \quad \ldots\ldots\ldots.. (i)$$

where
$$\left.\begin{aligned} z &= \rho\cos\varphi \\ r &= \rho\sin\varphi \end{aligned}\right\} \quad \ldots\ldots\ldots (ii)$$

We have transformation from (x,y) to (r,θ) as

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = \frac{\partial^2 w}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 w}{\partial\theta^2} + \frac{1}{r}\frac{\partial w}{\partial\theta}$$

Now if we take transformation from (z,r) to (ρ,φ), then

$$\Rightarrow \frac{\partial^2 w}{\partial z^2} + \frac{\partial^2 w}{\partial r^2} = \frac{\partial^2 w}{\partial\rho^2} + \frac{1}{\rho^2}\frac{\partial^2 w}{\partial\varphi^2} + \frac{1}{\rho}\frac{\partial w}{\partial\varphi}$$

Also $$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial\rho}\cdot\frac{\partial\rho}{\partial r} + \frac{\partial w}{\partial\varphi}\cdot\frac{\partial\varphi}{\partial r}$$

Where $\rho^2 = z^2 + r^2$, $\tan\varphi = \dfrac{r}{z}$

$$\Rightarrow 2\rho\frac{\partial\rho}{\partial r} = 2r \ \Rightarrow \frac{\partial\rho}{\partial r} = \frac{r}{\rho} = \frac{\rho\sin\varphi}{\rho} = \sin\varphi$$

$$\& \quad \sec^2\varphi\cdot\frac{\partial\varphi}{\partial r} = \frac{1}{z} \quad \Rightarrow \frac{\partial\varphi}{\partial r} = \frac{\cos^2\varphi}{z} = \frac{\cos^2\varphi}{\rho\cos\varphi} = \frac{\cos\varphi}{\rho}$$

$$\Rightarrow \frac{\partial w}{\partial r} = \frac{\partial w}{\partial\rho}\cdot\sin\varphi + \frac{\partial w}{\partial\varphi}\cdot\frac{\cos\varphi}{\rho} \quad \ldots\ldots\ldots\ldots (iv)$$

Substituting (*iii*) & (*iv*) in (*i*), we have

$$\nabla^2 w = \left(\frac{\partial^2 w}{\partial z^2} + \frac{\partial^2 w}{\partial r^2}\right) + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} + \frac{1}{r}\frac{\partial w}{\partial r}$$

$$= \frac{\partial^2 w}{\partial \rho^2} + \frac{1}{\rho^2}\frac{\partial^2 w}{\partial \varphi^2} + \frac{1}{\rho}\frac{\partial w}{\partial \rho} + \frac{1}{\rho^2 \sin^2\varphi}\cdot\frac{\partial^2 w}{\partial \theta^2} + \frac{1}{\rho\sin\varphi}\left(\frac{\partial w}{\partial \rho}\sin\varphi + \frac{\partial w}{\partial \varphi}\frac{\cos\varphi}{\rho}\right)$$

$$= \frac{\partial^2 w}{\partial \rho^2} + \frac{1}{\rho^2}\cdot\frac{\partial^2 w}{\partial \varphi^2} + \frac{1}{\rho}\cdot\frac{\partial w}{\partial \rho} + \frac{1}{\rho^2 \sin^2\varphi}\cdot\frac{\partial^2 w}{\partial \theta^2} + \frac{1}{\rho}\cdot\frac{\partial w}{\partial \rho} + \frac{\cot\varphi}{\rho^2}\cdot\frac{\partial w}{\partial \varphi}$$

$$= \frac{\partial^2 w}{\partial \rho^2} + \frac{1}{\rho^2}\cdot\frac{\partial^2 w}{\partial \varphi^2} + \frac{2}{\rho}\cdot\frac{\partial w}{\partial \rho} + \frac{1}{\rho^2 \sin^2\varphi}\cdot\frac{\partial^2 w}{\partial \theta^2} + \frac{\cot\varphi}{\rho^2}\cdot\frac{\partial w}{\partial \varphi}$$

❖ ***Question***

If u & v are functions of x & y defined by the equations

$$xy + uv = 1\ ,\quad xu + yv = 1$$

then find $\dfrac{\partial^2 u}{\partial x^2}$.

Solution

$$y\,dx + x\,dy + v\,du + u\,dv = 0 \ \ldots\ldots\ldots\ (i)$$

$$u\,dx + v\,dy + x\,du + y\,dv = 0 \ \ldots\ldots..\ (ii)$$

Eliminating dv between (i) & (ii)

$$\left(y^2 - u^2\right)dx + \left(xy - uv\right)dy + \left(vy - ux\right)du = 0$$

$$\Rightarrow\ du = \frac{u^2 - y^2}{vy - ux}dx + \frac{uv - xy}{vy - ux}dy$$

$$\Rightarrow\ \frac{\partial u}{\partial x} = \frac{u^2 - y^2}{vy - ux} = \frac{u^2 - y^2}{1 - 2ux} \qquad \text{(using given eq.)}$$

$$\Rightarrow\ \frac{\partial^2 u}{\partial x^2} = \frac{(1-2ux)\cdot 2u\cdot\frac{\partial u}{\partial x} - (u^2 - y^2)\left[(-2u) - 2x\frac{\partial u}{\partial x}\right]}{\left(1-2ux\right)^2}$$

❖ ***Question***

Find $\dfrac{\partial^2 w}{\partial x^2},\ \dfrac{\partial^2 w}{\partial y^2}$ when

i) $w = \dfrac{1}{\sqrt{x^2 + y^2}}$

ii) $w = \tan^{-1}\dfrac{y}{x}$

iii) $w = e^{x^2 - y^2}$

❖ ***Question***

Show that the following functions are harmonic in x & y

i) $e^x \cos y$

ii) $x^3 - 3xy^2$

iii) $\log\sqrt{x^2 + y^2}$

…………………………………………

❖ *Sufficient Condition for the Validity of Reversal in the Order of Derivation*

We now prove two theorems which lay sufficient conditions for the equality of f_{xy} and f_{yx}.

❖ *Schawarz's Theorem*

If (a,b) be a point of the domain of a function $f(x,y)$ such that

i) $f_x(x,y)$ exists in a certain nhood of (a,b).

ii) $f_{xy}(x,y)$ is continuous at (a,b).

then $f_{yx}(a,b)$ exists and is equal to $f_{xy}(a,b)$.

Proof

The given conditions imply that there exists a certain nhood of (a,b) at every point (x,y) of which $f_x(x,y)$, $f_y(x,y)$ and $f_{xy}(x,y)$ exist. Let $(a+h,b+k)$ be any point of this nhood. We write

$$\phi(h,k) = f(a+h,b+k) - f(a+h,b) - f(a,b+k) + f(a,b)$$
$$g(y) = f(a+h,y) - f(a,y)$$

so that $\phi(h,k) = g(b+k) - g(b)$ ……….. (*i*)

$\because$ f_y exists in a nhood of (a,b), the function $g(y)$ in derivable in $[b,b+k]$, and, therefore, by applying the M.V. theorem to the expression on R.H.S of (i), we have

$$\phi(h,k) = kg'(b+\theta k) \qquad (0<\theta<1)$$
$$= k\left(f_y(a+h,b+\theta k) - f_y(a,b+\theta k)\right) \text{ ……….. } (ii)$$

Again since f_{xy} exists in a nhood of (a,b), the function $f_y(x,b+\theta k)$ of x is derivable w.r.t. x in interval $(a,a+h)$ and, therefore, by applying the M.V. theorem to the right of (ii), we have

$$\phi(h,k) = hk\, f_{xy}(a+\theta' h,b+\theta k) \qquad (0<\theta'<1)$$

or $$\frac{1}{k}\left(\frac{f(a+h,b+k) - f(a,b+k)}{h} - \frac{f(a+h,b) - f(a,b)}{h}\right) = f_{xy}(a+\theta' h,b+\theta k)$$

Since $f_x(x,y)$ exists in a nhood of (a,b), this gives when $h \to 0$,

$$\frac{f_x(a,b+k) - f_x(a,b)}{k} = \lim_{h\to 0} f_{xy}(a+\theta' h,b+\theta k)$$

Let, now, $k \to 0$. Since $f_{xy}(x,y)$ is continuous at (a,b), we obtain

$$f_{yx}(a,b) = \lim_{k\to 0}\lim_{h\to 0} f_{xy}(a+\theta' h,b+\theta k) = f_{xy}(a,b)$$

❖ *Young's Theorem*

If (a,b) be a point of the domain of definition of a function $f(x,y)$ such that $f_x(x,y)$ and $f_y(x,y)$ are both differentiable at (a,b), then

$$f_{xy}(a,b) = f_{yx}(a,b)$$

Proof

The differentiability of f_x and f_y at (a,b) implies that they exist in a certain nhood of (a,b) and that f_{xx}, f_{yx}, f_{xy}, f_{yy} exist at (a,b).

Let $(a+h,b+h)$ be a point of this nhood. We write

$$\phi(h,h) = f(a+h,b+h) - f(a+h,b) - f(a,b+h) + f(a,b)$$
$$g(y) = f(a+h,y) - f(a,y)$$

so that $\phi(h,h) = g(b+h) - g(b)$ ………. (*i*)

Since f_y exists in a nhood of (a,b), the function $g(y)$ is derivable in $(b,b+h)$, and, therefore, by applying the M.V. theorem to the expression on the right of (i), we have

$$\phi(h,h) = h\,g'(b+\theta h) \qquad (0<\theta<1)$$

$$= h\left(f_y(a+h,b+\theta h) - f_y(a,b+\theta h)\right) \text{ ……….. } (ii)$$

Since $f_y(x,y)$ is differentiable at (a,b), we have, by definition,

$$f_y(a+h,b+\theta h) - f_y(a,b) = h\, f_{xy}(a,b) + \theta h\, f_{yy}(a,b)$$

$$+ h\varphi_1(h,h) + \theta h \psi_1(h,h) \text{ …… } (iii)$$

and $f_y(a,b+\theta h) - f_y(a,b) = \theta h\, f_{yy}(a,b) + \theta h \psi_2(h,h)$ ……… (iv)

where φ_1, ψ_1, ψ_2 all $\to 0$ as $h \to 0$

From (ii), (iii) and (iv), we obtain

$$\frac{\phi(h,h)}{h^2} = f_{xy}(a,b) + \phi_1(h,h) + \theta\psi_1(h,h) - \theta\psi_2(h,h) \text{ ….…. } (v)$$

By a similar argument and on considering

$$g(x) = f(x,b+k) - f(x,b)$$

We can show that

$$\frac{\phi(h,h)}{h^2} = f_{yx}(a,b) + \psi_3(h,h) + \theta'\varphi_2(h,h) - \theta'\varphi_3(h,h) \text{ ……….. } (vi)$$

where φ_2, φ_3, ψ_3 all $\to 0$ as $h \to 0$

Equating the right hand side of (v) and (vi) and making $h \to 0$, we obtain

$$f_{xy}(a,b) = f_{yx}(a,b)$$

………………………………….

❖ *Maxima and Minima for Functions of Two Variables*

Let $(x_0.y_0)$ be the point of the domain of a function $f(x,y)$, then $f(x_0,y_0)$ said to an extreme value of the function $f(x,y)$, if the expression

$$\Delta f = f(x_0+h, y_0+h) - f(x_0,y_0)$$

preserves its sign for all h and k.

The extreme value of $f(x_0,y_0)$ being called a maximum or a minimum value according as this difference is positive or negative respectively.

Necessary Condition

The Necessary Condition for $f(x_0,y_0)$ to be an extreme value of function $f(x,y)$ is that $f_x(x_0,y_0)=0=f_y(x_0,y_0)$, provided that these partial derivatives exist.

It is to be noted that it is impossible to determine the nature of a critical point by studying the function $f(x,y_0)$ and $f(x_0,y)$.

e.g. Let $f(x,y)=1+x^2-y^2$

then $f(0,y)=1-y^2 \quad \Rightarrow \quad f'(0,y)=-2y=0 \quad \Rightarrow \quad (0,0)$ is a turning point.

Now $f''(0,y)=-2 \quad \Rightarrow \quad (0,0)$ is a point of maximum value.

But $f(x,0)=1+x^2$

$\Rightarrow \quad f'(x,0)=2x=0 \quad \Rightarrow \quad x=0 \quad \Rightarrow \quad (0,0)$ is the critical point

$\Rightarrow \quad f''(x,0)=2>0 \quad \Rightarrow \quad (0,0)$ is the maximum value

Hence we fail to decide the nature of the critical point in this way.

Sufficient Condition

Let $z=f(x,y)$ be defined and have continuous 1st and 2nd order partial derivatives in a domain D. Suppose (x_0,y_0) is a point of D for which f_x and f_y are both zero.

Let $A=f_{xx}(x_0,y_0)$, $B=f_{xy}(x_0,y_0)$, $C=f_{yy}(x_0,y_0)$,

then we have the following cases

i) $B^2-AC<0$ and $A+C<0 \quad \Rightarrow$ relative maximum at (x_0,y_0).

ii) $B^2-AC<0$ and $A+C>0 \quad \Rightarrow$ relative minimum at (x_0,y_0)

iii) $B^2-AC>0 \quad \Rightarrow$ saddle point at (x_0,y_0)

iv) $B^2-AC=0 \quad \Rightarrow$ nature of the critical point is undetermined

Proof

By the application of M.V. theorem for function of two variables we have

$$\Delta f = hf_x(x_0+\theta h, y_0+\theta k)+kf_y(x_0+\theta h, y_0+\theta k) \qquad (0<\theta<1)$$

$$= h\left[f_x(x_0+\theta h, y_0+\theta k)-f_x(x_0,y_0)\right]+k\left[f_y(x_0+\theta h, y_0+\theta k)-f_y(x_0,y_0)\right]$$

(it is because $f_x(x_0,y_0)=f_y(x_0,y_0)=0$, a turning point)

$$= h\left[\theta h f_{xx}(x_0,y_0)+\theta k f_{yx}(x_0,y_0)+\varepsilon_1\theta h+\varepsilon_2\theta k\right]$$

$$+k\left[\theta h f_{xy}(x_0,y_0)+\theta k f_{yy}(x_0,y_0)+\varepsilon_3\theta h+\varepsilon_4\theta k\right]$$

where $\varepsilon_1,\varepsilon_2,\varepsilon_3$ & $\varepsilon_4 \to 0$ as $h,k\to 0$

$$\Delta f = h^2 f_{xx}(x_0,y_0)+2hk\, f_{xy}(x_0,y_0)+k^2 f_{yy}(x_0,y_0)+\varepsilon_1 h^2+(\varepsilon_2+\varepsilon_3)hk+\varepsilon_4 k^2$$

$$\Rightarrow \quad \Delta f = h^2A+2hkB+k^2C+\varepsilon_1 h^2+(\varepsilon_2+\varepsilon_3)hk+\varepsilon_4 k^2$$

The sign of Δf depends upon the quadratic $d^2 f = h^2A+2hk\,B+k^2C$

i & ii) Let $B^2-AC<0$, $(A\neq 0)$

$$\Rightarrow \quad d^2 f = \frac{1}{A}\left(h^2A+2hk\,AB+k^2AC\right)$$

$$= \frac{1}{A}\left(h^2A^2 + 2hk\,AB + k^2B^2 + (k^2AC - k^2B^2)\right)$$
$$= \frac{1}{A}\left((hA + kB)^2 + k^2(AC - B^2)\right)$$

Since $(hA+kB)^2$ is positive and $AC - B^2$ (supposed) is +ive, therefore the sign of $d^2 f$ depends upon the sign of A.

$\Rightarrow\ \Delta f > 0$ if $A > 0$ & $\Delta f < 0$ if $A < 0$

Again, since $B^2 - AC < 0 \Rightarrow B^2 < AC \Rightarrow AC > 0$

$\Rightarrow$ A and C are either both +ive or both –ive.

If $A > 0$, $C > 0$ then $A + C > 0$ and if $A < 0$, $C < 0$ then $A + C < 0$.

Hence we have the following result

a) $\Delta f > 0$ when $A + C > 0 \Rightarrow (x_0, y_0)$ is a point of minimum value.

b) $\Delta f < 0$ when $A + C < 0 \Rightarrow (x_0, y_0)$ is a point of maximum value.

iii) Let $B^2 - AC > 0$, then

$$d^2 f = \frac{1}{A}\left((hA + kB)^2 + k^2(AC - B^2)\right)$$
$$= \frac{1}{A}\left((hA + kB)^2 - k^2(B^2 - AC)\right)$$

which may be +ive or –ive for certain value of h & k, therefore (x_0, y_0) is a saddle point.

iv) Let $B^2 - AC = 0$, $A \neq 0$

$$\Rightarrow\ d^2 f = \frac{1}{A}\left(hA + kB\right)^2$$

which may vanish for certain values of h and k, implies that nature of the point remain undetermined.

❖ *Question*

Test for maxima and minima

$$z = 1 - x^2 - y^2$$

Solution

$$\frac{\partial z}{\partial x} = -2x = 0 \quad \Rightarrow \quad x = 0$$
$$\frac{\partial z}{\partial y} = -2y = 0 \quad \Rightarrow \quad y = 0$$

$\Rightarrow$ $(0,0)$ is the only critical point.

$$A = \frac{\partial^2 z}{\partial x^2} = -2 \quad , \quad B = \frac{\partial^2 z}{\partial x\,\partial y} = 0 \quad , \quad C = \frac{\partial^2 z}{\partial y^2} = -2$$

$B^2 - AC = 0 - 4 = -4 < 0$ and $A + C = -2 - 2 = -4 < 0$

$\Rightarrow$ the function has maximum value at $(0,0)$.

❖ *Question*

Test for maxima and minima

$$z = x^3 - 3xy^2$$

Solution

$$\frac{\partial z}{\partial y} = 3x^2 - 3y^2 = 0 \quad \Rightarrow \quad x = -y \ \ \& \ \ x = y$$
$$\frac{\partial z}{\partial y} = -6xy = 0 \quad \Rightarrow \quad xy = 0$$

$\Rightarrow$ $(0,0)$ is the critical point.

$$A = \frac{\partial^2 z}{\partial x^2} = 6x = 0 \quad \text{at } (0,0)$$

$$B = \frac{\partial^2 z}{\partial x \partial y} = -6y = 0 \quad \text{at } (0,0)$$

$$C = \frac{\partial^2 z}{\partial y^2} = -6x = 0 \quad \text{at } (0,0)$$

$B^2 - 4AC = 0$ also $A + C = 0$

Therefore we need further consideration for the nature of point

$$\Delta z = z(0+h, 0+k) - z(0,0)$$
$$= z(h,k) - z(0,0)$$
$$= h^3 - 2hk^2$$

For $h = k$

$$\Delta z = h^3 - 3h^3 = -2h^3$$
$$\Rightarrow \Delta z > 0 \text{ if } h < 0 \quad \& \quad \Delta z < 0 \text{ if } h > 0$$

Hence $(0,0)$ is a saddle point.

❖ *Question*

Examine the function

$$z = f(x,y) = x^2 y^2$$

Solution

$f_x = 0 \Rightarrow 2xy^2 = 0$

$f_y = 0 \Rightarrow 2yx^2 = 0$

implies that $(0,0)$ is the critical point

$A = f_{xx} = 2y^2 = 0$ at $(0,0)$

$B = f_{xy} = -4xy = 0$ at $(0,0)$

$C = f_{yy} = 2x^2 = 0$ at $(0,0)$

Since $B^2 - 4AC = 0$ and also $A + C = 0$

Therefore we need further consideration for the nature of point.

$$\Delta f = f(h,k) - f(0,0)$$
$$= h^2 k^2$$
$$\Delta f > 0 \quad \text{for all } h \ \& \ k$$

Hence $(0,0)$ is the point where function has minimum value.

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❖ *Lagrange's Multiplier (Maxima & Minima for Function with Side Condition)*

A problem of considerable importance for application is that of maximizing and minimizing of function (optimization) of several variables where the variables are related by one or more equations, which are turned as side condition. e.g. the problem of finding the radius of largest sphere inscribable in the ellipsoid $x^2+2y^2+3z^2=6$ is equivalent to minimizing the function $w=x^2+y^2+z^2$ with the side condition $x^2+2y^2+z^2=6$.

To handle such problem, we can, if possible, eliminate some of the variables by using the side conditions and reduce the problem to an ordinary maximum and minimum problem such as that consider previously.
This procedure is not always feasible and following procedure often is more convenient which treat the variable in more symmetrical manner, so that various simplifications may be possible.

Consider the problem of finding the extreme values of the function $f(x_1,x_2,....,x_n)$ when the variable are restricted by a certain number of side conditions say

$$g_1(x_1,x_2,....,x_n) = 0$$
$$g_2(x_1,x_2,....,x_n) = 0$$
$$\dots\dots\dots\dots$$
$$\dots\dots\dots\dots$$
$$\dots\dots\dots\dots$$
$$g_m(x_1,x_2,....,x_n) = 0$$

We then form the linear combination

$$\varphi(x_1,...,x_n)= f(x_1,...,x_n)+\lambda_1 g_1(x_1,...,x_n)+\lambda_2 g_2(x_2,...,x_n)+ ....+\lambda_m g_m(x_1,...,x_n)$$

where $\lambda_1,\lambda_2,....,\lambda_m$ are m constants.

We then differentiate φ w.r.t. each coordinate and consider the following system of $n+m$ equations.

$$D_r\varphi(x_1,x_2,....,x_n)=0 \quad , \quad r=1,2,....,n$$
$$g_k(x_1,x_2,....,x_n)=0 \quad , \quad k=1,2,....,m$$

Lagrange discovered that if the point $(x_1,x_2,....,x_n)$ is a solution of the extreme problem then it will also satisfy the system of $n+m$ equation.

In practise, we attempt to solve this system for $n+m$ unknowns, which are $\lambda_1,\lambda_2,....,\lambda_m$ & $x_1,x_2,....,x_n$

The point so obtain must then be tested to determine whether they yield a maximum, a minimum or neither.

The numbers $\lambda_1,\lambda_2,....,\lambda_m$, which are introduced only to help to solve the system for $x_1,x_2,....,x_n$ are known as Lagrange's multiplier. One multiplier is introduced for each side condition.

❖ *Question*

Find the critical points of $w=xyz$, subject to condition $x^2+y^2+z^2=1$.

Solution

We form the function

$$\varphi = xyz+\lambda(x^2+y^2+z^2-1)$$

then

$$\frac{\partial\varphi}{\partial x}= yz+2\lambda x=0$$
$$\frac{\partial\varphi}{\partial y}= xz+2\lambda y=0$$

$$\frac{\partial\varphi}{\partial z} = xy + 2\lambda z = 0$$

& $x^2 + y^2 + z^2 - 1 = 0$

Multiplying the first three equations by x, y & z respectively, adding and using the fourth equation, we find

$$\lambda = -\frac{3xyz}{2}$$

using this relation we find that $(0,0,\pm1)$, $(0,\pm1,0)$, $(\pm1,0,0)$ and $\left(\pm\frac{1}{\sqrt{3}},\pm\frac{1}{\sqrt{3}},\pm\frac{1}{\sqrt{3}}\right)$ are the critical points.

❖ ***Question***

Find the critical points of $w = xyz$, where $x^2 + y^2 = 1$ & $x - z = 0$. Also test for maxima and minima.

Solution

Consider $F = xyz + \lambda_1\left(x^2 + y^2 + 1\right) + \lambda_2\left(x - z\right)$

For the critical points, we have

$F_x = yz + 2\lambda_1 x + \lambda_2 = 0$ ……….. (i)

$F_y = xz + 2\lambda_1 y = 0$ …………...….. (ii)

$F_z = xy - \lambda_2 = 0$ …………...…….. (iii)

and $x^2 + y^2 = 1$ ………...……… (iv)

$x - z = 0$ …………………. (v)

From (iii), $\lambda_2 = xy$ & from (ii) $\lambda_1 = -\frac{xz}{2y}$

Use these values in equation (i) to have

$$yz - \frac{x^2 z}{y} + xy = 0$$

$$\Rightarrow\ y^2 z - x^2 z + xy^2 = 0$$

$\because$ $x = z$ from (v)

$\therefore$ $y^2 x - x^3 + xy^2 = 0 \quad \Rightarrow\ 2xy^2 - x^3 = 0$

But $y^2 = 1 - x^2$, from (iv)

$\therefore$ $2x\left(1 - x^2\right) - x^3 = 0 \quad \Rightarrow\ 2x - 3x^3 = 0 \quad \Rightarrow\ x = 0\ ,\ \pm\sqrt{\frac{2}{3}}$

This implies the critical points are $\left(\pm\sqrt{\frac{2}{3}},\frac{1}{\sqrt{3}},\pm\sqrt{\frac{2}{3}}\right)$, $\left(\pm\sqrt{\frac{2}{3}},-\frac{1}{\sqrt{3}},\pm\sqrt{\frac{2}{3}}\right)$, $(0,1,0)$, $(0,-1,0)$

$A = F_{xx} = 2\lambda_1$

$B = F_{xy} = z$

$C = F_{yy} = 2\lambda_1$

$B^2 - AC = z^2 - 4\lambda_1^2$

$$= z^2 - 4\frac{x^2 z^2}{4y^2} = \frac{z^2\left(y^2 - x^2\right)}{y^2}$$

(i) At $\left(\pm\sqrt{\frac{2}{3}},\frac{1}{\sqrt{3}},\pm\sqrt{\frac{2}{3}}\right)$, we have

$$B^2 - AC = \frac{\frac{2}{3}\left(\frac{1}{3} - \frac{2}{3}\right)}{\frac{1}{3}} < 0$$

$$\& \quad A = F_{xx} = 2\lambda_1 = -\frac{xz}{y} = -\left(\frac{\frac{2}{3}}{\frac{1}{\sqrt{3}}}\right) < 0$$

$\Rightarrow$ function has maximum value at $\left(\pm\sqrt{\frac{2}{3}}, \frac{1}{\sqrt{3}}, \pm\sqrt{\frac{2}{3}}\right)$

Similarly, we can show that F is also maximum at $(0,-1,0)$ and is minimum at remaining points. (Check yourself)

❖ ***Question***

Find the point of the curve

$$x^2 - xy + y^2 - z^2 = 1 \ , \ \ x^2 + y^2 = 1$$

which is nearest to the origin.

Solution

Let a point on a given curve be (x, y, z)

Implies that we are to minimize the function

$$f = d^2 = x^2 + y^2 + z^2$$

subject to the conditions

$$x^2 - xy + y^2 - z^2 = 1$$
$$x^2 + y^2 = 1$$

Consider

$$F = x^2 + y^2 + z^2 + \lambda_1\left(x^2 - xy + y^2 - z^2 - 1\right) + \lambda_2\left(x^2 + y^2 - 1\right)$$

For the critical points

$$F_x = 2x(1 + \lambda_1 + \lambda_2) - \lambda_1 y = 0 \ \ldots\ldots\ldots\ldots \ (i)$$
$$F_y = 2y(1 + \lambda_1 + \lambda_2) - \lambda_1 x = 0 \ \ldots\ldots\ldots\ldots \ (ii)$$
$$F_z = 2z(1 - \lambda_1) = 0 \ \ldots\ldots\ldots\ldots \ (iii)$$
$$x^2 - xy + y^2 - z^2 = 1 \ \ldots\ldots\ldots\ldots \ (iv)$$
$$x^2 + y^2 = 1 \ \ldots\ldots\ldots\ldots\ldots \ (v)$$

From equation (iii), we have

$$z = 0 \quad \text{and} \quad \lambda_1 = 1$$

Put $z = 0$ in equation (iv), gives

$$x^2 - xy + y^2 - 1 = 0$$
$$\Rightarrow \ xy = x^2 + y^2 - 1$$
$$\Rightarrow \ xy = 0 \quad \text{by} \ (v)$$
$$\Rightarrow \ x = 0 \ \text{ or } \ y = 0 \ \text{ or both are zero.}$$

$z = 0$, $x = 0$ in (v) gives, $y^2 = 1 \ \Rightarrow \ y = \pm 1$

$\Rightarrow$ $(0, \pm 1, 0)$ are the critical points.

$z = 0$, $y = 0 \ \Rightarrow \ x = \pm 1 \ \Rightarrow \ (\pm 1, 0, 0)$ are the critical points.

We can not take $x = 0$, $y = 0$ at the same time, because it gives $(0,0,0)$ which is origin itself as a critical point.

$\because$ $d^2 = 1$ at all these four points.

$\therefore$ these are the required point at which function is nearest to origin.

❖ ***Question***

Find the point on the curve

$$x^2 + y^2 + z^2 = 1$$

which is farthest from the point $(1,2,3)$

Solution

We are the maximize the function

$$f = (x-1)^2 + (y-2)^2 + (z-3)^2$$

subject to the condition

$$x^2 + y^2 + z^2 = 1$$

Let

$$F = (x-1)^2 + (y-2)^2 + (z-3)^2 + \lambda\left(x^2 + y^2 + z^2 - 1\right)$$

For the critical points, we have

$$x - 1 + \lambda x = 0 \;\ldots\ldots\ldots\ldots\; (i)$$

$$y - 2 + \lambda y = 0 \;\ldots\ldots\ldots\ldots\; (ii)$$

$$z - 3 + \lambda z = 0 \;\ldots\ldots\ldots\ldots\; (iii)$$

$$\& \quad x^2 + y^2 + z^2 = 1 \;\ldots\ldots\ldots\ldots\; (iv)$$

$$\Rightarrow \; x = \frac{1}{1+\lambda} \;,\; y = \frac{2}{1+\lambda} \;,\; z = \frac{1}{3+\lambda}$$

Putting in (iv)

$$\left(\frac{1}{1+\lambda}\right)^2 (1+4+9) = 1 \quad \Rightarrow \; (1+\lambda)^2 = 14 \quad \Rightarrow \; \lambda = -1 \pm \sqrt{14}$$

$$\Rightarrow \; x = \frac{1}{\pm\sqrt{14}} \;,\; y = \frac{2}{\pm\sqrt{14}} \;,\; z = \frac{3}{\pm\sqrt{14}}$$

$\Rightarrow$ critical points are

$$\left(\pm\frac{1}{\sqrt{14}}, \pm\frac{2}{\sqrt{14}}, \pm\frac{3}{\sqrt{14}}\right)$$

Its clear that the required point which is farthest from the point $(1,2,3)$ is $\left(-\frac{1}{\sqrt{14}}, -\frac{2}{\sqrt{14}}, -\frac{3}{\sqrt{14}}\right)$

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❖ *Directional Derivative*

i) Let $f: V \to \mathbb{R}$, where $V \subset \mathbb{R}^n$, is nhood of $\underline{a} \in \mathbb{R}^n$. Then the directional derivative $D_\beta f$ at $\underline{a}$ in the direction of $\underline{\beta} \in \mathbb{R}^n$, is defined by the limit, if it exists,

$$D_\beta f(\underline{a}) = \lim_{h\to 0} \frac{f\left(\underline{a} + h\underline{\beta}\right) - f(\underline{a})}{h}$$

ii) The directional derivative of $f\left(x_1, x_2, ..., x_i, ..., x_n\right)$ at $\underline{a} = \left(a_1, a_2, ..., a_i, ..., a_n\right)$ in the direction of the unit vector $\left(0,0,...,1,0,0,...,0\right)$ is called partial derivative of f at $\underline{a}$ w.r.t. the *i*th component x_i and is denoted by

$$D_i f(\underline{a}) \quad \text{or} \quad \frac{\partial f(\underline{a})}{\partial x} \quad \text{or} \quad f_{x_i}(a)$$

where $D_i f(\underline{a}) = \lim_{h\to 0} \dfrac{f\left(a_1, a_2, ..., a_i + h, ..., a_n\right) - f\left(a_1, a_2, ..., a_i, ..., a_n\right)}{h}$

❖ *Example*

Let $f(x,y) = x^2 + y^2 + x + y$, then f has a directional derivative in every direction and at every point in $\mathbb{R}^2$.

Since, if $\beta = \left(a,b\right) \in \mathbb{R}^2$, we have

$$\begin{aligned} D_\beta f\left(x,y\right) &= \lim_{h\to 0} \frac{\left(x+ha\right)^2 + \left(y+hb\right)^2 + \left(x+ha\right) + \left(y+hb\right) - x^2 - y^2 - x - y}{h} \\ &= \lim_{h\to 0}\left(2ax + 2by + ha^2 + hb^2 + a + b\right) \\ &= 2\left(ax + by\right) + a + b \end{aligned}$$

❖ *Exercise*

Let $\quad f(x,y) = \begin{cases} \dfrac{xy\left(x^2 - y^2\right)}{x^4 + y^4} & ; \quad x^4 + y^4 \neq 0 \\ 0 & ; \quad \left(x,y\right) \neq (0,0) \end{cases}$

Note that if $\beta = \left(a,b\right) \in \mathbb{R}^2$,

$$\begin{aligned} D_\beta f(0,0) &= \lim_{h\to 0} \frac{\left(0+ah\right)\left(0+bh\right)\left[\left(0+ah\right)^2 - \left(0+bh\right)^2\right]}{h\left[\left(0+ah\right)^4 + \left(0+bh\right)^4\right]} \\ &= \lim_{h\to 0} \frac{ab\left(a^2 - b^2\right)}{h\left(a^4 + b^4\right)} \end{aligned}$$

This limit obviously exists only if $\beta = \left(1,0\right)$ or $\left(0,1\right)$. Hence the directional derivatives of f at $\left(0,0\right)$ that exists are the partial derivatives f_x and f_y given by $f_x = 0$, $f_y = 0$.

❖ *Example*

Let

$$f(x,y) = \begin{cases} \dfrac{xy^2}{x^4 + y^4} & ; \quad \left(x,y\right) = \left(0,0\right) \\ 0 & ; \quad \left(x,y\right) \neq \left(0,0\right) \end{cases}$$

It is discontinuous at $\left(0,0\right)$. To see it, note that

$$\lim_{(x,y)\to(0,0)} f\left(x,y\right) \text{ is zero along } y = 0 \text{ and is } \frac{1}{2} \text{ along } y^2 = x.$$

However, if $\beta=(a,b)$, then

$$f_\beta(0,0)=\lim_{h\to 0}\frac{(0+ah)(0+bh)^2}{h\left[(0+ah)^2+(0+bh)^4\right]}$$

$$=\lim_{h\to 0}\frac{ah\cdot b^2h^2}{h\left[a^2h^2+b^4h^4\right]}\quad=\lim_{h\to 0}\frac{ab^2}{a^2+h^2b^4}$$

$$=\begin{cases} b^2/a & , \ a\neq 0 \\ 0 & , \ a=0\end{cases}$$

Hence the directional derivative of f at $(0,0)$ exists in every direction.

❖ *Question*

Let $z=f(x,y)$, $x=u^2-v^2$, $y=2uv$. Then show that

$$\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2=\frac{1}{4(u^2+v^2)}\left\{\left(\frac{\partial z}{\partial u}\right)^2+\left(\frac{\partial z}{\partial v}\right)^2\right\}$$

Solution

We have

$$\frac{\partial x}{\partial u}=2u\quad,\quad\frac{\partial x}{\partial v}=-2v\quad,\quad\frac{\partial y}{\partial u}=2v\quad,\quad\frac{\partial y}{\partial v}=2v$$

Also

$$1=2u\frac{\partial u}{\partial x}-2v\frac{\partial v}{\partial x}\quad,\quad 0=2u\frac{\partial u}{\partial y}-2v\frac{\partial v}{\partial y}$$

and

$$0=2v\frac{\partial u}{\partial x}+2u\frac{\partial v}{\partial x}\quad,\quad 1=2v\frac{\partial u}{\partial y}+2u\frac{\partial v}{\partial y}$$

Solving these four equations for $\frac{\partial u}{\partial x},\frac{\partial v}{\partial x},\frac{\partial u}{\partial y}$ & $\frac{\partial v}{\partial y}$, we get

$$\frac{\partial u}{\partial x}=\frac{u}{2(u^2+v^2)}\quad,\quad\frac{\partial v}{\partial x}=\frac{-v}{2(u^2+v^2)}$$

$$\frac{\partial u}{\partial y}=\frac{v}{2(u^2+v^2)}\quad,\quad\frac{\partial v}{\partial y}=\frac{u}{2(u^2+v^2)}$$

And

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}$$

$$=\frac{1}{2(u^2+v^2)}\left[u\cdot\frac{\partial z}{\partial u}-v\cdot\frac{\partial z}{\partial v}\right]$$

&

$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}$$

$$=\frac{1}{2(u^2+v^2)}\left[v\cdot\frac{\partial z}{\partial u}+u\cdot\frac{\partial z}{\partial v}\right]$$

Hence

$$\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2=\frac{1}{4(u^2+v^2)}\left\{\left(\frac{\partial z}{\partial u}\right)^2+\left(\frac{\partial z}{\partial v}\right)^2\right\}$$

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❖ ***Question***

Let $f:\mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(x,y)=\begin{cases}\dfrac{xy}{x^2+y^2} & ;\ (x,y)\neq(0,0)\\ 0 & ;\ (x,y)=(0,0)\end{cases}$$

Show that f_x, f_y exist at $(0,0)$ but f is discontinuous at $(0,0)$.

Solution

$$f_\beta(0,0)=\lim_{h\to 0}\frac{(ah)(bh)}{h\left[(ah)^2+(bh)^2\right]} \qquad \text{where } \beta=(a,b)$$

$$=\lim_{h\to 0}\frac{ab}{h\left(a^2+b^2\right)}$$

Which exists only when $\beta=(1,0)$ or $(0,1)$.

$\Rightarrow$ f_x & f_y exist at $(0,0)$

Now

$$\lim_{(x,y)\to(0,0)} f(x,y)=\lim_{(x,y)\to(0,0)}\frac{xy}{x^2+y^2}$$

Let $y=mx$, then

$$\lim_{(x,y)\to(0,0)}\frac{xy}{x^2+y^2}=\lim_{x\to 0}\frac{mx^2}{x^2+m^2x^2}$$

$$=\lim_{x\to 0}\frac{m}{1+m^2}$$

Which is different for different m.

$\Rightarrow$ $f(x,y)$ is discontinuous at $(0,0)$.

❖ ***Question***

Let $f:\mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(x,y)=\begin{cases}\dfrac{x^2y}{x^4+y^2} & ;\ (x,y)\neq(0,0)\\ 0 & ;\ (x,y)=(0,0)\end{cases}$$

Show that f_x, f_y exist at $(0,0)$ but f is discontinuous at $(0,0)$.

Solution

$$f_\beta(0,0)=\lim_{h\to 0}\frac{\left(a^2h^2\right)(bh)}{h\left[a^4h^4+b^2h^2\right]} \quad , \quad \beta=(a,b)$$

$$=\lim_{h\to 0}\frac{a^2b}{a^4h^2+b^2}$$

$$=\begin{cases}\dfrac{a^2}{b} & ,\ b\neq 0\\ 0 & ,\ b=0\end{cases}$$

Now $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ is zero along $x=0$ and is $\dfrac{1}{2}$ along $y=x^2$

$\Rightarrow$ it is discontinuous at $(0,0)$.

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❖ *Question*

Find the greatest volume of the box contained in the ellipsoid $3x^2+2y^2+z^2=18$, when each of its edges is parallel to one of the coordinate axes.

Solution

$$V = \text{volume of the box} = (12x)(2y)(2z) = 8xyz$$

We need to find maximum of V subject to $3x^2+2y^2+z^2-18=0$

Consider $\varphi(x,y,z) = 8xyz + \lambda(3x^2+2y^2+z^2-18) = 0$

Then

$$\varphi_x = 8yz + 6\lambda x = 0$$
$$\varphi_y = 8xz + 4\lambda y = 0$$
$$\varphi_z = 8xy + 2\lambda z = 0$$
$$\Rightarrow\ 4xyz + 3\lambda x^2 = 0$$
$$2xyz + \lambda y^2 = 0$$
$$4xyz + \lambda z^2 = 0$$
$$\Rightarrow\ \lambda(3x^2 - 2y^2) = 0$$
$$\lambda(3x^2 - z^2) = 0$$
$$\Rightarrow\ x^2 = \frac{2y^2}{3} = \frac{z^2}{3}$$

Substituting these values in

$$3x^2+2y^2+z^2-18=0$$

We get

$$3x^2+3x^2+3x^2=18 \qquad \Rightarrow\ 9x^2=18$$
$$\Rightarrow\ x=\sqrt{2}\ ,\ \ y=\sqrt{3}\ \text{ and }\ z=\sqrt{6}$$

Which gives

$$f(x,y,z) = 8xyz = 48$$

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❖ *Definition*

If $f:\mathbb{R}^n \to \mathbb{R}$, $\underline{a} \in \mathbb{R}^n$ then

$$\nabla f(\underline{a}) = \sum_{k=1}^{n} \frac{\partial f(\underline{a})}{\partial x_k} = \frac{\partial f(\underline{a})}{\partial x_1} + \frac{\partial f(\underline{a})}{\partial x_2} + ... + \frac{\partial f(\underline{a})}{\partial x_n}$$

❖ *Definition*

Let $f:G \to \mathbb{R}$, G is an open set in $\mathbb{R}^n$.

i) f is said to have a local maximum at $\underline{a} \in G$, if there is a nhood $V_\varepsilon(\underline{a})$ such that $f(\underline{x}) \le f(\underline{a}) \ \forall \ \underline{x} \in V_\varepsilon$.

ii) f is said to have a local minimum at $\underline{a} \in G$, if there is a nhood $V_\varepsilon(\underline{a})$ such that $f(\underline{x}) \ge f(\underline{a}) \ \forall \ \underline{x} \in V_\varepsilon$.

❖ *Theorem*

Let $f:G \to \mathbb{R}$, G is an open set in $\mathbb{R}^n$. If f has a local extremum at $\underline{a} \in G$, then $\nabla f(\underline{a}) = 0$.

Proof

It is clear that $\nabla f(\underline{a}) = 0$ iff $\dfrac{\partial(\underline{a})}{\partial x_i} = 0$, $i = 1,2,3,....,n$

Write $f(x_i + t) = f(x_1, x_2, ..., x_i + t, ..., x_n) = f(\underline{x})$

If f has a local maximum at $\underline{a}$, then

$$\frac{f(a_i + t) - f(a_i)}{t} \le 0 \quad \text{if} \quad t > 0$$

$$\Rightarrow \lim_{t \to 0} \frac{f(a_i + t) - f(a_i)}{t} \le 0 \quad \text{if} \quad t > 0$$

So that $\dfrac{\partial f(\underline{a})}{\partial x_i} \le 0$

Similarly,

$$\lim_{t \to 0} \frac{f(a_i + t) - f(a_i)}{t} \ge 0 \quad \text{if} \quad t < 0$$

So that $\dfrac{\partial f(\underline{a})}{\partial x_i} \ge 0$

Hence $\dfrac{\partial f(\underline{a})}{\partial x_i} = 0$, $i = 1,2,3,....,n$

$$\Rightarrow \nabla f(\underline{a}) = 0$$

Note

There are situations when $\nabla f(\underline{a}) = 0$ but f has no local maximum or minimum at $\underline{a}$. If so and if the sign of $f(\underline{x}) - f(\underline{a})$ depends upon the direction of $\underline{x}$ and $\underline{a}$, f is said to have a saddle point at a.

={ END }=

Maxima and Minima for Functions with side conditions. Lagrange's Multiplier.

Question

Find the critical points of $w = xyz$ subject to the condition $x^2 + y^2 + z^2 = 1$.

Solution

We form the function

$$j = f + l\,g = xyz + l\,(x^2 + y^2 + z^2 - 1)$$

and obtain four equations

$$\frac{\partial j}{\partial x} = yz + 2\,l\,x = 0$$

$$\frac{\partial j}{\partial y} = xz + 2\,l\,y = 0$$

$$\frac{\partial j}{\partial z} = xy + 2\,l\,z = 0$$

& $\quad x^2 + y^2 + z^2 - 1 = 0$

Multiplying the first three equations by x, y, z respectively, adding and using in fourth equation we find $\;l = -\dfrac{3xyz}{2}$.

Using this relation we have $(0,0,\pm 1)$, $(0,\pm 1,0)$, $(\pm 1,0,0)$, and $\left(\pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}\right)$ as the critical points.

Question

Find the critical points of the function $\quad z = x^2 + 24xy + 8y^2$ where $x^2 + y^2 = 25$. Test for maxima & minima.

Solution

$$F(x, y, l) = x^2 + 24xy + 8y^2 + l\,(x^2 + y^2 - 25)$$

$F_x = 2x + 24y + 2\,l\,x = 0$ ……………. (i)

$F_y = 24x + 16y + 2\,l\,y = 0$ …………… (ii)

& $\qquad x^2 + y^2 - 25 = 0$ ………………….. (iii)

$(i) \;\Rightarrow\; (1 + l)x + 12y = 0$ ……….…. (iv)

$(ii) \;\Rightarrow\; 12x + (8 + l)y = 0$ ………..…. (v)

Multiplying equation (iv) by 12, (v) by $(1 + l)$ and adding

$$12(1 + l)x + 144y = 0$$

$$\underset{-}{}12(1 + l)x \underset{-}{+} (1 + l)(8 + l)y = 0$$

$$144y - (1 + l)(8 + l)y = 0$$

$\Rightarrow\; y = 0 \quad$ or $\quad l^2 + 9l - 136 = 0$

$\Rightarrow\; y = 0,\quad l = 8,\ -17$

From (ii), $\quad y = 0 \quad \Rightarrow \quad x = 0$

$\because\; (0,0)$ does not satisfy (iii) $\;\therefore\;$ It is not a critical point.

$l = 8 \quad \Rightarrow\; x = -\dfrac{4y}{3} \qquad$ form (iv)

Put this value of x in (iii)

$\Rightarrow\ \frac{16y^2}{9}+y^2=25 \quad \Rightarrow\ y=\pm 3$

$\Rightarrow\ (-4,3)$ & $(4,-3)$ are the critical points.

Similarly when $l=-17$, we have $x=\frac{3y}{4}$ from (iv)

And putting the value of x in (iii) we get $y=\pm 4$

$\Rightarrow\ (\pm 3,\pm 4)$ are the other two critical points.

$A=F_{xx}=2+2l$

$B=F_{xy}=24$

$C=F_{yy}=16+2l$

When $l=8$

$A=2+16=18$, $B=24$, $C=16+16=32$

and so $B^2-AC=576-576=0$

$F(x,y,l)=x^2+24xy+8y^2+8(x^2+y^2-25)$ when $l=8$

$\Rightarrow\ F(x,y)=9x^2+24xy+16y^2-200$

At $(-4,3)$

$$\begin{aligned}\Delta F &= F(-4+h,3+h)-F(-4,3)\\ &=9(-4+h)^2+24(-4+h)(3+h)+16(3+h)^2-200\\ &\qquad -9(-4)^2+24(-4)(-3)-16(3)^2+200\\ &=9(16-8h+h^2)+24(h^2-h-12)+16(9+6h+h^2)\\ &\qquad -144+288-144\\ &=144-72h+9h^2+24h^2-24h-288+144+96h+26h^2\\ &\qquad -144+288-144\\ &=49h^2\geq 0\end{aligned}$$

$\Rightarrow\ (-4,3)$ is the point of minimum value.

Similarly $(4,-3)$ gives a point of minimum value.

And when $l=-17$, $(\pm 3,\pm 4)$ are the point of maximum value.

Question

Find the critical points of $w=x+z$, where $x^2+y^2+z^2=1$. Test for a maxima and minima.

Solution

Consider the function

$F(x,y,z)=x+z+l(x^2+y^2+z^2-1)$

$F_x=1+2lx$, $F_y=2ly$, $F_z=1+2lz$

For critical points, we have

$1+2lx=0$ …………..…. (i)

$2ly=0$ ………………..…. (ii)

$1+2lz=0$ …………….. (iii)

and $x^2+y^2+z^2=1$ ………… (iv)

Solving these equations we have $l=\pm\frac{1}{\sqrt{2}}$

$l=\frac{1}{\sqrt{2}}$ gives $\left(\frac{-1}{\sqrt{2}},0,\frac{-1}{\sqrt{2}}\right)$ as the critical point.

$l=-\frac{1}{\sqrt{2}}$ gives $\left(\frac{1}{\sqrt{2}},0,\frac{1}{\sqrt{2}}\right)$ as the critical point.

$A = F_{xx} = 2l$, $B = F_{xy} = 0$, $C = F_{yy} = 2l$

i) $l = \frac{1}{\sqrt{2}} \Rightarrow A = \sqrt{2}$, $B = 0$, $C = \sqrt{2}$

so $B^2 - AC = 0 - 2 < 0$ and $A > 0$

$\Rightarrow \left(\frac{-1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}} \right)$ is a point of relative minimum value.

ii) $l = -\frac{1}{\sqrt{2}} \Rightarrow A = -\sqrt{2}$, $B = 0$, $C = -\sqrt{2}$

so $B^2 - AC = 0 - 2 < 0$ and $A < 0$

$\Rightarrow \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right)$ is a point of relative maximum value.

Question

Find the critical points of $w = xyz$ where $x^2 + y^2 = 1$ & $x - z = 0$. Test for the maxima and minima.

Solution

Consider $F = xyz + l_1(x^2 + y^2 - 1) + l_2(x - z)$

For critical points

$F_x = yz + 2l_1 x + l_2 = 0$ ………… (i)

$F_y = xz + 2l_1 y = 0$ …………….. (ii)

$F_z = xy - l_2 = 0$ ……………….. (iii)

& $x^2 + y^2 = 1$ ………..……. (iv)

$x - z = 0$ …………….……. (v)

From (iii) $l_2 = xy$ and from (ii) $l_1 = -\frac{xz}{2y}$

Putting in (i), we get

$$yz - \frac{x^2 z}{y} + xy = 0$$

$$\Rightarrow y^2 z - x^2 z + xy^2 = 0$$

$\because x = z$ form (iv) $\therefore y^2 x - x^3 + xy^2 = 0$

$$\Rightarrow 2xy^2 - x^3 = 0$$

But from (iv), $y^2 = 1 - x^2 \Rightarrow 2x(1 - x^2) - x^3 = 0$

$$\Rightarrow 3x^3 - 2x = 0 \Rightarrow x(3x^2 - 2) = 0 \Rightarrow x = 0, \pm\sqrt{\frac{2}{3}}$$

$\Rightarrow$ The critical points are $\left(\pm\sqrt{\frac{2}{3}}, \frac{1}{\sqrt{3}}, \pm\sqrt{\frac{2}{3}} \right)$, $\left(\pm\sqrt{\frac{2}{3}}, \frac{-1}{\sqrt{3}}, \pm\sqrt{\frac{2}{3}} \right)$, $(0,1,0)$ and $(0,-1,0)$.

$A = F_{xx} = 2l_1$, $B = F_{xy} = z$, $C = F_{yy} = 2l_1$

$B^2 - AC = z^2 - 4l_1^2$

From (ii) $l_1^2 = \frac{x^2 z^2}{4y^2} \Rightarrow B^2 - AC = z^2 - \frac{z^2 x^2}{y^2} = \frac{z^2(y^2 - x^2)}{y^2}$

i) At $\left(\pm\sqrt{\frac{2}{3}}, \frac{1}{\sqrt{3}}, \pm\sqrt{\frac{2}{3}} \right)$, we have $B^2 - AC = \frac{\frac{2}{3}\left(\frac{1}{3} - \frac{2}{3}\right)}{1/3} < 0$

And $A = F_{xx} = 2l_1 = -\frac{xz}{y} = \frac{-2/3}{1/\sqrt{3}} < 0$

$\Rightarrow$ Function is maximum at $\left(\pm\sqrt{\frac{2}{3}}, \frac{1}{\sqrt{3}}, \pm\sqrt{\frac{2}{3}}\right)$.

Similarly we can show that w is maximum at $(0,-1,0)$ and minimum at $\left(\pm\sqrt{\frac{2}{3}}, \frac{-1}{\sqrt{3}}, \pm\sqrt{\frac{2}{3}}\right)$ & $(0,1,0)$.

Question

Find the point to the curves $x^2 - xy + y^2 - z^2 = 1$, $x^2 + y^2 = 1$ nearest to the origin $(0,0,0,)$.

Solution

Let (x, y, z) be a point on the curve. Then its distance from the origin is given by $\sqrt{x^2 + y^2 + z^2}$

We are to minimize $f = d^2 = x^2 + y^2 + z^2$

subject to the conditions $x^2 - xy + y^2 - z^2 = 1$, $x^2 + y^2 = 1$

Consider

$$F = x^2 + y^2 + z^2 + l_1\left(x^2 - xy + y^2 - z^2 - 1\right) + l_2\left(x^2 + y^2 - 1\right)$$

$$F_x = 2x + (2x - y)\,l_1 + 2\,l_2 x$$

$$F_y = 2y + (2y - x)\,l_1 + 2\,l_2 y$$

$$F_z = 2z + l_1(-2z)$$

For critical points, we have

$$2x\left(1 + l_1 + l_2\right) - l_1 y = 0 \quad \ldots\ldots\ldots\ldots \ (i)$$

$$2y(1 + l_1 + l_2) - l_1 x = 0 \quad \ldots\ldots\ldots\ldots (ii)$$

$$2z(1 - l_1) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \ (iii)$$

$$x^2 - xy + y^2 - z^2 - 1 = 0 \quad \ldots\ldots\ldots\ldots \ (iv)$$

$$x^2 + y^2 - 1 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \ (v)$$

From (iii), we have $z = 0$ & $l_1 = 1$.

$z = 0$ in (iv) gives $x^2 - xy + y^2 - 1 = 0 \Rightarrow xy = x^2 + y^2 - 1$

But $x^2 + y^2 - 1 = 0 \Rightarrow xy = 0$

$\Rightarrow x = 0$ or $y = 0$ or *both are zero.*

We can not take $x = 0$, $y = 0$ at a same time because it gives $(0,0,0)$ which is origin itself.

$z = 0$, $x = 0$ in (v) $\Rightarrow y^2 = 1 \Rightarrow y = \pm 1$

$\Rightarrow (0, \pm 1, 0)$ are the critical points

& $z = 0$, $y = 0$ in (v) $\Rightarrow x^2 = 1 \Rightarrow x = \pm 1$

$\Rightarrow (\pm 1, 0, 0)$ are the other critical points.

$\because$ $f = d^2 = 1$ at these four points

$\therefore$ These are the required points at which function is nearest to origin.

Remarks

Question

Find the shortest distance from the origin to the curve

$$x^2+8xy+7y^2=225$$

Solution

We are to find the minimum value of $f=d^2=x^2+y^2$

subject to the condition $x^2+8xy+7y^2=225$.

Consider $F=x^2+y^2+l(x^2+8xy+7y^2-225)$

$F_x=2x+l(2x+8y)$

$F_y=2y+l(8x+14y)$

For critical points

$$x+l(x+4y)=0 \;\ldots\ldots\ldots\ldots\ldots\; (i)$$

$$y+l(4x+7y)=0 \;\ldots\ldots\ldots\ldots\; (ii)$$

$$x^2+8xy+7y^2-225=0 \;\ldots\ldots\; (iii)$$

$(i) \Rightarrow (1+l)x+4ly=0 \quad \Rightarrow \dfrac{x}{y}=-\dfrac{4l}{1+l}$

$(ii) \Rightarrow 4lx+(1+7l)y=0 \quad \Rightarrow \dfrac{x}{y}=-\dfrac{1+7l}{4l}$

$\Rightarrow \dfrac{x}{y}=-\dfrac{4l}{1+l}=-\dfrac{1+7l}{4l} \qquad \Rightarrow 16l^2=(1+l)(1+7l)$

$\Rightarrow 16l^2=1+l+7l+7l^2 \quad \Rightarrow 9l^2-8l-1=0$

$\Rightarrow (l-1)(9l+1)=0 \qquad \Rightarrow l=1,\ -\dfrac{1}{9}$

$l=1 \Rightarrow \dfrac{x}{y}=-2 \Rightarrow x=-2y$

Putting this value of x in equation (iii) we have

$$(-2y)^2+8(-2y)y+7y^2=225$$

$$\Rightarrow 4y^2-16y^2+7y^2=225 \quad \Rightarrow -5y^2=225$$

which gives imaginary values of y.

$l=-\dfrac{1}{9} \quad \Rightarrow \dfrac{x}{y}=-\dfrac{4\left(-\frac{1}{9}\right)}{1-\frac{1}{9}}=\dfrac{\frac{4}{9}}{\frac{8}{9}}=\dfrac{1}{2} \quad \Rightarrow y=2x$

Putting in (iii), we have

$$x^2+8x(2x)+7(2x)^2=225$$

$$\Rightarrow x^2+16x^2+28x^2=225$$

$$\Rightarrow 45x^2=225 \Rightarrow x^2=5 \Rightarrow x=\pm\sqrt{5}$$

$x=\sqrt{5} \quad \Rightarrow y=2\sqrt{5}$

& $\quad x=-\sqrt{5} \Rightarrow y=-2\sqrt{5}$

$\therefore$ The critical points are $\left(\sqrt{5},2\sqrt{5}\right)$ & $\left(-\sqrt{5},-2\sqrt{5}\right)$.

$$d^2_{(\pm\sqrt{5},\pm 2\sqrt{5})}=25$$

$\Rightarrow$ Shortest distance $=$ $d=5$

Remarks

Question

Find a point (x,y,z) on the sphere $x^2+y^2+z^2=1$ which is farthest from the point $(1,2,3)$.

Solution

We are to maximize

$$f(x,y,z)=(x-1)^2+(y-2)^2+(z-3)^2$$

subject to the condition $x^2+y^2+z^2=1$

Let $F=(x-1)^2+(y-2)^2+(z-3)^2+l\,(x^2+y^2+z^2-1)$

For critical points

$$F_x=2(x-1)+2\,l\,x=0$$
$$F_y=2(y-2)+2\,l\,y=0$$
$$F_z=2(z-3)+2\,l\,z=0 \quad \text{and} \quad x^2+y^2+z^2=1$$
$$\Rightarrow\ x-1+l\,x=0 \ \ldots\ldots\ldots\ldots\ldots\ (i)$$
$$y-2+l\,y=0 \ \ldots\ldots\ldots\ldots\ldots\ (ii)$$
$$z-3+l\,z=0 \ \ldots\ldots\ldots\ldots\ldots\ (iii)$$
$$x^2+y^2+z^2=1 \ \ldots\ldots\ldots\ldots\ (iv)$$
$$\Rightarrow\ x=\frac{1}{1+l},\quad y=\frac{2}{1+l},\quad z=\frac{3}{1+l}$$

Putting in (iv)

$$\left(\frac{1}{1+l}\right)^2+\left(\frac{2}{1+l}\right)^2+\left(\frac{3}{1+l}\right)^2=1$$
$$\Rightarrow\ 14=(1+l)^2$$
$$\Rightarrow\ l+1=\pm\sqrt{14} \quad \Rightarrow\ l=-1\pm\sqrt{14}$$
$$\Rightarrow\ x=\frac{1}{\pm\sqrt{14}},\quad y=\frac{2}{\pm\sqrt{14}},\quad z=\frac{3}{\pm\sqrt{14}}$$

Clearly $\left(\frac{-1}{\sqrt{14}},\frac{-2}{\sqrt{14}},\frac{-3}{\sqrt{14}}\right)$ is the point which is farthest from $(1,2,3)$.

Question

Find the extreme values of $z=6-4x-3y$, provided x & y satisfy $x^2+y^2=1$.

Solution

Define $F=6-4x-3y+l\left(x^2+y^2-1\right)$

For critical points, we have

$$F_x=-4+2\,l\,x=0 \ \ldots\ldots\ldots\ldots\ (i)$$
$$F_y=-3+2\,l\,y=0 \ \ldots\ldots\ldots\ldots\ (ii)$$

and $x^2+y^2=1 \ \ldots\ldots\ldots\ldots\ (iii)$

From (i) and (ii) we have $x=\frac{2}{l}$, $y=\frac{3}{2l}$

Putting these values in (iii) we get $l=\pm\frac{5}{2}$

$$l=\frac{5}{2} \quad \Rightarrow\ x=\frac{2}{5/2}=\frac{4}{5} \quad \& \quad y=\frac{3}{2\cdot 5/2}=\frac{3}{5}$$

$l=-\frac{5}{2} \quad \Rightarrow \quad x=\frac{2}{-5/2}=-\frac{4}{5} \quad \& \quad y=\frac{3}{2\cdot(-5/2)}=-\frac{3}{5}$

$\Rightarrow \left(\frac{4}{5},\frac{3}{5}\right)$ & $\left(-\frac{4}{5},-\frac{3}{5}\right)$ are the critical points.

$A=F_{xx}=2l\,, \quad B=F_{xy}=0\,, \quad C=F_{yy}=2l$

$\Rightarrow \; B^2-AC=0-4l^2=-4\left(\pm\frac{5}{2}\right)^2=-25<0$

$\Rightarrow$ F is maximum or minimum at the critical points.

Now at $\left(\frac{4}{5},\frac{3}{5}\right)$, we have $A=5>0$

And at $\left(-\frac{4}{5},-\frac{3}{5}\right)$, we have $A=-5<0$

$\Rightarrow$ The function is min. at $\left(\frac{4}{5},\frac{3}{5}\right)$ and max. at $\left(-\frac{4}{5},-\frac{3}{5}\right)$.

Question

Find the critical point of $f(x,y)=x^2+2y^2+2xy+2x+3y$. Where $x^2-y=1$. Test for maxima and minima.

Solution

Define $F=x^2+2y^2+2xy+2x+3y+l(x^2-y-1)$

For critical points, we have

$F_x=2x+2y+2+2lx=0$ ………. (*i*)

$F_y=4y+2x+3-l=0$ ………… (*ii*)

and $x^2-y-1=0$ ………………. (*iii*)

From (*i*) $\quad l=\frac{-x-y-1}{x}$

From (*ii*) $\quad l=2x+4y+3$

$\Rightarrow \; \frac{-x-y-1}{x}=2x+4y+3$

$\Rightarrow \; -x-y-1=2x^2+4xy+3x$

$\Rightarrow \; 2x^2+4x+4xy+y+1=0$

But from (*iii*) $\quad x^2=1+y$

$\Rightarrow \; 2(1+y)+4x+4xy+y+1=0$

$\Rightarrow \; 4x+4xy+3y+3=0$

$\Rightarrow \; 4x(1+y)+3(y+1)=0$

$\Rightarrow \; (y+1)(4x+3)=0$

$\Rightarrow$ Either $y=-1$ or $x=-\frac{3}{4}$

If $y=-1$, we get $x^2=0$ from (*iii*)

$\Rightarrow$ $(0,-1)$ is a critical point and $l=-1$ in this case.

If $x=-\frac{3}{4}$, we get $\frac{9}{16}-1=y$ i.e. $y=-\frac{7}{16}$

$\Rightarrow \left(-\frac{3}{4},-\frac{7}{16}\right)$ is the other critical point and $l=-\frac{1}{4}$ in this. case.

Now $A = F_{xx} = 2 + 2l$, $B = F_{xy} = 2$, $C = F_{yy} = 4$

$\Rightarrow B^2 - AC = 4 - 4(2 + 2l) = -4 - 8l$

$l = -1 \Rightarrow B^2 - AC = 4 > 0 \Rightarrow f$ is neither maximum nor minimum at $(0,1)$.

$$l = -\frac{1}{4} \Rightarrow B^2 - AC = -4 - 8\left(-\frac{1}{4}\right) = -4 + 2 = -2 < 0$$

and $A = 2 + 2l = 2 + 2\left(-\frac{1}{4}\right) = 2 - \frac{1}{2} > 0$

$\Rightarrow \left(-\frac{3}{4}, -\frac{7}{16}\right)$ is the point of minimum value.

Question

Find the critical points of $z = x^2 + y^2$ when $x^3 + y^3 = 6xy$, Also test for maxima and minima.

Solution

Define $F = x^2 + y^2 + l(x^3 + y^3 - 6xy)$

For critical points we have

$F_x = 2x + 3l x^2 - 6l y = 0$ ………… (i)

$F_y = 2y + 3l y^2 - 6l x = 0$ ……….. (ii)

and $x^3 + y^3 - 6xy = 0$ …………… (iii)

from (i) $l = \dfrac{-2x}{3x^2 - 6y}$

from (ii) $l = \dfrac{-2y}{3y^2 - 6x}$

$$\Rightarrow \frac{-2x}{3x^2 - 6y} = \frac{-2y}{3y^2 - 6x}$$

$\Rightarrow x(3y^2 - 6x) = y(3x^2 - 6y)$

$\Rightarrow x(y^2 - 2x) = y(x^2 - 2y)$

$\Rightarrow xy^2 - 2x^2 = x^2y - 2y^2$

$\Rightarrow x^2y - xy^2 + 2x^2 - 2y^2 = 0$

$\Rightarrow xy(x - y) + 2(x - y)(x + y) = 0$

$\Rightarrow (x - y)(2x + 2y + xy) = 0$

$\Rightarrow$ Either $x - y = 0$ or $2x + 2y + xy = 0$

If $x - y = 0$ then (iii) becomes $x^3 + x^3 - 6x^2 = 0$

$\Rightarrow 2x^3 - 6x^2 = 0 \Rightarrow x^2(x - 3) = 0$

$\Rightarrow x = 0,\ 3$

$\Rightarrow x = 0,\ y = 0$ & $x = 3,\ y = 3$

$\Rightarrow (0,0)$ & $(3,3)$ are the critical points.

At $(0,0)$, $l = \dfrac{-2x}{3x^2 - 6y} = \dfrac{-2x}{3x^2 - 6x}$ $\because x - y = 0 \Rightarrow x = y$

$$= \frac{-2}{3x - 6} = \frac{-2}{3(0) - 6} = \frac{1}{3}$$

And at $(3,3)$, $l = -\dfrac{2}{3}$

$A = F_{xx} = 2 + 6l\,x$
$B = F_{xy} = -6l$
$C = F_{yy} = 2 + 6l\,y$

At $(0,0)$, we have $A = 2$, $B = -2$, $C = 2$

And $\therefore\ B^2 - AC = 0$

Consider $\Delta z = z(h,h) - z(0,0) = h^2 + h^2 = 2h^2 \geq 0$

$\Rightarrow$ $(0,0)$ is the point of minimum value.

At $(3,3)$, we have $A = 2 + 6\left(-\frac{2}{3}\right)(3) = -10$

$B = -6\left(-\frac{2}{3}\right) = 4$

$C = 2 + 6\left(-\frac{2}{3}\right)(3) = -10$

and $\therefore\ B^2 - AC = 16 - 100 < 0$ and $A = -10 < 0$

$\Rightarrow$ $(3,3)$ is a point of maximum value.

Question

Find the points in the plane $2x + 3y - z = 5$ nearest to the origin.

Solution

We are to minimize $f = d^2 = x^2 + y^2 + z^2$

subject to $2x + 3y - z - 5 = 0$.

Define $F = x^2 + y^2 + z^2 + l\,(2x + 3y - z - 5)$

$F_x = 2x + 2l = 0$ ……………..... (i)

$F_y = 2y + 3l = 0$ ……………... (ii)

$F_z = 2z - l = 0$ ………………..(iii)

and $2x + 3y - z - 5 = 0$ …...….. (iv)

$x = -l$, $y = \frac{-3l}{2}$, $z = \frac{l}{2}$ from (i), (ii) & (iii) resp.

(iv) becomes $-2l - \frac{9l}{2} - \frac{l}{2} - 5 = 0$

$\Rightarrow 4l + 9l + l = -10$

$\Rightarrow l = -\frac{10}{14} = -\frac{5}{7}$

$\Rightarrow x = \frac{5}{7}$, $y = \frac{15}{14}$, $z = -\frac{5}{14}$

$\Rightarrow \left(\frac{5}{7}, \frac{15}{14}, \frac{-5}{14}\right)$ is the critical point.

$A = F_{xx} = 2$, $B = F_{xy} = 0$, $C = F_{yy} = 2$

$B^2 - AC = 0 - 4 < 0$ and $A = 2 > 0$

$\Rightarrow$ F is relative minimum at $\left(\frac{5}{7}, \frac{15}{14}, \frac{-5}{14}\right)$ so this is the required point.

Maxima and Minima for Functions of Two Variable

Remarks

Question

Test for maxima and minima

(*i*) $z = 1 - x^2 - y^2$ (*ii*) $z = x^2 + y^2$

(*iii*) $z = xy$ (*iv*) $z = x^3 - 3xy^2$

(*v*) $z = x^2 y^2$ (*vi*) $z = 4 - y^2$

Solution

(*i*) $z = 1 - x^2 - y^2$

$$\frac{\partial z}{\partial x} = -2x \ , \quad \frac{\partial z}{\partial y} = -2y$$

For critical points $\frac{\partial z}{\partial x} = 0 = \frac{\partial z}{\partial y}$

$\Rightarrow \ x = 0, \ y = 0 \quad \Rightarrow \ (0,0)$ is the critical point.

$$A = \frac{\partial^2 z}{\partial x^2} = -2, \quad B = \frac{\partial^2 z}{\partial x \partial y} = 0, \quad C = \frac{\partial^2 z}{\partial y^2} = -2$$

$B^2 - AC = 0 - 4 = -4 < 0$ and $A + C = -2 - 2 = -4 < 0$

$\Rightarrow$ $(0,0)$ is the point of maximum value

and maximum value of z at $(0,0)$ is 1.

(*ii*) *Do yourself as above*

(*iii*) $z = xy$

$$\frac{\partial z}{\partial x} = y \ , \quad \frac{\partial z}{\partial y} = x$$

For critical points $\frac{\partial z}{\partial x} = 0 = \frac{\partial z}{\partial y}$

$\Rightarrow \ y = 0$ and $x = 0 \quad \Rightarrow \ (0,0)$ is the critical point.

$$A = \frac{\partial^2 z}{\partial x^2} = 0, \quad B = \frac{\partial^2 z}{\partial x \partial y} = 1, \quad C = \frac{\partial^2 z}{\partial y^2} = 0$$

$B^2 - AC = (1)^2 - (0)(0) = 1 > 0$

Therefore $(0,0)$ is a saddle point.

(*iv*) $z = x^3 - 3xy^2$

$$\frac{\partial z}{\partial x} = 0 \quad \Rightarrow 3x^2 - 3y^2 = 0 \quad \Rightarrow \ x = -y \ \ \& \ \ x = y$$

$$\frac{\partial z}{\partial y} = 0 \quad \Rightarrow -6xy = 0 \quad \Rightarrow \ xy = 0$$

$\Rightarrow$ either $x = 0$ or $y = 0$ or *both are zero*

$\Rightarrow$ $(0,0)$ is the only critical point.

$$A = \frac{\partial^2 z}{\partial x^2} = 6x = 0 \qquad \text{at } (0,0)$$

$$B = \frac{\partial^2 z}{\partial x \partial y} = -6y = 0 \qquad \text{at } (0,0)$$

$$C = \frac{\partial^2 z}{\partial y^2} = -6x = 0 \qquad \text{at } (0,0)$$

$\Rightarrow\ B^2 - AC = 0 \quad \text{and} \quad A + C = 0$

so we need further consideration for the nature of point.

$$\begin{aligned}\Delta z &= z(0+h, 0+k) - z(0,0)\\ &= z(h,k) - z(0,0)\\ &= z(h,k) \ = h^3 - 3hk\end{aligned}$$

For $h = k$ we have

$$\Delta z = h^3 - 3h^3 = -2h^3 \left| \begin{array}{l} > 0 \quad if \quad h < 0 \\ < 0 \quad if \quad h > 0 \end{array}\right.$$

$\Rightarrow$ $(0,0)$ is a saddle point.

(v) $z = f(x,y) = x^2y^2$

$f_x = 0 \ \Rightarrow\ 2xy^2 = 0, \qquad f_y = 0 \ \Rightarrow\ 2x^2y = 0$

$\Rightarrow$ $(0,0)$ is the critical point.

$$\begin{aligned}A &= f_{xx} = 2y^2 = 0 \quad \text{at } (0,0)\\ B &= f_{xy} = 4xy = 0 \quad \text{at } (0,0)\\ C &= f_{yy} = 2x^2 = 0 \quad \text{at } (0,0)\end{aligned}$$

$\Rightarrow\ B^2 - AC = 0 \quad \text{and} \quad A + C = 0$

so we need further consideration

$$\begin{aligned}\Delta f &= f(x_0 + h, y_0 + h) - f(x_0, y_0)\\ &= f(h,k) - f(0,0) = h^2k^2\end{aligned}$$

If $h = k$, we have

$$\Delta f = h^4 \geq 0 \quad \forall\ h$$

Thus $(0,0)$ is the point of minimum value.

Question

Find the critical points of the following functions and test for maxima and minima.

(a) $z = \sqrt{1 - x^2 - y^2}$

(b) $z = 2x^2 - xy - 3y^2 - 3x + 7y$

(c) $z = 1 + x^2 + y^2$

(d) $z = x^2 - 5xy - y^2$

(e) $z = x^2 - 2xy + y^2$

(f) $z = x^3 - 3xy^2 + y^3$

Solution

(a) $z = \sqrt{1 - x^2 - y^2}$

$$\frac{\partial z}{\partial x} = \frac{1}{2}\left(1 - x^2 - y^2\right)^{-\frac{1}{2}}(-2x) = \frac{-x}{\sqrt{1 - x^2 - y^2}} = 0 \quad \Rightarrow\ x = 0$$

$$\frac{\partial z}{\partial y} = \frac{-y}{\sqrt{1 - x^2 - y^2}} = 0 \quad \Rightarrow\ y = 0$$

$\Rightarrow$ $(0,0)$ is the only critical point.

$$\frac{\partial^2 z}{\partial x^2} = \frac{-\left[\sqrt{1 - x^2 - y^2} - x\cdot\left(\dfrac{-x}{\sqrt{1 - x^2 - y^2}}\right)\right]}{1 - x^2 - y^2}$$

$$= \frac{-\left[1-x^2-y^2+x^2\right]}{\left(1-x^2-y^2\right)^{3/2}} = \frac{-1+y^2}{\left(1-x^2-y^2\right)^{3/2}}$$

$$\Rightarrow \; A = \frac{\partial^2 z}{\partial x^2} = -1 \quad \text{at } (0,0)$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{-y}{\sqrt{1-x^2-y^2}}\right)$$

$$= -y \cdot \left(-\frac{1}{2}\right)\left(1-x^2-y^2\right)^{-\frac{3}{2}}(-2x) = \frac{-xy}{\left(1-x^2-y^2\right)^{\frac{3}{2}}}$$

$$\Rightarrow \; B = \frac{\partial^2 z}{\partial x \partial y} = 0 \quad \text{at } (0,0)$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{-\left[\left(1-x^2-y^2\right)^{\frac{1}{2}}(1) - y\left(\frac{-y}{\sqrt{1-x^2-y^2}}\right)\right]}{1-x^2-y^2} = \frac{-1+x^2}{\left(1-x^2-y^2\right)^{\frac{3}{2}}}$$

$$\Rightarrow \; C = \frac{\partial^2 z}{\partial y^2} = -1 \quad \text{at } (0,0)$$

$$\Rightarrow \; B^2 - AC = 0-(-1)(-1) = -1 < 0 \quad \text{and} \quad A+C = -1-1 = -2 < 0$$

$\Rightarrow$ z has a relative maxima at $(0,0)$.

(*b*) $z = 2x^2 - xy - 3y^2 - 3x + 7y$

$$\frac{\partial z}{\partial x} = 4x - y - 3, \qquad \frac{\partial z}{\partial y} = -x - 6y + 7$$

For critical points $\frac{\partial z}{\partial x} = 0$, $\frac{\partial z}{\partial y} = 0$

$\Rightarrow$ $4x - y - 3 = 0$ …................ (*i*)

& $x + 6y - 7 = 0$ …………… (*ii*)

Multiplying equation (*i*) by 6 and adding in (*ii*)

$$\begin{aligned} 24x - 6y - 18 &= 0 \\ x + 6y - 7 &= 0 \\ \hline 25x \quad - 25 &= 0 \end{aligned}$$

$\Rightarrow x = 1 \quad \Rightarrow y = 1$

$\Rightarrow$ $(1,1)$ is the critical point

$$A = \frac{\partial^2 z}{\partial x^2} = 4, \quad B = \frac{\partial^2 z}{\partial x \partial y} = -1, \quad C = \frac{\partial^2 z}{\partial y^2} = -6$$

$$B^2 - AC = (-1)^2 - (-4)(-6) = 25 > 0$$

$\Rightarrow$ There is a saddle point at $(1,1)$.

(*c*) $z = 1 + x^2 + y^2$

$$\frac{\partial z}{\partial x} = 2x, \qquad \frac{\partial z}{\partial y} = 2y$$

For critical points $\frac{\partial z}{\partial x} = 0$, $\frac{\partial z}{\partial y} = 0$ $\Rightarrow$ $(0,0)$ is the critical point.

$$A=\frac{\partial^2 z}{\partial x^2}=2, \quad B=\frac{\partial^2 z}{\partial x\partial y}=0, \quad C=\frac{\partial^2 z}{\partial y^2}=2$$

$\Rightarrow$ $B^2-AC=(0)^2-(2)(2)=-4<0$ and $A+C=2+2=4>0$

$\Rightarrow$ The function has a relative minima at $(0,0)$.

(*d*) $z=x^2-5xy-y^2$

$$\frac{\partial z}{\partial x}=2x-5y, \qquad \frac{\partial z}{\partial y}=-5x-2y$$

$$\frac{\partial z}{\partial x}=0 \quad \Rightarrow\ 2x-5y=0 \ \ldots\ldots\ldots\ (i)$$

$$\frac{\partial z}{\partial y}=0 \quad \Rightarrow\ -5x-2y=0 \ \ldots\ldots\ldots\ (ii)$$

(*i*) and (*ii*) gives $(0,0)$ is the critical point.

$$A=\frac{\partial^2 z}{\partial x^2}=2, \quad B=\frac{\partial^2 z}{\partial x\partial y}=-5, \quad C=\frac{\partial^2 z}{\partial y^2}=-2$$

$\Rightarrow$ $B^2-AC=(-5)^2-(2)(-2)=25+4=29>0$

$\Rightarrow$ There is a saddle point at $(0,0)$.

(*e*) $z=x^2-2xy+y^2$

$$\frac{\partial z}{\partial x}=2x-2y, \qquad \frac{\partial z}{\partial y}=2y-2x$$

$$\frac{\partial z}{\partial x}=0,\ \frac{\partial z}{\partial y}=0 \quad \Rightarrow\ x-y=0 \quad \Rightarrow\ x=y$$

$\Rightarrow$ Every point on the line $y=x$ is a critical point.

$$A=\frac{\partial^2 z}{\partial x^2}=2, \quad B=\frac{\partial^2 z}{\partial x\partial y}=-2, \quad C=\frac{\partial^2 z}{\partial y^2}=2$$

$\Rightarrow$ $B^2-AC=(-2)^2-(2)(2)=4-4=0$

Consider $\quad \Delta z=z(x+h,y+k)-z(x,y)$

$\because\ x=y \quad \therefore\ \Delta z=z(x+h,x+k)-z(x,x)$

$$=(x+h)^2-2(x+h)(x+k)+(x+k)^2$$

$$=\left[(x+h)-(x+k)\right]^2\geq 0$$

$\Rightarrow$ Each point on the line $y=x$ gives a relative minimum.

(*f*) $z=x^3-3xy^2+y^3$

$$\frac{\partial z}{\partial x}=3x^2-3y^2, \qquad \frac{\partial z}{\partial y}=-6xy+3y^2$$

$$\frac{\partial z}{\partial x}=0 \quad \Rightarrow\ 3x^2-3y^2=0 \ \ldots\ldots\ldots\ldots\ (i)$$

$$\frac{\partial z}{\partial y}=0 \quad \Rightarrow\ -6xy+3y^2=0 \ \ldots\ldots\ldots\ (ii)$$

From (*i*) and (*ii*), we have

$$3x^2-6xy=0 \quad \Rightarrow\ x(x-2y)=0 \quad \Rightarrow\ x=0,\ x=2y$$

Now $x=0 \ \Rightarrow\ y=0$

And $x=2y \ \Rightarrow\ (2y)^2-y^2=0 \ \Rightarrow\ y=0$

Hence (0,0) is the only critical point.

$$A=\frac{\partial^2 z}{\partial x^2}=6x=0 \qquad \text{at } (0,0)$$

$$B=\frac{\partial^2 z}{\partial x\partial y}=-6y=0 \qquad \text{at } (0,0)$$

$$C=\frac{\partial^2 z}{\partial y^2}=-6x+6y=0 \qquad \text{at } (0,0)$$

$\Rightarrow\ B^2-AC=0$

Consider $\Delta z=z(h,k)-z(0,0)$

$$=h^3-3hk^2+k^3 \quad =h^3-3h^3+h^3 \quad \text{when } h=k$$

$$=-h^3\left|\begin{array}{l}<0 \quad when\ \ h>0\\ >0 \quad when\ \ h<0\end{array}\right.$$

$\Rightarrow$ There is a saddle point at $(0,0)$

Note : (i) If for a point $A=B=C=0$ and $\Delta z\geq 0$, then z is minimum at that point and if $\Delta z\leq 0$, then z is maximum at that point.

(ii) If A, B, C are not zero and $B^2-AC=0$ then z is neither maximum nor minimum.

Question

Find the critical points of the following functions and test for maxima and minima.

(a) $z=x^3-2xy^2+y^3$

(b) $z=x^3+y^3-3x-12y+20$

(c) $z=x^3+y^3-63(x+y)+12xy$

(d) $z=xy(a-x-y)$

(e) $z=x^2-2xy+y^2+x^3-y^3+25$

(f) $z=x^2y^2-5x^2-8xy-5y^2$

(g) $z=2(x-y)^2-x^4-y^4$

(h) $z=2(x-y)^3-(x^4-y^4)$

(i) $z=x^2-5xy-y^3$

Solution

(***a***) $z=x^3-2xy^2+y^3$

$$\frac{\partial z}{\partial x}=3x^2-2y^2, \qquad \frac{\partial z}{\partial y}=-4xy+3y^2$$

$$\frac{\partial z}{\partial x}=0 \quad \Rightarrow\ 3x^2-2y^2=0 \ \ldots\ldots\ldots\ldots\ldots.. (i)$$

$$\frac{\partial z}{\partial y}=0 \quad \Rightarrow\ -4xy+3y^2=0 \ \ldots\ldots\ldots\ldots (ii)$$

Adding (i) and (ii), we get

$$3x^2-4xy+y^2=0 \quad \Rightarrow\ 3x^2-3xy-xy+y^2=0$$

$$\Rightarrow\ 3x(x-y)-y(x-y)=0 \quad \Rightarrow\ (x-y)(3x-y)=0$$

If $x-y=0$, then $x=y$ in (i) gives

$$3x^2-2x^2=0 \quad \Rightarrow\ x=0 \quad \Rightarrow\ y=0.$$

And if $3x-y=0$, then $y=3x$ in (i) gives

$$3x^2-2(3x)^2=0 \quad \Rightarrow\ x=0 \quad \Rightarrow\ y=0$$

$\Rightarrow$ $(0,0)$ is the only critical point.

$$A=\frac{\partial^2 z}{\partial x^2}=6x=0 \qquad \text{at } (0,0)$$

$$B=\frac{\partial^2 z}{\partial x\partial y}=-4y=0 \qquad \text{at } (0,0)$$

$$C=\frac{\partial^2 z}{\partial y^2}=6y-4x=0 \qquad \text{at } (0,0)$$

$\Rightarrow$ $A=B=C=0$ at $(0,0)$ and hence $B^2-AC=0$

Now consider $\Delta z=z(h,k)-z(0,0)$

$$=h^3-2hk^2+k^3$$

$$=h^3-2h^3+h^3=0 \qquad \text{when } h=k$$

$\Rightarrow$ The nature of the point is undetermined.

(***b***) $z=x^3+y^3-3x-12y+20$

$$\frac{\partial z}{\partial x}=3x^2-3\ , \qquad \frac{\partial z}{\partial y}=3y^2-12$$

$$\frac{\partial z}{\partial x}=0 \quad \Rightarrow \quad x^2-1=0$$

$$\frac{\partial z}{\partial y}=0 \quad \Rightarrow \quad y^2-4=0$$

$\Rightarrow$ $x=\pm 1$, $y=\pm 2$, and the critical points are

$$(1,2),\ (1,-2),\ (-1,2),\ (-1,-2)$$

$$A=\frac{\partial^2 z}{\partial x^2}=6x, \quad B=\frac{\partial^2 z}{\partial x\partial y}=0, \quad C=\frac{\partial^2 z}{\partial y^2}=6y$$

$\Rightarrow$ $B^2-AC=-36xy$

$B^2-AC=-36(1)(2)=-72<0$ at $(1,2)$

$B^2-AC=-36(1)(-2)=72>0$ at $(1,-2)$

$B^2-AC=-36(-1)(2)=72>0$ at $(-1,2)$

$B^2-AC=-36(-1)(-2)=-72<0$ at $(-1,-2)$

$\Rightarrow$ There is a saddle point at $(1,-2)$ and $(-1,2)$.

$B^2-AC<0$ while $A=6>0$ at $(1,2)$

and $A=-6<0$ at $(-1,-2)$

$\Rightarrow$ z has relative minima at $(1,2)$ & relative maxima at $(-1,-2)$.

(***c***) $z=x^3+y^3-63(x+y)+12xy$

$$\frac{\partial z}{\partial x}=3x^2-63+12y, \qquad \frac{\partial z}{\partial y}=3y^2-63+12x$$

For critical points $\frac{\partial z}{\partial x}=0$, $\frac{\partial z}{\partial y}=0$.

$\Rightarrow$ $3x^2+12y-63=0$ ……………… (i)

& $3y^2+12x-63=0$ ……………….. (ii)

Subtracting (ii) from (i), we get

$$3x^2-3y^2+12y-12x=0$$

$$\Rightarrow\ x^2-y^2+4(y-x)=0$$

$$\Rightarrow\ (x-y)(x+y)-4(x-y)=0$$

$$\Rightarrow\ (x-y)(x+y-4)=0$$

If $x-y=0$ then (i) gives $3x^2+12x-63=0$

$$\Rightarrow x^2+4x-21=0$$

$$\Rightarrow (x+7)(x-3)=0 \quad \Rightarrow x=-7,3$$

$\Rightarrow$ The critical points are $(-7,-7)$ & $(3,3)$.

If $x+y-4=0$ then $x=4-y$

Put this value of x in (ii), we have

$$3y^2+12(4-y)-63=0$$

$$\Rightarrow y^2+4(4-y)-21=0$$

$$\Rightarrow y^2-4y-5=0$$

$$\Rightarrow (y-5)(y+1)=0 \quad \Rightarrow y=5,\ -1$$

$y=5 \Rightarrow x=-1$ & $y=-1 \Rightarrow x=5$

$\Rightarrow (-1,5)$ and $(5,-1)$ are the other two critical points.

$$A=\frac{\partial^2 z}{\partial x^2}=6x, \quad B=\frac{\partial^2 z}{\partial x\partial y}=12, \quad C=\frac{\partial^2 z}{\partial y^2}=6y$$

$\Rightarrow B^2-AC=(12)^2-36xy=144-36xy$

At $(-7,-7)$, we have

$$B^2-AC=144-36(-7)(-7)<0 \quad \text{and} \quad A<0$$

$\Rightarrow (-7,7)$ is a point of relative maximum value.

At $(3,3)$, we have

$$B^2-AC=144-36(3)(3)=144-324<0 \quad \text{and} \quad A>0.$$

$\Rightarrow (3,3)$ is a point of relative minimum value.

At $(-1,5)$, we have

$$B^2-AC=144-36(-1)(5)>0$$

$\Rightarrow (-1,5)$ is a saddle point.

At $(-5,1)$, we have

$$B^2-AC=144-(-5)(1)>0$$

$\Rightarrow (-5,1)$ is also a saddle point.

(***d***) $z=xy(a-x-y)=axy-x^2y-xy^2$

$$\frac{\partial z}{\partial x}=ay-2xy-y^2$$

$$\frac{\partial z}{\partial y}=ax-x^2-2xy$$

$\frac{\partial z}{\partial x}=0 \Rightarrow ay-2xy-y^2=0$ ………………. (i)

$\frac{\partial z}{\partial y}=0 \Rightarrow ax-x^2-2xy=0$ ……………… (ii)

Subtracting (i) and (ii)

$$\begin{array}{r} ay-2xy-y^2=0 \\ \underline{\overset{-}{}ax\overset{+}{-}2xy\overset{+}{-}x^2=0} \\ ay-ax-y^2+x^2=0 \end{array}$$

$$\Rightarrow (x^2-y^2)-a(x-y)=0$$

$$\Rightarrow (x-y)(x+y)-a(x-y)=0$$

$$\Rightarrow (x-y)(x+y-a)=0$$

If $x-y=0 \Rightarrow x=y$ then (i) give

$$ax-2x^2-x^2=0 \quad \Rightarrow ax-3x^2=0$$

$$\Rightarrow x(a-3x)=0 \Rightarrow x=0\,,\ a/3$$

$\Rightarrow (0,0)$ & $\left(a/3\,,a/3\right)$ are the critical points.

If $x+y-a=0$ then $y=a-x$ and (i) gives

$$a(a-x)-2x(a-x)-(a-x)^2=0$$

$$\Rightarrow a^2-ax-2ax+2x^2-a^2-x^2+2ax=0$$

$$\Rightarrow x^2-ax=0 \Rightarrow x(x-a)=0 \Rightarrow x=0,\ a$$

$\Rightarrow (0,a)$ & $(a,0)$ are the other two critical points.

$$A=\frac{\partial^2 z}{\partial x^2}=-2y, \quad B=\frac{\partial^2 z}{\partial x\partial y}=a-2x-2y, \quad C=\frac{\partial^2 z}{\partial x\partial y}=-2x$$

$$\Rightarrow B^2-AC=(a-2x-2y)^2-4xy$$

At $(0,0)$, we have $B^2-AC=a^2>0 \Rightarrow (0,0)$ is a saddle point.

At $\left(a/3\,,a/3\right)$, we have

$$B^2-AC=\left(a-2a/3-2a/3\right)^2-4\left(a/3\right)\left(a/3\right)$$

$$=\frac{a^2}{9}-\frac{4a^2}{9}<0 \quad \text{and} \quad A<0$$

$\Rightarrow \left(\dfrac{a}{3},\dfrac{a}{3}\right)$ is a point of maximum value.

At $(0,a)$, we have $B^2-AC=(a-2a)^2-4(0)(a)=a^2>0$

$\Rightarrow (0,a)$ is a saddle point.

At $(a,0)$, we have $B^2-AC=(a-2a)^2-4(a)(0)=a^2>0$

$\Rightarrow (a,0)$ is also a saddle point.

(*e*) $z=x^2-2xy+y^2+x^3-y^3+25$

$$\frac{\partial z}{\partial x}=2x-2y+3x^2$$

$$\frac{\partial z}{\partial y}=-2x+2y-3y^2$$

$$\frac{\partial z}{\partial x}=0 \Rightarrow 3x^2+2x-2y=0 \ \ldots\ldots\ldots\ldots\ldots\ (i)$$

$$\frac{\partial z}{\partial y}=0 \Rightarrow 3y^2+2x-2y=0 \ \ldots\ldots\ldots\ldots\ldots(ii)$$

Subtracting (i) and (ii), we have

$$3x^2-3y^2=0$$

$$\Rightarrow 3(x-y)(x+y)=0$$

$x-y=0 \Rightarrow x=y$, using in (i) we have

$$3x^2+2x-2x=0 \quad \Rightarrow x=0$$

And $x+y=0 \Rightarrow x=-y$, using in (i) we have

$$3x^2+2x+2x=0$$

$$\Rightarrow 3x^2+4x=0 \Rightarrow x(3x+4)=0$$

$$\Rightarrow x=0,\ x=-4/3$$

$x=0 \quad \Rightarrow \quad y=0$ and $x=-\frac{4}{3} \Rightarrow y=\frac{4}{3}$

$\Rightarrow$ The critical points are $(0,0)$ & $\left(-\frac{4}{3},\frac{4}{3}\right)$

$$A=\frac{\partial^2 z}{\partial x^2}=2+6x$$

$$B=\frac{\partial^2 z}{\partial x \partial y}=-2$$

$$C=\frac{\partial^2 z}{\partial y^2}=2-6y$$

$$B^2-AC=4-(2+6x)(2-6y)$$

At $(0,0)$, we have $B^2-AC=4-4=0 \Rightarrow$ Nature undetermined

At $\left(-\frac{4}{3},\frac{4}{3}\right)$, we have

$$B^2-AC=4-(2-8)(2-8)=4-(-6)(-6)<0 \quad \text{and} \quad A<0$$

$\therefore$ Relative maximum at $\left(-\frac{4}{3},\frac{4}{3}\right)$.

(*f*) $\quad z=x^2y^2-5x^2-8xy-5y^2$

$$\frac{\partial z}{\partial x}=2xy^2-10x-8y$$

$$\frac{\partial z}{\partial y}=2x^2y-10y-8x$$

For critical points, we have

$$xy^2-5x-4y=0 \quad \ldots\ldots\ldots\ldots (i)$$

$$x^2y-5y-4x=0 \quad \ldots\ldots\ldots\ldots (ii)$$

Adding (*i*) and (*ii*), we have

$$xy^2+x^2y-9x-9y=0$$

$$\Rightarrow xy(y+x)-9(x+y)=0$$

$$\Rightarrow (x+y)(xy-9)=0$$

$x+y=0 \Rightarrow y=-x$ in (*i*) gives

$$x^3-5x+4x=0$$

$$\Rightarrow x^3-x=0 \quad \Rightarrow x(x-1)(x+1)=0$$

$$\Rightarrow x=0,1,-1$$

$x=0 \quad \Rightarrow y=0$

$x=1 \quad \Rightarrow y=-1$

$x=-1 \Rightarrow y=1$

$\Rightarrow (0,0), (1,-1), (-1,1)$ are the critical points.

If $xy-9=0$, then $y=\frac{9}{x}$ in (*i*) gives $x^2-9=0 \Rightarrow x=\pm 3$

$x=3 \Rightarrow y=3$ and $x=-3 \Rightarrow y=-3$

$\Rightarrow (3,3)$ & $(-3,-3)$ are also the critical points.

$$A=\frac{\partial^2 z}{\partial x^2}=2y^2-10, \quad B=\frac{\partial^2 z}{\partial x \partial y}=4xy-8, \quad C=\frac{\partial^2 z}{\partial y^2}=2x^2-10$$

$$B^2-AC=(4xy-8)^2-(2y^2-10)(2x^2-10)$$

At $(0,0)$, we have

$$B^2 - AC = 64-(-10)(-10) < 0 \quad \text{and} \quad A = -10 < 0$$

$\Rightarrow$ $(0,0)$ is the point of maximum value.

At $(1,-1)$, we have

$$B^2 - AC = (-4-8)^2 - (2-10)(2-10) = 144-64 > 0$$

$\Rightarrow$ $(1,-1)$ is a saddle point.

At $(-1,1)$, we have

$$B^2 - AC = (-4-8)^2 - (2-10)(2-10) = 144-64 > 0$$

$\Rightarrow$ $(-1,1)$ is a saddle point.

At $(3,3)$, we have

$$B^2 - AC = (36-8)^2 - (18-10)(18-10) = (24)^2 - 64 > 0$$

$\Rightarrow$ $(3,3)$ is a saddle point.

At $(-3,-3)$, we have

$$B^2 - AC = (36-8)^2 - (8)(8) > 0$$

$\Rightarrow$ $(-3,-3)$ is again a saddle point.

(**g**) $z = 2(x-y)^2 - x^4 - y^4$

$$\frac{\partial z}{\partial x} = 4(x-y) - 4x^3$$

$$\frac{\partial z}{\partial y} = -4(x-y) - 4y^3$$

For critical points

$$\frac{\partial z}{\partial x} = 0 \;\Rightarrow\; x - y - x^3 = 0 \;\ldots\ldots\ldots\ldots\ldots\; (i)$$

$$\frac{\partial z}{\partial y} = 0 \;\Rightarrow\; -x + y - y^3 = 0 \;\ldots\ldots\ldots\ldots\; (ii)$$

Addition of (i) and (ii) gives

$$x^3 + y^3 = 0$$

$$\Rightarrow (x+y)(x^2 - xy + y^2) = 0$$

$\Rightarrow$ $x + y = 0$ or $x^2 - xy + y^2 = 0$ which gives imaginary values.

$x + y = 0 \Rightarrow y = -x$ in (i) gives

$$x + x - x^3 = 0 \Rightarrow 2x - x^3 = 0$$

$$\Rightarrow x(2 - x^2) = 0 \Rightarrow x = 0, \pm\sqrt{2}$$

$x = 0 \Rightarrow y = 0$

$x = \sqrt{2} \Rightarrow y = -\sqrt{2}$

$x = -\sqrt{2} \Rightarrow y = \sqrt{2}$

$\Rightarrow$ The critical points are $(0,0)$, $\left(\sqrt{2}, -\sqrt{2}\right)$, $\left(-\sqrt{2}, \sqrt{2}\right)$.

$$A = \frac{\partial^2 z}{\partial x^2} = 4 - 12x^2, \quad B = \frac{\partial^2 z}{\partial x \partial y} = -4, \quad C = \frac{\partial^2 z}{\partial y^2} = 4 - 12y^2$$

$$B^2 - AC = 16 - (4-12x^2)(4-12y^2)$$

At $(0,0)$, we have $B^2 - AC = 0$

Consider $\Delta z = z(h,k) - z(0,0)$

$$= 2(h-k)^2 - h^4 - k^4 = -2h^4 \le 0 \quad \text{if } h = k$$

$\Rightarrow$ $(0,0)$ is the points of maximum value.

At $\left(\sqrt{2},-\sqrt{2}\right)$, we have

$$B^2 - AC = 16-(4-24)(4-24)$$
$$= 16-(-20)(-20) < 0 \quad \text{and} \quad A < 0.$$

$\Rightarrow \left(\sqrt{2},-\sqrt{2}\right)$ is a point of maximum value.

At $\left(-\sqrt{2},\sqrt{2}\right)$, we have

$$B^2 - AC = 16-(4-24)(4-24) < 0 \quad \text{and} \quad A < 0.$$

$\Rightarrow \left(-\sqrt{2},\sqrt{2}\right)$ is also a point of maximum vale.

(***h***) $z = 2(x-y)^3 - (x^4 - y^4)$

$$\frac{\partial z}{\partial x} = 6(x-y)^2 - 4x^3 = 0 \quad \ldots\ldots\ldots\ldots\ldots (i)$$

$$\frac{\partial z}{\partial y} = -6(x-y)^2 + 4y^3 = 0 \quad \ldots\ldots\ldots\ldots (ii)$$

Adding (*i*) and (*ii*), we get

$$y^3 - x^3 = 0 \;\Rightarrow\; (y-x)(y^2+xy+x^2) = 0$$

$y - x = 0 \;\Rightarrow\; y = x$ in (*i*) gives

$$4x^3 = 0 \;\Rightarrow\; x = 0 \;\Rightarrow\; y = 0$$

$x^2 + xy + y^2 = 0$ gives imaginary values

$\Rightarrow (0,0)$ is the only critical point

$$A = \frac{\partial^2 z}{\partial x^2} = 12(x-y) - 12x^2$$

$$B = \frac{\partial^2 z}{\partial x \partial y} = -12(x-y)$$

$$C = \frac{\partial^2 z}{\partial y^2} = 12(x-y) + 12y^2$$

at $(0,0)$, $A = B = C = 0 \;\Rightarrow\; B^2 - AC = 0$

Consider $\Delta z = z(h,h) - z(0,0) = 0$

$\Rightarrow$ Nature undecided.

(***i***) $z = x^2 - 5xy - y^3$

$$\frac{\partial z}{\partial x} = 2x - 5y = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (i)$$

$$\frac{\partial z}{\partial y} = -5x - 3y^2 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (ii)$$

From (*i*) $\quad y = \dfrac{2x}{5}$

(*ii*) becomes $\quad -5x - 3\left(\dfrac{4x^2}{25}\right) = 0$

$$\Rightarrow -125x - 12x^2 = 0$$

$$\Rightarrow 12x^2 + 125x = 0$$

$$\Rightarrow x(12x+125) = 0 \;\Rightarrow\; x = 0,\; -\frac{125}{12}$$

Remarks

$x=0 \Rightarrow y=0 \quad \& \quad x=-\frac{125}{12} \Rightarrow y=\frac{2}{5}\left(-\frac{125}{12}\right)=-\frac{25}{6}$

$\Rightarrow (0,0) \;\&\; \left(-\frac{125}{12},-\frac{25}{6}\right)$ are the critical points

$A=\frac{\partial^2 z}{\partial x^2}=2, \quad B=\frac{\partial^2 z}{\partial x \partial y}=-5, \quad C=\frac{\partial^2 z}{\partial y^2}=-6y$

$B^2-AC=25+12y$

At $(0,0)$, we have $B^2-AC=25>0 \Rightarrow (0,0)$ is a saddle point.

At $\left(-\frac{125}{12},-\frac{25}{6}\right)$, we have

$$B^2-AC=25+12\left(-\frac{25}{6}\right)=-25<0 \quad \text{and} \quad A=2>0$$

$\therefore \left(-\frac{125}{12},-\frac{25}{6}\right)$ is a point of maximum value.

Chapter 6 – Riemann-Stieltjes Integral.

Subject: *Real Analysis (Mathematics)* **Level:** *M.Sc.*
Source: *Syyed Gul Shah (Chairman, Department of Mathematics, US Sargodha)*
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➢ Introduction

In elementary treatment of Integral Calculus the subject of integration is treated as inverse of differentiation. The subject arose in connection with the determination of areas of plane regions and was based on the notion of the limit of a type of sum when the number of terms in the sum tends to infinity and each term tends to zero. In fact the name Integral Calculus has its origin in this process of summation. It was only afterwards that it was seen that the subject of integration can also be viewed from the point of the inverse of differentiation.

➢ Partition

Let $[a,b]$ be a given interval. A finite set $P=\{a=x_0,x_1,x_2,....,x_k,....,x_n=b\}$ is said to be a partition of $[a,b]$ which divides it into n such intervals

$$[x_0,x_1],[x_1,x_2],[x_2,x_3],......,[x_{n-1},x_n]$$

Each sub-interval is called a *component* of the partition.

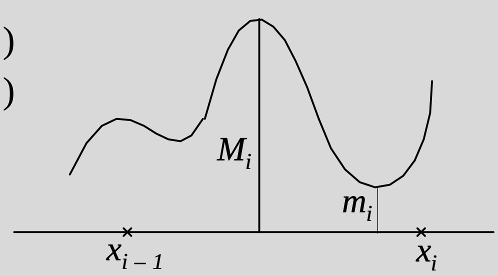
Obviously, corresponding to different choices of the points x_i we shall have different partition.

The maximum of the length of the components is defined as the *norm* of the partition.

➢ Riemann Integral

Let f be a real-valued function defined and bounded on $[a,b]$. Corresponding to each partition P of $[a,b]$, we put

$$M_i=\sup f(x) \qquad (x_{i-1}\leq x\leq x_i)$$
$$m_i=\inf f(x) \qquad (x_{i-1}\leq x\leq x_i)$$



We define upper and lower sums as

$$U(P,f)=\sum_{i=1}^{n}M_i\,\Delta x_i$$

and $$L(P,f)=\sum_{i=1}^{n}m_i\,\Delta x_i$$

where $\Delta x_i=x_i-x_{i-1} \quad (i=1,2,....,n)$

and finally $$\int_a^{\bar{b}} f\,dx=\inf U(P,f) \;\;\ldots\ldots\ldots\ldots\ldots\; (i)$$

$$\int_{\underline{a}}^{b} f\,dx=\sup L(P,f) \;\;\ldots\ldots\ldots\ldots\ldots (ii)$$

Where the infimum and the supremum are taken over all partitions P of $[a,b]$.

Then $\int_a^{\bar{b}} f\,dx$ and $\int_{\underline{a}}^{b} f\,dx$ are called the upper and lower Riemann Integrals of f over $[a,b]$ respectively.

In case the upper and lower integrals are equal, we say that f is Riemann-Integrable on $[a,b]$ and we write $f\in\boldsymbol{R}$, where $\boldsymbol{R}$ denotes the set of Riemann integrable functions.

The common value of (i) and (ii) is denoted by $\int_a^b f\,dx$ or by $\int_a^b f(x)\,dx$.

Which is known as the Riemann integral of f over $[a,b]$.

➢ *Theorem*

The upper and lower integrals are defined for every bounded function f.

Proof

Take M and m to be the upper and lower bounds of $f(x)$ in $[a,b]$.

$$\Rightarrow \quad m \le f(x) \le M \qquad (a \le x \le b)$$

Then $M_i \le M$ and $m_i \ge m$ $\qquad (i = 1,2,.....,n)$

Where M_i and m_i denote the supremum and infimum of $f(x)$ in $\left(x_{i-1}, x_i\right)$ for certain partition P of $\left[a,b\right]$.

$$\Rightarrow \; L\left(P,f\right) = \sum_{i=1}^{n} m_i\,\Delta x_i \ge \sum_{i=1}^{n} m\,\Delta x_i \qquad (\Delta x_i = x_{i-1} - x_i)$$

$$\Rightarrow \; L\left(P,f\right) \ge m \sum_{i=1}^{n} \Delta x_i$$

But $\displaystyle\sum_{i=1}^{n} \Delta x_i = (x_1 - x_0) + (x_2 - x_1) + (x_3 - x_2) + ... + (x_n - x_{n-1})$

$$= x_n - x_0 = b - a$$

$$\Rightarrow \; L\left(P,f\right) \ge m(b-a)$$

Similarity $U\left(P,f\right) \le M(b-a)$

$$\Rightarrow \; m(b-a) \le L\left(P,f\right) \le U\left(P,f\right) \le M(b-a)$$

Which shows that the numbers $L\left(P,f\right)$ and $U\left(P,f\right)$ form a bounded set.

$\Rightarrow$ The upper and lower integrals are defined for every bounded function f. ⊙

➢ *Riemann-Stieltjes Integral*

It is a generalization of the Riemann Integral. Let $\alpha(x)$ be a monotonically increasing function on $[a,b]$. $\alpha(a)$ and $\alpha(b)$ being finite, it follows that $\alpha(x)$ is bounded on $[a,b]$. Corresponding to each partition P of $[a,b]$, we write

$$\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1})$$

(Difference of values of α at x_i & x_{i-1})

$\because \;\; \alpha(x)$ is monotonically increasing.

$\therefore \;\; \Delta\alpha_i \ge 0$

Let f be a real function which is bounded on $[a,b]$.

Put
$$U\left(P,f,\alpha\right) = \sum_{i=1}^{n} M_i\,\Delta\alpha_i$$

$$L\left(P,f,\alpha\right) = \sum_{i=1}^{n} m_i\,\Delta\alpha_i$$

Where M_i and m_i have their usual meanings.

Define

$$\overline{\int_a^b} f\,d\alpha = \inf U\left(P,f,\alpha\right) \;\ldots\ldots\ldots\ldots\ldots\; (i)$$

$$\underline{\int_a^b} f\,d\alpha = \sup L\left(P,f,\alpha\right) \;\ldots\ldots\ldots\ldots\ldots\; (ii)$$

Where the infimum and supremum are taken over all partitions of $[a,b]$.

If $\overline{\int_a^b} f\,d\alpha = \underline{\int_a^b} f\,d\alpha$, we denote their common value by $\int_a^b f\,d\alpha$ or $\int_a^b f(x)\,d\alpha(x)$.

This is the Riemann-Stieltjes integral or simply the Stieltjes Integral of f w.r.t. α over $[a,b]$.

If $\int_a^b f\,d\alpha$ exists, we say that f is integrable w.r.t. α, in the Riemann sense, and write $f \in \boldsymbol{R}(\alpha)$.

➢ *Note*

The Riemann-integral is a special case of the Riemann-Stieltjes integral when we take $\alpha(x) = x$.

$\because$ The integral depends upon f, α, a and b but not on the variable of integration.

$\therefore$ We can omit the variable and prefer to write $\int_a^b f\,d\alpha$ instead of $\int_a^b f(x)\,d\alpha(x)$.

In the following discussion f will be assume to be real and bounded, and α monotonically increasing on $[a,b]$.

➢ *Refinement of a Partition*

Let P and P^* be two partitions of an interval $[a,b]$ such that $P \subset P^*$ i.e. every point of P is a point of P^*, then P^* is said to be a *refinement* of P.

➢ *Common Refinement*

Let P_1 and P_2 be two partitions of $[a,b]$. Then a partition P^* is said to be their *common refinement* if $P^* = P_1 \cup P_2$.

➢ *Theorem*

If P^* is a refinement of P, then

$$L(P, f, \alpha) \le L(P^*, f, \alpha) \quad \ldots\ldots\ldots\ldots\ldots\ldots (i)$$

and $$U(P, f, \alpha) \ge U(P^*, f, \alpha) \quad \ldots\ldots\ldots\ldots\ldots\ldots (ii)$$

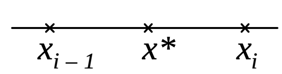
Proof

Let us suppose that P^* contains just one point x^* more than P such that $x_{i-1} < x^* < x_i$ where x_{i-1} and x_i are two consecutive points of P.

Put

$$w_1 = \inf f(x) \qquad (x_{i-1} \le x \le x^*)$$

$$w_2 = \inf f(x) \qquad (x^* \le x \le x_i)$$



It is clear that $w_1 \ge m_i$ & $w_2 \ge m_i$ where $m_i = \inf f(x)$, $(x_{i-1} \le x \le x_i)$.

Hence

$$\begin{aligned} L(P^*, f, \alpha) - L(P, f, \alpha) &= w_1\left[\alpha(x^*) - \alpha(x_{i-1})\right] + w_2\left[\alpha(x_i) - \alpha(x^*)\right] \\ &\quad - m_i\left[\alpha(x_i) - \alpha(x_{i-1})\right] \\ &= w_1\left[\alpha(x^*) - \alpha(x_{i-1})\right] + w_2\left[\alpha(x_i) - \alpha(x^*)\right] \\ &\quad - m_i\left[\alpha(x_i) - \alpha(x^*) + \alpha(x^*) - \alpha(x_{i-1})\right] \\ &= (w_1 - m_i)\left[\alpha(x^*) - \alpha(x_{i-1})\right] + (w_2 - m_i)\left[\alpha(x_i) - \alpha(x^*)\right] \end{aligned}$$

$\because\ \alpha$ is a monotonically increasing function.

$\therefore\ \alpha(x^*)-\alpha(x_{i-1})\geq 0 \quad , \quad \alpha(x_i)-\alpha(x^*)\geq 0$

$$\Rightarrow\ L(P^*, f, \alpha)-L(P, f, \alpha)\ \geq\ 0$$

$$\Rightarrow\ L(P, f, \alpha)\ \leq\ L(P^*, f, \alpha) \qquad \text{which is } (i)$$

If P^* contains k points more than P, we repeat this reasoning k times and arrive at (i).

Now put

$$W_1=\sup f(x) \qquad (x_{i-1}\leq x\leq x^*)$$

and $$W_2=\sup f(x) \qquad (x^*\leq x\leq x_i)$$

Clearly $\quad M_i\geq W_1 \quad \& \quad M_i\geq W_2$

Consider

$$\begin{aligned} U(P, f, \alpha)-U(P^*, f, \alpha) &= M_i\left[\alpha(x_i)-\alpha(x_{i-1})\right] \\ &\qquad -W_1\left[\alpha(x^*)-\alpha(x_{i-1})\right]-W_2\left[\alpha(x_i)-\alpha(x^*)\right] \\ &= M_i\left[\alpha(x_i)-\alpha(x^*)+\alpha(x^*)-\alpha(x_{i-1})\right] \\ &\qquad -W_1\left[\alpha(x^*)-\alpha(x_{i-1})\right]-W_2\left[\alpha(x_i)-\alpha(x^*)\right] \\ &= (M_i-W_1)\left[\alpha(x^*)-\alpha(x_{i-1})\right]+(M_i-W_2)\left[\alpha(x_i)-\alpha(x^*)\right]\ \geq\ 0 \\ &\qquad (\because\ \alpha \text{ is } \uparrow) \end{aligned}$$

$$\Rightarrow\ U(P, f, \alpha)\ \geq\ U(P^*, f, \alpha) \qquad \text{which is } (ii)$$

⊙

➢ ***Theorem***

Let f be a real valued function defined on $[a,b]$ and α be a monotonically increasing function on $[a,b]$. Then

$$\sup L(P, f, \alpha)\ \leq\ \inf U(P, f, \alpha)$$

i.e. $$\underline{\int_a^b} f\,d\alpha\ \leq\ \overline{\int_a^b} f\,d\alpha$$

Proof

Let P^* be the common refinement of two partitions P_1 and P_2. Then

$$L(P_1, f, \alpha)\ \leq\ L(P^*, f, \alpha)\ \leq\ U(P^*, f, \alpha)\ \leq\ U(P_2, f, \alpha)$$

Hence $\quad L(P_1, f, \alpha)\ \leq\ U(P_2, f, \alpha)$ …………. (i)

If P_2 is fixed and the supremum is taken over all P_1 then (i) gives

$$\underline{\int_a^b} f\,d\alpha\ \leq\ U(P_2, f, \alpha)$$

Now take the infimum over all P_2

$$\Rightarrow\ \underline{\int_a^b} f\,d\alpha\ \leq\ \overline{\int_a^b} f\,d\alpha$$ ⊙

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> ➢ ***Theorem (Condition of Integrability or Cauchy's Criterion for Integrability.)***
>
> $f \in \boldsymbol{R}(\alpha)$ on $[a,b]$ iff for every $\varepsilon > 0$ there exists a partition P such that
> $$U(P,f,\alpha) - L(P,f,\alpha) < \varepsilon$$

Proof

Let $\quad U(P,f,\alpha) - L(P,f,\alpha) < \varepsilon$ ………… (i)

Then $\quad L(P,f,\alpha) \le \underline{\int_a^b} f\,d\alpha \le \overline{\int_a^b} f\,d\alpha \le U(P,f,\alpha)$

$$\Rightarrow \underline{\int_a^b} f\,d\alpha - L(P,f,\alpha) \ge 0 \quad \text{and} \quad U(P,f,\alpha) - \overline{\int_a^b} f\,d\alpha \ge 0$$

Adding these two results, we have

$$\underline{\int_a^b} f\,d\alpha - \overline{\int_a^b} f\,d\alpha - L(P,f,\alpha) + U(P,f,\alpha) \ge 0$$

$$\Rightarrow \overline{\int_a^b} f\,d\alpha - \underline{\int_a^b} f\,d\alpha \le U(P,f,\alpha) - L(P,f,\alpha) < \varepsilon \qquad \text{from } (i)$$

i.e. $\quad 0 \le \overline{\int_a^b} f\,d\alpha - \underline{\int_a^b} f\,d\alpha < \varepsilon \quad$ for every $\varepsilon > 0$.

$$\Rightarrow \overline{\int_a^b} f\,d\alpha = \underline{\int_a^b} f\,d\alpha \qquad \text{i.e.} \quad f \in \boldsymbol{R}(\alpha)$$

Conversely, let $f \in \boldsymbol{R}(\alpha)$ and let $\varepsilon > 0$

$$\Rightarrow \overline{\int_a^b} f\,d\alpha = \underline{\int_a^b} f\,d\alpha = \int_a^b f\,d\alpha$$

Now $\quad \overline{\int_a^b} f\,d\alpha = \inf U(P,f,\alpha) \quad$ and $\quad \underline{\int_a^b} f\,d\alpha = \sup L(P,f,\alpha)$

There exist partitions P_1 and P_2 such that

$$U(P_2,f,\alpha) - \int_a^b f\,d\alpha < \frac{\varepsilon}{2} \quad \text{…………… } (ii)$$

and $\quad \int_a^b f\,d\alpha - L(P_1,f,\alpha) < \dfrac{\varepsilon}{2}$ …………… (iii)

$U(P_2,f,\alpha) - \varepsilon/2 < \int f\,d\alpha$

$\int f\,d\alpha < L(P_1,f,\alpha) + \varepsilon/2$

We choose P to be the common refinement of P_1 and P_2.

Then

$$U(P,f,\alpha) \le U(P_2,f,\alpha) < \int_a^b f\,d\alpha + \frac{\varepsilon}{2} < L(P_1,f,\alpha) + \varepsilon \le L(P,f,\alpha) + \varepsilon$$

So that

$$U(P,f,\alpha) - L(P,f,\alpha) < \varepsilon \qquad \odot$$

➢ ***Theorem***

a) If $U(P,f,\alpha)-L(P,f,\alpha)<\varepsilon$ holds for some P and some ε, then it holds (with the same ε) for every refinement of P.

b) If $U(P,f,\alpha)-L(P,f,\alpha)<\varepsilon$ holds for $P=\{x_0,....,x_n\}$ and s_i,t_i are arbitrary points in $[x_{i-1},x_i]$, then

$$\sum_{i=1}^{n}\left|f(s_i)-f(t_i)\right|\Delta\alpha_i<\varepsilon$$

c) If $f\in\boldsymbol{R}(\alpha)$ and the hypotheses of (*b*) holds, then

$$\left|\sum_{i=1}^{n}f(t_i)\Delta\alpha_i-\int_a^b f\,d\alpha\right|<\varepsilon$$

Proof

a) Let P^* be a refinement of P. Then

$$L(P,f,\alpha)\ \le\ L(P^*,f,\alpha)$$

$$\text{and}\quad U(P^*,f,\alpha)\ \le\ U(P,f,\alpha)$$

$$\Rightarrow\ L(P,f,\alpha)+U(P^*,f,\alpha)\ \le\ L(P^*,f,\alpha)+U(P,f,\alpha)$$

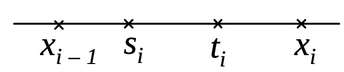
$$\Rightarrow\ U(P^*,f,\alpha)-L(P^*,f,\alpha)\ \le\ U(P,f,\alpha)-L(P,f,\alpha)$$

$$\because\ U(P,f,\alpha)-L(P,f,\alpha)<\varepsilon$$

$$\therefore\ U(P^*,f,\alpha)-L(P^*,f,\alpha)<\varepsilon$$

b) $P=\{x_0,....,x_n\}$ and s_i, t_i are arbitrary points in $[x_{i-1},x_i]$.

$$\Rightarrow\ f(s_i)\text{ and }f(t_i)\text{ both lie in }[m_i,M_i].$$



$$\Rightarrow\ \left|f(s_i)-f(t_i)\right|\ \le\ M_i-m_i$$

$$\Rightarrow\ \left|f(s_i)-f(t_i)\right|\Delta\alpha_i\ \le\ M_i\Delta\alpha_i-m_i\Delta\alpha_i$$

$$\Rightarrow\ \sum_{i=1}^{n}\left|f(s_i)-f(t_i)\right|\Delta\alpha_i\ \le\ \sum_{i=1}^{n}M_i\,\Delta\alpha_i-\sum_{i=1}^{n}m_i\,\Delta\alpha_i$$

$$\Rightarrow\ \sum_{i=1}^{n}\left|f(s_i)-f(t_i)\right|\Delta\alpha_i\ \le\ U(P,f,\alpha)-L(P,f,\alpha)$$

$$\because\ U(P,f,\alpha)-L(P,f,\alpha)\ <\ \varepsilon$$

$$\therefore\ \sum_{i=1}^{n}\left|f(s_i)-f(t_i)\right|\Delta\alpha_i\ <\ \varepsilon$$

c) $\because\ m_i\ \le\ f(t_i)\ \le\ M_i$

$$\therefore\ \sum m_i\,\Delta\alpha_i\ \le\ \sum f(t_i)\Delta\alpha_i\ \le\ \sum M_i\,\Delta\alpha_i$$

$$\Rightarrow\ L(P,f,\alpha)\ \le\ \sum f(t_i)\Delta\alpha_i\ \le\ U(P,f,\alpha)$$

and also $L(P,f,\alpha)\ \le\ \int_a^b f\,d\alpha\ \le\ U(P,f,\alpha)$

Using (*b*), we have

$$\left|\sum f(t_i)\Delta\alpha_i-\int_a^b f\,d\alpha\right|\ <\ \varepsilon$$

⊙

☙----------------------------❧

➢ *Theorem*

If f is continuous on $[a,b]$ then $f \in \boldsymbol{R}(\alpha)$ on $[a,b]$.

Proof

Let $\varepsilon > 0$ be given. Choose $\beta > 0$ so that

$$[\alpha(b) - \alpha(a)]\beta < \varepsilon$$

f is continuous on $[a,b]$ $\Rightarrow$ f is uniformly continuous on $[a,b]$.

$\Rightarrow$ There exists a $\delta > 0$ such that

$$|f(s) - f(t)| < \beta \quad \text{if } x \in [a,b],\ t \in [a,b] \text{ and } |x - t| < \delta \quad \ldots\ldots\ldots..(i)$$

If P is any partition of $[a,b]$ such that $\Delta x_i < \delta$ for all i

then (i) implies that $\quad M_i - m_i \le \beta \quad , \quad (i = 1, 2, \ldots., n)$

$$\begin{aligned}\Rightarrow U(P, f, \alpha) - L(P, f, \alpha) &= \sum M_i\,\Delta\alpha_i - \sum m_i\,\Delta\alpha_i \\ &= \sum (M_i - m_i)\Delta\alpha_i \\ &\le \beta \sum \Delta\alpha_i = \beta[\alpha(b) - \alpha(a)] < \varepsilon\end{aligned}$$

$\Rightarrow$ $f \in \boldsymbol{R}(\alpha)$ by Cauchy Criterion. ⊙

➢ *Theorem*

If f is monotonic on $[a,b]$, and if α is continuous on $[a,b]$, then $f \in \boldsymbol{R}(\alpha)$.

(Monotonicity of α still assumed.)

Proof

Let $\varepsilon > 0$ be a given positive number.

For any positive integer n, choose a partition $P = \{x_0, x_1, \ldots.., x_n\}$ of $[a,b]$ such that

$$\Delta\alpha_i = \frac{\alpha(b) - \alpha(a)}{n} \quad , \quad i = 1, 2, \ldots., n$$

This is possible because α is continuous and monotonic increasing on the closed interval $[a,b]$ and thus assumes every value between its bounds, $\alpha(a)$ and $\alpha(b)$.

Let f be monotonic increasing on $[a,b]$, so that its lower and upper bounds m_i, M_i in $[x_{i-1}, x_i]$ are given by

$$m_i = f(x_{i-1}) \quad , \quad M_i = f(x_i) \quad , \quad i = 1, 2, \ldots., n$$

$$\begin{aligned}\therefore\ U(P, f, \alpha) - L(P, f, \alpha) &= \sum_{i=1}^{n} (M_i - m_i)\Delta\alpha_i \\ &= \frac{\alpha(b) - \alpha(a)}{n} \sum_{i=1}^{n} [f(x_i) - f(x_{i-1})] \\ &= \frac{\alpha(b) - \alpha(a)}{n} [f(b) - f(a)] \\ &< \varepsilon \qquad \text{if } n \text{ is taken large enough.}\end{aligned}$$

$\Rightarrow$ $f \in R(\alpha)$ on $[a,b]$.

⊙

Note: $f \in \boldsymbol{R}(\alpha)$ when either

i) f is continuous and α is monotonic, or

ii) f is monotonic and α is continuous, of course α is still monotonic.

➢ *Properties of Integral*

i) If $f \in \boldsymbol{R}(\alpha)$ on $[a,b]$, then $cf \in \boldsymbol{R}(\alpha)$ for every constant c and

$$\int_a^b cf\,d\alpha = c\int_a^b f\,d\alpha .$$

Proof

$\because\ f \in \boldsymbol{R}(\alpha)$

$\therefore\ \exists$ a partition P such that

$$U(P,f,\alpha) - L(P,f,\alpha) < \varepsilon \quad , \text{ where } \varepsilon \text{ is an arbitrary +ive number.}$$

Now $U(P,cf,\alpha) = \sum_{i=1}^{n} cM_i\,\Delta\alpha_i = c\sum_{i=1}^{n} M_i\,\Delta\alpha_i$

& $L(P,cf,\alpha) = \sum_{i=1}^{n} cm_i\,\Delta\alpha_i = c\sum_{i=1}^{n} m_i\,\Delta\alpha_i$

$$\Rightarrow\ U(P,f,\alpha) - L(P,f,\alpha) = c\left[\sum M_i\,\Delta\alpha_i - \sum m_i\,\Delta\alpha_i\right]$$
$$= c\left[U(P,f,\alpha) - L(P,f,\alpha)\right]$$
$$< c\varepsilon = \varepsilon_1$$

$\Rightarrow\ cf \in \boldsymbol{R}(\alpha)$

$\because\ U(P,cf,\alpha) = c\left[U(P,f,\alpha)\right]$ & $L(P,cf,\alpha) = c\left[L(P,f,\alpha)\right]$

$\therefore\ \inf U(P,cf,\alpha) = c\left[\inf U(P,f,\alpha)\right]$ & $\sup L(P,cf,\alpha) = c\left[\sup L(P,f,\alpha)\right]$

where infimum and supremum are taken over all P on $[a,b]$.

$$\Rightarrow\ \int_a^{\bar{b}} cf\,d\alpha = c\int_a^{\bar{b}} f\,d\alpha \quad \& \quad \int_{\underline{a}}^{b} cf\,d\alpha = c\int_{\underline{a}}^{b} f\,d\alpha$$

$$\because\ \int_a^{\bar{b}} cf\,d\alpha = \int_{\underline{a}}^{b} cf\,d\alpha \quad \text{and} \quad \int_a^{\bar{b}} f\,d\alpha = \int_{\underline{a}}^{b} f\,d\alpha$$

$$\therefore\ \int_a^b cf\,d\alpha = c\int_a^b f\,d\alpha$$ ⊙

ii) If $f_1 \in \boldsymbol{R}(\alpha)$ and $f_2 \in \boldsymbol{R}(\alpha)$ on $[a,b]$, then $f_1 + f_2 \in \boldsymbol{R}(\alpha)$ and

$$\int_a^b (f_1 + f_2)\,d\alpha = \int_a^b f_1\,d\alpha + \int_a^b f_2\,d\alpha .$$

Proof

If $f = f_1 + f_2$ and P is any partition of $[a,b]$, we have

$$m_i' + m_i'' \le m_i \le M_i \le M_i' + M_i''$$

where M_i', m_i', M_i'', m_i'' and M_i, m_i are the bounds of f_1, f_2 and f respectively in $[x_{i-1}, x_i]$.

Multiplying throughout by $\Delta\alpha_i$ and adding the inequalities for $i = 1,2,....,n$, we get

$$L(P,f_1,\alpha) + L(P,f_2,\alpha) \le L(P,f,\alpha) \le U(P,f,\alpha) \le U(P,f_1,\alpha) + U(P,f_2,\alpha) \quad ....(i)$$

Since $f_1 \in \boldsymbol{R}(\alpha)$ and $f_2 \in \boldsymbol{R}(\alpha)$ on $[a,b]$ therefore $\exists$ $\varepsilon > 0$ and there are partitions P_1 and P_2 such that

$$\left.\begin{aligned} U(P_1,f_1,\alpha) - L(P_1,f_1,\alpha) &< \varepsilon \\ \text{and}\quad U(P_2,f_2,\alpha) - L(P_2,f_2,\alpha) &< \varepsilon \end{aligned}\right\} \quad .... (ii)$$

These inequalities hold if P_1 and P_2 are replaced by their common refinement P.

$(ii) \Rightarrow \left[U(P, f_1,\alpha)+U(P, f_2,\alpha)\right]-\left[L(P, f_1,\alpha)+L(P, f_2,\alpha)\right] < 2\varepsilon$

Using (i) we have

$$U(P, f,\alpha)-L(P, f,\alpha) < 2\varepsilon$$

which proves that $f \in \boldsymbol{R}(\alpha)$ on $[a,b]$

With the same partition P, we have

$$U(P, f_1,\alpha) < \int_a^b f_1\, d\alpha + \varepsilon$$

and $$U(P, f_2,\alpha) < \int_a^b f_2\, d\alpha + \varepsilon$$

Hence (i) implies that

$$\int_a^b f\, d\alpha \le U(P, f,\alpha) < \int_a^b f_1\, d\alpha + \int_a^b f_2\, d\alpha + 2\varepsilon$$

$\because$ ε is arbitrary, we conclude that

$$\int_a^b f\, d\alpha \le \int_a^b f_1\, d\alpha + \int_a^b f_2\, d\alpha$$

Similarly if we consider the lower sums we arrive at

$$\int_a^b f\, d\alpha \ge \int_a^b f_1\, d\alpha + \int_a^b f_2\, d\alpha$$

Combining the above two results, we have

$$\int_a^b f\, d\alpha = \int_a^b f_1\, d\alpha + \int_a^b f_2\, d\alpha$$ ⊙

iii) If $f_1(x) \le f_2(x)$ on $[a,b]$, then

$$\int_a^b f_1\, d\alpha \le \int_a^b f_2\, d\alpha$$

Proof

Let $f(x) \ge 0$, then $M_i \ge 0 \quad \Rightarrow U(P, f,\alpha) \ge 0$

and $\therefore \int_a^b f\, d\alpha \ge 0$

$$\because f_1 \le f_2 \qquad \therefore f_2 - f_1 \ge 0$$

$$\Rightarrow \int_a^b (f_2 - f_1)\, d\alpha \ge 0 \qquad \Rightarrow \int_a^b f_2\, d\alpha - \int_a^b f_1\, d\alpha \ge 0$$

$$\Rightarrow \int_a^b f_1\, d\alpha \le \int_a^b f_2\, d\alpha$$ ⊙

➢ ***Note***

(i) $(f+g)(x) = f(x)+g(x) \le \sup f + \sup g$

$\Rightarrow \sup(f+g) \le \sup f + \sup g$

(ii) $(f+g)(x) = f(x)+g(x) \ge \inf f + \inf g$

$\Rightarrow \inf(f+g) \ge \inf f + \inf g$

iv) If $f \in \boldsymbol{R}(\alpha)$ on $[a,b]$ and if $a<c<b$, then $f \in \boldsymbol{R}(\alpha)$ on $[a,c]$ and on $[c,b]$ and

$$\int_a^b f\,d\alpha = \int_a^c f\,d\alpha + \int_c^b f\,d\alpha$$

Proof

Since $f \in \boldsymbol{R}(\alpha)$ on $[a,b]$, therefore for $\varepsilon>0$, $\exists$ a partition P such that

$$U(P,f,\alpha)-L(P,f,\alpha) < \varepsilon$$

Let P^* be the refinement of P such that $P^* = P \cup \{c\}$

$$\therefore\ L(P,f,\alpha) \le L(P^*,f,\alpha) \le U(P^*,f,\alpha) \le U(P,f,\alpha) \quad \ldots\ldots\ldots\ldots (i)$$

$$\Rightarrow\ U(P^*,f,\alpha)-L(P^*,f,\alpha) \le U(P,f,\alpha)-L(P,f,\alpha) < \varepsilon \quad \ldots\ldots\ldots (ii)$$

Let P_1, P_2 denote the sets of points of P^* between $[a,c],[c,b]$ respectively.

Clearly P_1, P_2 are partitions of $[a,c],[c,b]$ respectively and $P^* = P_1 \cup P_2$.

Also $\quad U(P^*,f,\alpha) = U(P_1,f,\alpha) + U(P_2,f,\alpha) \quad \ldots\ldots\ldots (iii)$

and $\quad L(P^*,f,\alpha) = L(P_1,f,\alpha) + L(P_2,f,\alpha) \quad \ldots\ldots\ldots\ldots (iv)$

$$\therefore\ \{U(P_1,f,\alpha)-L(P_1,f,\alpha)\} + \{U(P_2,f,\alpha)-L(P_2,f,\alpha)\}$$

$$= U(P^*,f,\alpha) - L(P^*,f,\alpha) < \varepsilon$$

Since each bracket on the left is non-negative, it follows that

$$U(P_1,f,\alpha)-L(P_1,f,\alpha) < \varepsilon$$

and $\quad U(P_2,f,\alpha)-L(P_2,f,\alpha) < \varepsilon$

$\Rightarrow\ f \in \boldsymbol{R}(\alpha)$ on $[a,c]$ and on $[c,b]$.

We know that for any functions f_1 and f_2, if $f = f_1 + f_2$, then

$$\inf f \ \ge\ \inf f_1 + \inf f_2$$

and $\quad \sup f \ \le\ \sup f_1 + \sup f_2$

Now for any partitions P_1, P_2 of $[a,c],[c,b]$ respectively, if $P^* = P_1 \cup P_2$, then

$$U(P^*,f,\alpha) = U(P_1,f,\alpha) + U(P_2,f,\alpha)$$

Hence on taking the infimum for all partitions, we get

$$\overline{\int_a^b} f\,d\alpha \ \ge\ \overline{\int_a^c} f\,d\alpha + \overline{\int_c^b} f\,d\alpha$$

But since $f \in \boldsymbol{R}(\alpha)$ on $[a,c],[c,b],[a,b]$

$$\therefore\ \int_a^b f\,d\alpha \ \ge\ \int_a^c f\,d\alpha + \int_c^b f\,d\alpha \quad \ldots\ldots\ldots\ldots (v)$$

Again $\quad L(P^*,f,\alpha) = L(P_1,f,\alpha) + L(P_2,f,\alpha)$

and on taking the supremum for all partitions, we get

$$\underline{\int_a^b} f\,d\alpha \ \le\ \underline{\int_a^c} f\,d\alpha + \underline{\int_c^b} f\,d\alpha$$

But since $f \in \boldsymbol{R}(\alpha)$ on $[a,c],[c,b],[a,b]$

$$\therefore\ \int_a^b f\,d\alpha \ \le\ \int_a^c f\,d\alpha + \int_c^b f\,d\alpha \quad \ldots\ldots\ldots\ldots\ldots (vi)$$

(v) and (vi) imply that

$$\int_a^b f\,d\alpha \ = \ \int_a^c f\,d\alpha + \int_c^b f\,d\alpha$$

⊙

v) If $f \in \boldsymbol{R}(\alpha)$ on $[a,b]$ and $|f(x)| \le M$ on $[a,b]$, then

$$\left|\int_a^b f\,d\alpha\right| \le M\left[\alpha(b)-\alpha(a)\right]$$

Proof

We know that

$$\int_a^b f\,d\alpha \le U(P,f,\alpha)$$

$$= \sum M_i\,\Delta\alpha_i \le M\sum \Delta\alpha_i$$

But

$$\sum \Delta\alpha_i = \alpha(b)-\alpha(a)$$

$$\Rightarrow \left|\int_a^b f\,d\alpha\right| \le M\left[\alpha(b)-\alpha(a)\right]$$

⊙

vi) If $f \in \boldsymbol{R}(\alpha_1)$ and $f \in \boldsymbol{R}(\alpha_2)$, then $f \in \boldsymbol{R}(\alpha_1+\alpha_2)$ and

$$\int_a^b f\,d(\alpha_1+\alpha_2) = \int_a^b f\,d\alpha_1 + \int_a^b f\,d\alpha_2$$

and if $f \in \boldsymbol{R}(\alpha)$ and c is a positive constant, then $f \in \boldsymbol{R}(c\alpha)$ and

$$\int_a^b f\,d(c\alpha) = c\int_a^b f\,d\alpha$$

Proof

Since $f \in \boldsymbol{R}(\alpha_1)$ and $f \in \boldsymbol{R}(\alpha_2)$, therefore for $\varepsilon > 0$, there exists partitions P_1, P_2 of $[a,b]$ such that

$$U(P_1,f,\alpha_1) - L(P_1,f,\alpha_1) < \frac{\varepsilon}{2}$$

$$\text{and} \quad U(P_2,f,\alpha_2) - L(P_2,f,\alpha_2) < \frac{\varepsilon}{2}$$

Let $P = P_1 \cup P_2$

$$\left.\begin{aligned} \therefore\; & U(P,f,\alpha_1) - L(P,f,\alpha_1) < \frac{\varepsilon}{2} \\ \& \; & U(P,f,\alpha_2) - L(P,f,\alpha_2) < \frac{\varepsilon}{2} \end{aligned}\right\} \ldots\ldots\ldots\ldots (i)$$

Let m_i , M_i be bounds of f in $\left[x_{i-1}, x_i\right]$

Take $\alpha = \alpha_1 + \alpha_2$

$$\Rightarrow \Delta\alpha_i = \Delta\alpha_{1i} + \Delta\alpha_{2i}$$

$$\begin{aligned} \therefore\; U(P,f,\alpha) &= \sum M_i\,\Delta\alpha_i \\ &= \sum M_i\left(\Delta\alpha_{1i} + \Delta\alpha_{2i}\right) \\ &= U(P,f,\alpha_1) + U(P,f,\alpha_2) \end{aligned}$$

Similarly

$$L(P,f,\alpha) = L(P,f,\alpha_1) + L(P,f,\alpha_2)$$

$$\therefore\; U(P,f,\alpha) - L(P,f,\alpha) = U(P,f,\alpha_1) - L(P,f,\alpha_1) + U(P,f,\alpha_2) - L(P,f,\alpha_2)$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \qquad \text{by } (i)$$

$$\Rightarrow f \in \boldsymbol{R}(\alpha) \quad \text{where} \quad \alpha = \alpha_1 + \alpha_2$$

To prove the second part, we notice that

$$\int_a^b f\,d\alpha = \inf U(P,f,\alpha)$$
$$= \inf\{U(P,f,\alpha_1)+U(P,f,\alpha_2)\}$$
$$\geq \inf U(P,f,\alpha_1)+\inf U(P,f,\alpha_2)$$
$$= \int_a^b f\,d\alpha_1+\int_a^b f\,d\alpha_2 \quad \ldots\ldots\ldots\ldots\ldots (ii)$$

Similarly by taking the supremum of lower sum of partition we arrive that

$$\int_a^b f\,d\alpha \leq \int_a^b f\,d\alpha_1+\int_a^b f\,d\alpha_2 \quad \ldots\ldots\ldots\ldots\ldots (iii)$$

From (*ii*) and (*iii*)

$$\int_a^b f\,d\alpha = \int_a^b f\,d\alpha_1+\int_a^b f\,d\alpha_2$$

i.e. $$\int_a^b f\,d(\alpha_1+\alpha_2) = \int_a^b f\,d\alpha_1+\int_a^b f\,d\alpha_2 \qquad \because\ \alpha=\alpha_1+\alpha_2$$

Now $\because\ f\in \boldsymbol{R}(\alpha)$ $\therefore$ for $\varepsilon>0$, $\exists$ a partition P of $[a,b]$ such that

$$U(P,f,\alpha)-L(P,f,\alpha) < \varepsilon \quad \ldots\ldots\ldots\ldots (iv)$$

Let $\alpha'=c\alpha$ then $\Delta\alpha_i'=\Delta(c\alpha_i)=c\Delta\alpha_i$

$$\Rightarrow U(P,f,\alpha')=\sum M_i\,\Delta\alpha_i'$$
$$=\sum M_i\,(c\Delta\alpha_i)$$
$$=c\sum M_i\,\Delta\alpha_i$$
$$=c\,U(P,f,\alpha)$$

Similarly, $L(P,f,\alpha')=c\,L(P,f,\alpha)$

$$\Rightarrow U(P,f,\alpha')-L(P,f,\alpha')=c\{U(P,f,\alpha)-L(P,f,\alpha)\}<c\varepsilon \qquad \text{by } (iv)$$

$$\Rightarrow f\in \boldsymbol{R}(\alpha') \quad \text{where } \alpha'=c\alpha$$

Also $$\int_a^b f\,d\alpha'=\inf U(P,f,\alpha')$$
$$=\inf\ c\,U(P,f,\alpha)$$
$$=c\inf U(P,f,\alpha)$$
$$=c\int_a^b f\,d\alpha$$

and

$$\int_a^b f\,d\alpha'=\sup L(P,f,\alpha')$$
$$=\sup cU(P,f,\alpha)$$
$$=c\sup U(P,f,\alpha)$$
$$=c\int_a^b f\,d\alpha$$

Hence

$$\int_a^b f\,d\alpha'=c\int_a^b f\,d\alpha \quad \text{where } \alpha'=c\alpha$$

⊙

➢ *Lemma*

If M & m are the supremum and infimum of f and M', m' are the supremum & infimum of $|f|$ on $[a,b]$ then $M'-m' \le M-m$.

Proof

Let $x_1, x_2 \in [a,b]$, then

$$\big||f(x_1)|-|f(x_2)|\big| \le |f(x_1)-f(x_2)| \ \ldots\ldots\ldots\ldots\ldots\ldots (A)$$

$\because$ M and m denote the supremum and infimum of $f(x)$ on $[a,b]$

$\therefore$ $f(x) \le M$ & $f(x) \ge m$ $\quad \forall\ x \in [a,b]$

$\because$ $x_1, x_2 \in [a,b]$

$\therefore$ $f(x_1) \le M$ and $f(x_2) \ge m$

$\Rightarrow$ $f(x_1) \le M$ and $-f(x_2) \le -m$

$\Rightarrow$ $f(x_1)-f(x_2) \le M-m$ ……………... (i)

Interchanging x_1 & x_2, we get

$$-\big[f(x_1)-f(x_2)\big] \le M-m \ \ldots\ldots\ldots\ldots (ii)$$

(i) & (ii) $\Rightarrow$ $|f(x_1)-f(x_2)| \le M-m$

$$\Rightarrow \big||f(x_1)|-|f(x_2)|\big| \le M-m \quad \text{by eq. } (A) \ \ldots\ldots\ldots\text{(I)}$$

$\because$ M' and m' denote the supremum and infimum of $|f(x)|$ on $[a,b]$

$\therefore$ $|f(x)| \le M'$ and $|f(x)| \ge m'$ $\quad \forall\ x \in [a,b]$

$\Rightarrow$ $\exists\ \varepsilon > 0$ such that

$$|f(x_1)| > M'-\varepsilon \ \ldots\ldots\ldots\ldots (iii)$$

$$\text{and} \quad |f(x_2)| < m'+\varepsilon \ \Rightarrow\ -|f(x_2)|+\varepsilon > -m' \ \ldots\ldots\ldots\ldots (iv)$$

From (iii) and (iv), we get

$$|f(x_1)|-|f(x_2)|+\varepsilon > M'-m'-\varepsilon$$

$$\Rightarrow 2\varepsilon + |f(x_1)|-|f(x_2)| > M'-m'$$

$\because$ ε is arbitrary $\therefore$ $M'-m' \le |f(x_1)|-|f(x_2)|$ …………..…. (v)

Interchanging x_1 & x_2, we get

$$M'-m' \le -\big(|f(x_1)|-|f(x_2)|\big) \ \ldots\ldots\ldots\ldots\ldots (vi)$$

Combining (v) and (vi), we get

$$M'-m' \le \big||f(x_1)|-|f(x_2)|\big| \ \ldots\ldots\ldots\ldots\ldots \text{(II)}$$

From (I) and (II), we have the require result

$$M'-m' \le M-m$$

⊙

➢ *Theorem*

If $f \in \boldsymbol{R}(\alpha)$ on $[a,b]$, then $|f| \in \boldsymbol{R}(\alpha)$ on $[a,b]$ and $\left|\int_a^b f\,d\alpha\right| \le \int_a^b |f|\,d\alpha$.

Proof

$\because$ $f \in \boldsymbol{R}(\alpha)$

$\therefore$ given $\varepsilon > 0$ $\exists$ a partition P of $[a,b]$ such that

$$U(P,f,\alpha)-L(P,f,\alpha) < \varepsilon$$

$$\text{i.e.} \quad \sum M_i\,\Delta\alpha_i - \sum m_i\,\Delta\alpha_i = \sum(M_i-m_i)\Delta\alpha_i < \varepsilon$$

Where M_i and m_i are supremum and infimum of f on $[x_{i-1},x_i]$

Now if M_i' and m_i' are supremum and infimum of $|f|$ on $[x_{i-1},x_i]$ then

$$M_i'-m_i' \le M_i-m_i$$

$$\Rightarrow \sum (M_i' - m_i')\Delta\alpha_i \le \sum (M_i - m_i)\Delta\alpha_i$$
$$\Rightarrow U(P,|f|,\alpha) - L(P,|f|,\alpha) \le U(P,f,\alpha) - L(P,f,\alpha) < \varepsilon$$
$$\Rightarrow |f| \in \boldsymbol{R}(\alpha).$$

Take $c = +1$ or -1 to make $c\int f\,d\alpha \ge 0$

Then $\left|\int_a^b f\,d\alpha\right| = c\int_a^b f\,d\alpha$ .... (i)

Also $c\,f(x) \le |f(x)| \quad \forall\ x\in[a,b]$

$$\Rightarrow \int_a^b c\,f\,d\alpha \le \int_a^b |f|\,d\alpha \Rightarrow c\int_a^b f\,d\alpha \le \int_a^b |f|\,d\alpha \text{ .... } (ii)$$

From (i) and (ii), we have

$$\left|\int_a^b f\,d\alpha\right| \le \int_a^b |f|\,d\alpha$$ ⊙

> ### ➢ *Theorem*
> If $f\in\boldsymbol{R}(\alpha)$ on $[a,b]$, then $f^2\in\boldsymbol{R}(\alpha)$ on $[a,b]$.

Proof

$\because\ f\in\boldsymbol{R}(\alpha) \Rightarrow |f|\in\boldsymbol{R}(\alpha)$

$\Rightarrow |f(x)| < M \quad \forall\ x\in[a,b]$

$\because\ f\in\boldsymbol{R}(\alpha)$ $\therefore$ given $\varepsilon>0$, $\exists$ a partition P of $[a,b]$ such that

$$U(P,f,\alpha) - L(P,f,\alpha) < \varepsilon/2M \text{ .... } (i)$$

If M_i & m_i denote the sup. & inf. of f on $[x_{i-1},x_i]$ then M_i^2 & m_i^2 are the sup. & inf. of f^2 on $[x_{i-1},x_i]$.

$$\Rightarrow U(P,f^2,\alpha) - L(P,f^2,\alpha) = \sum (M_i^2 - m_i^2)\Delta\alpha_i$$
$$= \sum (M_i + m_i)(M_i - m_i)\Delta\alpha_i$$

$\because\ f(x) \le |f(x)| \le M \quad \forall\ x\in[a,b]$

and $f^2 = |f|^2$

$\therefore\ M_i \le M$ & $m_i \le M$

$$\Rightarrow U(P,f^2,\alpha) - L(P,f^2,\alpha) \le \sum (M+M)(M_i - m_i)\Delta\alpha_i$$
$$= 2M\sum (M_i - m_i)\Delta\alpha_i$$
$$= 2M\left[U(P,f,\alpha) - L(P,f,\alpha)\right] < 2M\cdot\frac{\varepsilon}{2M} = \varepsilon$$

$\Rightarrow f^2\in\boldsymbol{R}(\alpha)$ ⊙

> ### ➢ *Corollary*
> If $f\in\boldsymbol{R}(\alpha)$ & $g\in\boldsymbol{R}(\alpha)$ on $[a,b]$ then $fg\in\boldsymbol{R}(\alpha)$ on $[a,b]$.

Proof

$\because\ f\in\boldsymbol{R}(\alpha)$, $g\in\boldsymbol{R}(\alpha)$

$\therefore\ f+g\in\boldsymbol{R}(\alpha)$, $f-g\in\boldsymbol{R}(\alpha)$

$\Rightarrow (f+g)^2\in\boldsymbol{R}(\alpha)$, $(f-g)^2\in\boldsymbol{R}(\alpha)$

$\Rightarrow (f+g)^2 - (f-g)^2\in\boldsymbol{R}(\alpha) \Rightarrow 4fg\in\boldsymbol{R}(\alpha)$

and ultimately

$$fg\in\boldsymbol{R}(\alpha) \text{ on } [a,b]$$ ⊙

> **Theorem**
>
> Assume α increases monotonically and $\alpha' \in \boldsymbol{R}$ on $[a,b]$. Let f be bounded real function on $[a,b]$. Then $f \in \boldsymbol{R}(\alpha)$ iff $f\alpha' \in \boldsymbol{R}$. In that case
>
> $$\int_a^b f\,d\alpha = \int_a^b f(x)\cdot\alpha'(x)\,dx$$

Proof

$\because$ $\alpha' \in \boldsymbol{R}$ on $[a,b]$

$\therefore$ given $\varepsilon > 0$ $\exists$ a partition P of $[a,b]$ such that

$$U(P,\alpha') - L(P,\alpha') < \varepsilon \quad \ldots\ldots\ldots\ldots\ldots (i)$$

The Mean-value theorem furnishes point $t_i \in [x_{i-1}, x_i]$ such that

$$\begin{aligned}\Delta\alpha_i &= \alpha(x_i) - \alpha(x_{i-1})\\ &= \alpha'(t_i)\Delta x_i \quad \text{for } i = 1,2,\ldots,n \quad \ldots\ldots\ldots\ldots (ii)\end{aligned}$$

If $s_i \in [x_{i-1}, x_i]$, then form (i) we have

$$\left|\sum \alpha'(s_i)\Delta x_i - \sum \alpha'(t_i)\Delta x_i\right| < \varepsilon \qquad |\text{ Previously proved at page 6}$$

$$\Rightarrow \sum\left|\alpha'(s_i) - \alpha'(t_i)\right|\Delta x_i < \varepsilon \quad \ldots\ldots\ldots\ldots\ldots (iii)$$

Put $M = \sup|f(x)|$ and consider

$$\begin{aligned}&\left|\sum f(s_i)\Delta\alpha_i - \sum f(s_i)\alpha'(s_i)\Delta x_i\right| \quad \ldots\ldots\ldots\ldots\ldots\ldots (A)\\ &= \left|\sum f(s_i)\alpha'(t_i)\Delta x_i - \sum f(s_i)\alpha'(s_i)\Delta x_i\right| \qquad \text{by } (ii)\\ &= \left|\sum f(s_i)\big(\alpha'(t_i) - \alpha'(s_i)\big)\Delta x_i\right|\\ &\le \left|\sum M\left(\alpha'(t_i) - \alpha'(s_i)\right)\right|\Delta x_i\\ &\le M\varepsilon \qquad \ldots\ldots\ldots\ldots\ldots\ldots (iv) \qquad \text{by } (iii)\end{aligned}$$

$\Rightarrow \sum f(s_i)\Delta\alpha_i \le \sum f(s_i)\alpha'(s_i)\Delta x_i + M\varepsilon$ for all choices of $s_i \in [x_{i-1}, x_i]$

$\Rightarrow U(P, f, \alpha) \le U(P, f\alpha') + M\varepsilon$

The same arguments leads from (A) to

$$U(P, f\alpha') \le U(P, f, \alpha) + M\varepsilon$$

Thus $\left|U(P, f, \alpha) - U(P, f\alpha')\right| \le M\varepsilon \quad \ldots\ldots\ldots\ldots\ldots (v)$

$\because$ (i) remains true if P is replaced by any refinement

$\therefore$ (v) also remains true

$$\Rightarrow \left|\overline{\int_a^b} f\,d\alpha - \overline{\int_a^b} f(x)\alpha'(x)\,dx\right| \le M\varepsilon$$

$\because$ ε was arbitrary

$$\therefore \overline{\int_a^b} f\,d\alpha = \overline{\int_a^b} f(x)\alpha'(x)\,dx \qquad \text{for any bounded } f.$$

Using the same argument, we can prove from (iv) by considering the infimum of $|f(x)|$ that

$$\underline{\int_a^b} f\,d\alpha = \underline{\int_a^b} f(x)\alpha'(x)\,dx$$

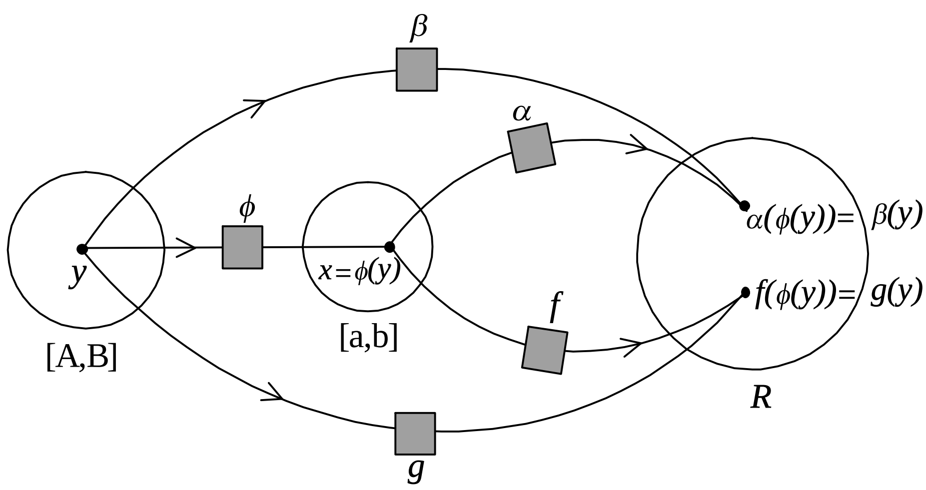
Hence

$$\overline{\int_a^b} f\,d\alpha = \underline{\int_a^b} f\,d\alpha \quad \Leftrightarrow \quad \overline{\int_a^b} f(x)\alpha'(x)\,dx = \underline{\int_a^b} f(x)\alpha'(x)\,dx$$

Equivalently $f \in \boldsymbol{R}(\alpha) \Leftrightarrow f\alpha' \in \boldsymbol{R}(\alpha)$. ⊙

> ➢ ***Theorem (Change of Variable)***
> Suppose φ is a strictly increasing continuous function that maps an interval $[A,B]$ onto $[a,b]$. Suppose α is monotonically increasing on $[a,b]$ and $f \in \mathbf{R}(\alpha)$ on $[a,b]$. Define β and g on $[A,B]$ by
> $$\beta(y)=\alpha(\varphi(y)) \quad , \quad g(y)=f(\varphi(y))$$
> then $g \in \mathbf{R}(\beta)$ and $\int_A^B g\,d\beta = \int_a^b f\,d\alpha$.

Proof



To each partition $P=\{x_0,.....,x_n\}$ of $[a,b]$ corresponds a partition $Q=\{y_0,.....,y_n\}$ of $[A,B]$ because φ maps $[A,B]$ onto $[a,b]$.

$\Rightarrow \; x_i=\varphi(y_i)$

All partitions of $[A,B]$ are obtained in this way.

$\because$ The value taken by f on $[x_{i-1},x_i]$ are exactly the same as those taken by g on $[y_{i-1},y_i]$, we see that

$$U(Q,g,\beta)=U(P,f,\alpha)$$
$$\text{and} \quad L(Q,g,\beta)=L(P,f,\alpha)$$

$\because \; f \in R(\alpha)$ on $[a,b]$

$\therefore$ given $\varepsilon>0$, we have

$$U(P,f,\alpha)-L(P,f,\alpha) < \varepsilon$$
$$\Rightarrow U(Q,g,\beta)-L(Q,g,\beta) < \varepsilon$$

$\Rightarrow \; g \in \mathbf{R}(\beta)$ and $\int_A^B g\,d\beta = \int_a^b f\,d\alpha$ ⊙

INTEGRATION AND DIFFERENTIATION

> **Theorem (Ist Fundamental Theorem of Calculus)**
>
> Let $f \in \boldsymbol{R}$ on $[a,b]$. For $a \le x \le b$, put $F(x) = \int_a^x f(t)\,dt$, then F is continuous on $[a,b]$; furthermore, if f is continuous at point x_0 of $[a,b]$, then F is differentiable at x_0, and $F'(x_0) = f(x_0)$.

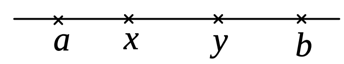
Proof

$\because\ f \in \boldsymbol{R}$

$\therefore\ f$ is bounded.

Let $|f(t)| \le M$ for $t \in [a,b]$

If $a \le x < y \le b$, then



$$|F(y) - F(x)| = \left| \int_a^y f(t)\,dt - \int_a^x f(t)\,dt \right|$$

$$= \left| \int_a^x f(t)\,dt + \int_x^y f(t)\,dt - \int_a^x f(t)\,dt \right|$$

$$= \left| \int_x^y f(t)\,dt \right| \le \int_x^y |f(t)|\,dt \le M \int_x^y dt = M(y-x)$$

$$\Rightarrow |F(y) - F(x)| < \varepsilon \quad \text{for } \varepsilon > 0 \text{ provided } M|y-x| < \varepsilon$$

i.e. $|F(y) - F(x)| < \varepsilon$ whenever $|y-x| < \dfrac{\varepsilon}{M}$

This proves the continuity (and, in fact, uniform continuity) of F on $[a,b]$.

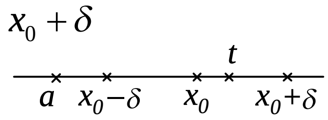
Next, we have to prove that if f is continuous at $x_0 \in [a,b]$ then F is differentiable at x_0 and $F'(x_0) = f(x_0)$

$$\text{i.e.} \quad \lim_{t \to x_0} \frac{F(t) - F(x_0)}{t - x_0} = f(x_0)$$

Suppose f is continuous at x_0. Given $\varepsilon > 0$, $\exists\ \delta > 0$ such that

$$|f(t) - f(x_0)| < \varepsilon \quad \text{if} \quad |t - x_0| < \delta \quad \text{where } t \in [a,b]$$

$$\Rightarrow f(x_0) - \varepsilon < f(t) < f(x_0) + \varepsilon \quad \text{if} \quad x_0 - \delta < t < x_0 + \delta$$



$$\Rightarrow \int_{x_0}^t \big(f(x_0) - \varepsilon\big)\,dt < \int_{x_0}^t f(t)\,dt < \int_{x_0}^t \big(f(x_0) + \varepsilon\big)\,dt$$

$$\Rightarrow \big(f(x_0) - \varepsilon\big) \int_{x_0}^t dt < \int_{x_0}^t f(t)\,dt < \big(f(x_0) + \varepsilon\big) \int_{x_0}^t dt$$

$$\Rightarrow \big(f(x_0) - \varepsilon\big)(t - x_0) < F(t) - F(x_0) < \big(f(x_0) + \varepsilon\big)(t - x_0)$$

$$\Rightarrow f(x_0) - \varepsilon < \frac{F(t) - F(x_0)}{t - x_0} < f(x_0) + \varepsilon$$

$$\Rightarrow \left| \frac{F(t) - F(x_0)}{t - x_0} - f(x_0) \right| < \varepsilon$$

$$\Rightarrow \lim_{t \to x_0} \frac{F(t) - F(x_0)}{t - x_0} = f(x_0)$$

$$\Rightarrow F'(x_0) = f(x_0)$$

⊙

℘ -------------------------------- ℘

➢ Theorem (IInd Fundamental Theorem of Calculus)

If $f\in \boldsymbol{R}$ on $[a,b]$ and if there is a differentiable function F on $[a,b]$ such that $F' = f$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Proof

$\because$ $f\in \boldsymbol{R}$ on $[a,b]$

$\therefore$ given $\varepsilon>0$, $\exists$ a partition P of $[a,b]$ such that

$$U(P,f) - L(P,f) < \varepsilon$$

$\because$ F is differentiable on $[a,b]$

$\therefore$ $\exists$ $t_i \in [x_{i-1},x_i]$ such that

$$F(x_i) - F(x_{i-1}) = F'(t_i)\Delta x_i$$

$$\Rightarrow F(x_i) - F(x_{i-1}) = f(t_i)\Delta x_i \quad \text{for } i = 1,2,\ldots.,n \quad \because F' = f$$

$$\Rightarrow \sum_{i=1}^{n} f(t_i)\Delta x_i = F(b) - F(a)$$

$$\Rightarrow \left| F(b) - F(a) - \int_a^b f(x)\,dx \right| < \varepsilon$$

$\because$ if $f\in \boldsymbol{R}(\alpha)$ then $\left|\sum f(t_i)\Delta\alpha_i - \int_a^b f\,d\alpha\right| < \varepsilon$

$\because$ ε is arbitrary

$$\therefore \int_a^b f(x)\,dx = F(b) - F(a)$$ ⊙

➢ Theorem (Integration by Parts)

Suppose F and G are differentiable function on $[a,b]$, $F' = f \in \boldsymbol{R}$ and $G' = g \in \boldsymbol{R}$ then

$$\int_a^b F(x)g(x)\,dx = F(b)G(b) - F(a)G(a) - \int_a^b f(x)G(x)\,dx$$

Proof

Put $H(x) = F(x)G(x)$

$\Rightarrow H' = F'(x)G(x) + F(x)G'(x) = h$

Now $\because$ $H\in \boldsymbol{R}$ and $h\in \boldsymbol{R}$ on $[a,b]$

$\therefore$ By applying the fundamental theorem of calculus to H and its derivative h, we have

$$\int_a^b h\,dx = H(b) - H(a)$$

$$\Rightarrow \int_a^b \left[F'(x)G(x) + F(x)G'(x)\right]dx = H(b) - H(a)$$

$$\Rightarrow \int_a^b f(x)G(x)\,dx + \int_a^b F(x)g(x)\,dx = F(b)G(b) - F(a)G(a)$$

$$\Rightarrow \int_a^b F(x)g(x)\,dx = F(b)G(b) - F(a)G(a) - \int_a^b f(x)G(x)\,dx$$ ⊙

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➢ ***Question***

Show that the function f defined on $[0,1]$ by

$$f(x)=\begin{cases}1 & ; \ x \text{ is rational}\\ 0 & ; \ x \text{ is irrational}\end{cases}$$

is not integrable on $[0,1]$

Solution

For any partition P of $[0,1]$, $m_k=0$, $M_k=1$

$$\Rightarrow S(P,f)=\sum_{k=1}^{n} M_k\,\Delta x_k=\sum_{k=1}^{n}\Delta x_k=1-0=1$$

and $\quad L(P,f)=\sum_{k=1}^{n} m_k\,\Delta x_k=0$

so that $\quad \overline{\int_0^1} f\,dx=1 \quad , \quad \underline{\int_0^1} f\,dx=0$

i.e. $\quad \overline{\int_0^1} f\,dx\neq\underline{\int_0^1} f\,dx \quad \Rightarrow f$ is not integrable on $[0,1]$. ⊙

➢ ***Question***

Show that $f(x)=\sin x$ is Riemann integrable over $\left[0,\frac{\pi}{2}\right]$.

Solution

Take $P=\left\{0,\frac{\pi}{2n},\frac{\pi}{n},\frac{3\pi}{2n},.....,\frac{n\pi}{2n}\right\}$ by dividing $\left[0,\frac{\pi}{2}\right]$ into n equal parts.

Then $M_k=\sin\frac{k\pi}{2n}$, $m_k=\sin\frac{(k-1)\pi}{2n}$

$$\Rightarrow S(P,f)-L(P,f)=\sum\left(\sin\frac{k\pi}{2n}-\sin\frac{(k-1)\pi}{2n}\right)\frac{\pi}{2n}$$

$$\leq\frac{\pi}{2n} < \varepsilon \qquad \text{for } n>n_0=\frac{\pi}{2\varepsilon}$$

$\Rightarrow f$ is Riemann integrable over $\left[0,\frac{\pi}{2}\right]$. ⊙

➢ ***Question***

Show that $f(x)=\begin{cases}1/x & ; \ x \text{ is rational}, \ 0<x\leq 1\\ 0 & ; \ x \text{ is irrational}\end{cases}$

is integrable on $[0,1]$.

Solution

f is continuous at each irrational. And rational numbers are dense in $[0,1]$.

Also $L(P,f)=0$ for any partition P of $[0,1]$ so that $\underline{\int_0^1} f\,dx=0$

$\because f\geq 0 \quad \therefore S(P,f)\geq 0 \quad \Rightarrow \overline{\int_0^1} f\,d\alpha\geq 0$ ………… (i)

$\because$ There are only finite number of points $\frac{p}{q}$ (rationals) for which $f\left(\frac{p}{q}\right)=\frac{q}{p}\geq\frac{\varepsilon}{2}$

$\therefore$ Suppose $f(x)\geq\frac{\varepsilon}{2}$ for k values of x in $[0,1]$

Take P_1 such that $|P_1|<\frac{\varepsilon}{2k}$.

Consider $S(P_1,f)=\sum_{i=1}^{n}M_i(x_i-x_{i-1})$

There are at most k values for which $\frac{\varepsilon}{2}\leq M_i\leq 1$. For all other values $M_i>\frac{\varepsilon}{2}$.

$$\Rightarrow S(P_1,f)=\sum_{k\text{ values}}M_i(x_i-x_{i-1})+\sum_{\textit{other}\text{ values}}M_i(x_i-x_{i-1})$$

$$\leq\frac{\varepsilon}{2k}\cdot k+\frac{\varepsilon}{2}\sum(x_i-x_{i-1})\ <\ \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$$

$\because$ ε is arbitrary

$\therefore$ $S(P_1,f)\leq 0$ and $\overline{\int_0^1} f\,dx\leq 0$ ………… (ii)

By (i) and (ii), we have

$$\overline{\int_0^1} f\,dx=0$$

Hence $\int_0^1 f\,dx=0$ ⊙

➢ *Note*

If f is integrable then $|f|$ is also integrable but the converse is false.

For example, let f be a function defined on $[a,b]$ by

$$f(x)=\begin{cases}1 & ;\ x\in\mathbb{Q}\cap[a,b]\\ -1 & ;\ \text{otherwise}\end{cases}$$

Then $|f|$ is Riemann-integrable but f is not.

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Chapter 7 – Functions of Bounded Variation.

Subject: *Real Analysis* ***Level:*** *M.Sc.*

Source: *Syed Gul Shah (Chairman, Department of Mathematics, US Sargodha)*

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We shall now discuss the concept of functions of bounded variation which is closely associated to the concept of monotonic functions and has wide application in mathematics. These functions are used in Riemann-Stieltjes integrals and Fourier series.

Let a function f be defined on an interval $[a,b]$ and $P=\{a=x_0,x_1,........,x_n=b\}$ be a partition of $[a,b]$. Consider the sum $\sum_{i=1}^{n}|f(x_i)-f(x_{i-1})|$. The set of these sums is infinite. It changes when we make a refinement in a partition. If this set of sums is bounded above then the function f is said to be a *bounded variation* and the supremum of the set is called the *total variation* of the function f on $[a,b]$, and is denoted by $V(f;a,b)$ or $V_f(a,b)$ and it is also affiliated as $V(f)$ or V_f.

Thus

$$V(f;a,b)=\sup\sum_{i=1}^{n}|f(x_i)-f(x_{i-1})|$$

The supremum being taken over all the partition of $[a,b]$.

Hence the function f is said to be of *bounded variation* on $[a,b]$ if, and only, if its total variation is finite i.e. $V(f;a,b)<\infty$.

Note

Since for $x\le c\le y$, we have

$$|f(y)-f(x)|\le|f(y)-f(c)|+|f(c)-f(x)|$$

Therefore the sum $\sum|f(x_i)-f(x_{i-1})|$ can not be decrease (it can, in fact only increase) by the refinement of the partition.

Theorem

A bounded monotonic function is a function of bounded variation.

Proof

Suppose a function f is monotonically increasing on $[a,b]$ and P is any partition of $[a,b]$ then

$$\sum_{i=1}^{n}|f(x_i)-f(x_{i-1})| = \sum_{i=1}^{n}\big(f(x_i)-f(x_{i-1})\big) = f(b)-f(a)$$

$$\therefore\ V(f;a,b)=\sup\sum|f(x_i)-f(x_{i-1})| = f(b)-f(a)\quad\text{(finite)}$$

Hence the function f is of bounded variation on $[a,b]$.

Similarly a monotonically decreasing bounded function is of bounded variation with total variation $=f(a)-f(b)$.

Thus for a bounded monotonic function f

$$V(f)=|f(b)-f(a)|$$ ❑

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Example

A continuous function may not be a function of bounded variation.

e.g. Consider a function f, where

$$f(x)\begin{cases} x\sin\dfrac{\pi}{x} & ; \text{ when } 0<x\le 1 \\ 0 & ; \text{ when } x=0 \end{cases}$$

It is clear that f is continuous on $[0,1]$.

Let us choose the partition $P=\left\{0,\dfrac{2}{2n+1},\dfrac{2}{2n-1},\ldots\ldots,\dfrac{2}{5},\dfrac{2}{3},1\right\}$

Then

$$\sum\left|f(x_i)-f(x_{i-1})\right|=\left|f(1)-f\left(\frac{2}{3}\right)\right|+\left|f\left(\frac{2}{3}\right)-f\left(\frac{2}{5}\right)\right|+\ldots\ldots+\left|f\left(\frac{2}{2n+1}\right)-f(0)\right|$$

$$=\left|\sin\pi-\frac{2}{3}\sin\left(\frac{3\pi}{2}\right)\right|+\left|\frac{2}{3}\sin\left(\frac{3\pi}{2}\right)-\frac{2}{5}\sin\left(\frac{5\pi}{2}\right)\right|+\ldots\ldots\ldots$$

$$\ldots\ldots+\left|\frac{2}{2n+1}\sin\left(\frac{(2n+1)\pi}{2}\right)-0\right|$$

$$=\frac{2}{3}+\left(\frac{2}{3}+\frac{2}{5}\right)+\left(\frac{2}{5}+\frac{2}{7}\right)+\ldots\ldots\ldots\ldots+\left(\frac{2}{2n-1}+\frac{2}{2n+1}\right)+\frac{2}{2n+1}$$

$$=\left(2\left(\frac{2}{3}\right)+2\left(\frac{2}{5}\right)+2\left(\frac{2}{7}\right)+\ldots\ldots\ldots\ldots+2\left(\frac{2}{2n+1}\right)\right)$$

$$=4\left(\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\ldots\ldots\ldots\ldots+\frac{1}{2n+1}\right)$$

Since the infinite series $\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\ldots\ldots\ldots$ is divergent, therefore its partial sums sequence $\{S_n\}$, where $S_n=\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\ldots\ldots+\frac{1}{2n+1}$, is not bounded above.

Thus $\sum\left|f(x_i)-f(x_{i-1})\right|$ can be made arbitrarily large by taking n sufficiently large.

$\Rightarrow V(f;0,1)\to\infty$ and so f is not of bounded variation. ❑

Remarks

A function of bounded variation is not necessarily continuous.

e.g. the step-function $f(x)=[x]$, where $[x]$ denotes the greatest integer not greater than x, is a function of bounded variation on $[0,2]$ but is not continuous.

Theorem

If the derivative of the function f exists and is bounded on $[a,b]$, then f is of bounded variation on $[a,b]$.

Proof

$\because$ f' is bounded on $[a,b]$

$\therefore$ $\exists$ k such that $|f'(x)|\le k \quad \forall\ x\in[a,b]$.

Let P be any partition of the interval $[a,b]$ then

$$\sum\left|f(x_i)-f(x_{i-1})\right|=\sum\left|x_i-x_{i-1}\right|f'(c)\ , \quad c\in[a,b] \quad \text{(by M.V.T)}$$

$$\le k|b-a|$$

$\Rightarrow V(f;a,b)$ is finite. $\Rightarrow$ f is of bounded variation. ❑

Note

Boundedness of f' is a sufficient condition for $V(f)$ to be finite and is not necessary.

☙ Theorem

A function of bounded variation is necessarily bounded.

Proof

Suppose f is of bounded variation on $[a,b]$.

For any $x\in[a,b]$, consider the partition $\{a,x,b\}$, consisting of just three points then

$$|f(x)-f(a)|+|f(b)-f(x)| \le V(f;a,b)$$

$$\Rightarrow |f(x)-f(a)| \le V(f;a,b)$$

Again

$$\begin{aligned}|f(x)| &= |f(a)+f(x)-f(a)|\\ &\le |f(a)|+|f(x)-f(a)|\\ &\le |f(a)|+V\big(f;a,b\big) < \infty\end{aligned}$$

$\Rightarrow$ f is bounded on $[a,b]$. ❑

☙ Properties of functions of bounded variation

1) The sum (difference) of two functions of bounded variation is also of bounded variation.

Proof

Let f and g be two functions of bounded variation on $[a,b]$. Then for any partition P of $[a,b]$ we have

$$\begin{aligned}\sum\big|(f+g)(x_i)-(f+g)(x_{i-1})\big| &= \sum\big|\{f(x_i)+g(x_i)\}-\{f(x_{i-1})+g(x_{i-1})\}\big|\\ &= \sum\big|f(x_i)-f(x_{i-1})+g(x_i)-g(x_{i-1})\big|\\ &\le \sum\big|f(x_i)-f(x_{i-1})\big|+\sum\big|g(x_i)-g(x_{i-1})\big|\\ &\le V(f;a,b)+V(g;a,b)\end{aligned}$$

$$\Rightarrow V\big(f+g;a,b\big) \le V(f;a,b)+V(g;a,b)$$

This show that the function $f+g$ is of bounded variation.

Similarly it can be shown that $f-g$ is also of bounded variation.

i.e. $V(f-g) \le V(f)+V(g)$ ❑

Note

(i) If f and g are monotonic increasing on $[a,b]$ then $(f-g)$ is of bounded variation on $[a,b]$.

(ii) If c is constant, the sums $\sum|f(x_i)-f(x_{i-1})|$ and therefore the total variation function, $V(f)$ is same for f and $f-c$.

2) The product of two functions of bounded variation is also of bounded variation.

Proof

Let f and g be two functions of bounded variation on $[a,b]$.

$\Rightarrow$ f and g are bounded and $\exists$ a number k such that

$$|f(x)| \le k \quad \& \quad |g(x)| \le k \qquad \forall\ x\in[a,b].$$

For any partition P of $[a,b]$ we have

$$
\begin{aligned}
&\sum\left|(fg)(x_i)-(fg)(x_{i-1})\right| \\
&=\sum\left|f(x_i)g(x_i)-f(x_{i-1})g(x_{i-1})\right| \\
&=\sum\left|f(x_i)g(x_i)-f(x_i)g(x_{i-1})+f(x_i)g(x_{i-1})-f(x_{i-1})g(x_{i-1})\right| \\
&=\sum\left|f(x_i)\{g(x_i)-g(x_{i-1})\}+g(x_{i-1})\{f(x_i)-f(x_{i-1})\}\right| \\
&\le\sum\left|f(x_i)\right|\left|g(x_i)-g(x_{i-1})\right|+\sum\left|g(x_{i-1})\right|\left|f(x_i)-f(x_{i-1})\right| \\
&\le k\sum\left|g(x_i)-g(x_{i-1})\right|+k\sum\left|f(x_i)-f(x_{i-1})\right| \\
&\le k\,V(g)+k\,V(f)
\end{aligned}
$$

$\Rightarrow$ fg is of bounded variation on $[a,b]$. ❑

✍ Note

Theorems like the above, could not be applied to quotients of functions because the reciprocal of a function of bounded variation need not be of bounded variation.

e.g. if $f(x)\to 0$ as $x\to x_0$, then $\dfrac{1}{f(x)}$ will not be bounded and therefore can not be of bounded variation on any interval which contains x_0.

Therefore to consider quotient, we avoid functions whose values becomes arbitrarily close to zero.

3) If f is a function of bounded variation on $[a,b]$ and if $\exists$ a positive number k such that $|f(x)| \ge k \quad \forall\ x\in[a,b]$ then $\dfrac{1}{f}$ is also of bounded variation on $[a,b]$.

Proof

For any partition P of $[a,b]$, we have

$$
\begin{aligned}
\sum\left|\frac{1}{f}(x_i)-\frac{1}{f}(x_{i-1})\right| &=\sum\left|\frac{1}{f(x_i)}-\frac{1}{f(x_{i-1})}\right| \\
&=\sum\left|\frac{f(x_{i-1})-f(x_i)}{f(x_i)\,f(x_{i-1})}\right| \\
&\le\frac{1}{k^2}\sum\left|f(x_{i-1})-f(x_i)\right| \le \frac{1}{k^2}V\left(f;a,b\right)
\end{aligned}
$$

$\Rightarrow$ $\dfrac{1}{f}$ is of bounded variation on $[a,b]$. ❑

4) If f is of bounded variation on $[a,b]$, then it is also of bounded variation on $[a,c]$ and $[c,b]$, where c is a point of $[a,b]$, and conversely. Also

$$V(f;a,b)=V(f;a,c)+V(f;c,b).$$

Proof

a) Let, first, f be of bounded variation on $[a,b]$.

Take $P_1=\{a=x_0,x_1,.....,x_m=c\}$ & $P_2=\{c=y_0,y_1,.....,y_n=b\}$ any two partitions of $[a,c]$ and $[c,b]$ respectively.

Evidently, $P=P_1\cup P_2=\{a=x_0,....,x_m,y_0,....,y_n=b\}$ is a partition of $[a,b]$.

We have

$$\left\{\sum_{i=1}^{m}\left|f(x_i)-f(x_{i-1})\right|+\sum_{i=1}^{n}\left|f(y_i)-f(y_{i-1})\right|\right\} \le V\left(f;a,b\right)$$

$$\Rightarrow \sum_{i=1}^{m} \left| f(x_i) - f(x_{i-1}) \right| \le V\left(f;a,b\right)$$

and $$\sum_{i=1}^{n} \left| f(y_i) - f(y_{i-1}) \right| \le V\left(f;a,b\right)$$

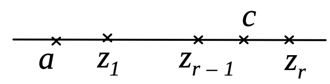
$\Rightarrow$ f is of bounded variation on $[a,c]$ and $[c,b]$ both.

b) Let, now, f be of bounded variation on $[a,c]$ and $[c,b]$ both.

Let $P = \{a = z_0, z_1, \ldots\ldots, z_n = b\}$ be a partition of $[a,b]$.

If it does not contain the point c, let us consider the partition $P^* = P \cup \{c\}$

Let $c \in [z_{r-1}, z_r]$ i.e. $z_{r-1} \le c \le z_r$, $r < n$



Then

$$\sum_{i=1}^{n} \left| f(z_i) - f(z_{i-1}) \right| = \sum_{i=1}^{r-1} \left| f(z_i) - f(z_{i-1}) \right| + \left| f(z_r) - f(z_{r-1}) \right| + \sum_{i=r+1}^{n} \left| f(z_i) - f(z_{i-1}) \right|$$

$$\le \sum_{i=1}^{r-1} \left| f(z_i) - f(z_{i-1}) \right| + \left| f(c) - f(z_{r-1}) \right|$$

$$+ \left| f(z_r) - f(c) \right| + \sum_{i=r+1}^{n} \left| f(z_i) - f(z_{i-1}) \right|$$

$$\le V\left(f;a,c\right) + V\left(f;c,b\right)$$

$\Rightarrow$ f is of bounded variation on $[a,b]$ if it is of bounded variation on $[a,c]$ & $[c,b]$ both, then

$$V\left(f;a,b\right) \le V\left(f;a,c\right) + V\left(f;c,b\right) \quad \ldots\ldots\ldots\ldots (i)$$

Now let $\varepsilon > 0$ be any arbitrary number.

Since $V\left(f;a,c\right)$ and $V\left(f;c,b\right)$ are the total variation of f on $[a,c]$ & $[c,b]$ respectively therefore $\exists$ partition $P_1 = \{a = x_0, x_1, x_2, \ldots\ldots, x_m = c\}$ and $P_2 = \{c = y_0, y_1, y_3, \ldots\ldots, y_n = b\}$ of $[a,c]$ & $[c,b]$ respectively such that

$$\sum_{i=1}^{m} \left| f(x_i) - f(x_{i-1}) \right| > V\left(f;a,c\right) - \frac{\varepsilon}{2} \quad \ldots\ldots\ldots\ldots\ldots\ldots (ii)$$

& $$\sum_{i=1}^{n} \left| f(y_i) - f(y_{i-1}) \right| > V\left(f;c,b\right) - \frac{\varepsilon}{2} \quad \ldots\ldots\ldots\ldots\ldots (iii)$$

Adding (ii) and (iii) we get

$$\sum_{i=1}^{m} \left| f(x_i) - f(x_{i-1}) \right| + \sum_{i=1}^{n} \left| f(y_i) - f(y_{i-1}) \right| > V\left(f;a,c\right) + V\left(f;c,b\right) - \varepsilon$$

$$\Rightarrow V\left(f;a,b\right) > V\left(f;a,c\right) + V\left(f;c,b\right) - \varepsilon$$

But ε is arbitrary positive number therefore we get

$$V\left(f;a,b\right) \ge V\left(f;a,c\right) + V\left(f;c,b\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots (iv)$$

From (i) and (iv), we get

$$V\left(f;a,b\right) = V\left(f;a,c\right) + V\left(f;c,b\right)$$ ❑

Variation Function

Let f be a function of bounded variation on $[a,b]$ and x is a point of $[a,b]$. Then the total variation of f is $V\left(f;a,x\right)$ on $[a,x]$, which is clearly a function of x, is called the *total variation function* or simply the *variation function* of f and is denoted by $V_f(x)$, and when there is no scope for confusion, it is simply written as $V(x)$.

Thus $V_f(x) = V\left(f;a,x\right) \quad ; \quad (a \le x \le b)$

If x_1, x_2 are two points of the interval $[a,b]$ such that $x_2 > x_1$, then

$$\begin{aligned} 0 \le \left| f(x_2) - f(x_1) \right| &\le V\left(f;x_1,x_2\right) \\ &= V\left(f;a,x_1\right) - V\left(f;a,x_2\right) \\ &= V_f(x_2) - V_f(x_1) \end{aligned}$$

$$\Rightarrow V_f(x_2) \ge V_f(x_1)$$

implies that the variation function is monotonically increasing function on $[a,b]$.

CHARACTERIZATION OF FUNCTIONS OF BOUNDED VARIATION

Theorem

A function of bounded variation is expressible as the difference of two monotonically increasing function.

Proof

We have

$$\begin{aligned} f(x) &= \frac{1}{2}\left(V(x) + f(x)\right) - \frac{1}{2}\left(V(x) - f(x)\right) \\ &= G(x) - H(x) \qquad \text{(say)} \end{aligned}$$

We shall prove that these two functions $G(x)$ and $H(x)$ are monotonically increasing on $[a,b]$.

Now, if $x_2 > x_1$, we have

$$\begin{aligned} G(x_2) - G(x_1) &= \frac{1}{2}\left[V(x_2) - V(x_1) + f(x_2) - f(x_1)\right] \\ &= \frac{1}{2}\left[V(f;x_1,x_2) - \left(f(x_1) - f(x_2)\right)\right] \end{aligned}$$

Since $V(f;x_1,x_2) \ge f(x_1) - f(x_2)$

$$\Rightarrow G(x_2) - G(x_1) \ge 0 \qquad \text{i.e.} \quad G(x_2) \ge G(x_1)$$

so that the function $G(x)$ is monotonically increasing on $[a,b]$.

Again, we have

$$\begin{aligned} H(x_2) - H(x_1) &= \frac{1}{2}\left[\left(V(x_2) - V(x_1)\right) - \left(f(x_2) - f(x_1)\right)\right] \\ &= \frac{1}{2}\left[V(f;x_1,x_2) - \left(f(x_2) - f(x_1)\right)\right] \end{aligned}$$

so that as before

$$H(x_2) - H(x_1) \ge 0 \quad \text{i.e.} \ H(x_2) \ge H(x_1).$$

i.e. $H(x)$ is also monotonically increasing function.

Hence the result. ❑

Note

A function $f(x)$ is of bounded variation over the interval $[a,b]$ iff it can be expressed as the difference of two monotonically functions.

✍ Theorem

Let f be of bounded variation on $[a,b]$. Let V be defined on $[a,b]$ as follows:

$$V(x)=V_f(x)=V\big(f;a,x\big) \quad \text{if} \quad a<x\le b \ , \ V(a)=0 .$$

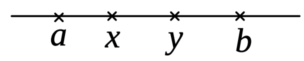
Then

i) V is an increasing function on $[a,b]$.

ii) $\big(V-f\big)$ is an increasing function on $[a,b]$.

Proof

If $a<x<y\le b$, we can write



$$V(f;a,y)=V(f;a,x)-V(f;x,y)$$

$$\Rightarrow V(y)-V(x)=V(f;x,y)$$

$$\because \ V(f;x,y)\ \ge 0$$

$$\therefore \ V(y)-V(x)\ \ge 0 \quad \Rightarrow V(x)\le V(y)$$ and (*i*) holds.

To prove (*ii*), let $D(x)=V(x)-f(x)$ if $x\in[a,b]$.

Then, if $a\le x<y\le b$, we have

$$D(y)-D(x)=\big[V(y)-V(x)\big]-\big[f(y)-f(x)\big]$$

$$=V(f;x,y)-\big[f(y)-f(x)\big]$$

But from the definition of $V(f;x,y)$, it follows that

$$f(y)-f(x)\ \le\ V(f;x,y)$$

This means that $D(y)-D(x)\ \ge\ 0$ and (*ii*) holds. ❑

✍ Theorem

If c be any point of $[a,b]$, then $V(x)$ is continuous at c if and only if $f(x)$ is continuous at c.

i.e. A point of continuity of $f(x)$ is also a point of continuity of $V(x)$ and conversely.

Proof

Firstly suppose that $V(x)$ is continuous at c.

Let $\varepsilon>0$ be given, then $\exists$ $\delta>0$ such that

$$\big|V(x)-V(c)\big|<\varepsilon \quad \text{for} \quad \big|x-c\big|<\delta \quad \ldots\ldots\ldots\ldots (i)$$

Also, we have

$$\big|f(x)-f(c)\big|\le V(x)-V(c) \quad \text{if} \quad x>c \quad \ldots\ldots\ldots\ldots\ldots (ii)$$

And

$$\big|f(x)-f(c)\big|\le V(c)-V(x) \quad \text{if} \quad x<c \quad \ldots\ldots\ldots\ldots\ldots (iii)$$

From (*i*), (*ii*) and (*iii*), we deduce that

$$\big|f(x)-f(c)\big|\le\big|V(x)-V(c)\big|<\varepsilon \quad \text{for} \quad \big|x-c\big|<\delta$$

Which shows that $f(x)$ is continuous at c.

Now suppose that c is a point of continuity of $f(x)$ and let $\varepsilon>0$ be given, then $\exists$ $\delta>0$ such that

$$\big|f(x)-f(c)\big|<\frac{\varepsilon}{2} \quad \text{for} \quad \big|x-c\big|<\delta$$

Also $\exists$ a partition $P=\big\{c=y_0,y_1,\ldots\ldots,y_{q-1},y_q,\ldots\ldots,y_n=b\big\}$ of $[c,b]$ such that

$$\sum_{q=1}^{n}\big|f(y_q)-f(y_{q-1})\big|>V(f;c,b)-\frac{1}{2}\varepsilon \quad \ldots\ldots\ldots\ldots\ldots (iv)$$

Since as a result of introducing addition points to the partition P, the corresponding sum of the moduli of the differences of the function values at end points will not be decreased, therefore we may assume that

$$0 < y_1 - c < \delta$$

so that $\left| f(y_1) - f(c) \right| < \dfrac{\varepsilon}{2}$ ……………. (v)

Thus (iv) becomes

$$V(f;c,b) - \frac{1}{2}\varepsilon < \frac{1}{2}\varepsilon + \sum_{q=2}^{n} \left| f(y_q) - f(y_{q-1}) \right| < \frac{1}{2}\varepsilon + V(f;y_1,b)$$

$$\Rightarrow V(f;c,b) - V(f;y_1,b) < \varepsilon$$

$$\Rightarrow V(y_1) - V(c) < \varepsilon$$

Thus for $0 < y_1 - c < \delta$, we have $0 < V(y_1) - V(c) < \varepsilon$

$$\therefore \lim_{x\to c+0} V(x) = V(c)$$

Similarly, we can have

$$\lim_{x\to c-0} V(x) = V(c)$$

Which shows that $V(x)$ is continuous at c. ❑

Note

$V(x)$ is continuous in $[a,b]$ iff $f(x)$ is continuous in $[a,b]$.

Corollary

A function f is of bounded variation on $[a,b]$ iff there is a bounded increasing function g on $[a,b]$ such that for any two points x' and x'' in $[a,b]$, $x' < x''$, we have

$$\left| f(x'') - f(x') \right| \le g(x'') - g(x')$$

Moreover, if g is continuous at x', so is f.

Proof

Take $g(x) = \begin{cases} V_a^x & , \ a < x \le b \\ 0 & , \ x = a \end{cases}$

Then g is increasing and bounded on $[a,b]$.

Also, $\left| f(x') - f(x'') \right| \le V_{x'}^{x''}(f) = g(x'') - g(x')$

Which also yields that if g is continuous at x', so is f. ❑

Question

Show that the function f defined by

$$f(x) = \begin{cases} x^2 \sin\dfrac{1}{x} & ; \ x \ne 0 \\ 0 & ; \ x = 0 \end{cases}$$

is of bounded variation on $[0,1]$.

Solution

f is differentiable on $[0,1]$ and $f'(x) = 2x\sin\dfrac{1}{x} - \sin x$ for $0 \le x \le 1$.

Also

$$\left| f'(x) \right| \le \left| 2x\sin\frac{1}{x} \right| + \left| \sin x \right| \le 2 + 1 = 3$$

i.e. $f'(x)$ is bounded on $[0,1]$

Hence f is of bounded variation on $[0,1]$. ❑

Question

Show that $g(x)=\begin{cases} x\cos\dfrac{\pi x}{2} & , \ 0<x\leq 1 \\ 0 & , \ x=0 \end{cases}$ is not of bounded variation on $[0,1]$

Solution

Let $P=\left\{0,\dfrac{1}{2n},\dfrac{1}{2n-1},.....,\dfrac{1}{3},\dfrac{1}{2},1\right\}$ be a partition of $[0,1]$.

Then

$$\sum\left|f(x_i)-f(x_{i-1})\right|$$
$$=\left|f(1)-f\left(\frac{1}{2}\right)\right|+\left|f\left(\frac{1}{2}\right)-f\left(\frac{1}{3}\right)\right|+\left|f\left(\frac{1}{3}\right)-f\left(\frac{1}{4}\right)\right|+......+\left|f\left(\frac{1}{2n}\right)-f(0)\right|$$
$$=\left|\cos\frac{\pi}{2}-\frac{1}{2}\cos\frac{\pi}{4}\right|+\left|\frac{1}{2}\cos\frac{\pi}{4}-\frac{1}{3}\cos\frac{\pi}{6}\right|+\left|\frac{1}{3}\cos\frac{\pi}{6}-\frac{1}{4}\cos\frac{\pi}{8}\right|+.....+\left|\frac{1}{2n}\cos\frac{\pi}{4n}-0\right|$$
$$=2\left(\frac{1}{2}\cos\frac{\pi}{4}\right)+2\left(\frac{1}{3}\cos\frac{\pi}{6}\right)+2\left(\frac{1}{4}\cos\frac{\pi}{8}\right)+.....+2\left(\frac{1}{2n}\cos\frac{\pi}{4n}\right)$$
$$=2\left(\frac{1}{2}\cos\frac{\pi}{4}+\frac{1}{3}\cos\frac{\pi}{6}+\frac{1}{4}\cos\frac{\pi}{8}+......+\frac{1}{2n}\cos\frac{\pi}{4n}\right)$$

which is not bounded.

Hence $f(x)$ is not of bounded variation on $[0,1]$. ❑

Alternative

We have

$$\left|g(x_{k+1})-g(x_k)\right|+\left|g(x_k)-g(x_{k-1})\right|$$
$$=\left|\frac{1}{k+1}\cos\frac{(k+1)\pi}{2}-\frac{1}{k}\cos\frac{k\pi}{2}\right|+\left|\frac{1}{k}\cos\frac{k\pi}{2}-\frac{1}{k-1}\cos\frac{(k-1)\pi}{2}\right|$$
$$=\begin{cases} \dfrac{2}{k} & ; \ \text{if } k \text{ is even} \\ \dfrac{1}{k+1}+\dfrac{1}{k-1} & ; \ \text{if } k \text{ is odd} \end{cases}$$

$$\Rightarrow V_a^b(g) \leq \sum_{k=1}^{n}\frac{1}{k} \leq \sum_{k=1}^{\infty}\frac{1}{k}$$

$\because \displaystyle\sum_{k=1}^{\infty}\frac{1}{k}$ is divergent $\therefore V_a^b(g)$ is not finite.

Hence g is not of bounded variation. ❑

References: (1) *Lectures & Notes (Year 2003-04)*
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Chapter 8 – Improper Integrals.

Subject: *Real Analysis (Mathematics)* **Level:** *M.Sc.*

Source: *Syed Gul Shah (Chairman, Department of Mathematics, US Sargodha)*

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We discussed Riemann-Stieltjes's integrals of the form $\int_a^b f\,d\alpha$ under the restrictions that both f and α are defined and bounded on a finite interval $[a,b]$. To extend the concept, we shall relax these restrictions on f and α.

➢ ***Definition***

The integral $\int_a^b f\,d\alpha$ is called an improper integral of first kind if $a=-\infty$ or $b=+\infty$ or *both* i.e. one or both integration limits is infinite.

➢ ***Definition***

The integral $\int_a^b f\,d\alpha$ is called an improper integral of second kind if $f(x)$ is unbounded at one or more points of $a\le x\le b$. Such points are called singularities of $f(x)$.

➢ ***Notations***

We shall denote the set of all functions f such that $f\in R(\alpha)$ on $[a,b]$ by $R(\alpha;a,b)$. When $\alpha(x)=x$, we shall simply write $R(a,b)$ for this set. The notation $\alpha\uparrow$ on $[a,\infty)$ will mean that α is monotonically increasing on $[a,\infty)$.

➢ ***Definition***

Assume that $f\in R(\alpha;a,b)$ for every $b\ge a$. Keep a,α and f fixed and define a function I on $[a,\infty)$ as follows:

$$I(b)=\int_a^b f(x)\,d\alpha(x) \quad \text{if} \quad b\ge a \;\ldots\ldots\ldots\ldots\; (i)$$

The function I so defined is called an infinite (or an improper) integral of first kind and is denoted by the symbol $\int_a^\infty f(x)\,d\alpha(x)$ or by $\int_a^\infty f\,d\alpha$.

The integral $\int_a^\infty f\,d\alpha$ is said to converge if the limit

$$\lim_{b\to\infty} I(b) \;\ldots\ldots\ldots\ldots\; (ii)$$

exists (finite). Otherwise, $\int_a^\infty f\,d\alpha$ is said to diverge.

If the limit in (*ii*) exists and equals A, the number A is called the value of the integral and we write $\int_a^\infty f\,d\alpha = A$

➢ ***Example***

Consider $\int_1^b x^{-p}\,dx$.

$\int_1^b x^{-p}\,dx=\dfrac{\left(1-b^{1-p}\right)}{p-1}$ if $p\neq 1$, the integral $\int_1^\infty x^{-p}\,dx$ diverges if $p<1$. When $p>1$, it converges and has the value $\dfrac{1}{p-1}$.

If $p=1$, we get $\int_1^b x^{-1}\,dx=\log b\to\infty$ as $b\to\infty$. $\Rightarrow \int_1^\infty x^{-1}\,dx$ diverges.

➢ *Example*

Consider $\int_0^b \sin 2\pi x\,dx$

$$\because \int_0^b \sin 2\pi x\,dx = \frac{(1-\cos 2\pi b)}{2\pi} \to \infty \quad \text{as} \quad b \to \infty.$$

$\therefore$ the integral $\int_0^\infty \sin 2\pi x\,dx$ diverges.

➢ *Note*

If $\int_{-\infty}^{a} f\,d\alpha$ and $\int_{a}^{\infty} f\,d\alpha$ are both convergent for some value of a, we say that the integral $\int_{-\infty}^{\infty} f\,d\alpha$ is convergent and its value is defined to be the sum

$$\int_{-\infty}^{\infty} f\,d\alpha = \int_{-\infty}^{a} f\,d\alpha + \int_{a}^{\infty} f\,d\alpha$$

The choice of the point a is clearly immaterial.

If the integral $\int_{-\infty}^{\infty} f\,d\alpha$ converges, its value is equal to the limit: $\lim_{b\to+\infty} \int_{-b}^{b} f\,d\alpha$.

➢ *Theorem*

Assume that $\alpha \uparrow$ on $[a,+\infty)$ and suppose that $f \in R(\alpha;a,b)$ for every $b \geq a$. Assume that $f(x) \geq 0$ for each $x \geq a$. Then $\int_a^\infty f\,d\alpha$ converges if, and only if, there exists a constant $M > 0$ such that

$$\int_a^b f\,d\alpha \leq M \quad \text{for every} \quad b \geq a.$$

Proof

We have $I(b) = \int_a^b f(x)\,d\alpha(x), \quad b \geq a$

$\Rightarrow I \uparrow$ on $[a,+\infty)$

Then $\lim_{b\to+\infty} I(b) = \sup\{I(b) \,|\, b \geq a\} = M > 0$ and the theorem follows

$\Rightarrow \int_a^b f\,d\alpha \leq M$ for every $b \geq a$ whenever the integral converges.

➢ *Theorem: (Comparison Test)*

Assume that $\alpha \uparrow$ on $[a,+\infty)$. If $f \in R(\alpha;a,b)$ for every $b \ge a$, if $0 \le f(x) \le g(x)$ for every $x \ge a$, and if $\int_a^\infty g\,d\alpha$ converges, then $\int_a^\infty f\,d\alpha$ converges and we have

$$\int_a^\infty f\,d\alpha \le \int_a^\infty g\,d\alpha$$

Proof

Let $I_1(b) = \int_a^b f\,d\alpha$ and $I_2(b) = \int_a^b g\,d\alpha$, $b \ge a$

$\because\ 0 \le f(x) \le g(x)$ for every $x \ge a$

$\therefore\ I_1(b) \le I_2(b)$ ………………… (i)

$\because\ \int_a^\infty g\,d\alpha$ converges $\therefore\ \exists$ a constant $M > 0$ such that

$$\int_a^\infty g\,d\alpha \le M \quad , \quad b \ge a \ \ldots\ldots\ldots\ldots\ldots\ldots (ii)$$

From (i) and (ii) we have $I_1(b) \le M$, $b \ge a$.

$\Rightarrow \lim_{b\to\infty} I_1(b)$ exists and is finite.

$\Rightarrow \int_a^\infty f\,d\alpha$ converges.

Also $\lim_{b\to\infty} I_1(b) \le \lim_{b\to\infty} I_2(b) \le M$

$\Rightarrow \int_a^\infty f\,d\alpha \le \int_a^\infty g\,d\alpha$.

➢ *Theorem (Limit Comparison Test)*

Assume that $\alpha \uparrow$ on $[a,+\infty)$. Suppose that $f \in R(\alpha;a,b)$ and that $g \in R(\alpha;a,b)$ for every $b \ge a$, where $f(x) \ge 0$ and $g(x) \ge 0$ if $x \ge a$. If

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$$

then $\int_a^\infty f\,d\alpha$ and $\int_a^\infty g\,d\alpha$ both converge or both diverge.

Proof

For all $b \ge a$, we can find some $N > 0$ such that

$$\left| \frac{f(x)}{g(x)} - 1 \right| < \varepsilon \qquad \forall\ x \ge N \ \text{ for every } \ \varepsilon > 0.$$

$$\Rightarrow 1 - \varepsilon < \frac{f(x)}{g(x)} < 1 + \varepsilon$$

Let $\varepsilon = \frac{1}{2}$, then we have

$$\frac{1}{2} < \frac{f(x)}{g(x)} < \frac{3}{2}$$

$\Rightarrow g(x) < 2f(x)$ ……….(i) and $2f(x) < 3g(x)$ ……….(ii)

From (i) $\int_a^\infty g\,d\alpha < 2\int_a^\infty f\,d\alpha$

$\Rightarrow \int_a^\infty g\,d\alpha$ converges if $\int_a^\infty f\,d\alpha$ converges and $\int_a^\infty f\,d\alpha$ diverges if $\int_a^\infty f\,d\alpha$ diverges.

From (ii) $2\int_a^\infty f\,d\alpha < 3\int_a^\infty g\,d\alpha$

$\Rightarrow \int_a^\infty f\,d\alpha$ converges if $\int_a^\infty g\,d\alpha$ converges and $\int_a^\infty g\,d\alpha$ diverges if $\int_a^\infty f\,d\alpha$ diverges.

$\Rightarrow$ The integrals $\int_a^\infty f\,d\alpha$ and $\int_a^\infty g\,d\alpha$ converge or diverge together.

➢ Note

The above theorem also holds if $\lim_{x\to\infty}\frac{f(x)}{g(x)} = c$, provided that $c \neq 0$. If $c = 0$, we can only conclude that convergence of $\int_a^\infty g\,d\alpha$ implies convergence of $\int_a^\infty f\,d\alpha$.

➢ Example

For every real p, the integral $\int_1^\infty e^{-x}x^p\,dx$ converges.

This can be seen by comparison of this integral with $\int_1^\infty \frac{1}{x^2}dx$.

Since $\lim_{x\to\infty}\frac{f(x)}{g(x)} = \lim_{x\to\infty}\frac{e^{-x}x^p}{1/x^2}$ where $f(x) = e^{-x}x^p$ and $g(x) = \frac{1}{x^2}$.

$$\Rightarrow \lim_{x\to\infty}\frac{f(x)}{g(x)} = \lim_{x\to\infty} e^{-x}x^{p+2} = \lim_{x\to\infty}\frac{x^{p+2}}{e^x} = 0$$

and $\because \int_1^\infty \frac{1}{x^2}dx$ is convergent

$\therefore$ the given integral $\int_1^\infty e^{-x}x^p\,dx$ is also convergent.

➢ Theorem

Assume $\alpha \uparrow$ on $[a,+\infty)$. If $f \in R(\alpha; a,b)$ for every $b \geq a$ and if $\int_a^\infty |f|\,d\alpha$ converges, then $\int_a^\infty f\,d\alpha$ also converges.

Or: An absolutely convergent integral is convergent.

Proof

If $x \geq a$, $\pm f(x) \leq |f(x)|$

$\Rightarrow |f(x)| - f(x) \geq 0$

$\Rightarrow 0 \leq |f(x)| - f(x) \leq 2|f(x)|$

$\Rightarrow \int_a^\infty (|f| - f)\, d\alpha$ converges.

Subtracting from $\int_a^\infty |f|\, d\alpha$ we find that $\int_a^\infty f\, d\alpha$ converges.

($\because$ Difference of two convergent integrals is convergent)

➢ *Note*

$\int_a^\infty f\, d\alpha$ is said to converge absolutely if $\int_a^\infty |f|\, d\alpha$ converges. It is said to be convergent conditionally if $\int_a^\infty f\, d\alpha$ converges but $\int_a^\infty |f|\, d\alpha$ diverges.

➢ *Remark*

Every absolutely convergent integral is convergent.

➢ *Theorem*

Let f be a positive decreasing function defined on $[a,+\infty)$ such that $f(x)\to 0$ as $x\to+\infty$. Let α be bounded on $[a,+\infty)$ and assume that $f\in R(\alpha;a,b)$ for every $b\geq a$. Then the integral $\int_a^\infty f\, d\alpha$ is convergent.

Proof

Integration by parts gives

$$\int_a^b f\, d\alpha = \left| f(x)\cdot\alpha(x)\right|_a^b - \int_a^b \alpha(x)\, df$$

$$= f(b)\cdot\alpha(b) - f(a)\cdot\alpha(a) + \int_a^b \alpha\, d(-f)$$

It is obvious that $f(b)\alpha(b)\to 0$ as $b\to+\infty$

($\because$ α is bounded and $f(x)\to 0$ as $x\to+\infty$)

and $f(a)\alpha(a)$ is finite.

$\therefore$ the convergence of $\int_a^b f\, d\alpha$ depends upon the convergence of $\int_a^b \alpha\, d(-f)$.

Actually, this integral converges absolutely. To see this, suppose $|\alpha(x)|\leq M$ for all $x\geq a$ ($\because$ $\alpha(x)$ is given to be bounded)

$$\Rightarrow \int_a^b |\alpha(x)|\, d(-f) \leq \int_a^b M\, d(-f)$$

But $\int_a^b M\, d(-f) = M\left| -f\right|_a^b = M\, f(a) - M\, f(b) \to M\, f(a)$ as $b\to\infty$.

$\Rightarrow \int_a^\infty M\, d(-f)$ is convergent.

$\because$ $-f$ is an increasing function.

$\therefore \int_a^\infty |\alpha|\, d(-f)$ is convergent. (Comparison Test)

$\Rightarrow \int_a^\infty f\, d\alpha$ is convergent.

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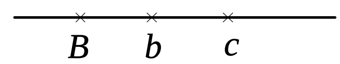
➢ *Theorem (Cauchy condition for infinite integrals)*

Assume that $f \in R(\alpha; a, b)$ for every $b \ge a$. Then the integral $\int_a^\infty f\, d\alpha$ converges if, and only if, for every $\varepsilon > 0$ there exists a $B > 0$ such that $c > b > B$ implies

$$\left| \int_b^c f(x)\, d\alpha(x) \right| < \varepsilon$$

Proof

Let $\int_a^\infty f\, d\alpha$ be convergent. Then $\exists \; B > 0$ such that



$$\left| \int_a^b f\, d\alpha - \int_a^\infty f\, d\alpha \right| < \frac{\varepsilon}{2} \quad \text{for every } \; b \ge B \quad \ldots\ldots\ldots(i)$$

Also for $c > b > B$,

$$\left| \int_a^c f\, d\alpha - \int_a^\infty f\, d\alpha \right| < \frac{\varepsilon}{2} \quad \ldots\ldots\ldots\ldots\ldots\ldots (ii)$$

Consider

$$\left| \int_b^c f\, d\alpha \right| = \left| \int_a^c f\, d\alpha - \int_a^b f\, d\alpha \right|$$

$$= \left| \int_a^c f\, d\alpha - \int_a^\infty f\, d\alpha + \int_a^\infty f\, d\alpha - \int_a^b f\, d\alpha \right|$$

$$\le \left| \int_a^c f\, d\alpha - \int_a^\infty f\, d\alpha \right| + \left| \int_a^\infty f\, d\alpha - \int_a^b f\, d\alpha \right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

$$\Rightarrow \left| \int_b^c f\, d\alpha \right| < \varepsilon \quad \text{when } \; c > b > B.$$

Conversely, assume that the Cauchy condition holds.

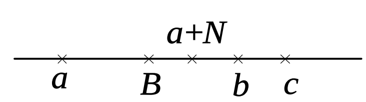
Define $a_n = \int_a^{a+n} f\, d\alpha$ if $n = 1, 2, \ldots\ldots$

The sequence $\{a_n\}$ is a Cauchy sequence $\Rightarrow$ it converges.

Let $\lim_{n\to\infty} a_n = A$

Given $\varepsilon > 0$, choose B so that $\left| \int_b^c f\, d\alpha \right| < \frac{\varepsilon}{2}$ if $c > b > B$.

and also that $|a_n - A| < \frac{\varepsilon}{2}$ whenever $a + n \ge B$.



Choose an integer N such that $a + N > B$ i.e. $N > B - a$

Then, if $b > a + N$, we have

$$\left| \int_a^b f\, d\alpha - A \right| = \left| \int_a^{a+N} f\, d\alpha - A + \int_{a+N}^b f\, d\alpha \right|$$

$$\le |a_N - A| + \left| \int_{a+N}^b f\, d\alpha \right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

$$\Rightarrow \int_a^\infty f\, d\alpha = A$$

This completes the proof.

➢ Remarks

It follows from the above theorem that convergence of $\int_a^\infty f\,d\alpha$ implies $\lim_{b\to\infty}\int_b^{b+\varepsilon} f\,d\alpha = 0$ for every fixed $\varepsilon > 0$.

However, this does not imply that $f(x)\to 0$ as $x\to\infty$.

➢ Theorem

Every convergent infinite integral $\int_a^\infty f(x)\,d\alpha(x)$ can be written as a convergent infinite series. In fact, we have

$$\int_a^\infty f(x)\,d\alpha(x) = \sum_{k=1}^\infty a_k \quad \text{where } a_k = \int_{a+k-1}^{a+k} f(x)\,d\alpha(x) \;\ldots\ldots\ldots\; (1)$$

Proof

$\because \int_a^\infty f\,d\alpha$ converges, the sequence $\left\{\int_a^{a+n} f\,d\alpha\right\}$ also converges.

But $\int_a^{a+n} f\,d\alpha = \sum_{k=1}^n a_k$. Hence the series $\sum_{k=1}^\infty a_k$ converges and equals $\int_a^\infty f\,d\alpha$.

➢ Remarks

It is to be noted that the convergence of the series in (1) does not always imply convergence of the integral. For example, suppose $a_k = \int_{k-1}^{k} \sin 2\pi x\,dx$. Then each $a_k = 0$ and $\sum a_k$ converges.

However, the integral $\int_0^\infty \sin 2\pi x\,dx = \lim_{b\to\infty}\int_0^b \sin 2\pi x\,dx = \lim_{b\to\infty}\dfrac{1-\cos 2\pi b}{2\pi}$ diverges.

IMPROPER INTEGRAL OF THE SECOND KIND

➢ Definition

Let f be defined on the half open interval $(a,b]$ and assume that $f \in R(\alpha; x, b)$ for every $x \in (a,b]$. Define a function I on $(a,b]$ as follows:

$$I(x) = \int_x^b f\,d\alpha \quad \text{if } x \in (a,b] \;\ldots\ldots\ldots\; (i)$$

The function I so defined is called an improper integral of the second kind and is denoted by the symbol $\int_{a+}^b f(t)\,d\alpha(t)$ or $\int_{a+}^b f\,d\alpha$.

The integral $\int_{a+}^b f\,d\alpha$ is said to converge if the limit

$$\lim_{x\to a+} I(x) \;\ldots\ldots(ii) \quad \text{exists (finite).}$$

Otherwise, $\int_{a+}^b f\,d\alpha$ is said to diverge. If the limit in (*ii*) exists and equals A, the number A is called the value of the integral and we write $\int_{a+}^b f\,d\alpha = A$.

Similarly, if f is defined on $[a,b)$ and $f \in R(\alpha; a, x)$ $\forall$ $x \in [a,b)$ then $I(x) = \int_a^x f\, d\alpha$ if $x \in [a,b)$ is also an improper integral of the second kind and is denoted as $\int_a^{b-} f\, d\alpha$ and is convergent if $\lim_{x \to b-} I(x)$ exists (finite).

➢ ***Example***

$f(x) = x^{-p}$ is defined on $(0,b]$ and $f \in R(x,b)$ for every $x \in (0,b]$.

$$I(x) = \int_x^b x^{-p}\, dx \quad \text{if} \quad x \in (0,b]$$

$$= \int_{0+}^b x^{-p}\, dx = \lim_{\varepsilon \to 0} \int_{0+\varepsilon}^b x^{-p}\, dx$$

$$= \lim_{\varepsilon \to 0} \left| \frac{x^{1-p}}{1-p} \right|_\varepsilon^b = \lim_{\varepsilon \to 0} \frac{b^{1-p} - \varepsilon^{1-p}}{1-p} \quad , \quad (p \neq 1)$$

$$= \begin{cases} \textit{finite} \ , & p < 1 \\ \textit{infinite} \ , & p > 1 \end{cases}$$

When $p = 1$, we get $\int_\varepsilon^b \frac{1}{x}\, dx = \log b - \log \varepsilon \to \infty$ as $\varepsilon \to 0$.

$\Rightarrow$ $\int_{0+}^b x^{-1}\, dx$ also diverges.

Hence the integral converges when $p < 1$ and diverges when $p \geq 1$.

➢ ***Note***

If the two integrals $\int_{a+}^c f\, d\alpha$ and $\int_c^{b-} f\, d\alpha$ both converge, we write

$$\int_{a+}^{b-} f\, d\alpha = \int_{a+}^c f\, d\alpha + \int_c^{b-} f\, d\alpha$$

The definition can be extended to cover the case of any finite number of sums. We can also consider mixed combinations such as

$$\int_{a+}^b f\, d\alpha + \int_b^\infty f\, d\alpha \quad \text{which can be written as} \quad \int_{a+}^\infty f\, d\alpha .$$

➢ ***Example***

Consider $\int_{0+}^\infty e^{-x} x^{p-1}\, dx$, $(p > 0)$

This integral must be interpreted as a sum as

$$\int_{0+}^\infty e^{-x} x^{p-1}\, dx = \int_{0+}^1 e^{-x} x^{p-1}\, dx + \int_1^\infty e^{-x} x^{p-1}\, dx$$

$$= I_1 + I_2 \ \ldots\ldots\ldots\ldots\ldots\ldots\ (i)$$

I_2, the second integral, converges for every real p as proved earlier.

To test I_1, put $t = \frac{1}{x}$ $\Rightarrow$ $dx = -\frac{1}{t^2}\, dt$

$$\Rightarrow\ I_1 = \lim_{\varepsilon\to 0}\int_{\varepsilon}^{1} e^{-x}x^{p-1}\,dx \ = \lim_{\varepsilon\to 0}\int_{1/\varepsilon}^{1} e^{-\frac{1}{t}}t^{1-p}\left(-\frac{1}{t^2}dt\right) \ = \lim_{\varepsilon\to 0}\int_{1}^{1/\varepsilon} e^{-\frac{1}{t}}t^{-p-1}\,dt$$

Take $f(t) = e^{-\frac{1}{t}}\, t^{-p-1}$ and $g(t) = t^{-p-1}$

Then $\lim\limits_{t\to\infty}\dfrac{f(t)}{g(t)} = \lim\limits_{t\to\infty}\dfrac{e^{-\frac{1}{t}}\cdot t^{-p-1}}{t^{-p-1}} = 1$ and since $\int_1^{\infty} t^{-p-1}\,dt$ converges when $p>0$

$\therefore\ \int_1^{\infty} e^{-\frac{1}{t}}\, t^{-p-1}\,dt$ converges when $p>0$

Thus $\int_{0+}^{\infty} e^{-x}\, x^{p-1}\,dx$ converges when $p>0$.

When $p>0$, the value of the sum in (i) is denoted by $\Gamma(p)$. The function so defined is called the Gamma function.

➤ *Note*

The tests developed to check the behaviour of the improper integrals of Ist kind are applicable to improper integrals of IInd kind after making necessary modifications.

➤ *A Useful Comparison Integral*

$$\int_a^b \frac{dx}{(x-a)^n}$$

We have, if $n \neq 1$,

$$\int_{a+\varepsilon}^{b} \frac{dx}{(x-a)^n} = \left|\frac{1}{(1-n)(x-a)^{n-1}}\right|_{a+\varepsilon}^{b}$$

$$= \frac{1}{(1-n)}\left(\frac{1}{(b-a)^{n-1}} - \frac{1}{\varepsilon^{n-1}}\right)$$

Which tends to $\dfrac{1}{(1-n)(b-a)^{n-1}}$ or $+\infty$ according as $n<1$ or $n>1$, as $\varepsilon\to 0$.

Again, if $n=1$,

$$\int_{a+\varepsilon}^{b} \frac{dx}{x-a} = \log(b-a) - \log\varepsilon \to +\infty \quad \text{as} \quad \varepsilon\to 0.$$

Hence the improper integral $\int_a^b \dfrac{dx}{(x-a)^n}$ converges iff $n<1$.

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➢ *Question*

Examine the convergence of

(*i*) $\displaystyle\int_0^1 \frac{dx}{x^{1/3}\left(1+x^2\right)}$ (*ii*) $\displaystyle\int_0^1 \frac{dx}{x^2(1+x)^2}$ (*iii*) $\displaystyle\int_0^1 \frac{dx}{x^{1/2}\left(1-x\right)^{1/3}}$

Solution

(*i*) $\displaystyle\int_0^1 \frac{dx}{x^{1/3}\left(1+x^2\right)}$

Here '0' is the only point of infinite discontinuity of the integrand.
We have

$$f(x)=\frac{1}{x^{1/3}\left(1+x^2\right)}$$

Take $g(x)=\dfrac{1}{x^{1/3}}$

Then $\displaystyle\lim_{x\to 0}\frac{f(x)}{g(x)}=\lim_{x\to 0}\frac{1}{1+x^2}=1$

$\Rightarrow \int_0^1 f(x)\,dx$ and $\int_0^1 g(x)\,dx$ have identical behaviours.

$\because \displaystyle\int_0^1 \frac{dx}{x^{1/3}}$ converges $\therefore \displaystyle\int_0^1 \frac{dx}{x^{1/3}\left(1+x^2\right)}$ also converges.

(*ii*) $\displaystyle\int_0^1 \frac{dx}{x^2(1+x)^2}$

Here '0' is the only point of infinite discontinuity of the given integrand.
We have

$$f(x)=\frac{1}{x^2(1+x)^2}$$

Take $g(x)=\dfrac{1}{x^2}$

Then $\displaystyle\lim_{x\to 0}\frac{f(x)}{g(x)}=\lim_{x\to 0}\frac{1}{(1+x)^2}=1$

$\Rightarrow \int_0^1 f(x)\,dx$ and $\int_0^1 g(x)\,dx$ behave alike.

But $n=2$ being greater than 1, the integral $\int_0^1 g(x)\,dx$ does not converge. Hence the given integral also does not converge.

(*iii*) $\displaystyle\int_0^1 \frac{dx}{x^{1/2}\left(1-x\right)^{1/3}}$

Here '0' and '1' are the two points of infinite discontinuity of the integrand.
We have

$$f(x)=\frac{1}{x^{1/2}\left(1-x\right)^{1/3}}$$

We take any number between 0 and 1, say $1/2$, and examine the convergence of the improper integrals $\int_0^{1/2} f(x)\,dx$ and $\int_{1/2}^1 f(x)\,dx$.

To examine the convergence of $\int_0^{1/2} \frac{1}{x^{1/2}(1-x)^{1/3}}dx$, we take $g(x)=\frac{1}{x^{1/2}}$

Then

$$\lim_{x\to 0}\frac{f(x)}{g(x)}=\lim_{x\to 0}\frac{1}{(1-x)^{1/3}}=1$$

$\because \int_0^{1/2}\frac{1}{x^{1/2}}dx$ converges $\therefore \int_0^{1/2}\frac{1}{x^{1/2}(1-x)^{1/3}}dx$ is convergent.

To examine the convergence of $\int_{1/2}^1 \frac{1}{x^{1/2}(1-x)^{1/3}}dx$, we take $g(x)=\frac{1}{(1-x)^{1/3}}$

Then

$$\lim_{x\to 1}\frac{f(x)}{g(x)}=\lim_{x\to 1}\frac{1}{x^{1/2}}=1$$

$\because \int_{1/2}^1\frac{1}{(1-x)^{1/3}}dx$ converges $\because \int_{1/2}^1\frac{1}{x^{1/2}(1-x)^{1/3}}dx$ is convergent.

Hence $\int_0^1 f(x)\,dx$ converges.

➢ *Question*

Show that $\int_0^1 x^{m-1}(1-x)^{n-1}dx$ exists iff m, n are both positive.

Solution

The integral is proper if $m\geq 1$ and $n\geq 1$.

The number ‘0’ is a point of infinite discontinuity if $m<1$ and the number ‘1’ is a point of infinite discontinuity if $n<1$.

Let $m<1$ and $n<1$.

We take any number, say $1/2$, between 0 & 1 and examine the convergence of the improper integrals $\int_0^{1/2} x^{m-1}(1-x)^{n-1}dx$ and $\int_{1/2}^1 x^{m-1}(1-x)^{n-1}dx$ at ‘0’ and ‘1’ respectively.

Convergence at 0:

We write

$$f(x)=x^{m-1}(1-x)^{n-1}=\frac{(1-x)^{n-1}}{x^{1-m}} \quad \text{and take} \quad g(x)=\frac{1}{x^{1-m}}$$

Then $\frac{f(x)}{g(x)}\to 1$ as $x\to 0$

As $\int_0^{1/2}\frac{1}{x^{1-m}}dx$ is convergent at 0 iff $1-m<1$ i.e. $m>0$

We deduce that the integral $\int_0^{1/2} x^{m-1}(1-x)^{n-1}dx$ is convergent at 0, iff m is +ive.

Convergence at 1:

We write $f(x)=x^{m-1}(1-x)^{n-1}=\dfrac{x^{m-1}}{(1-x)^{1-n}}$ and take $g(x)=\dfrac{1}{(1-x)^{1-n}}$

Then $\dfrac{f(x)}{g(x)}\to 1$ as $x\to 1$

As $\int\limits_{1/2}^{1}\dfrac{1}{(1-x)^{1-n}}dx$ is convergent, iff $1-n<1$ i.e. $n>0$.

We deduce that the integral $\int\limits_{1/2}^{1}x^{m-1}\left(1-x\right)^{n-1}dx$ converges iff $n>0$.

Thus $\int\limits_{0}^{1}x^{m-1}\left(1-x\right)^{n-1}dx$ exists for positive values of m, n only.

It is a function which depends upon m & n and is defined for all positive values of m & n. It is called Beta function.

➢ ***Question***

Show that the following improper integrals are convergent.

(i) $\int\limits_{1}^{\infty}\sin^2\dfrac{1}{x}\,dx$ (ii) $\int\limits_{1}^{\infty}\dfrac{\sin^2 x}{x^2}\,dx$ (iii) $\int\limits_{0}^{1}\dfrac{x\log x}{(1+x)^2}\,dx$ (iv) $\int\limits_{0}^{1}\log x\cdot\log(1+x)\,dx$

Solution

(*i*) Let $f(x)=\sin^2\dfrac{1}{x}$ and $g(x)=\dfrac{1}{x^2}$

then $\lim\limits_{x\to\infty}\dfrac{f(x)}{g(x)}=\lim\limits_{x\to\infty}\dfrac{\sin^2\frac{1}{x}}{\frac{1}{x^2}}=\lim\limits_{y\to 0}\left(\dfrac{\sin y}{y}\right)^2=1$

$\Rightarrow$ $\int\limits_{1}^{\infty}f(x)\,dx$ and $\int\limits_{1}^{\infty}\dfrac{1}{x^2}\,dx$ behave alike.

$\because$ $\int\limits_{1}^{\infty}\dfrac{1}{x^2}\,dx$ is convergent $\therefore$ $\int\limits_{1}^{\infty}\sin^2\dfrac{1}{x}\,dx$ is also convergent.

(*ii*) $\int\limits_{1}^{\infty}\dfrac{\sin^2 x}{x^2}\,dx$

Take $f(x)=\dfrac{\sin^2 x}{x^2}$ and $g(x)=\dfrac{1}{x^2}$

$\sin^2 x\le 1$ $\Rightarrow$ $\dfrac{\sin^2 x}{x^2}\le\dfrac{1}{x^2}$ $\forall$ $x\in\left(1,\infty\right)$

and $\int\limits_{1}^{\infty}\dfrac{1}{x^2}\,dx$ converges $\therefore$ $\int\limits_{1}^{\infty}\dfrac{\sin^2 x}{x^2}\,dx$ converges.

➢ ***Note***

$\int\limits_{0}^{1}\dfrac{\sin^2 x}{x^2}\,dx$ is a proper integral because $\lim\limits_{x\to 0}\dfrac{\sin^2 x}{x^2}=1$ so that '0' is not a point of infinite discontinuity. Therefore $\int\limits_{0}^{\infty}\dfrac{\sin^2 x}{x^2}\,dx$ is convergent.

(*iii*) $\int_0^1 \frac{x \log x}{(1+x)^2}\, dx$

$\because\ \log x < x\ ,\quad x \in (0,1)$

$\therefore\ x \log x < x^2$

$\Rightarrow\ \frac{x \log x}{(1+x)^2} < \frac{x^2}{(1+x)^2}$

Now $\int_0^1 \frac{x^2}{(1+x)^2}\, dx$ is a proper integral.

$\therefore\ \int_0^1 \frac{x \log x}{(1+x)^2}\, dx$ is convergent.

(*iv*) $\int_0^1 \log x \cdot \log(1+x)\, dx$

$\because\ \log x < x \quad \therefore\ \log(x+1) < x+1$

$\Rightarrow\ \log x \cdot \log(1+x) < x(x+1)$

$\because\ \int_0^1 x(x+1)\, dx$ is a proper integral $\quad \therefore\ \int_0^1 \log x \cdot \log(1+x)\, dx$ is convergent.

➢ ***Note***

(*i*) $\int_0^a \frac{1}{x^p} dx$ diverges when $p \geq 1$ and converges when $p < 1$.

(*ii*) $\int_a^\infty \frac{1}{x^p}\, dx$ converges iff $p > 1$.

UNIFORM CONVERGENCE OF IMPROPER INTEGRALS

➢ ***Definition***

Let f be a real valued function of two variables x & y, $x \in [a, +\infty)$, $y \in S$ where $S \subset \mathbb{R}$. Suppose further that, for each y in S, the integral $\int_a^\infty f(x,y)\, d\alpha(x)$ is convergent. If F denotes the function defined by the equation

$$F(y) = \int_a^\infty f(x,y)\, d\alpha(x) \quad \text{if} \quad y \in S$$

the integral is said to converge *pointwise* to F on S

➢ ***Definiton***

Assume that the integral $\int_a^\infty f(x,y)\, d\alpha(x)$ converges pointwise to F on S. The integral is said to converge Uniformly on S if, for every $\varepsilon > 0$ there exists a $B > 0$ (depending only on ε) such that $b > B$ implies

$$\left| F(y) - \int_a^b f(x,y)\, d\alpha(x) \right| < \varepsilon \qquad \forall\ y \in S.$$

(Pointwise convergence means convergence when y is fixed but uniform convergence is for every $y \in S$).

➢ Theorem (Cauchy condition for uniform convergence.)

The integral $\int_a^\infty f(x,y)\,d\alpha(x)$ converges uniformly on S, iff, for every $\varepsilon>0$ there exists a $B>0$ (depending on ε) such that $c>b>B$ implies

$$\left|\int_b^c f(x,y)\,d\alpha(x)\right| < \varepsilon \quad \forall\ y\in S.$$

Proof

Proceed as in the proof for Cauchy condition for infinite integral $\int_a^\infty f\,d\alpha$.

➢ Theorem (Weierstrass M-test)

Assume that $\alpha\uparrow$ on $[a,+\infty)$ and suppose that the integral $\int_a^b f(x,y)\,d\alpha(x)$ exists for every $b\ge a$ and for every y in S. If there is a positive function M defined on $[a,+\infty)$ such that the integral $\int_a^\infty M(x)\,d\alpha(x)$ converges and $|f(x,y)|\le M(x)$ for each $x\ge a$ and every y in S, then the integral $\int_a^\infty f(x,y)\,d\alpha(x)$ converges uniformly on S.

Proof

$\because\ |f(x,y)|\le M(x)$ for each $x\ge a$ and every y in S.

$\therefore$ For every $c\ge b$, we have

$$\left|\int_b^c f(x,y)\,d\alpha(x)\right|\le\int_b^c |f(x,y)\,d\alpha(x)|\le\int_b^c M\,d\alpha \ \ldots\ldots\ldots\ (i)$$

$\because\ I=\int_a^\infty M\,d\alpha$ is convergent

$\therefore$ given $\varepsilon>0$, $\exists\ B>0$ such that $b>B$ implies

$$\left|\int_a^b M\,d\alpha - I\right| < \varepsilon/2 \ \ldots\ldots\ldots\ (ii)$$

Also if $c>b>B$, then

$$\left|\int_a^c M\,d\alpha - I\right| < \varepsilon/2 \ \ldots\ldots\ldots\ (iii)$$

Then $$\left|\int_b^c M\,d\alpha\right| = \left|\int_a^c M\,d\alpha - \int_a^b M\,d\alpha\right|$$

$$= \left|\int_a^c M\,d\alpha - I + I - \int_a^b M\,d\alpha\right|$$

$$\le \left|\int_a^c M\,d\alpha - I\right| + \left|\int_a^b M\,d\alpha - I\right| < \varepsilon/2 + \varepsilon/2 = \varepsilon \qquad \text{(By } ii\ \&\ iii\text{)}$$

$$\Rightarrow \left|\int_b^c f(x,y)\,d\alpha(x)\right| < \varepsilon, \quad c>b>B\ \&\ \text{for each } y\in S$$

Cauchy condition for convergence (uniform) being satisfied.

Therefore the integral $\int_a^\infty f(x,y)\,d\alpha(x)$ converges uniformly on S.

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➢ *Example*

Consider $\int_0^\infty e^{-xy}\sin x\,dx$

$$\left|e^{-xy}\sin x\right| \le \left|e^{-xy}\right| = e^{-xy} \qquad \left(\because \left|\sin x\right|\le 1\right)$$

and $e^{-xy} \le e^{-xc}$ if $c \le y$

Now take $M(x) = e^{-cx}$

The integral $\int_0^\infty M(x)\,dx = \int_0^\infty e^{-cx}\,dx$ is convergent & converging to $\frac{1}{c}$.

$\therefore$ The conditions of M-test are satisfied and $\int_0^\infty e^{-xy}\sin x\,dx$ converges uniformly on $[c,+\infty)$ for every $c>0$.

➢ *Theorem (Dirichlet's test for uniform convergence)*

Assume that α is bounded on $[a,+\infty)$ and suppose the integral $\int_a^b f(x,y)\,d\alpha(x)$ exists for every $b \ge a$ and for every y in S. For each fixed y in S, assume that $f(x,y) \le f(x',y)$ if $a \le x' < x < +\infty$. Furthermore, suppose there exists a positive function g, defined on $[a,+\infty)$, such that $g(x)\to 0$ as $x\to+\infty$ and such that $x \ge a$ implies

$$\left|f(x,y)\right| \le g(x) \quad \text{for every } y \text{ in } S.$$

Then the integral $\int_a^\infty f(x,y)\,d\alpha(x)$ converges uniformly on S.

Proof

Let $M>0$ be an upper bound for $|\alpha|$ on $[a,+\infty)$.

Given $\varepsilon>0$, choose $B>a$ such that $x \ge B$ implies

$$g(x) < \frac{\varepsilon}{4M}$$

($\because$ $g(x)$ is +ive and $\to 0$ as $x\to\infty$ $\therefore$ $\left|g(x)-0\right| < \frac{\varepsilon}{4M}$ for $x \ge B$)

If $c>b$, integration by parts yields

$$\int_b^c f\,d\alpha = \left| f(x,y)\cdot\alpha(x)\right|_b^c - \int_b^c \alpha\,df$$

$$= f(c,y)\alpha(c) - f(b,y)\alpha(b) + \int_b^c \alpha\,d(-f) \quad \ldots\ldots\ldots\ldots (i)$$

But, since $-f$ is increasing (for each fixed y), we have

$$\left|\int_b^c \alpha\,d(-f)\right| \le M\int_b^c d(-f) \qquad (\because \text{ upper bound of } |\alpha| \text{ is } M)$$

$$= M\,f(b,y) - M\,f(c,y) \quad \ldots\ldots\ldots\ldots\ldots (ii)$$

Now if $c>b>B$, we have from (i) and (ii)

$$\left|\int_b^c f\,d\alpha\right| \le \left| f(c,y)\alpha(c) - f(b,y)\alpha(b)\right| + \left|\int_b^c \alpha\,d(-f)\right|$$

$$\le |\alpha(c)||f(c,y)| + |f(b,y)||\alpha(b)| + M\left|f(b,y) - f(c,y)\right|$$

$$\le |\alpha(c)||f(c,y)| + |\alpha(b)||f(b,y)| + M|f(b,y)| + M|f(c,y)|$$

$$\leq M\,g(c) + M\,g(b) + M\,g(b) + M\,g(c)$$
$$= 2M\left[g(b) + g(c)\right]$$
$$< 2M\left[\frac{\varepsilon}{4M} + \frac{\varepsilon}{4M}\right] = \varepsilon$$

$$\Rightarrow \left|\int_b^c f\,d\alpha\right| < \varepsilon \quad \text{for every } y \text{ in } S.$$

Therefore the Cauchy condition is satisfied and $\int_a^\infty f(x,y)\,d\alpha(x)$ converges uniformly on S.

➢ *Example*

Consider $\int_0^\infty \frac{e^{-xy}}{x}\sin x\,dx$

Take $\alpha(x) = \cos x$ and $f(x,y) = \frac{e^{-xy}}{x}$ if $x > 0,\ y \geq 0$.

If $S = [0,+\infty)$ and $g(x) = \frac{1}{x}$ on $[\varepsilon,+\infty)$ for every $\varepsilon > 0$ then

i) $f(x,y) \leq f(x',y)$ if $x' \leq x$ and $\alpha(x)$ is bounded on $[\varepsilon,+\infty)$.

ii) $g(x) \to 0$ as $x \to +\infty$

iii) $|f(x,y)| = \left|\frac{e^{-xy}}{x}\right| \leq \frac{1}{x} = g(x) \quad \forall\ y \in S$.

So that the conditions of Dirichlet's theorem are satisfied.
Hence

$$\int_\varepsilon^\infty \frac{e^{-xy}}{x}\sin x\,dx = +\int_\varepsilon^\infty \frac{e^{-xy}}{x}\,d(-\cos x) \text{ converges uniformly on } [\varepsilon,+\infty) \text{ if } \varepsilon > 0.$$

$\because \lim_{x\to 0}\frac{\sin x}{x} = 1 \qquad \therefore \int_0^\varepsilon e^{-xy}\frac{\sin x}{x}dx$ converges being a proper integral.

$\Rightarrow \int_0^\infty e^{-xy}\frac{\sin x}{x}dx$ also converges uniformly on $[0,+\infty)$.

➢ *Remarks*

Dirichlet's test can be applied to test the convergence of the integral of a product. For this purpose the test can be modified and restated as follows:

Let $\phi(x)$ be bounded and monotonic in $[a,+\infty)$ and let $\phi(x) \to 0$, when $x \to \infty$. Also let $\int_a^X f(x)\,dx$ be bounded when $X \geq a$.

Then $\int_a^\infty f(x)\phi(x)\,dx$ is convergent.

➢ *Example*

Consider $\int_0^\infty \frac{\sin x}{x}dx$

$\because \frac{\sin x}{x} \to 1$ as $x \to 0$.

$\therefore$ 0 is not a point of infinite discontinuity.

Now consider the improper integral $\int_1^\infty \frac{\sin x}{x} dx$.

The factor $\frac{1}{x}$ of the integrand is monotonic and $\to 0$ as $x \to \infty$.

Also $\left|\int_1^X \sin x\, dx\right| = \left|-\cos X + \cos(1)\right| \le \left|\cos X\right| + \left|\cos(1)\right| < 2$

So that $\int_1^X \sin x\, dx$ is bounded above for every $X \ge 1$.

$\Rightarrow \int_1^\infty \frac{\sin x}{x} dx$ is convergent. Now since $\int_0^1 \frac{\sin x}{x} dx$ is a proper integral, we see that $\int_0^\infty \frac{\sin x}{x} dx$ is convergent.

➢ *Example*

Consider $\int_0^\infty \sin x^2\, dx$.

We write $\sin x^2 = \frac{1}{2x} \cdot 2x \cdot \sin x^2$

Now $\int_1^\infty \sin x^2\, dx = \int_1^\infty \frac{1}{2x} \cdot 2x \cdot \sin x^2\, dx$

$\frac{1}{2x}$ is monotonic and $\to 0$ as $x \to \infty$.

Also $\left|\int_1^X 2x \sin x^2\, dx\right| = \left|-\cos X^2 + \cos(1)\right| < 2$

So that $\int_1^X 2x \sin x^2\, dx$ is bounded for $X \ge 1$.

Hence $\int_1^\infty \frac{1}{2x} \cdot 2x \cdot \sin x^2\, dx$ i.e. $\int_1^\infty \sin x^2\, dx$ is convergent.

Since $\int_0^1 \sin x^2\, dx$ is only a proper integral, we see that the given integral is convergent.

➢ *Example*

Consider $\int_0^\infty e^{-ax} \frac{\sin x}{x} dx$, $a > 0$

Here e^{-ax} is monotonic and bounded and $\int_0^\infty \frac{\sin x}{x} dx$ is convergent.

Hence $\int_0^\infty e^{-ax} \frac{\sin x}{x} dx$ is convergent.

➢ *Example*

Show that $\int_0^\infty \frac{\sin x}{x}dx$ is not absolutely convergent.

Solution

Consider the proper integral $\int_0^{n\pi} \frac{|\sin x|}{x}dx$

where n is a positive integer. We have

We need not take $|x|$ because $x \geq 0$.

$$\int_0^{n\pi} \frac{|\sin x|}{x}dx = \sum_{r=1}^{n} \int_{(r-1)\pi}^{r\pi} \frac{|\sin x|}{x}dx$$

Put $x = (r-1)\pi + y$ so that y varies in $[0,\pi]$.

We have $|\sin[(r-1)\pi + y]| = |(-1)^{r-1}\sin y| = \sin y$

$$\therefore \int_{(r-1)\pi}^{r\pi} \frac{|\sin x|}{x}dx = \int_0^{\pi} \frac{\sin y}{(r-1)\pi + y}dy$$

$\because$ $r\pi$ is the max. value of $[(r-1)\pi + y]$ in $[0,\pi]$

$$\therefore \int_0^{\pi} \frac{\sin y}{(r-1)\pi + y}dy \geq \frac{1}{r\pi}\int_0^{\pi} \sin y\, dy = \frac{2}{r\pi}$$

[$\because$ Division by max. value will lessen the value

$$\Rightarrow \int_0^{n\pi} \frac{|\sin x|}{x}dx \geq \sum_1^n \frac{2}{r\pi} = \frac{2}{\pi}\sum_1^n \frac{1}{r}$$

$\because$ $\sum_1^n \frac{1}{r} \to \infty$ as $n \to \infty$, we see that

$$\int_0^{n\pi} \frac{|\sin x|}{x}dx \to \infty \quad \text{as } n \to \infty.$$

Let, now, X be any real number.

There exists a +tive integer n such that $n\pi \leq X < (n+1)\pi$.

We have $\int_0^{X} \frac{|\sin x|}{x}dx \geq \int_0^{n\pi} \frac{|\sin x|}{x}dx$

Let $X \to \infty$ so that n also $\to \infty$. Then we see that $\int_0^{X} \frac{|\sin x|}{x}dx \to \infty$

So that $\int_0^{\infty} \frac{|\sin x|}{x}dx$ does not converge.

➢ *Questions*

Examine the convergence of

$$(i)\ \int_1^\infty \frac{x}{(1+x)^3}dx \qquad (ii)\ \int_1^\infty \frac{1}{(1+x)\sqrt{x}}dx \qquad (iii)\ \int_1^\infty \frac{dx}{x^{1/3}(1+x)^{1/2}}$$

Solution

(***i***) Let $f(x) = \frac{x}{(1+x)^3}$ and take $g(x) = \frac{x}{x^3} = \frac{1}{x^2}$

As $\lim\limits_{x\to\infty} \frac{f(x)}{g(x)} = \lim\limits_{x\to\infty} \frac{x^3}{(1+x)^3} = 1$

Therefore the two integrals $\int_1^{\infty}\frac{x}{(1+x)^3}dx$ and $\int_1^{\infty}\frac{1}{x^2}dx$ have identical behaviour for convergence at ∞.

$\because \int_1^{\infty}\frac{1}{x^2}dx$ is convergent $\therefore \int_1^{\infty}\frac{x}{(1+x)^3}dx$ is convergent.

(***ii***) Let $f(x)=\frac{1}{(1+x)\sqrt{x}}$ and take $g(x)=\frac{1}{x\sqrt{x}}=\frac{1}{x^{3/2}}$

We have $\lim\limits_{x\to\infty}\frac{f(x)}{g(x)}=\lim\limits_{x\to\infty}\frac{x}{1+x}=1$

and $\int_1^{\infty}\frac{1}{x^{3/2}}dx$ is convergent. Thus $\int_1^{\infty}\frac{1}{(1+x)\sqrt{x}}dx$ is convergent.

(***iii***) Let $f(x)=\frac{1}{x^{1/3}(1+x)^{1/2}}$

we take $g(x)=\frac{1}{x^{1/3}\cdot x^{1/2}}=\frac{1}{x^{5/6}}$

We have $\lim\limits_{x\to\infty}\frac{f(x)}{g(x)}=1$ and $\int_1^{\infty}\frac{1}{x^{5/6}}dx$ is convergent $\therefore \int_1^{\infty}f(x)\,dx$ is convergent.

➢ *Question*

Show that $\int_{-\infty}^{\infty}\frac{1}{1+x^2}dx$ is convergent.

Solution

We have

$$\int_{-\infty}^{\infty}\frac{1}{1+x^2}dx=\lim_{a\to\infty}\left[\int_{-a}^{0}\frac{1}{1+x^2}dx+\int_{0}^{a}\frac{1}{1+x^2}dx\right]$$

$$=\lim_{a\to\infty}\left[\int_{0}^{a}\frac{1}{1+x^2}dx+\int_{0}^{a}\frac{1}{1+x^2}dx\right]=2\lim_{a\to\infty}\left[\int_{0}^{a}\frac{1}{1+x^2}dx\right]$$

$$=2\lim_{a\to\infty}\left|\tan^{-1}x\right|_0^a=2\left(\frac{\pi}{2}\right)=\pi$$

therefore the integral is convergent.

➢ *Question*

Show that $\int_0^{\infty}\frac{\tan^{-1}x}{1+x^2}dx$ is convergent.

Solution

$\because (1+x^2)\cdot\frac{\tan^{-1}x}{(1+x^2)}=\tan^{-1}x\to\frac{\pi}{2}$ as $x\to\infty$

$\int_0^{\infty}\frac{\tan^{-1}x}{1+x^2}dx$ & $\int_0^{\infty}\frac{1}{1+x^2}dx$ behave alike.

Here $f(x)=\frac{\tan^{-1}x}{1+x^2}$

and $g(x)=1+x^2$

$\because \int_0^{\infty}\frac{1}{1+x^2}dx$ is convergent $\therefore$ A given integral is convergent.

➢ *Question*

Show that $\int_0^{\infty}\frac{\sin x}{(1+x)^{\alpha}}dx$ converges for $\alpha>0$.

Solution

$\int_0^{\infty}\sin x\,dx$ is bounded because $\int_0^{X}\sin x\,dx\le 2 \quad \forall\ x>0$.

Furthermore the function $\frac{1}{(1+x)^{\alpha}}$, $\alpha>0$ is monotonic on $[0,+\infty)$.

$\Rightarrow$ the integral $\int_0^{\infty}\frac{\sin x}{(1+x)^{\alpha}}dx$ is convergent.

➢ *Question*

Show that $\int_0^{\infty}e^{-x}\cos x\,dx$ is absolutely convergent.

Solution

$\because\ \left|e^{-x}\cos x\right|<e^{-x}$ and $\int_0^{\infty}e^{-x}\,dx=1$

$\therefore$ the given integral is absolutely convergent. (comparison test)

➢ *Question*

Show that $\int_0^{1}\frac{e^{-x}}{\sqrt{1-x^4}}dx$ is convergent.

Solution

$\because\ e^{-x}<1$ and $1+x^2>1$

$$\therefore\ \frac{e^{-x}}{\sqrt{1-x^4}}<\frac{1}{\sqrt{(1-x^2)(1+x^2)}}<\frac{1}{\sqrt{1-x^2}}$$

Also
$$\int_0^{1}\frac{1}{\sqrt{1-x^2}}\,dx=\lim_{\varepsilon\to 0}\int_0^{1-\varepsilon}\frac{1}{\sqrt{1-x^2}}\,dx$$
$$=\lim_{\varepsilon\to 0}\sin^{-1}(1-\varepsilon)=\frac{\pi}{2}$$

$\Rightarrow \int_0^{1}\frac{e^{-x}}{\sqrt{1-x^4}}dx$ is convergent. (by comparison test)

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