Sequence & Series Handout

Course Code & Title: MTH104 Calculus and Analytic Geometry
Course Instructor: Dr. Atiq ur Rehman
Course URL: https://www.mathcity.org/atiq/fa24-mth104

Infinite sequences are fundamental in computer science for modeling ongoing processes, such as data streams or algorithmic computations, that have no predefined end. They are essential in fields like functional programming, automata theory, and formal verification to represent and analyze continuous or iterative operations.

The first rigorous treatment of sequences was made by A. Cauchy (1789-1857) and George Cantor (1845-1918). A sequence (of real numbers, of sets, of functions, of anything) is simply a list.

There is a first element in the list, a second element, a third element, and so on continuing in an order forever. In mathematics a finite list is not called a sequence (some authors considered it finite sequence); a sequence must continue without interruption. Formally it is defined as follows:

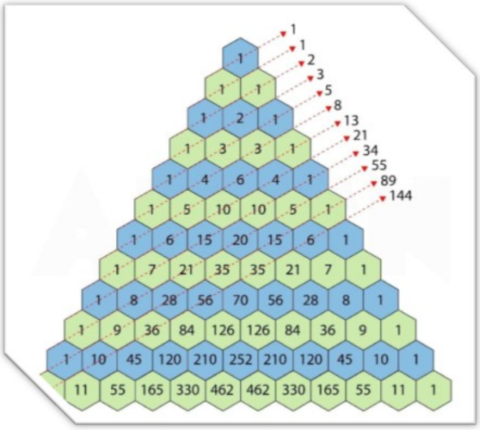


Figure 1: Fibonacci numbers in the Pascal's Triangle.

Definition 1: Sequence

A function whose domain is the set of natural numbers and range is a subset of real numbers is called infinite sequence or sequence.

A sequence is usually denoted as $\{a_n\}_{n=1}^{\infty}$ or $\{a_n : n \in \mathbb{N}\}$ or $\{a_1, a_2, a_3, \ldots\}$ or simply as $\{a_n\}$ or by (a_n). But it is not limited to above notations only.

The values a_n are called the terms or the elements of the sequence $\{a_n\}$.

Examples:

(i) $\{n\} = \{1, 2, 3, \ldots\}$.

(ii) $\left\{\frac{1}{n}\right\} = \left\{1, \frac{1}{2}, \frac{1}{3}, \ldots\right\}$.

(iii) $\left\{(-1)^{n+1}\right\} = \{1, -1, 1, -1, \ldots\}$.

(iv) $\{2, 3, 5, 7, 11, \ldots\}$, a sequence of positive prime numbers.

(v) $\{F_n\}$ such that $F_1 = 1$, $F_2 = 1$ and $F_{n+2} = F_{n+1} + F_n$.

Definition 2: Increasing, Decreasing, Monotone Sequence

A sequence $\{a_n\}$ is said to be an increasing sequence if $a_{n+1} \geq a_n \quad \forall\ n \geq 1$.

A sequence $\{a_n\}$ is said to be a decreasing sequence if $a_{n+1} \leq a_n \quad \forall\ n \geq 1$.

A sequence $\{a_n\}$ is said to be monotonic sequence if it is either increasing or decreasing.

An $\{a_n\}$ is called strictly increasing or strictly decreasing according as $a_{n+1} > a_n$ or $a_{n+1} < a_n$ $\forall\ n \geq 1$.

Examples:

(i) $\{n\} = \{1, 2, 3, \ldots\}$ is an increasing sequence (also it is strictly increasing).

(ii) $\left\{\frac{1}{n}\right\}$ is a decreasing sequence. (also it is strictly decreasing).

(iii) $\{1, 1, 2, 2, 3, 3, \ldots\}$ is increasing sequence but it is not strictly increasing.

(iv) $\{\cos n\pi\} = \{-1, 1, -1, 1, \ldots\}$ is neither increasing nor decreasing.

The sequence

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \ldots$$

is getting closer and closer to the number 0. We say that this sequence converges to 0 or that the limit of the sequence is the number 0. How should this idea be properly defined? The study of convergent sequences was undertaken and developed in the eighteenth century without any precise definition. The closest one might find to a definition in the early literature would have been something like A sequence $\{s_n\}$ converges to a number L if the terms of the sequence get closer and closer to L. However, this is too vague and too weak to serve as definition but a rough guide for the intuition, this is misleading in other respects. What about the sequence 0.1, 0.01, 0.02, 0.001, 0.002, 0.0001, 0.0002, 0.00001, 0.00002, ...? Surely this should converge to 0 but the terms do not get steadily “closer and closer” but back off a bit at each second step. The definition that captured the idea in the best way was given by Augustin Cauchy in the 1820s. He found a formulation that expressed the idea of “arbitrarily close” using inequalities.

Definition 3: Limit of the Sequence

A sequence $\{a_n\}$ of real numbers is said to convergent to limit ‘L’ as $n \to \infty$, if for every real number $\varepsilon > 0$, there exists a positive integer n_0, depending on ε, so that

$$|a_n - L| < \varepsilon \quad \text{whenever} \quad n > n_0.$$

It is written as $\lim_{n\to\infty} a_n = L$ or $\lim a_n = L$.

A sequence that converges is said to be convergent. A sequence that fails to converge is said to divergent

Our aim is not here to prove the limits by definition but just for interest of the reader, we are giving here example:

Example: Prove that $\lim_{n\to\infty} \frac{3n+2}{n+1} = 3$ (by definition).

Solution. Let$\varepsilon > 0$ be given. Now consider

$$\left|\frac{3n+2}{n+1} - 3\right| = \left|\frac{3n+2-3n-3}{n+1}\right|$$
$$= \left|\frac{-1}{n+1}\right| = \frac{1}{n+1} < \frac{1}{n} \quad (\because\ n+1 > n > 0)$$

Now if we take n_0 such that $\frac{1}{n_0} < \varepsilon$ (or $n_0 > \frac{1}{\varepsilon}$), then the above expression gives us

$$\left|\frac{3n+2}{n+1} - 3\right| < \varepsilon \quad \text{whenever} \quad n \geq n_0.$$

Hence, we conclude that $\lim_{n\to\infty} \frac{3n+2}{n+1} = 3$.

Please note that if $\lim a_n = L$, then $\lim a_{n+p} = L$ for $p \in \mathbb{N}$.

Sum, Product and Quotient of Limit of Sequence

Let a and b be fixed real numbers if $\{s_n\}$ and $\{t_n\}$ converge to s and t respectively, then
(i) $\{as_n + bt_n\}$ converges to as + bt.
(ii) $\{s_n t_n\}$ converges to st.
(iii) $\left\{\frac{s_n}{t_n}\right\}$ converges to $\frac{s}{t}$, provided $t_n \neq 0$ for all n and $t \neq 0$.

Questions

Prove that
(a) $\left\{\frac{2n}{n+1}\right\}$ converges to 2.
(b) $\left\{\frac{n^2-1}{2n^2+3}\right\}$ converges to $\frac{1}{2}$.
(c) $\left\{\frac{1}{3^n}\right\}$ converges to 0.

Remark:

Questions

1. If $\{s_n\}$ converges to 5, $\{t_n\}$ converges to -7 and $\{u_n\}$ converges to 1. then find

$$\lim \frac{3s_n - 2u_n}{t_n u_n}.$$

2. Let $\{t_n\}$ be a positive term sequence. Find the limit of the sequence if

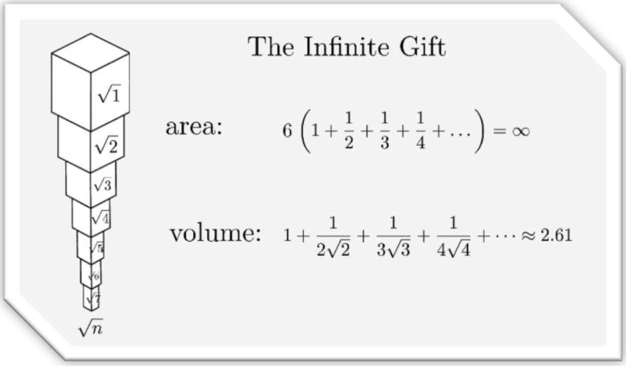
$$4t_{n+1} = \frac{2}{5} - 3t_n \quad \text{for all} \quad n \geq 1.$$

3. Let $\{u_n\}$ be a sequence of positive numbers. Then find the limit of the sequence if

$$u_{n+1} = \frac{1}{u_n} + \frac{1}{4}u_{n-1} \quad \text{for} \quad n \geq 1.$$

4. The Fibonacci numbers are: $F_1 = F_2 = 1$, and for every $n \geq 3$, F_n is defined by the recurrence relation $F_n = F_{n-1} + F_{n-2}$. Find the $\lim\limits_{n\to\infty} \frac{F_n}{F_{n-1}}$ (this limit is known as golden number).

I'm standing 5 m from a wall. I jump half the distance (2.5 m) towards the wall. I halve the distance again (1.25 m) and continue getting closer to the wall by stepping half the remaining distance each time. Do I ever reach the wall? Zeno, the 5th century BCE Greek philosopher, proposed a similar question in his famous Paradoxes (search for Zeno's paradox).



The first known example of an infinite sum was when Greek mathematician Archimedes showed in the 3rd century BCE that the area of a segment of a parabola is 4/3 the area of a triangle with the same base. The notation he used was different, of course, and some of the approach was more geometric than algebraic, but his approach of summing infinitely small quantities was quite remarkable for the time.

Mathematicians Madhava from Kerala, India studied infinite series around 1350 CE. Among his many contributions, he discovered the infinite series for the trigonometric functions of sine, cosine, tangent and arctangent, and many methods for calculating the circumference of a circle In the 17th century, James Gregory (1638-1675) worked in the new decimal system on infinite series and published several Maclaurin series. In 1715, a general method for constructing the Taylor series for all functions for which they exist was provided by Brook Taylor (1685-1731). Leonhard Euler (1707-1783) derived series for sine, cosine, exp, log, etc., and he also discovered relationships between them. He also introduced sigma notation ($\sum$) for sums of series.

Infinite Series

Let $\{a_n\}$ be a given sequence. Then a sum of the form

$$a_1 + a_2 + a_3 + \dots$$

is called an infinite series.

Another way of writing this infinite series is $\sum_{n=1}^{\infty} a_n$ or $\displaystyle\sum_{n=1}^{\infty} a_n$ or simply $\sum a_n$.

Definition 5: Convergence and divergence of the series

A series $\sum_{n=1}^{\infty} a_n$ is said to be convergent if the sequence $\{s_n\}$, where $s_n = \sum_{k=1}^{n} a_k$, is convergent.

If the sequence $\{s_n\}$ diverges then the series is said to be diverge.

Remarks:

For a series $\sum_{n=1}^{\infty} a_n$, the sequence $\{s_n\}$, where $s_n = \sum_{k=1}^{n} a_k$, is called the sequence of partial sum of the series. The numbers a_n are called terms and s_n are called partial sums. One can note that

$$\begin{aligned}
s_1 &= a_1 \\
s_2 &= a_1 + a_2 \\
s_3 &= a_1 + a_2 + a_3 \quad \text{and} \\
s_n &= a_1 + a_2 + \dots + a_n \text{ or } s_n = s_{n-1} + a_n.
\end{aligned}$$

If the sequence $\{s_n\}$ has a limit s (that is, converges to s), we say that the series converges and write

$$\sum_{n=1}^{\infty} a_n = s, \qquad \text{or we may write,} \quad \sum_{i=1}^{\infty} a_i = \lim_{n\to\infty} s_n.$$

The number s is called the sum or value of the series but it should be clearly understood that the 's' is the limit of the sequence of sums and is not obtained simply by addition.

Also note that the behaviour of the series remain unchanged by addition or deletion of the first finite terms. Just as a sequence may be indexed such that its first element is not a_n , but is a_0 , or a_5 or a_{99}, we will denote the series having these numbers as their first element by the symbols

$$\sum_{n=0}^{\infty} a_n \quad \text{or} \quad \sum_{n=5}^{\infty} a_n \quad \text{or} \quad \sum_{n=99}^{\infty} a_n.$$

Examples:

(i) Consider a series $\sum_{n=1}^{\infty} \frac{1}{2^n}$. Its sequence of partial sum is $\left\{\frac{1}{2} + \frac{1}{2^2} + +\frac{1}{2^3} + \ldots \frac{1}{2^n}\right\}$. Now

$$\begin{aligned} s_n &= \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots \frac{1}{2^n} \\ &= 1 - \frac{1}{2^n} \quad \text{(by the formula of sum of geometric progression)} \end{aligned}$$

As $\lim s_n = 1$, thus $\sum_{n=1}^{\infty} \frac{1}{2^n} = 1$, that is, given series is convergent.

(ii) Consider a series $\sum_{n=1}^{\infty} n$. Its sequence of partial sum is $\{1 + 2 + 3 + \ldots n\}$. Now

$$\begin{aligned} s_n &= 1 + 2 + 3 + \ldots n \\ &= \sum_{k=1}^{n} k = \frac{n(n+1)}{2} \quad \text{(by the formula sum of arithmetic progression)} \end{aligned}$$

As $\lim s_n = \lim \frac{n(n+1)}{2} = \infty$, thus the given series is divergent.

(iii) It is hard to prove that a series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent because its sequence of partial sum $\left\{1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots \frac{1}{n^2}\right\}$ has a finite limit.

(iv) It is hard to prove that a series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent because its sequence of partial sum $\left\{1 + \frac{1}{2} + \frac{1}{3} + \ldots \frac{1}{n}\right\}$ has no limit.

Questions

Prove that the $\left\{\frac{1}{3}\left[\frac{1}{2} - \frac{1}{3n+2}\right]\right\}$ is the sequence of partial sum of the series $\sum_{n=1}^{\infty} \frac{1}{9n^2 + 3n - 2}$. Also find its sum.

Remarks:

(1) Let $\{a_n\}$ be a convergent sequence, then $\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n-1}$.

(2) Using the fact that $s_n = s_{n-1} + a_n$, it is easy to conclude, if $\sum_{n=1}^{\infty} a_n$ converges then $\lim_{n\to\infty} a_n = 0$.

Basic Divergent Test

If $\lim_{n\to\infty} a_n \neq 0$, then $\sum a_n$ is divergent.

Examples:

(1) Is the series $\sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)$ is convergent or divergent?

Assume

$$a_n = 1+\frac{1}{n}.$$

Now we have

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(1+\frac{1}{n}\right) = 1 \neq 0.$$

Hence $\sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)$ is divergent (by basic divergent test)

(2) Show that the series $\sqrt{\frac{1}{4}}+\sqrt{\frac{2}{6}}+\sqrt{\frac{3}{8}}+\ldots$ is divergent.

The above series can be written as $\sum_{n=1}^{\infty}\sqrt{\frac{n}{2(n+1)}}$.

Then take

$$a_n = \sqrt{\frac{n}{2(n+1)}}.$$

As we have

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\sqrt{\frac{n}{2(n+1)}} = \frac{1}{\sqrt{2}} \neq 0.$$

Hence the given series is divergent by basic comparison test.

Remark:

Consider the series $\sum\frac{1}{n}$. We know that the sequence $\{s_n\}$, where $s_n = 1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}$, is divergent therefore $\sum\frac{1}{n}$ is divergent series, although $\lim_{n\to\infty} a_n = 0$.

Exercise 7

Use basic divergent test to prove that $\sum\frac{4n^2-n^3}{10+2n^3}$ is divergent.

$\alpha-$Series Test or ($p-$Series test)

The series $\sum\frac{1}{n^\alpha}$ is convergent if $\alpha > 1$ and diverges if $\alpha \leq 1$.

Example:

Find the value of m, for which the series $\sum\frac{1}{n^{m^2-1}}$ is convergent.

Solution. Given: $\sum\frac{1}{n^{m^2-1}}$

The series $\sum\frac{1}{n^\alpha}$ is convergent $\alpha > 1$.

Comparing it with given series, we get $\alpha = m^2-1 > 1$.

$$\Longrightarrow m^2 > 2$$
$$\Longrightarrow \pm m > \sqrt{2}$$
$$\Longrightarrow m < -\sqrt{2} \text{ or } m > \sqrt{2}.$$

Thus $m \in (-\infty, -\sqrt{2}) \cup (\sqrt{2}, \infty)$.

Basic Comparison Test

Suppose $\sum a_n$ and $\sum b_n$ are infinite series such that $a_n > 0$, $b_n > 0$ for all n. Also suppose that for some integer k, we have

$$a_n \leq b_n \quad \text{for all} \quad n \geq k.$$

(i) If $\sum b_n$ is convergent, then $\sum a_n$ is convergent.

(ii) If $\sum a_n$ is divergent, then $\sum b_n$ is divergent.

Questions

1. Use basic comparison test to prove that $\sum \frac{1}{1+n^2}$ is convergent.
2. Use basic comparison test to prove that $\sum \frac{n+5}{n^2+4n}$ is divergent.
3. Test the series for convergence or divergence: $\sum \frac{1}{n^{\frac{1}{2}} + n^{\frac{3}{2}}}$.
4. Use basic comparison test to prove that $\sum \frac{1}{n!}$ is convergent.

Limit Comparison Test

Let $a_n > 0$, $b_n > 0$ and

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L, \text{ where } L \geq 0.$$

(i) If $L \neq 0$, then the series $\sum a_n$ and $\sum b_n$ behave alike.
(ii) If $\lambda = 0$ and if $\sum b_n$is convergent, then $\sum a_n$ is convergent. If $\sum a_n$ is divergent then $\sum b_n$ is divergent.

Questions

1. Use limit comparison test to prove that $\sum \frac{\ln n}{n}$ is divergent.
2. Use limit comparison test to prove that $\sum \frac{1}{n!}$ is convergent.

Definition 11: Alternating series

A series in which successive terms have opposite signs is called an alternating series.

Example:

A series $\sum \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$ is an alternating series.

Alternating Series Test or Leibniz Test

Let $\{a_n\}$ be a decreasing sequence of positive numbers such that $\lim_{n\to\infty} a_n = 0$ then the alternating series $\sum_{n=1}^{\infty} (-1)^{n+1} a_n = a_1 - a_2 + a_3 - a_4 + \ldots$ converges.

Remark: The series $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ and $\sum_{n=1}^{\infty} (-1)^{n} a_n$ are different series, but they have the same convergence behavior. That is, both converge or both diverge (whether absolutely or conditionally).

Example: Consider an alternating series $\sum \frac{(-1)^{n+1}}{n}$

Here $a_n = \frac{1}{n}$ and

$$\lim a_n = \lim \frac{1}{n} = 0.$$

By alternating series test, the given series is convergent.

Definition 13: Absolute Convergence

A series $\sum a_n$ is said to be absolutely convergent if $\sum |a_n|$ converges.

Definition 14: Conditionally Convergence

If $\sum a_n$ converges and $\sum |a_n|$ diverges the series $\sum a_n$ is called conditionally convergent.

Example:

Consider series $\sum \frac{(-1)^n}{n}$ and $\sum \frac{(-1)^n}{n^2}$. Both the series are convergent by using Leibniz test (alternating series test), please note

$$\sum \left|\frac{(-1)^n}{n}\right| = \sum \frac{1}{n}$$
$$\sum \left|\frac{(-1)^n}{n^2}\right| = \sum \frac{1}{n^2}$$

As we know $\sum \frac{1}{n}$ is divergent and $\sum \frac{1}{n^2}$ is convergent, thus we conclude $\sum \frac{1}{n}$ is conditionally convergent and $\sum \frac{1}{n^2}$ is absolutely convergent.

Root Test

Let $\lim_{n\to\infty} |a_n|^{1/n} = p$. Then $\sum a_n$ converges absolutely if $p < 1$ and it diverges if $p > 1$.

Questions

1. Use root test to prove that $\sum(-1)^n \left(\frac{n+1}{5n+3}\right)^n$ is convergent.

Ratio Test

For a series $\sum a_n$, consider

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L.$$

(i) The series $\sum a_n$ converges if $L < 1$, and

(ii) it diverges, if $L > 1$.

The nature cannot be concluded if $L = 1$.

Example:

Consider a series $\sum \frac{1}{n!}$.

Here $a_n = \frac{1}{n!}$, now

$$\begin{aligned} L &= \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \\ &= \lim_{n\to\infty} \left|\frac{1}{(n+1)!} \times n!\right| \\ &= \lim_{n\to\infty} \frac{n!}{(n+1)n!} \\ &= \lim_{n\to\infty} \frac{1}{n+1} \\ &= 0 < 1 \end{aligned}$$

By ratio test, we conclude $\sum \frac{1}{n!}$ is convergent.

Definition 17: Power Series

A series of the form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots$$

where x is a variable and the coefficients a_n are constants (real numbers), is known as a power series in x.

Examples:

(i) $\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \ldots$. Since this series is a geometric series with ratio $r = |x|$, we know that it converges if $|x| < 1$ and diverges if $|x| \geq 1$.

(ii) A polynomial function can be easily represented as power series, let $p(x) = x^3 - 2x^2 + 3x + 5$, then p can be represented as a power series as

$$p(x) = 5 + 3x + (-2)x^2 + 1 \cdot x^3 + 0 \cdot x^4 + 0 \cdot x^5 + \ldots$$

(iii) $\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \cdots + \frac{1}{n!}x^n + \cdots$

Exercise:

Find all values of x for which the following power series is absolutely convergent (or convergent):

$$1+\frac{1}{5}x+\frac{2}{5^2}x^2+\cdots+\frac{n}{5^n}x^n+\cdots$$

Solution.

If we let $u_n=\dfrac{n}{5^n}x^n=n\dfrac{x^n}{5^n}$, then

$$\begin{aligned}L&=\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right|\\&=\lim_{n\to\infty}\left|\frac{(n+1)x^{n+1}}{5^{n+1}}\cdot\frac{5^n}{nx^n}\right|\\&=\lim_{n\to\infty}\left|\frac{(n+1)x}{5n}\right|=\lim_{n\to\infty}\left(\frac{n+1}{5n}\right)|x|=\frac{1}{5}|x|\end{aligned}$$

By the Ratio Test, the series is absolutely convergent (or convergent) if $L<1$, that is

$$\frac{1}{5}|x|<1,\quad\Longrightarrow |x|<5,\quad\Longrightarrow -5<x<5.$$

The series diverges if $\frac{1}{5}|x|>1$, that is, if $x>5$ or $x<-5$.

If $L=1$, that is, $\frac{1}{5}|x|=1$, the Ratio Test gives no information and hence the numbers 5 and -5 require special consideration. Substituting 5 for x in the power series we obtain

$$1+1+2+3+\cdots+n+\cdots$$

which is divergent. If we let $x=-5$ we obtain

$$1-1+2-3+\cdots+n(-1)^n+\cdots$$

which is also divergent. Consequently, the power series is absolutely convergent (or convergent) for every x in the open interval $(-5,5)$.

Exercise:

Find all values of x for which the following power series is absolutely convergent:

$$1+\frac{1}{1!}x+\frac{1}{2!}x^2+\cdots+\frac{1}{n!}x^n+\cdots$$

Solution.

If we let

$$u_n=\frac{1}{n!}x^n=\frac{x^n}{n!}$$

then

$$\begin{aligned}L&=\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right|\\&=\lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)!}\cdot\frac{n!}{x^n}\right|\\&=\lim_{n\to\infty}\left|\frac{x}{n+1}\right|=\lim_{n\to\infty}\frac{1}{n+1}|x|=0\end{aligned}$$

Since $L=0<1$ for every value of x, it follows from the Ratio Test that the power series is absolutely convergent for all real numbers.

Exercise:

Find all values of x for which $\sum n! x^n$ is convergent.

Solution.

Let $u_n = n!x^n$. If $x \neq 0$, then

$$\begin{aligned} L &= \lim_{n\to\infty} \left| \frac{u_{n+1}}{u_n} \right| \\ &= \lim_{n\to\infty} \left| \frac{(n+1)!x^{n+1}}{n!x^n} \right| \\ &= \lim_{n\to\infty} |(n+1)x| = \lim_{n\to\infty} (n+1)|x| = \infty. \end{aligned}$$

By the Ratio Test, the series diverges. Hence the power series is convergent only if $x = 0$.

Radius of Convergence

If $\sum a_n x^n$ is a power series, then precisely one of the following is true.

(i) The series converges only if $x = 0$.

(ii) The series is absolutely convergent for all x.

(iii) There is a positive number R such that the series is absolutely convergent if $|x| < R$ and divergent if $|x| > R$.

The number R is called radius of convergence of power series.

Formula for Radius of Convergence

If R represent the radius of convergence of power series $\sum a_n x^n$, then

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$$

OR

$$R = \frac{1}{\lim_{n\to\infty} \sqrt[n]{|a_n|}}$$

Exercise:

Find the radius of convergence: $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n^2+4} x^n$.

Solution.

Take $a_n = \frac{1}{n^2+4}$ and let R represents the radius of convergence, then

$$\begin{aligned} R &= \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n\to\infty} \left| \frac{1}{n^2+4} \cdot [(n+1)^2+4] \right| \\ &= \lim_{n\to\infty} \left| \frac{n^2+2n+1+4}{n^2+4} \right| = \lim_{n\to\infty} \left| \frac{n^2+2n+5}{n^2+4} \right| \\ &= \lim_{n\to\infty} \left| \frac{1+2/n+5/n^2}{1+4/n^2} \right| \\ &= \left| \frac{1+0+0}{1+0} \right| = 1 \end{aligned}$$

Hence radius of convergence is 1.

Remark:

Maclaurin series is a special case of a power series.

Definition 20: Power Series Centered at a Point

A series of the form

$$\sum_{n=0}^{\infty} a_n(x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + \dots$$

is a power series centered at $x = c$.

Exercise:

Determine the radius of convergence and interval of convergence for the following power series.

$$\sum_{n=1}^{\infty} \frac{(-1)^n n}{4^n}(x+3)^n.$$

Solution.

Suppose $a_n = \frac{(-1)^n n}{4^n}(x+3)^n$, and take

$$\begin{aligned}
L &= \left|\frac{a_{n+1}}{a_n}\right| \\
&= \lim_{n\to\infty}\left|\frac{(-1)^{n+1}(n+1)(x+3)^{n+1}}{4^{n+1}} \frac{4^n}{(-1)^n(n)(x+3)^n}\right| \\
&= \lim_{n\to\infty}\left|\frac{-(n+1)(x+3)}{4n}\right| \\
&= |x+3| \lim_{n\to\infty}\frac{n+1}{4n} \\
&= \frac{1}{4}|x+3|
\end{aligned}$$

By ratio rest, given series is convergent if $L < 1$, that is,

$$\begin{aligned}
&\frac{1}{4}|x+3| < 1 \\
\implies &|x+3| < 4 \quad \cdots\cdots(*)
\end{aligned}$$

Thus the radius of convergence $R = 4$. and to find interval of convergence, from $(*)$, we have

$$\begin{aligned}
&-4 < x+3 < 4 \\
\implies &-7 < x < 1
\end{aligned}$$

When $L = 1$, we need to check the series specifically for $x = -7$ and $x = 1$:

For $x = -7$, the series becomes

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{(-1)^n n}{4^n}(-4)^n &= \sum_{n=1}^{\infty}\frac{(-1)^n n}{4^n}(-1)^n 4^n \\
&= \sum_{n=1}^{\infty}(-1)^n(-1)^n n \qquad (-1)^n(-1)^n = (-1)^{2n} = 1 \\
&= \sum_{n=1}^{\infty} n.
\end{aligned}$$

This is divergent. Hence given series is divergent at $x = -7$.

Now for $x = 1$, the series become

$$\sum_{n=1}^{\infty} \frac{(-1)^n n}{4^n}(4)^n = \sum_{n=1}^{\infty} (-1)^n n.$$

This is also divergent. Hence interval of convergence is $(-7, 1)$.

Remark: Taylor series is the special case of above series.