SUBJECTIVE

Mathematics

Hand written Notes

12

For Intermediate Students

Salient Features

- Summary
- Solved Examples
- Solved Exercises

Written by:
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0314-3072609

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Mathematics

12

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- Summary
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Thank You!

DEDICATION

Dedicated to my Students

CONTENTS

UNIT

1

Functions and Limits

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Chapter # 1

Functions and Limits

* Theory

- **Function:** A function f from a set X to a set Y is a ruler or a correspondences that assigns to each element x in X a unique element y in Y.

- **Domain:** Let "f" be a function from a set X to a set Y. Then the Set X is called domain of "f".

- **Range:** Let "f" be a function to a set X to a set Y. Then the set correspondings element y in Y is called range of 'f'.

- **Independent Variable:** If "y" is a function of "x" i.e; $y = f(x)$ then x is called independant Variable of "f".

- **Dependent Variable:** If "y" is a function of "x" i.e; $y = f(x)$ then y is called dependent variable of "f".

- **Real Valued function:** A function in which variables are real numbers is called real valued function.

Example #1

Given $f(x) = x^3 - 2x^2 + 4x - 1$.
find:

(i)

$f(0)$

Solution:

$f(x) = x^3 - 2x^2 + 4x - 1$

$\boxed{x = 0}$

$f(0) = (0)^3 - 2(0)^2 + 4(0) - 1$

$= 0 - 0 + 0 - 1$

$\boxed{f(0) = -1}$

* * *

(ii)

$f(1)$

Solution:

$f(x) = x^3 - 2x^2 + 4x - 1$

$\boxed{x = 1}$

$f(2) = (1)^3 - 2(1)^2 + 4(1) - 1$

$= 1 - 2 + 4 - 1$

$\boxed{f(1) = 2}$

* * *

(iii)

$f(-2)$

Solution:

$f(x) = x^3 - 2x^2 + 4x - 1$

$\boxed{x = -2}$

$f(-2) = (-2)^3 - 2(-2)^2 + 4(-2) - 1$

$= -8 - 2(4) - 8 - 1$

$= -8 - 8 - 8 - 1$

$\boxed{f(-2) = -25}$

* * *

(iv)

$f(1+x)$

Solution

$f(x) = x^3 - 2x^2 + 4x - 1$

$\boxed{x = 1+x}$

$f(1+x) = (1+x)^3 - 2(1+x)^2 + 4(1+x) - 1$

$\because (a+b)^2 = a^3 + b^3 + 3a^2b + 3ab^2$

$\because (a+b)^2 = a^2 + b^2 + 2ab$

$f(1+x) = (1)^3 + (x)^3 + 3(1)^2(x) + 3(1)(x)^2 - 2(1 + x^2 + 2x) + 4 + 4x - 1$

$f(1+x) = \cancel{1} + x^3 + 3x + 3x^2 - 2 - 2x^2 - \cancel{4x} + 4 + \cancel{4x} - \cancel{1}$

$\boxed{f(1+x) = x^3 + x^2 + 3x + 2}$

* * *

(v)

$f\left[\frac{1}{x}\right]$

Solution:

$f(x) = x^3 - 2x^2 + 4x - 1$

$\boxed{x = \frac{1}{x}}$

$f\left(\frac{1}{x}\right) = \left(\frac{1}{x}\right)^3 - 2\left(\frac{1}{x}\right)^2 + 4\left(\frac{1}{x}\right) - 1$

$\boxed{f\left[\frac{1}{x}\right] = \frac{1}{x^3} - \frac{2}{x^2} + \frac{4}{x} - 1}$

* * *

Example #2

Let $f(x) = x^2$. Find domain and range of f.

Solution:

$f(x) = x^2$

Domain f:

Set of all real numbers.

Range f: Set of all non-negative real numbers.

• ——— •

Example #3

Let $f(x) = \frac{x}{x^2-4}$. Find domain and range of f.

Solution:

$$f(x) = \frac{x}{x^2-4}$$

Domain f: $\mathbb{R} - \{2, -2\}$

Range f: Set of real numbers.

• ——— •

Example #4

Let $f(x) = \sqrt{x^2-9}$. Find domain and range of f.

Solution:

$$f(x) = \sqrt{x^2-9}$$
$$x^2-9 \geq 0$$
$$x^2 \geq 9$$
$$x = \pm 3$$

Domain f: $(-\infty, -3] \cup [3, \infty)$

Range f: $[0, \infty)$

• ——— •

Example #5

Find domain and range of function $f(x) = x^2+1$.

Solution:

$$f(x) = x^2+1$$

Domain:

Set of all real numbers.

Range:

Set of all non-negative real numbers except the point $0 \leq y < 1$.

• ——— •

Example #6

Find domain and range of function defined by:

$$f(x) = \begin{cases} x & \text{when } 0 \leq x \leq 1 \\ x-1 & \text{when } 1 < x \leq 2 \end{cases}$$

Solution:

$$f(x) = \begin{cases} x & \text{when } 0 \leq x \leq 1 \\ x-1 & \text{when } 1 \nless x \leq 2 \end{cases}$$

Domain f:

$$[0, 1] \cup [1, 2]$$
$$= [0\ 2]$$

Range f:

$$[0, 1] \cup [0, 1]$$
$$= [0, 1)$$

• ——— •

TYPES OF Functions

Some important types of functions are given below:

Algebric Functions

Algebric functions are those functions which are defined by algebric expressions. We classify algebric functions as follows:

(i) Polynomial Function: A function P of the form $p(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \cdots\cdots + a_1 x^1 + a_0$ for all x, where the coefficient $a_n, a_{n-1}, a_{n-2}, \ldots\ldots, a_2, a_1, a_0$ are real numbers and the exponents are non-negative integers, is called Polynomial function.
If $a_n \neq 0$ then $P(x)$ is called polynomial function of degree n and a_n is the leading coeffient of $P(x)$.

Example: $P(x) = 2x^4 - 3x^3 + 2x - 1$ is a polynomial function of degree 4 with leading coeffient 2.

(ii) Linear Function: If the degree of polynomial function is 1, then it is called linear function. A linear function is of the form: $f(x) = ax + b$, $(a \neq b)$, a, b real numbers.

Example: $f(x) = 3x + 4$ or $y = 3x + 4$ is a linear function. Its domain and range are the set of real numbers.

(iii) Identity Function: For any set X, a function $I : X \to X$ of the form $I(x) = x \; \forall \, x \in X$ is called identity function. Its domain and range is the set X itself. In particular, if $X = R$ then $Ix = x$ for all $x \in R$, is the identity function.

(iv) Constant Function: Let X and Y be the set of real numbers.

A function $C : X \to Y$ defined by $C(x) = a \; \forall \; x \in X$, $a \in Y$ and fixed is called constant function.

Example: $C : R \to R$ defined by $C(x) = 2 \;\; x \in R$ is a constant function.

(v) **Rational Function:** A function $R(x)$ of the form $\frac{P(x)}{Q(x)}$, where both $P(x)$ and $Q(x)$ are polynomial function and $Q(x) \neq 0$ is called a rational function.

The domain of a rational function $R(x)$ is the set of all real numbers x for which $Q(x) \neq 0$.

- **Exponential Function:** A Function, in which variable appears as exponents power is called an exponential function. The functions, $y = e^{ax}$, $y = e^{x}$, $y = 2^{x} = e^{x \ln 2}$ e.t.c are exponential functions of x.

- **Logarithmic Function:** If $x = a^{y}$, then $y = \log_a x$, where $a > 0$, $a \neq 1$ is called logarithmic function of x.

(i) **Common logarithm:** If $a = 10$, then we have $\log_{10} x$ (written as $\lg x$) which is known as the common logarithm of x.

(ii) **Natural logarithm:** If $a = e$, then we have $\log_e x$ (written as $\ln x$) which is known as natural logarithm of x.

Hyperbolic functions

- $\sinh x = \frac{e^{x} - e^{-x}}{2}$
- $\cosh x = \frac{e^{x} + e^{-x}}{2}$
- $\tanh x = \frac{e^{x} - e^{-x}}{e^{x} + e^{-x}}$
- $\operatorname{cosech} x = \frac{2}{e^{x} - e^{-x}}$
- $\operatorname{sech} x = \frac{2}{e^{x} + e^{-x}}$
- $\coth x = \frac{e^{x} + e^{-x}}{e^{x} - e^{-x}}$

- **Explicit Functions:** If y is easily expressed in term of the independent variable x, then y is called an explicit function of x.

Example: (i) $y = x^2 + 2x - 1$ (ii) $y = \sqrt{x-1}$ are explicit function of x.

Symbolically it can be written as $y = f(x)$.

- **Implicit Function:** If x and y are so mixed up and y cannot be expressed in term of the variable of x, then y is called an implicit function of x.

Example:

(i) $x^2 + xy + y^2 = 2$ (ii) $\dfrac{xy^2 - y + 9}{xy} = 1$ are implicit function of y and x. Symbolically it is written as $f(x, y) = 0$.

- **Parametric Function:** Sometimes, a curve is described by expressing both x and y as function of a third variable 't' or 'θ' which is called a parameter. The equations of the type $x = f(t)$ and $y = g(t)$ are called the parametric equation of the curve.

Example: $x = at^2$, $y = at$, $x = a\cos t$, $y = a\sin t$

Here the variable t or θ is called parameter.

- **Even Function:** A function f is said to be an even if $f(-x) = f(x)$ for every number x in the domain of f.

Example: $f(x) = x^2$ and $f(x) = \cos x$ are even function of x.

$f(-x) = (-x)^2 = x^2$

$f(-x) = \cos(-x) = \cos x = f(x)$.

- **Odd Function:** A function f is said to be an odd if $f(-x) = -f(x)$ for every number x in the domain of f.

Example: $f(x) = x^3$ and $f(x) = \sin x$ are odd function of x.

Here $f(-x) = (-x)^3 = x^3 = -f(x)$

$f(-x) = \sin(-x) = -\sin x = -f(x)$

Example #1

Show that the parametric equations $x = a\cos t$ and $y = a\sin t$ represent the equation of circle $x^2+y^2=a^2$

Solution;

$x = a\cos t$ — ①

$y = a\sin t$ — ②

Squaring and Adding eq ① and ②

$$x^2+y^2 = (a\cos t)^2 + (a\sin t)^2$$

$$x^2+y^2 = a^2\cos^2 t + a^2\sin^2 t$$

$$x^2+y^2 = a^2(\cos^2 t + \sin^2 t)$$

$\because \cos^2\theta + \sin^2\theta = 1$

$$x^2+y^2 = a^2(1)$$

$\boxed{x^2+y^2 = a^2}$ which is equation of circle.

Hence proved.

Example #2

Prove the identies:

(i)

$$\cosh^2 x - \sinh^2 x = 1$$

Solution:

$\cosh^2 x - \sinh^2 x = 1$

L.H.S. $= \cosh^2 x - \sinh^2 x$

$$= \left[\frac{e^x+e^{-x}}{2}\right]^2 - \left[\frac{e^x-e^{-x}}{2}\right]^2$$

$$= \frac{e^{2x}+e^{-2x}+2e^x.e^{-x}}{4} - \frac{e^{2x}+e^{-2x}-2e^x.e^{-x}}{4}$$

$$= \frac{e^{2x}+e^{-2x}+2}{4} - \frac{e^{2x}+e^{-2x}-2}{4}$$

$$= \frac{\cancel{e^{2x}}+\cancel{e^{-2x}}+2-\cancel{e^{2x}}-\cancel{e^{-2x}}+2}{4}$$

$$= \frac{\cancel{4}}{\cancel{4}}$$

$= 1 =$ R.H.S.

L.H.S. = R.H.S.

(ii)

$\cosh^2 x + \sinh^2 x = \cosh 2x$

Solution:

$\cosh^2 x + \sinh^2 x = \cosh 2x$

L.H.S =

$= \cosh^2 x + \sinh^2 x$

$= \left[\frac{e^x+e^{-x}}{2}\right]^2 + \left[\frac{e^x-e^{-x}}{2}\right]^2$

$= \frac{e^{2x}+e^{-2x}+2e^x.e^{-x}}{4} + \frac{e^{2x}+e^{-2x}-2e^x.e^{-x}}{4}$

$= \frac{e^{2x}+e^{-2x}+2}{4} + \frac{e^{2x}+e^{-2x}-2}{4}$

$= \frac{e^{2x}+e^{-2x}+\cancel{2}+e^{2x}+e^{-2x}-\cancel{2}}{4}$

$= \frac{2e^{2x}+2e^{-2x}}{4}$

$= \cancel{2}\left[\frac{e^{2x}+e^{-2x}}{\cancel{4}\,2}\right]$

$= \frac{e^{2x}+e^{-2x}}{2}$

$= \cosh 2x = R.H.S$

L.H.S. = R.H.S.

* * *

Example #3

Determine whether the functions following are even or odd:

(a)

$f(x) = 3x^4 - 2x^2 + 7$

Solution

$f(x) = 3x^4 - 2x^2 + 7$

$\boxed{x = -x}$

$f(-x) = 3(-x)^4 - 2(-x)^2 + 7$

$f(-x) = 3x^4 - 2x^2 + 7$

$f(-x) = f(x)$

So, f is an even function.

* * *

(b)

$f(x) = \frac{3x}{x^2+1}$

Solution:

$f(x) = \frac{3x}{x^2+1}$

$\boxed{x = -x}$

$f(-x) = \frac{3(-x)}{(-x)^2+1}$

$f(-x) = \frac{-3x}{x^2+1}$

$f(-x) = -f(x)$

So, f is an odd function.

* * *

(c)

$f(x) = \sin x + \cos x$

Solution

$f(x) = \sin x + \cos x$

$\boxed{x = -x}$

$f(-x) = \sin(-x) + \cos(-x)$

$f(-x) = -\sin x + \cos x$

$f(-x) \neq f(x)$

so, f is neither even or nor odd function.

* * *

Exercise # 1·1

Question #1

Given that:

(a) $f(x) = x^2 - x$

Find (i)

$f(-2)$

Solution:

$f(-2)$

$f(x) = x^2 - x$ — ①

Replace x by -2

$f(-2) = (-2)^2 - (-2)$

$= 4 + 2$

$\boxed{f(-2) = 6}$

(ii)

$f(0)$

Solution

$f(x) = x^2 - x$

Replace "x" by 0

$f(0) = (0)^2 - 0$

$\boxed{f0 = 0}$

(iii)

$f(x-1)$

Solution:

$f(x) = x^2 - x$

Replace "x" by $(x-1)$

$f(x-1) = (x-1)^2 - x$

$f(x-1) = x^2 + 1 - 2x - x$

$\boxed{f(x-1) = x^2 - 3x + 1}$

(iv)

$f(x^2+4)$

Solution:

$f(x) = x^2 - x$

Replace "x" by (x^2+4)

$f(x^2+4)^2 = (x^2+4)^2 - x$

$f(x^2+4)^2 = x^4 + 16 + 8x^2 - x$

$\boxed{f(x^2+4) = x^4 + 8x^2 - x + 16}$

(b) $f(x) = \sqrt{x+4}$

(i)

$f(-2)$

Solution:

$f(-2)$

$f(x) = \sqrt{x+4}$

Replace "x" by "-2"

$f(-2) = \sqrt{-2+4}$

$\boxed{f(-2) = \sqrt{2}}$

(ii)

$f(0)$

Solution

$f(0)$

$f(x) = \sqrt{x+4}$

Replace "x" by "0"

$f(0) = \sqrt{0+4}$

$= \sqrt{4}$

$\boxed{f(0) = 2}$

(iii)

$f(x-1)$

Solution

$f(x) = \sqrt{x+4}$

Replace "x" by $x-1$

$f(x-1) = \sqrt{x-1+4}$

$\boxed{f(x-1) = \sqrt{x+3}}$

(iv)

$f(x^2+4)$

Solution:

$f(x) = \sqrt{x+4}$

Replace "x" by (x^2+4)

$f(x^2+4) = \sqrt{x^2+4+4}$

$\boxed{f(x^2+4) = \sqrt{x^2+8}}$

Question #2

Find $\frac{f(a+h)-f(a)}{h}$ and simplify where,

(i)

$f(x) = 6x-9$

Solution:

$f(x) = 6x-9$

$\because f(a+h)$

$= 6(a+h)-9$

$f(a+h) = 6a+6h-9$

$\because f(a)$

$f(a) = 6a-9$

$$\frac{f(a+h)-fa}{h} = \frac{6a+6h-9-(6a-9)}{h}$$

$$\frac{f(a+h)-f(a)}{h} = \frac{\cancel{6a}+6h-\cancel{9}-\cancel{6a}+\cancel{9}}{h}$$

$$\frac{f(a+h)-f(a)}{h} = \frac{6\cancel{h}}{\cancel{h}}$$

$$\boxed{\frac{f(a+h)-f(a)}{h} = 6}$$

(ii)

$f(x) = \sin x$

Solution:

$f(x) = \sin x$

$\because f(a+h)$

$f(a+h) = \sin(a+h)$

$\because f(a)$

$f(a) = \sin a$

$$\frac{f(a+h)-f(a)}{h} = \frac{\sin(a+h)-\sin a}{h}$$

$\because \sin P - \sin Q = 2\cos\left[\frac{P+Q}{2}\right]\sin\left[\frac{P-Q}{2}\right]$

$$= \frac{2\cos\left[\frac{a+h+a}{2}\right]\sin\left[\frac{\cancel{a}+h-\cancel{a}}{2}\right]}{h}$$

$$\frac{f(a+h)-f(a)}{h} = \frac{2\cos\left[\frac{2a+h}{2}\right]\sin\left[\frac{h}{2}\right]}{h}$$

$$\frac{f(a+h)-f(a)}{h} = \frac{2}{h}\cos\left(\frac{\cancel{2}a}{\cancel{2}}+\frac{h}{2}\right)\sin\frac{h}{2}$$

$$\boxed{\frac{f(a+h)-f(a)}{h} = \frac{2}{h}\cos\left[a+\frac{h}{2}\right]\sin\frac{h}{2}}$$

(iii)

$f(x) = x^3+2x^2-1$

Solution:

$f(x) = x^3+2x^2-1$

$\because f(a+h)$

$f(a+h) = (a+h)^3+2(a+h)^2-1$

$\because (a+b)^3 = a^3+b^3+3a^2b+3ab^2$

$f(a+h) = a^3+h^3+3a^2h+3ah^2+2(a^2+h^2+2ah)-1$

$f(a+h) = a^3+h^3+3a^2h+3ah^2+2a^2+2h^2+4ah-1$

$\because f(a)$

$f(a) = a^3+2a^2-1$

$$\frac{f(a+h)-f(a)}{h} = \frac{a^3+h^3+3a^2h+3ah^2+2a^2+2h^2+4ah-1-(a^3+2a^2-1)}{h}$$

$$\frac{f(a+h)-f(a)}{h} = \frac{\cancel{a^3}+h^3+3a^2h+3ah^2+\cancel{2a^2}+2h^2+4ah-\cancel{1}-\cancel{a^3}-\cancel{2a^2}+\cancel{1}}{h}$$

$$\frac{f(a+h)-f(a)}{h} = \frac{h^3+(3a^2h+3ah^2+2h^2+4ah}{h}$$

$= h^3+(3a^2$

$$\frac{f(a+h)-f(a)}{h} = \frac{\cancel{h}(h^2+3a^2+3ah+2h+4a)}{\cancel{h}}$$

$$\boxed{\frac{f(a+h)-f(a)}{h} = h^2+(3a+2)h+3a^2+4a}$$

(iv)

$f(x) = \cos x$

Solution

$f(x) = \cos x$

$\because f(a+h)$

$f(a+h) = \cos(a+h)$

$\because f(a)$

$f(a) = \cos a$

$$\frac{f(a+h)-f(a)}{h} = \frac{\cos(a+h)-\cos a}{h}$$

$$\because \cos P - \cos Q = -2\sin\left[\frac{P+Q}{2}\right]\sin\left[\frac{P-Q}{2}\right]$$

$$\frac{f(a+h)-f(a)}{h} = \frac{-2\sin\left[\frac{a+h+a}{2}\right]\sin\left[\frac{\cancel{a}+h-\cancel{a}}{2}\right]}{h}$$

$$\frac{f(a+h)-f(a)}{h} = \frac{-2\sin\left[\frac{2a+h}{2}\right]\sin\left[\frac{h}{2}\right]}{h}$$

$$\frac{f(a+h)-f(a)}{h} = \frac{-2}{h}\sin\left[\frac{\cancel{2}a}{\cancel{2}}+\frac{h}{2}\right]\sin\left[\frac{h}{2}\right]$$

$$\boxed{\frac{f(a+h)-f(a)}{h} = -\frac{2}{h}\sin\left[a+\frac{h}{2}\right]\sin\frac{h}{2}}$$

* * *

Question # 3

Express the following:

(a)

The perimeter P of the square as a function of its area A.

Solution:

One side $= x$

$P = x+x+x+x$

$P = 4x$ —— ①

Area $A = x^2$

$\sqrt{A} = x$

put in ①

$P = x$

$\boxed{P = 4\sqrt{A}}$

* * *

(b)

The area A of circle as a function of its circumference C.

Solution:

Area $A = \pi r^2$ —— ①

Circumference

$C = 2\pi r$

$r = \frac{C}{2\pi}$ —— ②

put in eq ①

$A = \pi\left(\frac{C}{2\pi}\right)^2$

$A = \frac{\cancel{\pi}\cdot C^2}{4\pi^{\cancel{2}}}$

$\boxed{A = \frac{C^2}{4\pi}}$

* * *

(c)

The volume V of a cube as a function of area A of its base.

Solution:

Volume $V = x\cdot x\cdot x$

$V = x^3$ —— ①

Area $A = x^2$

$\sqrt{A} = x$

put in eq ①

$V = (\sqrt{A})^3$ $\boxed{V = A^{3/2}}$

* * *

Question #4

Find the range and domain of the function g defined below.

(i)

$g(x) = 2x - 5$

Solution:

$g(x) = 2x - 5$

Domain: set of real number.

Range: Set of real number.

(ii)

$g(x) = \sqrt{x^2 - 4}$

Solution:

$g(x) = \sqrt{x^2 - 4}$

$x^2 - 4 \geqslant 0$

$x^2 \geqslant 4$

$x \geqslant \pm 2$

Domain: $(-\infty, -2] \cup [2, \infty)$

Range: $x = \pm 2$ in $g(x)$

$= \sqrt{(\pm 2)^2 - 4} \Rightarrow \sqrt{4 - 4} = 0$

Range = $[0, \infty)$

(iii)

$g(x) = \sqrt{x+1}$

Solution:

$x + 1 \geqslant 0$

$x \geqslant -1$

Domain: $[-1, \infty)$

$\sqrt{-1 + 1} = 0$

Range: $[0, \infty)$

(iv)

$g(x) = |x - 3|$

Solution:

$g(x) = |x - 3|$

Domain: $(-\infty, \infty)$

Range: $[0, \infty)$

(V)

$$g(x) = \begin{cases} 6x+7 & , x \leq -2 \\ 4-3 & , -2 < x \end{cases}$$

Solution:

Domain: $(-\infty, \infty)$

Range:

(vi)

$$g(x) = \begin{cases} x-1 & x < 3 \\ 2x+1 & 3 \leq x \end{cases}$$

Solution:

Domain: $(-\infty, \infty)$

Range: $(-\infty, 2] \cup [7, \infty)$

(vii)

$g(x) = \dfrac{x^2 + 3x + 2}{x + 1}, x \neq -1$

Solution:

Domain: $R - \{-1\}$

Range: $\mathbb{R} - \{1\}$

(viii)

$g(x) = \dfrac{x^2 - 16}{x - 4}, x \neq 4$

Solution:

Domain: $\mathbb{R} - \{4\}$

Range: $\mathbb{R} - \{8\}$

Question #5

Given $f(x) = x^3 - ax^2 + bx + 1$
If $f(2) = -3$ and $f(-1) = 0$
Find the value of 'a' and 'b'.

Solution:

$f(x) = x^3 - ax^2 + bx + 1$

$\boxed{x = 2}$

$f(2) = (2)^3 - a(2)^2 + b(2) + 1$

$f(2) = 8 - 4a + 2b + 1$

$f(2) = -4a + 2b + 9$

$\because f(2) = -3$

$-3 = -4a + 2b + 9$

$-3 - 9 = -4a + 2b$

$-12 = -4a + 2b$

Divided by '2'

$-6 = -2a + b$

$2a - b = 6$ —— ①

$\boxed{x = -1}$

$f(-1) = (-1)^3 - a(-1)^2 + b(-1) + 1$

$f(-1) = -\cancel{1} - a - b + \cancel{1}$

$f(-1) = -a - b$

$f(-1) = 0$

$0 = -a - b$

$0 = -(a + b)$

$a + b = 0$ —— ②

Add eq ① and ②

$$\begin{array}{r} 2a - \cancel{b} = 6 \\ a + \cancel{b} = 0 \\ \hline 3a = 6 \end{array}$$

$a = \frac{\cancel{6}^{2}}{\cancel{3}} \Rightarrow \boxed{a = 2}$

put in eq ②

$2 + b = 0$

$\boxed{b = -2}$

Question #6

A stone falls from a height of 60m on the ground, the height h after x second is approximately given by $h(x) = 40 - 10x^2$

(i)

What is the height of stone when:

(a)

$x = 1$ sec

Solution:

$h(x) = 40 - 10x^2$

$x = 1$

$h(1) = 40 - 10(1)^2$

$h(1) = 40 - 10$

$\boxed{h(1) = 30m}$

(b)

$x = 1.5$ sec

Solution:

$h(x) = 40 - 10x^2$

$x = 1.5$

$h(1.5) = 40 - 10(1.5)^2$

$h(1.5) = 40 - 10(2.25)$

$h(1.5) = 40 - 22.5$

$\boxed{h(1.5) = 17.5m}$

(C)

$x = 1 \cdot 7$ sec

Solution:

$h(x) = 40 - 10x^2$

$\boxed{x = 1 \cdot 7}$

$h(1 \cdot 7) = 40 - 10(1 \cdot 7)^2$

$h(1 \cdot 7) = 40 - 10(2 \cdot 89)$

$h(1 \cdot 7) = 40 - 28 \cdot 9$

$\boxed{h(1 \cdot 7) = 11 \cdot 1 m}$

* * *

(ii)

When does the stone strike the ground?

Solution:

$h(x) = 0$

$40 - 10x^2 = 0$

$-10x^2 = -40$

$10x^2 = 40$

$x^2 = \frac{\cancel{40}^{\,4}}{\cancel{10}}$

$x^2 = 4$

$\boxed{x = 2 sec}$

* * *

Question #7

Show that perimeter equation:

(i)

$x = at^2$, $y = 2at$ which represent the parabola $y^2 = 4ax$

Solution:

$x = at^2$ —①

$y = 2at$ —②

$t = \frac{y}{2a}$

put in eq ①

$x = a\left(\frac{y}{2a}\right)^2$

$x = a\frac{y^2}{4a^2}$

$x = \frac{y^2}{4a}$

$4ax = y^2$

$\boxed{y^2 = 4ax}$

* * *

(ii)

$x = a\cos\theta$, $y = b\sin\theta$ which represent the equation of ellipse

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Solution:

$x = a\cos\theta$ —①

$y = b\sin\theta$ —②

from eq ①

$\frac{x}{a} = \cos\theta$ —③

from eq ②

$\frac{y}{b} = \sin\theta$ —④

Squaring and Adding eq ③ and ④

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \sin^2\theta + \cos^2\theta$

$\because \sin^2\theta + \cos^2\theta = 1$

$\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1}$

* * *

(iii)

$x = a\sec\theta$, $y = b\tan\theta$ represent the equation of hyperabola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Solution:

$$x = a\sec\theta \text{ — ①}$$

$$y = y\,b\tan\theta \text{ — ②}$$

From eq ①

$$\frac{x}{a} = \sec\theta \text{ — ③}$$

From eq ②

$$\frac{y}{b} = \tan\theta \text{ — ④}$$

Subtract and squaring eq ③ and ④

$$\frac{x^2}{b^2} - \frac{y^2}{b^2} = \sec^2\theta - \tan^2\theta$$

$\because 1+\tan^2\theta = \sec^2\theta$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1+\cancel{\tan^2\theta} - \cancel{\tan^2\theta}$$

$$\boxed{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}$$

Question #8

Prove the identities:

(i)

$$\sinh 2x = 2\sinh x\cosh x$$

Solution:

$\sinh 2x = 2\sinh x \cosh x$

R.H.S = $2\sinh x\cosh x$

$$= 2\left[\frac{e^x - e^{-x}}{2}\right]\left[\frac{e^x+e^{-x}}{2}\right]$$

$$= \frac{\cancel{2}(e^x - e^{-x})(e^x+e^{-x})}{\cancel{2}\times 2}$$

$$= \frac{(e^x)^2 - (e^{-x})^2}{2} \quad \because (a-b)(a+b) = a^2 - b^2$$

$$= \frac{e^{2x} - e^{-2x}}{2}$$

$= \sinh 2x$ L.H.S.

L.H.S. = R.H.S.

(ii)

$$\text{Sech}^2 x = 1 - \tanh^2 x$$

Solution:

$\text{Sech}^2 x = 1 - \tanh^2 x$

R.H.S = $1 - \tanh^2 x$

$$= 1 - \left[\frac{e^x - e^{-x}}{e^x + e^{-x}}\right]^2$$

$$= 1 - \frac{(e^x - e^{-x})^2}{(e^x + e^{-x})^2}$$

$$= \frac{(e^x + e^{-x})^2 - (e^x - e^{-x})^2}{(e^x + e^{-x})^2}$$

$\because (a+b)^2 = a^2+b^2+2ab$

$$= \frac{(e^{2x} + e^{-2x} + 2e^x.e^{-x}) - (e^{2x} + e^{-2x} - 2e^x.e^{-x})}{(e^x + e^{-x})^2}$$

$\because e^x.e^{-x} = 1$

$$= \frac{e^{2x} + e^{-2x} + 2(1) - e^{2x} - e^{-2x} + 2(1)}{(e^x + e^{-x})^2}$$

$$= \frac{\cancel{e^{2x}} + \cancel{e^{-2x}} + 2 - \cancel{e^{2x}} - \cancel{e^{-2x}} + 2}{(e^x + e^{-x})^2}$$

$$= \frac{4}{(e^x + e^{-x})^2}$$

$$= \left[\frac{2}{e^x + e^{-x}}\right]^2$$

$= \text{Sech}^2 x$ = L.H.S.

L.H.S = R.H.S.

(iii)

$\text{Cosech}^2 x = \cot^2 x - 1$

Solution:

$\text{Cosech}^2 x = \cot^2 x - 1$

R.H.S. $= \cot^2 x - 1$

$= \left[\frac{e^x+e^{-x}}{e^x-e^{-x}}\right]^2 - 1$

$= \frac{(e^x+e^{-x})^2}{(e^x-e^{-x})^2} - 1$

$= \frac{(e^x+e^{-x})^2-(e^x-e^{-x})^2}{(e^x-e^{-x})^2}$

$= \frac{(e^{2x}+e^{-2x}+2e^x.e^{-x})-(e^{2x}+e^{-2x}-2e^x.e^{-x})}{(e^x-e^{-x})^2}$

$= \frac{\cancel{e^{2x}}+\cancel{e^{-2x}}+2-\cancel{e^{2x}}-\cancel{e^{-2x}}+2}{(e^x-e^{-x})^2}$

$= \frac{4}{(e^x-e^{-x})^2}$

$= \left[\frac{2}{e^x-e^{-x}}\right]^2$

$= \text{cosech}^2 x$ = L.H.S.

L.H.S. = R.H.S.

Question # 9

Determine whether the given function f is even or odd.

(i)

$f(x) = x^3+x$

Solution

$f(x) = x^3+x$ —— ①

put $x = -x$ in eq ①

$f(-x) = (-x)^3+(-x)$

$f(-x) = -x^3-x$

$f(-x) = -(x^3+x)$

$f(-x) = -f(x)$

Thus f is an odd function.

(ii)

$f(x) = (x+2)^2$

Solutions

$f(x) = (x+2)^2$

$f(x) = x^2+4x+4$ —— ①

put $x = -x$ in eq ①

$f(-x) = (-x)^2+4(-x)+4$

$f(-x) = x^2-4x+4$

$f(-x) = (x-2)^2$

$f(-x) \neq f(x)$

Thus f is neither even nor odd function.

(iii)

$f(x) = x\sqrt{x^2+5}$

Solution

$f(x) = x\sqrt{x^2+5}$ —— ①

put $x = -x$ in eq ①

$f(-x) = -x\sqrt{(-x)^2+5}$

$f(-x) = -x\sqrt{x^2+5}$

$f(-x) = -(x\sqrt{x^2+5})$

$f(-x) = -f(x)$

Thus f is an odd function.

(iv)

$$f(x) = \frac{x-1}{x+1}$$

Solution:

$$f(x) = \frac{x-1}{x+1} \quad \text{——} ①$$

put $x = -x$ in eq ①

$$f(-x) = \frac{-x-1}{-x+1}$$

$$f(-x) = \frac{-(x+1)}{-(x-1)}$$

$$f(-x) = \frac{x+1}{x-1}$$

$$f(-x) \neq f(x)$$

Thus f is neither even nor odd function.

(v)

$$f(x) = x^{\frac{2}{3}} + 6$$

Solution

$$f(x) = x^{\frac{2}{3}} + 6 \quad \text{——} ①$$

put $x = -x$ in eq ①

$$f(-x) = -x^{\frac{2}{3}} + 6$$

$$f(-x) = \left[(-x)^2\right]^{\frac{1}{3}} + 6$$

$$f(-x) = (x^2)^{1/3} + 6$$

$$f(-x) = x^{\frac{2}{3}} + 6$$

$$f(-x) = f(x)$$

Thus f is even function.

(vi)

$$f(x) = \frac{x^3 - x}{x^2 + 1}$$

Solution:

$$f(x) = \frac{x^3 - x}{x^2 + 1} \quad \text{——} ①$$

put $x = -x$ in eq ①

$$f(-x) = \frac{(-x)^3 - (-x)}{(-x)^2 + 1}$$

$$f(-x) = \frac{-x^3 + x}{x^2 + 1}$$

$$f(-x) = \frac{-(x^3 - x)}{x^2 + 1}$$

$$f(-x) = -f(x)$$

Thus f is an odd function.

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* Theory

Example #1

Let the real valued function f and g be defined as:

$f(x) = 2x+1$ and $g(x) = x^2-1$

Obtained the expression for:

(i) $f\,g(x)$ (ii) $g\,f(x)$ (iii) $f^2(x)$ (iv) $g^2(x)$.

Solution:

(i) $f\,g(x) = f[g(x)]$

$= f(x^2-1)$

$f\,g(x) = 2(x^2-1)+1$

$f\,g(x) = 2x^2-2+1$

$$\boxed{f\,g(x) = 2x^2-1}$$

(ii) $g\,f(x) = g[f(x)]$

$g\,f(x) = g(2x+1)$

$g\,f(x) = (2x+1)^2-1$

$g\,f(x) = 4x^2+1+4x-1$

$$\boxed{g\,f(x) = 4x^2+4x}$$

(iii) $f^2(x) = f[f(x)]$

$= f(2x+1)$

$f^2(x) = 2(2x+1)+1$

$= 4x+2+1$

$$\boxed{f^2(x) = 4x+3}$$

(iv) $g^2(x) = g[g(x)]$

$g^2(x) = g(x^2-1)$

$g^2(x) = (x^2-1)^2-1$

$g^2(x) = x^4+1-2x^2-1$

$$\boxed{g^2(x) = x^4-2x^2}$$

Example #2

Let $f: R \to R$ be the function defined by: $f(x) = 2x+1$. Find $f^{-1}(x)$

Solution:

$f(x) = 2x+1$

$\because f(x) = y$

$y = 2x+1$

$2x = y-1$

$x = \frac{y-1}{2}$

$\because x = f^{-1}(y)$

$f^{-1}(y) = \frac{y-1}{2}$

Replace y by x

$$\boxed{f^{-1}(x) = \frac{x-1}{2}}$$

Example #3

Without finding the inverse, state the domain and range of f^{-1} where $f(x) = 2+\sqrt{x+1}$

Solution:

$f(x) = 2+\sqrt{x+1}$

Domain f :-

$x-1 \geq 0$

$x \geq 1$

$Df = [1, \infty)$

Range f :-

put $x=1$ in $f(x)$

$f(1) = 2 + \sqrt{1-1}$

$= 2 + \sqrt{0}$

$f(1) = 2$

$Rf = [2, \infty)$

Hence Domain $f^{-1} = [2, +\infty)$

Range $f^{-1} = [1, +\infty)$

Exercise # 1.2

Question #1

The real valued functions f and g are defined below. Find

(a) $fog(x)$ (b) $gof(x)$ (c) $fof(x)$ (d) $gog(x)$

(i)

$f(x) = 2x+1$; $g(x) = \frac{3}{x-1}$, $x \neq 1$

Solution:

(a) $fog(x) = f[g(x)]$

$= f\left[\frac{3}{x-1}\right]$

$= 2\left[\frac{3}{x-1}\right] + 1$

$= \frac{6}{x-1} + 1$

$= \frac{6+x-1}{x-1}$

$\boxed{fog(x) = \frac{5+x}{x-1}}$

(b) $gof(x) = g[f(x)]$

$= g(2x+1)$

$= \frac{3}{2x+1-1}$

$\boxed{gof(x) = \frac{3}{2x}}$

(c) $fof(x) = f[f(x)]$

$= f(2x+1)$

$= 2(2x+1)+1$

$= 4x+2+1$

$\boxed{fof(x) = 4x+3}$

(d) $gog(x) = g[g(x)]$

$= g\left[\frac{3}{x-1}\right]$

$= \frac{3}{\left(\frac{3}{x-1}\right)-1}$

$= \frac{3}{\frac{3-x+1}{x-1}}$

$\boxed{gog(x) = \frac{3(x-1)}{4-x}}$

(ii)

$f(x) = \sqrt{x+1}$; $g(x) = \frac{1}{x^2}, x \neq 0$

Solution:

(a) $f \circ g(x) = f[g(x)]$

$= f\left(\frac{1}{x^2}\right)$

$= \sqrt{\frac{1}{x^2} + 1}$

$= \sqrt{\frac{1+x^2}{x^2}}$

$\boxed{f \circ g(x) = \frac{\sqrt{1+x^2}}{x}}$

(b) $g \circ f(x) = g[f(x)]$

$= g(\sqrt{x+1})$

$= \frac{1}{(\sqrt{x+1})^2}$

$\boxed{g \circ f(x) = \frac{1}{x+1}}$

(c) $f \circ f(x) = f[f(x)]$

$= f(\sqrt{x+1})$

$\sqrt{\sqrt{x+1} + 1}$

$\boxed{f \circ f(x) = \sqrt{\sqrt{x+1} + 1}}$

(d) $g \circ g(x) = g[g(x)]$

$= g\left(\frac{1}{x^2}\right)$

$= \frac{1}{\left(\frac{1}{x^2}\right)^2}$

$= \frac{1}{\frac{1}{x^4}}$

$\boxed{g \circ g(x) = x^4}$

(iii)

$f(x) = \frac{1}{\sqrt{x-1}}, x \neq 1$; $g(x) = (x^2+1)^2$

Solution:

(a) $f \circ g(x) = f[g(x)]$

$= f(x^2+1)^2$

$= \frac{1}{\sqrt{(x^2+1)^2 - 1}}$

$= \frac{1}{\sqrt{x^4 + \cancel{1} + 2x^2 - \cancel{1}}}$

$= \frac{1}{\sqrt{x^4 + 2x^2}}$

$= \frac{1}{\sqrt{x^2(x^2+2)}}$

$\boxed{f \circ g(x) = \frac{1}{x\sqrt{x^2+2}}}$

(b) $g \circ f(x) = g[f(x)]$

$= g\left(\frac{1}{\sqrt{x-1}}\right)$

$= \left[\left(\frac{1}{\sqrt{x-1}}\right)^2 + 1\right]^2$

$= \left[\frac{1}{x-1} + 1\right]^2$

$= \left[\frac{\cancel{1} + x - \cancel{1}}{x-1}\right]^2$

$\boxed{g \circ f(x) = \left[\frac{x}{x-1}\right]^2}$

(c) $f \circ f(x) = f[f(x)]$

$= f\left(\frac{1}{\sqrt{x-1}}\right)$

$= \frac{1}{\sqrt{\frac{1}{\sqrt{x-1}} - 1}}$

$= \frac{1}{\sqrt{\frac{1 - \sqrt{x-1}}{\sqrt{x-1}}}}$

$= \boxed{\sqrt{\frac{x-1}{1 - \sqrt{x-1}}}}$

(d) $g \circ g(x) = g[g(x)]$

$= g[(x^2+1)^2]$

$= [(x^2+1)^2 + 1]^2$

$= (x^4 + 1 + 2x^2 + 1)^2$

$\boxed{g \circ g(x) = (x^4 + 2x^2 + 2)^2}$

(iv)

$f(x) = 3x^4 - 2x^2 \quad ; \quad g(x) = \frac{2}{\sqrt{x}} \quad , x \neq 0$

Solution:

(a) $fog(x) = f[g(x)]$

$= f\left(\frac{2}{\sqrt{x}}\right)$

$= 3\left(\frac{2}{\sqrt{x}}\right)^4 - 2\left(\frac{2}{\sqrt{x}}\right)^2$

$= 3\left(\frac{16}{x^2}\right) - 2\left[\frac{4}{x}\right]$

$= \frac{48}{x^2} \quad \frac{-8}{x}$

$= \frac{48 - 8x}{x^2}$

$$\boxed{f[g(x)] = \frac{8(6-x)}{x^2}}$$

(b) $gof(x) = g[f(x)]$

$= g[3x^4 - 2x^2]$

$= \frac{2}{\sqrt{3x^4 - 2x^2}}$

$= \frac{2}{\sqrt{x^2(3x^2 - 2)}}$

$$\boxed{gof(x) = \frac{2}{x\sqrt{3x^2 - 2}}}$$

(c) $fof(x) = f[f(x)]$

$= f(3x^4 - 2x^2)$

$= 3(3x^4 - 2x^2)^4 - 2(3x^4 - 2x^2)$

$$\boxed{f(f(x)) = 3(3x^4 - 2x^2)^4 - 2(3x^4 - 2x^2)}$$

(d) $gog(x) = g[g(x)]$

$= g\,\frac{2}{\sqrt{x}}$

$= \frac{2}{\sqrt{\frac{2}{\sqrt{x}}}}$

$= \frac{2}{\frac{\sqrt{2}}{\sqrt{\sqrt{x}}}}$

$= \frac{2\sqrt{\sqrt{x}}}{\sqrt{2}}$

$= \frac{\sqrt{2}\cdot\sqrt{2}\sqrt{\sqrt{x}}}{\sqrt{2}}$

$= \sqrt{2}\sqrt{\sqrt{x}}$

$$\boxed{g(g(x)) = \sqrt{2\sqrt{x}}}$$

Question #2

For the real valued function, f is defined below, find

(a) $f^{-1}(x)$ (b) $f^{-1}(-1)$ and Verify $f(f^{-1}(x)) = f^{-1}(f(x)) = x$

(i)

$f(x) = -2x + 8$

Solution: (a) $= f^{-1}(x)$

$f(x) = y$

$x = f^{-1}(y)$

$f(x) = -2x + 8$

$y = -2x + 8$

$x = \frac{y - 8}{-2}$

$f^{-1}(y) = \frac{y - 8}{-2}$

Replace 'y' by 'x'

$$\boxed{f^{-1}(x) = \frac{8 - x}{2}}$$

(b) $f^{-1}(-1) = \frac{8 - (-1)}{2}$

$= \frac{8 + 1}{2}$

$$\boxed{f^{-1}(-1) = \frac{9}{2}}$$

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(c)

$f[f^{-1}(x)] = f^{-1}[f(x)] = x$

$f[f^{-1}(x)] = f\left[\frac{8-x}{2}\right]$

$= -\cancel{2}\left[\frac{8-x}{\cancel{2}}\right] + 8$

$= -(8-x)+8$

$= -\cancel{8}+x+\cancel{8}$

$= x$ —— ①

$f^{-1}[f(x)] = f^{-1}(-2x+8)$

$= \frac{8-(-2x+8)}{2}$

$= \frac{\cancel{8}+2x-\cancel{8}}{2}$

$= \frac{\cancel{2}x}{\cancel{2}}$

$= x$ —— ②

From eq ① and ②

$f[f^{-1}(x)] = f^{-1}[f(x)] = x$

• ——— •

(ii)

$f(x) = 3x^3+7$

Solution:

(a) $f^{-1}(x)$

$f(x) = y$

$x = f^{-1}(y)$

$f(x) = 3x^3+7$

$y = 3x^3+7$ $\because f(x) = y$

$\frac{y-7}{3} = x^3$

$x = \left[\frac{y-7}{3}\right]^{1/3}$ $\because x = f^{-1}(y)$

$f^{-1}(y) = \left[\frac{y-7}{3}\right]^{1/3}$

Replace 'y' by 'x'

$\boxed{f^{-1}(x) = \left[\frac{x-7}{3}\right]^{1/3}}$

(b) $f^{-1}(-1)$

$f^{-1}(-1) = \left[\frac{-1-7}{3}\right]^{1/3}$

$\boxed{f^{-1}(-1) = \left[\frac{-8}{3}\right]^{1/3}}$

(c) $f[f^{-1}(x)] = f^{-1}[f(x)] = x$

$f[f^{-1}(x)] = f\left[\frac{x-7}{3}\right]^{1/3}$

$= 3\left(\frac{x-7}{3}\right)^{3\times 1/3} + 7$

$= \cancel{3}\left(\frac{x-7}{\cancel{3}}\right) + 7$

$= x - \cancel{7} + \cancel{7}$

$f[f^{-1}(x)] = x$ —— ①

$f^{-1}[f(x)] = f^{-1}(3x^3+7)$

$= \left[\frac{3x^3+\cancel{7}-\cancel{7}}{3}\right]^{1/3}$

$= \left[\frac{\cancel{3}x^3}{\cancel{3}}\right]^{1/3}$

$= (x^3)^{1/3}$

$f^{-1}[f(x)] = x$ —— ②

From eq ① and ②

$f[f^{-1}(x)] = f^{-1}[f(x)] = x$

• ——— •

(iii)

$f(x) = (-x+9)^3$

Solution:

$f(x) = (-x+9)^3$

$f(x) = y$

$x = f^{-1} y$

(a) $f^{-1}(x) = ?$

$f(x) = (-x+9)^3$

$y = (-x+9)^3$

$y^{1/3} = -x+9$

$x = 9 - y^{1/3}$

$\because f^{-1}(y) = x$

$f^{-1} y = 9 - y^{1/3}$

Replace "y" by "x"

$f^{-1}(x) = 9 - x^{1/3}$

(b) $f^{-1}(-1) = 9 - (-1)^{1/3}$

$= 9 - (-1)^{1/3}$

(c) $= f[f^{-1}(x)] = f^{-1}[f(x)] = x$

$f(f^{-1}(x)] = [-(9 - x^{1/3}) + 9]^3$

$= [-\cancel{9} + x^{1/3} + \cancel{9}]^3$

$= (x^{1/3})^3$

$f[f^{-1}(x)] = x$

$f^{-1}(f(x)] = 9 - (-x+9)^{\cancel{3} \times \frac{1}{\cancel{3}}}$

$= \cancel{9} + x - \cancel{9}$

$= x$

$f[f^{-1}(x)] = f^{-1}[f(x)] = x$

Hence proved.

(iv)

$f(x) = \frac{2x+1}{x-1}, \quad x > 1$

Solution:

$f(x) = y$

$x = f^{-1}(y)$

(a) $f^{-1}(x) = ?$

$f(x) = \frac{2x+1}{x-1}$

$y(x-1) = 2x+1$

$xy - y = 2x+1$

$xy - 2x = y+1$

$x(y-2) = y+1$

$x = \frac{y+1}{y-2} \quad f^{-1}(y) = \frac{y+1}{y-2}$

Replace "y" by "x"

$f^{-1}(x) = \frac{x+1}{x-2}$

(b) $f^{-1}(-1) = \frac{-1+1}{-1-2} \Rightarrow \frac{0}{-3}$

$= 0$

(c) $= f[f^{-1}(x)] = f^{-1}[f(x)] = x$

$$f[f^{-1}(x)] = \frac{2\left(\frac{x+1}{x-2}\right)+1}{\frac{x+1}{x-2} - 1} = \frac{\frac{2x+\cancel{2}+x-\cancel{2}}{\cancel{x-2}}}{\frac{\cancel{x}+1-\cancel{x}+2}{\cancel{x-2}}}$$

$$= \frac{\cancel{3}x}{\cancel{3}} = x$$

$$f^{-1}[f(x)] = \frac{\frac{2x+1}{x-1} + 1}{\frac{2x+1}{x-1} - 2} \Rightarrow \frac{\frac{2x+\cancel{1}+x-\cancel{1}}{\cancel{x-1}}}{\frac{\cancel{2x}+1-\cancel{2x}+2}{\cancel{x-1}}}$$

$$= \frac{\cancel{3}x}{\cancel{3}} = x$$

$f[f^{-1}(x)] = f^{-1}[f(x)] = x$

Question # 3

Without finding the inverse, state the domain and range of f^{-1}.

(i)

$f(x) = \sqrt{x+2}$

Solution:

$f(x) = \sqrt{x+2}$

Daimain f: $x+2 \geq 0$

$x \geq -2$

$= [-2, \infty)$

Range f: $\sqrt{-2+2}$

$= 0$

$= [0, \infty)$

Domain $f^{-1} = [0, \infty)$

Range f^{-1}: $[-2, \infty)$

(ii)

$f(x) = \frac{1}{x+3}, x \neq -3$

Solution:

$f(x) = \frac{1}{x+3}$

Domain f: $\mathbb{R} - \{4\}$

Range f: $\mathbb{R} - \{1\}$

Domain f^{-1}: $\mathbb{R} - \{1\}$

Range f^{-1}: $\mathbb{R} - \{4\}$

(iii)

$f(x) = \frac{x-1}{x-4}, x \neq 4$

Solution:

$f(x) = \frac{x-1}{x-4}$

Domain f:

$= \mathbb{R} - \{-3\}$

Range f: $= \mathbb{R} - \{0\}$

Domain f^{-1}: $\mathbb{R} - \{0\}$

Range f^{-1}: $\mathbb{R} - \{-3\}$

(iv)

$f(x) = (x-5)^2, x \geq 5$

Solution:

$f(x) = (x-5)^2$

Domain f:

$= [5, \infty)$

Range f: $(5-5)^2 = 0$

$= [0, \infty)$

Domain f^{-1}: $[0, \infty)$

Range f^{-1}: $[5, \infty)$

* Theory

* Limit of a Function

Let a function $f(x)$ be defined in an open interval near the number 'a' (need not at a) If, as x approaches 'a' from both the left and right sides of 'a', $f(x)$ approaches a specific number 'L', then 'L' is called limit of $f(x)$ as x approches a.

Symbolically it is written as: $\lim_{x \to a} f_{(x)} = L$ read as "limit of $f(x)$, as $x \to a$, is L".

* Theorems on Limits of Function

Let f and g be two functions, for which

$\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$, then

- **Theorem 1:** The limit of the sum of two functions is equal to the sum of their limits.

$$\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x) = L + M$$

- **Theorem 2:** The limit of the difference of two functions is equal to the difference of their limits.

$$\lim_{x \to a} [f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x) = L - M$$

- **Theorem 3.** If 'k' is any real number, then

$$\lim_{x \to a} [k \cdot f(x)] = k \cdot \lim_{x \to a} f(x) = k \cdot L$$

- **Theorem 4:** The Limit of the product of the functions is equal to the product of their limits.

$$\lim_{x\to a}[f(x)\cdot g(x)] = \left[\lim_{x\to a} f(x)\right]\cdot\left[\lim_{x\to a} g(x)\right] = L\cdot M$$

- **Theorem 5:** The limit of the quotient of the functions is equal to the quotient of their limits provided the limit of denomenator is non-zero.

$$\lim_{x\to a}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{L}{M} \text{ , } g(x)\neq 0 \text{ , } M\neq 0$$

- **Theorem 6:** Limit of $[f(x)]^n$ where n is an integers

$$\lim_{x\to a}[f(x)]^n = \left[\lim_{x\to a} f(x)\right]^n = L^n.$$

Example #1

If $P(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots\ldots + a_1 x + a_0$ is polynomial function of degree n, then show that. $\lim_{x\to c} P(x) = P(c)$

Solution:

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots\cdots a_1 x + a_0$$

Taking $\lim_{x\to c}$ on both sides

$$\lim_{x\to c} P(x) = \lim_{x\to c}(a_n x^n + a_{n-1}x^{n-1} + \ldots\ldots a_1 x + a_0)$$

Using theorem on limits, we have

$$\lim_{x\to c} P(x) = a_n \lim_{x\to c} x^n + a_{n-1}\lim_{x\to c} x^{n-1} + \ldots\ldots a_1 \lim_{x\to c} x + \lim_{x\to c} a_0$$

$$= a_n c^n + a_{n-1} c^{n-1} + \cdots\cdots a_1 c + a_0$$

$$\boxed{\lim_{x\to c} P(x) = P(c)}$$

$$\lim_{x\to a}\frac{x^n-a^n}{x-a}=na^{n-1}$$, where n is an integer and $a>0$

Case I : Suppose n is a positive integer.

By subtituting $x=a$, we get $\left[\frac{0}{0}\right]$ form, So we make factor as follows.

$$L.H.S. = \lim_{x\to a}\frac{x^n-a^n}{x-a}$$

$$= \lim_{x\to a}\frac{(x-a)(x^{n-1}+ax^{n-2}+a^2x^{n-3}+\cdots\cdots a^{n-1})}{(x-a)}$$

$$= \lim_{x\to a}(x^{n-1}+ax^{n-2}+a^2x^{n-3}+\cdots\cdots+a^{n-1})$$

$$= a^{n-1}+a.a^{n-2}+a^2.a^{n-3}+\cdots\cdots+a^{n-1}$$

$$= a^{n-1}+a^{n-2+1}+a^{n-3+2}+\cdots+a^{n-1}$$

$$= a^{n-1}+a^{n-1}+a^{n-1}+\cdots\cdots+a^{n-1} \quad (n \text{ terms})$$

$$= na^{n-1} = R.H.S$$

$$L.H.S. = R.H.S.$$

Case II

Suppose n is a negative integer say $(n=-m)$. where m is a positive integer

$$L.H.S. = \lim_{x\to a}\frac{x^n-a^n}{x-a}$$

$$= \lim_{x\to a}\frac{x^{-m}-a^{-m}}{x-a}$$

$$= \lim_{x\to a}\frac{1}{x-a}\left[\frac{1}{x^m}-\frac{1}{a^m}\right]$$

$$= \lim_{x\to a}\frac{1}{x-a}\left[\frac{a^m-x^m}{x^m a^m}\right]$$

$$= \lim_{x\to a}\frac{-1}{x^m-a^m}\lim_{x\to a}\left[\frac{x^m-a^m}{x-a}\right]$$

$$= \frac{-1}{a^m a^m}(ma^{m-1})$$

$$= -ma^{m-1-m-m}$$

$$= -ma^{-m-1}$$

$$\because \boxed{n = -m}$$

$$= na^{n-1} = R.H.S.$$

$L.H.S = R.H.S.$ Hence proved.

* * *

Prove that $\lim\limits_{x\to 0} \dfrac{\sqrt{x+a}-\sqrt{a}}{x} = \dfrac{1}{2\sqrt{a}}$

Solution:

By substituting $x=0$, we have $\left[\frac{0}{0}\right]$ form, so rationalizing the numerator.

$$L.H.S = \lim_{x\to 0} \frac{\sqrt{a+a}-\sqrt{a}}{x}$$

$$= \lim_{x\to 0} \frac{\sqrt{x+a}-\sqrt{a}}{x} \times \frac{\sqrt{x+a}+\sqrt{a}}{\sqrt{x+a}+\sqrt{a}}$$

$$= \lim_{x\to 0} \frac{(\sqrt{x+a})^2-(\sqrt{a})^2}{x(\sqrt{x+a}+\sqrt{a})}$$

$$= \lim_{x\to 0} \frac{x+\cancel{a}-\cancel{a}}{x(\sqrt{x+a}+\sqrt{a})}$$

$$= \lim_{x\to 0} \frac{\cancel{x}}{\cancel{x}(\sqrt{x+a}+\sqrt{a})}$$

$$= \lim_{x\to 0} \frac{1}{\sqrt{x+a}+\sqrt{a}}$$

$$= \frac{1}{\sqrt{0+a}+\sqrt{a}}$$

$$= \frac{1}{\sqrt{a}+\sqrt{a}}$$

$$= \frac{1}{2\sqrt{a}} = R.H.S.$$

$L.H.S. = R.H.S.$ Hence proved.

* * *

Example #1 Evaluate

(i)

$$\lim_{x\to 1} \frac{x^2-1}{x^2-x}$$

Solution:

$$= \lim_{x\to 1} \frac{x^2-1}{x^2-x} \qquad \left[\frac{0}{0} \text{ form}\right]$$

$$\because a^2-b^2 = (a+b)(a-b)$$

$$= \lim_{x\to 1} \frac{(x+1)\cancel{(x-1)}}{x\cancel{(x-1)}}$$

$$= \lim_{x\to 1} \frac{x+1}{x}$$

$$= \frac{1+1}{1}$$

$$= \frac{2}{1}$$

$$= \boxed{2}$$

* * *

(ii)

$$\lim_{x\to 3} \frac{x-3}{\sqrt{x}-\sqrt{3}}$$

Solution:

$$= \lim_{x\to 3} \frac{x-3}{\sqrt{x}-\sqrt{3}} \qquad \left[\frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to 3} \frac{x-3}{\sqrt{x}-\sqrt{3}} \times \frac{\sqrt{x}+\sqrt{3}}{\sqrt{x}+\sqrt{3}}$$

$$\because (a-b)(a+b) = a^2-b^2$$

$$= \lim_{x\to 3} \frac{(x-3)(\sqrt{x}+\sqrt{3})}{(\sqrt{x})^2-(\sqrt{3})^2}$$

$$= \lim_{x\to 3} \frac{\cancel{(x-3)}(\sqrt{x}+\sqrt{3})}{\cancel{(x-3)}}$$

$$= \lim_{x\to 3} (\sqrt{x}+\sqrt{3})$$

$$= \sqrt{3}+\sqrt{3}$$

$$= \boxed{2\sqrt{3}}$$

* * *

Example #2

Evaluate $\lim\limits_{x \to +\infty} \dfrac{5x^4-10x^2+1}{-3x^3+10x^2+50}$

Solution:

$$\lim_{x \to +\infty} \frac{5x^4-10x^2+1}{-3x^3+10x^2+50} \quad \left[\frac{\infty}{\infty} \text{ form}\right]$$

Dividing up and down by 'x^3', we get

$$= \lim_{x \to +\infty} \frac{\frac{5x^4}{x^3} - \frac{10x^2}{x^3} + \frac{1}{x^3}}{\frac{-3x^3}{x^3} + \frac{10x^2}{x^3} + \frac{50}{x^3}}$$

$$= \lim_{x \to +\infty} \frac{5x - \frac{10}{x} + \frac{1}{x^3}}{-3 + \frac{10}{x} + \frac{50}{x^3}}$$

$$= \frac{5\infty - \frac{10}{\infty} + \frac{1}{\infty}}{-3 + \frac{10}{\infty} + \frac{50}{\infty}}$$

$$= \frac{\infty - 0 + 0}{-3+0+0}$$

$$= \frac{\infty}{-3}$$

$$= \boxed{\infty}$$

• ——— •

Example #3

Evaluate $\lim\limits_{x \to \infty} \dfrac{4x^4-5x^3}{3x^5+2x^2+1}$

Solution:

$$\lim_{x \to \infty} \frac{4x^4-5x^3}{3x^5+2x^2+1}$$

Since $x<0$, so dividing up and down by $(-x)^5 = -x^5$, we get

$$= \lim_{x \to \infty} \frac{\frac{4x^4}{-x^5} - \frac{5x^3}{-x^5}}{\frac{3x^5}{-x^5} + \frac{2x^2}{-x^5} + \frac{1}{-x^5}}$$

$$= \lim_{x \to \infty} \frac{\frac{-4}{x} + \frac{5}{x^2}}{-3 - \frac{2}{x^3} - \frac{1}{x^5}}$$

$$= \frac{\frac{-4}{\infty} + \frac{5}{\infty}}{-3 - \frac{2}{\infty} - \frac{1}{\infty}}$$

$$= \frac{0+0}{-3-0-0}$$

$$= \frac{0}{-3}$$

$$= \boxed{0}$$

• ——— •

Example #4

Evaluate

$$\lim_{x \to -\infty} \frac{2-3x}{\sqrt{3+4x^2}}$$

Solution:

$$\lim_{x \to \infty} \frac{2-3x}{\sqrt{3+4x^2}}$$

Since $x<0$, so dividing up and down by $\sqrt{x^2} = |x| = -x$

$$= \lim_{x \to -\infty} \frac{\frac{2-3x}{-x}}{\frac{\sqrt{3+4x^2}}{\sqrt{x^2}}}$$

$$= \lim_{x \to -\infty} \frac{\frac{2}{-x} - \frac{3x}{-x}}{\sqrt{\frac{3}{x^2} + \frac{4x^2}{x^2}}}$$

$$= \lim_{x \to -\infty} \frac{\frac{-2}{x} + 3}{\sqrt{\frac{3}{x^2} + 4}}$$

$$= \frac{\frac{-2}{-\infty} + 3}{\sqrt{\frac{3}{-\infty} + 4}}$$

$$= \frac{-0+3}{\sqrt{0+4}}$$

$$= \frac{3}{\sqrt{4}}$$

$$= \boxed{\frac{3}{2}}$$

(ii)

$$\lim_{x \to \infty} \frac{2-3x}{\sqrt{3+4x^2}}$$

Solution:

$$\lim_{x \to \infty} \frac{2-3x}{\sqrt{3+4x^2}}$$

Since $x > 0$, so dividing up and down by

$$\sqrt{x^2} = |x| = x$$

$$= \lim_{x \to \infty} \frac{\frac{2-3x}{x}}{\frac{\sqrt{3+4x^2}}{\sqrt{x^2}}}$$

$$= \lim_{x \to \infty} \frac{\frac{2}{x} - \frac{3x}{x}}{\sqrt{\frac{3}{x^2} + \frac{4x^2}{x^2}}}$$

$$= \lim_{x \to \infty} \frac{\frac{2}{x} - 3}{\sqrt{\frac{3}{x^2} + 4}}$$

$$= \frac{\frac{2}{\infty} - 3}{\sqrt{\frac{3}{\infty} + 4}}$$

$$= \frac{0-3}{\sqrt{0+4}}$$

$$= \frac{-3}{\sqrt{4}}$$

$$= \boxed{\frac{-3}{2}}$$

$$\lim_{n \to \infty} \left[1 + \frac{1}{n}\right]^n = e$$

Solution:

Using Binomial theorem, we have

$$\because (1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \frac{n(n-1)(n-2)(n-3)}{4!}x^4 + \cdots$$

$$\left[1+\frac{1}{n}\right]^n = 1 + \cancel{n}\left[\frac{1}{\cancel{n}}\right] + \frac{n(n-1)}{2!}\left[\frac{1}{n}\right]^2 + \frac{n(n-1)(n-2)}{3!}\left[\frac{1}{n}\right]^3 + \cdots$$

$$= 1 + 1 + \frac{1}{2!}\frac{n(n-1)}{n^2} + \frac{1}{3!}\frac{n(n-1)(n-2)}{n^3} + \cdots$$

$$= 2 + \frac{1}{2!} \cdot \frac{n}{n} \cdot \frac{n-1}{n} + \frac{1}{3!}\frac{n}{n} \cdot \frac{n-1}{n} \cdot \frac{n-2}{n} + \cdots$$

Taking $\lim_{n \to +\infty}$ both sides

$$\lim_{n \to +\infty}\left[1+\frac{1}{n}\right]^n = \lim_{n \to +\infty}\left[2 + \frac{1}{2!}\left[1-\frac{1}{n}\right] + \frac{1}{3!}\left[1-\frac{1}{n}\right]\left[1-\frac{2}{n}\right] + \cdots\right.$$

$$= 2 + \frac{1}{2!}\left[1-\frac{1}{\infty}\right] + \frac{1}{3!}\left[1-\frac{1}{\infty}\right]\left[1-\frac{2}{\infty}\right] + \cdots$$

$$= 2 + \frac{1}{2!}(1-0) + \frac{1}{3!}(1-0)(1-0) + \cdots$$

$$= 2 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots$$

$$= 2 + 0.5 + 0.166667 + 0.0416667 + \cdots$$

$$= 2.718281\ldots$$

$$\boxed{\lim_{n \to +\infty}\left[1+\frac{1}{n}\right]^n = e} \quad \because e \approx 2.7182$$

Hence proved

$$\lim_{x\to 0}(1+x)^{x}=e$$

Solution: we know that

$$\lim_{n\to\infty}\left[1+\frac{1}{n}\right]^{n}=e$$

put $\frac{1}{n}=x \Rightarrow n=\frac{1}{x}$

$n\to\infty$ then $x\to 0$

$$\lim_{n\to\infty}\left[1+\frac{1}{n}\right]^{n}=e$$

$$\boxed{\lim_{x\to 0}\left[1+x\right]^{\frac{1}{x}}=e}$$

Hence proved.

•——•

$$\lim_{x\to 0}\frac{a^{x}-1}{x}=\log_{e}a.$$

Solution:

$$\lim_{x\to 0}\frac{a^{x}-1}{x}=\log_{e}a$$

L.H.S.

$$=\lim_{x\to 0}.\frac{a^{x}-1}{x}$$

$a^{x}-1=y$

$a^{x}=y+1$

In logarithmic form

$\log_{a}(1+y)=x$

$x\to 0 \Rightarrow y\to 0$

$$=\lim_{y\to 0}\frac{y}{\log_{a}(1+y)}$$

$$=\lim_{y\to 0}\frac{1}{\frac{1}{y}\log_{a}(1+y)}$$

$$=\lim_{y\to 0}\frac{1}{\log_{a}(1+y)^{1/y}}\quad \because n\log m=\log m^{n}$$

$$=\frac{1}{\log_{a}\left[\lim_{y\to 0}(1+y)^{1/y}\right]}$$

$$\because \lim_{y\to 0}(1+y)^{\frac{1}{y}}=e$$

$$=\frac{1}{\log_{a}e}$$

$$\because \log_{e}a=\frac{1}{\log_{a}e}$$

$=\log_{e}a=R.H.S.$

$\boxed{L.H.S.=R.H.S.}$ Hence proved.

•——•

$$\lim_{x\to 0}\frac{e^{x}-1}{x}=\log_{e}e=1.$$

Solution:

We know that

$$\lim_{x\to 0}\frac{a^{x}-1}{x}=\log_{e}a$$

put $a=e$:

$$\lim_{x\to 0}\frac{e^{x}-1}{x}=\log_{e}e=1$$

Hence proved.

•——•

Example #5

Express each limit in terms of the number "e".

(a)

$$\lim_{n\to\infty}\left[1+\frac{3}{n}\right]^{2n}$$

Solution:

$$=\lim_{n\to\infty}\left[1+\frac{3}{n}\right]^{2n}$$

$$\lim_{n\to\infty}\left[1+\frac{1}{n}\right]^{n}=e$$

$$= \left[\lim_{n\to\infty}\left[1+\frac{3}{n}\right]^{n}\right]^{2}$$

$$= \left[\lim_{n\to+\infty}\left[1+\frac{3}{n}\right]^{\frac{n}{3}}\right]^{2\times 3}$$

$$= \boxed{e^6}$$

• ——— •

(b)

$$\lim_{h\to\infty}(1+2h)^{\frac{2}{h}}$$

Solution:

$$\lim_{h\to 0}(1+2h)^{2/h}$$

$$\lim_{x\to 0}(1+x)^{\frac{1}{x}} = e$$

$$= \left[\lim_{h\to 0}(1+2h)^{\frac{1}{2h}}\right]^{2}$$

$$= \boxed{e^2}$$

• ——— •

*The Sandwitch Theorem

Let f, g and h be functions such that $f(x) \le g(x) \le h(x)$ $\forall$ x in some open interval containing "c" except possible at c itself.

If $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} h(x) = L$, then

$$\lim_{x\to c} g(x) = L$$

• ——— •

Example #6

Evaluate

$$\lim_{\theta\to 0}\frac{\sin 7\theta}{\theta}$$

Solution:

$$\lim_{\theta\to 0}\frac{\sin 7\theta}{\theta} \qquad \left[\frac{0}{0}\text{ form}\right]$$

Multiply and divided by "7"

$$= \lim_{\theta\to 0}\frac{\sin 7\theta}{\theta}\times\frac{7}{7}$$

$$= 7\lim_{\theta\to 0}\frac{\sin 7\theta}{7\theta}$$

$$= 7(1) \qquad \because \lim_{\theta\to 0}\sin\frac{\theta}{\theta} = 1$$

$$= \boxed{7}$$

• ——— •

Example #7

Evalute $\lim_{\theta\to 0}\frac{1-\cos\theta}{\theta}$

Solution:

$$\lim_{\theta\to 0}\frac{1-\cos\theta}{\theta} \qquad \left[\frac{0}{0}\text{ form}\right]$$

$$= \lim_{\theta\to 0}\frac{1-\cos\theta}{\theta}\times\frac{1+\cos\theta}{1+\cos\theta}$$

$$\because (a-b)(a+b) = a^2-b^2$$

$$= \lim_{\theta\to 0}\frac{1-\cos^2\theta}{\theta(1+\cos\theta)}$$

$$= \lim_{\theta\to 0}\frac{\sin^2\theta}{\theta(1+\cos\theta)} \qquad \because 1-\cos^2\theta = \sin^2\theta$$

$$= \lim_{\theta\to 0}\frac{\sin\theta\cdot\sin\theta}{\theta\cdot(1+\cos\theta)}$$

$$= \lim_{\theta\to 0}\frac{\sin\theta}{\theta}\cdot\lim_{\theta\to 0}\frac{\sin\theta}{1+\cos\theta}$$

$$\because \lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$$

$$= (1)\cdot\frac{\sin 0}{1+\cos 0}$$

$$= 1\cdot\frac{0}{1+1}$$

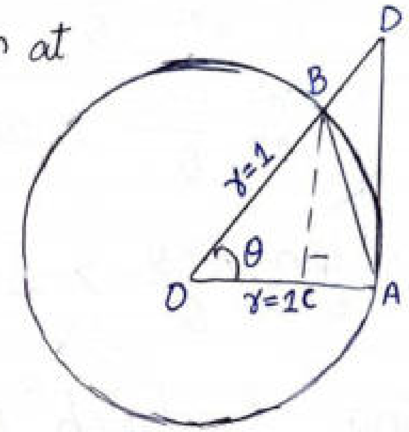
$$= \frac{0}{2}$$

$$= \boxed{0}$$

• ——— •

If θ is Measured in Radian, then $\lim\limits_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$.

Consider sector AOB of a circle with centre O and radius $r = 1$ subtending angle θ radian at O such that $0 < \theta < \frac{\pi}{2}$



Area of $\Delta OAB = \frac{1}{2}(OA \times BC)$

$= \frac{1}{2}(1 \times \sin\theta)$

$= \frac{1}{2}\sin\theta$

$\because$ In rt ΔOCB, $\sin\theta = \frac{BC}{OB} = BC$ $\because OB = 1$ Unit

Area of sector $OAB = \frac{1}{2}r^2\theta$

$= \frac{1}{2}(1)^2\theta$

$= \frac{1}{2}\theta$

Area of $\Delta AOD = \frac{1}{2}(OA \times AD)$

$= \frac{1}{2}(1 \times \tan\theta)$

$= \frac{1}{2}\tan\theta$

$\because$ In rt ΔOAD, $\tan\theta = \frac{AD}{OA} = AD$ $OA = 1$ unit

From figure, we see that:

Area of $\Delta OAB <$ Area of sector $OAB <$ Area of ΔOAD

$$\frac{1}{2}\sin\theta < \frac{1}{2}\theta < \frac{1}{2}\tan\theta$$

$$\sin\theta < \theta < \tan\theta$$

Dividing by $\sin\theta$

$$1 < \frac{\theta}{\sin\theta} < \frac{\cancel{\sin\theta}}{\cos\theta}\cdot\frac{1}{\cancel{\sin\theta}}$$

$\because 0 < \theta < \frac{\pi}{2}$, so sin is positive

$$1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}$$

$$1 > \frac{\sin\theta}{\theta} > \cos\theta$$

Taking $\lim\limits_{\theta \to 0}$ of each term

$$\lim_{\theta \to 0} 1 > \lim_{\theta \to 0} \frac{\sin\theta}{\theta} > \lim_{\theta \to 0} \cos\theta$$

$$1 > \lim_{\theta \to 0} \frac{\sin\theta}{\theta} > \cos 0$$

$$1 > \lim_{\theta \to 0} \frac{\sin\theta}{\theta} > 1$$

Using Sandwitch theorem

$$\boxed{\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1}$$ Hence proved.

Exercise # 1.3

Question #1

Evaluate each Limit by using theorems of Limits:

(i)

$$\lim_{x \to 3} (2x+4)$$

Solution:

$$\lim_{x \to 3} (2x+4)$$

$$= \lim_{x \to 3} (2x) + \lim_{x \to 3} (4)$$

$$= 2 \lim_{x \to 3} (x) + 4$$

$$= 2(3) + 4$$

$$= 6 + 4$$

$$= \boxed{10}$$

(ii)

$$\lim_{x \to 1} (3x^2 - 2x + 4)$$

Solution:

$$\lim_{x \to 1} (3x^2 - 2x + 4)$$

$$= \lim_{x \to 1} (3x^2) - \lim_{x \to 1} (2x) + \lim_{x \to 1} (4)$$

$$= 3 \lim_{x \to 1} (x^2) - 2 \lim_{x \to 1} (x) + 4$$

$$= 3(1)^2 - 2(1) + 4$$

$$= 3 - 2 + 4$$

$$= \boxed{5}$$

(iii)

$$\lim_{x\to 3} \sqrt{x^2+x+4}$$

Solution:

$$\lim_{x\to 3} \sqrt{x^2+x+4}$$

$$= \left[\lim_{x\to 3} (x^2+x+4)\right]^{1/2}$$

$$= \left[\lim_{x\to 3} (x^2) + \lim_{x\to 3} (x) + \lim_{x\to 3} (4)\right]^{1/2}$$

$$= [(3)^2 + (3) + 4]^{1/2}$$

$$= [9+3+4]^{1/2}$$

$$= [16]^{1/2}$$

$$= [4^2]^{1/2}$$

$$= \boxed{4}$$

(iv)

$$\lim_{x\to 2} x\sqrt{x^2-4}$$

Solution:

$$\lim_{x\to 2} x\sqrt{x^2-4}$$

$$= \left[\lim_{x\to 2} (x)\right] \left[\lim_{x\to 2} (x^2-4)\right]^{1/2}$$

$$= (2)\left(\lim_{x\to 2} (x^2) - \lim_{x\to 2} (4)\right)^{1/2}$$

$$= 2\,[(2)^2 - (4)]^{1/2}$$

$$= 2\,(4-4)^{1/2}$$

$$= 2\,(0)^{1/2}$$

$$= 2\,(0)$$

$$= \boxed{0}$$

(v)

$$\lim_{x\to 2} \left(\sqrt{x^3+1} - \sqrt{x^2+5}\right)$$

Solution:

$$\lim_{x\to 2} \left(\sqrt{x^3+1} - \sqrt{x^2+5}\right)$$

$$= \lim_{x\to 2} \sqrt{x^3+1} - \lim_{x\to 2} \sqrt{x^2+5}$$

$$= \left[\lim_{x\to 2} (x^3) + \lim_{x\to 2} (1)\right]^{1/2} - \left[\lim_{x\to 2} (x^2) + \lim_{x\to 2} (5)\right]^{1/2}$$

$$= [(2)^3 + 1]^{1/2} - [(2)^2 + 5]^{1/2}$$

$$= [8+1]^{1/2} - [4+5]^{1/2}$$

$$= \sqrt{9} - \sqrt{9}$$

$$= \boxed{0}$$

(vi)

$$\lim_{x\to -2} \frac{2x^3+5x}{3x-2}$$

Solution:

$$\lim_{x\to -2} \frac{2x^3+5x}{3x-2}$$

$$= \frac{\lim_{x\to -2} (2x^3+5x)}{\lim_{x\to -2} (3x-2)}$$

$$= \frac{2\lim_{x\to -2} (x^3) + 5\lim_{x\to -2} (x)}{3\lim_{x\to -2} (x) - \lim_{x\to -2} (2)}$$

$$= \frac{2(-2)^3 + 5(-2)}{3(-2) - 2}$$

$$= \frac{2(-8) - 10}{-6-2} = \frac{-16-10}{-8} = \frac{\overset{13}{\cancel{-26}}}{\underset{4}{\cancel{-8}}} = \boxed{\frac{13}{4}}$$

Question #2

Evaluate each limit by using algebric techniques.

(i)

$$\lim_{x\to -1} \frac{x^3-x}{x+1}$$

Solution:

$$\lim_{x\to -1} \frac{x^3-x}{x+1} \quad \left[\frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to -1} \frac{x(x^2-1)}{(x+1)}$$

$\because a^2-b^2=(a+b)(a-b)$

$$= \lim_{x\to -1} \frac{x\,\cancel{(x+1)}(x-1)}{\cancel{(x+1)}}$$

$$= \lim_{x\to -1} x(x-1)$$

$$= -1\,(-1-1)$$

$$= -1(-2)$$

$$= \boxed{2}$$

(ii)

$$\lim_{x\to 0} \frac{3x^3+4x}{x^2+x}$$

Solution:

$$\lim_{x\to 0} \frac{3x^3+4x}{x^2+x} \quad \left[\frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to 0} \frac{\cancel{x}(3x^2+4)}{\cancel{x}(x+1)}$$

$$= \lim_{x\to 0} \frac{3x^2+4}{x+1}$$

$$= \frac{3\,(0)^2+4}{0+1}$$

$$= \frac{0+4}{1}$$

$$= \frac{4}{1} = \boxed{4}$$

(iii)

$$\lim_{x\to 2} \frac{x^3-8}{x^2+x-6}$$

Solution:

$$\lim_{x\to 2} \frac{x^3-8}{x^2+x-6} \quad \left[\frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to 2} \frac{(x)^3-(2)^3}{x^2+x-6}$$

$\because a^3-b^3=(a-b)(a^2+ab+b^2)$

$$= \lim_{x\to 2} \frac{(x-2)(x^2+2x+4)}{x^2+3x-2x-6}$$

$$= \lim_{x\to 2} \frac{(x-2)(x^2+2x+4)}{x(x+3)-2(x+3)}$$

$$= \lim_{x\to 2} \frac{\cancel{(x-2)}(x^2+2x+4)}{(x+3)\cancel{(x-2)}}$$

$$= \frac{\lim\limits_{x\to 2}(x^2+2x+4)}{\lim\limits_{x\to 2}(x+3)}$$

$$= \frac{\lim\limits_{x\to 2}(x)^2+2\lim\limits_{x\to 2}(x)+\lim\limits_{x\to 2}(4)}{\lim\limits_{x\to 2}(x)+\lim\limits_{x\to 2}(3)}$$

$$= \frac{(2)^2+2(2)+4}{2+3}$$

$$= \frac{4+4+4}{5}$$

$$= \boxed{\frac{12}{5}}$$

(iv)

$$\lim_{x\to 1} \frac{x^3-3x^2+3x-1}{x^3-x}$$

Solutions:

$$= \lim_{x\to 1} \frac{x^3-3x^2+3x-1}{x^3-x} \quad \left[\frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to 1} \frac{x^3-1^3-3x(x-1)}{x(x^2-1)}$$

$\because a^3-b^3-3ab(a-b)=(a-b)^3$

$\because a^2-b^2=(a+b)(a-b)$

$$= \lim_{x\to 1} \frac{(x-1)^{\cancel{3}2}}{x(x+1)\cancel{(x-1)}}$$

$$= \frac{\lim_{x\to 1} (x-1)^2}{\lim_{x\to 1} x(x+1)}$$

$$= \frac{(1-1)^2}{1(1+1)}$$

$$= \frac{0}{1(2)}$$

$$= \frac{0}{2}$$

$$= \boxed{0}$$

(v)

$$\lim_{x\to -1} \left[\frac{x^3+x^2}{x^2-1}\right]$$

Solution:

$$\lim_{x\to -1} \left[\frac{x^3+x^2}{x^2-1}\right] \quad \left[\frac{0}{0} \text{ form}\right]$$

$\because a^2-b^2=(a+b)(a-b)$

$$= \lim_{x\to -1} \frac{x^2\cancel{(x+1)}}{\cancel{(x+1)}(x-1)}$$

$$= \frac{\lim_{x\to -1} x^2}{\lim_{x\to -1} (x-1)}$$

$$= \frac{(-1)^2}{(-1-1)}$$

$$= \boxed{-\frac{1}{2}}$$

(vi)

$$\lim_{x\to 4} \frac{2x^2-32}{x^3-4x^2}$$

Solution:

$$\lim_{x\to 4} \frac{2x^2-32}{x^3-4x^2} \quad \left[\frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to 4} \frac{2(x^2-16)}{x^2(x-4)}$$

$$= \lim_{x\to 4} \frac{2(x^2-4^2)}{x^2(x-4)}$$

$\because a^2-b^2=(a-b)(a+b)$

$$= \lim_{x\to 4} \frac{2\cancel{(x-4)}(x+4)}{x^2\cancel{(x-4)}}$$

$$= \lim_{x\to 4} \frac{2(x+4)}{x^2}$$

$$= \frac{\lim_{x\to 4} 2(x+4)}{\lim_{x\to 4} (x^2)}$$

$$= \frac{2(4+4)}{(4)^2}$$

$$= \frac{2(8)}{16}$$

$$= \frac{\cancel{16}}{\cancel{16}}$$

$$= \boxed{1}$$

(vii)

$$\lim_{x\to 2} \frac{\sqrt{x}-\sqrt{2}}{x-2}$$

Solution:

$$\lim_{x\to 2} \frac{\sqrt{x}-\sqrt{2}}{x-2} \quad \left[\frac{0}{0} \text{ form}\right]$$

$$\lim_{x\to 2} \frac{\sqrt{x}-\sqrt{2}}{x-2} \times \frac{\sqrt{x}+\sqrt{2}}{\sqrt{x}+\sqrt{2}}$$

$$= \lim_{x \to 2} \frac{(\sqrt{x})^2 - (\sqrt{2})^2}{(x-2)(\sqrt{x}+\sqrt{2})}$$

$$= \lim_{x \to 2} \frac{(x-2)}{(x-2)(\sqrt{x}+\sqrt{2})}$$

$$= \lim_{x \to 2} \frac{1}{(\sqrt{x}+\sqrt{2})}$$

$$= \frac{1}{\sqrt{2}+\sqrt{2}}$$

$$= \boxed{\frac{1}{2\sqrt{2}}}$$

(viii)

$$\lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

Solution:

$$\lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \qquad \left[\frac{0}{0} \text{ form}\right]$$

$$= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \times \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}$$

$\because (a-b)(a+b) = a^2 - b^2$

$$= \lim_{h \to 0} \frac{(\sqrt{x+h})^2 - (\sqrt{x})^2}{h(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h \to 0} \frac{\cancel{x}+h-\cancel{x}}{h(\sqrt{x+h}+\sqrt{x})}$$

$$\lim_{h \to 0} \frac{\cancel{h}}{\cancel{h}(\sqrt{x+h}+\sqrt{x})}$$

$$= \lim_{h \to 0} \frac{1}{\sqrt{x+h}+\sqrt{x}}$$

$$= \frac{1}{\sqrt{x+0}+\sqrt{x}}$$

$$= \frac{1}{\sqrt{x}+\sqrt{x}}$$

$$= \boxed{\frac{1}{2\sqrt{x}}}$$

(ix)

$$\lim_{x \to a} \frac{x^n - a^n}{x^m - a^m}$$

Solution

$$\lim_{x \to a} \frac{x^n - a^n}{x^m - a^m} \qquad \left[\frac{0}{0} \text{ form}\right]$$

Dividing up and down by $(x-a)$

$$= \lim_{x \to a} \frac{\frac{x^n - a^n}{x-a}}{\frac{x^m - a^m}{x-a}}$$

$$= \frac{\lim_{x \to a} \frac{x^n - a^n}{x-a}}{\lim_{x \to a} \frac{x^m - a^m}{x-a}}$$

$$\because \lim_{x \to a} \frac{x^n - a^n}{x-a} = na^{n-1}$$

$$= \frac{na^{n-1}}{ma^{m-1}}$$

$$= \frac{n}{m} a^{n-\cancel{1}-m+\cancel{1}}$$

$$= \boxed{\frac{n}{m} a^{n-m}}$$

Question #3

Evaluate the following Limits:

(i)

$$\lim_{x\to 0} \frac{\sin 7x}{x}$$

Solution:

$$\lim_{x\to 0} \frac{\sin 7x}{x} \qquad \left[\frac{0}{0} \text{ form}\right]$$

Multiply and divided by '7'

$$= \lim_{x\to 0} \frac{\sin 7x}{7x} \cdot 7$$

$$= 7 \lim_{x\to 0} \frac{\sin 7x}{7x}$$

$$\because \lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$$

$$= 7\ (1)$$

$$= \boxed{7}$$

(ii)

$$\lim_{x\to 0} \frac{\sin x^\circ}{x}$$

Solution:

$$\lim_{x\to 0} \frac{\sin x^\circ}{x} \qquad \left[\frac{0}{0} \text{ form}\right]$$

$$\because 1^\circ = \frac{\pi}{180} \text{ rad} \Rightarrow x^\circ = \frac{x\pi}{180} \text{ rad}$$

$$= \lim_{x\to 0} \frac{\sin \frac{x\pi}{180}}{x}$$

$$= \lim_{x\to 0} \frac{\sin x\frac{\pi}{180}}{x\cdot\frac{\pi}{180}} \cdot \frac{\pi}{180}$$

$$= (1)\cdot\frac{\pi}{180} \qquad \because \lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$$

$$= \boxed{\frac{\pi}{180}}$$

(iii)

$$\lim_{\theta\to 0} \frac{1-\cos\theta}{\sin\theta}$$

Solution:

$$\lim_{\theta\to 0} \frac{1-\cos\theta}{\sin\theta} \qquad \left[\frac{0}{0} \text{ form}\right]$$

$$= \lim_{\theta\to 0} \frac{1-\cos\theta}{\sin\theta} \times \frac{1+\cos\theta}{1+\cos\theta}$$

$$\because (a-b)(a+b) = a^2 - b^2$$

$$= \lim_{\theta\to 0} \frac{1-\cos^2\theta}{\sin\theta\,(1+\cos\theta)}$$

$$\because 1-\cos^2\theta = \sin^2\theta$$

$$= \lim_{\theta\to 0} \frac{\sin^{\cancel{2}}\theta}{\cancel{\sin\theta}\,(1+\cos\theta)}$$

$$= \lim_{\theta\to 0} \frac{\sin\theta}{1+\cos\theta}$$

$$= \frac{\sin 0}{1+\cos 0}$$

$$= \frac{0}{1+1}$$

$$= \frac{0}{2}$$

$$= \boxed{0}$$

(iv)

$$\lim_{x\to \pi} \frac{\sin x}{\pi - x}$$

Solution:

$$\lim_{x\to \pi} \frac{\sin x}{\pi - x} \qquad \left[\frac{0}{0} \text{ form}\right]$$

Let

$\theta = \pi - x$

$x = \pi - \theta$

$\theta = \pi - \pi = 0$

$\because \sin(\pi - x) = \sin x$

$= \lim\limits_{\theta \to 0} \frac{\sin(\pi - x)}{\theta}$

$= \lim\limits_{\theta \to 0} \frac{\sin\theta}{\theta}$

$\because \lim\limits_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$

$= \boxed{1}$

(V)

$\lim\limits_{x \to 0} \frac{\sin ax}{\sin bx}$

Solution:

$\lim\limits_{x \to 0} \frac{\sin ax}{\sin bx}$ $\left[\frac{0}{0} \text{ form}\right]$

$= \dfrac{\lim\limits_{x \to 0} \frac{\sin ax}{ax} \cdot ax}{\lim\limits_{x \to 0} \frac{\sin bx}{bx} \cdot bx}$

$\because \lim\limits_{x \to 0} \frac{\sin bx}{bx} = 1$

$= \frac{1 \cdot ax}{1 \cdot bx}$

$= \frac{ax}{bx}$

$= \boxed{\frac{a}{b}}$

(Vi)

$\lim\limits_{x \to 0} \frac{x}{\tan x}$

Solution:

$\lim\limits_{x \to 0} \frac{x}{\tan x}$ $\left[\frac{0}{0} \text{ form}\right]$

$= \lim\limits_{x \to 0} \dfrac{x}{\frac{\sin x}{\cos x}}$

$= \lim\limits_{x \to 0} \dfrac{\cos x}{\frac{\sin x}{x}}$

$= \dfrac{\lim\limits_{x \to 0} \cos x}{\lim\limits_{x \to 0} \frac{\sin x}{x}}$

$\because \lim\limits_{x \to 0} \frac{\sin x}{x} = 1$

$= \frac{\cos 0}{1}$

$= \cos 0$

$= \boxed{1}$

(Vii)

$\lim\limits_{x \to 0} \frac{1 - \cos 2x}{x^2}$

Solution:

$\lim\limits_{x \to 0} \frac{1 - \cos 2x}{x^2}$ $\left[\frac{0}{0} \text{ form}\right]$

$\because \cos 2x = 1 - 2\sin^2 x$

$\because 2\sin^2 x = 1 - \cos 2x$

$= \lim\limits_{x \to 0} \frac{2\sin^2 x}{x^2}$

$= 2 \lim\limits_{x \to 0} \frac{\sin^2 x}{x^2}$

$= 2\left[\lim\limits_{x \to 0} \frac{\sin x}{x}\right]^2$

$= 2\,(1)^2$

$= \boxed{2}$

(viii)

$$\lim_{x\to 0} \frac{1-\cos x}{\sin^2 x}$$

Solution:

$$\lim_{x\to 0} \frac{1-\cos x}{\sin^2 x} \qquad \left[\frac{0}{0} \text{ form}\right]$$

$\because \sin^2 x = 1-\cos^2 x$

$$= \lim_{x\to 0} \frac{1-\cos x}{1-\cos^2 x}$$

$\because a^2-b^2 = (a-b)(a+b)$

$$= \lim_{x\to 0} \frac{\cancel{1-\cos x}}{(\cancel{1-\cos x})(1+\cos x)}$$

$$= \lim_{x\to 0} \frac{1}{1+\cos x}$$

$$= \frac{1}{1+\cos 0}$$

$$= \frac{1}{1+1}$$

$$= \boxed{\frac{1}{2}}$$

* * *

(ix)

$$\lim_{\theta\to 0} \frac{\sin^2\theta}{\theta}$$

Solution:

$$\lim_{\theta\to 0} \frac{\sin^2\theta}{\theta} \qquad \left[\frac{0}{0} \text{ form}\right]$$

$$= \lim_{\theta\to 0} \frac{\sin\theta}{\theta} \cdot \sin\theta$$

$$= \lim_{\theta\to 0} \frac{\sin\theta}{\theta} \cdot \lim_{\theta\to 0} \sin\theta$$

$\because \lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$

$$= 1 \cdot \sin 0$$

$$= 1 \cdot (0)$$

$$= \boxed{0}$$

* * *

(x)

$$\lim_{x\to 0} \frac{\sec x - \cos x}{x}$$

Solution:

$$\lim_{x\to 0} \frac{\sec x - \cos x}{x} \qquad \left[\frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to 0} \frac{\frac{1}{\cos x} - \cos x}{x}$$

$$= \lim_{x\to 0} \frac{\frac{1-\cos^2 x}{\cos x}}{x}$$

$$= \lim_{x\to 0} \frac{1-\cos^2 x}{x \cos x}$$

$\because 1-\cos^2 x = \sin^2 x$

$$= \lim_{x\to 0} \frac{\sin^2 x}{x(\cos x)}$$

$$= \lim_{x\to 0} \frac{\sin x}{x} \cdot \lim_{x\to 0} \frac{\sin x}{\cos x}$$

$\because \lim_{x\to 0} \frac{\sin x}{x} = 1$

$$= 1 \cdot \frac{\sin 0}{\cos 0}$$

$$= 1 \cdot \frac{0}{1}$$

$$= 1 \cdot 0$$

$$= \boxed{0}$$

* * *

(xi)

$$\lim_{\theta\to 0} \frac{1-\cos p\theta}{1-\cos q\theta}$$

Solution:

$$\lim_{\theta\to 0} \frac{1-\cos p\theta}{1-\cos q\theta} \qquad \left[\frac{0}{0} \text{ form}\right]$$

$\because 1-2\sin^2\frac{\theta}{2} = \cos\theta$

$\because 1-\cos\theta = 2\sin^2\frac{\theta}{2}$

$$\lim_{\theta\to 0} \frac{2\sin^2 p\frac{\theta}{2}}{2\sin^2 q\frac{\theta}{2}}$$

$$= \left[\lim_{\theta\to0} \frac{\sin p\theta/2}{\sin q\frac{\theta}{2}}\right]^2$$

$$= \left[\lim_{\theta\to0} \frac{\frac{\sin p\frac{\theta}{2}}{p\frac{\theta}{2}}\cdot p\frac{\theta}{2}}{\frac{\sin q\frac{\theta}{2}}{q\frac{\theta}{2}}\cdot q\frac{\theta}{2}}\right]^2$$

$\because \lim_{\theta\to0}\frac{\sin\theta}{\theta}=1$

$$= \left[\frac{1\times p\cancel{\frac{\theta}{2}}}{1\times q\cancel{\frac{\theta}{2}}}\right]^2$$

$$= \left[\frac{p}{q}\right]^2$$

$$= \boxed{\frac{p^2}{q^2}}$$

(xii)

$$\lim_{\theta\to0}\frac{\tan\theta-\sin\theta}{\sin^3\theta}$$

Solution:

$$\lim_{\theta\to0}\frac{\tan\theta-\sin\theta}{\sin^3\theta} \qquad \left[\frac{0}{0}\text{ form}\right]$$

$$= \lim_{\theta\to0}\frac{\frac{\sin\theta}{\cos\theta}-\sin\theta}{\sin^3\theta}$$

$$= \lim_{\theta\to0}\frac{\frac{\sin\theta-\sin\theta\cos\theta}{\cos\theta}}{\sin^3\theta}$$

$$= \lim_{\theta\to0}\frac{\cancel{\sin\theta}(1-\cos\theta)}{\cos\theta\cdot\sin^3\theta}$$

$$= \lim_{\theta\to0}\frac{1-\cos\theta}{\cos\theta\cdot\sin^2\theta}$$

$$= \lim_{\theta\to0}\frac{1-\cos\theta}{\cos\theta\cdot(1-\cos^2\theta)} \qquad \because \sin^2\theta = 1-\cos^2\theta$$

$$= \lim_{\theta\to0}\frac{\cancel{1-\cos\theta}}{\cos\theta\cdot\cancel{(1-\cos\theta)}(1+\cos\theta)} \qquad \because a^2-b^2=(a-b)(a+b)$$

$$= \lim_{\theta\to0}\frac{1}{\cos\theta(1+\cos\theta)}$$

$$= \frac{1}{\cos0(1+\cos0)}$$

$$= \frac{1}{1(1+1)} = \frac{1}{1(2)}$$

$$= \boxed{\frac{1}{2}}$$

Question # 4

Express each Limit in terms of e:

(i)

$$\lim_{n\to+\infty}\left[1+\frac{1}{n}\right]^{2n}$$

Solution:

$$\lim_{n\to+\infty}\left[1+\frac{1}{n}\right]^{2n}$$

$= \left[\lim_{n\to+\infty}\left(1+\frac{1}{n}\right)^n\right]^2$

$\because \lim_{n\to+\infty}\left[1+\frac{1}{n}\right]^n = e$

$= (e)^2$

$= \boxed{e^2}$

(ii)

$\lim_{n\to+\infty}\left[1+\frac{1}{n}\right]^{\frac{n}{2}}$

Solution:

$\lim_{n\to+\infty}\left[1+\frac{1}{n}\right]^{\frac{n}{2}}$

$= \left[\lim_{n\to+\infty}\left[1+\frac{1}{n}\right]^n\right]^{\frac{1}{2}}$

$\because \lim_{n\to+\infty}\left[1+\frac{1}{n}\right]^n = e$

$= e^{1/2}$

$= \boxed{\sqrt{e}}$

(iii)

$\lim_{n\to+\infty}\left[1-\frac{1}{n}\right]^n$

Solution:

$\lim_{n\to+\infty}\left[1-\frac{1}{n}\right]^n$

$= \left[\lim_{n\to+\infty}\left[1+\frac{1}{-n}\right]^{-n}\right]^{-1}$

$\because \lim_{n\to+\infty}\left[1+\frac{1}{n}\right]^n = e$

$= e^{-1}$

$= \boxed{\frac{1}{e}}$

(iv)

$\lim_{n\to+\infty}\left[1+\frac{1}{3n}\right]^n$

Solution:

$\lim_{n\to+\infty}\left[1+\frac{1}{3n}\right]^n$

$= \left[\lim_{n\to+\infty}\left[1+\frac{1}{3n}\right]^{3n}\right]^{\frac{1}{3}}$

$\because \lim_{n\to+\infty}\left[1+\frac{1}{n}\right]^n = e$

$= \boxed{e^{1/3}}$

(v)

$\lim_{n\to+\infty}\left[1+\frac{4}{n}\right]^n$

Solution:

$\lim_{n\to+\infty}\left[1+\frac{4}{n}\right]^n$

$= \left[\lim_{n\to+\infty}\left[1+\frac{4}{n}\right]^{\frac{n}{4}}\right]^4$

$\because \lim_{n\to+\infty}\left[1+\frac{1}{n}\right]^n = e$

$= \boxed{e^4}$

(vi)

$\lim_{x\to 0}(1+3x)^{\frac{2}{x}}$

Solution:

$\lim_{x\to 0}(1+3x)^{\frac{2}{x}}$

$= \left[\lim_{x\to 0}(1+3x)^{1/x}\right]^2$

$= \left[\lim_{x\to 0}(1+3x)^{\frac{1}{3x}}\right]^{2\times 3}$

$\because \lim_{x\to 0}(1+x)^{\frac{1}{x}} = e$

$= \boxed{e^6}$

(vii)

$$\lim_{x\to 0} (1+2x^2)^{\frac{1}{x^2}}$$

Solution:

$$\lim_{x\to 0} (1+2x^2)^{\frac{1}{x^2}}$$

$$= \left[\lim_{x\to 0} (1+2x^2)^{\frac{1}{2x^2}}\right]^2$$

$$\because \lim_{x\to 0} (1+x)^{\frac{1}{x}} = e$$

$$= (e)^2$$

$$= \boxed{e^2}$$

(viii)

$$\lim_{h\to 0} (1-2h)^{\frac{1}{h}}$$

Solution:

$$\lim_{h\to 0} (1-2h)^{\frac{1}{h}}$$

$$= \left[\lim_{h\to 0} (1-2h)^{\frac{1}{-2h}}\right]^{-2}$$

$$\because \lim_{x\to 0} (1+x)^{\frac{1}{x}} = e$$

$$= e^{-2}$$

$$= \boxed{\frac{1}{e^2}}$$

(ix)

$$\lim_{x\to \infty}\left[\frac{x}{1+x}\right]^x$$

Solution:

$$\lim_{x\to \infty}\left[\frac{x}{1+x}\right]^x$$

$$= \lim_{x\to \infty}\left[\frac{1+x}{x}\right]^{-x}$$

$$= \lim_{x\to \infty}\left[\frac{1}{x}+\frac{x}{x}\right]^{-x}$$

$$= \lim_{x\to \infty}\left[1+\frac{1}{x}\right]^{-x}$$

$$= \left[\lim_{x\to \infty}\left[1+\frac{1}{x}\right]^{x}\right]^{-1}$$

$$\because \lim_{n\to \infty}\left[1+\frac{1}{n}\right]^{n} = e$$

$$= e^{-1}$$

$$= \boxed{\frac{1}{e}}$$

(x)

$$\lim_{x\to 0} \frac{e^{1/x}-1}{e^{\frac{1}{x}}+1}, \quad x<0$$

Solution:

$$\lim_{x\to 0} \frac{e^{\frac{1}{x}}-1}{e^{\frac{1}{x}}+1}$$

$$\because \lim_{x\to 0} \frac{1}{x} = -\infty$$

$$= \frac{\lim_{x\to 0}\left(e^{\frac{1}{x}}-1\right)}{\lim_{x\to 0}\left(e^{\frac{1}{x}}+1\right)}$$

$$= \frac{e^{-\infty}-1}{e^{-\infty}+1}$$

$$\because e^{-\infty} = 0$$

$$= \frac{0-1}{0+1}$$

$$= \frac{-1}{1}$$

$$= \boxed{-1}$$

(xi)

$$\lim_{x\to 0}\frac{e^{\frac{1}{x}}-1}{e^{\frac{1}{x}}+1} \quad , x>0$$

Solution:

$$\lim_{x\to 0}\frac{e^{\frac{1}{x}}-1}{e^{\frac{1}{x}}+1} \quad ; x>0$$

$$\because \lim_{x\to 0}\frac{1}{x}=+\infty$$

$$=\lim_{x\to 0}\frac{e^{\frac{1}{x}}\left[1-\frac{1}{e^{\frac{1}{x}}}\right]}{e^{\frac{1}{x}}\left[1+\frac{1}{e^{\frac{1}{x}}}\right]}$$

$$=\lim_{x\to 0}\frac{1-e^{-\frac{1}{x}}}{1+e^{-\frac{1}{x}}}$$

$$=\frac{1-e^{-\frac{1}{0}}}{1+e^{-\frac{1}{0}}}$$

$$=\frac{1-e^{-\infty}}{1+e^{-\infty}} \qquad \because e^{-\infty}=0$$

$$=\frac{1-0}{1+0}$$

$$=\frac{1}{1}$$

$$=\boxed{1}$$

*Theory

• The Left Hand Limit:

$\lim_{x\to c^-} f(x) = L$ is read as the limit of $f(x)$ is equal to L as x approaches c the left i.e; for all x sufficiently close to c, but less than c, the value of $f(x)$ can be made as close as we please to L.

• The Right Hand Limit:

$\lim_{x\to c^+} f(x) = M$ is read as the limit of $f(x)$ is equal to M as x approaches c from the right i.e., for all x sufficiently close to c, but greater than c, the value of $f(x)$ can be made as close as we please to M.

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Example #1

Determine whether $\lim_{x \to 2} f(x)$ and $\lim_{x \to 4} f(x)$ exist, when

$$f(x) = \begin{cases} 2x+1 & \text{if } 0 \leq x \leq 2 \\ 7-x & \text{if } 2 \leq x \leq 4 \\ x & \text{if } 4 \leq x \leq 6 \end{cases}$$

(i)

Solution:

$$f(x) = \begin{cases} 2x+1 & \text{if } 0 \leq x \leq 2 \\ 7-x & \text{if } 2 \leq x \leq 4 \\ x & \text{if } 4 \leq x \leq 6 \end{cases}$$

Left Hand Limit

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2} (2x+1)$$

$$= 2(2)+1$$

$$= 4+1$$

$$= 5$$

Right Hand Limit

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2} (7-x)$$

$$= 7-2$$

$$= 5$$

Left Hand Limit = Right Hand Limit

$\therefore \lim_{x \to 2} f(x)$ exist and equal to 5.

(ii)

Left Hand Limit.

$$\lim_{x \to 4^-} f(x) = \lim_{x \to 4} (7-x)$$

$$= 7-4$$

$$= 3$$

Right Hand Limit

$$\lim_{x \to 4^+} f(x) = \lim_{x \to 4} (x)$$

$$= 4$$

As Left Hand limit $\neq$ Right Hand Limit

$\therefore \lim_{x \to 4} f(x)$ does not exist.

• ——— •

• Continuous Function:

A function f is said to be continuous at a number 'c' if and only if the following three conditions are satisfied.

(i) $f(c)$ is defined

(ii) $\lim_{x \to c} f(x)$ exist

(iii) $\lim_{x \to c} f(x) = f(c)$

• Discontinuous Function:

If one or more of the above three conditions fail to hold at 'c' then the function f is said to be discontinous at 'c'

• ——— •

Example #2

Discuss the continuity of $f(x) = \dfrac{x^2-1}{x-1}, \forall x \in R$

Solution:

$$f(x) = \frac{x^2-1}{x-1}$$

f(x) is not defined at $x=1$

so, f(x) is discontinous at $x=1$

$$\lim_{x\to 1} f(x) = \lim_{x\to 1}\frac{x^2-1}{x-1}$$

$$= \lim_{x\to 1}\frac{(x+1)\cancel{(x-1)}}{\cancel{(x-1)}}$$

$$= \lim_{x\to 1}(x+1)$$

$$= (1+1)$$

$$\lim_{x\to 1} f(x) = 2 \text{ (finite)}$$

Therefore f(x) is continous at any other number $x \neq 1$

Example #3

For $f(x)=3x^2-5x+4$, discuss continuity of f at $x=1$.

Solution:

$$f(x) = 3x^2-5x+4$$

$$f(1) = 3(1)^2-5(1)+4$$

$$= 3-5+4$$

$$= 2$$

$$\lim_{x\to 1} f(x) = \lim_{x\to 1}(3x^2-5x+4)$$

$$= 3(1)^2-5(1)+4$$

$$= 3-5+4$$

$$= 2$$

$$\lim_{x\to 1} f(x) = f(1)$$

f(x) is continuous at $x=1$

Example #4

Discuss the Continuity of f(x) and g(x) at $x=3$.

(a)

$$f(x) = \begin{cases} \frac{x^2-9}{x-3} & \text{if } x \neq 3 \\ 6 & \text{if } x = 3 \end{cases}$$

Solution:

$$f(x) = \begin{cases} \frac{x^2-9}{x-3} & \text{if } x \neq 3 \\ 6 & \text{if } x = 3 \end{cases}$$

$$f(3) = 6$$

The function f is defined at $x=3$

$$\lim_{x\to 3} f(x) = \lim_{x\to 3}\frac{x^2-9}{x-3}$$

$$= \lim_{x\to 3}\frac{(x+3)\cancel{(x-3)}}{\cancel{(x-3)}}$$

$$= \lim_{x\to 3}(x+3)$$

$$= 3+3$$

$$\lim_{x\to 3} f(x) = 6$$

$$\lim_{x\to 3} f(x) = f(3)$$

f(x) is continuous at $x=3$

(b)

$$g(x) = \frac{x^2-9}{x-3} \text{ if } x \neq 3$$

Solution:

$$g(x) = \frac{x^2-9}{x-3} \text{ if } x \neq 3$$

As g(x) is not defined at $x=3$

$\Rightarrow$ g(x) is discontinuous at $x=3$

Example #5

Discuss the continuity of f at 3, when $f(x) = \begin{cases} x-1 & \text{if } x < 3 \\ 2x+1 & \text{if } 3 \geqslant x \end{cases}$

Solution:

$$f(x) = \begin{cases} x-1 & \text{if } x < 3 \\ 2x+1 & \text{if } 3 \geqslant x \end{cases}$$

(i) $f(3) = 2(3) + 1$

$= 6 + 1$

$f(3) = 7$ (defined)

condition (i) is satisfied

(ii) L·H·L: $\lim\limits_{x \to 3^-} f(x) = \lim\limits_{x \to 3} (x-1)$

$= 3 - 1$

$\lim\limits_{x \to 3^-} f(x) = 2$

R·H·L:

: $\lim\limits_{x \to 3^+} f(x) = \lim\limits_{x \to 3} (2x+1)$

$= 2(3)+1$

$= 6+1$

$\lim\limits_{x \to 3^+} f(x) = 7$

L·H·L ≠ R·H·L

so, $\lim\limits_{x \to 3} f(x)$ does not exist

Condition (ii) is not satisfied

f(x) is discontinuous at x = 3

• —— •

Exercise # 1·4

Question #1

Determine the left hand limit and the right hand limit, then find the limit of the following function when $x \to c$:

(i)

$f(x) = 2x^2 + x - 5$, $c = 1$

Solution:

$f(x) = 2x^2 + x - 5$

Left Hand Limit

L·H·L = $\lim\limits_{x \to 1^-} (f(x))$

$= \lim\limits_{x \to 1^-} (2x^2 + x - 5)$

$= 2(1)^2 + 1 - 5$

$= 2 + 1 - 5$

$= -2$

Right Hand Limit

R·H·L = $\lim\limits_{x \to 1^+} f(x)$

$= \lim\limits_{x \to 1^+} (2x^2 + x - 5)$

$= 2(1)^2 + 1 - 5$

$= 2 + 1 - 5$

$= -2$

Left Hand Limit = Right Hand Limit = -2

So, $\lim\limits_{x \to 1} f(x) = -2$

• —— •

(ii)

$$f(x) = \frac{x^2-9}{x-3} \quad , c = -3$$

Solution:

$$f(x) = \frac{x^2-9}{x-3}$$

Left Hand Limit

$$L.H.L = \lim_{x \to -3^-} f(x)$$

$$= \lim_{x \to -3^-} \left[\frac{x^2-9}{x-3}\right]$$

$$= \frac{(-3)^2-9}{-3-3}$$

$$= \frac{9-9}{-6}$$

$$= \frac{0}{-6}$$

$$L.H.L = 0$$

Right Hand Limit

$$R.H.L = \lim_{x \to -3^+} f(x)$$

$$= \lim_{x \to -3^+} \frac{x^2-9}{x-3}$$

$$= \frac{(-3)^2-9}{-3-3}$$

$$= \frac{9-9}{-6}$$

$$= \frac{0}{-6}$$

$$R.H.L = 0$$

Left Hand Limit = Right Hand Limit = 0

So, $\lim_{x \to -3} f(x) = 0$

(iii)

$$f(x) = |x-5| \quad , c = 5$$

Solution:

$$f(x) = |x-5|$$

Left Hand Limit

$$L.H.L = \lim_{x \to 5^-} f(x)$$

$$= \lim_{x \to 5^-} -(x-5)$$

$$= -5+5 = 0$$

Right Hand Limit

$$R.H.L = \lim_{x \to 5^+} f(x)$$

$$= \lim_{x \to 5^+} (x-5)$$

$$= 5-5$$

$$= 0$$

As

Left Hand Limit = Right Hand Limit.

So, $\lim_{x \to 5} f(x) = 0$

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Question #2

Discuss the continuity of $f(x)$ at $x=c$:

(i) $f(x) = \begin{cases} 2x+5 & \text{if } x \leq 2 \\ 4x+1 & \text{if } x > 2 \end{cases}, c = 2$

Solution:

$f(x) = \begin{cases} 2x+5 & \text{if } x \leq 2 \\ 4x+1 & \text{if } x > 2 \end{cases}$

(i) $f(2) = 2(2)+5$

$= 4+5$

$= 9$ (defined)

(ii) $L.H.L = \lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (2x+5)$

$= 2(2)+5$

$= 4+5$

$= 9$

$R.H.L = \lim_{x \to 2^+} f(x) = \lim_{x \to 2} (4x+1)$

$= 4(2)+1$

$= 8+1$

$= 9$

$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x)$

so, $\lim_{x \to 2} f(x) = 9$

(iii) $\lim_{x \to 2} f(x) = f(2)$

$f(x)$ is continuous at $x=2$

. ——— .

(ii) $f(x) = \begin{cases} 3x-1 & \text{if } x < 1 \\ 4 & \text{if } x = 1 \\ 2x & \text{if } x > 1 \end{cases}, c = 1$

Solution:

$f(x) = \begin{cases} 3x-1 & \text{if } x < 1 \\ 4 & \text{if } x = 1 \\ 2x & \text{if } x > 1 \end{cases}, c = 1$

(i) $f(1) = 4$ (defined)

(ii) $L.H.L = \lim_{x \to 1^-} f(x) = \lim_{x \to 1} (3x-1)$

$= 3(1)-1$

$= 3-1$

$\lim_{x \to 1} f(x) = 2$

$R.H.L = \lim_{x \to 1^+} f(x) = \lim_{x \to 1} (2x)$

$= 2(1)$

$\lim_{x \to 1} f(x) = 2$

$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) =$

so, $\lim_{x \to 1} f(x) = 2$

(iii) $\lim_{x \to 1} f(x) \neq f(1)$

$f(x)$ is discontinuous at $x=1$

. ——— .

Question #3

$$f(x) = \begin{cases} 3x & \text{if } x \leq -2 \\ x^2-1 & \text{if } -2 < x < 2 \\ 3 & \text{if } x \geqslant 2 \end{cases}$$

Discuss continuity at $x=2$ and $x=-2$

Solution:

$$f(x) = \begin{cases} 3x & \text{if } x \leq -2 \\ x^2-1 & \text{if } -2 < x < 2 \\ 3 & \text{if } x \geqslant 2 \end{cases}$$

$\boxed{\text{When } x=2}$

(i) $f(2) = 3$ (defined)

(ii) $L.H.L = \lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (x^2-1)$

$= (2)^2 - 1$

$= 4 - 1$

$\lim_{x\to 2^-} f(x) = 3$

$R.H.L = \lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (3)$

$= 3$

As $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x) = 3$

So, $\lim_{x\to 2} f(x) = 3$

(iii)

$\lim_{x\to 2} f(x) = f(2)$

Hence $f(x)$ is continuous at $x=2$

$\boxed{\text{When } x=-2}$

(i) $f(-2) = 3(-2) = -6$ (defined)

(ii)

$L.H.L = \lim_{x\to -2^-} f(x) = \lim_{x\to -2^-} (3x)$

$= 3(-2)$

$\lim_{x\to -2^-} f(x) = -6$

$R.H.L. =$

$\lim_{x\to -2^+} f(x) = \lim_{x\to -2} (x^2-1)$

$= (-2)^2 - 1$

$= 4 - 1$

$\lim_{x\to -2^+} f(x) = 3$

As $L.H.L = R.H.L$

So, $\lim_{x\to -2} f(x)$ does not exist.

$f(x)$ is discontinuous at $x=-2$

Question #4

If $f(x) = \begin{cases} x+2 & , x \leq -1 \\ C+2 & , x > -1 \end{cases}$,

find 'c' so that $\lim_{x\to -1} f(x)$ exist.

Solution:

$$f(x) = \begin{cases} x+2 & , x \leq -1 \\ C+2 & , x > -1 \end{cases}$$

Left Hand Limit:

$L.H.L = \lim_{x\to -1^-} f(x) = \lim_{x\to -1^-} (x+2)$

$= -1+2$

$\lim_{x\to -1} f(x) = 1$

Right Hand Limit:

$R.H.L = \lim_{x\to -1^+} f(x) = \lim_{x\to -1}(c+2)$

$= c+2$

Since $\lim_{x\to -1^-} f(x)$ exist, so

$\lim_{x\to -1^-} f(x) = \lim_{x\to -1^+} f(x)$

$1 = c+2$

$1-2 = c$

$-1 = c$

$\boxed{c = -1}$

Question #5

Find the values of m and n, so that given function f is continuous at x = 3.

(i)

$$f(x) = \begin{cases} mx & \text{if } x<3 \\ n & \text{if } x=3 \\ -2x+9 & \text{if } x>3 \end{cases}$$

Solution:

$f(3) = n$

$L.H.L = \lim_{x\to 3^-} f(x) = \lim_{x\to 3}(mx)$

$\lim_{x\to 3^-} f(x) = 3m$

$R.H.L = \lim_{x\to 3^+} f(x) = \lim_{x\to 3}(-2x+9)$

$= -2(3)+9$

$= -6+9$

$\lim_{x\to 3^+} f(x) = 3$

Since $f(x)$ is continuous at $x=3$, therefore

$\lim_{x\to 3} f(x) = f(3)$

$\Rightarrow \lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x) = f(3)$

$3m = 3 = n$

$3m = 3 \quad ; \quad 3=n$

$m = \frac{3}{3} \quad , \quad \boxed{n=3}$

$\boxed{m=1} \quad ,$

(ii)

$$f(x) = \begin{cases} mx & \text{if } x<3 \\ x^2 & \text{if } x\geq 3 \end{cases}$$

Solution:

$f(3) = (3)^2$

$= 9$ (defined)

$L.H.L = \lim_{x\to 3^-} f(x) = \lim_{x\to 3}(mx)$

$\lim_{x\to 3^-} f(x) = 3m$

$R.H.L = \lim_{x\to 3^+} f(x) = \lim_{x\to 3^+}(x^2)$

$= (3)^2$

$\lim_{x\to 3^+} f(x) = 9$

Since $f(x)$ is continuous at $x = 3$, therefore $\lim\limits_{x\to3} f(x)$ exists.

$$\lim_{x\to3^-} f(x) = \lim_{x\to3^+} f(x)$$

$$3m = 9$$

$$m = \frac{\cancel{9}\,3}{\cancel{3}}$$

$$\boxed{m = 3}$$

Question #6

If $f(x) = \begin{cases} \dfrac{\sqrt{2x+5} - \sqrt{x+7}}{x-2} & , x \neq 2 \\ k & , x = 2 \end{cases}$

Find the value of k so that f is continuous at $x = 2$.

Solution:

Since $f(x)$ is Continuous at $x = 2$ therefore

$$\lim_{x\to2} f(x) = f(2)$$

$$\lim_{x\to2} \frac{\sqrt{2x+5} - \sqrt{x+7}}{x-2} = k \qquad \left[\frac{0}{0} \text{ form}\right]$$

$$\lim_{x\to2} \frac{\sqrt{2x+5} - \sqrt{x+7}}{x-2} \times \frac{\sqrt{2x+5} + \sqrt{x+7}}{\sqrt{2x+5} + \sqrt{x+7}} = k$$

$$\because (a-b)(a+b) = a^2 - b^2$$

$$\lim_{x\to2} \frac{(\sqrt{2x+5})^2 - (\sqrt{x+7})^2}{x-2\,(\sqrt{2x+5} + \sqrt{x+7})} = k$$

$$\lim_{x\to2} \frac{2x+5-x-7}{x-2\,(\sqrt{2x+5} + \sqrt{x+7})} = k$$

$$\lim_{x\to 2} \frac{\cancel{(x-2)}}{\cancel{(x-2)}(\sqrt{2x+5}+\sqrt{x+7})} = k$$

$$\lim_{x\to 2} \frac{1}{\sqrt{2x+5}+\sqrt{x+7}} = k$$

$$\frac{1}{\sqrt{2(2)+5}+\sqrt{2+7}} = k$$

$$\frac{1}{\sqrt{4+5}+\sqrt{9}} = k$$

$$\frac{1}{\sqrt{9}+\sqrt{9}} = k$$

$$\frac{1}{3+3} = k$$

$$\frac{1}{6} = k$$

$$\boxed{k = \frac{1}{6}}$$

UNIT
2

Differentiation

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Unit no 2

Differentiation

* Theory

*** Derivative of a function:-** Let f be the real valued function continous in the interval $(x, x_1) \subseteq D_f$ (the f domain), then difference quotient $\frac{f(x_1) - f(x)}{x_1 - x}$

• Differentiation:- The derivative of f w.r.t 'x' at x and is denoted by $f'(x)$. The domain of f' consist of all x for which the limit exits. If $x \in D_f$ and $f'(x)$ exist then f is said to be differentiable at x. This process of finding f' is called differentiation.

Name of Mathematics	Leibniz	Newton	Lagrange	Cauchy
Notation used for derivatives.	$\frac{dy}{dx}$ or $\frac{df}{dx}$	$\overset{\bullet}{f}(x)$	$f'(x)$	$Df(x)$

* Finding $f'(x)$ from differentiation of Derivative:-

- **Step I :** Find $f(x+\delta x)$
- **Step II :** Simplify $f(x+\delta x) - f(x)$
- **Step III :** Divide $f(x+\delta x) - f(x)$ by δx to get $\frac{f(x+\delta x) - f(x)}{\delta x}$ and simplify it.
- **Step IV :** Find $\lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$

Example #1

Find the derivative of the following function by defination:-

(a)

$f(x) = c$

Solution:

$f(x) = c$ —— ①

$f(x+\delta x) = c$ —— ②

Subtract eq ② by ①

$f(x+\delta x) - f(x) = c - c$

$f(x+\delta x) - f(x) = 0$

Divided by δx both sides

$$\frac{f(x+\delta x) - f(x)}{\delta x} = \frac{0}{\delta x}$$

Appling $\lim_{\delta x \to 0}$ both sides

$$\lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x} = \lim_{\delta x \to 0} \frac{0}{\delta x}$$

$$\frac{f(x+0) - f(x)}{0} = 0$$

$f(x) = 0$

$$\boxed{f'(x) = 0 \text{ that is } \frac{d}{dx}(c) = 0}$$

(b)

$f(x) = x^2$

$f(x) = x^2$ —— ①

$f(x+\delta x) = (x+\delta x)^2$ —— ②

Subtract eq ② by ①

$f(x+\delta x) - f(x) = (x+\delta x)^2 - x^2$

$f(x+\delta x) - f(x) = \cancel{x^2} + \delta x^2 + 2x\delta x - \cancel{x^2}$

$f(x+\delta x) - f(x) = 2x\delta x + \delta x^2$

$f(x+\delta x) - f(x) = \delta x(2x+\delta x)$

Divided by δx both sides

$$\frac{f(x+\delta x) - f(x)}{\delta x} = \frac{\cancel{\delta x}(2x+\delta x)}{\cancel{\delta x}}$$

Appling $\lim_{\delta x \to 0}$ both sides

$$\lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x} = \lim_{\delta x \to 0} (2x+\delta x)$$

$f(x) = 2x + 0$

$$\boxed{f'(x) = 2x}$$

Example #2

Find the derivative of $\sqrt{x}$ at $x = a$ from first principles.

Solution:

Let $f(x) = \sqrt{x}$

$f(x) = x^{1/2}$ —— ①

$f(x+\delta x) = (x+\delta x)^{1/2}$ —— ②

Subtract eq ② by ①

$f(x+\delta x) - f(x) = (x+\delta x)^{1/2} - x^{1/2}$

$f(x+\delta x) - f(x) = x^{1/2}\left(1+\frac{\delta x}{x}\right)^{1/2} - x^{1/2}$

$f(x+\delta x) - f(x) = x^{1/2}\left[\left(1+\frac{\delta x}{x}\right)^{1/2} - 1\right]$

Using Binomial theorem

$$f(x+\delta x) - f(x) = x^{1/2}\left[1+\frac{1}{2}\left(\frac{\delta x}{x}\right) + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}\left(\frac{\delta x}{x}\right)^2 + \cdots - 1\right]$$

$$f(x+\delta x) - f(x) = x^{1/2}\frac{\delta x}{x}\left[\frac{1}{2} + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}\left(\frac{\delta x}{x}\right) + \cdots\right]$$

Divided by δx both sides

$$\frac{f(x+\delta x) - f(x)}{\delta x} = x^{1/2-1}\left[\frac{1}{2} + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2}\left(\frac{\delta x}{x}\right) + \cdots\right]$$

Appling $\lim_{\delta x \to 0}$ both sides.

$$\lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x} = \lim_{\delta x \to 0} x^{-\frac{1}{2}}\left[\frac{1}{2} + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}\left(\frac{\delta x}{x}\right) + \cdots\right]$$

$f'(x) = x^{-\frac{1}{2}}\left(\frac{1}{2}\right)$

$f'(x) = \frac{1}{2\sqrt{x}}$

$\boxed{x = a}$

$$\boxed{f'(a) = \frac{1}{2\sqrt{a}}}$$

Example #3

If $y = \frac{1}{x^2}$ then find $\frac{dy}{dx}$ at $x = -1$ by ab-initio method.

$y = \frac{1}{x^2}$

$y + \delta y = \frac{1}{(x+\delta x)^2}$

$\cancel{y} + \delta y - \cancel{y} = \frac{1}{(x+\delta x)^2} - \frac{1}{x^2}$

$$\delta y = \frac{x^2 - (x+\delta x)^2}{x^2(x+\delta x)^2}$$

$$\delta y = \frac{\cancel{x^2} - \cancel{x^2} - \delta x^2 - 2x\delta x}{x^2(x+\delta x)^2}$$

$$\delta y = \frac{-\delta x(2x+\delta x)}{x^2(x+\delta x)^2}$$

$$\frac{\delta y}{\delta x} = \frac{-(2x+\delta x)}{x^2(x+\delta x)^2}$$

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{-(2x+\delta x)}{x^2(x+\delta x)^2}$$

$$\frac{dy}{dx} = \frac{-2x+0}{x^2(x+0)^2} \Rightarrow \frac{-2x}{x^2(x^2)} \Rightarrow \frac{-2x}{x^{43}} \Rightarrow \boxed{\frac{-2}{x^3}}$$

$\boxed{x=-1}$

$\frac{dy}{dx}=\frac{-2}{(-1)^3}\Rightarrow\frac{-2}{-1}$

$\boxed{\frac{dy}{dx}=2}$

Example # 4

Find the derivative of $x^{\frac{2}{3}}$ and also calculate the value of derivative at $x=8$.

Solution:

Let $y=x^{\frac{2}{3}}$

$y+\delta y=(x+\delta x)^{\frac{2}{3}}$

$\cancel{y}+\delta y-\cancel{y}=(x+\delta x)^{\frac{2}{3}}-x^{\frac{2}{3}}$

$\delta y=x^{\frac{2}{3}}\left[\left(1+\frac{\delta x}{x}\right)^{\frac{2}{3}}-1\right]$

$\delta y=x^{\frac{2}{3}}\left[\cancel{1}+\frac{2}{3}\left(\frac{\delta x}{x}\right)+\frac{\frac{2}{3}\left(\frac{2}{3}-1\right)}{2!}\left(\frac{\delta x}{x}\right)^2+---\cancel{-1}\right]$

$\delta y=x^{\frac{2}{3}}\left[\frac{2}{3}\left(\frac{\delta x}{x}\right)+\frac{\frac{2}{3}\left(\frac{2}{3}-1\right)}{2!}\left(\frac{\delta x}{x}\right)^2+---\right]$

$\delta y=x^{\frac{2}{3}}\left(\frac{\delta x}{x}\right)\left[\frac{2}{3}+\frac{\frac{2}{3}\left(\frac{2}{3}-1\right)}{2!}\left(\frac{\delta x}{x}\right)+---\right]$

$\frac{\delta y}{\delta x}=x^{\frac{2}{3}-1}\left[\frac{2}{3}+\frac{\frac{2}{3}\left(\frac{2}{3}-1\right)}{2!}\left(\frac{\delta x}{x}\right)+---\right]$

$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\lim_{\delta x\to 0}x^{-\frac{1}{3}}\left[\frac{2}{3}+\frac{\frac{2}{3}\left(\frac{2}{3}-1\right)}{2!}\left(\frac{\delta x}{x}\right)+---\right]$

$\frac{dy}{dx}=\frac{2}{3x^{\frac{1}{3}}}$

$\frac{dy}{dx}=\frac{2}{3(8)^{1/3}}\Rightarrow\boxed{f'(8)=\frac{1}{3}}$

- **Sum or Difference theorem:-**

If "f" and "g" are differentiable at x,

$\frac{d}{dx}\left[f(x)\pm g(x)\right]=\frac{d}{dx}f(x)\pm\frac{d}{dx}g(x)=f'(x)\pm g'(x$

- **Product Theorem:-** If "f" and "g" are differentiable at x then.

$\frac{d}{dx}\left[f(x)g(x)\right]=\frac{d}{dx}\left[f(x)\right]g(x)+f(x)\frac{d}{dx}g(x)=f'(x)\,g$
$+f(x)\,g'(x)$

Example # 5

Find the derivative of x^3+2x+3.

Solution:

Let $y=x^3+2x+3$ —— ①

$y+\delta y=(x+\delta x)^3+2(x+\delta x)+3$ —— ②

Subtract eq ② by ①

$\cancel{y}+\delta y-\cancel{y}=(x+\delta x)^3+2(x+\delta x)+3-(x^3+2x+3)$

$\because (a+b)^3=a^3+b^3+3a^2b+3ab^2$

$\delta y=\cancel{x^3}+\delta x^3+3x^2\delta x+3x\delta x^2+\cancel{2x}+2\delta x+\cancel{3}-\cancel{x^3}-\cancel{2x}-\cancel{3}$

$\delta y=\delta x^3+3x^2\delta x+3x\delta x^2+2\delta x$

$\delta y=\delta x\left(\delta x^2+3x^2+3x\delta x+2\right)$

Divided by δx both sides

$\frac{\delta y}{\delta x}=\frac{\cancel{\delta x}\left(\delta x^2+3x^2+3x\delta x+2\right)}{\cancel{\delta x}}$

Apply $\lim_{\delta x\to 0}$ both sides

$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\lim_{\delta x\to 0}\left(\delta x^2+3x^2+3x\delta x+2\right)$

$\frac{dy}{dx}=(0)^2+3x^2+3x(0)+2$

$\boxed{\frac{dy}{dx}=3x^2+2}$

- **Power Rule:-**

$\frac{d}{dx}(x^n)=nx^{n-1}$, where n is rational number.

- **Derivative of constant:-**

$\frac{d}{dx}(c)=0$ derivative of constant is zero.

- **Quotient Theorem:-** If "f" and "g" are differentiable at "x" then

$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]=\frac{g(x)\frac{d}{dx}\left[f(x)\right]-f(x)\frac{d}{dx}g(x)}{\left[g(x)\right]^2}$

Exercise # 2·1

Question #1

Find the definition, the derivative w.r.t 'x' of the following functions defined as:

(i)

$2x^2+1$

Solution:

Let

$y = 2x^2+1$ —— ①

$y+\delta y = 2(x+\delta x)^2+1$ —— ②

$y+\delta y = (2x+2\delta x)^2+1$

Subtract eq ② by ①

$\cancel{y}+\delta y-\cancel{y} = 2(x+\delta x)^2+1-(2x^2+1)$

$\delta y = 2(x^2+2x\delta x+\delta x^2)+\cancel{1}-2x^2-\cancel{1}$

$\delta y = \cancel{2x^2}+4x\delta x+2\delta x^2-\cancel{2x^2}$

$\delta y = 2\delta x\,(2x+\delta x)$

Divided by δx both sides

$$\frac{\delta y}{\delta x} = \frac{2\cancel{\delta x}(2x+\delta x)}{\cancel{\delta x}}$$

$$\frac{\delta y}{\delta x} = 2(2x+\delta x)$$

Appling $\lim\limits_{\delta x\to 0}$ both sides

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x} = \lim_{\delta x\to 0} 2(2x+\delta x)$$

$$\frac{dy}{dx} = 2(2x+0)$$

$$\boxed{\frac{dy}{dx} = 4x}$$

• —— •

(ii)

$2-\sqrt{x}$

Solution:

Let

$y = 2-\sqrt{x}$ —— ①

$y+\delta y = 2-\sqrt{x+\delta x}$ —— ②

Subtract eq ② by ①

$\cancel{y}+\delta y-\cancel{y} = 2-\sqrt{x+\delta x}-(2-\sqrt{x})$

$\delta y = \cancel{2}-\sqrt{x+\delta x}-\cancel{2}+\sqrt{x}$

$$\delta y = \sqrt{x}-\sqrt{x+\delta x} \times \frac{\sqrt{x}+\sqrt{x+\delta x}}{\sqrt{x}+\sqrt{x+\delta x}}$$

$$\delta y = \frac{(\sqrt{x}-\sqrt{x+\delta x})(\sqrt{x}+\sqrt{x+\delta x})}{\sqrt{x}+\sqrt{x+\delta x}}$$

$$\delta y = \frac{(\sqrt{x})^2-(\sqrt{x+\delta x})^2}{\sqrt{x}+\sqrt{x+\delta x}}$$

$$\delta y = \frac{\cancel{x}-\cancel{x}-\delta x}{\sqrt{x}+\sqrt{x+\delta x}}$$

$$\delta y = \frac{-\delta x}{\sqrt{x}+\sqrt{x+\delta x}}$$

Divided by δx both side

$$\frac{\delta y}{\delta x} = \frac{-1}{\sqrt{x}+\sqrt{x+\delta x}}$$

Appling $\lim_{\delta x\to 0}$ both side

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x} = \lim_{\delta x\to 0}\frac{-1}{\sqrt{x}+\sqrt{x+\delta x}}$$

$$\frac{dy}{dx} = \frac{-1}{\sqrt{x}+\sqrt{x}}$$

$$\boxed{\frac{dy}{dx} = \frac{-1}{2\sqrt{x}}}$$

(iii)

$$\frac{1}{\sqrt{x}}$$

Solution:

Let $y = \frac{1}{\sqrt{x}}$ —— ①

$y+\delta y = \frac{1}{\sqrt{x+\delta x}}$ —— ②

Subtract eq ② by ①

$$\cancel{y}+\delta y-\cancel{y} = \frac{1}{\sqrt{x+\delta x}} - \frac{1}{\sqrt{x}}$$

$$\delta y = \frac{\sqrt{x} - \sqrt{x+\delta x}}{(\sqrt{x})(\sqrt{x+\delta x})} \times \frac{\sqrt{x}+\sqrt{x+\delta x}}{\sqrt{x}+\sqrt{x+\delta x}}$$

$$\delta y = \frac{(\cancel{\sqrt{x}})^2 - (\cancel{\sqrt{x+\delta x}})^2}{(\sqrt{x})(\sqrt{x+\delta x})(\sqrt{x}+\sqrt{x+\delta x})}$$

$$\delta y = \frac{\cancel{x}-\cancel{x}-\delta x}{(\sqrt{x})(\sqrt{x+\delta x})(\sqrt{x}+\sqrt{x+\delta x})}$$

$$\delta y = \frac{-\delta x}{(\sqrt{x})(\sqrt{x+\delta x})(\sqrt{x}+\sqrt{x+\delta x})}$$

Divided by δx both sides

$$\frac{\delta y}{\delta x} = \frac{-1}{(\sqrt{x})(\sqrt{x+\delta x})(\sqrt{x}+\sqrt{x+\delta x})}$$

Appling $\lim\limits_{\delta x\to 0}$ both sides.

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x} = \lim_{\delta x\to 0}\frac{-1}{(\sqrt{x})(\sqrt{x+\delta x})(\sqrt{x}+\sqrt{x+\delta x})}$$

$$\frac{dy}{dx} = \frac{-1}{(\sqrt{x})(\sqrt{x})(\sqrt{x}+\sqrt{x})}$$

$$\frac{dy}{dx} = \frac{-1}{x\cdot 2\sqrt{x}}$$

$$\boxed{\frac{dy}{dx} = \frac{-1}{2x^{3/2}}}$$

(iv)

$$\frac{1}{x^3}$$

Solution:

Let $y = \frac{1}{x^3}$ —— ①

$y+\delta y = \frac{1}{(x+\delta x)^3}$ —— ②

Subtract eq ② by ①

$$\cancel{y}+\delta y-\cancel{y} = \frac{1}{(x+\delta x)^3} - \frac{1}{x^3}$$

$$\delta y = \frac{x^3-(x+\delta x)^3}{x^3(x+\delta x)^3}$$

$\because (a+b)^3 = a^3+b^3+3a^2b+3ab^2$

$$\delta y = \frac{x^3-(x^3+\delta x^3+3x^2\delta x+3x\delta x^2)}{x^3(x+\delta x)^3}$$

$$\delta y = \frac{\cancel{x^3}-\cancel{x^3}-\delta x^3-3x^2\delta x-3x\delta x^2}{x^3(x+\delta x)^3}$$

$$\delta y = \frac{-\delta x^3-3x^2\delta x-3x\delta x^2}{x^3(x+\delta x)^3}$$

$$\delta y = \frac{\delta x\left[-\delta x^2-3x^2-3x\delta x\right]}{x^3(x+\delta x)^3}$$

Divided by δx both sides

$$\frac{\delta y}{\delta x} = \frac{-\delta x^2-3x^2-3x\delta x}{x^3(x+\delta x)^3}$$

Appling $\lim\limits_{\delta x\to 0}$ both sides

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x} = \lim_{\delta x\to 0}\frac{-\delta x^2-3x^2-3x\delta x}{x^3(x+\delta x)^3}$$

$$\frac{dy}{dx} = \frac{-(0)^2-3x^2-3x(0)}{x^3(x+0)^3}$$

$$\frac{dy}{dx} = \frac{-3x^2}{(x^3)(x^3)}$$

$$\frac{dy}{dx} = \frac{-3\cancel{x^2}}{x^{\cancel{6}4}}$$

$$\boxed{\frac{dy}{dx} = \frac{-3}{x^4}}$$

(V)

$$\frac{1}{x-a}$$

Solution:

Let $y = \frac{1}{x-a}$ —— ①

$y+\delta y = \frac{1}{x+\delta x-a}$ —— ②

Subtract eq ② by ①

$$y+\delta y-y = \frac{1}{x+\delta x-a} - \frac{1}{x-a}$$

$$\delta y = \frac{(x-a)-(x+\delta x-a)}{(x+\delta x-a)(x-a)}$$

$$\delta y = \frac{\cancel{x}-\cancel{a}-\cancel{x}-\delta x+\cancel{a}}{(x+\delta x-a)(x-a)}$$

$$\delta y = \frac{-\delta x}{(x+\delta x-a)(x-a)}$$

Divided by δx both sides

$$\frac{\delta y}{\delta x} = \frac{-1}{(x+\delta x-a)(x-a)}$$

Appling $\lim_{\delta x\to 0}$ both sides

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x} = \lim_{\delta x\to 0}\frac{-1}{(x+\delta x-a)(x-a)}$$

$$\frac{dy}{dx} = \frac{-1}{(x+0-a)(x-a)}$$

$$\frac{dy}{dx} = \frac{-1}{(x-a)(x-a)}$$

$$\boxed{\frac{dy}{dx} = \frac{-1}{(x-a)^2}}$$

• ——— •

(Vi)

$$x(x-3)$$

Solution:

Let

$y = x(x-3) \Rightarrow y = x^2-3x$ —— ①

$y+\delta y = (x+\delta x)^2-3(x+\delta x)$ —— ②

Subtract eq ② by ①

$$\cancel{y}+\delta y-\cancel{y} = (x+\delta x)(x+\delta x)-3 - (x^2-3x)$$

$$\delta y = \cancel{x^2}+2x\delta x-\cancel{3x}+\delta x^2-3\delta x-\cancel{x^2}+\cancel{3x}$$

$$\delta y = 2x\delta x+\delta x^2-3\delta x$$

$$\delta y = \delta x(2x+\delta x-3)$$

Divided by δx both sides

$$\frac{\delta y}{\delta x} = \frac{\cancel{\delta x}(2x+\delta x-3)}{\cancel{\delta x}}$$

$$\frac{\delta y}{\delta x} = (2x+\delta x-3)$$

Appling $\lim_{\delta x\to 0}$ both sides

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x} = \lim_{\delta x\to 0}(2x+\delta x-3)$$

$$\frac{dy}{dx} = (2x+0-3)$$

$$\boxed{\frac{dy}{dx} = 2x-3}$$

• ——— •

(Vii)

$$\frac{2}{x^4}$$

Solution:

Let $y = \frac{2}{x^4}$

$y = 2x^{-4}$ —— ①

$y+\delta y = 2(x+\delta x)^{-4}$ —— ②

Subtract eq ② by ①

$$\cancel{y}+\delta y-\cancel{y} = 2(x+\delta x)^{-4} - 2x^{-4}$$

$$\delta y = 2x^{-4}\left[1+\frac{\delta x}{x}\right]^{-4} - 2x^{-4}$$

$$\delta y = 2x^{-4}\left[\left(1+\frac{\delta x}{x}\right)^{-4} - 1\right]$$

Binomial theorem

$$\delta y = 2x^{-4}\left[\cancel{1} + -4\left(\frac{\delta x}{x}\right) + \frac{-4(-4-1)}{2!}\left(\frac{\delta x}{x}\right)^2 + \cdots \cancel{-1}\right]$$

$$\delta y = 2x^{-4}\left[-4\left(\frac{\delta x}{x}\right) + 10\left(\frac{\delta x}{x}\right)^2 + \cdots\right]$$

$$\delta y = 2x^{-4}\left(\frac{\delta x}{x}\right)\left[-4 + 10\left(\frac{\delta x}{x}\right) + \cdots\right]$$

Divided by δx both sides

$$\frac{\delta y}{\delta x} = 2x^{-4-1}\left[-4 + 10\left(\frac{\delta x}{x}\right) + \cdots\right]$$

$$\frac{\delta y}{\delta x} = 2x^{-5}\left[-4 + 10\left(\frac{\delta x}{x}\right) + \cdots\right]$$

Appling $\lim\limits_{\delta x \to 0}$ both sides

$$\lim_{\delta x \to 0}\frac{\delta y}{\delta x} = \lim_{\delta x \to 0} 2x^{-5}\left[-4 + 10\left(\frac{\delta x}{x}\right) + \cdots\right]$$

$$\frac{dy}{dx} = 2x^{-5}(-4)$$

$$\frac{dy}{dx} = -8x^{-5}$$

$$\boxed{\frac{dy}{dx} = \frac{-8}{x^5}}$$

(ix)

$$x^{3/2}$$

Solution:

Let

$$y = x^{3/2} \quad \text{——} ①$$

$$y + \delta y = (x+\delta x)^{3/2} \quad \text{——} ②$$

Subtract eq ② by ①

$$y + \delta y - y = (x+\delta x)^{3/2} - x^{3/2}$$

$$\delta y = x^{3/2}\left(1+\frac{\delta x}{x}\right)^{3/2} - x^{3/2}$$

$$\delta y = x^{3/2}\left[\left(1+\frac{\delta x}{x}\right)^{3/2} - 1\right]$$

Binomial theorem

(viii)

$$(x+4)^{1/3}$$

Solution:

Let

$$y = (x+4)^{1/3} \quad \text{——} ①$$

$$y + \delta y = (x+\delta x+4)^{1/3} \quad \text{——} ②$$

~~Divided~~ Subtract eq ② by ①

$$y + \delta y - y = (x+\delta x+4)^{1/3} - (x+4)^{1/3}$$

$$\delta y = (x+4)^{1/3}\left[1+\frac{\delta x}{x}\right]^{1/3} - (x+4)^{1/3}$$

$$\delta y = (x+4)^{1/3}\left[\left(1+\frac{\delta x}{x}\right)^{1/3} - 1\right]$$

Binomial theorem

$$\delta y = (x+4)^{1/3}\left[\cancel{1} + \frac{1}{3}\left(\frac{\delta x}{x}\right) + \frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}\left(\frac{\delta x}{x}\right)^2 + \cdots \cancel{-1}\right]$$

$$\delta y = (x+4)^{1/3}\left[\frac{1}{3}\left(\frac{\delta x}{x}\right) + \frac{\frac{-2}{9}}{2}\left(\frac{\delta x}{x}\right)^2 + \cdots\right]$$

$$\delta y = (x+4)^{1/3}\left(\frac{\delta x}{x}\right)\left[\frac{1}{3} - \frac{1}{9}\left[\frac{\delta x}{x}\right]^2 + \cdots\right]$$

Divided by δx both sides

$$\frac{\delta y}{\delta x} = (x+4)^{1/3-1}\left[\frac{1}{3} - \frac{1}{9}\left(\frac{\delta x}{x}\right) + \cdots\right]$$

Appling $\lim\limits_{\delta x \to 0}$ both sides

$$\lim_{\delta x \to 0}\frac{\delta y}{\delta x} = \lim_{\delta x \to 0}(x+4)^{1/3-1}\left[\frac{1}{3} - \frac{1}{9}\left(\frac{\delta x}{x}\right)^2 + \cdots\right]$$

$$\frac{dy}{dx} = (x+4)^{\frac{-2}{3}}\left(\frac{1}{3}\right)$$

$$\boxed{\frac{dy}{dx}) = \frac{1}{3(x+4)^{2/3}}}$$

(X)

$$x^{5/2}$$

Solution:

$$\text{Let } y = x^{5/2} \quad \text{——} ①$$

$$y + \delta y = (x+\delta x)^{5/2} \quad \text{——} ②$$

Subtract eq ② by ①

$$y + \delta y - y = (x+\delta x)^{5/2} - x^{5/2}$$

$$\delta y = x^{5/2}\left[1+\frac{\delta x}{x}\right]^{5/2} - x^{5/2}$$

$$\delta y = x^{5/2}\left[\left(1+\frac{\delta x}{x}\right)^{5/2} - 1\right]$$

$$\delta y = x^{3/2}\left[\cancel{1}+\frac{3}{2}\left(\frac{\delta x}{x}\right)+\frac{\frac{3}{2}\left(\frac{3}{2}-1\right)}{2!}\left(\frac{\delta x}{x}\right)^2+\cdots\cancel{-1}\right]$$

$$\delta y = x^{3/2}\left[\frac{3}{2}\left(\frac{\delta x}{x}\right)+\frac{3/4}{2}\left(\frac{\delta x}{x}\right)^2+\cdots\right]$$

$$\delta y = x^{3/2}\left(\frac{\delta x}{x}\right)\left[\frac{3}{2}+\frac{3}{8}\left(\frac{\delta x}{x}\right)+\cdots\right]$$

Divided by δx both sides

$$\frac{\delta y}{\delta x} = x^{3/2-1}\left[\frac{3}{2}+\frac{3}{8}\left(\frac{\delta x}{x}\right)+\cdots\right]$$

Appling $\lim\limits_{\delta x\to 0}$ both sides

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x} = \lim_{\delta x\to 0} x^{\frac{1}{2}}\left[\frac{3}{2}+\frac{3}{8}\left(\frac{\delta x}{x}\right)+\cdots\right]$$

$$\frac{dy}{dx} = x^{1/2}\left(\frac{3}{2}\right)$$

$$\boxed{\frac{dy}{dx} = \frac{3}{2}\sqrt{x}}$$

(xi)

x^m , $m \in N$

Solution:

Let $y = x^m$ ——— ①

$y+\delta y = (x+\delta x)^m$ ——— ②

Subtract eq ② by ①

$$y+\delta y-y = (x+\delta x)^m - x^m$$

$$\delta y = x^m\left(1+\frac{\delta x}{x}\right)^m - x^m$$

$$\delta y = x^m\left[\left(1+\frac{\delta x}{x}\right)^m - 1\right]$$

Binomial theorem

$$\delta y = x^m\left[\cancel{1}+m\left(\frac{\delta x}{x}\right)+\frac{m(m-1)}{2!}\left(\frac{\delta x}{x}\right)^2+\cdots\cancel{-1}\right]$$

$$\delta y = x^m\left[m\left(\frac{\delta x}{x}\right)+\frac{m(m-1)}{2!}\left(\frac{\delta x}{x}\right)^2+\cdots\right]$$

$$\delta y = x^m\left(\frac{\delta x}{x}\right)\left[m+\frac{(m(m-1)}{2!}\left(\frac{\delta x}{x}\right)+\cdots\right]$$

Divided by δx both sides

$$\frac{\delta y}{\delta x} = x^{m-1}\left[m+\frac{(m(m-1)}{2!}\,\frac{\delta x}{x}+\cdots\right]$$

Appling $\lim\limits_{\delta x\to 0}$ both sides

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x} = \lim_{\delta x\to 0} x^{m-1}\left[m+\frac{m(m-1)}{2!}\left(\frac{\delta x}{x}\right)\cdots\right]$$

Binomial theorem

$$\delta y = x^{5/2}\left[\cancel{1}+\frac{5}{2}\left(\frac{\delta x}{x}\right)+\frac{\frac{5}{2}\left(\frac{5}{2}-1\right)}{2!}\left(\frac{\delta x}{x}\right)^2+\cdots\cancel{-1}\right]$$

$$\delta y = x^{5/2}\left[\frac{5}{2}\left(\frac{\delta x}{x}\right)+\frac{\frac{5}{2}\left(\frac{5}{2}-1\right)}{2}\left(\frac{\delta x}{x}\right)^2+\cdots\right]$$

$$\delta y = x^{5/2}\left(\frac{\delta x}{x}\right)\left[\frac{5}{2}+\frac{\frac{5}{2}\left(\frac{5}{2}-1\right)}{2}\left(\frac{\delta x}{x}\right)+\cdots\right]$$

Divided by δx both sides

$$\frac{\delta y}{\delta x} = x^{5/2-1}\left[\frac{5}{2}+\frac{\frac{5}{2}\left(\frac{5}{2}-1\right)}{2}\left(\frac{\delta x}{x}\right)+\cdots\right]$$

Appling $\lim\limits_{\delta x\to 0}$ both sides.

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x} = \lim_{\delta x\to 0} x^{3/2}\left[\frac{5}{2}+\frac{\frac{5}{2}\left(\frac{5}{2}-1\right)}{2}\left(\frac{\delta x}{x}\right)+\cdots\right]$$

$$\frac{dy}{dx} = x^{3/2}\left(\frac{5}{2}\right)$$

$$\boxed{\frac{dy}{dx} = \frac{5x^{3/2}}{2}}$$

(xii)

$\frac{1}{x^m}$, $m \in N$

Solution:

Let $y = \frac{1}{x^m}$

$y = x^{-m}$ ——— ①

$y+\delta y = (x+\delta x)^{-m}$ ——— ②

Subtract eq ② by ①

$$y+\delta y-y = (x+\delta x)^{-m} - x^m$$

$$\delta y = x^{-m}\left(1+\frac{\delta x}{x}\right)^{-m} - x^{-m}$$

$$\delta y = x^{-m}\left[\left(1+\frac{\delta x}{x}\right)^{-m} - 1\right]$$

Binomial theorem

$$\delta y = x^{-m}\left[\cancel{1}-m\left(\frac{\delta x}{x}\right)+\frac{-m(-m-1)}{2!}\left(\frac{\delta x}{x}\right)^2+\cdots\cancel{-1}\right]$$

$$\delta y = x^{-m}\left[-m\left\{\frac{\delta x}{x}\right\}+\frac{-m(-m-1)}{2!}\left(\frac{\delta x}{x}\right)^2+\cdots\right]$$

$$\delta y = x^{-m}\left(\frac{\delta x}{x}\right)\left[-m+\frac{-m(-m-1)}{2!}\left(\frac{\delta x}{x}\right)+\cdots\right]$$

Divided by δx both sides

$$\frac{\delta y}{\delta x} = x^{-m-1}\left[-m+\frac{-m(-m-1)}{2!}\left(\frac{\delta x}{x}\right)+\cdots\right]$$

$$\frac{dy}{dx} = x^{m-1}(m)$$

$$\boxed{\frac{dy}{dx} = mx^{m-1}}$$

(xiii)

x^{40}

Solution:

Let $y = x^{40}$ —— ①

$y + \delta y = (x+\delta x)^{40}$ —— ②

Subtract eq ② by ①

$$y + \delta y - y = (x+\delta x)^{40} - x^{40}$$

$$\delta y = x^{40}\left(1+\frac{\delta x}{x}\right)^{40} - x^{40}$$

$$\delta y = x^{40}\left[\left(1+\frac{\delta x}{x}\right)^{40} - 1\right]$$

Using Binomial theorem

$$\delta y = x^{40}\left[\cancel{1} + 40\left(\frac{\delta x}{x}\right) + \frac{40(40-1)}{2!}\left(\frac{\delta x}{x}\right)^2 + \cdots - \cancel{1}\right]$$

$$\delta y = x^{40}\left[40\left(\frac{\delta x}{x}\right) + \frac{40(39)}{2!}\left(\frac{\delta x}{x}\right)^2 + \cdots\right]$$

$$\delta y = x^{40}\left(\frac{\delta x}{x}\right)\left[40 + \frac{40(39)}{2!}\left(\frac{\delta x}{x}\right) + \cdots\right]$$

Divided by δx both sides

$$\frac{\delta y}{\delta x} = x^{40-1}\left[40 + \frac{40(39)}{2!}\left(\frac{\delta x}{x}\right) + \cdots\right]$$

Appling $\lim_{\delta x \to 0}$ both sides

$$\lim_{\delta x \to 0}\frac{\delta y}{\delta x} = \lim_{\delta x \to 0} x^{39}\left[40 + \frac{40(39)}{2!}\left(\frac{\delta x}{x}\right) + \cdots\right]$$

$$\frac{dy}{dx} = x^{39}.(40)$$

$$\boxed{\frac{dy}{dx} = 40x^{39}}$$

Appling $\lim_{\delta x \to 0}$ both sides.

$$\lim_{\delta x \to 0}\frac{\delta y}{\delta x} = \lim_{\delta x \to 0} x^{-m-1}\left[-m + \frac{-m(-m-1)}{2!}\left(\frac{\delta x}{x}\right) + \cdots\right]$$

$$\frac{dy}{dx} = x^{-m-1}(-m)$$

$$\frac{dy}{dx} = -mx^{-m-1} \Rightarrow -mx^{-(m+1)}$$

$$\boxed{\frac{dy}{dx} = \frac{-m}{x^{m+1}}}$$

(xiv)

x^{-100}

Solution:

Let $y = x^{-100}$

$$y + \delta y = (x+\delta x)^{-100}$$

$$y + \delta y - y = (x+\delta x)^{-100} - x^{-100}$$

$$\delta y = x^{-100}\left(1+\frac{\delta x}{x}\right)^{-100} - x^{-100}$$

$$\delta y = x^{-100}\left[\left(1+\frac{\delta x}{x}\right)^{-100} - 1\right]$$

$$\delta y = x^{-100}\left[\cancel{1} - 100\left(\frac{\delta x}{x}\right) + \frac{-100(-100-1)}{2!}\left(\frac{\delta x}{x}\right)^2 + \cdots - \cancel{1}\right]$$

$$\delta y = x^{-100}\left[-100\left(\frac{\delta x}{x}\right) + \frac{-100(100-1)}{2!}\left(\frac{\delta x}{x}\right)^2 + \cdots\right]$$

$$\delta y = x^{-100}\frac{\delta x}{x}\left[-100 + \frac{-100(-100-1)}{2!}\left(\frac{\delta x}{x}\right) + \cdots\right]$$

$$\frac{\delta y}{\delta x} = x^{-100-1}\left[-100 + \frac{-100(-100-1)}{2!}\left(\frac{\delta x}{x}\right) + \cdots\right]$$

$$\lim_{\delta x \to 0}\frac{\delta y}{\delta x} = \lim_{\delta x \to 0} x^{-101}\left[-100 + \frac{-100(-100-1)}{2!}\left(\frac{\delta x}{x}\right) + \cdots\right]$$

$$\frac{dy}{dx} = x^{-101}(-100)$$

$$\frac{dy}{dx} = -100x^{-101}$$

$$\boxed{\frac{dy}{dx} = \frac{-100}{x^{101}}}$$

Question #2

Find $\frac{dy}{dx}$ from first principles if:

(i)

$\sqrt{x+2}$

Solution:

Let $y = \sqrt{x+2}$ — ①

$y+\delta y = \sqrt{x+\delta x+2}$ — ②

Subtract eq ② by ①

$y+\delta y-y = \sqrt{x+\delta x+2} - \sqrt{x+2}$

$\delta y = \sqrt{x+\delta x+2} - \sqrt{x+2}$

Divided by δx both sides

$$\frac{\delta y}{\delta x} = \frac{\sqrt{x+\delta x+2} - \sqrt{x+2}}{\delta x}$$

Appling $\lim\limits_{\delta x \to 0}$ both sides

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x} = \lim_{\delta x\to 0}\frac{\sqrt{x+\delta x+2}-\sqrt{x+2}}{\delta x}\times\frac{\sqrt{x+\delta x+2}+\sqrt{x+2}}{\sqrt{x+\delta x+2}+\sqrt{x+2}}$$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x} = \lim_{\delta x\to 0}\frac{(\sqrt{x+\delta x+2})^2-(\sqrt{x+2})^2}{\delta x\sqrt{x+\delta x+2}+\sqrt{x+2}}$$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x} = \lim_{\delta x\to 0}\frac{\cancel{x}+\delta x+\cancel{2}-\cancel{x}-\cancel{2}}{\delta x(\sqrt{x+\delta x+2})+\sqrt{x+2}}$$

$$\frac{dy}{dx} = \frac{0(1)}{0\sqrt{x+0+2}+\sqrt{x+2}}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{x+2}+\sqrt{x+2}}$$

$$\boxed{\frac{dy}{dx} = \frac{1}{2\sqrt{x+2}}}$$

(ii)

$\frac{1}{\sqrt{x+a}}$

Solution:

Let $y = \frac{1}{\sqrt{x+a}}$

$y = (x+a)^{-1/2}$ — ①

$y+\delta y = (x+\delta x+a)^{\frac{-1}{2}}$ — ②

Subtract eq ② by ①

$y+\delta y-y = (x+\delta x+a)^{\frac{-1}{2}} - (x+a)^{\frac{-1}{2}}$

$\delta y = (x+a)^{\frac{-1}{2}}\left(1+\frac{\delta x}{x}\right)^{\frac{-1}{2}} - (x+a)^{\frac{-1}{2}}$

$\delta y = (x+a)^{\frac{-1}{2}}\left[\left(1+\frac{\delta x}{x}\right)^{\frac{-1}{2}} - 1\right]$

Using Binomial theorem

$$\delta y = (x+a)^{\frac{-1}{2}}\left[\cancel{1}-\frac{1}{2}\left(\frac{\delta x}{x}\right)+\frac{\frac{-1}{2}\left(\frac{-1}{2}-1\right)}{2!}\left(\frac{\delta x}{x}\right)^2+\cdots\cancel{-1}\right]$$

$$\delta y = (x+a)^{\frac{-1}{2}}\left[\frac{-1}{2}\left(\frac{\delta x}{x}\right)+\frac{\frac{3}{4}}{2!}\left(\frac{\delta x}{x}\right)^2+\cdots\right]$$

$$\delta y = (x+a)^{\frac{-1}{2}}\left[\frac{-1}{2}\left(\frac{\delta x}{x}\right)+\frac{3}{8}\left(\frac{\delta x}{x}\right)^2+\cdots\right]$$

$$\delta y = (x+a)^{\frac{-1}{2}}\left(\frac{\delta x}{x}\right)\left[\frac{-1}{2}+\frac{3}{8}\left(\frac{\delta x}{x}\right)+\cdots\right]$$

Divided by δx both sides

$$\frac{\delta y}{\delta x} = (x+a)^{\frac{-1}{2}-1}\left[\frac{-1}{2}+\frac{3}{8}\left(\frac{\delta x}{x}\right)+\cdots\right]$$

Appling $\lim\limits_{\delta x\to 0}$ both sides

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x} = \lim_{\delta x\to 0}(x+a)^{\frac{-3}{2}}\left[\frac{-1}{2}+\frac{3}{8}\left(\frac{\delta x}{x}\right)+\cdots\right]$$

$$\frac{dy}{dx} = (x+a)^{\frac{-3}{2}}\left(\frac{-1}{2}\right)$$

$$\frac{dy}{dx} = \frac{-1}{2}(x+a)^{-3/2}$$

$$\boxed{\frac{dy}{dx} = \frac{-1}{2(x+a)^{3/2}}}$$

* Theory

Example #1

Find from definition the defferential Coeffient of $(ax+b)^n$ w.r.t "x" when n is a positive integers.

Solution:

let

$$y = (ax+b)^n \text{ ——— } ①$$

$$y+\delta y = [a(x+\delta x)+b]^n$$

$$y+\delta y = [ax+a\delta x+b]^n \text{ ——— } ②$$

Subtract eq ② by ①

$$\cancel{y}+\delta y-\cancel{y} = [ax+a\delta x+b]^n - (ax+b)^n$$

$$\delta y = [ax+b]^n \left[1+\frac{a\delta x}{ax+b}\right]^n - (ax+b)^n$$

$$\delta y = (ax+b)^n \left[\left(1+\frac{a\delta x}{ax+b}\right)^n - 1\right]$$

Using Binomial theorem

$$(1+x)^n = 1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+\cdots\cdots$$

$$\delta y = (ax+b)^n\left[\cancel{1}+n\left(\frac{a\delta x}{ax+b}\right)+\frac{n(n-1)}{2!}\left(\frac{a\delta x}{ax+b}\right)^2+\cdots\cdots-\cancel{1}\right]$$

$$\delta y = (ax+b)^n\left[n\left(\frac{a\delta x}{ax+b}\right)+\frac{n(n-1)}{2!}\left(\frac{a\delta x}{ax+b}\right)^2+\cdots\cdots\right]$$

$$\delta y = (ax+b)^n\left(\frac{a\delta x}{ax+b}\right)\left[n+\frac{n(n-1)}{2!}\left(\frac{a\delta x}{ax+b}\right)+\cdots\right]$$

Divided by δx on both sides

$$\frac{\delta y}{\delta x} = \frac{(ax+b)^n\left(\frac{a\cancel{\delta x}}{ax+b}\right)\left[n+\frac{n(n-1)}{2!}\left(\frac{a\delta x}{ax+b}\right)+\cdots\right]}{\cancel{\delta x}}$$

$$\frac{\delta y}{\delta x} = (ax+b)^n\frac{a}{(ax+b)}\left[n+\frac{n(n-1)}{2!}\left(\frac{a\delta x}{ax+b}\right)+\cdots\right]$$

Appling limit $\delta x\to 0$ both sides.

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x} = \lim_{\delta x\to 0}(ax+b)^{n-1}\cdot a\left[n+\frac{n(n-1)}{2!}\left(\frac{a\delta x}{ax+b}\right)+\cdots\right]$$

$$\frac{dy}{dx} = a(ax+b)^{n-1}\cdot n$$

$$\boxed{\frac{dy}{dx} = na(ax+b)^{n-1}}$$

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Example #2

Find form first principles, the derivative of $\frac{1}{(ax+b)^n}$ w.r.t 'x'.

Solution:

Let

$$y = \frac{1}{(ax+b)^n}$$

$$y = (ax+b)^{-n} \text{ —— ①}$$

$$y+\delta y = (a(x+\delta x)+b)^{-n}]$$

$$y+\delta y = (ax+\delta ax+b)^{-n} \text{ —— ②}$$

Subtract eq ② by ①

$$\cancel{y}+\delta y-\cancel{y} = (ax+\delta ax+b)^{-n} - (ax+b)^{-n}$$

$$\delta y = (ax+b)^{-n}\left[1+\frac{\delta xa}{(ax+b)}\right]^{-n} - (ax+b)^{-n}$$

$$\delta y = (ax+b)^{-n}\left[\left(1+\frac{\delta xa}{(ax+b)}\right)^{-n} - 1\right]$$

Using Binomial theorem

$$\delta y = (ax+b)^{-n}\left[\cancel{1} - n\left(\frac{\delta xa}{(ax+b)}\right) + \frac{-n(-n-1)}{2!}\left(\frac{\delta xa}{(ax+b)}\right)^2 + \cdots - \cancel{1}\right]$$

$$\delta y = (ax+b)^{-n}\left[-n\left(\frac{\delta xa}{(ax+b)}\right) + \frac{-n(-n-1)}{2!}\left(\frac{\delta xa}{(ax+b)}\right)^2 + \cdots\right]$$

$$\delta y = (ax+b)^{-n}\left(\frac{\delta xa}{(ax+b)}\right)\left[-n + \frac{-n(-n-1)}{2!}\left(\frac{\delta xa}{ax+b}\right) + \cdots\right]$$

Divided by δx both sides

$$\frac{\delta y}{\delta x} = (ax+b)^{-n-1}\cdot a\left[-n + \frac{-n(-n-1)}{2!}\left(\frac{\delta xa}{ax+b}\right) + \cdots\right]$$

Appling $\lim_{\delta x\to 0}$ both sides

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x} = \lim_{\delta x\to 0}(ax+b)^{-n-1}\cdot a\left[-n + \frac{-n(-n-1)}{2!}\left(\frac{\delta xa}{ax+b}\right) + \cdots\right]$$

$$\frac{dy}{dx} = (ax+b)^{-n-1}(-na)$$

$$\frac{dy}{dx} = -na(ax+b)^{-(n+1)}$$

$$\boxed{\frac{dy}{dx} = \frac{-na}{(ax+b)^{n+1}}}$$

Exercise # 2.2

Question #1

Find form first principle, the derivative of the following expression w.r.t to their respective independant variable.

(i)

$$(ax+b)^3$$

Let $y = (ax+b)^3$ —— ①

$y+\delta y = (a(x+\delta x)+b)^3$

$y+\delta y = (ax+a\delta x+b)^3$ —— ②

Subtract eq ② by ①

$\cancel{y}+\delta y-\cancel{y} = (ax+a\delta x+b)^3 - (ax+b)^3$

$\delta y = (ax+b)^3\left(1+\frac{a\delta x}{ax+b}\right)^3 - (ax+b)^3$

$\delta y = (ax+b)^3\left[\left(1+\frac{a\delta x}{ax+b}\right)^3 - 1\right]$

Using Binomial theorem

$\delta y = (ax+b)^3\left[\cancel{1}+3\left(\frac{a\delta x}{ax+b}\right)+\frac{3-1}{2!}3\left(\frac{a\delta x}{ax+b}\right)^2+\cdots\cdots\cancel{-1}\right]$

$\delta y = (ax+b)^3\left[3\left(\frac{a\delta x}{ax+b}\right)+\frac{\cancel{2}}{\cancel{2!}}3\left(\frac{a\delta x}{ax+b}\right)^2+\cdots\cdots\right]$

$\delta y = (ax+b)^3\left(\frac{a\delta x}{ax+b}\right)\left[3+3\left(\frac{a\delta x}{ax+b}\right)+\cdots\cdots\right]$

Divided by δx both sides

$\frac{\delta y}{\delta x} = (ax+b)^3\left(\frac{a}{a\cancel{x}+b}\right)\left[3+3\left(\frac{a\delta x}{ax+b}\right)+\cdots\cdots\right]$

Appling $\lim\limits_{\delta x\to 0}$ both sides

$\lim\limits_{\delta x\to 0}\frac{\delta y}{\delta x} = \lim\limits_{\delta x\to 0}(ax+b)^{3-1}\cdot a\left[3+3\left(\frac{\delta x}{ax+b}\right)+\cdots\cdots\right]$

$\frac{dy}{dx} = a\cdot(ax+b)^2\cdot 3$

$$\boxed{\frac{dy}{dx} = 3a\,(ax+b)^2}$$

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(ii)

$$(2x+3)^5$$

Solution:

Let $y = (2x+3)^5$ —— ①

$y + \delta y = (2(x+\delta x+3)^5$

$y + \delta y = (2x+2\delta x+3)^5$ —— ②

Subtract eq ② by ①

$y+\delta y - y = (2x+2\delta x+3)^5 - (gx+3)^5$

$\delta y = (2x+3)^5 \left(1+\frac{2\delta x}{2x+3}\right)^5 - (2x+3)^5$

$\delta y = (2x+3)^5 \left[\left(1+\frac{2\delta x}{2x+3}\right)^5 - 1\right]$

Using Binomial theorem

$\delta y = (2x+3)^5 \left[\cancel{1} + 5\left(\frac{2\delta x}{2x+3}\right) - \frac{5(5-1)}{2!}\left(\frac{2\delta x}{2x+3}\right)^2 + \cdots - \cancel{1}\right]$

$\delta y = (2x+3)^5 \left[5\left(\frac{2\delta x}{2x+3}\right) - \frac{\overset{10}{\cancel{20}}}{\cancel{2}}\left(\frac{2\delta x}{2x+3}\right)^2 + \cdots\right]$

$\delta y = (2x+3)^5 \left(\frac{2\delta x}{2x+3}\right)\left[5 - 10\left(\frac{2\delta x}{2x+3}\right) + \cdots\right]$

Divided by δx both sides

$\frac{\delta y}{\delta x} = (2x+3)^5 \cdot \frac{2}{2x+3}\left[5 - 10\left(\frac{2\delta x}{2x+3}\right) + \cdots\right]$

Appling $\lim\limits_{\delta x \to 0}$ both sides

$\lim\limits_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim\limits_{\delta x \to 0} (2x+3)^5 \cdot \frac{2}{2x+3} \cdot \left(5 - 10\left(\frac{2\delta x}{2x+3}\right) + \cdots\right)$

$\frac{dy}{dx} = (2x+3)^{5-1} \cdot 2(5)$

$\frac{dy}{dx} = (2x+3)^4 \cdot 10$

$$\boxed{\frac{dy}{dx} = 10(2x+3)^4}$$

(iii)

$$(3t+2)^{-2}$$

Solution:

Let $y = (3t+2)^{-2}$ —— ①

$y + \delta y = (3(t+\delta t)+2)^{-2}$

$y + \delta y = (3t+3\delta t+2)^{-2}$ —— ②

Subtract eq ② by ①

$y+\delta y - y = (3t+3\delta t+2)^{-2} - (3t+2)^{-2}$

$\delta y = (3t+2)^{-2}\left(1+\frac{3\delta t}{3t+2}\right)^{-2} - (3t+2)^{-2}$

$\delta y = (3t+2)^{-2}\left[\left(1+\frac{3\delta t}{3t+2}\right)^{-2} - 1\right]$

Using Binomial theorem

$\delta y = (3t+2)^{-2}\left[\cancel{1} - 2\left(\frac{3\delta t}{3t+2}\right) + \frac{-2(-2-1)}{2!}\left(\frac{3\delta t}{3t+2}\right)^2 + \cdots - \cancel{1}\right]$

$\delta y = (3t+2)^{-2}\left[-2\left(\frac{3\delta t}{3t+2}\right) + \frac{\overset{3}{\cancel{6}}}{\cancel{2}}\left(\frac{3\delta t}{3t+2}\right)^2 + \cdots\right]$

$\delta y = (3t+2)^{-2}\left(\frac{3\delta t}{3t+2}\right)\left[-2 + 3\left(\frac{3\delta t}{3t+2}\right) + \cdots\right]$

Divided by δx both sides

$\frac{\delta y}{\delta x} = (3t+2)^{-2}\left(\frac{3}{3t+2}\right)\left[-2 + 3\left(\frac{3\delta t}{3t+2}\right) + \cdots\right]$

Appling $\lim\limits_{\delta x \to 0}$ both sides.

$\lim\limits_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim\limits_{\delta x \to 0} (3t+2)^{-2-1} \cdot 3\left[-2 + 3\left(\frac{3\delta t}{3t+2}\right) + \cdots\right]$

$\frac{dy}{dx} = (3t+2)^{-3} \cdot 3(-2)$

$\frac{dy}{dx} = -6(3t+2)^{-3}$

$$\boxed{\frac{dy}{dx} = \frac{-6}{(3t+2)^3}}$$

(iv)

$$\frac{1}{(ax+b)^5}$$

Solution:

Let $y = \frac{1}{(ax+b)^5}$

$y = (ax+b)^{-5}$ ——— ①

$y+\delta y = (a(x+\delta x)+b)^{-5}$

$y+\delta y = (ax+a\delta x+b)^{-5}$ ——— ②

Subtract eq ② by ①

$\cancel{y}+\delta y-\cancel{y} = (ax+a\delta x+b)^{-5} - (ax+b)^{-5}$

$\delta y = (ax+b)^{-5}\left[1+\frac{a\delta x}{ax+b}\right)^{-5} - (ax+b)^{-5}$

$\delta y = (ax+b)^{-5}\left[\left(1+\frac{a\delta x}{ax+b}\right)^{-5} - 1\right]$

Using Binomial theorem

$\delta y = (ax+b)^{-5}\left[\cancel{1} - 5\left(\frac{a\delta x}{ax+b}\right) + \frac{-5(-5-1)}{2!}\left(\frac{a\delta x}{ax+b}\right)^2 + \cdots\cdots - \cancel{1}\right]$

$\delta y = (ax+b)^{-5}\left[-5\left(\frac{a\delta x}{ax+b}\right) + \frac{25}{2}\left(\frac{a\delta x}{ax+b}\right)^2 + \cdots\right]$

$\delta y = (ax+b)^{-5}\left[\frac{a\delta x}{ax+b}\right)\left[-5+\frac{25}{2}\left(\frac{a\delta x}{ax+b}\right)+\cdots\right]$

Divided by δx both sides

$\frac{\delta y}{\delta x} = (ax+b)^{-5}\left(\frac{a}{ax+b}\right)\left[-5+\frac{25}{2}\left(\frac{a\delta x}{ax+b}\right)+\cdots\right]$

Appling $\lim_{\delta x\to 0}$ both sides

$\lim_{\delta x\to 0}\frac{\delta y}{\delta x} = \lim_{\delta x\to 0}(ax+b)^{-5-1}\cdot a\left(-5+\frac{25}{2}\left(\frac{a\delta x}{ax+b}\right)+\cdots\right)$

$\frac{dy}{dx} = (ax+b)^{-6}\cdot a(-5)$

$\frac{dy}{dx} = -5a(ax+b)^{-6}$

$$\boxed{\frac{dy}{dx} = \frac{-5a}{(ax+b)^6}}$$

(V)

$$\frac{1}{(az-b)^7}$$

Solution:

Let $y = \frac{1}{(az-b)^7}$

$y = (az-b)^{-7}$ —— ①

$y + \delta y = (a(z+\delta z)-b)^{-7}$

$y + \delta y = (az + \delta az - b)^{-7}$ —— ②

Subtract eq ② by ①

$y + \delta y - y = (az + \delta az - b)^{-7} - (az-b)^{-7}$

$\delta y = (az-b)^{-7}\left(1 + \frac{\delta az}{az-b}\right)^{-7} - (az-b)^{-7}$

$\delta y = (az-b)^{-7}\left[\left(1+\frac{\delta az}{az-b}\right)^{-7} - 1\right]$

Using Binomial theorem

$\delta y = (az-b)^{-7}\left[\cancel{1} - 7\left(\frac{\delta az}{az-b}\right) + \frac{-7(-7-1)}{2!}\left(\frac{\delta az}{az-b}\right)^2 + ---- \cancel{-1}\right]$

$\delta y = (az-b)^{-7}\left[-7\left(\frac{\delta az}{az-b}\right) + \frac{49}{2}\left(\frac{\delta az}{az-b}\right)^2 + ----\right]$

$\delta y = (az-b)^{-7}\left(\frac{\delta az}{az-b}\right)\left[-7 + \frac{49}{2}\left(\frac{\delta az}{az-b}\right) + ...\right]$

Divided by δx both sides

$\frac{\delta y}{\delta z} = (az-b)^{-7}\left(\frac{a}{az-b}\right)\left[-7 + \frac{49}{2}\left(\frac{\delta az}{az-b}\right) + ...\right]$

Appling $\lim_{\delta z \to 0}$ both sides

$\lim_{\delta z \to 0}\frac{\delta y}{\delta z} = \lim_{\delta z \to 0}(az-b)^{-7}\left(\frac{a}{az-b}\right)\left[-7 + \frac{49}{2}\left(\frac{\delta az}{az-b}\right) + ...\right]$

$\frac{dy}{dz} = (az-b)^{-7-1)}\ a\ (-7)$

$\frac{dy}{dz} = -7a(az-b)^{-8}$

$$\boxed{\frac{dy}{dz} = \frac{-7a}{(az-b)^8}}$$

* Theory

Example #1

Calculate $\frac{d}{dx}\left(3x^{\frac{4}{3}}\right)$

Solution:

$$= \frac{d}{dx}\ 3x^{\frac{4}{3}}$$

$$= 3\ \frac{d}{dx}\ x^{\frac{4}{3}}$$

$$= \cancel{3} \times \frac{4}{\cancel{3}}\ x^{\frac{4}{3}-1}$$

$$= \boxed{4x^{\frac{1}{3}}}$$

Example #2

Find the derivative of $y = \frac{3}{4}x^4 + \frac{2}{3}x^3 + \frac{1}{2}x^2 + 2x + 5$ w.r.t 'x'

Solution:

$$y = \frac{3}{4}x^4 + \frac{2}{3}x^3 + \frac{1}{2}x^2 + 2x + 5$$

Differentiate w.r.t "x"

$$\frac{dy}{dx} = \frac{d}{dx}\left[\frac{3}{4}x^4 + \frac{2}{3}x^3 + \frac{1}{2}x^2 + 2x + 5\right]$$

$$\frac{dy}{dx} = \frac{3}{\cancel{4}}(\cancel{4}x^{4-1}) + \frac{2}{\cancel{3}}(\cancel{3}x^{3-1}) + \frac{1}{\cancel{2}}(\cancel{2}x^{2-1}) + 2(1) + 0$$

$$\boxed{\frac{dy}{dx} = 3x^3 + 2x^2 + x + 2}$$

Example #3

Find the derivative of $y = (x^2+5)(x^3+7)$ w.r.t 'x'

Solution:

$$y = (x^2+5)(x^3+7)$$

$$y = x^5 + 7x^2 + 5x^3 + 35$$

$$y = x^5 + 5x^3 + 7x^2 + 35$$

$$\frac{dy}{dx} = \frac{d}{dx}\left[x^5 + 5x^3 + 7x^2 + 35\right]$$

$$\frac{dy}{dx} = 5x^{5-1} + 5(3x^{3-1}) + 7(2x^{2-1}) + 0$$

$$\boxed{\frac{dy}{dx} = 5x^4 + 15x^4 + 14x}$$

Example #4

Find the derivative of $y = (2\sqrt{x}+2)(x-\sqrt{x})$ w.r.t 'x'

Solution:

$$y = (2\sqrt{x}+2)(x-\sqrt{x})$$

$$y = 2(\sqrt{x}+1)(\sqrt{x}\cdot\sqrt{x}-\sqrt{x})$$

$$y = 2(\sqrt{x}+1)\sqrt{x}(\sqrt{x}-1)$$

$$y = 2\sqrt{x}(\sqrt{x}+1)(\sqrt{x}-1)$$

$$y = 2\sqrt{x}(x-1) \qquad \because (a+b)(a-b) = a^2-b^2$$

$$y = 2\cdot x^{1/2}(x-1)$$

$$y = 2(x^{3/2} - x^{1/2})$$

Differentiate w.r.t "x"

$$\frac{dy}{dx} = \frac{d}{dx}\left[2(x^{3/2} - x^{1/2})\right]$$

$$\frac{dy}{dx} = \cancel{2}\left[\frac{3}{\cancel{2}}x^{\frac{3}{2}-1} - \frac{1}{\cancel{2}}x^{\frac{1}{2}-1}\right]$$

$$\frac{dy}{dx} = 3x^{\frac{1}{2}} - x^{\frac{-1}{2}}$$

$$\frac{dy}{dx} = 3\sqrt{x} - \frac{1}{\sqrt{x}}$$

$$\boxed{\frac{dy}{dx} = \frac{3x-1}{\sqrt{x}}}$$

Example #5

Find the derivative of
$y = (2\sqrt{x}+2)(x-\sqrt{x})$ w.r.t 'x'

Solution:

$y = (2\sqrt{x}+2)(x-\sqrt{x})$

$y = 2(\sqrt{x}+1)(x-\sqrt{x})$

Differentiate w.r.t 'x'

$\frac{dy}{dx} = 2\frac{d}{dx}\left[(\sqrt{x}+1)(x-\sqrt{x})\right]$

$\frac{dy}{dx} = 2\left[\frac{d}{dx}(\sqrt{x}+1)\right](x-\sqrt{x}) + (\sqrt{x}+1)\frac{d}{dx}(x-\sqrt{x})\Big]$

$\frac{dy}{dx} = 2\left[\frac{1}{2}x^{\frac{1}{2}-1}+0\right](x-\sqrt{x}) + (\sqrt{x}+1)\left[1-\frac{1}{2}x^{\frac{1}{2}-1}\right]$

$\frac{dy}{dx} = 2\left[\frac{1}{2\sqrt{x}}(x-\sqrt{x}) + (\sqrt{x}+1)\left[1-\frac{1}{2\sqrt{x}}\right]\right]$

$\frac{dy}{dx} = 2\left[\frac{x-\sqrt{x}}{2\sqrt{x}} + (\sqrt{x}+1)\left[\frac{2\sqrt{x}-1}{2\sqrt{x}}\right]\right]$

$\frac{dy}{dx} = 2\left[\frac{x-\sqrt{x}+2\sqrt{x}+2x-\sqrt{x}-1}{2\sqrt{x}}\right]$

$\frac{dy}{dx} = \frac{\cancel{2}}{\cancel{2}\sqrt{x}}\left[3x-2\cancel{\sqrt{x}}+2\cancel{\sqrt{x}}-1\right]$

$$\boxed{\frac{dy}{dx} = \frac{3x-1}{\sqrt{x}}}$$

Example #6

Find $\frac{dy}{dx}$ if $y = \frac{(\sqrt{x}+1)(x^{3/2}-1)}{x^{1/2}-1}, x \neq 1$

Solution:

$y = \frac{(\sqrt{x}+1)(x^{3/2}-1)}{x^{1/2}-1}$

$y = \frac{(\sqrt{x}+1)\left[(\sqrt{x})^3-(1)^3\right]}{\sqrt{x}-1}$

$\because a^3-b^3 = (a-b)(a^2+ab+b^2)$

$y = \frac{(\sqrt{x}+1)\cancel{(\sqrt{x}-1)}(x+1+\sqrt{x})}{\cancel{(\sqrt{x}-1)}}$

$y = (\sqrt{x}+1)(x+1+\sqrt{x})$

$y = (\sqrt{x}+1)(\sqrt{x}+1+x)$

$y = (\sqrt{x}+1)^2 + (\sqrt{x}+1)x$

$y = x+1+2\sqrt{x}+x\sqrt{x}+x$

$y = x^{3/2}+2\sqrt{x}+2x+1$

$\frac{dy}{dx} = \frac{d}{dx}\left[x^{3/2}+2x^{1/2}+2x+1\right]$

$\frac{dy}{dx} = \left[\frac{3}{2}x^{\frac{3}{2}-1}+\cancel{2}\cdot\frac{1}{\cancel{2}}x^{\frac{1}{2}-1}+2(1)+0\right]$

$\frac{dy}{dx} = \frac{3}{2}x^{\frac{1}{2}}+x^{-\frac{1}{2}}+2$

$$\boxed{\frac{dy}{dx} = \frac{3}{2}\sqrt{x}+\sqrt{x}+2}$$

Example #7

Differentiate $\frac{(\sqrt{x}+1)(x^{3/2}-1)}{x^{3/2}-x^{1/2}}$ w.r.t 'x'

Solution:

Let $y = \frac{(\sqrt{x}+1)(x^{3/2}-1)}{x^{3/2}-x^{1/2}}$

$y = \frac{(\sqrt{x}+1)\left[(\sqrt{x})^3-(1)^3\right]}{x\sqrt{x}-\sqrt{x}}$

$\because a^3-b^3 = (a-b)(a^2+ab+b^2)$

$y = \frac{(\sqrt{x}+1)(\sqrt{x}-1)(x+\sqrt{x}+1)}{\sqrt{x}(x-1)}$

$\because (a+b)(a-b) = a^2-b^2$

$y = \frac{\cancel{(x-1)}(x+\sqrt{x}+1)}{\sqrt{x}\cancel{(x-1)}}$

$y = \frac{x+\sqrt{x}+1}{\sqrt{x}}$

$\frac{dy}{dx} = \frac{d}{dx}\left[\frac{x+\sqrt{x}+1}{\sqrt{x}}\right]$

$\frac{dy}{dx} = \frac{\sqrt{x}\frac{d}{dx}(x+\sqrt{x}+1)-(x+\sqrt{x}+1)\frac{d}{dx}\sqrt{x}}{(\sqrt{x})^2}$

$\frac{dy}{dx} = \frac{\sqrt{x}\left[1+\frac{1}{2}x^{-\frac{1}{2}}+0\right]-(x+\sqrt{x}+1)\cdot\frac{1}{2}x^{-\frac{1}{2}}}{x}$

$\frac{dy}{dx} = \frac{\sqrt{x}\left[1+\frac{1}{2\sqrt{x}}\right]-(x+\sqrt{x}+1)\frac{1}{2\sqrt{x}}}{x}$

$\frac{dy}{dx} = \frac{\sqrt{x}\left(\frac{2\sqrt{x}+1}{2\sqrt{x}}\right)-\left[\frac{x+\sqrt{x}+1}{2\sqrt{x}}\right]}{x}$

$\frac{dy}{dx} = \frac{2x+\cancel{\sqrt{x}}-x-\cancel{\sqrt{x}}-1}{x\cdot 2\sqrt{x}}$

$$\boxed{\frac{dy}{dx} = \frac{x-1}{2x^{3/2}}}$$

Example #8

Differentiate $\frac{2x^3-3x^2+5}{x^2+1}$ w.r.t 'x'

Solution:

Let $y = \frac{2x^3-3x^2+5}{x^2+1}$

Differentiate w.r.t 'x'

$$\frac{dy}{dx} = \frac{(x^2+1)\frac{d}{dx}(2x^3-3x^2+5)-(2x^3-3x^2+5)\frac{d}{dx}(x^2+1)}{(x^2+1)^2}$$

$$\frac{dy}{dx} = \frac{(x^2+1)\ (6x^2-6x+0)-(2x^3-3x^2+5)(2x+0)}{(x^2+1)^2}$$

$$\frac{dy}{dx} = \frac{(x^2+1)(6x^2-6x)-(2x^3-3x^2+5)(2x)}{(x^2+1)^2}$$

$$\frac{dy}{dx} = \frac{6x^4+6x^2-6x^3-6x-(4x^4-6x^3+10x)}{(x^2+1)^2}$$

$$\frac{dy}{dx} = \frac{6x^4+6x^2\cancel{-6x^3}-6x-4x^4\cancel{+6x^3}-10x}{(x^2+1)^2}$$

$$\boxed{\frac{dy}{dx} = \frac{2x^4+6x^2-16x}{(x^2+1)^2}}$$

Exercise # 2.3

Differentiate w.r.t 'x'

Question #1

$x^4+2x^3+x^2$

Solution:

Let $y = x^4+2x^3+x^2$

Differentiate w.r.t 'x'

$$\frac{dy}{dx} = \frac{d}{dx}(x^4+2x^3+x^2)$$

$$\frac{dy}{dx} = 4x^{4-1}+2(3x^{3-1})+2x^{2-1}$$

$$\boxed{\frac{dy}{dx} = 4x^3+6x^2+2x}$$

Question #2

$x^{-3}+2x^{-\frac{3}{2}}+3$

Solution:

Let $y = x^{-3}+2x^{-\frac{3}{2}}+3$

Differentiate w.r.t 'x'

$$\frac{dy}{dx} = \frac{d}{dx}(x^{-3}+2x^{-\frac{3}{2}}+3)$$

$$\frac{dy}{dx} = -3x^{-3-1}+2\times-\frac{3}{2}x^{(-\frac{3}{2}-1)}+0$$

$$\frac{dy}{dx} = -3x^{-4}-3x^{-5/2}$$

$$\boxed{\frac{dy}{dx} = -3\left[\frac{1}{x^4}+\frac{1}{x^{5/2}}\right]}$$

Question #3

$$\frac{a+x}{a-x}$$

Solution:

Let $y = \frac{a+x}{a-x}$

Differentiate w.r.t "x"

$$\frac{dy}{dx} = \frac{(a-x)\frac{d}{dx}(a+x) - (a+x)\frac{d}{dx}(a-x)}{(a-x)^2}$$

$$\frac{dy}{dx} = \frac{(a-x)(1) - (a+x)(-1)}{(a-x)^2}$$

$$\frac{dy}{dx} = \frac{(a-x) + (a+x)}{(a-x)^2}$$

$$\frac{dy}{dx} = \frac{a-\cancel{x}+a+\cancel{x}}{(a-x)^2}$$

$$\boxed{\frac{dy}{dx} = \frac{2a}{(a-x)^2}}$$

Question #4

$$\frac{2x-3}{2x+1}$$

Solution:

Let $y = \frac{2x-3}{2x+1}$

Differentiate w.r.t "x"

$$\frac{dy}{dx} = \frac{d}{dx}\left[\frac{2x-3}{2x+1}\right]$$

$$\frac{dy}{dx} = \frac{2x+1\frac{d}{dx}(2x-3) - (2x-3)\frac{d}{dx}(2x+1)}{(2x+1)^2}$$

$$\frac{dy}{dx} = \frac{(2x+1)(2) - (2x-3)(2)}{(2x+1)^2}$$

$$\frac{dy}{dx} = \frac{\cancel{4x}+2-\cancel{4x}+6}{(2x+1)^2}$$

$$\boxed{\frac{dy}{dx} = \frac{8}{(2x+1)^2}}$$

Question #5

$$(x-5)(3-x)$$

Solution:

Let $y = (x-5)(3-x)$

Differentiate w.r.t "x"

$$\frac{dy}{dx} = \frac{d}{dx}(x-5)(3-x)$$

$$\frac{dy}{dx} = (x-5)\frac{d}{dx}(3-x) + (3-x)\frac{d}{dx}(x-5)$$

$$\frac{dy}{dx} = (x-5)(-1) + (3-x)(1)$$

$$= -x+5+3-x$$

$$\boxed{\frac{dy}{dx} = -2x+8}$$

Question #6

$$\left[\sqrt{x} - \frac{1}{\sqrt{x}}\right]^2$$

Solution:

Let $y = \left[\sqrt{x} - \frac{1}{\sqrt{x}}\right]^2$

$$y = (\sqrt{x})^2 + \left(\frac{1}{\sqrt{x}}\right)^2 - 2(\sqrt{x})\left(\frac{1}{\sqrt{x}}\right)$$

$$y = x + \frac{1}{x} - 2$$

$$y = x + x^{-1} - 2$$

Differentiate w.r.t "x"

$$\frac{dy}{dx} = \frac{d}{dx}\left(x + x^{-1} - 2\right)$$

$$\frac{dy}{dx} = (1 - x^{-1-1} - 0)$$

$$\frac{dy}{dx} = 1 - x^{-2}$$

$$\frac{dy}{dx} = 1 - \frac{1}{x^2}$$

$$\boxed{\frac{dy}{dx} = \frac{x^2-1}{x^2}}$$

Question #7

$$\frac{(1+\sqrt{x})(x-x^{3/2})}{\sqrt{x}}$$

Solution:

Let $y = \frac{(1+\sqrt{x})(x-x^{3/2})}{\sqrt{x}}$

$$y = \frac{(1+\sqrt{x})(x - x\cdot x^{1/2})}{\sqrt{x}}$$

$$y = \frac{(1+\sqrt{x})\,x(1-\sqrt{x})}{\sqrt{x}}$$

$$y = \frac{x\ (1-x)}{\sqrt{x}}$$

$$y = \frac{x-x^2}{\sqrt{x}}$$

$$y = \frac{x}{x^{1/2}} - \frac{x^2}{x^{1/2}}$$

$$y = x^{1/2} - x^{3/2}$$

Differentiate w.r.t "x"

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^{1/2} - x^{3/2}\right)$$

$$\frac{dy}{dx} = \left(\frac{1}{2}x^{\frac{1}{2}-1} - \frac{3}{2}x^{\frac{3}{2}-1}\right)$$

$$\frac{dy}{dx} = \frac{1}{2}x^{\frac{-1}{2}} - \frac{3}{2}x^{\frac{1}{2}}$$

$$\frac{dy}{dx} = \frac{1}{2x^{1/2}} - \frac{3}{2}\sqrt{x}$$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}} - \frac{3}{2}\sqrt{x}$$

$$\boxed{\frac{dy}{dx} = \frac{1}{2}\left[\frac{1}{\sqrt{x}} - 3\sqrt{x}\right]}$$

• ———— •

Question #9

$$\frac{x^2+1}{x^2-3}$$

Solution:

Let

$$y = \frac{x^2+1}{x^2-3}$$

Question #8

$$\frac{(x^2+1)^2}{x^2-1}$$

Solution:

Let $y = \frac{(x^2+1)^2}{(x^2-1)}$

Differentiate w.r.t "x"

$$\frac{dy}{dx} = \frac{d}{dx}\left[\frac{(x^2+1)^2}{x^2-1}\right]$$

$$\frac{dy}{dx} = \frac{(x^2-1)\ \frac{d}{dx}(x^2+1)^2 - (x^2+1)^2\frac{d}{dx}(x^2-1)}{(x^2-1)^2}$$

$$\frac{dy}{dx} = \frac{(x^2-1)\ 2(x^2+1)^{2-1}\frac{d}{dx}(x^2+1) - (x^2+1)^2\,(2x)}{(x^2-1)^2}$$

$$\frac{dy}{dx} = \frac{(x^2-1)\,2(x^2+1)\,(2x) - (x^2+1)^2\,2x}{(x^2-1)^2}$$

$$\frac{dy}{dx} = \frac{4x(x^2+1)(x^2-1) - 2x(x^2+1)^2}{(x^2-1)^2}$$

$$\frac{dy}{dx} = \frac{2x(x^2+1)\ [2(x^2-1) - (x^2+1)]}{(x^2+1)^2}$$

$$\frac{dy}{dx} = \frac{2x(x^2+1)\,[2x^2-2-x^2-1]}{(x^2-1)^2}$$

$$\boxed{\frac{dy}{dx} = \frac{2x(x^2+1)(x^2-3)}{(x^2-1)^2}}$$

• ———— •

Question #10

$$\frac{\sqrt{1+x}}{\sqrt{1-x}}$$

Solution:

Let $y = \frac{\sqrt{1+x}}{\sqrt{1-x}}$

$$y = \sqrt{\frac{1+x}{1-x}}$$

$$y = \left(\frac{1+x}{1-x}\right)^{1/2}$$

Differentiate w.r.t 'x'

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{x^2+1}{x^2-3}\right)$$

$$\frac{dy}{dx} = \frac{(x^2-3)\frac{d}{dx}(x^2+1) - (x^2-1)\frac{d}{dx}(x^2-3)}{(x^2-3)^2}$$

$$\frac{dy}{dx} = \frac{(x^2-3)(2x) - (x^2+1)\,2x}{(x^2-3)^2}$$

$$\frac{dy}{dx} = \frac{\cancel{2x^3} - 6x - \cancel{2x^3} - 2x}{(x^2-3)^2}$$

$$\boxed{\frac{dy}{dx} = \frac{-8x}{(x^2-3)^2}}$$

Question #11

$$\frac{2x-1}{\sqrt{x^2+1}}$$

Solution:

Let $y = \frac{2x-1}{\sqrt{x^2+1}}$

Differentiate w.r.t 'x'

$$\frac{dy}{dx} = \frac{d}{dx}\left[\frac{2x-1}{\sqrt{x^2+1}}\right]^1$$

$$\frac{dy}{dx} = \frac{(\sqrt{x^2+1})\frac{d}{dx}(2x-1) - (2x-1)\frac{d}{dx}(x^2+1)^{1/2}}{(\sqrt{x^2+1})^2}$$

$$\frac{dy}{dx} = \frac{(x^2+1)^{1/2}(2) - (2x-1)\frac{1}{2}(x^2+1)^{\frac{1}{2}-1}\frac{d}{dx}(x^2+1)}{(x^2+1)}$$

$$\frac{dy}{dx} = \frac{2(x^2+1)^{1/2} - (2x-1)\frac{1}{\cancel{2}}(x^2+1)^{-\frac{1}{2}}(\cancel{2}x)}{(x^2+1)}$$

$$\frac{dy}{dx} = \frac{2(x^2+1)^{1/2} - \frac{x(2x-1)}{(x^2+1)^{1/2}}}{(x^2+1)}$$

$$\frac{dy}{dx} = \frac{2((x^2+1)^{1/2})^2 - x(2x-1)}{(x^2+1)^{1/2}(x^2+1)}$$

$$\frac{dy}{dx} = \frac{\cancel{2x^2} + 2 - \cancel{2x^2} + x}{(x^2+1)^{3/2}}$$

$$\boxed{\frac{dy}{dx} = \frac{2+x}{(x^2+1)^{3/2}}}$$

Differentiate w.r.t 'x'

$$\frac{dy}{dx} = \frac{d}{dx}\left[\frac{1+x}{1-x}\right]^{1/2}$$

$$\frac{dy}{dx} = \frac{1}{2}\left[\frac{1+x}{1-x}\right]^{1/2-1}\frac{d}{dx}\left(\frac{1+x}{1-x}\right)$$

$$\frac{dy}{dx} = \frac{1}{2}\left[\frac{1+x}{1-x}\right]^{-\frac{1}{2}}\frac{(1-x)\frac{d}{dx}(1+x) - (1+x)\frac{d}{dx}(1-x)}{(1-x)^2}$$

$$\frac{dy}{dx} = \frac{1}{2}\sqrt{\frac{1-x}{1+x}} \cdot \frac{(1-x)(1) - (1+x)(-1)}{(1-x)^2}$$

$$\frac{dy}{dx} = \frac{1}{2}\sqrt{\frac{1-x}{1+x}} \cdot \frac{1-\cancel{x}+1+\cancel{x}}{(1-x)^2}$$

$$\frac{dy}{dx} = \frac{1}{\cancel{2}}\sqrt{\frac{1-x}{1+x}} \cdot \frac{\cancel{2}}{(1-x)^2}$$

$$\frac{dy}{dx} = \frac{(1-x)^{1/2}}{(1+x)^{1/2}} \cdot \frac{1}{(1-x)^2}$$

$$\frac{dy}{dx} = \frac{1}{(1+x)^{1/2}(1-x)^{2-\frac{1}{2}}}$$

$$\boxed{\frac{dy}{dx} = \frac{1}{\sqrt{1+x}\,(1-x)^{3/2}}}$$

Question #12

$$\sqrt{\frac{a-x}{a+x}}$$

Solution:

Let $y = \sqrt{\frac{a-x}{a+x}}$

$y = \left[\frac{a-x}{a+x}\right]^{1/2}$

$$\frac{dy}{dx} = \frac{d}{dx}\left[\frac{a-x}{a+x}\right]^{1/2}$$

$$\frac{dy}{dx} = \frac{1}{2}\left[\frac{a-x}{a+x}\right]^{1/2-1} \cdot \frac{d}{dx}\left[\frac{a-x}{a+x}\right]$$

$$\frac{dy}{dx} = \frac{1}{2}\left[\frac{a-x}{a+x}\right]^{-\frac{1}{2}} \cdot \frac{(a+x)\frac{d}{dx}(a-x) - (a-x)\frac{d}{dx}(a+x)}{(a+x)^2}$$

$$\frac{dy}{dx} = \frac{1}{2}\left[\frac{a+x}{a-x}\right]^{1/2} \cdot \frac{(a+x)(-1) - (a-x)(1)}{(a+x)^2}$$

$$\frac{dy}{dx} = \frac{1}{2}\left[\frac{a+x}{a-x}\right]^{1/2} \cdot \frac{-a-\cancel{x}-a+\cancel{x}}{(a+x)^2}$$

Question # 13

$$\frac{\sqrt{x^2+1}}{\sqrt{x^2-1}}$$

Solution:

Let $y = \frac{\sqrt{x^2+1}}{\sqrt{x^2-1}}$

$y = \sqrt{\frac{x^2+1}{x^2-1}}$

$y = \left(\frac{x^2+1}{x^2-1}\right)^{1/2}$

Differentiate w.r.t 'x'

$$\frac{dy}{dx} = \frac{d}{dx}\left[\frac{x^2+1}{x^2-1}\right]^{1/2}$$

$$\frac{dy}{dx} = \frac{1}{2}\left[\frac{x^2+1}{x^2-1}\right]^{\frac{1}{2}-1} \cdot \frac{d}{dx}\left(\frac{x^2+1}{x^2-1}\right)$$

$$\frac{dy}{dx} = \frac{1}{2}\left[\frac{x^2+1}{x^2-1}\right)^{\frac{-1}{2}} \cdot \frac{(x^2-1)\frac{d}{dx}(x^2+1)-(x^2-1)\frac{d}{dx}(x^2+1)}{(x^2-1)^2}$$

$$\frac{dy}{dx} = \frac{1}{2}\left[\frac{x^2-1}{x^2+1}\right]^{1/2} \cdot \frac{(x^2-1)(2x)-(x^2-1)(2x)}{(x^2-1)^2}$$

$$\frac{dy}{dx} = \frac{1}{2}\left[\frac{x^2-1}{x^2+1}\right]^{1/2} \cdot \frac{\cancel{2x^3}-2x-\cancel{2x^3}-2x}{(x^2-1)^2}$$

$$\frac{dy}{dx} = \frac{1}{\cancel{2}(x^2+1)^{1/2}} \cdot \frac{\overset{2}{\cancel{-4}}x}{(x^2-1)^{2-\frac{1}{2}}}$$

$$\boxed{\frac{dy}{dx} = \frac{-2x}{\sqrt{x^2+1}\,(x^2-1)^{3/2}}}$$

Question 15

$$\frac{x\sqrt{a+x}}{\sqrt{a-x}}$$

Solution:

Let $y = \frac{x\sqrt{a+x}}{\sqrt{a-x}}$

Differentiate w.r.t 'x'

$$\frac{dy}{dx} = \frac{d}{dx}\left[\frac{x\sqrt{a+x}}{\sqrt{a-x}}\right]$$

$$\frac{dy}{dx} = x\frac{d}{dx}\frac{(a+x)^{1/2}}{(a-x)^{1/2}} + \frac{(a+x)^{1/2}}{(a-x)^{1/2}}\frac{d}{dx}(x)$$

$$\frac{dy}{dx} = x\frac{(a-x)^{1/2}\frac{d}{dx}(a+x)^{1/2}-(a+x)\frac{d}{dx}(a-x)^{1/2}}{((a-x)^{1/2})^2} + \frac{(a+x)^{1/2}}{(a-x)^{1/2}}\cdot(1)$$

$$\frac{dy}{dx} = \frac{1}{\cancel{2}}\left[\frac{a+x}{a-x}\right]^{1/2} \cdot \frac{-\cancel{2}a}{(a+x)^2}$$

$$\frac{dy}{dx} = \frac{-a(a+x)^{1/2}}{(a-x)^{1/2}(a+x)^2}$$

$$\frac{dy}{dx} = \frac{-a}{\sqrt{a-x}\,(a+x)^{2-\frac{1}{2}}}$$

$$\boxed{\frac{dy}{dx} = \frac{-a}{\sqrt{a-x}\,(a+x)^{3/2}}}$$

Question # 14

$$\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}$$

Solution:

Let $y = \frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}} \times \frac{\sqrt{1+x}-\sqrt{1+x}}{\sqrt{1+x}-\sqrt{1+x}}$

$$y = \frac{(\sqrt{1+x}-\sqrt{1-x})^2}{(\sqrt{1+x})^{\cancel{2}}-(\sqrt{1+x})^{\cancel{2}}}$$

$$y = \frac{(\sqrt{1+x})^{\cancel{2}}+(\sqrt{1-x})^{\cancel{2}}-2(\sqrt{1+x})(\sqrt{1-x})}{\cancel{1}+x-\cancel{1}+x}$$

$$y = \frac{1+\cancel{x}+1-\cancel{x}-2\sqrt{(1+x)(1-x)}}{2x}$$

$$y = \frac{2-2\sqrt{(1)^2-x^2}}{2x}$$

$$y = \frac{\cancel{2}[1-\sqrt{1-x^2}]}{\cancel{2}x}$$

$$y = \frac{1-(1-x^2)^{1/2}}{x}$$

Differentiate w.r.t 'x'

$$\frac{dy}{dx} = \frac{d}{dx}\left[\frac{1-(1-x^2)^{1/2}}{x}\right]$$

$$\frac{dy}{dx} = \frac{x\frac{d}{dx}[1-(1-x^2)^{1/2}]-(1-(1-x^2)^{1/2})\frac{d}{dx}(x)}{x^2}$$

$$\frac{dy}{dx} = \frac{x(0-\frac{1}{2}(1-x^2)^{\frac{-1}{2}}\frac{d}{dx}(1-x^2)-(1-(1-x^2)^{1/2})(1)}{x^2}$$

$$\frac{dy}{dx}=\frac{x\,(a-x)^{1/2}\cdot\frac{1}{2}(a+x)^{\frac{1}{2}-1}\frac{d}{dx}(a+x)-(a+x)^{1/2}\cdot\frac{1}{2}(a-x)^{\frac{1}{2}-1}\frac{d}{dx}(a-x)}{(a-x)}+\frac{\sqrt{a+x}}{\sqrt{a-x}}$$

$$\frac{dy}{dx}=\frac{x\,(a-x)^{1/2}\cdot\frac{1}{2}(a+x)^{\frac{-1}{2}}-(a+x)^{\frac{1}{2}}\cdot\frac{1}{2}(a-x)^{\frac{-1}{2}}(-1)}{(a-x)}+\frac{\sqrt{a+x}}{\sqrt{a-x}}$$

$$\frac{dy}{dx}=\frac{x\cdot\frac{1}{2}\left[(a-x)^{1/2}\cdot(a+x)^{\frac{-1}{2}}+(a+x)^{\frac{1}{2}}(a-x)^{\frac{-1}{2}}\right]}{(a-x)}+\frac{\sqrt{a+x}}{\sqrt{a-x}}$$

$$\frac{dy}{dx}=\frac{x}{2(a-x)}\left[\frac{\sqrt{a-x}}{\sqrt{a+x}}+\frac{\sqrt{a+x}}{\sqrt{a-x}}\right]+\frac{\sqrt{a+x}}{\sqrt{a-x}}$$

$$\frac{dy}{dx}=\frac{x}{2(a-x)}\left[\frac{(\sqrt{a-x})^2+(\sqrt{a+x})^2}{(\sqrt{a+x})\;(\sqrt{a-x})}\right]+\frac{\sqrt{a+x}}{\sqrt{a-x}}$$

$$\frac{dy}{dx}=\frac{x}{2(a-x)}\left(\frac{a-\cancel{x}+a+\cancel{x}}{(\sqrt{a+x})(\sqrt{a-x})}\right]+\frac{\sqrt{a+x}}{\sqrt{a-x}}$$

$$\frac{dy}{dx}=\frac{x}{\cancel{2}(a-x)}\cdot\frac{\cancel{2}a}{(\sqrt{a+x})(\sqrt{a-x})}+\frac{\sqrt{a+x}}{\sqrt{a-x}}$$

$$\frac{dy}{dx}=\frac{ax}{(a-x)(\sqrt{a+x})(\sqrt{a-x})}+\frac{\sqrt{a+x}}{\sqrt{a-x}}$$

$$\frac{dy}{dx}=\frac{ax+(a-x)(\sqrt{a+x})^2}{(a-x)(\sqrt{a+x})(\sqrt{a-x})}$$

$$\frac{dy}{dx}=\frac{ax+(a-x)(a+x)}{(\sqrt{a+x})\;(a-x)^{3/2}}$$

$$\frac{dy}{dx}=\frac{ax+a^2-x^2}{(\sqrt{a+x})(a-x)^{3/2}}$$

$$\boxed{\frac{dy}{dx}=\frac{a^2-x^2+ax}{\sqrt{a+x}\;(a-x)^{3/2}}}$$

Question # 16

If $y=\sqrt{x}-\frac{1}{\sqrt{x}}$, show that

$$2x\frac{dy}{dx}+y=2\sqrt{x}$$

Solution:

$$y=\sqrt{x}-\frac{1}{\sqrt{x}}$$

$$y=x^{1/2}-\frac{1}{x^{1/2}}$$

$$y=x^{1/2}-x^{-\frac{1}{2}}$$

$$\frac{dy}{dx}=\frac{x\cdot\frac{1}{\cancel{2}}(1-x^2)^{\frac{-1}{2}}\cdot(-\cancel{2}x)-\left(1-(1-x^2)^{1/2}\right)}{x^2}$$

$$\frac{dy}{dx}=\frac{x^2(1-x^2)^{\frac{-1}{2}}-\left(1-(1-x^2)^{1/2}\right)}{x^2}$$

$$\frac{dy}{dx}=\frac{1}{x^2}\left[\frac{x^2}{\sqrt{1-x^2}}-1+(1-x^2)^{1/2}\right]$$

$$\frac{dy}{dx}=\frac{1}{x^2}\left[\frac{x^2-\sqrt{1-x^2}+\left((1-x^2)^{1/2}\right)^2}{(\sqrt{1-x^2})}\right]$$

$$\frac{dy}{dx}=\frac{1}{x^2}\left[\frac{x^2-\sqrt{1-x^2}+(1-x^2)}{\sqrt{1-x^2}}\right]$$

$$\frac{dy}{dx}=\frac{1}{x^2}\left[\frac{\cancel{x^2}-\sqrt{1-x^2}+1-\cancel{x^2}}{\sqrt{1-x^2}}\right]$$

$$\boxed{\frac{dy}{dx}=\frac{1-\sqrt{1-x^2}}{x^2\sqrt{1-x^2}}}$$

Question # 17

If $y=x^4+2x^2+2$, prove that

$$\frac{dy}{dx}=4x\cdot\sqrt{y-1}$$

Solution:

$$y=x^4+2x^2+2 \quad \text{———} \quad ①$$

Differentiate w.r.t 'x'

$$\frac{dy}{dx}=\frac{d}{dx}(x^4+2x^2+2)$$

$$\frac{dy}{dx}=4x^{4-1}+2(2x^{2-1})+0$$

$$\frac{dy}{dx}=4x^3+4x$$

$$\frac{dy}{dx}=4x(x^2+1) \quad \text{———} \quad ②$$

From eq ①

$$y-1=x^4+2x^2+2-1$$

$$y-1=x^4+2x^2+1$$

$$y-1=(x^2+1)^2$$

$$\sqrt{y-1}=\sqrt{(x^2+1)^2}$$

Differentiate w.r.t 'x'

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^{1/2} - x^{-1/2}\right)$$

$$\frac{dy}{dx} = \frac{1}{2}x^{\frac{1}{2}-1} - \left(\frac{-1}{2}x^{\frac{-1}{2}-1}\right)$$

$$\frac{dy}{dx} = \frac{1}{2}x^{\frac{-1}{2}} + \frac{1}{2}x^{\frac{-3}{2}}$$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}} + \frac{1}{2x\cdot\sqrt{x}}$$

According to given condition.

$$2x\frac{dy}{dx} + y = 2\sqrt{x}$$

L.H.S

$$= 2x\left[\frac{1}{2\sqrt{x}} + \frac{1}{2x\sqrt{x}}\right] + \sqrt{x} - \frac{1}{\sqrt{x}}$$

$$= \frac{\cancel{2}x}{\cancel{2}\sqrt{x}} + \frac{\cancel{2x}}{\cancel{2x}\sqrt{x}} + \sqrt{x} - \frac{1}{\sqrt{x}}$$

$$= \frac{x}{\sqrt{x}} + \frac{1}{\sqrt{x}} + \sqrt{x} - \frac{1}{\sqrt{x}}$$

$$= \frac{\sqrt{x}\cdot\cancel{\sqrt{x}}}{\cancel{\sqrt{x}}} + \cancel{\frac{1}{\sqrt{x}}} + \sqrt{x} - \cancel{\frac{1}{\sqrt{x}}}$$

$$= 2\sqrt{x} \quad R.H.S$$

Hence proved

$$2x\frac{dy}{dx} + y = 2\sqrt{x}$$

$$\sqrt{y-1} = x^2+1$$

put in eq ②

$$\boxed{\frac{dy}{dx} = 4x\sqrt{y-1}}$$

Hence proved.

Example 1

* Theory

Find the derivative of $(x^3+1)^9$ w.r.t 'x'?

Solution.

Let $y = (x^3+1)^9$

$y = u^9$

Differentiate w.r.t 'u'

$$\frac{dy}{du} = 9u^8$$

$u = x^3+1$

Differentiate w.r.t 'x'

$$\frac{du}{dx} = 3x^2$$

Using chain rule.

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$$

$$= 9u^8(3x^2)$$

$$= 9(x^3+1)^8(3x^2)$$

$$\boxed{\frac{dy}{dx} = 27x^2(x^3+1)^8}$$

Example #2

Differentiate $\sqrt{\frac{a-x}{a+x}}$ $(x \neq -a)$ w.r.t 'x'

Solution:

Let

$$y = \sqrt{\frac{a-x}{a+x}}$$

Differentiate w.r.t 'u'

$$y = (u)^{1/2}$$

$$\frac{dy}{du} = \frac{1}{2}u^{\frac{1}{2}-1}$$

$$\frac{dy}{du} = \frac{1}{2}u^{\frac{-1}{2}}$$

$$u = \frac{a-x}{a+x}$$

Differentiate w.r.t 'x'

$$\frac{du}{dx} = \frac{(a+x)\frac{d}{dx}(a-x) - (a-x)\frac{d}{dx}(a+x)}{(a+x)^2}$$

$$\frac{du}{dx} = \frac{(a+x)(-1) - (a-x)(1)}{(a+x)^2}$$

$$\frac{du}{dx} = \frac{-a-\cancel{x}-a+\cancel{x}}{(a+x)^2}$$

$$\frac{du}{dx} = \frac{-2a}{(a+x)^2}$$

Using chain rule

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$\frac{dy}{dx} = \frac{1}{2}u^{-\frac{1}{2}} \cdot \frac{-2a}{(a+x)^2}$$

$$\frac{dy}{dx} = \frac{1}{\cancel{2}}\left[\frac{a-x}{a+x}\right]^{-\frac{1}{2}} \times \frac{-\cancel{2}a}{(a+x)^2}$$

$$\frac{dy}{dx} = \frac{(a-x)^{-\frac{1}{2}}}{(a+x)^{-\frac{1}{2}}} \cdot \frac{-a}{(a+x)^2}$$

$$\boxed{\frac{dy}{dx} = \frac{-a}{(a-x)^{\frac{1}{2}}(a+x)^{3/2}}}$$

Example #3

Find $\frac{dy}{dx}$ if $y = \frac{\sqrt{a+x}+\sqrt{a-x}}{\sqrt{a+x}-\sqrt{a-x}}$, $(x \neq 0)$

Solution:

$$y = \frac{\sqrt{a+x}+\sqrt{a-x}}{\sqrt{a+x}-\sqrt{a-x}}$$

$$y = \frac{\sqrt{a+x}+\sqrt{a-x}}{\sqrt{a+x}-\sqrt{a-x}} \times \frac{\sqrt{a+x}-\sqrt{ax}}{\sqrt{a+x}-\sqrt{a-x}}$$

$$y = \frac{(\sqrt{a+x})^2 - (\sqrt{a-x})^2}{(\sqrt{a+x} - \sqrt{a-x})^2}$$

$$y = \frac{a+x-a+x}{(\sqrt{a+x})^2 + (\sqrt{a-x})^2 - 2(\sqrt{a+x})\sqrt{a-x})}$$

$$y = \frac{2x}{a+\cancel{x}+a-\cancel{x}-2\sqrt{a^2-x^2}}$$

$$y = \frac{2x}{2a-2\sqrt{a^2-x^2}} \Rightarrow \frac{\cancel{2}x}{\cancel{2}(a-\sqrt{a^2-x^2})}$$

$$y = \frac{x}{a-\sqrt{a^2-x^2}}$$

Differentiate w.r.t 'x'

$$\frac{dy}{dx} = \frac{d}{dx}\left[\frac{x}{a-\sqrt{a^2-x^2}}\right]$$

$$= \frac{(a-\sqrt{a^2-x^2})\frac{d}{dx}(x) - (x)\frac{d}{dx}(a-\sqrt{a^2+x^2})}{(a-\sqrt{a^2-x^2})^2}$$

$$\frac{dy}{dx} = \frac{a-\sqrt{a^2-x^2}\ (1) - x\left[0-\frac{d}{dx}(a^2-x^2)^{1/2}\right]}{(a-\sqrt{a^2-x^2})^2}$$

$$\frac{dy}{dx} = \frac{a-\sqrt{a^2-x^2} + x\,\frac{1}{2}(a^2-x^2)^{-\frac{1}{2}}\frac{d}{dx}(a^2-x^2)}{(a-\sqrt{a^2-x^2})^2}$$

$$= \frac{1}{(a-\sqrt{a^2-x^2})^2}\cdot\left[a-\sqrt{a^2-x^2} + \frac{x}{\cancel{2}\sqrt{a^2-x^2}}\cdot -\cancel{2}x\right]$$

$$\frac{dy}{dx} = \frac{1}{(a-\sqrt{a^2-x^2})^2}\cdot\left[\frac{a\sqrt{a^2-x^2}-(\sqrt{a^2-x^2})^2-x^2}{\sqrt{a^2-x^2}}\right]$$

$$\frac{dy}{dx} = \frac{1}{(a-\sqrt{a^2-x^2})^2}\left[\frac{a\sqrt{a^2-x^2}-(a^2-x^2)-x^2}{\sqrt{a^2-x^2}}\right]$$

$$\frac{dy}{dx} = \frac{1}{(a-\sqrt{a^2-x^2})^2}\left[\frac{a\sqrt{a^2+x^2}-a^2+\cancel{x^2}-\cancel{x^2}}{\sqrt{a^2-x^2}}\right]$$

$$\frac{dy}{dx} = \frac{1}{(a-\sqrt{a^2-x^2})^{\cancel{2}}}\cdot\frac{-a\,\cancel{(a-\sqrt{a^2-x^2})}}{\sqrt{a^2+x^2}}$$

$$\boxed{\frac{dy}{dx} = \frac{-a}{(a-\sqrt{a^2-x^2})\sqrt{a^2+x^2}}}$$

Example #4

Find $\frac{dy}{dx}$ if $y = (1+2\sqrt{x})^3 . x^{3/2}$

Solution:

$$y = (1+2\sqrt{x})^3 . x^{3/2}$$

$$y = (1+2\sqrt{x})^3 . (x^{1/2})^3$$

$$y = (1+2\sqrt{x})^3 (\sqrt{x})^3$$

Let

$$u = (1+2\sqrt{x})(\sqrt{x})$$

Differentiate w.r.t "x"

$$\frac{du}{dx} = (1+2\sqrt{x})\frac{d}{dx}\sqrt{x} + \sqrt{x}\frac{d}{dx}(1+2\sqrt{x})$$

$$y = u^3$$

Differentiate w.r.t "u"

$$\frac{dy}{du} = 3u^2$$

$$\frac{du}{dx} = (1+2\sqrt{x}) . \frac{1}{2}x^{-\frac{1}{2}} + \sqrt{x}\left[0+2 . \frac{1}{2}x^{-\frac{1}{2}}\right]$$

$$\frac{du}{dx} = (1+2\sqrt{x})\frac{1}{2\sqrt{x}} + 2\sqrt{x} . \frac{1}{2\sqrt{x}}$$

$$\frac{du}{dx} = \frac{1+2\sqrt{x}}{2\sqrt{x}} + \frac{2\sqrt{x}}{2\sqrt{x}}$$

$$\frac{du}{dx} = \frac{1+2\sqrt{x}+2\sqrt{x}}{2\sqrt{x}}$$

$$\frac{du}{dx} = \frac{1+4\sqrt{x}}{2\sqrt{x}}$$

Using chain rule.

$$\frac{dy}{dx} = \frac{dy}{du} . \frac{du}{dx}$$

$$\frac{dy}{dx} = 3u^2 . \frac{1+4\sqrt{x}}{2\sqrt{x}}$$

$$= 3(1+2\sqrt{x})\sqrt{x})^2 . \frac{1+4\sqrt{x}}{2\sqrt{x}}$$

$$\frac{dy}{dx} = 3(1+2\sqrt{x})^2(\sqrt{x})^2 . \frac{1+4\sqrt{x}}{2\sqrt{x}}$$

$$\boxed{\frac{dy}{dx} = \frac{3}{2}(1+2\sqrt{x})^2(\sqrt{x}+4x)}$$

Example #5

If $y = (ax+b)^n$ where n is negative integers. Find $\frac{dy}{dx}$ using quotient theorem.

Solution:

$$y = (ax+b)^n$$

Let $n = -m$

$$y = (ax+b)^{-m}$$

$$y = \frac{1}{(ax+b)^m}$$

Differentiate w.r.t "x"

$$\frac{dy}{dx} = \frac{d}{dx}\left[\frac{1}{(ax+b)^m}\right]$$

$$\frac{dy}{dx} = \frac{(ax+b)^m\frac{d}{dx}(1) - 1\frac{d}{dx}(ax+b)^m}{((ax+b)^m)^2}$$

$$\frac{dy}{dx} = \frac{(ax+b)^m(0) - 1 . m\underbrace{(ax+b)^{-m}}_{\frac{d}{dx}(ax+b)}}{(ax+b)^{2m}}$$

$$\frac{dy}{dx} = \frac{-m(ax+b)^{-m}(a)}{(ax+b)^{2m}}$$

$$\frac{dy}{dx} = -ma(ax+b)^{-2m-1+m}$$

$$\frac{dy}{dx} = -ma(ax+b)^{-m-1}$$

$\because n = -m$

$$\boxed{\frac{dy}{dx} = -na(ax+b)^{n-1}}$$

Example #6

Find $\frac{dy}{dx}$ if $y = x^n$ where $n = \frac{p}{q}$, $q \neq 0$

Solution:

$y = x^n$

$\because n = \frac{p}{q}$

$y = x^{\frac{p}{q}}$ —— ①

Taking q^{th} power both sides

$y^q = x^p$ —— ②

Differentiate eq ② w.r.t 'x'

$$\frac{d}{dx}(y^q) = \frac{d}{dx}(x^p)$$

$$\frac{d}{dy}(y^q)\frac{dy}{dx} = \frac{d}{dx}x^p \text{ —— ③}$$

Using chain rule.

$$q y^{q-1}\frac{dy}{dx} = p x^{p-1}$$

multiply by 'y' eq ③ both sides.

$$q \cdot y^q \frac{dy}{dx} = p y x^{p-1}$$

$$q \cdot x^p \frac{dy}{dx} = p x^{\frac{p}{q}} \cdot x^{p-1}$$

$$\frac{dy}{dx} = \frac{p}{q} \cdot \frac{1}{x^p} \cdot x^{\frac{p}{q}} x^{p-1}$$

$$= \frac{p}{q} \times x^{\frac{p}{q}+q-1-p}$$

$$= \frac{p}{q} x^{\frac{p}{q}-1}$$

$\because \frac{p}{q} = n$

$$\boxed{\frac{d}{dx}(x^n) = n x^{n-1}}$$

Example #1

Find $\frac{dy}{dx}$ if $x = at^2$ and $y = 2at$.

Solution:

$x = at^2$

Differentiate w.r.t 't'

$$\frac{dx}{dt} = \frac{d}{dt}(at^2)$$

$$\frac{dx}{dt} = 2at$$

$y = 2at$

Differentiate w.r.t 't'

$$\frac{dy}{dt} = \frac{d}{dt}(2at)$$

$$\frac{dy}{dt} = 2a$$

Using chain rule

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$

$$= 2a \cdot \frac{1}{2at}$$

$$\boxed{\frac{dy}{dx} = \frac{1}{t}}$$

Example #2

Find $\frac{dy}{dx}$ if $x = 1 - t^2$ and $y = 3t^2 - 2t^3$.

Solution.

$x = 1 - t^2$

Differentiate w.r.t 't'

$$\frac{dx}{dt} = \frac{d}{dt}(1-t^2)$$

$$\frac{dx}{dt} = -2t$$

$y = 3t^2 - 2t^3$

Differentiate w.r.t "x"

$$\frac{dy}{dt} = \frac{d}{dt}(3t^2 - 2t^3)$$

$$\frac{dy}{dt} = 6t - 6t^2$$

$$\frac{dy}{dt} = 6t(1-t)$$

Using chain rule.

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$

$$= \overset{3}{\cancel{6}}\cancel{t}(1-t) \cdot \frac{1}{-\cancel{2}\cancel{t}}$$

$$\frac{dy}{dx} = -3(1-t)$$

$$\boxed{\frac{dy}{dx} = 3(-1+t)}$$

Example #3

$\frac{dy}{dx}$ if $x = \frac{(1-t^2)}{1+t^2}$, $y = \frac{2t}{1+t^2}$

Solution:

$x = \frac{1-t^2}{1+t^2}$

Differentiate w.r.t "t"

$$\frac{dx}{dt} = \frac{d}{dt}\left[\frac{1-t^2}{1+t^2}\right]$$

$$= \frac{(1+t^2)\frac{d}{dt}(1-t^2) - (1-t^2)\frac{d}{dx}(1+t^2)}{(1+t^2)^2}$$

$$\frac{dx}{dt} = \frac{(1+t^2)(-2t) - (1-t^2)(2t)}{(1+t^2)^2}$$

$$\frac{dx}{dt} = \frac{-2t - \cancel{2t^3} - 2t + \cancel{2t^3}}{(1+t^2)^2}$$

$$\frac{dx}{dt} = \frac{-4t}{(1+t^2)^2}$$

$y = \frac{2t}{1+t^2}$

Differentiate w.r.t 't'

$$\frac{dy}{dt} = \frac{d}{dt}\left[\frac{2t}{1+t^2}\right]$$

$$= \frac{(1+t^2)\frac{d}{dt}(2t) - 2t\frac{d}{dt}(1+t^2)}{(1+t^2)^2}$$

$$\frac{dy}{dt} = \frac{(1+t^2)(2) - 2t(2t)}{(1+t^2)^2}$$

$$\frac{dy}{dt} = \frac{2+2t^2-4t^2}{(1+t^2)^2}$$

$$\frac{dy}{dt} = \frac{2-2t^2}{(1+t^2)^2}$$

$$\frac{dy}{dt} = \frac{2(1-t^2)}{(1+t^2)^2}$$

Using chain rule

$$\frac{dy}{dx} = \frac{2(1-t^2)}{\cancel{(1+t^2)^2}} \cdot \frac{\cancel{(1+t^2)^2}}{-\underset{2}{\cancel{4}}t}$$

$$\frac{dy}{dx} = \frac{-(1-t^2)}{2t}$$

$$\boxed{\frac{dy}{dx} = \frac{t^2-1}{2t}}$$

Example #1

Find $\frac{dy}{dx}$ if $x^2+y^2=4$

Solution:

$x^2+y^2=4$

Differentiate w.r.t "x"

$$\frac{d}{dx}(x^2+y^2) = \frac{d}{dx}(4)$$

$$2x+2y\frac{dy}{dx} = 0$$

$$2y\frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = \frac{-\cancel{2}x}{\cancel{2}y}$$

$$\boxed{\frac{dy}{dx} = \frac{-x}{y}}$$

Example #2

Find $\frac{dy}{dx}$ if $y^2+x^2-4x=5$

Solution.

$y^2+x^2-4x=5$

Differentiate w.r.t "x"

$$\frac{d}{dx}\left[y^2+x^2-4x\right] = \frac{d}{dx}(5)$$

$$2y\frac{dy}{dx}+2x-4=0$$

$$2y\frac{dy}{dx} = 4-2x$$

$$\frac{dy}{dx} = \frac{4-2x}{2y}$$

$$\frac{dy}{dx} = \frac{\cancel{2}(2-x)}{\cancel{2}y}$$

$$\boxed{\frac{dy}{dx} = \frac{2-x}{y}}$$

Example #3

Find $\frac{dy}{dx}$ if $y^2 - xy - x^2 + 4 = 0$

Solution:

$y^2 - xy - x^2 + 4 = 0$

Differentiate w.r.t 'x'

$$\frac{d}{dx}(y^2) - \frac{d}{dx}(xy) - \frac{d}{dx}(x^2) + \frac{d}{dx}(4) = 0$$

$$2y\frac{dy}{dx} - \left[x\frac{d}{dx}(y) + y\frac{d}{dx}(x)\right] - 2x + 0 = 0$$

$$2y\frac{dy}{dx} - \left[x\frac{dy}{dx} + y(1)\right] - 2x = 0$$

$$2y\frac{dy}{dx} - x\frac{dy}{dx} - y - 2x = 0$$

$$\frac{dy}{dx}\left[2y - x\right] = 2x + y$$

$$\boxed{\frac{dy}{dx} = \frac{2x + y}{2y - x}}$$

Example #4

Find $\frac{dy}{dx}$ if $y^3 - 2xy^2 + x^2y + 3x = 0$

Solution:

$y^3 - 2xy^2 + x^2y + 3x = 0$

Differentiate w.r.t 'x'

$$\frac{d}{dx}\left[y^3 - 2xy^2 + x^2y + 3x\right] = \frac{d}{dx}(0)$$

$$\frac{d}{dx}(y^3) - 2\frac{d}{dx}\left[xy^2\right) + \frac{d}{dx}(x^2y) + 3\frac{d}{dx}(x) = 0$$

$$3y^2\frac{dy}{dx} - 2\left[x\frac{d}{dx}(y^2) + y^2\frac{d}{dx}(x)\right] + \left[x^2\frac{d}{dx}(y) + y\frac{d}{dx}(x^2)\right] + 3(1) = 0$$

$$3y^2\frac{dy}{dx} - 2\left[x \cdot 2y\frac{dy}{dx} + y^2(1)\right] + \left[x^2\frac{dy}{dx} + y(2x)\right] + 3 = 0$$

$$3y^2\frac{dy}{dx} - 2\left[2xy\frac{dy}{dx} + y^2\right] + \left[x^2\frac{dy}{dx} + 2xy\right] + 3 = 0$$

$$3y^2\frac{dy}{dx} - 4xy\frac{dy}{dx} - 2y^2 + x^2\frac{dy}{dx} + 2xy + 3 = 0$$

$$\frac{dy}{dx}\left[3y^2 - 4xy + x^2\right] = 2y^2 - 2xy - 3$$

$$\boxed{\frac{dy}{dx} = \frac{2y^2 - 2xy - 3}{3y^2 - 4xy + x^2}}$$

Example #5

Differentiate $x^2 + \frac{1}{x^2}$ w.r.t $x - \frac{1}{x}$

Solution:

Let

$y = x^2 + \frac{1}{x^2}$

Differentiate w.r.t 'x'

$$\frac{dy}{dx} = \frac{d}{dx}\left[x^2 + \frac{1}{x^2}\right]$$

$$\frac{dy}{dx} = \frac{d}{dx}\left[x^2 + x^{-2}\right]$$

$$\frac{dy}{dx} = 2x - 2x^{-3}$$

$$\frac{dy}{dx} = 2x - \frac{2}{x^3}$$

$$\frac{dy}{dx} = 2\left[x - \frac{1}{x^3}\right]$$

$$\frac{dy}{dx} = 2\left[\frac{x^4 - 1}{x^3}\right]$$

$$\frac{dy}{dx} = \frac{2(x^2-1)(x^2+1)}{x^3}$$

$$u = x - \frac{1}{x}$$

$$u = x - x^{-1}$$

Differentiate w.r.t "x"

$$\frac{du}{dx} = \frac{d}{dx}(x - x^{-1})$$

$$\frac{du}{dx} = 1 - (-1x^{-2})$$

$$\frac{du}{dx} = 1 + \frac{1}{x^2}$$

$$\frac{du}{dx} = \frac{x^2+1}{x^2}$$

Using chain rule

$$\frac{dy}{du} = \frac{dy}{dx} \cdot \frac{dx}{du}$$

$$\frac{dy}{du} = \frac{2(x^2-1)\cancel{(x^2+1)}}{x^{\cancel{3}\,1}} \cdot \frac{\cancel{x^2}}{\cancel{(x^2+1)}}$$

$$\frac{dy}{du} = \frac{2(x^2-1)}{x}$$

$$\boxed{\frac{dy}{du} = 2\left[x - \frac{1}{x}\right]}$$

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Exercise # 2.4

Question # 1

Find $\frac{dy}{dx}$ by making suitable substitutions in the following functions defined as:

(i)

$$y = \sqrt{\frac{1-x}{1+x}}$$

Solution:

Let $u = \frac{1-x}{1+x}$ =>

Differentiate w.r.t "x"

$$\frac{du}{dx} = \frac{d}{dx}\left[\frac{1-x}{1+x}\right]$$

$$\frac{du}{dx} = \frac{(1+x)\frac{d}{dx}(1-x) - (1-x)\frac{d}{dx}(1+x)}{(1+x)^2}$$

$$\frac{du}{dx} = \frac{(1+x)(-1) - (1-x)(1)}{(1+x)^2}$$

$$\frac{du}{dx} = \frac{-1\cancel{-x}-1\cancel{+x}}{(1+x)^2}$$

$$\frac{du}{dx} = \frac{-2}{(1+x)^2}$$

$y = \sqrt{u}$

$y = u^{1/2}$

Differentiate w.r.t "u"

$$\frac{dy}{du} = \frac{d}{du}\left(u^{1/2}\right)$$

$$\frac{dy}{du} = \frac{1}{2}u^{\frac{1}{2}-1}$$

$$\frac{dy}{du} = \frac{1}{2}u^{-\frac{1}{2}}$$

$$\frac{dy}{du} = \frac{1}{2u^{1/2}}$$

$$\frac{dy}{du} = \frac{1}{2\sqrt{u}}$$

$$\frac{dy}{du} = \frac{1}{2\sqrt{\frac{1-x}{1+x}}}$$

$$\frac{dy}{du} = \frac{(1+x)^{1/2}}{2(1-x)^{1/2}}$$

Using Chain rule

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$\frac{dy}{dx} = \frac{(1+x)^{\frac{1}{2}}}{2(1-x)^{\frac{1}{2}}} \cdot \frac{-2}{(1+x)^2}$$

$$\frac{dy}{dx} = \frac{-1}{(1-x)^{\frac{1}{2}}(1+x)^{2-\frac{1}{2}}}$$

$$\boxed{\frac{dy}{dx} = \frac{-1}{\sqrt{1-x}\,(1+x)^{3/2}}}$$

(ii)

$$y = \sqrt{x + \sqrt{x}}$$

Solution:

Let $u = x + \sqrt{x}$ =>

Differentiate w.r.t "x"

$$\frac{du}{dx} = \frac{d}{dx}\left(x + x^{1/2}\right)$$

$$\frac{du}{dx} = 1 + \frac{1}{2}x^{\frac{1}{2}-1}$$

$$\frac{du}{dx} = 1 + \frac{1}{2}x^{-\frac{1}{2}}$$

$$\frac{du}{dx} = 1 + \frac{1}{2\sqrt{x}}$$

$$\frac{du}{dx} = \frac{2\sqrt{x}+1}{2\sqrt{x}}$$

$y = \sqrt{u}$

$y = u^{1/2}$

Differentiate w.r.t "u"

$$\frac{dy}{du} = \frac{d}{du}\left(u^{1/2}\right)$$

$$\frac{dy}{du} = \frac{1}{2}u^{\frac{1}{2}-1}$$

$$\frac{dy}{du} = \frac{1}{2}u^{-\frac{1}{2}}$$

$$\frac{dy}{du} = \frac{1}{2\sqrt{u}}$$

$$\frac{dy}{du} = \frac{1}{2\sqrt{x+\sqrt{x}}}$$

Using chain rule

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x+\sqrt{x}}} \cdot \frac{2\sqrt{x}+1}{2\sqrt{x}}$$

$$\boxed{\frac{dy}{dx} = \frac{2\sqrt{x}+1}{4\sqrt{x}\,\sqrt{x+\sqrt{x}}}}$$

(iii)

$$y = x\sqrt{\frac{a+x}{a-x}}$$

Solution:

Let

$$u = \frac{a+x}{a-x} \Rightarrow$$

Differentiate w.r.t 'x'

$$\frac{du}{dx} = \frac{d}{dx}\left[\frac{a+x}{a-x}\right]$$

$$\frac{du}{dx} = \frac{(a-x)\frac{d}{dx}(a+x) - (a+x)\frac{d}{dx}(a-x)}{(a-x)^2}$$

$$\frac{du}{dx} = \frac{(a-x)(1) - (a+x)(-1)}{(a-x)^2}$$

$$\frac{du}{dx} = \frac{a-\cancel{x}+a+\cancel{x}}{(a-x)^2}$$

$$\frac{du}{dx} = \frac{2a}{(a-x)^2}$$

$$y = x\sqrt{u}$$

$$y = xu^{1/2}$$

Differentiate w.r.t 'u'

$$\frac{dy}{du} = \frac{d}{du}\left(x \cdot u^{1/2}\right)$$

$$\frac{dy}{du} = x \cdot \frac{d}{du}u^{1/2} + u^{1/2}\frac{d}{dx}(x)$$

$$\frac{dy}{dx} = x \cdot \frac{1}{2}u^{\frac{1}{2}-1} + u^{1/2}(1)$$

$$\frac{dy}{du} = x \cdot \frac{1}{2}u^{\frac{-1}{2}} + u^{1/2}$$

$$\frac{dy}{du} = x \cdot \frac{1}{2\sqrt{u}} + u^{1/2}$$

$$\frac{dy}{du} = x \cdot \frac{1}{2\sqrt{\frac{a+x}{a-x}}} + \sqrt{\frac{a+x}{a-x}}$$

Using chain rule

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$\frac{dy}{dx} = \frac{x}{2\sqrt{\frac{a+x}{a-x}}} + \sqrt{\frac{a+x}{a-x}} \cdot \frac{2a}{(a-x)^2}$$

$$\frac{dy}{dx} = \frac{x}{\cancel{2}\sqrt{\frac{a+x}{a-x}}} \cdot \frac{\cancel{2}a}{(a-x)^2} + \frac{(a+x)^{1/2}}{(a-x)^{1/2}}$$

$$\frac{dy}{dx} = \frac{x(a-x)^{1/2} \cdot a}{\sqrt{a+x}\,(a-x)^2} + \frac{(a+x)^{1/2}}{(a-x)^{1/2}}$$

$$\frac{dy}{dx} = \frac{x \cdot a}{\sqrt{a+x}\,(a-x)^{2-\frac{1}{2}}} + \frac{(a+x)^{1/2}}{(a-x)^{1/2}}$$

$$\frac{dy}{dx} = \frac{xa}{\sqrt{a+x}\,(a-x)^{3/2}} + \frac{(a+x)^{1/2}}{(a-x)^{1/2}}$$

$$\frac{dy}{dx} = \frac{xa+(a+x)(a-x)}{\sqrt{a+x}\,(a-x)^{3/2}}$$

$$\boxed{\frac{dy}{dx} = \frac{a^2-x^2+ax}{\sqrt{a+x}\,(a-x)^{3/2}}}$$

(iv)

$$y = (3x^2-2x+7)^6$$

Solution:

Let

$$u = 3x^2-2x+7 \Rightarrow$$

Differentiate w.r.t 'x'

$$\frac{du}{dx} = \frac{d}{dx}(3x^2-2x+7)$$

$$\frac{du}{dx} = 3(2x^{2-1}) - 2(1) + 0$$

$$\frac{du}{dx} = 6x - 2$$

$$y = (u)^6$$

Differentiate w.r.t 'x'

$$\frac{dy}{du} = \frac{d}{du}(u)^6$$

$$\frac{dy}{du} = 6u^{6-1}$$

$$\frac{dy}{du} = 6u^5$$

$$\frac{dy}{du} = 6(3x^2-2x+7)^5$$

Chain rule

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$\frac{dy}{dx} = 6(3x^2-2x+7)^5 \cdot (6x-2)$$

$$\frac{dy}{dx} = 6(3x^2-2x+7)^5 \cdot 2(3x-1)$$

$$\boxed{\frac{dy}{dx} = 12(3x^2-2x+7)^5 \cdot (3x-1)}$$

* * *

(V)

$$\sqrt{\frac{a^2+x^2}{a^2-x^2}}$$

Solution:

$$\text{Let } u = \frac{a^2+x^2}{a^2-x^2} \Rightarrow$$

Differentiate w.r.t 'x'

$$\frac{du}{dx} = \frac{d}{dx}\left[\frac{a^2+x^2}{a^2-x^2}\right]$$

$$\frac{du}{dx} = \frac{(a^2-x^2)\frac{d}{dx}(a^2+x^2) - (a^2+x^2)\frac{d}{dx}(a^2-x^2)}{(a^2-x^2)^2}$$

$$\frac{du}{dx} = \frac{(a^2-x^2)(2x) - (a^2+x^2)(-2x)}{(a^2-x^2)^2}$$

$$\frac{du}{dx} = \frac{2a^2x - \cancel{2x^3} + 2a^2x + \cancel{2x^3}}{(a^2-x^2)^2}$$

$$y = \sqrt{u}$$

Differentiate w.r.t 'u'

$$\frac{dy}{du} = \frac{d}{du}\left(u^{1/2}\right)$$

$$\frac{dy}{du} = \frac{1}{2}u^{\frac{1}{2}-1}$$

$$\frac{dy}{du} = \frac{1}{2}u^{\frac{-1}{2}}$$

$$\frac{dy}{du} = \frac{1}{4\sqrt{u}}$$

$$\frac{du}{dx} = \frac{4a^2x}{(a^2-x^2)^2}$$

$$\frac{du}{dx} = \frac{4a^2x}{(a^2-x^2)^2}$$

$$\frac{dy}{du} = \frac{1}{2\sqrt{\frac{a^2+x^2}{a^2-x^2}}}$$

$$\frac{dy}{du} = \frac{(a^2-x^2)^{1/2}}{2(a^2+x^2)^{1/2}}$$

Chain rule

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$\frac{dy}{dx} = \frac{(a^2-x^2)^{1/2}}{2(a^2+x^2)^{1/2}} \cdot \frac{4a^2x}{(a^2-x^2)^2}$$

$$\frac{dy}{dx} = \frac{2a^2x}{\sqrt{a^2+x^2}\,(a^2-x^2)^{2-\frac{1}{2}}}$$

$$\boxed{\frac{dy}{dx} = \frac{2a^2x}{\sqrt{a^2+x^2}\,(a^2-x^2)^{\frac{3}{2}}}}$$

* * *

Question #2

Find $\frac{dy}{dx}$ if:

(i)

$$3x+4y+7=0$$

Solution:

$$3x+4y+7=0$$

Differentiate w.r.t 'x'

$$3\frac{d}{dx}(x)+4\frac{d}{dx}+\frac{d}{dx}(7)+0=0$$

$$3(1)+4\frac{dy}{dx}+0=0$$

$$3+4\frac{dy}{dx}=0$$

$$\boxed{\frac{dy}{dx} = \frac{-3}{4}}$$

* * *

(ii)

$$xy+y^2=2$$

Solution:

$$xy+y^2=2$$

Differentiate w.r.t 'x'

$$\frac{d}{dx}(xy)+\frac{d}{dx}(y^2)=\frac{d}{dx}(2)$$

$$x\cdot\frac{dy}{dx}+y\cdot\frac{d}{dx}(x)+2y\frac{dy}{dx}=0$$

$$x\frac{dy}{dx}+y+2y\frac{dy}{dx}=0$$

$$\frac{dy}{dx}\left[x+2y\right]+y=0$$

$$\frac{dy}{dx}\left[x+2y\right]=-y$$

$$\boxed{\frac{dy}{dx} = \frac{-y}{x+2y}}$$

* * *

(iii)

$$4x^2+2hxy+by^2+2gx+2fy+c=0$$

Solution:

$$4x^2+2hxy+by^2+2gx+2fy+c=0$$

Differentiate w.r.t 'x'

$$\frac{d}{dx}(4x^2)+2h\frac{d}{dx}(xy)+b\frac{d}{dx}(y^2)+2g\frac{d}{dx}(x)+2f\frac{d}{dx}(y)=0$$

$$4(2x)+2h\left[x\frac{d}{dx}(y)+y\frac{d}{dx}(x)\right]+b\,2y\cdot\frac{dy}{dx}+2g(1)+2f\frac{dy}{dx}=0$$

$$8x+2h\left[x\frac{dy}{dx}+y(1)\right]+2by\frac{dy}{dx}+2g+2f\frac{dy}{dx}=0$$

$$8x+2hx\frac{dy}{dx}+2hy+2by\frac{dy}{dx}+2g+2f\frac{dy}{dx}=0$$

$$8x+2hy+2g+2hx\frac{dy}{dx}+2by\frac{dy}{dx}+2f\frac{dy}{dx}=0$$

$$2(4x+hy+g)+2\left[hx+by+f\right]\frac{dy}{dx}=0$$

$$(4x+hy+g)+(hx+by+f)\frac{dy}{dx}=0$$

$$\boxed{\frac{dy}{dx} = -\frac{4x+hy+g}{hx+by+f}}$$

(iii)

$$x^2-4xy-5y=0$$

Solution:

$x^2-4xy-5y=0$

Differentiate w.r.t "x"

$$\frac{d}{dx}\left(x^2-4xy-5y\right)=0$$

$$\frac{d}{dx}(x^2)-4\frac{d}{dx}(xy)-5\frac{d}{dx}(y)=0$$

$$2x-4\left[x\frac{d}{dx}(y)+y\frac{d}{dx}(x)\right]-5\frac{dy}{dx}=0$$

$$2x-4\left[x\frac{dy}{dx}+y\,(1)\right]-5\frac{dy}{dx}=0$$

$$2x-4x\frac{dy}{dx}-4y-5\frac{dy}{dx}=0$$

$$-4x-5\frac{dy}{dx}+2x-4y=0$$

$$-\frac{dy}{dx}\left[4x+5\right]+2x-4y=0$$

$$2x-4y=\frac{dy}{dx}\left[4x+5\right]$$

$$\boxed{\frac{dy}{dx}=\frac{2x-4y}{4x+5}}$$

(v)

$$x\sqrt{1+y}+y\sqrt{1+x}=0$$

Solution:

$x\sqrt{1+y}+y\sqrt{1+x}=0$

Differentiate w.r.t 'x'

$$\frac{d}{dx}\left(x\sqrt{1+y}\right)+\frac{d}{dx}\left[y\sqrt{1+x}\right]=0$$

$$x\frac{d}{dx}(1+y)^{1/2}+\sqrt{1+y}\,\frac{d}{dx}(x)+y\frac{d}{dx}(1+x)^{1/2}+\sqrt{1+x}\,\frac{d}{dx}(y)=0$$

$$x\cdot\frac{1}{2}(1+y)^{\frac{-1}{2}}+\sqrt{1+y}\,(1)+y\cdot\frac{1}{2}(1+x)^{\frac{-1}{2}}+\sqrt{1+x}\,\frac{dy}{dx}=0$$

$$x\cdot\frac{1}{2\sqrt{1+y}}\frac{dy}{dx}+\sqrt{1+y}+\frac{y}{2\sqrt{1+x}}+\sqrt{1+x}\,\frac{dy}{dx}=0$$

$$\frac{x}{2\sqrt{1+y}}\frac{dy}{dx}+\sqrt{1+x}\,\frac{dy}{dx}=-\sqrt{1+y}-\frac{y}{2\sqrt{1+x}}$$

$$\left[\frac{x}{2\sqrt{1+y}}+\sqrt{1+x}\right]\frac{dy}{dx}=-\sqrt{1+y}-\frac{y}{2\sqrt{1+x}}$$

$$\left[\frac{x+2\sqrt{1+y}\,\sqrt{1+x}}{2\sqrt{1+y}}\right]\frac{dy}{dx}=-\left[\frac{2\sqrt{1+x}\,\sqrt{1+y}+y}{2\sqrt{1+x}}\right]$$

$$\frac{dy}{dx}=\frac{-2\sqrt{(1+x)(1+y)}+y}{\cancel{2}\sqrt{1+x}}\times\frac{\cancel{2}\sqrt{1+y}}{x+2\sqrt{(1+y)(1+x)}}$$

$$\boxed{\frac{dy}{dx}=\frac{\left(-2\sqrt{(1+x)(1+y)}+y\right)\sqrt{1+y}}{\sqrt{1+x}\left(x+2\sqrt{(1+y)(1+x)}\right)}}$$

(vi)

$$y(x^2-1)=x\sqrt{x^2+4}$$

Solution:

$y(x^2-1)=x\sqrt{x^2+4}$

$$\frac{d}{dx}\left(y\,(x^2-1)\right)=\frac{d}{dx}\left(x\sqrt{x^2+4}\right)$$

$$y\cdot\frac{d}{dx}(x^2-1)+(x^2-1)\frac{d}{dx}(y)=x\cdot\frac{d}{dx}(x^2+4)^{1/2}+\sqrt{x^2+4}\,\frac{d}{dx}(x)$$

$$y\,(2x)+(x^2-1)\frac{dy}{dx}=x\cdot\frac{1}{2}(x^2+4)^{\frac{1}{2}-1}\frac{d}{dx}(x^2+4)+\sqrt{x^2+4}\,(1)$$

$$2xy+(x^2-1)\frac{dy}{dx}=x\cdot\frac{1}{\cancel{2}}(x^2+4)^{\frac{-1}{2}}(\cancel{2}x)+\sqrt{x^2+4}$$

$$2xy+(x^2-1)\frac{dy}{dx}=\frac{x^2}{\sqrt{x^2+4}}+\sqrt{x^2+4}$$

$$2xy + (x^2-1)\frac{dy}{dx} = \frac{x^2+(x^2+4)}{\sqrt{x^2+4}}$$

$$2xy + (x^2-1)\frac{dy}{dx} = \frac{x^2+x^2+4}{\sqrt{x^2+4}}$$

$$(x^2-1)\frac{dy}{dx} = \frac{2x^2+4}{\sqrt{x^2+4}} - 2xy$$

$$(x^2-1)\frac{dy}{dx} = \frac{2x^2+4}{\sqrt{x^2+4}} - 2x\left[\frac{x\sqrt{x^2+4}}{x^2-1}\right]$$

$$(x^2-1)\frac{dy}{dx} = \frac{2x^2+4^{(x^2-1)}-2x^2(x^2+4)}{(\sqrt{x^2+4})(x^2-1)}$$

$$(x^2-1)\frac{dy}{dx} = \frac{\cancel{2x^4}-2x^2+4x^2-4-\cancel{2x^4}-8x^2}{(\sqrt{x^2+4})\ (x^2-1)}$$

$$\frac{dy}{dx} = \frac{-6x^2-4}{\sqrt{x^2+4}\ (x^2-1)^2}$$

$$\boxed{\frac{dy}{dx} = \frac{-2(3x^2+2)}{\sqrt{x^2+4}\ (x^2-1)^2}}$$

(ii)

$$x = \frac{a(1-t^2)}{1+t^2},\ y = \frac{2bt}{1+t^2}$$

Solution:

$$x = \frac{a(1-t^2)}{1+t^2} \quad \text{———} ①$$

$$y = \frac{2bt}{1+t^2} \quad \text{———} ②$$

Differentiate eq ① w.r.t 't'

$$\frac{dx}{dt} = \frac{d}{dt}\left[\frac{x(1-t^2)}{(1+t^2)}\right]$$

$$\frac{dx}{dt} = \frac{(1+t^2)a\frac{d}{dt}(1-t^2) - a(1-t^2)\frac{d}{dt}(1+t^2)}{(1+t^2)^2}$$

$$\frac{dx}{dt} = \frac{(1+t^2)\cdot a(-2t) - a(1-t^2)(2t)}{(1+t^2)^2}$$

$$\frac{dx}{dt} = \frac{-2at(1+t^2) - 2at(1-t^2)}{(1+t^2)^2}$$

Question #3

Find $\frac{dy}{dx}$ of the following parametric functions:

(i)

$$x = \theta + \frac{1}{\theta} \text{ and } y = \theta + 1$$

Solution:

$$x = \theta + \frac{1}{\theta}$$

$$x = \theta + \theta^{-1} \quad \text{———} ①$$

$$y = \theta + 1 \quad \text{———} ②$$

Differentiate eq ① w.r.t 'θ'

$$\frac{dx}{d\theta} = \frac{d}{d\theta}(\theta + \theta^{-1})$$

$$\frac{dx}{d\theta} = \frac{d}{d\theta}(\theta) + \frac{d}{d\theta}(\theta^{-1})$$

$$\frac{dx}{d\theta} = 1 + (-\theta^{-1-1})$$

$$= 1 - \theta^{-2}$$

$$\frac{dx}{d\theta} = 1 - \frac{1}{\theta^2}$$

$$\frac{dx}{d\theta} = \frac{\theta^2-1}{\theta^2}$$

Differentiat eq ② w.r.t "θ"

$$\frac{dy}{d\theta} = \frac{d}{d\theta}(\theta+1)$$

$$\frac{dy}{d\theta} = \frac{d}{d\theta}(\theta) + \frac{d}{d\theta}(1)$$

$$\frac{dy}{d\theta} = 1 + 0$$

$$\frac{dy}{d\theta} = 1$$

$$\frac{dx}{dt} = \frac{-2at - 2at^3 - 2at + 2at^3}{(1+t^2)^2}$$

$$\frac{dx}{dt} = \frac{-4at}{(1+t^2)^2}$$

Differentiate eq ② w.r.t 't'

$$\frac{dy}{dt} = \frac{d}{dt}\left(\frac{2bt}{1+t^2}\right)$$

$$\frac{dy}{dt} = \frac{(1+t^2)\frac{d}{dt}(2bt) - 2bt\frac{d}{dt}(1+t^2)}{(1+t^2)^2}$$

$$\frac{dy}{dt} = \frac{(1+t^2)(2b)\frac{d}{dt}(t) - 2bt(2t)}{(1+t^2)^2}$$

$$\frac{dy}{dt} = \frac{2b + 2bt^2 - 4bt^2}{(1+t^2)^2}$$

$$\frac{dy}{dt} = \frac{2b - 2bt^2}{(1+t^2)^2} \Rightarrow \frac{2b(1-t^2)}{(1+t^2)^2}$$

Using chain rule

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$

$$\frac{dy}{dx} = \frac{\cancel{2}b(1-t^2)}{\cancel{(1+t^2)^2}} \cdot \frac{\cancel{(1+t^2)^2}}{-\underset{2}{\cancel{4}}at}$$

$$\boxed{\frac{dy}{dx} = \frac{-b(1-t^2)}{2at}}$$

* * *

Using chain rule

$$\frac{dy}{dx} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dx}$$

$$\frac{dy}{dx} = 1 \cdot \frac{\theta^2}{\theta^2 - 1}$$

$$\boxed{\frac{dy}{dx} = \frac{\theta^2}{\theta^2 - 1}}$$

* * *

Question #4

Prove that $y\frac{dy}{dx} + x = 0$ **if**

$$x = \frac{1-t^2}{1+t^2}, \quad y = \frac{2t}{1+t^2}$$

Solution:

$$x = \frac{1-t^2}{1+t^2}$$

Differentiate w.r.t 't'

$$\frac{dx}{dt} = \frac{d}{dt}\left[\frac{1-t^2}{1+t^2}\right]$$

$$\frac{dx}{dt} = \frac{(1+t^2)\frac{d}{dt}(1-t^2) - (1-t^2)\frac{d}{dt}(1+t^2)}{(1+t^2)^2}$$

$$\frac{dx}{dt} = \frac{(1+t^2)(-2t) - (1-t^2)(2t)}{(1+t^2)^2}$$

$$\frac{dx}{dt} = \frac{-2t - 2t^3 - 2t + 2t^3}{(1+t^2)^2}$$

$$\frac{dx}{dt} = \frac{-4t}{(1+t^2)^2}$$

$$y = \frac{2t}{(1+t^2)}$$

Differentiate w.r.t 't'

$$\frac{dy}{dt} = \frac{d}{dt}\left(\frac{2t}{1+t^2}\right)$$

$$\frac{dy}{dt} = \frac{(1+t^2)\frac{d}{dt}(2t) - 2t\frac{d}{dt}(1+t^2)}{(1+t^2)^2}$$

$$\frac{dy}{dt} = \frac{(1+t^2)(2) - 2t(2t)}{(1+t^2)^2}$$

$$\frac{dy}{dt} = \frac{2+2t^2-4t^2}{(2+t^2)^2}$$

$$\frac{dy}{dt} = \frac{2-2t^2}{(1+t^2)^2}$$

$$\frac{dy}{dt} = \frac{2(1-t^2)}{(1+t^2)^2}$$

Using chain rule

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$

$$\frac{dy}{dx} = \frac{\cancel{2}(1-t^2)}{\cancel{(1+t^2)^2}} \cdot -\frac{\cancel{(1+t^2)^2}}{{}_2\cancel{4}t}$$

$$\frac{dy}{dx} = \frac{-(1-t^2)}{2t}$$

L.H.S

$$= y\frac{dy}{dx} + x$$

$$= \frac{\cancel{2t}}{(1+t^2)} \cdot \frac{-(1-t^2)}{\cancel{2t}} + \frac{1-t^2}{1+t^2}$$

$$= \frac{-(1-t^2)}{(2+t^2)} + \frac{(1-t^2)}{(1+t^2)}$$

$$= \frac{-(1-t^2)+(1-t^2)}{(1+t^2)}$$

$$= \frac{0}{(1+t^2)}$$

$= 0 =$ R.H.S.

Hence proved $y\frac{dy}{dx} + x = 0$

Question #5

Differentiate

(i)

$x^2 - \frac{1}{x^2}$ w.r.t x^4.

Solution:

Let

$$y = x^2 - \frac{1}{x^2}$$

$$y = x^2 - x^{-2}$$

Differentiate w.r.t 'x'

$$\frac{dy}{dx} = \frac{d}{dx}(x^2 - x^{-2})$$

$$\frac{dy}{dx} = 2x + 2x^{-3}$$

$$\frac{dy}{dx} = 2x + \frac{2}{x^3}$$

$$\frac{dy}{dx} = 2\left[x + \frac{1}{x^3}\right]$$

$$\frac{dy}{dx} = 2\left[\frac{x^4+1}{x^3}\right]$$

Let

$$u = x^4$$

Differentiate w.r.t 'x'

$$\frac{du}{dx} = \frac{d}{dx}(x^4)$$

$$\frac{du}{dx} = 4x^3$$

Using chain rule

$$\frac{dy}{du} = \frac{dy}{dx} \cdot \frac{dx}{du}$$

$$\frac{dy}{du} = \cancel{2}\left[\frac{x^4+1}{x^3}\right] \cdot \frac{1}{{}_2\cancel{4}x^3}$$

$$\boxed{\frac{dy}{du} = \frac{x^4+1}{2x^6}}$$

* * *

(ii)

$(1+x^2)^n$ w.r.t x^2.

Solution:

Let $y=(1+x^2)^n$

Differentiate w.r.t 'x'

$$\frac{dy}{dx}=\frac{d}{dx}(1+x^2)^n$$

$$\frac{dy}{dx}=n(1+x^2)^{n-1}\frac{d}{dx}(1+x^2)$$

$$\frac{dy}{dx}=n(1+x^2)^{n-1}(2x)$$

$$\frac{dy}{dx}=2xn(1+x^2)^{n-1}$$

Let

$u=x^2$

Differentiate w.r.t 'x'

$$\frac{du}{dx}=\frac{d}{dx}(x^2)$$

$$\frac{du}{dx}=2x$$

Using chain rule.

$$\frac{dy}{dx}=\frac{dy}{dx}\cdot\frac{dx}{du}$$

$$\frac{dy}{dx}=\cancel{2x}n(1+x^2)^{n-1}\cdot\frac{1}{\cancel{2x}}$$

$$\boxed{\frac{dy}{dx}=n(1+x^2)^{n-1}}$$

• ——— •

(iii)

$\frac{x^2+1}{x^2-1}$ w.r.t $\frac{x-1}{x+1}$

Solution:

$y=\frac{x^2+1}{x^2-1}$

Differentiate w.r.t 'x'

$$\frac{dy}{dx}=\frac{d}{dx}\left[\frac{x^2+1}{x^2-1}\right]$$

$$\frac{dy}{dx}=\frac{(x^2-1)\frac{d}{dx}(x^2+1)-(x^2+1)\frac{d}{dx}(x^2-1)}{(x^2-1)^2}$$

$$\frac{dy}{dx}=\frac{(x^2-1)(2x)-(x^2+1)(2x)}{(x^2-1)^2}$$

$$\frac{dy}{dx}=\frac{\cancel{2x^3}-2x-\cancel{2x^3}-2x}{(x^2-1)^2}$$

$$\frac{dy}{dx}=\frac{-4x}{(x^2-1)^2}$$

Let

$u=\frac{x-1}{x+1}$

Differentiate w.r.t 'x'

$$\frac{du}{dx}=\frac{d}{dx}\left(\frac{x-1}{x+1}\right)$$

$$\frac{du}{dx}=\frac{(x+1)\frac{d}{dx}(x-1)-(x-1)\frac{d}{dx}(x+1)}{(x+1)^2}$$

$$\frac{du}{dx}=\frac{(x+1)-(x-1)}{(x+1)^2}$$

$$\frac{du}{dx}=\frac{\cancel{x}+1-\cancel{x}+1}{(x+1)^2}$$

$$\frac{du}{dx}=\frac{2}{(x+1)^2}$$

Using chain rule

$$\frac{dy}{du}=\frac{dy}{dx}\cdot\frac{dx}{du}$$

$$\frac{dy}{du}=\frac{-\overset{2}{\cancel{4}}x}{(x^2-1)^2}\cdot\frac{(x+1)^2}{\cancel{2}}$$

$$\frac{dy}{du}=\frac{-2x\cancel{(x+1)^2}}{\cancel{(x+1)^2}(x-1)^2}$$

$$\boxed{\frac{dy}{du}=\frac{-2x}{(x-1)^2}}$$

(iv)

$\frac{ax+b}{cx+d}$ w.r.t $\frac{ax^2+b}{ax^2+d}$

Solution:

Let $y=\frac{ax+b}{cx+d}$

Differentiate w.r.t 'x'

$$\frac{dy}{dx}=\frac{d}{dx}\left(\frac{ax+b}{cx+d}\right)$$

$$\frac{dy}{dx}=\frac{(cx+d)\frac{d}{dx}(ax+b)-(ax+b)\frac{d}{dx}(cx+d)}{(cx+d)^2}$$

$$\frac{dy}{dx}=\frac{(cx+d)(a)-(ax+b)(c)}{(cx+d)^2}$$

$$\frac{dy}{dx}=\frac{\cancel{acx}+ad-\cancel{acx}-bc}{(cx+d)^2}$$

$$\frac{dy}{dx}=\frac{ad-bc}{(cx+d)^2}$$

Let $u=\frac{ax^2+b}{ax^2+d}$

Differentiate w.r.t 'x'

$$\frac{du}{dx}=\frac{d}{dx}\left(\frac{ax^2+b}{ax^2+d}\right)$$

$$\frac{du}{dx}=\frac{(ax^2+d)\frac{d}{dx}(ax^2+b)-(ax^2+b)\frac{d}{dx}(ax^2+d)}{(ax^2+d)^2}$$

$$\frac{du}{dx}=\frac{(ax^2+d)(2ax)-(ax^2+b)(2ax)}{(ax^2+d)^2}$$

$$\frac{du}{dx}=\frac{\cancel{2a^2x^3}+2axd-\cancel{2a^2x^3}-2abx}{(ax^2+d)^2}$$

$$\frac{du}{dx}=\frac{2ax(d-b)}{(ax^2+d)^2}$$

Using chain rule

$$\frac{dy}{du}=\frac{dy}{dx}\cdot\frac{dx}{du}$$

$$\frac{dy}{du}=\frac{ad-bc}{(cx+d)^2}\cdot\frac{(ax^2+d)^2}{2ax(d-b)}$$

$$\boxed{\frac{dy}{du}=\frac{(ad-bc)(ax^2+d)^2}{(cx+d)^2(2ax(d-b))}}$$

(v)

$\frac{x^2+1}{x^2-1}$ w.r.t x^3.

Solution:

Let $y=\frac{x^2+1}{x^2-1}$

$$\frac{dy}{dx}=\frac{d}{dx}\left[\frac{x^2+1}{x^2-1}\right]$$

$$\frac{dy}{dx}=\frac{(x^2-1)\frac{d}{dx}(x^2+1)-(x^2+1)\frac{d}{dx}(x^2-1)}{(x^2-1)^2}$$

$$\frac{dy}{dx}=\frac{(x^2-1)(2x)-(x^2+1)(2x)}{(x^2-1)^2}$$

$$\frac{dy}{dx}=\frac{\cancel{2x^3}-2x-\cancel{2x^3}-2x}{(x^2-1)^2}$$

$$\frac{dy}{dx}=\frac{-4x}{(x^2-1)^2}$$

Let $u=x^3$

Differentiate w.r.t 'x'

$$\frac{du}{dx}=\frac{d}{dx}(x^3)$$

$$=3x^{3-1}$$

$$\frac{du}{dx}=3x^2$$

Using chain rule

$$\frac{dy}{du}=\frac{dy}{dx}\cdot\frac{dx}{du}$$

$$\frac{dy}{du}=\frac{-4\cancel{x}}{(x^2-1)^2}\cdot\frac{1}{3x^{\cancel{2}}}$$

$$\boxed{\frac{dy}{du}=\frac{-4}{3x(x^2-1)^2}}$$

Theory

Example #1

Find the derivative of tanx from first principle.

Solution:

Let $y = \tan x$ —— ①

$y + \delta y = \tan(x + \delta x)$ —— ②

Equation ② − Eq ①

$\cancel{y} + \delta y - \cancel{y} = \tan(x + \delta x) - \tan x$

$\delta y = \tan(x + \delta x) - \tan x$

$\delta y = \frac{\sin(x + \delta x)}{\cos(x + \delta x)} - \frac{\sin x}{\cos x}$

$\delta y = \frac{\sin(x + \delta x)\cos x - \cos(x + \delta x)\sin x}{\cos(x + \delta x)\cos x}$

$\delta y = \frac{\sin(x + \delta x - x)}{\cos(x + \delta x)\cos x}$

$\delta y = \frac{\sin \delta x}{\cos(x + \delta x)(\cos x)}$

Divided by δx both sides

$\frac{\delta y}{\delta x} = \frac{\sin \delta x}{\delta x} \cdot \frac{1}{\cos(x + \delta x)\cos x}$

Appling $\lim_{\delta x \to 0}$ both sides

$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} . \frac{\sin \delta x}{\delta x} . \lim_{\delta x \to 0} \frac{1}{\cos(x + \delta x)\cos x}$

$\frac{dy}{dx} = 1 \cdot \frac{1}{(\cos x)(\cos x)}$

$\frac{dy}{dx} = \frac{1}{\cos^2 x}$

$$\boxed{\frac{dy}{dx} = \sec^2 x}$$

Example #2

Differentiation ab-initio w.r.t "x"?

(i)

Cos2x

Solution:

Let $y = \cos 2x$ —— ①

$y + \delta y = \cos 2(x + \delta x)$ —— ②

Subtract eq ② − eq ①

$\cancel{y} + \delta y - \cancel{y} = \cos 2(x + \delta x) - \cos 2x$

$\delta y = \cos(2x + 2\delta x) - \cos 2x$

$\because \cos A - \cos B = -2 \sin \frac{A+B}{2} \sin \frac{A-B}{2}$

$\delta y = -2 \sin \frac{(2x + 2\delta x + 2x)}{2} \sin\left(\frac{\cancel{2x} + 2\delta x - \cancel{2x}}{2}\right)$

$\delta y = -2 \sin \frac{4x + 2\delta x}{2} \sin \frac{\cancel{2}\delta x}{\cancel{2}}$

$\delta y = -2 \sin \frac{\cancel{2}(2x + \delta x)}{\cancel{2}} \sin \delta x$

$\delta y = -2 \sin(2x + \delta x) \sin \delta x$

Divided by δx both sides

$\frac{\delta y}{\delta x} = \frac{-2 \sin(2x + \delta x) \sin \delta x}{\delta x}$

Apply $\lim_{\delta x \to 0}$ both sides

$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} -2 \sin(2x + \delta x) \cdot \lim_{\delta x \to 0} \frac{\sin \delta x}{\delta x}$

$\frac{dy}{dx} = -2 \sin 2x \ (1)$

$$\boxed{\frac{dy}{dx} = -2 \sin 2x}$$

* * *

(ii)

$\sin\sqrt{x}$

Solution:

Let $y = \sin\sqrt{x}$ —— ①

$y + \delta y = \sin\sqrt{x+\delta x}$ —— ②

Subtract eq ② by ①

$y + \delta y - y = \sin\sqrt{x+\delta x} - \sin\sqrt{x}$

$\delta y = \sin\sqrt{x+\delta x} - \sin\sqrt{x}$

$\because \sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$

$$\delta y = 2\cos\frac{\sqrt{x+\delta x}+\sqrt{x}}{2}\sin\frac{\sqrt{x+\delta x}-\sqrt{x}}{2}$$

Divided by δx both sides

$$\frac{\delta y}{\delta x} = \frac{2\cos\frac{\sqrt{x+\delta x}+\sqrt{x}}{2}\sin\frac{\sqrt{x+\delta x}-\sqrt{x}}{2}}{\delta x}$$

As $(\sqrt{x+\delta x}+\sqrt{x})(\sqrt{x+\delta x}-\sqrt{x}) = \delta x$

$$\frac{\delta y}{\delta x} = \frac{2\cos\frac{\sqrt{x+\delta x}+\sqrt{x}}{2}\sin\frac{\sqrt{x+\delta x}-\sqrt{x}}{2}}{(\sqrt{x+\delta x}+\sqrt{x})(\sqrt{x+\delta x}-\sqrt{x})}$$

$$\frac{\delta y}{\delta x} = \frac{\cancel{2}\cos\frac{\sqrt{x+\delta x}+\sqrt{x}}{2}}{\sqrt{x+\delta x}+\sqrt{x}} \cdot \frac{\sin\frac{\sqrt{x+\delta x}-\sqrt{x}}{2}}{\cancel{2}\left(\frac{\sqrt{x+\delta x}-\sqrt{x}}{2}\right)}$$

Apply $\lim\limits_{\delta x\to 0}$ both sides

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x} = \lim_{\delta x\to 0}\frac{\cos\frac{\sqrt{x+\delta x}+\sqrt{x}}{2}}{\sqrt{x+\delta x}+\sqrt{x}}$$

$$\frac{dy}{dx} = \frac{\cos\frac{\sqrt{x}+\sqrt{x}}{2}}{\sqrt{x}+\sqrt{x}}$$

$$\frac{dy}{dx} = \frac{\cos\frac{\cancel{2}\sqrt{x}}{\cancel{2}}}{2\sqrt{x}}$$

$$\boxed{\frac{dy}{dx} = \frac{\cos\sqrt{x}}{2\sqrt{x}}}$$

(iii)

$\cot^2 x$

Solution:

Let $y = \cot^2 x$ —— ①

$y + \delta y = \cot^2(x+\delta x)$ —— ②

Subtract eq ② by eq ①

$y + \delta y - y = \cot^2(x+\delta x) - \cot^2 x$

$$\delta y = [\cot(x+\delta x) + \cot x]\,[\cot(x+\delta x) - \cot x]$$

$$= [\cot(x+\delta x) + \cot x]\left[\frac{\cos(x+\delta x)}{\sin(x+\delta x)} - \frac{\cos x}{\sin x}\right]$$

$$\delta y = [\cot(x+\delta x) + \cot x]\left[\frac{\sin x\cos(x+\delta x) - \cos x\sin(x+\delta x)}{\sin x\ \sin(x+\delta x)}\right]$$

$\because \sin\alpha\cos\beta - \cos\alpha\sin\beta = \sin(\alpha - \beta)$

$$\delta y = [\cot(x+\delta x) + \cot x]\left[\frac{\sin(\cancel{x} - \cancel{x} - \delta x)}{\sin x(\sin(x+\delta x)}\right]$$

$$\delta y = [\cot(x+\delta x) + \cot x]\ \frac{\sin(-\delta x)}{\sin x\ \sin(x+\delta x)}$$

Divided by δx both sides

$$\frac{\delta y}{\delta x} = \frac{[\cot(x+\delta x) + \cot x]\ \frac{\sin\delta x}{\sin x(\sin(x+\delta x)}}{\delta x}$$

Apply $\lim\limits_{\delta x\to 0}$ both sides

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x} = \lim_{\delta x\to 0}\cot(x+\delta x)+\cot x \cdot \lim_{\delta x\to 0}\frac{\sin\delta x}{\delta x}\cdot\lim_{\delta x\to 0}\frac{-1}{\sin x\,\sin(x+\delta x)}$$

$$\frac{dy}{dx} = (\cot x + \cot x) - \frac{1}{(\sin x)\sin x}\cdot(1)$$

$$\frac{dy}{dx} = \frac{-2\cot x}{\sin^2 x}$$

$$\boxed{\frac{dy}{dx} = -2\cot x\,\operatorname{cosec}^2 x}$$

Example #3

Differentiate $\sin^3 x$ w.r.t. $\cos^2 x$.

Solution:

Let $y = \sin^3 x$

Differentiate w.r.t 'x'

$$\frac{dy}{dx} = \frac{d}{dx}(\sin^3 x)$$

$$\frac{dy}{dx} = 3\sin^2 x \cdot \frac{d}{dx}(\sin x)$$

$$\frac{dy}{dx} = 3\sin^2 x \cos x$$

$u = \cos^2 x$

Differentiate w.r.t 'x'

$$\frac{du}{dx} = \frac{d}{dx}(\cos^2 x)$$

$$\frac{du}{dx} = 2\cos x \frac{d}{dx}(\cos x)$$

$$\frac{du}{dx} = +2\cos x(-\sin x)$$

$$\frac{dy}{du} = \frac{dy}{dx} \cdot \frac{dx}{du}$$

$$\frac{dy}{du} = 3\sin^{\cancel{2}} x \cancel{\cos x} \cdot \frac{1}{-2\cancel{\cos x}\cancel{\sin x}}$$

$$\boxed{\frac{dy}{du} = -\frac{3}{2}\sin x}$$

Proof (2)

Let $y = \cos^{-1} x$

$\cos y = x$

Differentiate w.r.t "x"

$$\frac{d}{dx}(\cos y) = \frac{d}{dx}(x)$$

$$-\sin y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{-1}{\sin y}$$

$\because \sin\theta = \sqrt{1-\cos^2\theta}$

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-\cos^2 y}}$$

$\because \cos^2 y = x^2$

$$\boxed{\frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}}}$$ **Hence Proved**

Proof (1)

Let $y = \sin^{-1} x$

$x = \sin y$

Differentiate w.r.t "x"

$$\frac{d}{dx}(x) = \frac{d}{dx}(\sin y)$$

$$1 = \cos y \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{\cos y}$$

$\because 1 - \sin^2\theta = \cos^2 x$

$\because \sqrt{1-\sin^2\theta} = \cos x$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-\sin^2 y}}$$

$\because x^2 = \sin^2 y$

$$\boxed{\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}}$$ **Hence proved**

Proof (3)

Let $y = \tan^{-1} x$

$\tan y = x$

Differentiate w.r.t "x"

$$\frac{d}{dx}(\tan y) = \frac{d}{dx}(x)$$

$$\sec^2 y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\sec^2 y}$$

$\because 1 + \tan^2\theta = \sec^2\theta$

$$\frac{dy}{dx} = \frac{1}{1+\tan^2 y}$$

$\because \tan^2 y = x^2$

$$\boxed{\frac{dy}{dx} = \frac{1}{1+x^2}}$$ **Hence proved.**

Proof (4)

Let $y = \operatorname{cosec}^{-1} x$

$\operatorname{cosec} y = x$

Differentiate w.r.t 'x'

$$\frac{d}{dx}(\operatorname{cosec} y) = \frac{d}{dx}(x)$$

$$-\operatorname{cosec} y \cot y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = -\frac{1}{\operatorname{cosec} y \cot y}$$

$$\frac{dy}{dx} = \frac{-1}{\operatorname{cosec} y\left(\sqrt{\operatorname{cosec}^2 y - 1}\right)}$$

$\because 1 + \cot^2\theta = \operatorname{cosec}^2\theta$

$\because \cot\theta = \sqrt{\operatorname{cosec}^2\theta - 1}$

$$\boxed{\frac{dy}{dx} = \frac{-1}{x\sqrt{x^2-1}}}$$

$\because \operatorname{cosec}^2 y = x^2$

Hence proved.

Proof (6)

Let $y = \cot^{-1} x$

$\cot y = x$

Differentiate w.r.t 'x'

$$\frac{d}{dx}(\cot y) = \frac{d}{dx}(x)$$

$$-\operatorname{cosec}^2 x \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{-1}{\operatorname{cosec}^2 x}$$

$$\frac{dy}{dx} = \frac{-1}{1+\cot^2 x}$$

$\because 1 + \cot^2 x = \operatorname{cosec}^2 x$

$$\boxed{\frac{dy}{dx} = \frac{-1}{1+x^2}}$$

$\because \cot^2 x = x^2$

Proof (5)

Let $y = \sec^{-1} x$

$\sec y = x$

Differentiate w.r.t 'x'

$$\frac{d}{dx}(\sec y) = \frac{d}{dx}(x)$$

$$\sec y \tan y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\sec y \tan y}$$

$$\frac{dy}{dx} = \frac{1}{\sec y\sqrt{\sec^2 y - 1}}$$

$\because \tan\theta = \sqrt{\sec^2\theta - 1}$

$$\frac{dy}{dx} = \frac{1}{\sec y\sqrt{x^2-1}}$$

$\because \sec^2 y = x^2$

$$\boxed{\frac{dy}{dx} = \frac{1}{x\sqrt{x^2-1}}}$$

Hence proved.

Example # 1

Find $\frac{dy}{dx}$ if $y = x\sin^{-1}\left(\frac{x}{a}\right) + \sqrt{a^2-x^2}$

Solution:

$$y = x\sin^{-1}\left(\frac{x}{a}\right) + \sqrt{a^2-x^2}$$

Differentiate w.r.t 'x'

$$\frac{dy}{dx} = \frac{d}{dx}\left[x\sin^{-1}\left(\frac{x}{a}\right) + \sqrt{a^2-x^2}\right]$$

$$\frac{dy}{dx} = \frac{d}{dx}\left[x\sin^{-1}\frac{x}{a}\right] + \frac{d}{dx}(a^2-x^2)^{1/2}$$

$$\frac{dy}{dx} = 1\cdot\sin^{-1}\frac{x}{a} + x\cdot\frac{1}{\sqrt{1-\left(\frac{x}{a}\right)^2}}\frac{d}{dx}\left(\frac{x}{a}\right) + \frac{1}{2}(a^2-x^2)^{\frac{1}{2}-1}\frac{d}{dx}(a^2-x^2$$

$$\frac{dy}{dx} = \sin^{-1}\frac{x}{a} + \frac{x}{\sqrt{1-\frac{x^2}{a^2}}}\left(\frac{1}{a}\right) + \frac{1}{\cancel{2}}(a^2-x^2)^{-\frac{1}{2}}(-\cancel{2}x)$$

$$\frac{dy}{dx} = \sin^{-1}\frac{x}{a} + x \cdot \frac{1}{\sqrt{\frac{a^2-x^2}{a^2}}}\left(\frac{1}{a}\right) - \frac{x}{\sqrt{a^2-x^2}}$$

$$\frac{dy}{dx} = \sin^{-1}\frac{x}{a} + x \frac{1}{\frac{\sqrt{a^2-x^2}}{a}}\left(\frac{1}{a}\right) - \frac{x}{\sqrt{a^2-x^2}}$$

$$\frac{dy}{dx} = \sin^{-1}\frac{x}{a} + x \frac{\cancel{a}}{\sqrt{a^2-x^2}}\frac{1}{\cancel{a}} - \frac{x}{\sqrt{a^2-x^2}}$$

$$\frac{dy}{dx} = \sin^{-1}\frac{x}{a} + \cancel{\frac{x}{\sqrt{a^2-x^2}}} - \cancel{\frac{x}{\sqrt{a^2-x^2}}}$$

$$\boxed{\frac{dy}{dx} = \sin^{-1}\frac{x}{a}}$$

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Example #2

If $y = \tan\left[2\tan^{-1}\frac{x}{2}\right]$, show that

$$\frac{dy}{dx} = \frac{4(1+y^2)}{(4+x^2)}$$

Solution:

$$y = \tan\left(2\tan^{-1}\frac{x}{2}\right)$$

Let $u = 2\tan^{-1}\frac{x}{2}$

$$y = \tan u$$

Differentiate w.r.t "u"

$$\frac{dy}{du} = \frac{d}{du}(\tan u)$$

$$= \sec^2 u \qquad \because 1+\tan^2\theta = \sec^2\theta$$

$$\frac{dy}{du} = 1+\tan^2 y$$

$$\frac{dy}{du} = 1+y^2$$

$$u = 2\tan^{-1}\frac{x}{2}$$

Differentiate w.r.t "x"

$$\frac{du}{dx} = \frac{d}{dx}\left(2\tan^{-1}\frac{x}{2}\right)$$

$$\frac{du}{dx} = 2 \cdot \frac{1}{1+\left(\frac{x}{2}\right)^2}\frac{d}{dx}\left(\frac{x}{2}\right)$$

$$\frac{du}{dx} = \cancel{2}\frac{1}{1+\frac{x^2}{4}} \cdot \frac{1}{\cancel{2}}$$

$$\frac{du}{dx} = \frac{1}{\frac{4+x^2}{4}}$$

$$\frac{du}{dx} = \frac{4}{4+x^2}$$

Using chain rule.

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$\frac{dy}{dx} = (1+y^2) \cdot \frac{4}{4+x^2}$$

$$\boxed{\frac{dy}{dx} = \frac{4(1+y^2)}{(4+x^2)}}$$ Hence proved.

Exercise # 2·5

Question # 1

Differentiate the following trigonometric functions from the first principle.

(i)

$\sin 2x$

Solution:

Let $y = \sin 2x$ —— ①

$y+\delta y = \sin 2(x+\delta x)$ —— ②

Subtract eq ② by ①

$\cancel{y}+\delta y-\cancel{y} = \sin 2(x+\delta x) - \sin 2x$

$\delta y = \sin(2x+2\delta x) - \sin 2x)$

$\because \sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$

$\delta y = 2\cos\left(\frac{2x+2\delta x+2x}{2}\right)\sin\left(\frac{\cancel{2x}+2\delta x-\cancel{2x}}{2}\right)$

$\delta y = 2\cos\left(\frac{4x+2\delta x}{2}\right)\sin\left(\frac{\cancel{2}\delta x}{\cancel{2}}\right)$

$\delta y = 2\cos\frac{\cancel{2}(2x+\delta x)}{\cancel{2}}\sin\delta x$

$\delta y = 2\cos(2x+\delta x)\sin\delta x$

Divided by δx both sides

$\frac{\delta y}{\delta x} = 2\cos(2x+\delta x)\frac{\sin\delta x}{\delta x}$

Apply $\lim_{\delta x\to 0}$ both sides

$\lim_{\delta x\to 0}\frac{\delta y}{\delta x} = \lim_{\delta x\to 0} 2\cos(2x+\delta x)\lim_{\delta x\to 0}\frac{\sin\delta x}{\delta x}$

$\frac{dy}{dx} = 2\cos(2x)\cdot(1)$

$$\boxed{\frac{dy}{dx} = 2\cos 2x}$$

(ii)

$\tan 3x$

Solution:

Let $y = \tan 3x$ —— ①

$y+\delta y = \tan 3(x+\delta x) \Rightarrow \tan(3x+3\delta x)$ —— ②

Subtract eq ② by ①

$\cancel{y}+\delta y-\cancel{y} = \tan(3x+3\delta x) - \tan 3x$

$\delta y = \frac{\sin(3x+3\delta x)}{\cos(3x+3\delta x)} - \frac{\sin 3x}{\cos 3x}$

$\delta y = \frac{\sin(3x+3\delta x)(\cos 3x) - \sin 3x(\cos 3x+3\delta}{\cos 3x\cdot\cos(3x+3\delta x)}$

$\because \sin\alpha\cos\beta - \cos\alpha\sin\beta = \sin(\alpha-\beta)$

$\delta y = \frac{\sin(\cancel{3x}+3\delta x-\cancel{3x})}{\cos 3x\cdot\cos(3x+3\delta x)}$

$\delta y = \frac{\sin 3\delta x}{\cos 3x\cdot\cos(3x+3\delta x)}$

Divided by δx and $\lim_{\delta x\to 0}$ both sides

$\lim_{\delta x\to 0}\frac{\delta y}{\delta x} = \lim_{\delta x\to 0}\frac{1}{\cos 3x\cos(3x+3\delta x)}\cdot\lim_{\delta x\to 0}\frac{3\sin 3\delta x}{3\delta x}$

$\frac{dy}{dx} = \frac{1}{\cos 3x\cos(3x+3\Omega)}\cdot 3(1)$

$\frac{dy}{dx} = \frac{3}{\cos^2 3x}$

$$\boxed{\frac{dy}{dx} = 3\sec^2 3x}$$

(iii)

$\sin 2x + \cos 2x$

Solution:

Let $y = \sin 2x + \cos 2x$ —— ①

$y + \delta y = \sin 2(x + \delta x) + \cos 2(x + \delta x)$

$y + \delta y = \sin(2x + 2\delta x) + \cos(2x + 2\delta x)$ —— ②

Subtract eq ② by ①

$y + \delta y - y = \sin(2x + 2\delta x) + \cos(2x + 2\delta x) - (\sin 2x + \cos 2x)$

$\delta y = \sin(2x + 2\delta x) + \cos(2x + 2\delta x) - \sin 2x - \cos 2x$

$\delta y = \sin(2x + 2\delta x) - \sin 2x + \cos(2x + 2\delta x) - \cos 2x$

$\because \sin A - \sin B = 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$

$\because \cos A - \cos B = -2\sin\left(\frac{A+B}{2}\right)\sin\frac{A-B}{2}$

$\delta y = 2\cos\left(\frac{2x + 2\delta x + 2x}{2}\right)\sin\left(\frac{\cancel{2x} + 2\delta x - \cancel{2x}}{2}\right) + -2\sin\left(\frac{2x + 2\delta x + 2x}{2}\right)\sin\left(\frac{\cancel{2x} + 2\delta x - \cancel{2x}}{2}\right)$

$\delta y = 2\cos\left(\frac{4x + 2\delta x}{2}\right)\sin\left(\frac{\cancel{2}\delta x}{\cancel{2}}\right) - 2\sin\left(\frac{4x + 2\delta x}{2}\right)\sin\left(\frac{\cancel{2}\delta x}{\cancel{2}}\right)$

$\delta y = 2\cos\left(\frac{4x + 2\delta x}{2}\right)\sin\delta x - 2\sin\left(\frac{4x + 2\delta x}{2}\right)\sin\delta x$

$\delta y = 2\cos\frac{\cancel{2}(2x + \delta x)}{\cancel{2}}\sin\delta x - 2\sin\frac{\cancel{2}(2x + \delta x)}{\cancel{2}}\sin\delta x$

Divided by δx and $\lim\limits_{\delta x \to 0}$ both sides

$\lim\limits_{\delta x \to 0}\frac{\delta y}{\delta x} = 2\lim\limits_{\delta x \to 0}\cos(2x + \delta x)\lim\limits_{\delta x \to 0}\frac{\sin\delta x}{\delta x} - 2\lim\limits_{\delta x \to 0}(2x + \delta x)\lim\limits_{\delta x \to 0}\frac{\sin\delta x}{\delta x}$

$\frac{dy}{dx} = 2\cos 2x(1) - 2\sin 2x\ (1)$

$$\boxed{\frac{dy}{dx} = 2\cos 2x - 2\sin 2x}$$

(iv)

$\cos x^2$

Solution:

Let $y = \cos x^2$ —— ①

$y + \delta y = \cos(x + \delta x)^2$ —— ②

Subtract eq ② by ①

$y + \delta y - y = \cos(x + \delta x)^2 - \cos x^2$

$\delta y = \cos(x+\delta x)^2 - \cos x^2$

$\because \cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$

$\delta y = -2\sin\frac{(x+\delta x)^2+x^2}{2}\sin\frac{(x+\delta x)^2-x^2}{2}$

$\delta y = -2\sin\frac{(x+\delta x)^2+x^2}{2}\sin\left(\frac{\cancel{x^2}+\delta x^2+2x\delta x-\cancel{x^2}}{2}\right)$

$\delta y = -2\sin\frac{(x+\delta x)^2+x^2}{2}\sin\frac{\delta x(\delta x+2x)}{2}$

$\delta y = -2\sin\frac{(x+\delta x)^2+x^2}{2}\frac{\sin\frac{\delta x(\delta x+2x)}{2}}{x+\frac{\delta x}{2}}\cdot\left(x+\frac{\delta x}{2}\right)$

$\frac{\delta y}{\delta x}\lim_{\delta x\to 0} = \lim_{\delta x\to 0}\left(-2\sin\frac{(x+\delta x)^2+x^2}{2}\right)\lim_{\delta x\to 0}\frac{\sin\frac{\delta x+2x}{2}\cdot\delta x}{\left(x+\frac{\delta x}{2}\right)\delta x}\cdot\lim_{\delta x\to 0}\left(x+\frac{\delta x}{2}\right)$

Divided by δx and apply $\lim_{\delta x\to 0}$ both sides

$\frac{dy}{dx} = -2\sin\frac{x^2+x^2}{2}(1)\cdot x$

$\frac{dy}{dx} = -2\sin\frac{\cancel{2}x^2}{\cancel{2}}(x)$

$$\boxed{\frac{dy}{dx} = -2x\sin x^2}$$

(V)

$\tan^2 x$

Solution:

Let $y = \tan^2 x$ ——— ①

$y+\delta y = \tan^2(x+\delta x)$ —— ②

Subtract eq ② by ①

$\cancel{y}+\delta y-\cancel{y} = \tan^2(x+\delta x) - \tan^2 x$

$\because a^2-b^2 = (a+b)(a-b)$

$\delta y = (\tan(x+\delta x)+\tan x)(\tan(x+\delta x)-\tan x)$

$\delta y = (\tan(x+\delta x)+\tan x)\left[\frac{\sin(x+\delta x)}{\cos(x+\delta x)} - \frac{\sin x}{\cos x}\right]$

$$\delta y = [\tan(x+\delta x) + \tan x]\left[\frac{\cos x \sin(x+\delta x) - \sin x(\cos(x+\delta x)}{\cos x \cdot \cos(x+\delta x)}\right]$$

$\because \sin\alpha\cos\beta - \cos\alpha\sin\beta = \sin(\alpha-\beta)$

$$\delta y = [\tan(x+\delta x)+\tan x]\left[\frac{\sin\ \cancel{x}+\delta x-\cancel{x}}{\cos x\ \cos(x+\delta x)}\right]$$

Divided by δx and $\lim\limits_{\delta x\to 0}$ both sides.

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x} = \lim_{\delta x\to 0}[\tan(x+\delta x)+\tan x]\lim_{\delta x\to 0}\frac{\sin\delta x}{\delta x}\cdot\lim_{\delta x\to 0}\frac{1}{\cos x\cos(x+\delta x)}$$

$$\frac{dy}{dx} = [\tan x+\tan x]\ (1)\cdot\frac{1}{(\cos x)(\cos x)}$$

$$\frac{dy}{dx} = \frac{2\tan x}{\cos^2 x}$$

$$\boxed{\frac{dy}{dx} = 2\tan x\sec^2 x}$$

(vi)

$\sqrt{\tan x}$

Solution:

Let

$y = \sqrt{\tan x}$ —— ①

$y+\delta y = \sqrt{\tan(x+\delta x)}$ —— ②

Subtract eq ② by ①

$$\cancel{y}+\delta y-\cancel{y} = \sqrt{\tan(x+\delta x)} - \sqrt{\tan x}$$

$$\delta y = \sqrt{\tan(x+\delta x)} - \sqrt{\tan x} \times \frac{\sqrt{\tan(x+\delta x)}+\sqrt{\tan x}}{\sqrt{\tan(x+\delta x)}+\sqrt{\tan x}}$$

$$\delta y = \frac{\left(\sqrt{\tan(x+\delta x)}\right)^2 - \left(\sqrt{\tan x}\right)^2}{\sqrt{\tan(x+\delta x)}+\sqrt{\tan x}}$$

$$\delta y = \frac{\tan(x+\delta x)-\tan x}{\sqrt{\tan(x+\delta x)}+\sqrt{\tan x}}$$

$$\delta y = \frac{1}{\sqrt{\tan(x+\delta x)}+\sqrt{\tan x}}\left[\frac{\sin(x+\delta x)}{\cos(x+\delta x)} - \frac{\sin x}{\cos x}\right]$$

$$\delta y = \frac{1}{\sqrt{\tan(x+\delta x)}+\sqrt{\tan x}}\left[\frac{\sin(x+\delta x)\cdot\cos x - \cos(x+\delta x)\sin x}{\cos x\ \cos(x+\delta x)}\right]$$

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$\because \sin\alpha\cos\beta - \cos\alpha\sin\beta = \sin(\alpha-\beta)$

$$\delta y = \frac{1}{\sqrt{\tan(x+\delta x)}+\sqrt{\tan x}} \cdot \left[\frac{\sin(\cancel{x}+\delta x-\cancel{x})}{\cos x \cdot \cos(x+\delta x)}\right]$$

Divided by δx and $\lim\limits_{\delta x\to 0}$ both sides.

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x} = \lim_{\delta x\to 0}\frac{1}{\sqrt{\tan(x+\delta x)}+\sqrt{\tan x}} \lim_{\delta x\to 0}\frac{1}{\cos x\cos(x+\delta x)} \cdot \lim_{\delta x\to 0}\frac{\sin\delta x}{\delta x}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{\tan x}+\sqrt{\tan x}} \cdot \frac{1}{\cos x\cos x} \quad (1)$$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{\tan x}} \cdot \frac{1}{\cos^2 x}$$

$$\boxed{\frac{dy}{dx} = \frac{\sec^2 x}{2\sqrt{\tan x}}}$$

(vii)

$\cos\sqrt{x}$

Solution:

Let $y = \cos\sqrt{x}$ —— ①

$y+\delta y = \cos\sqrt{x+\delta x}$ —— ②

Subtract eq ② by ①

$\cancel{y}+\delta y-\cancel{y} = \cos\sqrt{(x+\delta x)} - \cos\sqrt{x}$

$\because \cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$

$$\delta y = -2\sin\frac{\sqrt{x+\delta x}+\sqrt{x}}{2}\sin\frac{\sqrt{x+\delta x}-\sqrt{x}}{2}$$

Divided by δx both sides.

$$\frac{\delta y}{\delta x} = \frac{-2\sin\frac{\sqrt{x+\delta x}+\sqrt{x}}{2}\sin\frac{\sqrt{x+\delta x}-\sqrt{x}}{2}}{\delta x}$$

$\because (\sqrt{x+\delta x}+\sqrt{x})(\sqrt{x+\delta x}-\sqrt{x}) = \delta x$

$$\frac{\delta y}{\delta x} = \frac{-2\sin\frac{\sqrt{x+\delta x}+\sqrt{x}}{2}\sin\frac{\sqrt{x+\delta x}-\sqrt{x}}{2}}{(\sqrt{x+\delta x}+\sqrt{x})(\sqrt{x+\delta x}-\sqrt{x})}$$

$$\frac{\delta y}{\delta x} = \frac{-\cancel{2}\sin\frac{\sqrt{x+\delta x}+\sqrt{x}}{2}}{(\sqrt{x+\delta x}+\sqrt{x})} \cdot \frac{\sin\frac{\sqrt{x+\delta x}-\sqrt{x}}{2}}{\cancel{2}\,\frac{\sqrt{x+\delta x}-\sqrt{x}}{2}}$$

Apply $\underset{\delta x \to 0}{\text{limit}}$ both sides

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{-\sin \frac{\sqrt{x+\delta x}+\sqrt{x}}{2}}{\sqrt{x+\delta x}+\sqrt{x}} \cdot \lim_{\delta x \to 0} \frac{\sin \frac{\sqrt{x+\delta x}+\sqrt{x}}{2}}{\frac{\sqrt{x+\delta x}-\sqrt{x}}{2}}$$

$$\frac{dy}{dx} = \frac{-\sin \frac{\sqrt{x}+\sqrt{x}}{2}}{\sqrt{x}+\sqrt{x}} \quad (1)$$

$$\frac{dy}{dx} = \frac{-\sin \frac{\cancel{2}\sqrt{x}}{\cancel{2}}}{2\sqrt{x}}$$

$$\boxed{\frac{dy}{dx} = \frac{-\sin\sqrt{x}}{2\sqrt{x}}}$$

Question #2

Differentiate the following w.r.t the variable involved.

(i)

$x^2 \sec 4x.$

Solution:

Let $y = x^2 \sec 4x$

Differentiate w.r.t 'x'

$$\frac{dy}{dx} = \frac{d}{dx}(x^2 \sec 4x)$$

$$\frac{dy}{dx} = x^2 \frac{d}{dx}(\sec 4x) + \sec 4x \frac{d}{dx}(x^2)$$

$$\frac{dy}{dx} = x^2 \sec 4x \tan 4x \overset{\frac{d}{dx}(4x)}{+} \sec 4x \cdot 2x$$

$$\frac{dy}{dx} = x^2 \sec 4x \tan 4x \cdot 4 + \sec 4x \, 2x$$

$$\frac{dy}{dx} = 4x^2 \sec 4x \tan 4x + 2x \sec 4x$$

$$\boxed{\frac{dy}{dx} = 2x \sec 4x \left[2x \tan 4x + 1\right]}$$

(ii)

$\tan^3\theta \sec^2\theta$

Solution

Let $y = \tan^3\theta \sec^2\theta$

Differentiate w.r.t 'x'

$$\frac{dy}{dx} = \frac{d}{dx}(\tan^3\theta \sec^2\theta)$$

$$\frac{dy}{dx} = \tan^3\theta \frac{d}{dx}(\sec^2\theta) + \sec^2\theta \frac{d}{dx}(\tan^3\theta)$$

$$\frac{dy}{dx} = \tan^3\theta \cdot 2\sec\theta \frac{d}{dx}(\sec\theta) + \sec^2\theta \cdot 3\tan^2\theta \frac{d}{dx}(\tan\theta)$$

$$\frac{dy}{dx} = \tan^3\theta \; 2\sec\theta(\sec\theta \tan\theta) + \sec^2\theta \cdot 3\tan^2\theta \sec^2\theta$$

$$\frac{dy}{dx} = 2\sec^2\theta \tan^4\theta + 3\tan^2\theta \sec^4\theta$$

$$\boxed{\frac{dy}{dx} = \tan^2\theta \sec^2\theta \left(2\tan^2\theta + 3\sec^2\theta\right)}$$

(iii)

$(\sin 2\theta - \cos 3\theta)^2$

Solution:

Let $y = (\sin 2\theta - \cos 3\theta)^2$

Differentiate w.r.t "θ"

$$\frac{dy}{d\theta} = \frac{d}{d\theta}(\sin 2\theta - \cos 3\theta)^2$$

$$\frac{dy}{d\theta} = 2(\sin 2\theta - \cos 3\theta)\frac{d}{d\theta}(\sin 2\theta - \cos 3\theta)$$

$$\frac{dy}{d\theta} = 2(\sin 2\theta - \cos 3\theta)\left[\cos 2\theta \frac{d}{d\theta}(2\theta) - -\sin 3\theta \frac{d}{d\theta}(3\theta)\right]$$

$$\boxed{\frac{dy}{d\theta} = 2(\sin 2\theta - \cos 3\theta)\left[2\cos 2\theta + 3\sin 3\theta\right]}$$

(iv)

$\cos\sqrt{x} + \sqrt{\sin x}$

Solution:

Let $y = \cos\sqrt{x} + \sqrt{\sin x}$

Differentiate w.r.t "x"

$$\frac{dy}{dx} = \frac{d}{dx}\left[\cos\sqrt{x} + \sqrt{\sin x}\right]$$

$$\frac{dy}{dx} = -\sin x^{1/2}\frac{d}{dx}x^{1/2} + \frac{1}{2}(\sin x)^{-\frac{1}{2}}\frac{d}{dx}(\sin x)$$

$$\frac{dy}{dx} = -\frac{1}{2}\sin\sqrt{x}\,(x^{-\frac{1}{2}}) + \frac{1}{2}(\sin x)^{-\frac{1}{2}}(\cos x)$$

$$\frac{dy}{dx} = \frac{-\sin\sqrt{x}}{2\sqrt{x}} + \frac{\cos x}{2\sqrt{\sin x}}$$

$$\boxed{\frac{dy}{dx} = \frac{1}{2}\left[\frac{-\sin\sqrt{x}}{\sqrt{x}} + \frac{\cos x}{\sqrt{\sin x}}\right]}$$

Question #3

Find $\frac{dy}{dx}$ if:

(i)

$y = x\cos y$

Solution:

$y = x\cos y$

Differentiate w.r.t 'x'

$$\frac{dy}{dx} = \frac{d}{dx}(x\cos y)$$

$$\frac{dy}{dx} = x\frac{d}{dx}\cos y + \cos y\frac{d}{dx}(x)$$

$$\frac{dy}{dx} = x -\sin y\frac{d}{dx}(y) + \cos y\,(1)$$

$$\frac{dy}{dx} = -x\sin y\frac{dy}{dx} + \cos y$$

$$\frac{dy}{dx} + x\sin y\frac{dy}{dx} = \cos y$$

$$\frac{dy}{dx}\left[1 + x\sin y\right] = \cos y$$

$$\boxed{\frac{dy}{dx} = \frac{\cos y}{1 + x\sin y}}$$

(ii)

$x = y\sin y$

Solution

$x = y\sin y$

Differentiate w.r.t "y"

$$\frac{dx}{dy} = \frac{d}{dy}(y\sin y)$$

$$\frac{dx}{dy} = y\frac{d}{dy}(\sin y) + \sin y\frac{d}{dy}(y)$$

$$\frac{dx}{dy} = y(\cos y) + \sin y$$

$$\boxed{\frac{dy}{dx} = \frac{1}{y\cos y + \sin y}}$$

Question # 4

Find the derivative w.r.t 'x'

(i)

$$\cos\sqrt{\frac{1+x}{1+2x}}$$

Solution:

Let $y=\cos\sqrt{\frac{1+x}{1+2x}}$

Differentiate w.r.t 'x'

$$\frac{dy}{dx}=\frac{d}{dx}\left(\cos\frac{(1+x)^{1/2}}{(1+2x)^{1/2}}\right]$$

$$=-\sin\frac{(1+x)^{1/2}}{(1+2x)^{1/2}}\cdot\frac{d}{dx}\frac{(1+x)^{1/2}}{(1+2x)^{1/2}}$$

$$=-\sin\sqrt{\frac{1+x}{1+2x}}\cdot\frac{(1+2x)^{1/2}\frac{d}{dx}(1+x)^{1/2}-(1+x)^{1/2}\frac{d}{dx}(1+2x)^{1/2}}{((1+2x)^{1/2})^2}$$

$$=-\sin\sqrt{\frac{1+x}{1+2x}}\cdot\frac{(1+2x)^{1/2}\cdot\frac{1}{2}(1+x)^{-\frac{1}{2}}\frac{d}{dx}(1+x)-(1+x)^{1/2}\cdot\frac{1}{2}(1+2x)^{-\frac{1}{2}}\frac{d}{dx}(1+2x)}{(1+2x)}$$

$$=-\sin\sqrt{\frac{1+x}{1+2x}}\,\frac{\frac{1}{2}(1+2x)^{1/2}(1+x)^{-\frac{1}{2}}(1)-(1+x)^{1/2}\frac{1}{\cancel{2}}(1+2x)^{-\frac{1}{2}}(\cancel{2})}{(1+2x)}$$

$$=-\sin\sqrt{\frac{1+x}{1+2x}}\,\frac{\left\{\frac{\sqrt{1+2x}}{2\sqrt{1+x}}-\frac{\sqrt{1+x}}{\sqrt{1+2x}}\right]}{1+2x}$$

$$=-\sin\sqrt{\frac{1+x}{1+2x}}\cdot\frac{1}{1+2x}\left[\frac{1+2x-2(1+x)}{2\sqrt{1+x}\sqrt{1+2x}}\right]$$

$$=-\sin\sqrt{\frac{1+x}{1+2x}}\cdot\frac{1}{1+2x}\left\{\frac{1+\cancel{2x}-2-\cancel{2x}}{2\sqrt{1+x}(\sqrt{1+2x})}\right]$$

$$=\cancel{-}\sin\sqrt{\frac{1+x}{1+2x}}\,\frac{\cancel{-}1}{2\sqrt{1+x}(1+2x)^{1/2}(1+2x)}$$

$$\boxed{\frac{dy}{dx}=\frac{\sin\sqrt{\frac{1+x}{1+2x}}}{2\sqrt{1+x}\,(1+2x)^{3/2}}}$$

* * *

(ii)

$$\sin\sqrt{\frac{1+2x}{1+x}}$$

Solution:

Let $y=\sin\sqrt{\frac{1+2x}{1+x}}$

Differentiate w.r.t 'x'

$$\frac{dy}{dx}=\frac{d}{dx}\left[\sin\frac{(1+2x)^{1/2}}{(1+x)^{1/2}}\right]$$

$$\frac{dy}{dx}=\cos\frac{(1+2x)^{1/2}}{(1+x)^{1/2}}\cdot\frac{d}{dx}\frac{(1+2x)^{1/2}}{(1+x)^{1/2}}$$

$$=\cos\sqrt{\frac{1+2x}{1+x}}\cdot\frac{(1+x)^{1/2}\frac{d}{dx}(1+2x)^{1/2}-(1+2x)^{1/2}\frac{d}{dx}(1+x)^{1/2}}{((1+x)^{1/2})^2}$$

$$=\cos\sqrt{\frac{1+2x}{1+x}}\,\frac{(1+x)^{1/2}\cdot\frac{1}{2}(1+2x)^{1/2-1}\frac{d}{dx}(1+2x)-(1+2x)^{1/2}\cdot\frac{1}{2}(1+x)^{-\frac{1}{2}}\frac{d}{dx}(1+x)}{(1+x)}$$

$$=\cos\sqrt{\frac{1+2x}{1+x}}\cdot\frac{(1+x)^{1/2}\cdot\frac{1}{\cancel{2}}(1+2x)^{-\frac{1}{2}}(\cancel{2})-(1+2x)^{1/2}\cdot\frac{1}{2}(1+x)^{-\frac{1}{2}}}{(1+x)}$$

$$=\cos\sqrt{\frac{1+2x}{1+x}}\,\frac{1}{(1+x)}\left[\frac{\sqrt{1+x}}{\sqrt{1+2x}}-\frac{\sqrt{1+2x}}{2\sqrt{1+x}}\right]$$

$$=\cos\sqrt{\frac{1+2x}{1+x}}\cdot\frac{1}{1+x}\cdot\frac{2(1+x)-(1+2x}{2\sqrt{1+x}\sqrt{1+2x}}$$

$$=\cos\sqrt{\frac{1+2x}{1+x}}\cdot\frac{1}{1+x}\,\frac{2+\cancel{2x}-1-\cancel{2x}}{2\sqrt{1+x}\sqrt{1+2x}}$$

$$=\cos\sqrt{\frac{1+2x}{1+x}}\cdot\frac{1}{2(1+x)^{1/2}(1+x)\sqrt{1+2x}}$$

$$\boxed{\frac{dy}{dx}=\frac{\cos\sqrt{\frac{1+2x}{1+x}}}{2(1+x)^{3/2}\sqrt{1+2x}}}$$

* * *

Question #5

Differentiate:

(i)

$\sin x$ w.r.t $\cot x$

Solution:

Let

$y = \sin x$

Differentiate w.r.t 'x'

$$\frac{dy}{dx} = \frac{d}{dx}(\sin x)$$

$$\frac{dy}{dx} = \cos x$$

$u = \cot x$

Differentiate w.r.t 'x'

$$\frac{du}{dx} = \frac{d}{dx}(\cot x)$$

$$\frac{du}{dx} = -\operatorname{cosec}^2 x$$

Using chain rule

$$\frac{dy}{du} = \frac{dy}{dx} \cdot \frac{dx}{du}$$

$$\frac{dy}{du} = \cos x \cdot \frac{-1}{\operatorname{cosec}^2 x}$$

$$\boxed{\frac{dy}{du} = -\cos x \operatorname{cosec}^2 x}$$

(ii)

$\sin^2 x$ w.r.t $\cos^4 x$

Solution:

Let

$y = \sin^2 x$

Differentiate w.r.t 'x'

$$\frac{dy}{dx} = \frac{d}{dx}(\sin^2 x)$$

$$\frac{dy}{dx} = 2\sin x \frac{d}{dx}\sin x$$

$$\frac{dy}{dx} = 2\sin x \cos x$$

$u = \cos^4 x$

Differentiate w.r.t 'x'

$$\frac{du}{dx} = \frac{d}{dx}(\cos^4 x)$$

$$\frac{du}{dx} = 4\cos^3 x \frac{d}{dx}\cos x$$

$$\frac{du}{dx} = 4\cos^3 x(-\sin x)$$

Using chain rule

$$\frac{dy}{dx} = \frac{dy}{dx} \cdot \frac{dx}{du}$$

$$\frac{dy}{du} = \cancel{2}\cancel{\sin x}\cancel{\cos x} \cdot \frac{-1}{{}_{2}\cancel{4}\cancel{\cos^3}^{2} x \cancel{\sin x}}$$

$$\frac{dy}{du} = \frac{-1}{2\cos^2 x}$$

$$\boxed{\frac{dy}{du} = \frac{-1}{2}\sec^2 x}$$

Question #6

If $\tan y(1+\tan x) = 1-\tan x$, show that $\frac{dy}{dx} = -1$

Solution.

$$\tan y(1+\tan x) = 1-\tan x$$

$$\tan y = \frac{1-\tan x}{1+\tan x}$$

$$\tan y = \frac{1-\tan x}{1+1\cdot\tan x}$$

$$\tan y = \frac{\tan\frac{\pi}{4} - \tan x}{1+\tan\pi/4 \quad -\tan x}$$

$$\because \frac{\tan A - \tan B}{1 + \tan A \tan B} = \tan(A-B)$$

$$\tan y = \tan\left(\frac{\pi}{4} - x\right)$$

$$y = \frac{\pi}{4} - x$$

Differentiate w.r.t "x"

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\pi}{4} - x\right)$$

$$\frac{dy}{dx} = 0 - 1$$

$$\boxed{\frac{dy}{dx} = -1}$$

Question #7

If $y = \sqrt{\tan x + \sqrt{\tan x + \sqrt{\tan x}}} + \cdots \infty$

proved that $(2y-1)\frac{dy}{dx} = \sec^2 x.$

Solution:

Let

$$y = \sqrt{\tan x + \sqrt{\tan x + \sqrt{\tan x}}}$$

Taking square on both sides

$$y^2 = \tan x + \sqrt{\tan x + \sqrt{\tan x}}$$

$$y^2 = \tan x + y$$

$$\frac{d}{dx}(y^2) = \frac{d}{dx}(\tan x + y)$$

$$2y\frac{dy}{dx} = \sec^2 x + \frac{dy}{dx}$$

$$2y\frac{dy}{dx} - \frac{dy}{dx} = \sec^2 x$$

$$\boxed{(2y-1)\frac{dy}{dx} = \sec^2 x}$$

Hence proved.

Question #8

If $x = a\cos^3\theta$, $y = b\sin^3\theta$, show that

$$a\frac{dy}{dx} + b\tan\theta = 0$$

Solution:

$$x = a\cos^3\theta \quad — ①$$

$$y = b\sin^3\theta \quad — ②$$

Differentiate eq ① w.r.t "θ"

$$\frac{dx}{d\theta} = \frac{d}{d\theta}(a\cos^3\theta)$$

$$\frac{dx}{d\theta} = 3a\cos^2\theta\frac{d}{d\theta}(\cos\theta)$$

$$\frac{dx}{d\theta} = 3a\cos^2\theta(-\sin\theta)$$

$$\frac{dx}{d\theta} = -3a\cos^2\theta\sin\theta$$

Differentiate w.r.t "θ" in eq ②

$$\frac{dy}{d\theta} = \frac{d}{d\theta}(b\sin^3\theta)$$

$$\frac{dy}{d\theta} = 3b\sin^2\theta\frac{d}{d\theta}(\sin\theta)$$

$$\frac{dy}{d\theta} = 3b\sin^2\theta(\cos\theta)$$

Using chain rule.

$$\frac{dy}{dx} = \frac{dy}{d\theta}\cdot\frac{d\theta}{dx}$$

$$\frac{dy}{dx} = \cancel{3}b\sin^{\cancel{2}}\theta\cancel{\cos\theta}\cdot\frac{-1}{\cancel{3}a\cos^{\cancel{2}}\theta\cancel{\sin\theta}}$$

$$\frac{dy}{dx} = \frac{-b\sin\theta}{a\cos\theta} \Rightarrow \frac{-b}{a}\tan\theta$$

$$a\frac{dy}{dx} = -b\tan\theta$$

$$\boxed{a\frac{dy}{dx} + b\tan\theta = 0}$$ **Hence proved.**

Question #9

Find $\frac{dy}{dx}$ if $x = a(\cos t + \sin t)$, $y = a(\sin t - t\cos t)$

Solution:

$x = a(\cos t + \sin t)$ —— ①

$y = a(\sin t - t\cos t)$ —— ②

Differentiate w.r.t 't'

$$\frac{dx}{dt} = \frac{d}{dt}\left(a(\cos t + \sin t)\right)$$

$$\frac{dx}{dt} = a(-\sin t + \cos t)$$

$$\frac{dx}{dt} = a(\cos t - \sin t)$$

Differentiate w.r.t 't'

$$\frac{dy}{dt} = \frac{d}{dt} a(\sin t - t\cos t))$$

$$\frac{dy}{dt} = a\cos t - t\frac{d}{dx}\cos' t + \cos t\frac{d}{dt}(t)$$

$$\frac{dy}{dt} = a(\cos t - [t\sin t + \cos t]$$

$$\frac{dy}{dt} = a(\cos t + t\sin t - \cos t)$$

$$\frac{dy}{dt} = a t \sin t$$

Using chain rule

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$

$$\frac{dy}{dx} = \cancel{a} t\sin t \cdot \frac{1}{\cancel{a}(\cos t - \sin t)}$$

$$\boxed{\frac{dy}{dx} = \frac{t\sin t}{\cos t - \sin t}}$$

Question #10

Differentiate w.r.t 'x'

(i)

$\cos^{-1}\frac{x}{a}$

Solution:

Let $y = \cos^{-1}\frac{x}{a}$

Differentiate w.r.t 'x'

$$\frac{dy}{dx} = \frac{d}{dx}\left(\cos^{-1}\frac{x}{a}\right)$$

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-\left(\frac{x}{a}\right)^2}} \cdot \frac{d}{dx}\left(\frac{x}{a}\right)$$

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-\frac{x^2}{a^2}}} \cdot \frac{1}{a}$$

$$\frac{dy}{dx} = \frac{-1}{\sqrt{\frac{a^2-x^2}{a^2}}} \cdot \frac{1}{a}$$

$$\frac{dy}{dx} = \frac{-1}{\frac{\sqrt{a^2-x^2}}{a}} \cdot \frac{1}{a}$$

$$\frac{dy}{dx} = \frac{-\cancel{a}}{\sqrt{a^2-x^2}} \cdot \frac{1}{\cancel{a}}$$

$$\boxed{\frac{dy}{dx} = \frac{-1}{\sqrt{a^2-x^2}}}$$

(ii)

$\cot^{-1}\frac{x}{a}$

Solution:

Let $y = \cot^{-1}\frac{x}{a}$

Differentiate w.r.t 'x'

$$\frac{dy}{dx} = \frac{d}{dx}\left(\cot^{-1}\frac{x}{a}\right)$$

$$\frac{dy}{dx} = \frac{-1}{1+\left(\frac{x}{a}\right)^2} \cdot \frac{d}{dx}\left(\frac{x}{a}\right)$$

$$\frac{dy}{dx} = \frac{-1}{1+\frac{x^2}{a^2}} \cdot \frac{1}{a}$$

$$\frac{dy}{dx} = \frac{-1}{\frac{a^2+x^2}{a^2}} \cdot \frac{1}{a}$$

$$\frac{dy}{dx} = \frac{-a^{\cancel{2}}}{a^2+x^2} \cdot \frac{1}{\cancel{a}}$$

$$\boxed{\frac{dy}{dx} = \frac{-a}{a^2+x^2}}$$

(iii)

$$\frac{1}{a}\sin^{-1}\frac{a}{x}$$

Solution.

Let $y = \frac{1}{a}\sin^{-1}\frac{a}{x}$

Differentiate w.r.t 'x'

$$\frac{dy}{dx} = \frac{1}{\cancel{a}} \frac{1}{\sqrt{1-\left(\frac{a}{x}\right)^2}} \cancel{a} \cdot \frac{d}{dx}\left(\frac{1}{x}\right)$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-\frac{a^2}{x^2}}} \cdot -x^{-2}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{\frac{x^2-a^2}{x^2}}} \cdot \frac{-1}{x^2}$$

$$\frac{dy}{dx} = \frac{1}{\frac{\sqrt{x^2-a^2}}{x}} \cdot -\frac{1}{x^2}$$

$$\frac{dy}{dx} = \frac{\cancel{x}}{\sqrt{x^2-a^2}} \cdot \frac{-1}{x^{\cancel{2}}}$$

$$\boxed{\frac{dy}{dx} = \frac{-1}{x\sqrt{x^2-a^2}}}$$

(iv)

$$\sin^{-1}\sqrt{1-x^2}$$

Solution:

Let

$y = \sin^{-1}\sqrt{1-x^2}$

Differentiate w.r.t 'x'

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-\left(\sqrt{1-x^2}\right)^2}} \cdot \frac{d}{dx}\left(\sqrt{1-x^2}\right)$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-(1-x^2)}} \cdot \frac{1}{\cancel{2}\sqrt{1-x^2}} \cdot \frac{d}{dx}(-\cancel{2}x)$$

$$\frac{dy}{dx} = \frac{-x}{\left(\sqrt{\cancel{1}-\cancel{1}+x^2}\right)\left(\sqrt{1-x^2}\right)}$$

$$\frac{dy}{dx} = \frac{-x}{\sqrt{x^2}\left(\sqrt{1-x^2}\right)}$$

$$\frac{dy}{dx} = \frac{-\cancel{x}}{\cancel{x}\sqrt{1-x^2}}$$

$$\boxed{\frac{dy}{dx} = \frac{-1}{\sqrt{1-x^2}}}$$

(V)

$$\sec^{-1}\left[\frac{x^2+1}{x^2-1}\right]$$

Solution:

Let $y = \sec^{-1}\left(\frac{x^2+1}{x^2-1}\right)$

Differentiate w.r.t 'x'

$$\frac{dy}{dx} = \frac{d}{dx}\sec^{-1}\left(\frac{x^2+1}{x^2-1}\right)$$

$$\frac{dy}{dx} = \frac{1}{\frac{x^2+1}{x^2-1}\sqrt{\left(\frac{x^2+1}{x^2-1}\right)^2-1}} \cdot \frac{d}{dx}\left(\frac{x^2+1}{x^2-1}\right)$$

$$\frac{dy}{dx} = \frac{1}{\frac{x^2+1}{x^2-1}\sqrt{\frac{(x^2+1)^2-(x^2-1)^2}{(x^2-1)^2}}} \cdot \frac{(x^2-1)\frac{d}{dx}(x^2+1)-(x^2+1)\frac{d}{dx}(x^2-1)}{(x^2-1)^2}$$

$$\frac{dy}{dx}=\frac{1}{\frac{x^2+1}{x^2-1}\frac{\sqrt{(x^2+1)^2-(x^2-1)^2}}{(x^2-1)}}\cdot\frac{(x^2-1)(2x)-(x^2+1)(2x)}{(x^2-1)^2}$$

$$\frac{dy}{dx}=\frac{1}{\frac{x^2+1}{(x^2-1)^2}\sqrt{\cancel{x^4}+1+2x^2-\cancel{x^4}-\cancel{1}+2x^2}}\cdot\frac{\cancel{2x^3}-2x-\cancel{2x^3}-2x}{(x^2-1)^2}$$

$$\frac{dy}{dx}=\frac{\cancel{(x^2-1)^2}}{x^2+1\sqrt{4x^2}}\cdot\frac{-4x}{\cancel{(x^2-1)^2}}$$

$$\frac{dy}{dx}=\frac{-\overset{2}{\cancel{4x}}}{(x^2+1)\cancel{2x}}$$

$$\boxed{\frac{dy}{dx}=\frac{-2}{(x^2+1)}}$$

(vi)

$\cot^{-1}\frac{2x}{1-x^2}$

Solution:

Let $y=\cot^{-1}\frac{2x}{1-x^2}$

Differentiate w.r.t 'x'

$$\frac{dy}{dx}=\frac{d}{dx}\left[\cot^{-1}\frac{2x}{1-x^2}\right]$$

$$\frac{dy}{dx}=\frac{-1}{1+\left(\frac{2x}{1-x^2}\right)^2}\cdot\frac{d}{dx}\left(\frac{2x}{1-x^2}\right)$$

$$\frac{dy}{dx}=\frac{-1}{1+\frac{4x^2}{(1-x^2)^2}}\,\frac{(1-x^2)\frac{d}{dx}2x-2x\frac{d}{dx}(1-x^2)}{(1-x^2)^2}$$

$$\frac{dy}{dx}=\frac{-1}{\frac{(1-x^2)^2+4x^2}{(1-x^2)^2}}\cdot\frac{(1-x^2)\,2-2x(-2x)}{(1-x^2)^2}$$

$$\frac{dy}{dx}=\frac{-\cancel{(1-x^2)^2}}{1+x^4-2x^2+4x^2}\cdot\frac{2-2x^2+4x^2}{\cancel{(1-x^2)^2}}$$

$$\frac{dy}{dx}=\frac{-1}{x^4+2x^2+1}\cdot 2+2x^2$$

$$\frac{dy}{dx}=\frac{-1}{(x^2+1)^2}\cdot 2\cancel{(x^2+1)}$$

$$\boxed{\frac{dy}{dx}=\frac{-2}{1+x^2}}$$

(vii)

$\cos^{-1}\frac{1-x^2}{1+x^2}$

Solution:

Let $y=\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

Differentiate w.r.t 'x'

$$\frac{dy}{dx}=\frac{d}{dx}\left(\cos^{-1}\frac{1-x^2}{1+x^2}\right)$$

$$\frac{dy}{dx}=\frac{-1}{\sqrt{1-\left(\frac{1-x^2}{1+x^2}\right)^2}}\cdot\frac{d}{dx}\left(\frac{1-x^2}{1+x^2}\right)$$

$$\frac{dy}{dx}=\frac{-1}{\sqrt{\frac{(1+x^2)^2-(1-x^2)^2}{(1+x^2)^2}}}\,\frac{(1+x^2)\frac{d}{dx}(1-x^2)-(1-x^2)\frac{d}{dx}(1+x^2)}{(1+x^2)^2}$$

$$\frac{dy}{dx}=\frac{-1}{\frac{\sqrt{1+x^4+2x^2-x^4-1+2x^2}}{(1+x^2)}}\cdot\frac{(1+x^2)(-2x)-(1-x^2)(2x)}{(1+x^2)^2}$$

$$\frac{dy}{dx}=-\frac{\cancel{(1+x^2)}}{\sqrt{4x^2}}\cdot\frac{-2x-\cancel{2x^3}-2x+\cancel{2x^3}}{(1+x^2)^{\cancel{2}}}$$

$$\frac{dy}{dx}=-\frac{1}{\cancel{2x}}\cdot\frac{-\overset{2}{\cancel{4x}}}{1+x^2}$$

$$\boxed{\frac{dy}{dx}=\frac{+2}{1-x^2}}$$

Question #11

Show that $\frac{dy}{dx} = \frac{y}{x}$ if $\frac{y}{x} = \tan^{-1}\left(\frac{x}{y}\right)$

Solution:

$$\frac{y}{x} = \tan^{-1}\frac{x}{y}$$

$$y = x\tan^{-1}\frac{x}{y}$$

Differentiate w.r.t "x"

$$\frac{dy}{dx} = \frac{d}{dx}\left(x\tan^{-1}\frac{x}{y}\right)$$

$$\frac{dy}{dx} = x\frac{d}{dx}\left(\tan^{-1}\frac{x}{y}\right) + \tan^{-1}\frac{x}{y}\frac{d}{dx}(x)$$

$$\frac{dy}{dx} = x\cdot\frac{1}{1+\left(\frac{x}{y}\right)^2}\cdot\frac{d}{dx}\left(\frac{x}{y}\right) + \tan^{-1}\frac{x}{y}(1)$$

$$\frac{dy}{dx} = x\cdot\frac{1}{1+\frac{x^2}{y^2}}\cdot\frac{y\frac{d}{dx}(x) - x\frac{d}{dx}(y)}{y^2} + \tan^{-1}\frac{x}{y}$$

$$\frac{dy}{dx} = x\frac{1}{\frac{y^2+x^2}{y^2}}\cdot\frac{y - x\frac{dy}{dx}}{y^2} + \frac{y}{x}$$

$$\frac{dy}{dx} = x\frac{\cancel{y^2}}{x^2+y^2}\left[\frac{y - x\frac{dy}{dx}}{\cancel{y^2}} + \frac{y}{x}\right]$$

$$\frac{dy}{dx} = \frac{xy}{x^2+y^2} - \frac{x^2}{x^2+y^2}\frac{dy}{dx} + \frac{y}{x}$$

$$\frac{dy}{dx} + \frac{x^2}{x^2+y^2}\frac{dy}{dx} = \frac{xy}{x^2+y^2} + \frac{y}{x}$$

$$\left[1+\frac{x^2}{x^2+y^2}\right]\frac{dy}{dx} = \frac{x^2y + y(x^2+y^2)}{x(x^2+y^2)}$$

$$\left[\frac{x^2+y^2+x^2}{x^2+y^2}\right]\frac{dy}{dx} = \frac{x^2y + x^2y + y^3}{x(x^2+y^2)}$$

$$\frac{dy}{dx} = \frac{2x^2y+y^3}{x\cancel{(x^2+y^2)}}\cdot\frac{\cancel{(x^2+y^2)}}{2x^2+y^2}$$

$$\frac{dy}{dx} = \frac{y(2x^2+y^2)}{x(2x^2+y^2)}$$

$$\boxed{\frac{dy}{dx} = \frac{y}{x}}$$

Hence proved.

Question #12

If $y = \tan(P\tan^{-1}x)$, show that $(1+x^2)y_1 - P(1+y^2) = 0$.

Solution:

$$y = \tan(P\tan^{-1}x)$$

$$\tan^{-1}y = P\tan^{-1}x$$

$$\frac{d}{dx}(\tan^{-1}y) = P\frac{d}{dx}(\tan^{-1}x)$$

$$\frac{1}{1+y^2}\frac{dy}{dx} = P\cdot\frac{1}{1+x^2}$$

$$(1+x^2)\frac{dy}{dx} = P(1+y^2)$$

$$\boxed{(1+x^2)y_1 - P(1+y^2) = 0}$$

Hence proved.

---

* Theory

Example #1

Find $\frac{dy}{dx}$ if:

(i)

$$y = e^{x^2+1}$$

Solution:

Let $u = x^2+1$ — ①

$y = e^u$ — ②

From eq ①

$u = x^2+1$

$$\frac{du}{dx} = \frac{d}{dx}(x^2+1)$$

$$\frac{du}{dx} = 2x$$

From eq ②

Diff w.r.t "x"

$$\frac{dy}{dx} = \frac{d}{dx}(e^u)$$

$$= e^u \frac{du}{dx}$$

Using chain rule

$$\frac{dy}{dx} = e^{x^2+1} . (2x)$$

$$\boxed{\frac{dy}{dx} = 2x e^{x^2+1}}$$

(ii)

$$y = a^{\sqrt{x}}$$

Solution:

Let $u = \sqrt{x}$ — ①

$y = a^u$ — ②

From eq ①

$u = x^{1/2}$

Diff w.r.t "x"

$$\frac{du}{dx} = \frac{d}{dx}(x^{1/2})$$

$$= \frac{1}{2} x^{-\frac{1}{2}}$$

$$\frac{du}{dx} = \frac{1}{2\sqrt{x}}$$

From eq ②

Diff w.r.t "x"

$$\frac{dy}{dx} = \frac{d}{dx}(a^u)$$

$$= (a^u \ln a) . \frac{du}{dx}$$

$$\boxed{\because \frac{d}{dx}(a^x) = a^x \ln a}$$

$$\frac{d}{dx}(a^{\sqrt{x}}) = (a^{\sqrt{x}} \ln a) . \frac{1}{2\sqrt{x}}$$

$$= \frac{\ln a}{2} \cdot a^{\sqrt{x}} \cdot \frac{1}{\sqrt{x}}$$

Example #2

Differentiation $y = a^x$ w.r.t "x"; we have.

Solution:

$y = a^x$

$= e^{x \ln a}$

Diff w.r.t "x"

$$\frac{dy}{dx} = e^{x \ln a} \cdot \frac{d}{dx}(x \ln a)$$

$$= e^{x \ln a}(\ln a)$$

$$\boxed{\because e^{x \ln a} = a^x}$$

$$\boxed{\frac{dy}{dx} = a^x . (\ln a)}$$

Example #1

Find if $\frac{dy}{dx}$ $y = \log_{20}(ax^2+bx+c)$

Solution:

Let

$u = ax^2+bx+c$

$y = \log_{10}(ax^2+bx+c)$

Differentiate w.r.t "x" both sides

$$\frac{dy}{dx} = \frac{d}{dx}\left(\log_{10}(ax^2+bx+c)\right)$$

$$\frac{dy}{dx} = \frac{1}{ax^2+bx+c}\cdot\frac{1}{\ln_{10}}\frac{d}{dx}(ax^2+bx+c)$$

$$= \frac{1}{ax^2+bx+c}\cdot\frac{1}{\ln_{10}}(2ax+b)$$

$$\boxed{\frac{dy}{dx} = \frac{2ax+b}{(ax^2+bx+c)\ln_{10}}}$$

Example #2

Differentiate $\ln(x^2+2x)$ w.r.t "x".

Solution: $y = \ln(x^2+2x)$

Differentiate w.r.t "x" both sides.

$$\frac{dy}{dx} = \frac{d}{dx}\left(\ln(x^2+2x)\right)$$

$$\boxed{\because \frac{d}{dx}(\ln x) = \frac{1}{x}}$$

$$\frac{dy}{dx} = \frac{1}{x^2+2x}\frac{d}{dx}(x^2+2x)$$

$$\frac{dy}{dx} = \frac{1}{x^2+2x}\cdot(2x+2)$$

$$\boxed{\frac{dy}{dx} = \frac{2(x+1)}{x^2+2x}}$$

Example #1

Differentiate $y = e^{f(x)}$ w.r.t "x"

Solution:

$y = e^{f(x)}$

Taking logarithm of both sides.

$\ln y = \ln e^{f(x)}$

$\ln y = f(x)(\ln e)$

$\boxed{\because \ln e = 1}$

$\ln y = f(x)$

Differentiate w.r.t "x"

$$\frac{d}{dx}(\ln y) = \frac{d}{dx}fx$$

$$\frac{1}{y}\frac{dy}{dx} = f'(x)$$

$$\frac{dy}{dx} = y \times f'(x)$$

$$\frac{dy}{dx} = e^{f(x)} \times f'(x)$$

$$\boxed{\frac{d(e^{f(x)})}{dx} = e^{fx} \times f'(x)}$$

Example #3

$(\ln x)^x$ w.r.t "x". Differentiate

Solution:

Let $y = (\ln x)^x$

Taking logarithm on both sides

$\ln y = \ln[\ln x]^x$

$= x\ln(\ln x)$

Differentiate w.r.t "x"

$$\frac{1}{y}\frac{dy}{dx} = 1\cdot\ln(\ln x) + x\cdot\frac{1}{\ln x}\frac{d}{dx}(\ln x)$$

$$= \ln(\ln x) + \cancel{x}\cdot\frac{1}{\ln x}\cdot\frac{1}{\cancel{x}}$$

$$\frac{1}{y}\frac{dy}{dx} = \ln(\ln x) + \frac{1}{\ln x}$$

$$\frac{dy}{dx} = y\left[\ln(\ln x) + \frac{1}{\ln x}\right]$$

$$\boxed{\frac{dy}{dx} = (\ln x)^x\left[\ln(\ln x) + \frac{1}{\ln x}\right]}$$

• ——— •

Example #2

Find the derivation of

$$\frac{x\sqrt{x^2+3}}{x^2+1}$$

Solution:

Let $y = \frac{x\sqrt{x^2+3}}{x^2+1}$

Taking logarithm on both sides

$$\ln y = \ln\left[\frac{x\sqrt{x^2+3}}{x^2+1}\right]$$

$$= \ln x\sqrt{x^2+3} - \ln(x^2+1)$$

$$\ln y = \ln x + \frac{1}{2}\ln(x^2+3) - \ln(x^2+1)$$

Differentiate w.r.t "x"

$$\frac{d}{dx}(\ln y) = \frac{d}{dx}\left[\ln x + \frac{1}{2}\ln(x^2+3) - \ln(x^2+1)\right]$$

$$\frac{1}{y}\frac{dy}{dx} = \left[\frac{1}{x} + \frac{1}{2}\cdot\frac{1}{x^2+3}\frac{d}{dx}(x^2+3) - \frac{1}{(x^2+1)}\frac{d}{dx}(x^2+1)\right]$$

$$\frac{1}{y}\frac{dy}{dx} = \left[\frac{1}{x} + \frac{1}{\cancel{2}}\cdot\frac{1}{x^2+3}(\cancel{2}x) - \frac{1}{x^2+1}(2x)\right]$$

$$= \left[\frac{1}{x} + \frac{x}{x^2+3} - \frac{2x}{x^2+1}\right]$$

$$= \frac{(x^2+3)(x^2+1) + x^2(x^2+1) - 2x^2(x^2+3)}{x(x^2+3)(x^2+1)}$$

$$= \frac{\cancel{x^4} + x^2 + 3x^2 + 3 + \cancel{x^4} + x^2 - \cancel{2x^4} - 6x^2}{x(x^2+3)(x^2+1)}$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{3 - x^2}{x(x^2+3)(x^2+1)}$$

$$\frac{dy}{dx} = y\left[\frac{3-x^2}{x(x^2+3)(x^2+1)}\right]$$

$$\frac{dy}{dx} = \frac{\cancel{x}\sqrt{x^2+3}}{x^2+1}\cdot\frac{3-x^2}{\cancel{x}(x^2+3)(x^2+1)}$$

$$\boxed{\frac{dy}{dx} = \frac{3-x^2}{\sqrt{x^2+3}\,(x^2+1)^2}}$$

• ——— •

Example #1

Find $\frac{dy}{dx}$ if $y = \sinh 2x$

Solution:

$$y = \sinh 2x$$

Differentiate w.r.t "x"

$$\frac{dy}{dx} = \frac{d}{dx}(\sinh 2x)$$

$$= \cosh 2x\,\frac{d}{dx}(2x)$$

$$= \cosh 2x\,(2)$$

$$\boxed{\frac{dy}{dx} = 2\cosh 2x}$$

• ——— •

Example #2

Find $\frac{dy}{dx}$ if $y = \tanh(x^2)$

Solution:

$$y = \tanh x^2$$

$$\frac{dy}{dx} = \frac{d}{dx}\tanh x^2$$

$$= \operatorname{sech}^2 x^2\,\frac{d}{dx}(x^2)$$

$$\frac{dy}{dx} = \operatorname{sech}^2 x^2 \cdot 2x$$

$$\boxed{\frac{dy}{dx} = 2x\operatorname{sech}^2 x^2}$$

Example #1

Find $\frac{dy}{dx}$ if $y = \sinh^{-1}(ax+b)$

Solution:

$$y = \sinh^{-1}(ax+b)$$

Differentiate w.r.t "x"

$$\frac{dy}{dx} = \frac{d}{dx}\left(\sin^{-1}(ax+b)\right)$$

$$= \frac{1}{\sqrt{1+(ax+b)^2}}\,\frac{d}{dx}(ax+b)$$

$$= \frac{1}{\sqrt{1+(ax+b)^2}} \cdot (a(1)+0)$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1+(ax+b)^2}} \cdot a$$

$$\boxed{\frac{dy}{dx} = \frac{a}{\sqrt{1+(ax+b)^2}}}$$

Example #2

Find $\frac{dy}{dx}$ if $y = \cosh^{-1}(\sec x)$

Solution:

$$y = \cosh^{-1}(\sec x)$$

Differentiate w.r.t "x"

$$\frac{dy}{dx} = \frac{d}{dx}\left(\cosh^{-1}(\sec x)\right)$$

$$= \frac{1}{\sqrt{\sec^2 x - 1}}\,\frac{d}{dx}(\sec x)$$

$$\boxed{\because \sec^2 x - 1 = \tan^2 x}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{\tan^2 x}} \cdot \sec x \tan x$$

$$\frac{dy}{dx} = \frac{1}{\cancel{\tan x}} \cdot \sec x \cancel{\tan x}$$

$$\boxed{\frac{dy}{dx} = \sec x}$$

Exercise # 2.6

Question #1

Find $f'(x)$ if:

(i)

$f(x) = e^{\sqrt{x}-1}$

Solution:

$$y = e^{\sqrt{x}-1}$$

(ii)

$f(x) = x^3 e^{1/x}$, $(x \neq 0)$

Solution:

$$y = x^3 e^{1/x}$$

Differentiate w.r.t "x"

$$\frac{d}{dx}(f(x)) = \frac{d}{dx}\left(e^{\sqrt{x}-1}\right)$$

$$f'(x) = e^{\sqrt{x}-1}\frac{d}{dx}(\sqrt{x}-1)$$

$$f'(x) = e^{\sqrt{x}-1}\cdot\frac{1}{2\sqrt{x}}$$

$$\boxed{f'(x) = \frac{1}{2\sqrt{x}}\cdot e^{\sqrt{x}-1}}$$

(ii)

$$f(x) = \frac{e^x}{e^{-x}+1}$$

Solution:

$$f(x) = \frac{e^x}{e^{-x}+1}$$

Differentiate w.r.t "x"

$$\frac{d}{dx}(f(x)) = \frac{d}{dx}\left(\frac{e^x}{e^{-x}+1}\right)$$

$$f'(x) = \frac{(e^{-x}+1)\frac{d}{dx}(e^x) - e^x\frac{d}{dx}(e^{-x}+1)}{(e^{-x}+1)^2}$$

$$f'(x) = \frac{(e^{-x}+1)(e^x) - e^x(e^{-x})\frac{d}{dx}(-x)}{(e^{-x}+1)^2}$$

$$f'(x) = \frac{e^{-x+x} + e^x - e^{x-x}(-1)}{(e^{-x}+1)^2}$$

$$f'(x) = \frac{e^0 + e^x + e^0}{(e^{-x}+1)^2}$$

$$\boxed{\because e^0 = 1}$$

$$f'(x) = \frac{1 + e^x + 1}{(e^{-x}+1)^2}$$

$$\boxed{f'(x) = \frac{2+e^x}{(e^{-x}+1)^2}}$$

Differentiate w.r.t "x"

$$\frac{d}{dx}(f(x)) = \frac{d}{dx}(x^3\cdot e^{1/x})$$

$$f'(x) = x^3\frac{d}{dx}(e^{1/x}) + e^{1/x}\frac{d}{dx}(x^3)$$

$$f'(x) = x^3 e^{1/x}\frac{d}{dx}\left(\frac{1}{x}\right) + e^{1/x}(3x^2)$$

$$f'(x) = x^3 e^{1/x}\frac{d}{dx}(x^{-1}) + 3x^2 e^{1/x}$$

$$f'(x) = x^3 e^{1/x}.(-x^{-2}) + 3x^2 e^{1/x}$$

$$f'(x) = e^{1/x}\left[-x^3.x^{-2} + 3x^2\right]$$

$$f'(x) = e^{1/x}\left[-x^{3-2} + 3x^2\right]$$

$$f'(x) = e^{1/x}\left[-x + 3x^2\right]$$

$$f'(x) = x e^{1/x}\left[-1+3x\right]$$

$$\boxed{f'(x) = x e^{1/x}(3x-1)}$$

(iii)

$$f(x) = e^x(1+\ln x)$$

Solution:

$$f(x) = e^x(1+\ln x)$$

Differentiate w.r.t 'x'

$$\frac{d}{dx}(f(x)) = \frac{d}{dx}\, e^x(1+\ln x)$$

$$f'(x) = e^x\frac{d}{dx}(1+\ln x) + (1+\ln x)\frac{d}{dx}\, e^x$$

$$f'(x) = e^x.\ \frac{1}{x} + (1+\ln x)\, e^x$$

$$f'(x) = \frac{e^x}{x} + e^x + e^x\ln x$$

$$f'(x) = e^x\left(\frac{1}{x} + 1 + \ln x\right)$$

$$f'(x) = e^x\left(\frac{1+x+x\ln x}{x}\right)$$

$$\boxed{f'(x) = e^x\left[\frac{x(1+\ln x)+1}{x}\right]}$$

(V)

$$\ln(e^x + e^{-x})$$

Solution:

Let

$$f(x) = \ln(e^x + e^{-x})$$

Differentiate w.r.t "x"

$$\frac{d}{dx}(f'(x)) = \frac{d}{dx}\left[\ln(e^x + e^{-x})\right]$$

$$f'(x) = \frac{1}{e^x + e^{-x}} \frac{d}{dx}(e^x + e^{-x})$$

$$f'(x) = \frac{1}{e^x + e^{-x}} \cdot e^x + e^{-x} \frac{d}{dx}(-x)$$

$$f'(x) = \frac{1}{e^x + e^{-x}} \cdot (e^x - e^{-x})$$

$$f'(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$f'(x) = \frac{e^x - \frac{1}{e^x}}{e^x + \frac{1}{e^x}}$$

$$f'(x) = \frac{\frac{e^{2x} - 1}{e^x}}{\frac{e^{2x} + 1}{e^x}}$$

$$f'(x) = \frac{e^{2x} - 1}{\cancel{e^x}} \times \frac{\cancel{e^x}}{e^{2x} + 1}$$

$$\boxed{f'(x) = \frac{e^{2x} - 1}{e^{2x} + 1}}$$

(Vi)

$$f(x) = \frac{e^{ax} - e^{-ax}}{e^{ax} + e^{-ax}}$$

Solution:

$$f(x) = \frac{e^{ax} - e^{-ax}}{e^{ax} + e^{-ax}}$$

Differentiate w.r.t "x"

$$\frac{d}{dx}(f(x)) = \frac{d}{dx}\left(\frac{e^{ax} - e^{-ax}}{e^{ax} + e^{-ax}}\right)$$

$$f'(x) = \frac{(e^{ax} + e^{-ax}) \frac{d}{dx}(e^{ax} - e^{-ax}) - (e^{ax} - e^{-ax}) \frac{d}{dx}(e^{ax} + e^{-ax})}{(e^{ax} + e^{-ax})^2}$$

$$= \frac{(e^{ax} + e^{-ax})(ae^{ax} + ae^{-ax}) - (e^{ax} - e^{-ax})(ae^{ax} - ae^{-ax})}{(e^{ax} + e^{-ax})^2}$$

$$f'(x) = \frac{a[(e^{ax} + e^{-ax})(e^{ax} + e^{-ax}) - (e^{ax} - e^{-ax})(e^{ax} - e^{-ax})}{(e^{ax} + e^{-ax})^2}$$

$$f'(x) = \frac{a[(e^{ax} + e^{-ax})^2 - (e^{ax} - e^{-ax})^2]}{(e^{ax} + e^{-ax})^2}$$

$$f'(x) = \frac{a\,[e^{2ax} + e^{-2ax} + 2 - (e^{2ax} + e^{-2ax} - 2)]}{(e^{ax} + e^{-ax})^2}$$

$$f'(x) = \frac{a\,(\cancel{e^{2ax}} \cancel{+e^{-2ax}} + 2 - \cancel{e^{2ax}} - \cancel{e^{-2ax}} + 2)}{(e^{ax} + e^{-ax})^2}$$

$$\boxed{f'(x) = \frac{4a}{(e^{ax} + e^{-ax})^2}}$$

(vii)

$f(x) = \sqrt{\ln(e^{2x}+e^{-2x})}$

Solution:

$f(x) = \sqrt{\ln(e^{2x}+e^{-2x})}$

Differentiate w.r.t "x"

$$\frac{d}{dx}(f(x)) = \frac{d}{dx}\sqrt{\ln(e^{2x}+e^{-2x})}$$

$$f'(x) = \frac{1}{2}[\ln(e^{2x}+e^{-2x})]^{\frac{1}{2}-1}\frac{d}{dx}\ln(e^{2x}+e^{-2x})$$

$$f'(x) = \frac{1}{2}[\ln(e^{2x}+e^{-2x})^{\frac{-1}{2}} \cdot \frac{1}{e^{2x}+e^{-2x}}\left[e^{2x}\frac{d}{dx}(2x)+e^{-2x}\frac{d}{dx}(-2x)\right]$$

$$f'(x) = \frac{1}{2\sqrt{\ln(e^{2x}+e^{-2x})}} \cdot \frac{1}{e^{2x}+e^{-2x}}(2e^{2x}-2e^{-2x})$$

$$f'(x) = \frac{1}{\cancel{2}\sqrt{\ln(e^{2x}+e^{-2x})}} \cdot \frac{\cancel{2}(e^{2x}-e^{-2x})}{e^{2x}+e^{-2x}}$$

$$\boxed{f'(x) = \frac{e^{2x}-e^{-2x}}{(e^{2x}+e^{-2x})\sqrt{\ln(e^{2x}+e^{-2x})}}}$$

* * *

Question #2

Find $\frac{dy}{dx}$ if:

(i)

$y = x^2\ln\sqrt{x}$

Solution:

$y = x^2\ln\sqrt{x} \Rightarrow x^2\ln(x)^{1/2}$

$\because \log m^n = n\log m$

$y = \frac{1}{2}x^2\ln x$

Differentiate w.r.t "x"

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{1}{2}x^2\ln x\right)$$

(viii)

$\ln\sqrt{e^{2x}+e^{-2x}}$

Solution:

$f(x) = \ln\sqrt{e^{2x}+e^{-2x}}$

$f(x) = \ln(e^{2x}+e^{-2x})^{1/2}$

Differentiate w.r.t "x"

$\because \log m^n = n\log m$

$f(x) = \frac{1}{2}\ln(e^{2x}+e^{-2x})$

Differentiate w.r.t "x"

$$f'(x) = \frac{1}{2}\cdot\frac{1}{e^{2x}+e^{-2x}}\frac{d}{dx}(e^{2x}+e^{-2x})$$

$$f'(x) = \frac{1}{2(e^{2x}+e^{-2x})}\cdot\left(e^{2x}\frac{d}{dx}(2x)+e^{-2x}\frac{d}{dx}(-2x)\right)$$

$$= \frac{1}{2(e^{2x}+e^{-2x})}\cdot 2e^{2x}-2e^{-2x}$$

$$= \frac{1}{2(e^{2x}+e^{-2x})}\cdot 2(e^{2x}-e^{-2x})$$

$$f'(x) = \frac{\cancel{2}(e^{2x}-e^{-2x})}{\cancel{2}(e^{2x}+e^{-2x})}$$

$$f'(x) = \frac{e^{2x}-e^{-2x}}{e^{2x}+e^{-2x}}$$

$$\boxed{f'(x) = \tanh 2x}$$

* * *

(ii)

$y = x\sqrt{\ln x}$

Solution:

$y = x\sqrt{\ln x}$

$y = x(\ln x)^{1/2}$

Differentiate w.r.t "x"

$$\frac{dy}{dx} = x\frac{d}{dx}(\ln x)^{1/2} + (\ln x)^{1/2}\frac{d}{dx}(x)$$

$$= \cancel{x}\cdot\frac{1}{2}\cdot\frac{1}{\cancel{x}}(\ln x)^{\frac{-1}{2}} + \sqrt{\ln x}\cdot(1)$$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{\ln x}} + \sqrt{\ln x}$$

$$= \frac{1}{2}\left[x^2 \frac{d}{dx}(\ln x) + \ln x \frac{d}{dx}(x^2)\right]$$

$$= \frac{1}{2}\left[x^2 \cdot \frac{1}{x} + \ln x \cdot 2x\right]$$

$$= \frac{1}{2}\left[x + \ln x \cdot 2x\right]$$

$$= \left[\frac{x}{2} + \frac{\cancel{2}x\ln x}{\cancel{2}}\right]$$

$$\boxed{\frac{dy}{dx} = x\ln x + \frac{1}{2}x}$$

(iii)

$$y = \frac{x}{\ln x}$$

Solution:

$$y = \frac{x}{\ln x}$$

Differentiate w.r.t "x"

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{x}{\ln x}\right)$$

$$\frac{dy}{dx} = \frac{\ln x \frac{d}{dx}(x) - x\frac{d}{dx}(\ln x)}{(\ln x)^2}$$

$$\frac{dy}{dx} = \frac{\ln(x)\cdot(1) - \cancel{x}\left(\frac{1}{\cancel{x}}\right)}{(\ln x)^2}$$

$$\boxed{\frac{dy}{dx} = \frac{\ln x - 1}{(\ln x)^2}}$$

(V)

$$y = \ln\sqrt{\frac{x^2-1}{x^2+1}}$$

Solution:

$$y = \ln\left[\frac{x^2-1}{x^2+1}\right]^{1/2}$$

Differentiate w.r.t "x"

$$\frac{dy}{dx} = \frac{d}{dx}\left[\ln\left(\frac{x^2-1}{x^2+1}\right)^{1/2}\right]$$

$$\frac{dy}{dx} = \frac{1}{2}\cdot\left[\frac{1}{\frac{x^2-1}{x^2+1}} \cdot \frac{d}{dx}\left(\frac{x^2-1}{x^2+1}\right)\right]$$

$$\boxed{\frac{dy}{dx} = \frac{1+2\ln x}{2\sqrt{\ln x}}}$$

(iv)

$$y = x^2\ln\frac{1}{x}$$

Solution:

$$y = x^2\ln\frac{1}{x}$$

Differentiate w.r.t "x"

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^2\ln\frac{1}{x}\right)$$

$$\frac{dy}{dx} = x^2 \frac{d(\ln\frac{1}{x})}{dx} + \frac{d}{dx}x^2 \ln\frac{1}{x}$$

$$\frac{dy}{dx} = x^2 \cdot x\frac{d}{dx}(x^{-1}) + 2x\cdot\ln\frac{1}{x}$$

$$\frac{dy}{dx} = x^3\cdot(-x^{-2}) + 2x\ln\frac{1}{x}$$

$$\frac{dy}{dx} = -x^{3-2} + 2x\ln\frac{1}{x}$$

$$\frac{dy}{dx} = -x + 2x\ln\frac{1}{x}$$

$$\frac{dy}{dx} = x\left[-1 + 2\ln\frac{1}{x}\right]$$

$$\boxed{\frac{dy}{dx} = x\left[2\ln\frac{1}{x} - 1\right]}$$

(Vi)

$$y = \ln\left(x + \sqrt{x^2+1}\right)$$

Solution:

$$y = \ln\left(x + \sqrt{x^2+1}\right)$$

$$\frac{dy}{dx} = \frac{d}{dx}\left(\ln\left(x + \sqrt{x^2+1}\right)\right)$$

$$\frac{dy}{dx} = \frac{1}{x+\sqrt{x^2+1}} \cdot \frac{d}{dx}\left(x + \sqrt{x^2+1}\right)$$

$$\frac{dy}{dx} = \frac{1}{x+\sqrt{x^2+1}} \cdot \left[1 + \frac{1}{2\sqrt{x^2+1}}\frac{d}{dx}(x^2+1)\right]$$

$$\frac{dy}{dx} = \frac{1}{x+\sqrt{x^2+1}}\left[1 + \frac{1}{\cancel{2}\sqrt{x^2+1}} \cdot \cancel{2}x\right]$$

$$\frac{dy}{dx} = \frac{1}{2}\left[\frac{x^2+1}{x^2-1} \cdot \frac{(x^2+1)\frac{d}{dx}(x^2-1) - (x^2-1)\frac{d}{dx}(x^2+1)}{(x^2+1)^2}\right]$$

$$\frac{dy}{dx} = \frac{1}{2}\left[\frac{x^2+1}{x^2-1} \cdot \frac{(x^2+1)(2x) - (x^2-1)(2x)}{(x^2+1)^2}\right]$$

$$\frac{dy}{dx} = \frac{1}{2}\left[\frac{\cancel{x^2+1}}{x^2-1} \cdot \frac{\cancel{2x^3}+2x-\cancel{2x^3}+2x}{(x^2+1)^{\cancel{2}}}\right]$$

$$\frac{dy}{dx} = \frac{1}{2}\left[\frac{1}{x^2-1} \cdot \frac{\overset{2}{\cancel{4}}x}{x^2+1}\right]$$

$$\boxed{\frac{dy}{dx} = \frac{2x}{x^4-1}}$$

(vii)

$$y = \ln(9-x^2)$$

Solution:

$$y = \ln(9-x^2)$$

Differentiate w.r.t 'x'

$$\frac{dy}{dx} = \frac{d}{dx}\ln(9-x^2)$$

$$\frac{dy}{dx} = \frac{1}{9-x^2} \cdot \frac{d}{dx}(9-x^2)$$

$$\frac{dy}{dx} = \frac{1}{9-x^2}(-2x)$$

$$\boxed{\frac{dy}{dx} = \frac{-2x}{9-x^2}}$$

(ix)

$$y = e^{-x}(x^3+2x^2+1)$$

Solution:

$$y = e^{-x}(x^3+2x^2+1)$$

Differentiate w.r.t 'x'

$$\frac{dy}{dx} = \frac{d}{dx}\left[e^{-x}(x^3+2x^2+1)\right]$$

$$= e^{-x}\frac{d}{dx}(x^3+2x^2+1) + (x^3+2x^2+1)\frac{d}{dx}e^{-x}$$

$$\frac{dy}{dx} = e^{-x}(3x^2+4x) + (x^3+2x^2+1)e^{-x}\frac{d}{dx}(-x)$$

$$\frac{dy}{dx} = \frac{1}{x+\sqrt{x^2+1}}\left[1+\frac{x}{\sqrt{x^2+1}}\right]$$

$$\frac{dy}{dx} = \frac{1}{x+\sqrt{x^2+1}} \cdot \frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}}$$

$$\boxed{\frac{dy}{dx} = \frac{1}{\sqrt{x^2+1}}}$$

(viii)

$$y = e^{-2x}\sin 2x$$

Solution:

$$y = e^{-2x}\sin 2x$$

Differentiate w.r.t 'x'

$$\frac{dy}{dx} = \frac{d}{dx}(e^{-2x}\sin 2x)$$

$$= e^{-2x}\frac{d}{dx}(\sin 2x) + \sin 2x\frac{d}{dx}e^{-2x}$$

$$= e^{-2x}\cdot\cos 2x\frac{d}{dx}(2x) + \sin 2x\cdot e^{-2x}\frac{d}{dx}(-2x)$$

$$\frac{dy}{dx} = 2e^{-2x}\cos 2x - 2e^{-2x}\sin 2x$$

$$\boxed{\frac{dy}{dx} = 2e^{-2x}(\cos 2x - \sin 2x)}$$

(x)

$$y = xe^{\sin x}$$

Solution:

$$y = xe^{\sin x}$$

Differentiate w.r.t 'x'

$$\frac{dy}{dx} = \frac{d}{dx}(xe^{\sin x})$$

$$\frac{dy}{dx} = x\frac{d}{dx}e^{\sin x} + e^{\sin x}\frac{d}{dx}(x)$$

$$\frac{dy}{dx} = xe^{\sin x}\frac{d}{dx}(\sin x) + e^{\sin x}(1)$$

$$\frac{dy}{dx} = xe^{\sin x}\cos x + e^{\sin x}$$

$$\frac{dy}{dx} = e^{-x}(3x^2+4x) + (x^3+2x^2+1)\, e^{-x}\,(-1)$$

$$= e^{-x}(3x^2+4x - x^3-2x^2-1)$$

$$\frac{dy}{dx} = e^{-x}(-x^3+x^2+4x-1)$$

$$\boxed{\frac{dy}{dx} = -e^{-x}(x^3-x^2-4x+1)}$$

(xi)

$$y = 5e^{3x-4}$$

Solution:

$y = 5e^{3x-4}$

Differentiate w.r.t "x"

$$\frac{dy}{dx} = 5\frac{d}{dx}(e^{3x-4})$$

$$= 5e^{3x-4}\frac{d}{dx}(3x-4)$$

$$\frac{dy}{dx} = 5e^{3x-4}(3)$$

$$\boxed{\frac{dy}{dx} = 15e^{3x-4}}$$

(xiii)

$$(\ln x)^{\ln x}$$

Solution:

Let $y = (\ln x)^{\ln x}$

Taking log on both sides

$\ln y = \ln(\ln x)^{\ln x}$

Differentiate w.r.t "x"

$$\frac{d}{dx}(\ln y) = \frac{d}{dx}\left[\ln x \cdot \ln(\ln x)\right]$$

$$\frac{1}{y}\frac{dy}{dx} = \ln x\frac{d}{dx}\ln(\ln x) + \ln(\ln x)\frac{d}{dx}(\ln x)$$

$$\frac{1}{y}\frac{dy}{dx} = \cancel{\ln x}\cdot\frac{1}{\cancel{\ln x}}\frac{d}{dx}(\ln x) + \ln(\ln x)\,\frac{1}{x}$$

$$\boxed{\frac{dy}{dx} = e^{\sin x}(x\cos x + 1)}$$

(xii)

$$y = (x+1)^x$$

Solution:

$y = (x+1)^x$

Taking log on both sides.

$\ln y = \ln(x+1)^x$

$\because \log m^n = n\ln m$

$\ln y = x\ln(x+1)$

Differentiate w.r.t "x".

$$\frac{d}{dx}(\ln y) = \frac{d}{dx}\left[x\ln(x+1)\right]$$

$$\frac{1}{y}\frac{dy}{dx} = x\frac{d}{dx}\ln(x+1) + \ln(x+1)\frac{d}{dx}(x)$$

$$\frac{1}{y}\frac{dy}{dx} = x\cdot\frac{1}{x+1}\frac{d}{dx}(x+1) + \ln(x+1)\,(1)$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{x}{x+1}\cdot(1) + \ln(x+1)$$

$$\frac{dy}{dx} = y\left[\frac{x}{x+1} + \ln(x+1)\right]$$

$$\boxed{\frac{dy}{dx} = (x+1)^x\left[\frac{x}{x+1} + \ln(x+1)\right]}$$

(xiv)

$$y = \frac{\sqrt{x^2-1}\,(x+1)}{(x^3+1)^{3/2}}$$

Solution:

$$y = \frac{\sqrt{x^2-1}\;(x+1)}{(x^3+1)^{3/2}}$$

$\boxed{\because a^2-b^2 = (a+b)(a-b)}$

$$y = \frac{\sqrt{(x+1)(x-1)}\;(x+1)}{(x^3+1)^{3/2}}$$

$\boxed{\because a^3+b^3 = a^2-ab+b^2}$

$$y = \frac{\sqrt{(x+1)(x-1)}\;(x+1)}{(x+1)^{3/2}(x^2-x+1)^{3/2}}$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x} + \ln(\ln x)\,\frac{1}{x}$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x}\left[1 + \ln(\ln x)\right]$$

$$\frac{dy}{dx} = y\left[\frac{1+\ln(\ln x)}{x}\right]$$

$$\boxed{\frac{dy}{dx} = (\ln x)^{\ln x}\left[\frac{1+\ln(\ln x)}{x}\right]}$$

Question #3

Find $\frac{dy}{dx}$ if:

(i)

$y = \cosh 2x$

$y = \cosh 2x$

Differentiate w.r.t 'x'

$$\frac{dy}{dx} = \frac{d}{dx}\cosh 2x$$

$$= \sinh 2x\,\frac{d}{dx}(2x)$$

$$\boxed{\frac{dy}{dx} = 2\sinh 2x}$$

(ii)

$y = \sinh 3x$

Solution:

$y = \sinh 3x$

Differentiate w.r.t 'x'

$$\frac{dy}{dx} = \frac{d}{dx}\sinh 3x$$

$$\frac{dy}{dx} = \cosh 3x\,\frac{d}{dx}(3x)$$

$$\boxed{\frac{dy}{dx} = 3\cosh 3x}$$

$$y = \frac{(x-1)^{1/2}\,(x+1)^{3/2}}{(x+1)^{3/2}\,(x^2-x+1)^{3/2}}$$

$$y = \frac{(x-1)^{1/2}}{(x^2-x+1)^{3/2}}$$

Taking log on both sides

$$\ln y = \ln\left[\frac{(x-1)^{1/2}}{(x^2-x+1)^{3/2}}\right]$$

$$\ln y = \ln(x-1)^{1/2} - \ln(x^2-x+1)^{3/2}$$

$$\ln y = \frac{1}{2}\ln(x-1) - \frac{3}{2}\ln(x^2-x+1)$$

Differentiate w.r.t 'x'

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{2}\cdot\frac{1}{(x-1)}\frac{d}{dx}(x-1) - \frac{3}{2}\cdot\frac{1}{x^2-x+1}\frac{d}{dx}(x^2-x+1)$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{2(x-1)}(1) - \frac{3}{2}\cdot\frac{1}{(x^2-x+1)}\cdot(2x-1)$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{2(x-1)} - \frac{3(2x-1)}{2(x^2-x+1)}$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{(x^2-x+1) - 3(2x-1)(x-1)}{2(x-1)(x^2-x+1)}$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{x^2-x+1 - 3(2x^2-2x-x+1)}{2(x-1)(x^2-x+1)}$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{x^2-x+1-6x^2+6x+3x-3}{2(x-1)(x^2-x+1)}$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{-5x^2+8x-2}{2(x-1)(x^2-x+1)}$$

$$\frac{dy}{dx} = y - \left[\frac{5x^2-8x+2}{2(x-1)(x^2-x+1)}\right]$$

$$\frac{dy}{dx} = \frac{-(x-1)^{1/2}}{(x^2-x+1)^{3/2}}\cdot\frac{5x^2-8x+2}{2(x-1)(x^2-x+1)}$$

$$\frac{dy}{dx} = -\frac{5x^2-8x+2}{2(x-1)^{1-\frac{1}{2}}(x^2-x+1)^{3/2+1}}$$

$$\frac{dy}{dx} = -\frac{5x^2-8x+2}{2(x-1)^{\frac{1}{2}}(x^2-x+1)^{5/2}}$$

$$\boxed{\frac{dy}{dx} = -\frac{+5x^2-8x+2}{2\sqrt{x-1}\,(x^2-x+1)^{5/2}}}$$

(iii)

$y = \tanh^{-1}(\sin x)$

Solution:

$y = \tanh^{-1}(\sin x)$

Differentiate w.r.t "x"

$$\frac{dy}{dx} = \frac{d}{dx}\left[\tanh^{-1}(\sin x)\right]$$

$$\frac{dy}{dx} = \frac{1}{1-\sin^2 x} \cdot \frac{d}{dx}(\sin x)$$

$\because 1-\sin^2 x = \cos^2 x$

$$\frac{dy}{dx} = \frac{1}{\cos^{\cancel{2}} x} \cdot \cancel{\cos x}$$

$$\frac{dy}{dx} = \frac{1}{\cos x}$$

$$\boxed{\frac{dy}{dx} = \sec x}$$

* * *

(iv)

$y = \sinh^{-1}(x^3)$

Solution:

$y = \sinh^{-1}(x^3)$

Differentiate w.r.t "x"

$$\frac{dy}{dx} = \frac{d}{dx} \sinh^{-1}(x^3)$$

$$= \frac{1}{\sqrt{1+x^6}} \cdot \frac{d}{dx}(x^3)$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1+x^6}} \cdot 3x^2$$

$$\boxed{\frac{dy}{dx} = \frac{3x^2}{\sqrt{1-x^6}}}$$

* * *

(v)

$y = (\ln \tanh x)$

Solution:

$y = (\ln \tanh x)$

Differentiate w.r.t "x"

$$\frac{dy}{dx} = \frac{d}{dx}\left[\ln \tanh x\right]$$

$$\frac{dy}{dx} = \frac{1}{\tanh x} \cdot \frac{d}{dx}(\tanh x)$$

$$\frac{dy}{dx} = \frac{1}{\frac{\sinh x}{\cosh x}} \cdot \operatorname{sech}^2 x$$

$$\frac{dy}{dx} = \frac{\cosh x}{\sinh x} \cdot \operatorname{sech}^2 x$$

$$\frac{dy}{dx} = \frac{\cancel{\cosh x}}{\sinh x} \cdot \frac{1}{\cosh^{\cancel{2}} x}$$

$$\frac{dy}{dx} = \frac{1}{\sinh x \cosh x}$$

Multiply and divided by "2"

$$\frac{dy}{dx} = \frac{2}{2 \sinh x \cosh x}$$

$$\frac{dy}{dx} = \frac{2}{\sinh 2x}$$

$$\boxed{\frac{dy}{dx} = 2 \operatorname{cosech} 2x}$$

* * *

(vi)

$y = \sinh^{-1}\left(\frac{x}{2}\right)$

Differentiate w.r.t "x"

$$\frac{dy}{dx} = \frac{1}{\sqrt{1 + \frac{x^2}{4}}} \cdot \frac{d}{dx}\left(\frac{x}{2}\right)$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{\frac{4+x^2}{4}}} \cdot \frac{1}{2}$$

$$\frac{dy}{dx} = \frac{1}{\frac{\sqrt{4+x^2}}{2}} \cdot \frac{1}{2}$$

$$\frac{dy}{dx} = \frac{\cancel{2}}{\sqrt{4+x^2}} \cdot \frac{1}{\cancel{2}}$$

$$\boxed{\frac{dy}{dx} = \frac{1}{\sqrt{x^2+4}}}$$

* Theory

• Example #1

Find higher derivative of the polynomial.

$$f(x)=\frac{1}{12}x^4-\frac{1}{6}x^3+\frac{1}{4}x^2+2x+7$$

Solution:

$$f(x)=\frac{1}{12}x^4-\frac{1}{6}x^3+\frac{1}{4}x^2+2x+7$$

Diff w.r.t "x"

$$f'(x)=\frac{d}{dx}\left[\frac{1}{12}x^4-\frac{1}{6}x^3+\frac{1}{4}x^2+2x+7\right]$$

$$f'(x)=\frac{1}{\cancel{12}_3}(4x^3)-\frac{1}{\cancel{6}_2}(3x^2)+\frac{1}{\cancel{4}_2}(\cancel{2}x)+2(1)+0$$

$$f'(x)=\frac{1}{3}x^3-\frac{1}{2}x^2+\frac{1}{2}x+2$$

Diff w.r.t "x"

$$f''(x)=\frac{d}{dx}\left[\frac{1}{3}x^3-\frac{1}{2}x^2+\frac{1}{2}x+2\right]$$

$$=\frac{1}{\cancel{3}}(3x^2)-\frac{1}{\cancel{2}}(\cancel{2}x)+\frac{1}{2}(1)+0$$

$$f''(x)=x^2-x+\frac{1}{2}$$

Diff w.r.t "x"

$$f'''(x)=\frac{d}{dx}\left[x^2-x+\frac{1}{2}\right]$$

$$f'''(x)=2x-1+0$$

$$f'''(x)=2x-1$$

Diff w.r.t "x"

$$f^{iv}(x)=\frac{d}{dx}(2x-1)$$

$$f^{iv}(x)=2\ (1)$$

$$f^{iv}(x)=2$$

All other higher derivatives are zero.

• Example #2

Find $\frac{d^3y}{dx^3}$ if $y=\ln(x+\sqrt{x^2+a^2})$

Solution:

$$y=\ln\left(x+\sqrt{x^2+a^2}\right)$$

Diff w.r.t "x" both sides

$$\frac{dy}{dx}=\frac{1}{x+\sqrt{x^2+a^2}}\frac{d}{dx}\left(x+\sqrt{x^2+a^2}\right)$$

$$\frac{dy}{dx}=\frac{1}{x+\sqrt{x^2+a^2}}\cdot\left(1+\frac{1}{2\sqrt{x^2+a^2}}\cdot\frac{d}{dx}(x^2+a^2)\right)$$

$$\frac{dy}{dx}=\frac{1}{x+\sqrt{x^2+a^2}}\cdot\left[1+\frac{1\times\cancel{2}x}{\cancel{2}\sqrt{x^2+a^2}}\right]$$

$$\frac{dy}{dx}=\frac{1}{x+\sqrt{x^2+a^2}}\cdot\left[1+\frac{x}{\sqrt{x^2+a^2}}\right]$$

$$\frac{dy}{dx}=\frac{1}{\cancel{x+\sqrt{x^2+a^2}}}\cdot\frac{\cancel{\sqrt{x^2+a^2}+x}}{\sqrt{x^2+a^2}}$$

$$\frac{dy}{dx}=\frac{1}{\sqrt{x^2+a^2}}$$

Diff w.r.t "x"

$$\frac{d^2y}{dx^2}=\frac{d}{dx}(x^2+a^2)^{\frac{-1}{2}}$$

$$\frac{d^2y}{dx^2}=\frac{-1}{2}(x^2+a^2)^{\frac{-1}{2}-1}\cdot\frac{d}{dx}(x^2+a^2)$$

$$=\frac{-1}{\cancel{2}}(x^2+a^2)^{\frac{-3}{2}}\cdot\cancel{2}x$$

$$\frac{d^2y}{dx^2}=-\frac{x}{(x^2+a^2)^{3/2}}$$

Diff w.r.t "x"

$$\frac{d^3y}{dx^3}=-\frac{(x^2+a^2)^{3/2}\cdot\frac{d}{dx}(x)-x\frac{d}{dx}(x^2+a^2)^{3/2}}{((x^2+a^2)^{3/2})^2}$$

$$\frac{d^3y}{dx^3}=-\frac{1\cdot(x^2+a^2)^{3/2}-x\frac{3}{2}(x^2+a^2)^{\frac{3}{2}-1}\frac{d}{dx}(x^2+a^2)}{(x^2+a^2)^3}$$

$$\frac{d^3y}{dx^3}=-1\cdot\frac{(x^2+a^2)^{3/2}-x\frac{3}{\cancel{2}}(x^2+a^2)^{\frac{1}{2}}(\cancel{2}x)}{(x^2+a^2)^3}$$

$$\frac{d^3y}{dx^3}=\frac{-1(x^2+a^2)^{1/2}\left[(x^2+a^2)-3x^2\right]}{(x^2+a^2)^3}$$

$$\frac{d^3y}{dx^3}=-\frac{(x^2+a^2-3x^2)}{(x^2+a^2)^{3-\frac{1}{2}}}$$

$$=-\frac{(a^2-2x^2)}{(x^2+a^2)^{5/2}}$$

$$\boxed{\frac{d^3y}{dx^3}=\frac{2x^2-a^2}{(x^2+a^2)^{5/2}}}\quad \text{Ans.}$$

- ## Example #3

Find $\frac{d^2y}{dx^2}$ if $y^3+3ax^2+x^3=0$

Solution:

$y^3+3ax^2+x^3=0$ — Ⓐ

Diff w.r.t 'x'

$\frac{d}{dx}(y^3+3ax^2+x^3)=\frac{d}{dx}(0)$

$3y^2\frac{dy}{dx}+3a(2x)+3x^2=0$

Divided by 3

$y^2\frac{dy}{dx}+2ax+x^2=0$

$y^2\frac{dy}{dx}=-2ax-x^2$

$y^2\frac{dy}{dx}=-(2ax+x^2)$

$\frac{dy}{dx}=-\frac{(2ax+x^2)}{y^2}$

Diff w.r.t 'x'

$\frac{d^2y}{dx^2}=(-1)\frac{d}{dx}\left[\frac{2ax+x^2}{y^2}\right]$

$=-\frac{y^2\frac{d}{dx}(2ax+x^2)-(2ax+x^2)\frac{d}{dx}(y^2)}{(y^2)^2}$

$=-\frac{y^2(2a+2x)-(2ax+x^2)(2y\frac{dy}{dx})}{y^4}$ $\boxed{\because \frac{dy}{dx}=\frac{-(2ax+x^2)}{y^2}}$

$=-\frac{y^2(2(a+x)-(2ax+x^2)2\cancel{y}\times\left[-\frac{2ax+x^2}{y^{\cancel{2}}}\right]}{y^4}$

$\frac{d^2y}{dx^2}=-2\frac{\left[(a+x)y^2+\frac{(2ax+x^2)(2ax+x^2)}{y}\right]}{y^4}$

$=-2\frac{[(a+x)y^3+(2ax+x^2)^2}{y^4\cdot y}$ $\boxed{\because y^3=-3ax^2-x^3}$

$=-2\frac{[(a+x)(-3ax^2-x^3)+x^2(2a+x)^2]}{y^5}$

$=-\frac{2x^2[-(a+x)(3a+x)+(4a^2+x^2+4ax)]}{y^5}$

$=-\frac{2x^2[-(3a^2+4ax+x^2)+4a^2+x^2+4ax]}{y^5}$

$=-\frac{2x^2-3a^2\cancel{-4ax}\cancel{-x^2}+4a^2\cancel{+x^2}\cancel{+4ax}}{y^5}$

$\frac{d^2y}{dx^2}=-\frac{2x^2\{a^2\}}{y5}$ $\boxed{\frac{d^2y}{dx^2}=-\frac{2a^2x^2}{y^5}}$ Proved.

Example #4

If $x=(a(\theta-\sin\theta))$ $y=a(1+\cos\theta)$;

Then show that $y^2\frac{d^2y}{dx^2}+a=0$

Solution:

$x=a(\theta-\sin\theta)$ — ①

$y=a(1+\cos\theta)$ — ②

Diff w.r.t eq ①, ② 'θ'

$\frac{dx}{d\theta}=a(1+\cos\theta)$

$\frac{dy}{d\theta}=a(-\sin\theta)$

Using chain rule

$\frac{dy}{dx}=\frac{dy}{d\theta}\cdot\frac{d\theta}{dx}$

$=-\cancel{a}\sin\theta\cdot\frac{1}{\cancel{a}(1+\cos\theta)}$

$\frac{dy}{dx}=\frac{-\sin\theta}{1+\cos\theta}$

Diff w.r.t 'x'

$\frac{d^2y}{dx^2}=-\frac{d}{d\theta}\left[\frac{\sin\theta}{1+\cos\theta}\right]\frac{d\theta}{dx}$

$\frac{d^2y}{dx^2}=-\frac{(1+\cos\theta)\frac{d}{dx}(\sin\theta)-\sin\theta\frac{d}{dx}(1+\cos\theta)}{(1+\cos\theta)^2}\cdot\frac{d\theta}{dx}$

$\frac{d^2y}{dx^2}=-\frac{(1+\cos\theta)(\cos\theta)-\sin\theta(-\sin\theta)}{(1+\cos\theta)^2}\cdot\frac{d\theta}{dx}$

$\frac{d^2y}{dx^2}=-\frac{(\cos\theta+\cos^2\theta+\sin^2\theta)}{(1+\cos\theta)^2}\cdot\frac{d\theta}{dx}$

$\boxed{\because \cos^2\theta+\sin^2\theta=1}$

$\frac{d^2y}{dx^2}=-\frac{1+\cos\theta}{(1+\cos\theta)^2}\cdot\frac{d\theta}{dx}$

$\boxed{\because \frac{d\theta}{dx}=\frac{1}{a(1+\cos\theta)}}$

$\frac{d^2y}{dx^2}=-\frac{\cancel{1+\cos\theta}}{(1+\cos\theta)^2}\times\frac{1}{a(\cancel{1+\cos\theta})}$

$=-\frac{1}{a}\cdot\frac{1}{(1+\cos\theta)^2}$

$\boxed{\because 1+\cos\theta=\frac{y}{a}}$

$\frac{d^2y}{dx^2}=-\frac{1}{a}\cdot\frac{1}{(\frac{y}{a})^2}$

$=\frac{-1}{\cancel{a}}\cdot\frac{a^{\cancel{2}}}{y^2}$

$\frac{d^2y}{dx^2}=\frac{-a}{y^2}\Rightarrow$ $\boxed{y^2\frac{d^2y}{dx^2}+a=0}$ Proved.

Example #5

Find first four derivatives of cos (ax+b).

Solution:

Let $y = \cos(ax+b)$

Diff w.r.t "x"

$$\frac{dy}{dx} = \frac{d}{dx}(\cos(ax+b)$$

$$y_1 = -\sin(ax+b)\frac{d}{dx}(ax+b)$$

$$y_1 = -\sin(ax+b)(a)$$

$$y_1 = -a\sin(ax+b)$$

Diff w.r.t "x"

$$y_2 = \frac{d}{dx}[-a\sin(ax+b)]$$

$$y_2 = -a\cos(ax+b)\frac{d}{dx}(ax+b)$$

$$y_2 = -a^2\cos(ax+b)$$

Diff w.r.t "x"

$$y_3 = -a^2\frac{d}{dx}\cos(ax+b)$$

$$y_3 = -a^2(-\sin(ax+b)\frac{d}{dx}(ax+b))$$

$$y_3 = +a^2\sin(ax+b)(a)$$

$$y_3 = a^3\sin(ax+b)$$

Diff w.r.t "x"

$$y_4 = a^3\frac{d}{dx}\sin(ax+b)$$

$$y_4 = a^3.\cos(ax+b)\frac{d}{dx}(ax+b)$$

$$y_4 = a^3.\cos(ax+b)(a)$$

$$\boxed{y_4 = a^4\cos(ax+b)}$$

* * *

Example #6

If $y = e^{-ax}$ then show that $\frac{d^3y}{dx^3} + a^3y = 0$

Solution:

$$y = e^{-ax}$$

Diff w.r.t "x" both sides

$$\frac{dy}{dx} = \frac{d}{dx}(e^{-ax})$$

$$\frac{dy}{dx} = e^{-ax}.\frac{d}{dx}(-ax)$$

$$\frac{dy}{dx} = e^{-ax}.(-a)$$

$$\boxed{\because y = e^{-ax}}$$

$$\frac{dy}{dx} = -ay$$

Diff w.r.t "x"

$$\frac{d^2y}{dx^2} = \frac{d}{dx}[-ay]$$

$$\frac{d^2y}{dx^2} = -a\left(\frac{dy}{dx}\right)$$

$$= -a(-ay)$$

$$\boxed{\because \frac{dy}{dx} = -ay}$$

$$\frac{d^2y}{dx^2} = a^2y$$

Diff w.r.t "x"

$$\frac{d^3y}{dx^3} = a^2\frac{dy}{dx}$$

$$= a^2(-ay)$$

$$\frac{d^3y}{dx^3} = -a^3y$$

$$\boxed{\frac{d^3y}{dx^3} + a^3y = 0}$$ proved.

* * *

Example # 7

If $y = \sin^{-1}\left(\frac{x}{a}\right)$, then show that $y_2 = x(a^2-x^2)^{-\frac{3}{2}}$

Solution:

$$y = \sin^{-1}\left(\frac{x}{a}\right)$$

Diff w.r.t 'x' both sides.

$$y_1 = \frac{d}{dx}\left(\sin^{-1}\left(\frac{x}{a}\right)\right)$$

$$y_1 = \frac{1}{\sqrt{1-\left(\frac{x}{a}\right)^2}} \cdot \frac{d}{dx}\left(\frac{x}{a}\right)$$

$$y_1 = \frac{1}{\sqrt{1-\frac{x^2}{a^2}}} \cdot \frac{1}{a}$$

$$y_1 = \frac{1}{\sqrt{\frac{a^2-x^2}{a^2}}} \cdot \frac{1}{a}$$

$$y_1 = \frac{1}{\frac{\sqrt{a^2-x^2}}{a}} \cdot \frac{1}{a}$$

$$y_1 = \frac{\cancel{a}}{\sqrt{a^2-x^2}} \cdot \frac{1}{\cancel{a}}$$

$$y_1 = \frac{1}{\sqrt{a^2-x^2}} \Rightarrow (a^2-x^2)^{\frac{-1}{2}}$$

Diff w.r.t 'x' both sides

$$y_2 = \frac{d}{dx}(a^2-x^2)^{\frac{-1}{2}}$$

$$y_2 = \frac{-1}{2}(a^2-x^2)^{\frac{-1}{2}-1} \cdot \frac{d}{dx}(a^2-x^2)$$

$$y_2 = \frac{-1}{\cancel{2}}(a^2-x^2)^{\frac{-3}{2}} \cdot (-\cancel{2}x)$$

$$\boxed{y_2 = x\,(a^2-x^2)^{\frac{-3}{2}}}$$ Hence proved.

Exercise # 2.7

Question # 1

Find y_2 if:

(i)

$$y = 2x^5 - 3x^4 + 4x^3 + x - 2$$

Solution:

$$y = 2x^5 - 3x^4 + 4x^3 + x - 2$$

Differentiation w.r.t 'x'

$$y_1 = \frac{d}{dx}(2x^5 - 3x^4 + 4x^3 + x - 2)$$

$$y_1 = 10x^4 - 12x^3 + 12x^2 + 1 - 0$$

$$y_1 = 10x^4 - 12x^3 + 12x^2 + 1$$

Differentiation w.r.t 'x'

$$y_2 = \frac{d}{dx}(10x^4 - 12x^3 + 12x^2 + 1)$$

$$y_2 = 40x^3 - 36x^2 + 24x + 0$$

$$\boxed{y_2 = 40x^3 - 36x^2 + 24x}$$

(iii)

$$y = \sqrt{x} + \frac{1}{\sqrt{x}}$$

Solution:

$$y = \sqrt{x} + \frac{1}{\sqrt{x}}$$

$$y = x^{1/2} + \frac{1}{x^{1/2}}$$

$$y = x^{\frac{1}{2}} + x^{\frac{-1}{2}}$$

(ii)

$$y = (2x+5)^{3/2}$$

Solution:

$$y = (2x+5)^{3/2}$$

Differentiation w.r.t 'x'

$$y_1 = \frac{d}{dx}(2x+5)^{3/2}$$

$$y_1 = \frac{3}{2}(2x+5)^{\frac{3}{2}-1}\frac{d}{dx}(2x+5)$$

$$y_1 = \frac{3}{\cancel{2}}(2x+5)^{\frac{1}{2}}(\cancel{2})$$

$$y_1 = 3(2x+5)^{\frac{1}{2}}$$

Differentiation w.r.t 'x'

$$y_2 = \frac{d}{dx}\left[3(2x+5)^{\frac{1}{2}}\right]$$

$$= 3\cdot\left(\frac{1}{2}\right)(2x+5)^{\frac{1}{2}-1}\frac{d}{dx}(2x+5)$$

$$y_2 = \frac{3}{\cancel{2}}(2x+5)^{-\frac{1}{2}}(\cancel{2})$$

$$y_2 = 3(2x+5)^{\frac{-1}{2}}$$

$$\boxed{y_2 = \frac{3}{\sqrt{2x+5}}}$$

Differentiation w.r.t "x"

$y_1 = \frac{d}{dx}\left(x^{1/2} + x^{\frac{-1}{2}}\right)$

$y_1 = \frac{1}{2}x^{\frac{1}{2}-1} + \left(\frac{-1}{2}x^{\frac{-1}{2}-1}\right)$

$y_1 = \frac{1}{2}x^{\frac{-1}{2}} - \frac{1}{2}x^{-3/2}$

$y_1 = \frac{1}{2x^{1/2}} - \frac{1}{2x^{3/2}}$

Differentiation w.r.t "x"

$y_2 = \frac{d}{dx}\left[\frac{1}{2x^{1/2}} - \frac{1}{2x^{3/2}}\right]$

$y_2 = \frac{1}{2}\frac{d}{dx}\left(x^{-1/2} - x^{-3/2}\right]$

$y_2 = \frac{1}{2}\left[\frac{-1}{2}x^{\frac{-1}{2}-1} + \frac{3}{2}x^{\frac{-3}{2}-1}\right]$

$y_2 = \frac{1}{2}\left[\frac{-1}{2}x^{-\frac{3}{2}} + \frac{3}{2}x^{-\frac{5}{2}}\right]$

$y_2 = -\frac{1}{4}x^{-\frac{3}{2}} + \frac{3}{4}x^{-\frac{5}{2}}$

$y_2 = \frac{-1}{4}x \cdot x^{-\frac{5}{2}} + \frac{3}{4}x^{-\frac{5}{2}}$

$y_2 = \frac{-x}{4x^{5/2}} + \frac{3}{4x^{5/2}}$

$$\boxed{y_2 = \frac{-x+3}{4x^{5/2}}}$$

Question #2

Find y_2 if:

(i)

$y = x^2 \cdot e^{-x}$

Solution

$y = x^2 \cdot e^{-x}$

Differentiation w.r.t "x"

$y_1 = \frac{d}{dx}\left(x^2 \cdot e^{-x}\right)$

$y_1 = x^2 \cdot \frac{d}{dx}(e^{-x}) + e^{-x}\frac{d}{dx}(x^2)$

$y_1 = x^2(e^{-x})\frac{d}{dx}(-x) + e^{-x} \cdot 2x$

$y_1 = e^{-x}x^2(-1) + 2xe^{-x}$

$y_1 = e^{-x}\left[-x^2 + 2x\right]$

Differentiation w.r.t "x"

$y_2 = e^{-x}\frac{d}{dx}(-x^2+2x) + (-x^2+2x)\frac{d}{dx}e^{-x}$

$y_2\ e^{-x}(-2x)+2 + (-x^2+2x)e^{-x} \cdot \frac{d}{dx}(-x)$

$y_2 = e^{-x}(-2x+2) + (-x^2+2x)e^{-x}(-1)$

$y_2 = -2xe^{-x} + 2e^{-x} - e^{-2}2x + x^2e^{-x}$

$y_2 = e^{-x}\left[-2x+2-2x+x^2\right]$

$$\boxed{y_2 = e^{-x}\left[x^2 - 4x + 2\right]}$$

$y = \ln\left[\frac{2x+3}{3x+2}\right]$ (ii)

Solution:

$y = \ln\left[\frac{2x+3}{3x+2}\right]$

$$\boxed{\because \ln\frac{m}{n} = \ln m - \ln n}$$

$y = \ln(2x+3) - \ln(3x+2)$

$y_1 = \frac{d}{dx}\left[\ln(2x+3) - \ln(3x+2)\right]$

$y_1 = \frac{1}{2x+3}\frac{d}{dx}(2x+3) - \frac{1}{3x+2}\frac{d}{dx}(3x+2)$

$$y_1 = \frac{1}{2x+3}(2) - \frac{1}{3x+2}(3)$$

$$y_1 = \frac{2(3x+2) - 3(2x+3)}{(2x+3)(3x+2)}$$

$$y_1 = \frac{\cancel{6x}+4-\cancel{6x}-9}{(2x+3)(3x+2)}$$

$$y_1 = \frac{-5}{(2x+3)(3x+2)}$$

Differentiation w.r.t 'x'

$$y_2 = \frac{d}{dx}\frac{-5}{(2x+3)(3x+2)}$$

$$y_2 = \frac{(2x+3)(3x+2)\frac{d}{dx}(-5) - (-5)\frac{d}{dx}(2x+3)(3x+2)}{[(2x+3)(3x+2)]^2}$$

$$y_2 = \frac{(2x+3)(3x+2)(0) + 5\,(2x+3)\frac{d}{dx}(3x+2) + (3x+2)\frac{d}{dx}(2x+3)}{(2x+3)^2\,(3x+2)^2}$$

$$y_2 = \frac{0 + 5\,(2x+3)(3) + (3x+2)(2)}{(2x+3)^2\,(3x+2)^2}$$

$$y_2 = \frac{5(6x+9+6x+4)}{(2x+3)^2(3x+2)^2}$$

$$y_2 = \frac{5(12x+13)}{(2x+3)^2(3x+2)^2}$$

$$\boxed{y_2 = \frac{60x+65}{(2x+3)^2(3x+2)^2}}$$

Question #3

Find y_2 if:

(i)

$$x^2+y^2=a^2$$

Solution:

$$x^2+y^2=a^2$$

Differentiation w.r.t "x"

$$\frac{d}{dx}(x^2+y^2=a^2)$$

$$2x+2yy_1=0$$

$$y_1=\frac{-\cancel{2}x}{\cancel{2}y}$$

$$y_1=\frac{-x}{y}$$

Differentiation w.r.t "x"

$$y_2=\frac{d}{dx}\left[\frac{-x}{y}\right]$$

$$y_2=\frac{y\frac{d}{dx}(-x)-(-x)\frac{d}{dx}(y)}{y^2}$$

$$y_2=\frac{y(-1)+x\frac{dy}{dx}}{y_2}$$

$$\boxed{\because \frac{dy}{dx}=\frac{-x}{y}}$$

$$y_2=\frac{-1}{y_2}\left[y+x\left(\frac{-x}{y}\right)\right]$$

$$y_2=\frac{-1}{y^2}\left[y+\left(\frac{-x^2}{y}\right)\right]$$

$$y_2=\frac{-1}{y^2}\left[\frac{y^2-x^2}{y}\right]$$

$$y_2=\frac{-y^2+x^2}{y^3}$$

$$y_2=\frac{-(x^2+y^2)}{y^3}$$

$$\boxed{\because x^2+y^2=a^2}$$

$$\boxed{y_2=\frac{-a^2}{y^3}}$$

• ———— •

$x^3-y^3=a^3$ (ii)

Solution:

$$x^3-y^3=a^3$$

Differentiation w.r.t "x"

$$\frac{d}{dx}(x^3)-\frac{d}{dx}(y^3)=\frac{d}{dx}(a^3)$$

$$3x^2-3y^2y_1=0$$

$$y_1=\frac{-\cancel{3}x^2}{-\cancel{3}y^2}$$

$$y_1=\frac{x^2}{y^2}$$

Differentiation w.r.t "x"

$$y_2=\frac{y^2\frac{d}{dx}(x^2)-x^2\frac{d}{dx}(y^2)}{(y^2)^2}$$

$$y_2=\frac{y^2(2x)-x^2\cdot 2y\frac{dy}{dx}}{y^4}$$

$$\boxed{\because \frac{dy}{dx}=\frac{x^2}{y^2}}$$

$$y_2=\frac{1}{y^4}\left[2xy^2-2yx^2\left(\frac{x^2}{y^2}\right)\right]$$

$$y_2=\frac{1}{y^4}\left[\frac{2xy^4-2x^4y}{y^2}\right]$$

$$y_2=\frac{1}{y^4}\left[\frac{y(2xy^3-2x^4)}{y^2}\right]$$

$$y_2=\frac{y}{y^{65}}(2xy^3-2x^4)$$

$$y_2=\frac{-2x(-y^3+x^3)}{y^5}$$

$$y_2=\frac{-2x(x^3-y^3)}{y^5}$$

$$\boxed{\because x^3-y^3=a^3}$$

$$\boxed{y_2=\frac{-2xa^3}{y^5}}$$

• ———— •

(iii)

$x = a\cos\theta$; $y = a\sin\theta$

Solution:

$x = a\cos\theta$ — ①

$y = a\sin\theta$ — ②

Differentiation eq ① w.r.t 'θ'

$$\frac{dx}{d\theta} = \frac{d}{d\theta}(a\cos\theta)$$

$\boxed{\because \cos\theta = -\sin\theta}$

$$\frac{dx}{d\theta} = -a\sin\theta$$

Differentiation eq ② w.r.t 'θ'

$$\frac{dy}{d\theta} = \frac{d}{d\theta}(a\sin\theta)$$

$\boxed{\because \sin\theta = \cos\theta}$

$$\frac{dy}{d\theta} = a\cos\theta$$

Using chain rule

$$\frac{dy}{dx} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dx}$$

$$y_1 = \cancel{a}\cos\theta \cdot \frac{1}{-\cancel{a}\sin\theta}$$

$$y_1 = -\frac{\cos\theta}{\sin\theta}$$

$$y_1 = -\cot\theta$$

Differentiation w.r.t to 'x'

$$y_2 = \frac{d}{dx}(-\cot\theta)$$

$\boxed{\because \cot\theta = -\operatorname{cosec}^2\theta}$

$$y_2 = -(-\operatorname{cosec}^2\theta) \cdot \frac{d\theta}{dx}$$

$$y_2 = \operatorname{cosec}^2\theta \cdot \frac{-1}{a\sin\theta}$$

$$y_2 = \frac{1}{\sin^2\theta} \cdot \frac{-1}{a\sin\theta}$$

$$\boxed{y_2 = \frac{-1}{a\sin^3\theta}}$$

(iv)

$x = at^2$; $y = bt^4$

Solution:

$x = at^2$ — ①

$y = bt^4$ — ②

Differentiation eq ① w.r.t 't'

$$\frac{dx}{dt} = \frac{d}{dt}(at^2)$$

$$\frac{dx}{dt} = 2at$$

Differentiation eq ② w.r.t 't'

$$\frac{dy}{dt} = \frac{d}{dt}(bt^4)$$

$$\frac{dy}{dt} = 4bt^3$$

Using chain rule

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$

$$= {}^{2}\cancel{4}bt^{\cancel{3}2} \cdot \frac{1}{\cancel{2}a\cancel{t}}$$

$$y_1 = \frac{2bt^2}{a}$$

$$y_1 = \frac{2b}{a}\left(\frac{x}{a}\right)$$

$\boxed{\because t^2 = \frac{x}{a}}$

$$y_1 = \frac{2bx}{a^2}$$

Differentiation w.r.t 'x'

$$y_2 = \frac{d}{dx}\left(\frac{2bx}{a^2}\right)$$

$$y_2 = \frac{2b}{a^2}\frac{d}{dx}(x)$$

$$y_2 = \frac{2b}{a^2}(1)$$

$$\boxed{y_2 = \frac{2b}{a^2}}$$

(v)

$$x^2+y^2+2gx+2fy+c=0$$

Solution:

$$x^2+y^2+2gx+2fy+c=0$$

Differentiation w.r.t "x"

$$\frac{d}{dx}\left(x^2+y^2+2gx+2fy+c\right)=0$$

$$2x+2yy_1+2g+2fy_1+0=0$$

$$2x+2g+2yy_1+2fy_1=0$$

$$(2y+2f)y_1=-2x-2g$$

$$2y_1(y+f)=-2(x+g)$$

$$y_1=\frac{-\cancel{2}(x+g)}{\cancel{2}(y+f)}$$

$$y_1=\frac{-(x+g)}{y+f} \Rightarrow \frac{-x-g}{y+f}$$

Differentiation w.r.t "x"

$$y_2=\frac{d}{dx}\frac{-(x+g)}{(y+f)}$$

$$y_2=\frac{(y+f)\frac{d}{dx}-(x+g)-(-(x+g))\frac{d}{dx}(y+f)}{(y+f)^2}$$

$$y_2=\frac{(y+f)(-1)-(-x-g)\frac{dy}{dx}}{(y+f)^2}$$

$$\boxed{\because \frac{dy}{dx}=\frac{-x-g}{y+f}}$$

$$y_2=\frac{1}{(y+f)^2}\left[-(y+f)-(-x-g)\left(\frac{-x-g}{y+f}\right)\right]$$

$$y_2=\frac{1}{(y+f)^2}\left[\frac{-(y+f)^2-(-x-g)^2}{y+f}\right]$$

$$y_2=\frac{1}{(y+f)^3}\left[-(y^2+f^2+2fy)-(x^2+g^2+2gx)\right]$$

$$y_2=\frac{1}{(y+f)^3}\left[-y^2-f^2-2fy-x^2-g^2-2gx\right]$$

$$\boxed{\because c=-x^2-y^2-2gx-2fy}$$

$$\boxed{y_2=\frac{c-f^2-g^2}{(y+f)^3}}$$

Question #4

Find y_4 if:

(i)

$$y=\sin 3x$$

Solution:

$$y=\sin 3x$$

Diff w.r.t 'x'

$$y_2=\frac{d}{dx}(\sin 3x)$$

$$\boxed{\because \sin x=\cos x}$$

$$y_1=\cos 3x\,\frac{d}{dx}(3x)$$

$$y_1=3\cos 3x$$

Diff w.r.t "x"

$$y_2=\frac{d}{dx}(3\cos 3x)$$

$$\boxed{\because \cos x=-\sin x}$$

$$y_2=-3\sin 3x\,\frac{d}{dx}(3x)$$

$$y_2=-9\sin 3x$$

Diff w.r.t 'x'

$$y_3=\frac{d}{dx}(-9\sin 3x)$$

$$y_3=-9\cos 3x\,\frac{d}{dx}(3x)$$

$$y_3=-27\cos 3x$$

Diff w.r.t "x"

$$y_4=\frac{d}{dx}(-27\cos 3x)$$

$$y_4=-27(-\sin 3x)\,\frac{d}{dx}(3x)$$

$$\boxed{y_4=81\sin 3x}$$

(ii)

$y = \cos^3 x$

Solution:

$y = \cos^3 x$

Diff w.r.t 'x'

$\frac{dy}{dx} = \frac{d}{dx} \cos^3 x$

$= 3\cos^2 x \frac{d}{dx}(\cos x)$

$y_1 = 3\cos^2 x (-\sin x)$

$\boxed{\because \cos^2 x = 1 - \sin^2 x}$

$y_1 = 3(1-\sin^2 x)(-\sin x)$

$y_1 = -3\sin x + 3\sin^3 x$

Diff w.r.t 'x'

$y_2 = \frac{d}{dx}(-3\sin x + 3\sin^3 x)$

$y_2 = -3\cos x + 9\sin^2 x \frac{d}{dx}(\sin x)$

$y_2 = -3\cos x + 9\sin^2 x \cos x$

$\boxed{\because \sin^2 x = 1 - \cos^2 x}$

$y_2 = -3\cos x + 9(1-\cos^2 x)\cos x$

$y_2 = -3\cos x + 9\cos x - 9\cos^3 x$

$y_2 = 6\cos x - 9\cos^3 x$

Diff w.r.t 'x'

$y_3 = \frac{d}{dx}(6\cos x - 9\cos^3 x)$

$y_3 = -6\sin x - (27\cos^2 x) \frac{d}{dx}(\cos x)$

$y_3 = -6\sin x - 27(1-\sin^2 x)(-\sin x)$

$y_3 = -6\sin x + 27\sin x - 27\sin^3 x$

$y_3 = 21\sin x - 27\sin^3 x$

Diff w.r.t 'x'

$y_4 = \frac{d}{dx}(21\sin x - 27\sin^3 x)$

$y_4 = 21\cos x - 81\sin^2 x \frac{d}{dx}(\sin x)$

$y_4 = 21\cos x - 81(1-\cos^2 x)(\cos x)$

$y_4 = 21\cos x - 81\cos x + 81\cos^3 x$

$$\boxed{y_4 = -60\cos x + 81\cos^3 x}$$

(iii)

$\ln(x^2-9)$

Solution:

$y = \ln(x^2-9)$

$\boxed{\because a^2 - b^2 = (a-b)(a+b)}$

$y = \ln(x-3)(x+3)$

$\boxed{\because \ln(m)(n) = \ln m + \ln n}$

$y = \ln(x-3) + \ln(x+3)$

Diff w.r.t 'x'

$y_1 = \frac{d}{dx}[\ln(x-3)] + \frac{d}{dx}[\ln(x+3)]$

$y_1 = \frac{1}{x-3}\frac{d}{dx}(x-3) + \frac{1}{(x+3)}\frac{d}{dx}(x+3)$

$y_1 = \frac{1}{x-3} + \frac{1}{x+3}$

$y_1 = (x-3)^{-1} + (x+3)^{-1}$

Diff w.r.t 'x'

$y_2 = \frac{d}{dx}((x-3)^{-1} + (x+3)^{-1})$

$y_2 = -1(x-3)^{-1-1} - 1(x+3)^{-1-1}$

$y_2 = -1(x+3)^{-2} - 1(x+3)^{-2}$

Diff w.r.t 'x'

$y_3 = \frac{d}{dx}[-1(x-3)^{-2} - 1(x+3)^{-2}]$

$y_3 = 2(x-3)^{-3} + 2(x+3)^{-3}$

Diff w.r.t 'x'

$y_4 = \frac{d}{dx}[2(x-3)^{-3} + 2(x+3)^{-3}]$

$y_4 = -6(x-3)^{-3-1} - 6(x+3)^{-3-1}$

$y_4 = -6(x-3)^{-4} - 6(x+3)^{-4}$

$y_4 = \frac{-6}{(x-3)^4} - \frac{6}{(x+3)^4}$

$$\boxed{y_4 = -6\left[\frac{1}{(x-3)^4} + \frac{1}{(x+3)^4}\right]}$$

Question #5

If $x = \sin\theta$, $y = \sin m\theta$, show that $(1-x^2)y_2 - xy_1 + m^2y = 0$

Solution:

$$x = \sin\theta \text{ ——— } ①$$
$$y = \sin m\theta \text{ ——— } ②$$

From eq ①

$$x = \sin\theta$$
$$\theta = \sin^{-1}x$$

put in eq ②

$$y = \sin(m\sin^{-1}x)$$

Diff w.r.t 'x' both sides

$$y_1 = \frac{d}{dx}\left[\sin(m\sin^{-1}x)\right]$$

$$y_1 = \cos(m\sin^{-1}x)\frac{d}{dx}(m\sin^{-1}x)$$

$$y_1 = \cos(m\sin^{-1}x)\; m \cdot \frac{1}{\sqrt{1-x^2}}$$

$$\sqrt{1-x^2}\, y_1 = m\cos(m\sin^{-1}x)$$

Diff w.r.t 'x' both side

$$\frac{d}{dx}\sqrt{1-x^2}\, y_1 = m\frac{d}{dx}\cos(m\sin^{-1}x)$$

$$\sqrt{1-x^2}\frac{d}{dx}(y_1) + y_1\frac{d}{dx}\sqrt{1-x^2} = m - \sin(m\sin^{-1}x)\frac{d}{dx}(m\sin^{-1}x)$$

$$\sqrt{1-x^2}\; y_2 + y_1 \cdot \frac{1}{2\sqrt{1-x^2}}\frac{d}{dx}(1-x^2) = -m\sin(m\sin^{-1}x)\; m \cdot \frac{1}{\sqrt{1-x^2}}$$

$$\sqrt{1-x^2}\; y_2 + y_1 \cdot \frac{-\cancel{2}x}{\cancel{2}\sqrt{1-x^2}} = -m^2\sin(m\sin^{-1}x)\frac{1}{\sqrt{1-x^2}}$$

Multiply by $\sqrt{1-x^2}$ both sides

$$(1-x^2)y_2 + xy_1 = -m^2\sin(m\sin^{-1}x)$$

$$\boxed{\because y = \sin(m\sin^{-1}x)}$$

$$(1-x^2)y_2 - xy_1 = -m^2y$$

$$\boxed{(1-x^2)y_2 - xy_1 + m^2y = 0}$$ Hence proved.

•———•

Question #6

If $y = e^x\sin x$, show that $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$

Solution:

$$y = e^x\sin x$$

Diff w.r.t 'x' both sides

$$\frac{dy}{dx} = \frac{d}{dx}(e^x\sin x)$$

$$\frac{dy}{dx} = e^x\frac{d}{dx}\sin x + \sin x\frac{d}{dx}\cdot e^x$$

$$\frac{dy}{dx} = e^x\cos x + \sin x \cdot e^x\frac{d}{dx}(x)$$

$$\frac{dy}{dx} = e^x\cos x + e^x\sin x$$

$$\frac{dy}{dx} = e^x(\cos x + \sin x)$$

Diff w.r.t 'x' both sides

$$\frac{d^2y}{dx^2} = e^x(-\sin x + \cos x) + (\cos x + \sin x)\frac{d}{dx}(e^x)$$

$$= -e^x\sin x + e^x\cos x + (\cos x + \sin x)e^x$$

$$= -e^x\cancel{\sin x} + e^x\cos x + e^x\cos x + e^x\cancel{\sin x}$$

$$\frac{d^2y}{dx^2} = 2e^x\cos x$$

L.H.S =

$$= 2e^x\cos x - 2(e^x(\cos x + \sin x)) + 2(e^x\sin x)$$

$$= 2e^x\cancel{\cos x} - 2e^x\cancel{\cos x} - 2e^x\cancel{\sin x} + 2e^x\cancel{\sin x}$$

$$= 0 \quad R.H.S.$$

L.H.S. = R.H.S.

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Question #7

If $y = e^{ax}\sin bx$, show that $\frac{d^2y}{dx^2} - 2a\frac{dy}{dx} + (a^2+b^2)y = 0$

Solution:

$$y = e^{ax}\sin bx$$

Diff w.r.t "x" both sides

$$\frac{dy}{dx} = \frac{d}{dx}(e^{ax}\sin bx)$$

$$\frac{dy}{dx} = e^{ax}\cos bx \frac{d}{dx}(bx) + \sin bx \cdot e^{ax}\frac{d}{dx}(ax)$$

$$\frac{dy}{dx} = e^{ax}\cos bx \cdot b + \sin bx \cdot e^{ax}(a)$$

$$\frac{dy}{dx} = be^{ax}\cos bx + ae^{ax}\sin bx$$

$$\frac{d^2y}{dx^2} = b\frac{d}{dx}(e^{ax} \cdot \cos bx) + a\frac{d}{dx}(e^{ax}\sin bx)$$

$$\frac{d^2y}{dx^2} = b\left[e^{ax}(-\sin bx)\cdot b + \cos bx \cdot e^{ax}(a)\right] + a\left[e^{ax}(\cos bx)\cdot b + \sin bx \cdot e^{ax}\cdot(a)\right]$$

$$\frac{d^2y}{dx^2} = b^2(e^{ax} - \sin bx + \cos bx \cdot ae^{ax}) + abe^{ax}\cos bx + \sin bx \cdot a^2e^{ax}$$

$$\frac{d^2y}{dx^2} = -b^2e^{ax}\sin bx + abe^{ax}\cos bx + abe^{ax}\cos bx + a^2e^{ax}\sin bx$$

$$L.H.S. = \frac{d^2y}{dx^2} - 2a\frac{dy}{dx} + (a^2+b^2)y$$

$$-b^2e^{ax}\sin bx + abe^{ax}\cos bx + abe^{ax}\cos bx + a^2e^{ax}\sin bx - 2a(be^{ax}\cos bx + ae^{ax}\sin bx) + (a^2+b^2)e^{ax}\sin bx$$

$$\cancel{-b^2e^{ax}}\sin bx + abe^{ax}\cos bx + abe^{ax}\cos bx + a^2e^{ax}\sin bx - 2abe^{ax}\cos bx - 2a^2e^{ax}\sin bx + a^2e^{ax}\sin bx + \cancel{b^2e^{ax}\sin bx}$$

$$2abe^{ax}\cancel{\cos}bx - 2ab\cancel{e^{ax}}\cos bx + 2a^2\cancel{e^{ax}}\sin bx - 2a^2e^{ax}\cancel{\sin}bx = 0 \quad R.H.S.$$

$$L.H.S. = R.H.S.$$

Question #8

If $y = (\cos^{-1}x)^2$, prove that $(1-x^2)y_2 - xy_1 - 2 = 0$

Solution:

$$y = (\cos^{-1}x)^2$$

Diff w.r.t 'x' both sides

$$y_1 = 2(\cos^{-1}x)^{2-1}\cdot\frac{d}{dx}(\cos^{-1}x)$$

$$y_1 = 2(\cos^{-1}x)\cdot\frac{-1}{\sqrt{1-x^2}}$$

$$\sqrt{1-x^2}\,y_1 = -2\cos^{-1}x$$

Diff w.r.t 'x' both sides

$$\frac{d}{dx}\sqrt{1-x^2}\cdot y_1 = -2\frac{d}{dx}(\cos^{-1}x)$$

$$\sqrt{1-x^2}\cdot y_2 + \frac{d}{dx}\sqrt{1-x^2}\;y_1 = -2\,\frac{-1}{\sqrt{1-x^2}}$$

$$\sqrt{1-x^2}\,y_2 + y_1\,\frac{1}{2\sqrt{1-x^2}}\,\frac{d}{dx}(1-x^2) = \frac{2}{\sqrt{1-x^2}}$$

$$\sqrt{1-x^2}\,y_2 + y_1\cdot\frac{1}{\cancel{2}\sqrt{1-x^2}} - \cancel{2}x = \frac{2}{\sqrt{1-x^2}}$$

$$\sqrt{1-x^2}\,y_2 + y_1\,\frac{-x}{\sqrt{1-x^2}} = \frac{2}{\sqrt{1-x^2}}$$

Multiply by $\sqrt{1-x^2}$ both sides

$$(1-x^2)y_2 + (-xy_1) = 2$$

$$\boxed{(1-x^2)y_2 - xy_1 - 2 = 0}$$

Hence proved.

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Question #9

If $y = a(\cos\ln x) + b\sin(\ln x)$, prove that $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$

Solution:

$$y = a(\cos\ln x) + b\sin(\ln x)$$

Diff w.r.t 'x' both sides

$$\frac{dy}{dx} = \frac{d}{dx}(a\cos\ln x) + \frac{d}{dx}(b\sin\ln x)$$

$$\frac{dy}{dx} = a(-\sin\ln x)\frac{d}{dx}(\ln x) + b\cos\ln x\,\frac{d}{dx}(\ln x)$$

$$\frac{dy}{dx} = -a\sin\ln x\cdot\frac{1}{x} + b\cos\ln x\cdot\frac{1}{x}$$

$$\frac{dy}{dx} = \frac{1}{x}\left[-a\sin\ln x + b\cos\ln x\right]$$

$$x\cdot\frac{dy}{dx} = -a\sin\ln x + b\cos\ln x$$

Diff w.r.t 'x' both sides

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx}(1) = -a\cos\ln x\left(\frac{1}{x}\right) - b\sin\ln x\left(\frac{1}{x}\right)$$

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} = -\frac{1}{x}\left[a\cos\ln x + b\sin\ln x\right]$$

$$\boxed{\because\ y = a\cos\ln x + b\sin\ln x}$$

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} = -\frac{1}{x}(y)$$

Multiply by 'x' both sides

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} = -y$$

$$\boxed{x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0}$$ proved.

•——•

✶ Theory

Maclaurin's Series

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots\cdots \frac{f^{(n)}(0)}{n!} + \cdots\cdots$$

Example #1 Expand $f(x) = \frac{1}{1+x}$ in the Maclaurin Series.

Solution:

$$f(x) = \frac{1}{1+x} \Rightarrow \boxed{x=0} \quad f(0) = \frac{1}{1+0} = 1$$

Differentiation w.r.t 'x' both side

$$f'(x) = \frac{d}{dx}\left[\frac{1}{1+x}\right]$$

$$f'(x) = -1(1+x)^{-1-1}$$

$\boxed{x=0}$

$$f'(0) = -1(1+0)^{-2}$$

$$\boxed{f'(0) = -1}$$

$$f''(x) = +2(1+x)^{-2-1}$$

$$f''(x) = +2(1+x)^{-3}$$

$\boxed{x=0}$

$$f''(0) = +2(1+0)^{-3}$$

$$\boxed{f''(0) = +2}$$

$$f'''(x) = -6(1+x)^{-3-1}$$

$$f'''(x) = -6(1+x)^{-4}$$

$\boxed{x=0}$

$$f'''(0) = -6(1+0)^{-4}$$

$$\boxed{f'''(0) = -6}$$

Using Maclaurin Series

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots\cdots$$

$$(1+x)^{-1} = 1 + (-1)x + \frac{(+2)}{2!}x^2 + \frac{(-6)}{3!}x^3 + \cdots\cdots$$

$$(1+x)^{-1} = 1 - x + \frac{\cancel{2}}{\cancel{2}\cdot1}x^2 + \frac{-\cancel{6}3}{\cancel{3}\cdot\cancel{2}}x^3 + \cdots$$

$$\boxed{(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots\cdots}$$

✶ Taylor Series Expansion of function:

If f is defined in the interval containing 'a' and its derivatives of all order exist $\boxed{x=a}$ then we expand f(x) as

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{\lfloor 2}(x-a)^2 + \cdots\cdots \frac{f^{n}(a)}{\lfloor n}(x-a)^n + \cdots$$

Example #1 Find the taylor series of expansion of $\ln(1+x)$ at $x=2$.

Solution:

let $f(x) = \ln(1+x)$ $\quad x = 2 = a$

$$\Rightarrow f(2) = \ln(1+2) = \ln(3)$$

$$f'(x) = \frac{1}{(1+x)} \Rightarrow (1+x)^{-1}, \quad f'(2) = \frac{1}{1+2} = \frac{1}{3}$$

$$f''(x) = -1(1+x)^{-2}, \quad f''(2) = -(1+2)^{-2} \Rightarrow -\frac{1}{3^2} = \frac{-1}{9}$$

$$f'''(x) = 2(1+x)^{-3}, \quad f''(2) = 2(1+2)^{-3} = \frac{2}{3^3} = \frac{2}{27}$$

Taylor's series expension of 'f' at $x=a$ is

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

$$\ln(1+x) = \ln 3 + \frac{1}{3}(x-2) - \frac{\frac{1}{9}}{2\cdot1}(x-2)^2 + \frac{\frac{2}{27}}{3\cdot2\cdot1}(x-2)^3 + \cdots$$

$$= \ln 3 + \frac{x-2}{3} - \frac{(x-2)^2}{2\times9} + \frac{\cancel{2}(x-2)^3}{\cancel{6}_{3}\times27} + \cdots$$

$$\boxed{\ln(1+x) = \ln 3 + \frac{x-2}{3} - \frac{(x-2)^2}{2\,3^2} + \frac{(x-2)^3}{3\cdot3^3} + \cdots}$$

Example #2 Find the Maclaurin Series for sinx.

Solution:

put $\boxed{x=0}$ $f(x) = \sin x$

$f(0) = \sin 0 = 0$

Differentiation w.r.t 'x' b/s

$\boxed{x=0}$ $f'(x) = \cos x$

$f'(0) = \cos 0 = 1$

$\boxed{x=0}$ $f''(x) = -\sin x$

$f'(0) = -\sin 0 = 0$

$\boxed{x=0}$ $f'''(x) = -\cos x$

$f'''(0) = -\cos 0 = -1$

$\boxed{x=0}$ $f^{iv}(x) = -(-\sin x) \Rightarrow \sin x$

$f^{iv}(0) = \sin 0 = 0$

$\boxed{x=0}$ $f^{v}(x) = \cos x$

$f^{v}(0) = \cos 0 = 1$

Using Maclaurin Series

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{iv}(0)}{4!}x^4 + \frac{f^{v}(0)}{5!}x^5 + \cdots\cdots$$

$$\sin x = 0 + 1\cdot x + \frac{0}{2!}x^2 + \frac{(-1)}{3!}x^3 + \frac{0}{4!}x^4 + \frac{1}{5!}x^5 + \cdots\cdots$$

$$\boxed{\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\cdots}$$

Example #3: Expand a^x in the Maclaurin Series.

Solution:

Let $f(x) = a^x$ $f(0) = a^0 = 1$

Differentiation w.r.t 'x' both sides.

$f'(x) = a^x(\ln a)$ $\boxed{x=0}$ $f'(0) = a^0(\ln a) \Rightarrow \ln a$

$f''(x) = a^x(\ln a)^2$ $\boxed{x=0}$ $f''(0) = a^0(\ln a)^2 \Rightarrow (\ln a)^2$

$f'''(x) = a^x(\ln a)^3$ $\boxed{x=0}$ $f'''(0) = a^0(\ln a)^3 \Rightarrow (\ln a)^3$

$f^{n}(x) = a^n(\ln a)^n$ $\boxed{x=0}$ $f^{n}(0) = a^0(\ln a)^n \Rightarrow (\ln a)^n$

Using Maclaurin Series

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots\ldots + \frac{f^n(0)}{n!}x^n + \ldots$$

$$a^x = 1 + (\ln a)x + \frac{(\ln a)^2}{2!}x^2 + \frac{(\ln a)^3}{3!}x^3 + \ldots\ldots \frac{(\ln a)^n}{n!}x^n + \ldots$$

$$\boxed{a^x = 1 + (\ln a)x + \frac{(\ln a)^2}{2!}x^2 + \frac{(\ln a)^3}{3!}x^3 + \ldots\ldots}$$

Example #4: Expand $(1+x)^n$ in the Maclaurin Series.

Solution:

Let $f(x) = (1+x)^n$

put $x=0$ $f(0) = (1+0)^n = 1$

Differentiation w.r.t "x" both sides

$f'(x) = n(1+x)^{n-1}$

$\boxed{x=0}$ $f'(0) = n(1+0)^{n-1} \Rightarrow n$

$f''(x) = n(n-1)(1+x)^{n-1-1}$

$f''(x) = n(n-1)(1+x)^{n-2}$

$\boxed{x=0}$ $f''(0) = n(n-1)(1+0)^{n-2} \Rightarrow n(n-1)$

$f'''(x) = n(n-1)(n-2)(1+x)^{n-2-1}$

$f'''(x) = n(n-1)(n-2)(1+x)^{n-3}$

$\boxed{x=0}$

$f'''(0) = n(n-1)(n-2)(1+0)^{n-3} \Rightarrow n(n-1)(n-2)$

Using Maclaurin Series

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{iv}(0)}{4!}x^4 + \ldots\ldots$$

$$\boxed{(1+x)^n = 1 + n.x + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!} + \ldots\ldots}$$

Taylor's Theorem:

$$f(x+h) = f(x) + f'(x)h + \frac{f''(x)}{\lfloor\underline{2}}h^2 + \frac{f'''(x)}{\lfloor\underline{3}}h^3 + \ldots\ldots \frac{f^n(x)}{\lfloor\underline{n}}h^n + \ldots\ldots$$

Example #2: Use the Taylor's Series expansion to find the value of $\sin 31^\circ$.

Solution:-

Let

$f(x) = \sin x$

Differentiation w.r.t "x" both sides

$f'(x) = \cos x$

$f''(x) = -\sin x$

$f'''(x) = -\cos x$

$f^{iv}(x) = -(-\sin x) = \sin x$

Using Taylor's Series

$$f(x+h) = f(x) + f'(x)h + \frac{f''(x)}{\lfloor 2}h^2 + \frac{f'''(x)}{\lfloor 3}h^3 + \cdots\cdots$$

$$\sin(x+h) = \sin x + \cos x \cdot h + \frac{(-\sin x)}{\lfloor 2}h^2 + \frac{(-\cos x)}{\lfloor 3}h^3 + \cdots\cdots$$

$$\sin(x+h) = \sin x + \cos x h - \frac{\sin x}{\lfloor 2}h^2 - \frac{\cos x}{\lfloor 3}h^3 + \cdots\cdots$$

$$\sin 31^\circ \Rightarrow x = 30^\circ, \ h = 1^\circ$$

$$\sin(30^\circ + 1^\circ) = \sin 30^\circ + \cos 30^\circ 1^\circ - \frac{\sin 30^\circ}{\lfloor 2}(1^\circ)^2 - \frac{\cos 30^\circ}{\lfloor 3}(1^\circ)^3 + \cdots\cdots$$

$$\sin 31^\circ = (0.5) + (0.8660)(0.0174) - \frac{(0.5)}{2}(0.0174)^2 - \frac{(0.866)}{3.2}(0.0174)^3 + \cdots\cdots$$

$$\boxed{\because 1^\circ = \frac{\pi}{180} = 0.0174}$$

$$\sin 31^\circ = 0.5 + 0.0150 - 0.000075 - 0.00000676$$

$$\boxed{\sin 31^\circ = 0.5150}$$

Example #3 Prove that $e^{x+h} = e^x\left\{1 + h + \frac{h^2}{\lfloor 2} + \frac{h^3}{\lfloor 3} + \cdots\cdots\right\}$

Solution: Let $f(x) = e^x$

$f(x+h) = e^{x+h}$

Differentiation w.r.t "x" both sides

$f'(x) = e^x \frac{d}{dx}(x) = e^x(1) = e^x$

$f''(x) = e^x$

$f'''(x) = e^x$

Using Taylor's Series

$$f(x+h) = f(x) + f'(x)h + \frac{f''(x)}{\lfloor 2}h^2 + \frac{f'''(x)}{\lfloor 3}h^3 + \cdots\cdots$$

$$e^{x+h} = e^x + e^x h + \frac{e^x}{\lfloor 2}h^2 + \frac{e^x}{\lfloor 3}h^3 + \frac{e^x}{\lfloor 4}h^3 + \cdots\cdots$$

$\boxed{e^{x+h} = e^{x}\left\{1 + h + \frac{h^2}{\underline{|2}} + \frac{h^3}{\underline{|3}} + - - - - - \right\}}$ Hence proved.

• ——— •

Exercise # 2·8

Question #1:

Apply the Maclaurin Series expansion to prove that:

(i)

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + - - - - -$$

Solution:-

$L{\cdot}H{\cdot}S = \ln(1+x)$

Let $f(x) = \ln(1+x)$

$f(0) = \ln(1+0) = 0$

Differentiation w.r.t 'x' both sides

$f'(x) = \frac{1}{1+x} = (1+x)^{-1}$

$\boxed{x=0}$

$f'(0) = (1+0)^{-1} = 1$

$f''(x) = -1(1+x)^{-1-1}$

$f''(x) = -1(1+x)^{-2}$

$\boxed{x=0}$

$f''(0) = -1(1+0)^{-2} \Rightarrow -1$

$f'''(x) = 2(1+x)^{-2-1}$

$f'''(x) = 2(1+x)^{-3}$

$\boxed{x=0}$

$f'''(0) = 2(1+0)^{-3} \Rightarrow 2$

$f^{iv}(x) = -6(1+x)^{-4}$

$\boxed{x=0}$

$f^{iv}(0) = -6(1+0)^{-4} \Rightarrow -6$

Using Maclaurin Series

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{\lfloor 2}x^2 + \frac{f'''(0)}{\lfloor 3}x^3 + \frac{f^{iv}(0)}{\lfloor 4}x^4 + \cdots$$

$$\ln(1+x) = 0 + 1\cdot x + \frac{(-1)}{2\cdot 1}x^2 + \frac{(2)}{3\cdot 2\cdot 1}x^3 + \frac{(-6)}{4\cdot 3\cdot 2\cdot 1}x^4 + \cdots$$

$$\boxed{\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots} \quad R\cdot H\cdot S\cdot$$

$L\cdot H\cdot S\cdot = R\cdot H\cdot S\cdot$

(ii)

$$\cos x = 1 - \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} - \frac{x^6}{\lfloor 6} + \cdots$$

Solution:-

$L\cdot H\cdot S\cdot = \cos x$

$f(x) = \cos x$

$\boxed{x=0}$ $f(0) = \cos 0 = 1$

Differentiation w.r.t 'x'

$f'(x) = -\sin x$

$\boxed{x=0}$ $f'(0) = -\sin 0 = 0$

$f''(x) = -\cos x$

$\boxed{x=0}$ $f''(0) = -\cos 0 = -1$

$f'''(x) = -(-\sin x) = \sin x$

$\boxed{x=0}$ $f''(0) = \sin 0 = 0$

$f^{iv}(x) = \cos x$

$\boxed{x=0}$ $f^{iv}(0) = \cos 0 = 1$

$f^{v}(x) = -\sin x$

$\boxed{x=0}$ $f^{v}(0) = -\sin 0 = 0$

$f^{vi}(x) = -\cos x$

$\boxed{x=0}$ $f^{vi}(0) = -\cos 0 = -1$

Using Maclaurin Series

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{\lfloor 2}x^2 + \frac{f'''(0)}{\lfloor 3}x^3 + \frac{f^{iv}(0)}{\lfloor 4}x^4 + \frac{f^{v}(0)}{\lfloor 5}x^5 + \frac{f^{vi}(0)}{\lfloor 6}x^6 + \cdots$$

$$\cos x = 1 + 0\cdot x + \frac{(-1)}{\lfloor 2}x^2 + \frac{0}{\lfloor 3}x^3 + \frac{1}{\lfloor 4}x^4 + \frac{0}{\lfloor 5}x^5 + \frac{-1}{\lfloor 6}x^6 + \cdots$$

$$\cos x = 1 - \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} - \frac{x^6}{\lfloor 6} + \cdots = R\cdot H\cdot S\cdot$$

$L\cdot H\cdot S\cdot = R\cdot H\cdot S\cdot$

$$\sqrt{1+x} = 1+\frac{x}{2}-\frac{x^2}{8}+\frac{x^3}{16}$$ (iii)

Solution:

L.H.S. = $\sqrt{1+x}$

$f(x) = \sqrt{1+x} \Rightarrow (1+x)^{1/2}$

$\boxed{x=0}$

$f(0) = (1+0)^{1/2} \Rightarrow 1$

Differentiation w.r.t 'x' both sides

$f'(x) = \frac{1}{2}(1+x)^{\frac{1}{2}-1}$

$f'(x) = \frac{1}{2}(1+x)^{\frac{-1}{2}}$

$\boxed{x=0}$

$f'(0) = \frac{1}{2}(1+0)^{\frac{-1}{2}}$

$f'(0) = \frac{1}{2}$

$f''(x) = \frac{-1}{4}(1+x)^{\frac{-1}{2}-1}$

$f''(x) = \frac{-1}{4}(1+x)^{\frac{-3}{2}}$

$\boxed{x=0}$

$f''(0) = \frac{-1}{4}(1+0)^{\frac{-3}{2}} \Rightarrow -\frac{1}{4}$

$f'''(x) = \frac{3}{8}(1+x)^{\frac{-3}{2}-1}$

$f'''(x) = \frac{3}{8}(1+x)^{\frac{-5}{2}}$

$\boxed{x=0}$

$f'''(0) = \frac{3}{8}(1+0)^{\frac{-5}{8}} \Rightarrow \frac{3}{8}$

Using Maclaurin Series

$$f(x) = f(0)+f'(0)x+\frac{f''(0)}{\underline{|2}}x^2+\frac{f'''(0)}{\underline{|3}}x^3+\text{-----}$$

$$\sqrt{1+x} = 1+\frac{1}{2}\cdot x+\frac{-1/4}{2\cdot 1}x^2+\frac{3/8}{3\cdot 2\cdot 1}x^3+\text{----}$$

$$\sqrt{1+x} = 1+\frac{x}{2}-\frac{x^2}{8}+\frac{x^3}{16}+\text{-------} = R.H.S.$$

L.H.S. = R.H.S

(iv)

$$e^x = 1+x+\frac{x^2}{\underline{|2}}+\frac{x^3}{\underline{|3}}+\text{......}$$

Solution:

L.H.S = e^x

$f(x) = e^x$

$\boxed{x=0}$

$f(0) = e^0 = 1$

Differentiation w.r.t 'x' both sides

$f'(x) = e^x \frac{d}{dx}(x)$

$f'(x) = e^x \cdot (1) \Rightarrow e^x$

$\boxed{x=0}$

$f'(0) = e^0 = 1$

$f''(x) = e^x$

$\boxed{x=0}$

$f''(0) = e^0 = 1$

$f'''(x) = e^x$

$\boxed{x=0}$

$f'''(0) = e^0 = 1$

Using Maclaurin Series

$$f(x) = f(0)+f'(0)(x)+\frac{f''(0)}{\underline{|2}}x^2+\frac{f'''(0)}{\underline{|3}}x^3+\text{----}$$

$$e^x = 1+1\cdot x+\frac{1}{\underline{|2}}x^2+\frac{1}{\underline{|3}}x^3+\text{-----}$$

$$e^x = 1+x+\frac{x^2}{\underline{|2}}+\frac{x^3}{\underline{|3}}+\text{......} = R.H.S.$$

L.H.S. = R.H.S.

(V)

$$e^{2x} = 1 + 2x + \frac{4x^2}{\lfloor 2} + \frac{8x^3}{\lfloor 3} + \cdots\cdots$$

Solution:

L.H.S. $= e^{2x}$

$\underset{\boxed{x=0}}{f(x)} = e^{2x}$

$f(0) = e^{2(0)} \Rightarrow e^0 \Rightarrow 1$

Differentiation w.r.t 'x' both sides

$f'(x) = e^{2x} \cdot \frac{d}{dx}(2x)$

$= e^{2x} \cdot 2$

$\underset{\boxed{x=0}}{f'(x)} = 2 \cdot e^{2x}$

$f'(0) = 2e^{2(0)} \Rightarrow 2 \cdot e^0 \Rightarrow 2 \cdot 1 \Rightarrow 2$

$f''(x) = 2 \cdot e^{2x} \cdot \frac{d}{dx}(2x)$

$\underset{\boxed{x=0}}{f''(x)} = 4e^{2x}$

$f''(0) = 4e^{2(0)} \Rightarrow 4$

$f'''(x) = 4e^{2x} \cdot \frac{d}{dx}(2x)$

$\underset{\boxed{x=0}}{f'''(x)} = 8e^{2x}$

$f'''(0) = 8e^{2(0)} \Rightarrow 8e^0$

$= 8 \cdot 1 \Rightarrow 8$

Using Maclaurin Series

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{\lfloor 2}x^2 + \frac{f'''(0)}{\lfloor 3}x^3 + \cdots\cdots$$

$$f(x) = 1 + 2x + \frac{4}{\lfloor 2}x^2 + \frac{8}{\lfloor 3}x^3 + \cdots\cdots$$

$$e^{2x} = 1 + 2x + \frac{4x^2}{\lfloor 2} + \frac{8x^3}{\lfloor 3} + \cdots\cdots = \text{R.H.S.}$$

L.H.S. = R.H.S.

Question #2

Show that: $\cos(x+h) = \cos x - h\sin x - \frac{h^2}{\lfloor 2}\cos x + \frac{h^3}{\lfloor 3}\sin x + \cdots\cdots$

and evaluate cos 61°.

Solution:

Let

$$f(x+h) = \cos(x+h)$$

$$f(x) = \cos x$$

$$f'(x) = -\sin x$$

$$f''(x) = -\cos x$$

$$f'''(x) = -(-\sin x) = \sin x$$

Using Taylor's Series

$$f(x+h) = f(x) + f'(x)h + \frac{f''(x)}{\lfloor 2}h^2 + \frac{f'''(x)}{\lfloor 3}h^3 + \text{------}$$

$$\cos(x+h) = \cos x + (-\sin x)h + \frac{(-\cos x)}{\lfloor 2}h^2 + \frac{\sin x}{\lfloor 3}h^3 + \text{---}$$

$$\cos(x+h) = \cos x - h\sin x - \frac{h^2\cos x}{\lfloor 2} + \frac{h^3}{\lfloor 3}\sin x + \text{----}$$

Cos 61° $\quad x = 60° \quad , \quad h = 1°$

$$\cos(60°+1°) = \cos 60° - 1°\sin 60° - \frac{(1°)^2}{\lfloor 2}\cos 60° + \frac{(1°)^3}{\lfloor 3}\sin 60° + \text{----}$$

$$\because 1° = \frac{\pi}{180} = 0.0174$$

$$\cos 61° = 0.5 - (0.0174)(0.8660) - \frac{(0.0174)^2}{2\cdot 1}(0.5) + \frac{(0.0174)^3}{3\cdot 2\cdot 1}(0.8660) + \text{------}$$

$$\cos 61° = 0.5 - 0.0151 - 0.00007569 + 0.00000076$$

$$\boxed{\cos 61° = 0.4848}$$

Hence proved.

Question #3:

Show that:

$$2^{x+h} = 2^x\left\{1 + (\ln 2)h + \frac{(\ln 2)^2}{\lfloor 2}h^2 + \frac{(\ln 2)^3}{\lfloor 3}h^3 + \text{.....}\right\}$$

Solution:

Let

$f(x+h) = 2^{x+h}$

$f(x) = 2^{x}$

Differentiation w.r.t 'x' both sides.

$f'(x) = 2^{x}(\ln 2)$

$f''(x) = 2^{x}(\ln 2)(\ln 2)$

$= 2^{x}(\ln 2)^{2}$

$f'''(x) = 2^{x}(\ln 2)^{2}(\ln 2)$

$= 2^{x}(\ln 2)^{3}$

Using Taylor's Series

$$f(x+h) = f(x) + f'(x)h + \frac{f''(x)}{\underline{|2}}h^{2} + \frac{f'''(x)}{\underline{|3}}h^{3} + \text{----}$$

$$2^{x+h} = 2^{x} + 2^{x}(\ln 2)h + \frac{2^{x}(\ln 2)^{2}}{\underline{|2}}h^{2} + \frac{2^{x}(\ln 2)^{3}}{\underline{|3}}h^{3} + \text{-------}$$

$$\boxed{2^{x+h} = 2^{x}\left\{1 + (\ln 2)h + \frac{(\ln 2)^{2}}{\underline{|2}}h^{2} + \frac{(\ln 2)^{3}}{\underline{|3}}h^{3} + \text{------}\right\}}$$ Hence proved.

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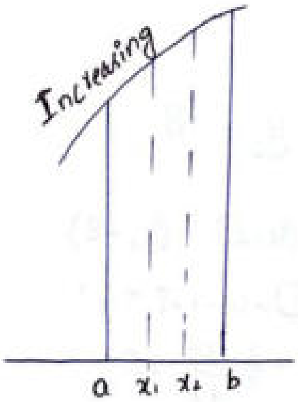
* Theory

Definitions:-

• Increasing function:

f is increasing on the interval (a, b) if $f(x_2) > f(x_1)$ whenever $x_2 > x_1$



$f(x_2) > f(x_1)$ if $x_2 > x_1$

• ——— •

• Stationary Point:-

Any point where f is neither increasing nor decreasing is called stationary point.

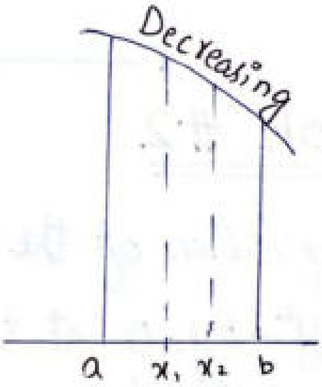
Stationary Point

$f'(x) = 0$

• ——— •

• Decreasing Function:-

f is decreasing on the interval (a, b) if $f(x_2) < f(x_1)$ whenever $x_2 > x_1$



$f(x_2) < f(x_1)$ if $x_2 > x_1$

• ——— •

• Critical Value and Critical Point:-

If $c \in D_f$ and $f'(c) = 0$ or $f'(c)$ does not exist, then the number c is called Critical value for f while the point $(c, f(c))$ on the graph of f is named as a Critical point.

• ——— •

• Relative Maxima:-

A function f is said to have relative maxima/maximum at $x = c \in [a,b]$ if

(i) There exist interval $[a,c]$ in which 'f' increase

(ii) There exist interval $[c,d]$ in which 'f' decrease.

• Relative Minima:-

A function f is said to have relative minima/minimum at $x = c \in [a,b]$ if

(i) There exist interval $[a,c]$ in which 'f' decrease

(ii) There exist interval $[c,d]$ in which 'f' increase.

Example #2

Find the equation of the tangent to curve $x^2 - y^2 - 6y = 0$ at the point whose abscissa is 4.

Solution:

$$x^2 - y^2 - 6y = 0 \text{ —— } ①$$

put $\boxed{x=4}$ in eq ①

$$16 - y^2 - 6y = 0$$

$$y^2 + 6y - 16 = 0$$

By Quadratic formula

$$y = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$y = \frac{-6 \pm \sqrt{(6)^2 - 4(1)(-16)}}{2(1)}$$

$$y = \frac{-6 \pm \sqrt{36+64}}{2}$$

$$y = \frac{-6 \pm \sqrt{100}}{2}$$

$$y = \frac{-6 \pm 10}{2}$$

$$y_1 = \frac{-6+10}{2} \qquad y_2 = \frac{-6-10}{2}$$

$$y_1 = \frac{4}{\cancel{2}} \qquad y_2 = \frac{-\cancel{16}\,8}{\cancel{2}}$$

$$y_1 = 2 \qquad y_2 = -8$$

Thus point are $(4,2)$ $(4,-8)$

Differentiate eq ① w.r.t 'x'

$$2x - 2y\frac{dy}{dx} - 6\frac{dy}{dx} = 0$$

$$-2\frac{dy}{dx}(y+3) = -2x$$

$$2\frac{dy}{dx}(y+3) = 2x$$

Divided by '2'

$$\frac{dy}{dx} = \frac{x}{y+3} \quad \text{slope}$$

The slope of tangent ① at $(4,2)$

$$= \frac{4}{2+3} = \frac{4}{5}$$

$$y - y_1 = m(x - x_1)$$

$$y - 2 = \frac{4}{5}(x-4)$$

$$5y - 10 = 4x - 16$$

$$5y = 4x - 16 + 10$$

$$5y = 4x - 6$$

The slope of tangent ① at $(4,-8)$

$$= \frac{4}{-8+3} = \frac{4}{-5}$$

$y - y_2 = m(x - x_2)$

$y - (-8) = -\frac{4}{5}(x - 4)$

$y + 8 = -\frac{4}{5}(x - 4)$

$5(y + 8) = -4(x - 4)$

$5y + 40 = -4x + 16$

$4x + 5y + 40 - 16 = 0$

$\boxed{4x + 5y + 24 = 0}$

Example #2

Determine the intervals in which f is increasing or its decreasing
$f(x) = x^3 - 6x^2 + 9x$

Solution

$f'(x) = 3x^2 - 12x + 9$

$= 3(x^2 - 4x + 3)$

$= 3(x^2 - 3x - x + 3)$

$f'(x) = 3(x(x-3) - 1(x-3))$

$f'(x) = 3((x-3)(x-1)]$

For stationary point

$f'(x) = 0$

$3(x-1)(x-3) = 0$

$x - 1 = 0$ | $x - 3 = 0$

$\boxed{x = 1}$ | $\boxed{x = 3}$

For increasing:

$f'(x) > 0$

$3(x-1)(x-3) > 0$

$(x-1)(x-3) > 0$

$x - 1 > 0$ | $x - 3 > 0$

$x > 1$ | $x > 3$

Hence the function is increase $(-\infty, 1)\ (3, \infty)$

For decreasing.

$f'(x) < 0$

$(x-1)(x-3) < 0$

$x - 1 < 0$ | $x - 3 < 0$

$x < 1$ | $x < 3$

$1 < x < 3$

Hence the function Decreasing $(1, 3)$

Example #1

Determine the value of x for which f defined as $f(x) = x^2 + 2x - 3$ is

Solution:

$f(x) = x^2 + 2x - 3$ — ①

$f'(x) = 2x + 2$

for stationary point

$f'(x) = 0$

(iii)

$2x + 2 = 0$

$2(x + 1) = 0$

$x + 1 = 0$

$\boxed{x = -1}$

put in ①

$f(-1) = (-1)^2 + 2(-1) - 3$

$= 1 - 2 - 3$

$f(-1) = -4$

Hence the function is neither decreasing nor increasing at

(i) For increasing.

$f'(x) > 0$

$2x + 2 > 0$

$2x > -2$

$x > -1$

Hence the function is increasing on interval $(-1, \infty)$

(ii) For Decreasing.

$f'(x) < 0$

$2x + 2 < 0$

$2x < -2$

$x = -1$

Hence the function is decreasing in the interval $(-\infty, -1)$

Example #1

Examine the function defined as $f(x) = x^3 - 6x^2 + 9x$ for extreme values.

Solution.

$f'(x) = 3x^2 - 12x + 9$ — ①

$= 3(x^2 - 4x + 3)$

$= 3(x^2 - 9x - x + 3)$

$f'(x) = 3(x(x-3) - 1(x-3))$

$f'(x) = 3(x-1)(x-3)$

for stationary point

$f'(x) = 0$

Example #2

Examine the function defined as $f(x) = 1+x^3$ for extreme value

Solution:

$f(x) = 1+x^3$

Differentiate w.r.t 'x'

$f'(x) = 3x^2 + 0$

$f'(x) = 3x^2$

for stationary point

$f'(x) = 0$

$3x^2 = 0$

$\boxed{x = 0}$

$f''(x) = 6x$ — ①

put $\boxed{x=0}$ in eq ①

$f''(0) = 6(0) = 0$

The second derivative does not help in determing the extreme value

$f'(0-\varepsilon) = 3(0-\varepsilon)^2 = 3\varepsilon^2 > 0$

$f'(0+\varepsilon) = 3(0+\varepsilon)^2 = 3\varepsilon^2 > 0$

• ——— •

$3(x-1)(x-3) = 0$

$(x-1)(x-3) = 0$

$x-1=0$ | $x-3=0$

$\boxed{x=1}$ | $\boxed{x=3}$

$f''(x) = 3(2x-4)$

$f''(x) = 6(x-2)$ — ②

put $\boxed{x=1}$ in eq ②

$f''(1) = 6(1-2)$

$= 6(-1)$

$= -6 < 0$

Relative maxima

$f(1) = (1)^3 - 6(1)^2 + 9(1)$

$= 1-6+9$

$= 4$

put $\boxed{x=3}$ in eq ②

$f''(3) = 6(3-2)$

$= 6(1)$

$= 6 > 0$

Relative minimum

$f(3) = (3)^3 - 6(3)^2 + 9(3)$

$= 27-54+27$

$f(3) = 0$

• ——— •

Example #3

Discuss the function defined as $f(x) = \sin x + \frac{1}{2\sqrt{2}} \cos 2x$ for extreme values in the interval $[0, 2\pi]$.

Solution:

$$f(x) = \sin x + \frac{1}{2\sqrt{2}} \cos 2x$$

Differentiate w.r.t "x"

$$f'(x) = \cos x + \frac{1}{\cancel{2}\sqrt{2}} - \cancel{2}\sin 2x$$

$\because \sin 2x = 2 \sin x \cos x$

$$= \cos x - \frac{1}{\sqrt{2}}(2 \sin x \cos x)$$

$$= \cos x - \sqrt{2} \sin x \cos x$$

$$f'(x) = \cos x \left[1 - \sqrt{2} \sin x\right]$$

For stationary point

$$f'(x) = 0$$

$$\cos x(1 - \sqrt{2} \sin x) = 0$$

$$\cos x = 0$$

$$x = \frac{\pi}{2}, \frac{3\pi}{2}$$

$$1 - \sqrt{2} \sin x = 0$$

$$-\sqrt{2} \sin x = -1$$

$$\sin x = \frac{-1}{-\sqrt{2}}$$

$$\sin x = \frac{1}{\sqrt{2}}$$

$$x = \frac{\pi}{4}, \frac{3\pi}{4}$$

$$f''(x) = -\sin x - \frac{1}{\sqrt{2}}(\cos 2x) \times 2$$

$$= -\sin x - \sqrt{2} \cos 2x$$

$$f''\left(\frac{\pi}{2}\right) = -\sin\frac{\pi}{2} - \sqrt{2}\cos\pi$$

$$= -1 - \sqrt{2} \times (-1) \Rightarrow \sqrt{2} - 1 > 0$$

$$f''\left(\frac{3\pi}{2}\right) = -\sin\frac{3\pi}{2} - \sqrt{2}\cos 3\pi \Rightarrow -1(-1) - \sqrt{2}(-1)$$

$$= 1 + \sqrt{2} > 0$$

Thus, $f(x)$ has minimum values for $x = \frac{\pi}{2}$, $x = \frac{3\pi}{2}$

$$f''\left(\frac{\pi}{4}\right) = -\sin\frac{\pi}{4} - \sqrt{2}\cos\frac{\pi}{2}$$

$$f''\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}} - \sqrt{2} \cdot \underline{\sqrt{2}} \cdot 0$$

$$f''\left(\frac{\pi}{4}\right) = \frac{-1}{2} < 0$$

$$f''\left(\frac{3\pi}{4}\right) = -\sin\frac{3\pi}{4} - \sqrt{2}\cos\frac{3\pi}{2}$$

$$f''\left(\frac{3\pi}{4}\right) = \frac{-1}{\sqrt{2}} - \sqrt{2} \cdot 0$$

$$f''\left(\frac{3\pi}{4}\right) = \frac{-1}{\sqrt{2}} < 0$$

Thus, $f(x)$ has maximum values for $x = \frac{\pi}{4}$ and $x = \frac{3\pi}{4}$.

Exercise # 2.9

Question #1

Determine the intervals in which f is increasing or decreasing for the domain metitioned in each case.

(i)

$f(x) = \sin x \; : \; x \in (-\pi, \pi)$

Solution:

$f(x) = \sin x$

Differentiate w.r.t "x"

$f'(x) = \cos x$

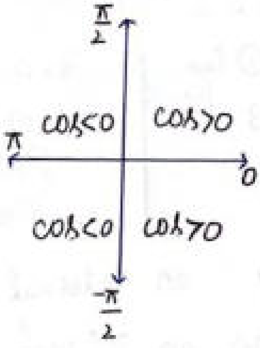
for stationary point

$f'(x) = 0$

$\cos x = 0$

$x = \cos^{-1}(0)$

$x = \frac{\pi}{2}, -\frac{\pi}{2}$



f is increasing on interval $\left[\frac{\pi}{2}, -\frac{\pi}{2}\right]$

and f is decreasing on interval

$\left[\frac{\pi}{2}, \pi\right] \cup \left[-\pi, -\frac{\pi}{2}\right]$

(ii)

$f(x) = \cos x \; : \; x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Solution:

$f(x) = \cos x$

Differentiate w.r.t "x"

$f'(x) = -\sin x$

for stationary point

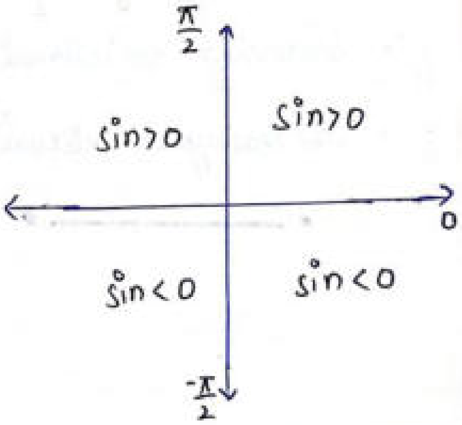
$f'(x) = 0$

$-\sin x = 0$

$\sin x = 0$

$x = \sin^{-1}(0)$

$x = 0$



Given sub-interval

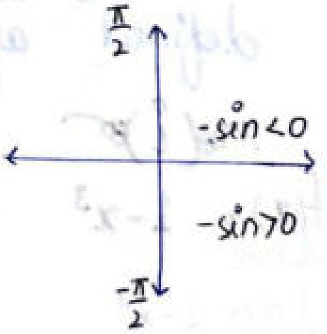
$\left(-\frac{\pi}{2}, 0\right), \left(0, \frac{\pi}{2}\right)$

f is increasing on interval

$\left[-\frac{\pi}{2}, 0\right]$

f is decreasing on interval

$\left[0, \frac{\pi}{2}\right]$



(iii)

$f(x) = 4 - x^2 : x \in (-2, 2)$

Solution:

$f(x) = 4 - x^2$

Differentiate w.r.t "x"

$\frac{d}{dx}(f(x)) = \frac{d}{dx}(4 - x^2)$

$f'(x) = 0 - 2x$

$f'(x) = -2x$

for stationary point

$f'(x) = 0$

$-2x = 0$

$x = \frac{0}{-2}$

$x = 0$

f is increasing on interval $(-2, 0)$

f is decreasing on interval $(0, 2)$.

(iv)

$f(x) = x^2 + 3x + 2 : x \in (-4, 1)$

Solution:

$f(x) = x^2 + 3x + 2$

Differentiate w.r.t "x"

$\frac{d}{dx}(f(x)) = \frac{d}{dx}(x^2 + 3x + 2)$

$f'(x) = 2x + 3(1) + 0$

$f'(x) = 2x + 3$ —— ①

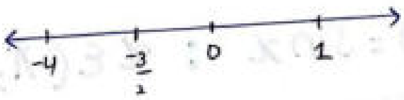
for stationary point

$f'(x) = 0$

$2x + 3 = 0$

$2x = -3$

$\boxed{x = \frac{-3}{2}}$



Two sub-interval

$\left[-4, -\frac{3}{2}\right]$, $\left[-\frac{3}{2}, 1\right]$

$\boxed{x = -3}$ in eq ① from $\left[-4, -\frac{3}{2}\right]$

$f'(-3) = 2(-3) + 3$

$f'(-3) = -6 + 3$

$f'(-3) = -3 < 0$

$x = 0$ in eq ① from $\left[-\frac{3}{2}, 1\right]$

$f'(0) = 2(0) + 3$

$f'(0) = 0 + 3$

$f'(0) = 3 > 0$

f is increasing on interval $\left[\frac{-3}{2}, 1\right]$.

f is decreasing on interval $\left[-4, -\frac{3}{2}\right]$

Question #2

Find the extreme value for the following functions defined as:

(i)

$f(x) = 1 - x^3$

Solution:

$f(x) = 1 - x^3$

Differentiate w.r.t "x"

$f'(x) = 0 - 3x^{3-1}$

$f'(x) = -3x^2$

for stationary value

$f'(x) = 0$

(ii)

$f(x) = x^2 - x - 2$

Solution:

$f(x) = x^2 - x - 2$ —— ①

Differentiate w.r.t "x"

$f'(x) = 2x - 1 - 0$

$f'(x) = 2x - 1$

for stationary value

$f'(x) = 0$

$-3x = 0$

$x^2 = \frac{0}{-3}$

$x^2 = 0$

$x = 0$

$f''(x) = -6x$

Using first derivative test

$f'(0-\varepsilon) = -3(0-\varepsilon)^2$

$f'(0-\varepsilon) = -3\varepsilon < 0$

$f'(0+\varepsilon) = -3(0+\varepsilon)^2$

$= -3\varepsilon^2 < 0$

put $x=0$ in eq ①

$f(x) = 1-(0)^2$

$= 1$

Point of inflexion (0,1)

* * *

(iii)

$f(x) = 5x^2 - 6x + 2$

Solution:

$f(x) = 5x^2 - 6x + 2$ —— ①

Differentiate w.r.t 'x'

$f'(x) = 5(2x^{2-1}) - 6(1) + 0$

$f'(x) = 10x - 6$

for stationary point

$f'(x) = 0$

$10x - 6 = 0$

$10x = 6$

$x = \frac{\cancel{6}^{3}}{\cancel{10}_{5}}$

$x = \frac{3}{5}$

Differentiate w.r.t 'x'

$f''(x) = 10$

$f''(\frac{3}{5}) = 10 > 0$ Relative minimum

put in eq ①

$f(\frac{3}{5}) = 5(\frac{3}{5})^2 - 6(\frac{3}{5}) + 2$

$= \cancel{5}(\frac{9}{\cancel{25}_{5}}) - \frac{18}{5} + 2$

$f(\frac{3}{5}) = \frac{9}{5} - \frac{18}{5} + 2$

$= \frac{9-18+10}{2}$

$\boxed{f(\frac{3}{5}) = \frac{1}{2}}$

$2x = 1$

$x = \frac{1}{2}$

Differentiate w.r.t 'x'

$f''(x) = 2$

$f''(\frac{1}{2}) = 2 > 0$

f is relative minimum at $\frac{1}{2}$

put in eq ①

$f(\frac{1}{2}) = (\frac{1}{2})^2 - (\frac{1}{2}) - 2$

$= \frac{1}{4} - \frac{1}{2} - 2$

$= \frac{1-2-8}{4}$

$\boxed{f(\frac{1}{2}) = \frac{-9}{4}}$

* * *

(iv)

$f(x) = 3x^2$

Solution:

$f(x) = 3x^2$

Differentiate w.r.t 'x'

$f'(x) = 3(2x^{2-1})$

$f'(x) = 6x$

for stationary point

$f'(x) = 0$

$6x = 0$

$x = 0$

Differentiate w.r.t 'x'

$f''(x) = 6$

$f''(0) = 6 > 0$ Relative minimum.

From eq ①

$f(x) = 3x^2$

$f(0) = 3(0)^2$

$\boxed{f(0) = 0}$

* * *

(V)

$f(x) = 3x^2 - 4x + 5$

Solution

$f(x) = 3x^2 - 4x + 5$ —— ①

Differentiate w.r.t "x"

$f'(x) = 3(2x^{2-1}) - 4(1) + 0$

$f'(x) = 6x - 4$

for stationary point

$f'(x) = 0$

$6x - 4 = 0$

$6x = 4$

$x = \frac{\cancel{4}^{2}}{\cancel{6}_{3}}$

$x = \frac{2}{3}$

Differentiate w.r.t "x"

$f''(x) = 6$

$f''\left(\frac{2}{3}\right) = 6 > 0$ Relative Minimum

put in eq ①

$f\left(\frac{2}{3}\right) = 3\left(\frac{2}{3}\right)^2 - 4\left(\frac{2}{3}\right) + 5$

$f\left(\frac{2}{3}\right) = \cancel{3}\left(\frac{4}{\cancel{9}_{3}}\right) - \frac{8}{3} + 5$

$f\left(\frac{2}{3}\right) = \frac{4}{3} - \frac{8}{3} + 5$

$f\left(\frac{2}{3}\right) = \frac{4 - 8 + 15}{3}$

$$\boxed{f\left(\frac{2}{3}\right) = \frac{11}{3}}$$

(Vii)

$f(x) = x^4 - 4x^2.$

Solution:

$f(x) = x^4 - 4x^2$ —— ①

Differentiate w.r.t "x"

$f'(x) = 4x^{4-1} - 4(2x^{2-1})$

$f'(x) = 4x^3 - 8x$

(Vi)

$f(x) = 2x^3 - 2x^2 - 36x + 3$

Solution:

$f(x) = 2x^3 - 2x^2 - 36x + 3$ —— ①

Differentiate w.r.t "x"

$f'(x) = 2(3x^{3-1}) - 2(2x^{2-1}) - 36(1) + 0$

$f'(x) = 6x^2 - 4x - 36$

for stationary point

$f'(x) = 0$

$6x^2 - 4x - 36 = 0$

for quadratic formula

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(6)(-36)}}{2(6)}$

$x = \frac{4 \pm \sqrt{16 + 864}}{12}$

$x = \frac{4 \pm \sqrt{880}}{12}$

$x = \frac{4 \pm 4\sqrt{55}}{12}$

$x = \frac{\cancel{4}(1 \pm \sqrt{55})}{\cancel{12}\,3} \Rightarrow x = \frac{1 \pm \sqrt{55}}{3}$

$f''(x) = 12x - 4$

for $x = \frac{1 + \sqrt{55}}{3}$

$f''\left(\frac{1+\sqrt{55}}{3}\right) = \overset{4}{\cancel{12}}\left(\frac{1+\sqrt{55}}{\cancel{3}}\right) - 4$

$= 4(1 + \sqrt{55}) - 4$

$= \cancel{4} + 4\sqrt{55} - \cancel{4}$

$f''\left(\frac{1+\sqrt{55}}{3}\right) = 4\sqrt{55} > 0$

Relative minimum

for $x = \frac{1 - \sqrt{55}}{3}$

$f''\left(\frac{1-\sqrt{55}}{3}\right) = \overset{4}{\cancel{12}}\frac{(1-\sqrt{55})}{\cancel{3}} - 4$

$= 4(1 - \sqrt{55}) - 4$

$= \cancel{4} - 4\sqrt{55} - \cancel{4}$

$f''\left(\frac{1-\sqrt{55}}{3}\right) = -4\sqrt{55} < 0$

Relative maximum

put $x = \frac{1 + \sqrt{55}}{3}$ in eq ①

$f\left(\frac{1+\sqrt{55}}{3}\right) = 2\left(\frac{1+\sqrt{55}}{3}\right)^3 - 2\left(\frac{1+\sqrt{55}}{3}\right)^2 - 36\left(\frac{1+\sqrt{55}}{3}\right) + 3$

for stationary value

$f'(x) = 0$

$4x^3 - 8x = 0$

$4x(x^2-2) = 0$

$4x = 0$

$x = \frac{0}{4}$

$\boxed{x=0}$

$x^2 - 2 = 0$

$x^2 = 2$

$\boxed{x = \pm\sqrt{2}}$

Differentiat w.r.t 'x'

$f''(x) = 4(3x^{3-1}) - 8(1)$

$f''(x) = 12x^2 - 8$

for $\boxed{x=0}$	for $x=\sqrt{2}$	for $x=-\sqrt{2}$
$f''(0) = 12(0)^2 - 8$	$f''(\sqrt{2}) = 12(\sqrt{2})^2 - 8$	$f(-\sqrt{2}) = 12(-\sqrt{2})^2 - 8$
$f''(0) = 0 - 8$	$= 12\times2 - 8$	$= 12\times2 - 8$
$f''(0) = -8 < 0$	$= 24 - 8$	$= 24 - 8$
Relative maxima	$f''(\sqrt{2}) = 16 > 0$	$f''(-\sqrt{2}) = 16 > 0$
	Relative minima	Relative minima

$\boxed{x=0}$ in eq ①

$f(0) = (0)^4 - 4(0)^2$

$f(0) = 0 - 0$

$\boxed{f(0) = 0}$

f has relative maximum at $x=0$ and $f(0) = 0$

put $\boxed{x=\sqrt{2}}$ in eq ①

$f(\sqrt{2}) = (\sqrt{2})^4 - 4(\sqrt{2})^2$

$= (2)^2 - 4(2)$

$= 4 - 8$

$\boxed{f(\sqrt{2}) = -4}$

f has relative minimum at $x = \sqrt{2}$ and $f(\sqrt{2}) = -4$

Put $\boxed{x=-\sqrt{2}}$ in eq ②

$f(-\sqrt{2}) = (-\sqrt{2})^2 - 4(-\sqrt{2})^2$

$= (2)^2 - 4(2)$

$= 4 - 8$

$\boxed{f(-\sqrt{2}) = -4}$

f has relative minimum at $x = -\sqrt{2}$ and $f(\sqrt{2}) = -4$

$$f\left(\frac{1+\sqrt{55}}{3}\right) = 2\frac{(1+\sqrt{55})^3}{(3)^3} - 2\frac{(1+\sqrt{55})^2}{(3)^2} - 36\left(\frac{1+\sqrt{55}}{3}\right) + 3$$

$\because (a+b)^3 = a^3 + b^3 + 3a^2b + 3ab^2$

$\because (a+b)^2 = a^2 + b^2 + 2ab$

$$= 2\frac{[(1)^3 + (\sqrt{55})^3 + 3(1)^2\sqrt{55} + 3(1)(\sqrt{55})^2]}{27} - 2\frac{(1)^2 + (\sqrt{55})^2 + 2(1)\sqrt{55})}{9}$$

$$- \frac{36 - 36\sqrt{55}}{3} + 3$$

$$= 2\left[\frac{1 + 55\sqrt{55} + 3\sqrt{55} + 3(55)}{27}\right] - 2\left[\frac{1+55+2\sqrt{55}}{9}\right] - \frac{36 - 36\sqrt{55}}{3} + 3$$

$$= \frac{2 + 110\sqrt{55} + 6\sqrt{55} + 330}{27} - \frac{2 - 110 - 4\sqrt{55}}{9} - \frac{36 - 36\sqrt{55}}{3} + 3$$

$$= \frac{2 + 110\sqrt{55} + 6\sqrt{55} + 330 - 6 - 330 - 12\sqrt{55} - 324 - 324\sqrt{55} + 81}{27}$$

$$= \frac{1}{27}\left[-220\sqrt{55} - 247\right]$$

$= -\frac{1}{27}\left[220\sqrt{55} + 247\right]$ f has relative minimum.

put $x = \frac{1-\sqrt{55}}{3}$ in eq ①

$$f\left(\frac{1-\sqrt{55}}{3}\right) = 2\left(\frac{1-\sqrt{55}}{3}\right)^3 - 2\left(\frac{1-\sqrt{55}}{3}\right)^2 - 36\left(\frac{1-\sqrt{55}}{3}\right) + 3$$

$\because (a^3 - b^3)^3 = a^3 - b^3 - 3a^2b + 3ab^2$

$\because (a-b)^2 = a^2 - 2ab + b^2$

$$= 2\left[\frac{(1)^3 - (\sqrt{55})^3 - 3(1)^2\sqrt{55} + 3(1)(\sqrt{55})^2}{27}\right] - 2\left[\frac{(1)^2 + (\sqrt{55})^2 - 2(\sqrt{55}}{9}\right.$$

$$- \frac{36 + 36\sqrt{55}}{3} + 3$$

$$= 2\left[\frac{1 - 55\sqrt{55} - 3\sqrt{55} + 3(55)}{27}\right] - 2\left[\frac{1+55-2\sqrt{55}}{9}\right] - \frac{36 + 36\sqrt{55}}{3} + 3$$

$$= \frac{2 - 110\sqrt{55} - 6\sqrt{55} + 330}{27} - \frac{2 - 110 + 4\sqrt{55}}{9} - \frac{36 + 36\sqrt{55}}{3} + 3$$

$$= \frac{2 - 110\sqrt{55} - 6\sqrt{55} + 330 - 6 - 330 + 12\sqrt{55} - 324 + 324\sqrt{55} + 81}{27}$$

$$= \frac{1}{27}\left[220\sqrt{55} - 247\right]$$

$= \frac{1}{27}\left[-247 + 220\sqrt{55}\right]$ f has relative maximum.

(viii)

$f(x) = (x-2)^2(x-1)$

Solution:

$f(x) = (x-2)^2(x-1)$ — ①

$= (x^2+4-4x)(x-1)$

$= x^3+4x-4x^2-x^2-4+4x$

$f(x) = x^3-5x^2+8x-4$

Differentiate w.r.t 'x'

$f'(x) = 3x^2-10x+8(1)-0$

$f'(x) = 3x^2-10x+8$

for stationary point

$f'(x) = 0$

$3x^2-10x+8 = 0$

$3x^2-6x-4x+8 = 0$

$3x(x-2)-4(x-2) = 0$

$(3x-4)(x-2) = 0$

$3x-4=0$

$3x=4$

$\boxed{x=\frac{4}{3}}$

$x-2=0$

$\boxed{x=2}$

Differentiate w.r.t 'x'

$f''(x) = 6x-10$

for $x=2$

$f''(2) = 6(2)-10$

$f''(2) = 12-10$

$\boxed{f''(2) = 2>0}$

Relative minima

for $x=\frac{4}{3}$

$f''(\frac{4}{3}) = 6(\frac{4}{3})-10$

$f''(\frac{4}{3}) = \frac{24}{3}-10$

$f''(\frac{4}{3}) = \frac{24-30}{3}$

$f''(\frac{4}{3}) = \frac{-\cancel{6}\,2}{\cancel{3}}$

$f''(\frac{4}{3}) = -2<0$

Relative maxima

put $x=2$ in eq ①

$f(2) = (2-2)^2(2-1)$

$= (0)^2(1)$

$f(2) = 0$

f has Relative Minimum at $x=2$ and $f(2)=0$

put $x=\frac{4}{3}$ in eq ①

$f(\frac{4}{3}) = (\frac{4}{3}-2)^2(\frac{4}{3}-1)$

$f(\frac{4}{3}) = (\frac{4-6}{3})^2(\frac{4-3}{3})$

$= (\frac{-2}{3})^2(\frac{1}{3})$

$f(\frac{4}{3}) = \frac{4}{9}(\frac{1}{3})$

$f(\frac{4}{3}) = \frac{4}{27}$

f has relative maxima at $x=\frac{4}{3}$ and $f(\frac{4}{3}) = \frac{4}{27}$

(ix)

$f(x) = 5+3x-x^3$

Solution:

$f(x) = 5+3x-x^3$

Differentiate w.r.t 'x'

$f'(x) = 0+3(1)-3x^{3-1}$

$f'(x) = 3-3x^2$

for stationary point

$f'(x) = 0$

$3-3x^2 = 0$

$-3x^2 = -3$

$x^2 = \frac{-\cancel{3}}{-\cancel{3}}$

$x^2 = 1$

$x = \pm 1$

Differentiate w.r.t 'x'

$f''(x) = 0-3(2x^{2-1})$

$f''(x) = -6x$

for $x=1$;

$f''(1) = -6(1)$

$f''(1) = -6<0$

Relative maxima

for $x=-1$

$f''(-1) = -6(-1)$

$f''(-1) = 6>0$

Relative minima

put $x=1$ in eq ①

$f(1) = 5+3(1)-(1)^3$

$f(1) = 5+3-1$

$f(1) = 7$

f has Relative maxima at $x=1$ and $f(1) = 7$

put $x=-1$ in eq ①

$f(-1) = 5+3(-1)-(-1)^3$

$f(-1) = 5-3-(-1)$

$f(-1) = 5-3+1$

$f(-1) = 3$

f has Relative minimum at $x=-1$ and $f(-1)=3$

Question #3

Find the maximum and minimum value at of the following defined by the following equation occuring in the interval $[0, 2\pi]$.

$f(x) = \sin x + \cos x$

Solution:

$f(x) = \sin x + \cos x$ —— ①

Differentiate w.r.t "x"

$f'(x) = \cos x - \sin x$

for stationary point

$f'(x) = 0$

$\cos x - \sin x = 0$

$\cos x = \sin x$

$1 = \frac{\sin x}{\cos x}$

$1 = \tan x$

$x = \tan^{-1}(1)$

$x = \frac{\pi}{4}, \frac{5\pi}{4}$

Differentiate w.r.t 'x'

$f''(x) = -\sin x - \cos x$

$f''(x) = -(\sin x + \cos x)$

$x = \frac{\pi}{4}$

$f''(\frac{\pi}{4}) = -[\sin\frac{\pi}{4} + \cos\frac{\pi}{4}]$

$= -[\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}]$

$= -\frac{2}{\sqrt{2}} \Rightarrow -\frac{\sqrt{2}\cdot\cancel{\sqrt{2}}}{\cancel{\sqrt{2}}}$

$f''\frac{\pi}{4} = -\sqrt{2}$

Relative maximum at $x = \frac{\pi}{4}$

$x = \frac{5\pi}{4}$

$f(\frac{5\pi}{4}) = -[\sin\frac{5\pi}{4} + \cos\frac{5\pi}{4}]$

$f(\frac{5\pi}{4}) = -[\frac{-1}{\sqrt{2}} - \frac{1}{\sqrt{2}}]$

$f(\frac{5\pi}{4}) = -(\frac{-2}{\sqrt{2}})$

$f(\frac{5\pi}{4}) = \frac{2}{\sqrt{2}} \Rightarrow \frac{\sqrt{2}\cdot\cancel{\sqrt{2}}}{\cancel{\sqrt{2}}}$

$f(\frac{5\pi}{4}) = \sqrt{2}$

Relative minimum at $x = \frac{5\pi}{4}$

put $\boxed{x = \frac{\pi}{4}}$ in eq ①

$f(\frac{\pi}{4}) = \sin\frac{\pi}{4} + \cos\frac{\pi}{4}$

$f(\frac{\pi}{4}) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}$

$f(\frac{\pi}{4}) = \frac{2}{\sqrt{2}}$

$f(\frac{\pi}{4}) = \frac{\cancel{\sqrt{2}}\cdot\sqrt{2}}{\cancel{\sqrt{2}}}$

$f\frac{\pi}{4} = \sqrt{2}$

f has Relative maximum at $x = \frac{\pi}{4}$ and $f(\frac{\pi}{4}) = \sqrt{2}$

put $\boxed{x = \frac{5\pi}{4}}$ in eq ①

$f(\frac{5\pi}{4}) = \sin(\frac{5\pi}{4}) + \cos(\frac{5\pi}{4})$

$f(\frac{5\pi}{4}) = \frac{-1}{\sqrt{2}} - \frac{1}{\sqrt{2}}$

$f(\frac{5\pi}{4}) = \frac{-2}{\sqrt{2}}$

$f(\frac{5\pi}{4}) = -\frac{\sqrt{2}\cdot\cancel{\sqrt{2}}}{\cancel{\sqrt{2}}}$

$f(\frac{5\pi}{4}) = -\sqrt{2}$

f has relative minimum at $x = \frac{5\pi}{4}$ and $f(\frac{5\pi}{4}) = -\sqrt{2}$.

Question #4

Show that $y = \frac{\ln x}{x}$ has maximum value at $x = e$.

Solution:

$$y = \frac{\ln x}{x}$$

Differentiate w.r.t 'x'

$$\frac{dy}{dx} = \frac{x\frac{d}{dx}(\ln x) - (\ln x)\frac{d}{dx}(x)}{x^2}$$

$$\frac{dy}{dx} = \frac{x\left(\frac{1}{x}\right) - \ln x\ (1)}{x^2}$$

$$\frac{dy}{dx} = \frac{1-\ln x}{x^2}$$

for stationary point

$$f'(x) = 0$$

$$\frac{1-\ln x}{x^2} = 0$$

$$1 - \ln x = 0$$

$$1 = \ln x$$

$$\because \ln e = 1$$

$$\ln e = \ln x$$

$$\boxed{x = e}$$

Differentiate w.r.t 'x'

$$\frac{d^2y}{dx^2} = \frac{x^2\frac{d}{dx}(1-\ln x) - (1-\ln x)\frac{d}{dx}(x^2)}{x^4}$$

$$= \frac{x^{x'}\cdot\left(-\frac{1}{\cancel{x}}\right) - (1-\ln x)(2x)}{x^4}$$

$$f''(x) = \frac{-\cancel{x}[1+2(1-\ln x)]}{x^{\cancel{4}\,3}}$$

$$f''(x) = -\frac{(1+2(1-\ln x)]}{x^3}$$

$$=$$

$$\boxed{x = e}$$

Question #5

Show that $y = x^x$ has minimum value at $x = \frac{1}{e}$

Solution:

$$y = x^x$$

Taking logarithm on both side

$$\ln y = \ln x^x$$

$$\because \ln m^n = n \ln m$$

$$\ln y = x \ln x$$

Differentiate w.r.t 'x'

$$\frac{d}{dx}(\ln y) = \frac{d}{dx}(x\cdot\ln x)$$

$$\frac{1}{y}\frac{dy}{dx} = x\cdot\frac{d}{dx}(\ln x) + \ln x\ \frac{d}{dx}(x)$$

$$\frac{1}{y}\frac{dy}{dx} = \cancel{x}\cdot\frac{1}{\cancel{x}} + \ln x\,(1)$$

$$\frac{1}{y}\frac{dy}{dx} = 1 + \ln x$$

$$\frac{dy}{dx} = y(1+\ln x)$$

$$\frac{dy}{dx} = x^x(1+\ln x)$$

for stationary point

$$f'(x) = 0$$

$$x^x(1+\ln x) = 0$$

$$x^x \neq 0$$

$$1 + \ln x = 0$$

$$\ln x = -1$$

$$\because \ln e = 1$$

$$\ln x = -\ln e$$

$$x = -e$$

$$f''(x) = -\frac{[(1+2)(1-\ln e)]}{e^3}$$

$$f''(x) = -\frac{(1+2)(1-1)}{e^3}$$

$$f''(x) = \frac{-[(1+2)(0)]}{e^3}$$

$$f''(x) = \frac{-1}{e^3} < 0$$

Hence proved

y has maximum at $x = e$

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$$x = e^{-1}$$

$$x = \frac{1}{e}$$

Differentiate w.r.t "x"

$$f''(x) = x^x \frac{d}{dx}(1+\ln x) + (1+\ln x)\frac{d}{dx}(x^x)$$

$$f''(x) = x^x\left[\frac{1}{x} + (1+\ln x)\, x^x (1+\ln x)\right]$$

$$f''(x) = x^x\left[\frac{1}{x} + (1+\ln x)^2\right]$$

$$\boxed{x = \frac{1}{e}}$$

$$= \left(\frac{1}{e}\right)^{1/e}\left[\frac{1}{\frac{1}{e}} + \left(1+\ln\frac{1}{e}\right)^2\right]$$

$$= \left(\frac{1}{e}\right)^{1/e}\left[e + (1+\ln e^{-1})^2\right]$$

$$= \left(\frac{1}{e}\right)^{1/e}\left[e + (1-\ln e)^2\right]$$

$$= \left(\frac{1}{e}\right)^{1/e}\left[e + (1-1)^2\right]$$

$$= \left(\frac{1}{e}\right)^{1/e}\left[e + 0\right]$$

$$= \left(\frac{1}{e}\right)^{1/e} e > 0$$

Hence proved.

y is minimum at $x = \frac{1}{e}$.

✱ Theory

Example # 1

Find the two positive integers whose sum is 9 and the product of one with the square of other will be maximum.

Solution:

Suppose the two integer 'x' and 'y'

$x + y = 9$

$y = 9 - x$ —— ①

According to given condition.

$f(x) = xy^2$

$f(x) = x(9-x)^2$

Differentiate w.r.t 'x'

$f'(x) = x\frac{d}{dx}(9-x)^2 + (9-x)\frac{d}{dx}(x)$

$f'(x) = x.2(9-x) + (9-x)^2(1)$

$f'(x) = x2(9-x) + (9-x)^2$

$= [9-x]\ (9-x-2x)$

$= (9-x)\ (9-3x)$

stationary point

$f'(x) = 0$ | $9-3x=0$

$9-x=0$ | $9=3x$

$\boxed{9=x}$ | $\boxed{x=3}$

$f''(x) = (9-x) -3 + (9-3x)(-1)$

$= -27+3x-9+3x$

$f''(x) = 6x-36$

$f''(3) = 6(3) - 36$

$= 18-36$

$= -18 < 0$ Relative maximum

put $x=3$ in eq ①

$y = 9-x$

$y = 9-3$

$\boxed{y=6}$

Example #2

What are the dimension of a box of a square base having largest volume if the sum of one side of the base and its height is 12cm.

Solution:

length = x

width = x

height = h

Volume $= V = x^2h$

$x+h = 12$

$h = 12-x$ —— ①

$V = x^2h$

$V = x^2(12-x)$

$V = 12x^2 - x^3$

Differentiate w.r.t 'x'

$f'(x) = 24x - 3x^2$

for stationary point

$f'(x) = 0$

Example #3

The perimeter of triangle is 20cm. If one side of length is 8cm. What are length of the other two sides of maximum area of triangle?

Solution:

let "x" and "y" be the required length of triangle

$\because$ Perimeter $= x+y+8$

P = sum of all sides

$20 = x+y+8$

$x+y = 20-8$

$x+y = 12$

$y = 12-x$ —— ①

$S = \frac{x+y+8}{2} \Rightarrow \frac{\overset{10}{\cancel{20}}}{\cancel{2}} \Rightarrow S = 10$

Area $= A = \sqrt{S(S-\cancel{x})(S-y)(S-8)}$

$A^2 = S(S-x)(S-y)(S-8)$

$f(x) = 10(10-x)(10-y)(10-8)$

$= 10(10-x)(10-y)(2)$

$f(x) = 20(10-x)(10-12-x)$

$f'(x) = 20\frac{d}{dx}(10-x)(10-2)$

$= 20(10-x-x+2)$

$= 20(12-2x)$

$f(x) = 40(6-x)$

$f''(x) = 40(-1)$

$f''(x) = -40$ —— ②

for stationary point

$f'(x) = 0$

$40(6-x) = 0$

$24x - 3x^2 = 0$

$3x(8-x) = 0$

$3x = 0$ | $8-x = 0$

$x = 0$ | $\boxed{x=8}$

$f''(x) = 24-6x$

$f''(8) = 24-6(8)$

$= 24-48$

$f''(8) = -24 < 0$ Relative maximum

put $\boxed{x=8}$ in eq ①

$h = 12-x$

$h = 12-8$

$\boxed{h = 4cm}$

Example #4

An open box of rectangular base is to be 24cm by 45cm cardboard by cutting square sheet of equal size from each corner Find the dimension of corner square to obtain a box largest possible.

Solution:

length $= 45cm$

Width $= 24cm$

let length of each sides of square be x cm. After cutting $x\,cm^2$ from each corner.

Length $= 45-2x$

Width box $= 24-2x$

height of box $= x$

Volume $= f(x) = (45-2x)(24-2x)(x)$

$f(x) = (45-2x)(24x-2x^2)$

$6 - x = 0$

$\boxed{x=6}$

put $\boxed{x=6}$ in eq ②

$f''(6) = -40 < 0$

Relative maximum

put $x = 6$ in eq ①

$y = 12-6$

$\boxed{y=6}$

Hence length of required sides are 6 cm.

* * *

Example #5

Find the point on graph of the curve $y = 4-x^2$ which is closest to the point (3,4).

Solution:

Let $P(x,y)$ be the required point

$y = 4-x^2$ — ①

$d = f(x) =$ distance b/w (x,y) and $(3,4)$

$d = f(x) = \sqrt{(x-3)^2+(y-4)^2}$

$d^2 = f(x) = (x-3)^2+(y-4)^2$

$= (x-3)^2 - (4-x^2-4)^2$

$= (x-3)^2 + x^4$

$f(x) = x^2+9-6x+x^4$

$f'(x) = 2x-6+4x^3$

$f''(x) = 12x+2$ — ②

for stationary point

$f'(x) = 0$

$4x^3+2x-6 = 0$

$2x^3+x-3 = 0$

$(x-1)(2x^2+2x+3) = 0$

$f(x) = 1080x - 90x - 48x^2 + 4x^3$

$f(x) = 4x^3 - 138x^2 + 1080x$

$f'(x) = 12x^2 - 276x + 1080$

$f''(x) = 24x - 276$ — ①

Stationary point

$f'(x) = 0$

$12(x^2-23x+90) = 0$

$x^2-5x-18x+90 = 0$

$x(x-5)-18(x-5) = 0$

$(x-5)(x-18) = 0$

$\boxed{x=5}$, $\boxed{x=18}$

put $x=5$ in eq ①

$f''(5) = 24(5)-276$

$= 120-276$

$= -156 < 0$

Thus $f(x)$ will be maximum.

Hence 5cm is required length of each sides of square corner cutt of it.

* * *

$x = 1$

$2(1)^3+1-3$

$2+1-3 = 0$

$3-3 = 0$

put $\boxed{x=1}$ in eq ②

$f''(1) = 12(1)+2 \Rightarrow 12+2 \Rightarrow 14 > 0$

Thus $f(x)$ has minimum value

put $\boxed{x=1}$ in eq ①

$y = 4-x^2$

$y = 4-(1)^2$

$y = 4-1$

$\boxed{y=3}$

Hence $P(1,3)$ is required point closest $(3,4)$.

Exercise # 2.10

Question # 1

Find two positive integers whose sum is 30 and their product will be maximum.

Solution:

Suppose two numbers are 'x' and 'y'

$x+y=30$

$y=30-x$ —— ①

According to given condition

Product $= f(x) = xy$

$f(x) = x(30-x)$

$f(x) = 30x - x^2$

Differentiate w.r.t 'x'

$f'(x) = 30-2x$

for stationary point

$f'(x) = 0$

$30-2x=0$

$30=2x$

$\frac{\cancel{30}^{15}}{\cancel{2}} = x$

$\boxed{x=15}$

Differentiate w.r.t 'x'

$f''(x) = -2$

$f''(15) = -2 < 0$

so, $f(x)$ is Relative maximum

Now. first number $= x = 15$

Second number $y = 30-x$

$= 30-15$

$\boxed{y = 15}$

Question # 2

Divide 20 into two parts so that the sum of their square will be minimum.

Solution:

Suppose two numbers are 'x' and 'y'

$x+y=20$

$y=20-x$ —— ①

According to Given condition.

$f(x) = x^2+y^2$

$f(x) = x^2+(20-x)^2$

$f(x) = x^2+400+x^2-40x$

$f(x) = 2x^2-40x+400$

Differentiate w.r.t 'x'

$f'(x) = 4x-40$

for stationary point

$f'(x) = 0$

$4x-40=0$

$4x=40$

$x = \frac{\cancel{40}^{10}}{\cancel{4}}$

$\boxed{x=10}$

Differentiate w.r.t 'x'

$f''(x) = 4$

$f''(10) = 4 > 0$

so, $f(x)$ is Relative minimum

put $\boxed{x=10}$ in eq ①

$y = 20-x$

$y = 20-10$

$\boxed{y=10}$

Question #3

Find two positive integers whose sum is 12 and the product of one with square with the other will be maximum.

Solution:

Suppose two numbers are 'x' and 'y'

$x+y=12$

$y=12-x$ —— ①

According to given condition

$f(x)=x^2y$

$f(x)=x^2(12-x)$

$f(x)=12x^2-x^3$

Differentiate w.r.t 'x'

$f'(x)=24x-3x^2$

for stationary point

$f'(x)=0$

$24x-3x^2=0$

$3x(8-x)=0$

$3x=0 \quad | \quad 8-x=0$

$\boxed{x=0} \quad | \quad \boxed{x=8}$

Differentiate w.r.t 'x'

$f''(x)=24-6x$

$f''(8)=24-6(8)$

$f''(8)=24-48$

$=-24<0$

so, f(x) is Relative maximum

put $\boxed{x=8}$ in eq ①

$y=12-x$

$y=12-8$

$\boxed{y=4}$

* * *

Question #4

The perimeter of triangle is 16 Centimeters. If one side is of length is 6cm, what are length of other side for maximum area of triangle?

Solution:

Suppose length of one side $= x$

2nd side $= 6cm$

3rd side $= 16-6-x$

$= 10-x$

so, sides are $x, 6, 10-x$

Suppose f(x) is represent square of area

$f(x)=S(S-a)(S-b)(S-c)$ —— ①

$\boxed{a=x}$, $\boxed{b=6}$, $\boxed{c=10-x}$

$S=\frac{a+b+c}{2} \Rightarrow \frac{x+6+10-x}{2} \Rightarrow \frac{\cancel{16}\,8}{\cancel{2}} \Rightarrow 8$

eq ① become.

$f(x)=8(8-x)(8-6)(8-(10-x))$

$=8(8-x)(2)(8-10+x)$

$=16(8-x)(-2+x)$

$f(x)=16(-16-x^2+2x+8x)$

$f(x)=16(-x^2+10x-16)$

Differentiate w.r.t 'x'

$f'(x)=16[-2x+10)$

for stationary point

$f'(x)=0$

$16(-2x+10)=0$

$-2x+10=0$

$10=2x$

$5\,\frac{\cancel{10}}{\cancel{2}}=x$

$\boxed{x=5}$

Differentiate w.r.t 'x'

$f''(x)=16(-2) \Rightarrow -32$

$f''(5)=-32<0$ Relative maximum

One side $= 5cm$

2nd side $= 6cm$

3rd side $= 10-x$

$=10-5$

$=5cm$

* * *

Question #5

Find the dimension of rectangle of largest area having perimeter 120 centimeters.

Solution:

Suppose length = x

width = y

Perimeter = 120cm

$2(x+y) = 120$

$x+y = \frac{\overset{60}{\cancel{120}}}{\cancel{2}}$

$x+y = 60$

$y = 60-x$ —— ①

According to given condition.

Area = $A = xy$

$A=(x(60-x))$

$A = 60x - x^2$

$\frac{dA}{dx} = 60-2x$

for stationary point

$\frac{dA}{dx} = 0$

$60-2x = 0$

$60 = 2x$

$\frac{\overset{30}{\cancel{60}}}{\cancel{2}} = x$

$\boxed{x = 30cm}$

$\frac{d^2A}{dx^2} = -2 < 0$

put $\boxed{x=30}$ in eq ①

$y = 60-x$

$y = 60-30$

$\boxed{y = 30cm}$

Length = x = 30cm

Width = y = 30cm

Question #6

Find the length of the sides of variable rectangle having area 36cm² when its perimeter is minimum.

Solution:

Suppose length = x

width = y

Area = xy

$36 = xy$

$y = \frac{36}{x}$ —— ①

According to given condition.

Perimeter = $2(x+y)$

$= 2\left(x + \frac{36}{x}\right)$

$= \cancel{2\left(\frac{x^2+36}{x}\right)}$

$P = 2x + 72x^{-1}$

Differentiate w.r.t 'x'

$\frac{dP}{dx} = 2 - 72x^{-2}$

$= 2 - \frac{72}{x^2}$

for stationary point

$\frac{dP}{dx} = 0$

$2 - \frac{72}{x^2} = 0$

$\frac{2x^2-72}{x^2} = 0$

$2x^2 - 72 = 0$

$2x^2 = 72$

$x^2 = \frac{\overset{36}{\cancel{72}}}{\cancel{2}}$

$x^2 = 36$

$\boxed{x = \pm 6}$

$\boxed{x = 6}$

Differentiate w.r.t 'x'

$\frac{d^2P}{dx^2} = 144x^{-3}$

$= \frac{144}{x^3}$

$= \frac{144}{(6)^3}$

$= \frac{144}{216}$

$\frac{d^2P}{dx^2} = \frac{2}{3} > 0$ Relative minimum

put in eq ①

$y = \frac{\overset{6}{\cancel{36}}}{\cancel{6}}$

$\boxed{y = 6}$ length. x = 6cm

Width = y = 6 cm

Question #7

A box with a square base and open top is to have a volume of 4cubic dm. Find the dimension of box which will require the least material.

Solution:
Suppose length = x
width = x
height = h

Volume = $V = x^2h$

$4 = x^2h$

$h = \frac{4}{x^2}$ —— ①

∵ S = Area of square + Area four vertically walls

$S = x^2 + 4hx$

$= x^2 + 4\left(\frac{4}{x^2}\right)x$

$= x^2 + \frac{16}{x}$

$S = x^2 + 16x^{-1}$

Differentiate w.r.t "x"

$f'(x) = 2x - 16x^{-2}$

$= 2x - \frac{16}{x^2}$

for stationary point

$f'(x) = 0$

$2x - \frac{16}{x^2} = 0$

$\frac{2x^3 - 16}{x^2} = 0$

$2x^3 - 16 = 0$

$2x^3 = 16$

$x^3 = 8$

$\boxed{x = 2}$

put $x=2$ in eq ①

$4 = x^2h$

$4 = (2)^2h$

$4 = 4h$

$\frac{\cancel{4}}{\cancel{4}} = h$

$\boxed{h = 1}$

Differentiate w.r.t "x"

$f''(x) = 2 + 32x^{-3}$

$= 2 + \frac{32}{x^3}$

$\boxed{x = 2}$

$f''(2) = 2 + \frac{32}{(2)^3}$

$= 2 + \frac{\cancel{32}^{4}}{\cancel{8}}$

$f''(2) = 6 > 0$

Relative minimum.

Length = 2dm
Width = 2dm
height = 1dm

Question #8

Find the dimension of a rectangular garden having perimeter 80meters. If its area is to be maximum.

Solution:
Suppose
length = x
width = y

Perimeter = 80m

$2(x+y) = 80$

$x + y = \frac{\cancel{80}}{\cancel{2}}$

$x + y = 40$

$y = 40 - x$ —— ①

According to given condition

Area = $A = xy$

$A = x(40-x)$

$A = 40x - x^2$

Differentiate w.r.t "x"

$\frac{dA}{dx} = 40 - 2x$

for stationary point

$\frac{dA}{dx} = 0$

$40 - 2x = 0$

$40 = 2x$

$\frac{^{20}\cancel{40}}{\cancel{2}} = x$

$\boxed{x = 20m}$

Differentiate w.r.t "x"

$\frac{d^2A}{dx^2} = -2$

$x = 20$

$\frac{d^2A}{dx^2} = -2 < 0$

Relative maximum

put $x=20$ in eq ①

$y = 40 - x$

$y = 40 - 20$

$\boxed{y = 20m}$

Length = 20m
Width = 20m

Question #11

Find the point on the curve $y = x^2 - 1$ that is closest to the point $(3, -1)$.

Solution:

$$y = x^2 - 1 \quad — ①$$

Let $P(x, y)$ be the required point

Let d = distance between (x, y) and $(3, -1)$

$$d = \sqrt{(x-3)^2 + (y+1)^2}$$

$$d^2 = (x-3)^2 + (y+1)^2$$

$$d^2 = x^2 + 9 - 6x + y^2 + 1 + 2y$$

$\because d^2 = f(x)$

$\because y = (x^2 - 1)$

$$f(x) = x^2 + 9 - 6x + (x^2-1)^2 + 1 + 2(x^2-1)$$

$$= x^2 + 9 - 6x + x^4 \cancel{+1} \cancel{-2x^2} \cancel{+1} + \cancel{2x^2} \cancel{-2}$$

$$f(x) = x^4 + x^2 - 6x + 9$$

Differentiate w.r.t 'x'

$$f'(x) = 4x^3 + 2x - 6$$

$$f''(x) = 12x^2 + 2 \quad — ①$$

for $x = 1$

$$\Rightarrow 2x^3 + x - 3 = 0$$

$$2(1)^3 + 1 - 3 = 0$$

$$2 + 1 - 3 = 0$$

$$3 - 3 = 0$$

$$0 = 0$$

put $\boxed{x = 1}$ in eq ②

$$f''(1) = 12(1)^2 + 2$$

$$= 12 + 2$$

$$= 14 > 0$$ f(x) has relative minimum

put $x = 1$ in eq ①

$$y = (1)^2 - 1$$

$$y = 1 - 1$$

$$\boxed{y = 0}$$

So, $(1, 0)$ is the required point closest to $(3, -1)$

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Question #12

Find the point on the curve $y=x^2+1$ that is closest to the point (18,1).

Solution:

$y = x^2+1$ —— ①

Let $P(x,y)$ be the required point

$\because d$ = distance between $P(x,y$ and $(18,1)$

$d = \sqrt{(x-18)^2+(y-1)^2}$

$d^2 = (x-18)^2+(y-1)^2$

$\because d^2 = f(x)$

$f(x) = (x-18)^2+(x^2+1-1)^2$

$f(x) = x^2+324-36x+x^4$

$f(x) = x^4+x^2-36x+324$

Differentiate w.r.t "x"

$f'(x) = 4x^3+2x-36$

$f''(x) = 12x^2+2$ —— ②

for stationary point

$f'(x) = 0$

$4x^3+2x-36=0$

$2x^3+x-18=0$

$(x-2)(2x^2+4x+9)=0$

put $\boxed{x=2}$ in eq ②

$f''(2) = 12(2)^2+12>0$

$f(x)$ has minimum value

put $\boxed{x=2}$ in eq ①

$y = (2)^2+1$

$y = 4+1$

$\boxed{y = 5}$

So, (2,5) is the closest point which is closest to (18,1).

UNIT

3

Integration

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Unit no 3

Integration

Theory:

• Differential of a function:

Let "f" be a differentiable functions define by the equation $y = f(x)$ and let δx be the arbitrary increment in x. Then the number $f'(x)\delta x$ is called the differential of the dependent variable "y" and is denoted by dy. Thus $dy = f'(x)\delta x$

Note:

i) The increment in the dependent variable "x" is equal to its differential dx i.e; $dx = \delta x$

ii) Instead of dy, we can write df, i.e., $df = f'(x)dx$ whele $f'(x)$ being coefficient of differential is called differential coefficient

Example # 1: Find δy and dy of the funtion defined as $f(x) = x^2$, when $x = 2$ and $dx = 0.01$.

Let $y = f(x)$

$$\frac{dy}{dx} = 2x$$

$$dy = 2x\,dx$$

Take $x = 2$ and $dx = 0.01$

$$dy = 2(2)(0.01)$$
$$= 0.04$$

Now we find δy

$$\delta y = (x+\delta x)^2 - y$$

$$\delta y = (x+\delta x)^2 - x^2$$

$$\therefore \delta x = dx = 0.01$$

$$\delta y = (2+0.01)^2 - (2)^2$$

$$\delta y = 4.0401 - 4$$

$$\delta y = 0.0401$$

Example # 2: Use differentials find $\frac{dy}{dx}$ when $\frac{y}{x} - \ln x = \ln c$.

$$\frac{y}{x} - \ln x = \ln c$$

$$d\left(\frac{y}{x} - \ln x\right) = d(\ln c)$$

$$d\left(\frac{y}{x}\right) - d(\ln x) = 0$$

$$\frac{x\,dy - y\,dx}{x^2} - \frac{1}{x}dx = 0$$

$$\frac{x\,dy - y\,dx}{x^2} = \frac{1}{x}dx$$

$$x\,dy - y\,dx = \frac{x^2}{x}dx$$

$$x\,dy - y\,dx = x\,dx$$

$$x\,dy = x\,dx + y\,dx$$

$$x\,dy = (x+y)\,dx$$

$$\frac{dy}{dx} = \frac{x+y}{x}$$

Example # 3: Use differentials to approximate the value of $\sqrt{17}$.

Let $f(x) = \sqrt{x}$

$$f'(x) = \frac{1}{2\sqrt{x}}$$

$$\Rightarrow f(x+\delta x) = \sqrt{x+\delta x}$$

Take $x = 16$ & $dx = \delta x = 1$

$$f(x) = \sqrt{16} = 4$$

$$f(x+\delta x) = \sqrt{16+1} = \sqrt{17}$$

$$f'(x) = \frac{1}{2\sqrt{16}} = \frac{1}{2(4)} = \frac{1}{8} = 0.125$$

Since we know that

$$f(x+\delta x) \approx f(x) + f'(x)\,dx$$

$$\sqrt{17} \approx 4 + 0.125(1) = 4.125$$

Example # 4: Use differential to approximate the value of $\sqrt[3]{8.6}$.

Let $f(x) = \sqrt[3]{x} = (x)^{1/3}$

$$f'(x) = \frac{1}{3}x^{1/3-1} = \frac{1}{3}x^{-2/3}$$

$$f'(x) = \frac{1}{3x^{2/3}}$$

And $f(x+\delta x) = \sqrt[3]{x+\delta x}$

$x = 8$ & $dx = \delta x = 0.6$

$$f(x) = (8)^{1/3} = (2^3)^{1/3} = 2$$

$$f'(x) = \frac{1}{3(8)^{2/3}} = \frac{1}{(2^3)^{2/3}}$$

$$f'(x) = \frac{1}{3(4)} = \frac{1}{12} = 0.0833$$

$$f(x+\delta x) = \sqrt[3]{8+0.6}$$

$$= \sqrt[3]{8.6}$$

We know that

$$f(x+\delta x) \approx f(x) + f'(x)\,dx$$

$$\sqrt[3]{8.6} \approx 2 + (0.833)(0.6)$$

$$\sqrt[3]{8.6} \approx 2 + 0.04998$$

$$\sqrt[3]{8.6} \approx 2.04998$$

$$\sqrt[3]{8.6} \approx 2.05$$

Example # 5:

Using differentials, find the approximate value of sin 46°

Let $f(x) = \sin x$

$f'(x) = \cos x$

And $f(x + \delta x) = \sin(x + \delta x)$

Take $x = 45°$ & $dx = \delta x = 1° = 0.01745$

$f(x) = \sin 45°$

$= 0.7071$

$f'(x) = \cos 45°$

$= 0.7071$

$f(x + \delta x) = \sin(45° + 1°)$

$= \sin 46°$

As we know that

$$f(x + \delta x) \approx f(x) + f'(x)dx$$

$\sin 46° \approx 0.7071 + (0.7071)(0.01745)$

$\sin 46° \approx 0.7071 + 0.0123$

$\sin 46° \approx 0.7194$

Example # 6:

The side of a cube is measured to be 20cm with a maximum error of 0.12cm in its measurement. Find the maximum error in the calculated volume of the cube.

Solution:

For a cube length of each side be x and V be the volume

So

$\text{Volume} = x \cdot x \cdot x$

$V = x^3$

$dV = d(x^3)$

$dV = 3x^2 dx$

Take

$x = 20$, $dx = 0.12$

$dV = 3(20)^2(0.12)$

$dV = 3(400)(0.12)$

$dV = 144 cm^3$

Exercise # 3·1

Question # 1

Find δy and dy in the following cases:

(i)

$y = x^2 - 1$ when x changes from 3 to 3·02.

Solution:

$$y = x^2 - 1 \quad \text{———} \quad ①$$

$$y + \delta y = (x+\delta x)^2 - 1 \quad \text{———} \quad ②$$

Equation ② − Equation ①

$$\cancel{y} + \delta y - \cancel{y} = (x+\delta x)^2 - 1 - (x^2 - 1)$$

$$\delta y = (x+\delta x)^2 - \cancel{1} - x^2 + \cancel{1}$$

$$\delta y = (x+\delta x)^2 - x^2$$

As x changes from 3 to 3·02

$x = 3$ $\quad \delta x = dx = 3·02 - 3 = 0·02$

$$\delta y = (3 + 0·02)^2 - (3)^2$$

$$\delta y = 9·1204 - 9$$

$$\boxed{\delta y = 0·1204}$$

$$y = x^2 - 1$$

Taking differential both sides

$$dy = d(x^2 - 1)$$

$$dy = d(x^2) - d(1)$$

$$dy = 2x\,dx - 0$$

$$dy = 2x\,dx$$

$$dy = 2(3)(0·02)$$

$$dy = 6 \cdot (0·02)$$

$$\boxed{dy = 0·12}$$

· ——— ·

(ii)

$y = x^2 + 2x$ when x changes from 2 to 1·8.

Solution.

$$y = x^2 + 2x \quad \text{———} \quad ①$$

$$y + \delta y = (x+\delta x)^2 + 2(x+\delta x) \quad \text{———} \quad ②$$

Equation ② − Equation ①

$$\cancel{y} + \delta y - \cancel{y} = (x+\delta x)^2 + 2(x+\delta x) - (x^2 + 2x)$$

$$\delta y = (x+\delta x)^2 + \cancel{2x} + 2\delta x - x^2 - \cancel{2x}$$

$$\delta y = (x+\delta x)^2 + 2\delta x - x^2$$

As x changes from 2 to 1·8

$x = 2$ $\quad \delta x = dx = 1·8 - 2 = -0·8$

$\delta y = (2-0.2)^2 + 2(-0.2) - (2)^2$

$\delta y = (1.8)^2 - 0.4 - 4$

$\delta y = 3.24 - 0.4 - 4$

$\boxed{\delta y = -1.16}$

$y = x^2 + 2x$

Taking differential on both sides

$d(y) = d(x^2 + 2x)$

$dy = d(x^2) + d(2x)$

$dy = 2x\,dx + 2\,dx$

$dy = (2x+2)dx$

$dy = (2(2)+2)(-0.2)$

$dy = (4+2)(-0.2)$

$dy = 6(-0.2)$

$\boxed{dy = -1.2}$

* * *

(iii)

$y = \sqrt{x}$ when x changes from 4 to 4.41.

Solution:

$y = \sqrt{x}$

$y + \delta y = \sqrt{x + \delta x}$

$\cancel{y} + \delta y - \cancel{y} = \sqrt{x + \delta x} - \sqrt{x}$

$\delta y = \sqrt{x + \delta x} - \sqrt{x}$

As x changes from 4 to 4.41

$x = 4 \quad \delta x = dx = 4.41 - 4 = 0.41$

$\delta y = \sqrt{4 + 0.41} - \sqrt{4}$

$\delta y = \sqrt{4.41} - 2$

$\delta y = 2.1 - 2$

$\boxed{\delta y = 0.2}$

$y = \sqrt{x}$

$y = (x)^{1/2}$

Taking differential on both sides

$d(y) = d(x)^{1/2}$

$dy = \frac{1}{2} x^{\frac{1}{2}-1} dx$

$dy = \frac{1}{2} x^{\frac{1-2}{2}} dx$

$dy = \frac{1}{2} x^{\frac{-1}{2}} dx$

$dy = \frac{1}{2x^{1/2}} dx$

$dy = \frac{1}{2\sqrt{x}} dx$

$dy = \frac{1}{2\sqrt{4}} (0.41)$

$dy = \frac{1}{2(2)} (0.41)$

$dy = \frac{0.41}{4}$

$\boxed{dy = 0.1025}$

* * *

Question #2

Using differentials find $\frac{dy}{dx}$ and $\frac{dx}{dy}$ in the following equation:

(i)

$xy + x = 4$

Solution:

$xy + x = 4$

Taking differentials on both sides

$d(xy+x) = d(4)$

$d(xy) + dx = 0$

$x\,dy + y\,dx + dx = 0$

$x\,dy + (y+1)\,dx = 0$

$x\,dy = -(y+1)\,dx$

$$\boxed{\frac{dy}{dx} = \frac{-(y+1)}{x}}$$

$$\boxed{\frac{dx}{dy} = \frac{-x}{y+1}}$$

(ii)

$x^2 + 2y^2 = 16$

Solution:

$x^2 + 2y^2 = 16$

Taking differentials on both sides

$d(x^2+2y^2) = d(16)$

$d(x^2) + d(2y^2) = 0$

$d(x^2) + 2(dy^2) = 0$

$2x^{2-1} dx + 2 \cdot 2y^{2-1} \cdot dy = 0$

$2x\,dx + 4y\,dy = 0$

$4y\,dy = -2x\,dx$

$$\frac{dy}{dx} = \frac{-\overset{1}{\cancel{2}}x}{\underset{2}{\cancel{4}}y}$$

$$\boxed{\frac{dy}{dx} = \frac{-x}{2y}}$$

$$\boxed{\frac{dx}{dy} = -\frac{2y}{x}}$$

(iii)

$x^4 + y^2 = xy^2$

Solution:

$x^4 + y^2 = xy^2$

Taking differentials on both sides

$d(x^4+y^2) = d(xy^2)$

$d(x^4) + d(y^2) = x\,d(y^2) + y^2\,dx$

$4x^{4-1} dx + 2y^{2-1} dy = x \cdot 2y^{2-1} dy + y^2\,dx$

$4x^3 dx + 2y\,dy = x \cdot 2y\,dy + y^2\,dx$

$4x^3 dx + 2y\,dy = 2xy\,dy + y^2\,dx$

$$2y\,dy - 2xy\,dy = y^2dx - 4x^3dx$$

$$dy(2y-2xy) = dx(y^2-4x^3)$$

$$\frac{dy}{dx} = \frac{y^2-4x^3}{2y-2xy}$$

$$\boxed{\frac{dy}{dx} = \frac{y^2-4x^3}{2y(1-x)}}$$

$$\boxed{\frac{dx}{dy} = \frac{2y(1-x)}{y^2-4x^3}}$$

* * *

(iv)

$xy - \ln x = c$

Solution:

$$xy - \ln x = c$$

Taking differentials on both sides

$$d(xy+\ln x) = d(c)$$

$$d(xy)+d(\ln x) = 0$$

$$x\,dy + y\,dx + \frac{1}{x}dx = 0$$

$$x\,dy + \left[y+\frac{1}{x}\right]dx = 0$$

$$x\,dy = -\left(y+\frac{1}{x}\right)dx$$

$$\frac{dy}{dx} = \frac{-\left(\frac{xy+1}{x}\right)}{x}$$

$$\frac{dy}{dx} = \frac{-(xy+1)}{x^2}$$

$$\boxed{\frac{dy}{dx} = \frac{(1-xy)}{x^2}}$$

$$\boxed{\frac{dx}{dy} = \frac{x^2}{(1-xy)}}$$

Question #3

Use differentials to appoximate the values of:

(i)

$$\sqrt[4]{17}$$

Solution:

Let

$$f(x) = \sqrt[4]{x}$$

$$f(x) = (x)^{1/4}$$

$$f'(x) = \frac{1}{4}x^{\frac{1}{4}-1}$$

$$f'(x) = \frac{1}{4}x^{\frac{-3}{4}}$$

$$f'(x) = \frac{1}{4x^{3/4}}$$

$$f(x+\delta x) = \frac{1}{\sqrt[4]{x+\delta x}}$$

$x = 16$, $dx = \delta x = 1$

$$f(x) = (16)^{1/4} = (2^4)^{1/4} = 2$$

$$f'(x) = \frac{1}{4(16)^{3/4}} = \frac{1}{4(2^4)^{3/4}} = \frac{1}{4(2)^3} = \frac{1}{4(8)} = \frac{1}{32} = 0.03125$$

$$f(x+\delta x) = \sqrt[4]{16+1} = \sqrt[4]{17}$$

$$f(x+\delta x) \simeq f(x)+f'(x)\,\delta x$$

$$\sqrt[4]{17} \simeq 2 + (0.03125)(1)$$

$$\sqrt[4]{17} \simeq 2 + 0.03125$$

$$\boxed{\sqrt[4]{17} \simeq 2.03125}$$

* * *

(ii)

$(31)^{1/5}$

Solution:

Let

$f(x) = (x)^{1/5}$

$f'(x) = \frac{1}{5} x^{\frac{1}{5}-1}$

$f'(x) = \frac{1}{5} x^{-\frac{4}{5}}$

$f'(x) = \frac{1}{5x^{4/5}}$

$f(x+\delta x) = (x+\delta x)^{1/5}$

$x = 32 \ , \quad \delta x = dx = -1$

$f(32) = (32)^{1/5} = (2^5)^{1/5} = 2$

$f'(32) = \frac{1}{5(32)^{4/5}} = \frac{1}{5(2^5)^{4/5}} = \frac{1}{5(16)}$

$= \frac{1}{80} = 0.0125$

$f(x+\delta x) = (32-1)^{1/5} = (31)^{1/5}$

$f(x+\delta x) \simeq f(x) + f'(x)\,\delta x$

$(31)^{1/5} \simeq 2 + (0.0125)(-1)$

$(32)^{1/5} \simeq 2 - 0.0125$

$\boxed{(32)^{1/5} \simeq 1.9875}$

(iii)

$\cos 29°$

Solution:

Let

$f(x) = \cos x$

$f'(x) = -\sin x$

$f(x+\delta x) = \cos(x+\delta x)$

$x = 30° \qquad dx = \delta x = -1° = -0.01745$

$f(x) = \cos 30° = 0.866$

$f'(x) = -\sin 30° = -0.5$

$f(x+\delta x) = \sin(30-1) = \cos 29°$

$f(x+\delta x) \simeq f(x) + f'(x)\,\delta x$

$\cos 29° \simeq (0.866) + (-0.5)(-0.01745)$

$\cos 29° \simeq 0.866 + 0.08725$

$\boxed{\cos 29° \simeq 0.8747}$

(iv)

$\sin 61°$

Solution:

Let

$f(x) = \sin x$

$f'(x) = \cos x$

$f(x+\delta x) = \sin(x+\delta x)$

$x = 60° \quad , \quad \delta x = dx = 1° = 0.01745$

$f(x) = \sin 60° = 0.866$

$f'(x) = \cos 60° = 0.5$

$f(x+\delta x) = \sin(60+1°) = \sin 61°$

$f(x+\delta x) \simeq f(x) + f'(x)\,\delta x$

$\sin 61° \simeq 0.866 + (0.5)(0.01745)$

$\sin 61° \simeq 0.866 + 0.872$

$\boxed{\sin 61° \simeq 0.8747}$

Question #4

Find the appoximate increase in the volume of a cube if the length of its each edge change from 5 to 5·02.

Solution:

Let

Length of each edge of a cube = x

Volume of cube = $x \cdot x \cdot x$.

$$V = x^3$$

Taking differentials on both sides

$$dV = d(x^3)$$

$$dV = 3x^{3-1}dx$$

$$dV = 3x^2dx$$

As x changes from 5 to 5·02

$x = 5$, $\delta x = dx = 5{\cdot}02 - 5 = 0{\cdot}02$

$$dV = 3(5)^2(0{\cdot}02)$$

$$dV = 3(25)(0{\cdot}02)$$

$$dV = 75(0{\cdot}02)$$

$$\boxed{dV = 1{\cdot}5 \text{ cubic units}}$$

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Theory

Integration:

The process of finding such a function whose derivative is given is called **anti-differentiation** or **integration**.

"c" is an arbitrary constant and it is not definite, so $\phi(x) + c$ is called the indefinite integral of $f(x)$, that is

$$\int f(x)\,dx = \phi(x) + c$$

- The function $f(x)$ is called the integrand
- The symbol $\int$ is called integral sign
- "c" is called the constant of integration.
- $\int \ldots dx$ indicates that integrand is to be integrated w.r.t x.

Theorems on Anti-Derivatives:

1- The integral of the product of a constant and function is equal to the product of the constant and the integral of the function. In symbols $\int a f(x)\,dx = a \int f(x)\,dx$ where a is a constant.

2- The integral of the sum (or difference) of two functions is equal to the sum (or difference) of their integral

In symbols. $\int [f_1(x) \pm f_2(x)]\,dx = \int f_1(x)\,dx \pm \int f_2(x)\,dx$

Anti-Deraivatives of $[f(x)]^n f'(x)$ and $[f(x)]^{-1} f'(x)$

1- $\int [f(x)]^n \cdot f'(x)\,dx = \frac{[f(x)]^{n+1}}{n+1} + c$, 2- $\int [f(x)]^{-1} f'(x)\,dx = \ln f(x) + c$

	General form	**Simple form**
1.	$\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)}+c, (n\neq -1)$	$\int x^n dx = \frac{x^{n+1}}{n+1}+c, (n\neq -1)$
2.	$\int \sin(ax+b)dx = -\frac{1}{a}\cos(ax+b)+c,$	$\int \sin x\, dx = -\cos x + c$
3.	$\int \cos(ax+b)dx = \frac{1}{a}\sin(ax+b)+c,$	$\int \cos x\, dx = \sin x + c$
4.	$\int \sec^2(ax+b)dx = \frac{1}{a}\tan(ax+b)+c,$	$\int \sec^2 x\, dx = \tan x + c$
5.	$\int \operatorname{cosec}^2(ax+b)dx = -\frac{1}{a}\cot(ax+b)+c,$	$\int \operatorname{cosec}^2 x\, dx = -\cot x + c$
6.	$\int \sec(ax+b)\tan(ax+b)dx = \frac{1}{a}\sec(ax+b)+c,$	$\int \sec x \tan x\, dx = \sec x + c$
7.	$\int \operatorname{cosec}(ax+b)\cot(ax+b)dx = -\frac{1}{a}\operatorname{cosec}(ax+b)+c,$	$\int \operatorname{cosec} x \cot x\, dx = \operatorname{cose}x+c$
8.	$\int e^{\lambda x+\mu}dx = \frac{1}{\lambda}\times e^{\lambda x+\mu}+c, (\lambda\neq 0)$	$\int e^x dx = e^x + c$
9.	$\int a^{\lambda x+\mu}dx = \frac{1}{\lambda \ln a}\cdot a^{\lambda x+\mu}+c,$ $(a>0, a\neq 1, \lambda\neq 0)$	$\int a^x dx = \frac{1}{\ln a}\cdot a^x + c,$ $(a>0, a\neq 1)$
10.	$\int \frac{1}{ax+b}dx = \int (ax+b)^{-1}dx$ $= \frac{1}{a}\ln(ax+b)+c$	$\int \frac{1}{x}dx = \ln\lvert x\rvert + c,$ $x\neq 0$
11.	$\int \tan(ax+b)dx = \frac{1}{a}\ln\lvert\sec(ax+b)\rvert+c,$ $= -\frac{1}{a}\ln\lvert\cos(ax+b)\rvert+c$	$\int \tan x\, dx = \ln\lvert\sec x\rvert+c$ $= -\ln\lvert\cos x\rvert+c$
12.	$\int \cot(ax+b)dx = \frac{1}{a}\ln\lvert\sin(ax+b)\rvert+c,$	$\int \cot x\, dx = \ln\lvert\sin x\rvert+c$
13.	$\int \sec(ax+b)dx = \frac{1}{a}\ln\lvert\sec(ax+b)+\tan(ax+b)\rvert+c,$	$\int \sec x\, dx = \ln\lvert\sec x+\tan x\rvert+c$
14.	$\int \operatorname{cosec}(ax+b)dx = \frac{1}{a}\ln\lvert\operatorname{cosec}(ax+b)\mp\cot(ax+b)\rvert+c,$	$\int \operatorname{cosec} x\, dx = \ln\lvert\operatorname{cosec} x - \cot x\rvert+c$

Examples:

1. $\int x^5 \, dx$

$= \frac{x^{5+1}}{5+1}$ $\quad \therefore \int x^n dx = \frac{x^{n+1}}{n+1} + C$

$= \frac{x^6}{6}$

$= \frac{1}{6} x^6 + C$

2. $\int \frac{1}{\sqrt{x^3}} \, dx$

$= \int \frac{1}{(x^{1/2})^3} dx$

$= \int x^{-3/2} dx$

$= \frac{x^{-3/2+1}}{-3/2+1}$ $\quad \therefore \int x^n dx = \frac{x^{n+1}}{n+1} + C$

$= \frac{x^{-1/2}}{-1/2}$

$= \frac{-2}{\sqrt{x}} + C$

3. $\int \frac{1}{(2x+3)^4} \, dx$

$= \int (2x+3)^{-4} dx$

$= \frac{1}{2} \cdot \frac{(2x+3)^{-4+1}}{-4+1}$ $\quad \therefore \int (ax+b)^n dx = \frac{1}{a} \frac{(ax+b)^{n+1}}{n+1} + C$

$= -\frac{(2x+3)^{-3}}{6}$

$= \frac{-1}{6(2x+3)^3} + C$

4. $\int \cos 2x \, dx$ $\quad \therefore \int \cos ax \, dx$

$= \frac{\sin 2x}{2} + c$ $\quad = \frac{\sin ax}{a} + C$

5. $\int \sin 3x \, dx$ $\quad \therefore \int \sin ax \, dx$

$= \frac{-\cos 3x}{3} + C$ $\quad = \frac{-\cos ax}{a} + C$

6. $\int \operatorname{cosec}^2 x \, dx$

$= -\cot x + C$

7. $\int \sec 5x \tan 5x \, dx$

$= \frac{\sec 5x}{5} + C$ $\quad \therefore \int \sec ax \tan ax \, dx = \frac{\sec ax}{a} + C$

8. $\int e^{ax} + b \, dx$

$= \frac{e^{ax+b}}{a} + C$

9. $\int 3^{\lambda x} \, dx$ $\quad \therefore \int a^{bx} dx = \frac{a^{bx}}{b \ln a} + C$

$= \frac{3^{\lambda x}}{\lambda \ln 3} + C$

10. $\int \frac{1}{(ax+b)} \, dx$

$= \int (ax+b)^{-1} dx$

$= \frac{1}{a} \ln|ax+b| + C$

Examples: Evaluate

i) $\int (x+1)(x-3) \, dx$

$= \int (x^2 - 3x + x - 3) dx$

$= \int (x^2 - 2x - 3) dx$

$= \int x^2 dx - 2\int x \, dx - 3\int 1 \, dx$

$= \frac{x^3}{3} - 2\frac{x^2}{2} - 3x$

$= \frac{x^3}{3} - x^2 - 3x + C$

ii) $\int x\sqrt{x^2-1}\,dx$

$= \int (x^2-1)^{1/2} \cdot x\,dx$

$= \frac{1}{2}\int (x^2-1)^{1/2} \cdot 2x\,dx$

$= \frac{1}{2}\frac{(x^2-1)^{1/2+1}}{1/2+1}$ $\because \int [f(x)]^n f'(x)dx = \frac{f(x)^{n+1}}{n+1}+c$

$= \frac{1}{2}\frac{(x^2-1)^{3/2}}{3/2}$

$= \frac{1}{2}\cdot\frac{2}{3}(x^2-1)^{3/2}$

$= \frac{1}{3}(x^2-1)^{3/2}+C$

iii) $\int \frac{x}{x+2}\,dx$

$= \int \frac{x}{x+2}\,dx$

$= \int 1-\frac{2}{x+2}\,dx$ $\quad \begin{array}{r|l} & 1 \\ x+2 & x \\ & \underline{\pm x \pm 2} \\ & -2 \end{array}$

$= \int 1dx - 2\int\frac{1}{x+2}\,dx$

$x - 2\ln|x+2|+C$

iv) $\int \frac{1}{\sqrt{x}(\sqrt{x}+1)}\,dx$

$= \int \frac{1}{\sqrt{x}(\sqrt{x}+1)}\,dx$

$= \int\left(\frac{1}{\sqrt{x}+1}\cdot\frac{1}{\sqrt{x}}\right)dx$

$= 2\int\left(\frac{1}{\sqrt{x}+1}\cdot\frac{1}{2\sqrt{x}}\right)dx$

$= 2\int\left(\frac{1/2\sqrt{x}}{\sqrt{x}+1}\right)dx$

$= 2\ln|\sqrt{x}+1|+c$

$\because \int\frac{f'(x)}{f(x)}\,dx = \ln f(x)+c$

v) $\int \frac{dx}{\sqrt{x+1}-\sqrt{x}}$

$= \int \frac{1}{\sqrt{x+1}-\sqrt{x}}\,dx$

$= \int\left[\frac{1}{\sqrt{x+1}-\sqrt{x}}\times\frac{\sqrt{x+1}+\sqrt{x}}{\sqrt{x+1}+\sqrt{x}}\right]dx$

$= \int\left[\frac{\sqrt{x+1}+\sqrt{x}}{(\sqrt{x+1})^2-(\sqrt{x})^2}\right]dx$

$= \int\left[\frac{(x+1)^{1/2}+(x)^{1/2}}{x+1-x}\right]dx$

$= \int\left[(x+1)^{1/2}+(x)^{1/2}\right]dx$

$= \int (x+1)^{1/2}dx + \int x^{1/2}\,dx$

$= \frac{(x+1)^{1/2+1}}{1/2+1} + \frac{x^{1/2+1}}{1/2+1}$

$= \frac{(x+1)^{3/2}}{3/2} + \frac{x^{3/2}}{3/2}$

$= \frac{2}{3}(x+1)^{3/2} + \frac{2}{3}(x)^{3/2}+C$

vi) $\int \frac{\sin x+\cos^3 x}{\cos^2 x\,\sin x}\,dx$

$= \int \frac{\sin x}{\cos^2 x\sin x}\,dx + \int\frac{\cos^3 x}{\cos^2 x\sin x}\,dx$

$= \int \sec^2 x\,dx + \int \cot x\,dx$

$= \tan x + \ln|\sin x| + C$

(vii) $\int \frac{3-\cos 2x}{1+\cos 2x}\,dx$ $\quad \begin{array}{r|l} & -1 \\ 1+\cos 2x & 3-\cancel{\cos 2x} \\ & \underline{\mp 1 \mp \cancel{\cos 2x}} \\ & 4 \end{array}$

$= \int\left(-1+\frac{4}{1+\cos 2x}\right)dx$

$= \int -1\,dx + 4\int\frac{1}{1+\cos 2x}\,dx$

$= -x + 4\int\frac{1}{2\cos^2 x}\,dx$ $\because 1+\cos 2x = 2\cos^2 x$

$= -x + \frac{4}{2}\int \sec^2 x\,dx$ $\because \int\sec^2 x\,dx = \tan x + c$

$= -x + 2\tan x + C$

Exercise # 3.2

Question # 1

Evaluate the following indefinite integral.

(i)

$$\int (3x^2 - 2x + 1)\,dx$$

Solution:

$$\int (3x^2 - 2x + 1)\,dx$$

$$= \int 3x^2 dx - \int 2x\,dx + \int 1\,dx$$

$$= 3\int x^2 dx - 2\int x\,dx + \int 1\,dx$$

$$\because \int x^n dx = \frac{x^{n+1}}{n+1} + C$$

$$= 3\,\frac{x^{2+1}}{2+1} - 2\cdot\frac{x^{1+1}}{1+1} + x + C$$

$$= \cancel{3}\cdot\frac{x^2}{\cancel{3}} - \cancel{2}\cdot\frac{x^2}{\cancel{2}} + x + C$$

$$= \boxed{x^2 - x^2 + x + C}$$

* * *

(ii)

$$\int \left[\sqrt{x} + \frac{1}{\sqrt{x}}\right] dx, \; (x > 0)$$

Solution:

$$= \int \left[\sqrt{x} + \frac{1}{\sqrt{x}}\right] dx$$

$$= \int \left[x^{1/2} + \frac{1}{(x)^{1/2}}\right] dx$$

$$= \int (x^{1/2} + x^{-1/2})\,dx$$

$$= \int x^{1/2} dx + \int x^{-1/2} dx$$

$$\because \int x^n dx = \frac{x^{n+1}}{n+1} + C$$

$$= \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + \frac{x^{-\frac{1}{2}+1}}{\frac{-1}{2}+1} + C$$

$$= \frac{x^{\frac{1+2}{2}}}{\frac{1+2}{2}} + \frac{x^{\frac{-1+2}{2}}}{\frac{-1+2}{2}} + C$$

$$= \frac{x^{3/2}}{\frac{3}{2}} + \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + C$$

$$= \frac{2x^{3/2}}{3} + 2x^{1/2} + C$$

$$= \boxed{\frac{2}{3}x^{3/2} + 2x^{1/2} + C}$$

* * *

(iii)

$$\int x(\sqrt{x} + 1)\,dx, \; (x > 0)$$

Solution:

$$= \int x(\sqrt{x} + 1)\,dx$$

$$= \int x\,(x^{1/2} + 1)\,dx$$

$$= \int (x^{3/2} + x)\,dx$$

$$= \int x^{3/2} dx + \int x \, dx$$

$$\because \int x^n dx = \frac{x^{n+1}}{n+1} + c$$

$$= \frac{x^{3/2+1}}{\frac{3}{2}+1} + \frac{x^{1+1}}{1+1} + c$$

$$= \frac{x^{\frac{3+2}{2}}}{\frac{3+2}{2}} + \frac{x^2}{2} + c$$

$$= \frac{x^{5/2}}{5/2} + \frac{x^2}{2} + c$$

$$= \boxed{\frac{2}{5} x^{5/2} + \frac{1}{2} x^2 + c}$$

* * *

(iv)

$$\int (2x+3)^{1/2} dx$$

Solution:

$$= \int (2x+3)^{1/2} dx$$

Multiply and divided by "2"

$$= \frac{1}{2} \int (2x+3)^{1/2} \cdot 2 \, dx$$

$$\because \int [f(x)]^n \cdot f'(x) = \frac{[f(x)]^{n+1}}{n+1} + c$$

$$= \frac{1}{2} \cdot \frac{(2x+3)^{1/2+1}}{\frac{1}{2}+1} + c$$

$$= \frac{1}{2} \cdot \frac{(2x+3)^{\frac{1+2}{2}}}{\frac{1+2}{2}} + c$$

$$= \frac{1}{2} \frac{(2x+3)^{3/2}}{3/2} + c$$

$$= \frac{1}{\cancel{2}} \cdot \frac{\cancel{2}}{3} (2x+3)^{3/2} + c$$

$$= \boxed{\frac{1}{3} (2x+3)^{3/2} + c}$$

* * *

(v)

$$\int (\sqrt{x} + 1)^2 dx \, , \, (x > 0)$$

Solution:

$$= \int (\sqrt{x} + 1)^2 dx$$

$$\because (a+b)^2 = a^2 + b^2 + 2ab$$

$$= \int \left[(\sqrt{x})^2 + (1)^2 + 2(\sqrt{x})(1) \right] dx$$

$$= \int (x + 1 + 2\sqrt{x}) \, dx$$

$$= \int x dx + \int 1 dx + \int 2 \cdot x^{1/2} dx$$

$$= \int x \, dx + \int 1 \, dx \cdot 2 \int x^{1/2} dx$$

$$\because \int x^n dx = \frac{x^{n+1}}{n+1} + c$$

$$= \frac{x^{1+1}}{1+1} + x + 2 \cdot \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + c$$

$$= \frac{x^2}{2} + x + 2 \frac{x^{3/2}}{\frac{3}{2}} + c$$

$$= \frac{x^2}{2} + x + 2 \cdot \frac{2}{3} x^{3/2} + c$$

$$= \boxed{\frac{1}{2} x^2 + x + \frac{4}{3} x^{3/2} + c}$$

* * *

(vi)

$$\int\left[\sqrt{x}-\frac{1}{\sqrt{x}}\right]^2 dx\ ,\ (x>0)$$

Solution:

$$\int\left[\sqrt{x}-\frac{1}{\sqrt{x}}\right]^2 dx$$

$\because (a-b)^2=a^2+b^2-2ab$

$$=\int (\sqrt{x})^2+\left(\frac{1}{\sqrt{x}}\right)^2-2(\sqrt{x})\left(\frac{1}{\sqrt{x}}\right)\Big]\,dx$$

$$=\int\left(x+\frac{1}{x}-2\right)dx$$

$$=\int\left(x+\frac{1}{x}'-2\right)dx$$

$$=\int x\,dx+\int\frac{1}{x}'dx-2\int 1\,dx$$

$\because \int x^n\,dx=\frac{x^{n+1}}{n+1}+C$

$$=\frac{x^{1+1}}{1+1}+\ln x-2x+C$$

$$=\boxed{\frac{x^2}{2}+\ln x-2x+C}$$

* * *

(vii)

$$\int\frac{3x+2}{\sqrt{x}}\,dx\ ,\ (x>0)$$

Solution:

$$=\int\frac{3x+2}{\sqrt{x}}\,dx$$

$$=\int\left[\frac{3x}{\sqrt{x}}+\frac{2}{\sqrt{x}}\right]dx$$

$$=\int\left(\frac{3\cdot\sqrt{x}\cdot\cancel{\sqrt{x}}}{\cancel{\sqrt{x}}}+\frac{2}{\sqrt{x}}\right)dx$$

$$=\int\left(3\sqrt{x}+\frac{2}{\sqrt{x}}\right)dx$$

$$=\int 3x^{1/2}dx+\int 2x^{-1/2}dx$$

$$=3\int x^{1/2}dx+2\int x^{-\frac{1}{2}}dx$$

$\because \int (x)^n\,dx=\frac{x^{n+1}}{n+1}+C$

$$=3\cdot\frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1}+2\cdot\frac{x^{-\frac{1}{2}+1}}{\frac{-1}{2}+1}+C$$

$$=3\cdot\frac{x^{3/2}}{\frac{3}{2}}+2\frac{x^{1/2}}{\frac{1}{2}}+C$$

$$=\cancel{3}\cdot\frac{2}{\cancel{3}}x^{3/2}+2\cdot\frac{2}{1}x^{1/2}+C$$

$$=\boxed{2x^{3/2}+4x^{1/2}+C}$$

* * *

(viii)

$$\int\frac{\sqrt{y}\,(y+1)}{y}\,dy\ ,\ (y>0)$$

Solution:

$$=\int\left(\frac{y^{1/2}\,(y+1)}{y}\right)dy$$

$$=\int\left[\frac{y^{3/2}+y^{1/2}}{y}\right]dy$$

$$=\int\left[\frac{y^{3/2}}{y}+\frac{y^{1/2}}{y}\right]dy$$

$= \int \left(y^{\frac{3}{2}-1} + y^{\frac{1}{2}-1}\right) dy$

$= \int \left(y^{1/2} + y^{\frac{-1}{2}}\right) dy$

$= \int y^{1/2} dy + y^{\frac{-1}{2}} dy$

$\because \int x^n dx = \frac{x^{n+1}}{n+1} + C$

$= \frac{y^{\frac{1}{2}+1}}{\frac{1}{2}+1} + \frac{y^{\frac{-1}{2}+1}}{\frac{-1}{2}+1} + C$

$= \frac{y^{3/2}}{\frac{3}{2}} + \frac{y^{1/2}}{\frac{1}{2}} + C$

$= \boxed{\frac{2}{3} y^{3/2} + 2 y^{1/2} + C}$

* * *

(ix)

$$\int \frac{(\sqrt{\theta} - 1)^2}{\sqrt{\theta}} d\theta \quad (\theta > 0)$$

Solution:

$= \int \frac{(\sqrt{\theta} - 1)^2}{\sqrt{\theta}} d\theta$

$\because (a-b)^2 = a^2 + b^2 - 2ab$

$= \int \left[\frac{(\sqrt{\theta})^2 + (1)^2 - 2(\sqrt{\theta})(1)}{\sqrt{\theta}}\right] d\theta$

$= \int \left(\frac{\theta + 1 - 2\sqrt{\theta}}{\sqrt{\theta}}\right) d\theta$

$= \int \left(\frac{\theta}{\sqrt{\theta}} + \frac{1}{\sqrt{\theta}} - 2\frac{\sqrt{\theta}}{\sqrt{\theta}}\right) d\theta$

$= \int \left[\frac{\sqrt{\theta} \cdot \sqrt{\theta}}{\sqrt{\theta}} + \frac{1}{\theta^{1/2}} - 2\right] d\theta$

$= \int \left[\theta^{1/2} + \theta^{-1/2} - 2\right) d\theta$

$= \int \theta^{1/2} d\theta + \int \theta^{\frac{-1}{2}} d\theta - 2\int 1 \, d\theta$

$\because \int x^n dx = \frac{x^{n+1}}{n+1} + C$

$= \frac{\theta^{\frac{1}{2}+1}}{\frac{1}{2}+1} + \frac{\theta^{\frac{-1}{2}+1}}{\frac{-1}{2}+1} - 2\theta + C$

$= \frac{\theta^{\frac{3}{2}}}{\frac{3}{2}} + \frac{\theta^{\frac{1}{2}}}{\frac{1}{2}} - 2\theta + C$

$= \boxed{\frac{2}{3}\theta^{\frac{3}{2}} + 2\theta^{\frac{1}{2}} - 2\theta + C}$

* * *

(x)

$$\int \frac{(1 - \sqrt{x})^2}{\sqrt{x}} dx, \quad (x > 0)$$

Solution:

$= \int \frac{(1 - \sqrt{x})^2}{\sqrt{x}} dx$

$\because (a-b)^2 = a^2 + b^2 - 2ab$

$= \int \left[\frac{(1)^2 + (\sqrt{x})^2 - 2(\sqrt{x})(1)}{\sqrt{x}}\right] dx$

$= \int \left(\frac{1 + x - 2\sqrt{x}}{\sqrt{x}}\right) dx$

$= \int \left[\frac{1}{\sqrt{x}} + \frac{x}{\sqrt{x}} - \frac{2\sqrt{x}}{\sqrt{x}}\right] dx$

$= \int \left[\frac{1}{x^{1/2}} + \frac{\sqrt{x} \cdot \sqrt{x}}{\sqrt{x}} - 2\right] dx$

$$= \int (x^{-1/2} + x^{1/2} - 2)\, dx$$

$$= \int x^{-\frac{1}{2}} dx + \int x^{\frac{1}{2}} dx - 2\int 1\, dx$$

$$\because \int x^n dx = \frac{x^{n+1}}{n+1} + C$$

$$= \frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} - 2x + C$$

$$= \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + \frac{x^{3/2}}{\frac{3}{2}} - 2x + C$$

$$= \boxed{2x^{\frac{1}{2}} + \frac{2}{3}x^{3/2} - 2x + C}$$

* * *

(xi)

$$\int \frac{e^{2x} + e^x}{e^x}\, dx$$

Solution:

$$= \int \frac{e^{2x} + e^x}{e^x}\, dx$$

$$= \int \left(\frac{e^{2x}}{e^x} + \frac{e^x}{e^x}\right) dx$$

$$= \int (e^x + 1)\, dx$$

$$= \int e^x dx + \int 1\, dx$$

$$= \boxed{e^x + x + c}$$

* * *

Question #2

Evaluate:

(i)

$$\int \frac{dx}{\sqrt{x+a} + \sqrt{x+b}}, \quad \begin{bmatrix} x+a > 0 \\ x+b > 0 \end{bmatrix}$$

Solution:

$$= \int \frac{dx}{\sqrt{x+a} + \sqrt{x+b}}$$

$$= \int \frac{1}{\sqrt{x+a} + \sqrt{x+b}} \times \frac{\sqrt{x+a} - \sqrt{x+b}}{\sqrt{x+a} - \sqrt{x+b}}\, dx$$

$$\because (a+b)(a-b) = a^2 - b^2$$

$$= \int \left[\frac{\sqrt{x+a} - \sqrt{x+b}}{(\sqrt{x+a})^2 - (\sqrt{x+b})^2}\right] dx$$

$$= \int \left[\frac{\sqrt{x+a} - \sqrt{x+b}}{\cancel{x}+a-\cancel{x}-b}\right] dx$$

$$= \int \left(\frac{\sqrt{x+a} - \sqrt{x+b}}{a-b}\right) dx$$

$$= \frac{1}{a-b} \int \left((x+a)^{1/2} - (x+b)^{1/2}\right) dx$$

$$= \frac{1}{a-b} \int (x+a)^{1/2} dx - \int (x+b)^{1/2} dx$$

$$\because \int x^n dx = \frac{x^{n+1}}{n+1} + C$$

$$= \frac{1}{a-b}\left[\frac{(x+a)^{\frac{1}{2}+1}}{\frac{1}{2}+1} - \frac{(x+b)^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right] + C$$

$$= \frac{1}{a-b}\left\{\frac{(x+a)^{\frac{3}{2}}}{\frac{3}{2}} - \frac{(x+b)^{\frac{3}{2}}}{\frac{3}{2}}\right\} + C$$

$$= \boxed{\frac{2}{3(a-b)}\left\{(x+a)^{3/2} - (x+b)^{3/2}\right\} + C}$$

* * *

(ii)

$$\int \frac{1-x^2}{1+x^2}\,dx$$

Solution:

$$\int \frac{1-x^2}{1+x^2}\,dx$$

$$\begin{array}{r|l} & -1 \\ \hline 1+x^2 & 1-x^2 \\ & \underline{\underset{+}{-}1 \mp x^2} \\ & 2 \end{array}$$

$$= \int \left(-1 + \frac{2}{1+x^2}\right) dx$$

$$= \int -1\,dx + 2\int \frac{1}{1+x^2}\,dx$$

$$\because \int \frac{1}{1+x^2} = \tan^{-1}x$$

$$= -x + 2\tan^{-1}x + C$$

$$= \boxed{2\,\text{Tan}^{-1}x - x + C}$$

(iii)

$$\int \frac{dx}{\sqrt{x+a}+\sqrt{x}}, \quad (x>0, a>0)$$

Solution:

$$= \int \frac{dx}{\sqrt{x+a}+\sqrt{x}}$$

$$= \int \left(\frac{1}{\sqrt{x+a}+\sqrt{x}} \times \frac{\sqrt{x+a}-\sqrt{x}}{\sqrt{x+a}-\sqrt{x}}\right) dx$$

$$\because (a+b)(a-b) = a^2 - b^2$$

$$= \int \left(\frac{\sqrt{x+a}-\sqrt{x}}{(\sqrt{x+a})^2 - (\sqrt{x})^2}\right) dx$$

$$= \int \left(\frac{\sqrt{x+a}-\sqrt{x}}{x+a-\cancel{x}}\right) dx$$

$$= \int \left(\frac{\sqrt{x+a}-\sqrt{x}}{a}\right) dx$$

$$= \frac{1}{a}\int \left((x+a)^{1/2} - x^{1/2}\right) dx$$

$$= \frac{1}{a}\left[\int (x+a)^{1/2}dx - \int x^{1/2}dx\right]$$

$$\because \int x^n dx = \frac{x^{n+1}}{n+1} + C$$

$$= \frac{1}{a}\left\{\frac{(x+a)^{1/2+1}}{\frac{1}{2}+1} - \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right\} + C$$

$$= \frac{1}{a}\left\{\frac{(x+a)^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right\} + C$$

$$= \frac{1}{a}\cdot\frac{2}{3}\left\{(x+a)^{3/2} - x^{3/2}\right\} + C$$

$$= \boxed{\frac{2}{3a}\left\{(x+a)^{3/2} - x^{3/2}\right\} + C}$$

(iv)

$$\int (a-2x)^{3/2}\,dx$$

Solution:

$$\int (a-2x)^{3/2}\,dx$$

Multiply and divided by '-2'

$$= -\frac{1}{2}\int (a-2x)^{3/2}\cdot -2\,dx$$

$$\because \int [f(x)]^n \cdot f'(x) = \frac{[f(x)]^{n+1}}{n+1} + C$$

$$= -\frac{1}{2}\cdot\frac{(a-2x)^{\frac{3}{2}+1}}{\frac{3}{2}+1} + C$$

$$= -\frac{1}{2}\cdot\frac{(a-2x)^{5/2}}{\frac{5}{2}} + C$$

$= -\frac{1}{\cancel{2}} \cdot \frac{\cancel{2}}{5} \; (a-2x)^{5/2} + C$

$= \boxed{-\frac{1}{5} \; (a-2x)^{5/2} + C}$

(V)

$$\int \frac{(1+e^x)^3}{e^x}\, dx$$

Solution:

$\int \frac{(1+e^x)^3}{e^x}\, dx$

$\because (a+b)^3 = a^3+b^3+3a^2b+3ab^2$

$= \int \frac{(1)^3+(e^x)^3+3(1)^2(e^x)+3(1)(e^x)^2}{e^x}\, dx$

$= \int \left(\frac{1+e^{3x}+3e^x+3e^{2x}}{e^x}\right) dx$

$= \int \left(\frac{1}{e^x} + \frac{e^{3x}}{e^x} + \frac{3e^x}{e^x} + \frac{3e^{2x}}{e^x}\right) dx$

$= \int \left(e^{-x} + e^{2x} + 3 + 3e^x\right) dx$

$= \int e^{-x} dx + \int e^{2x} dx + 3\int 1\, dx + \int 3e^x\, dx$

$= \frac{e^{-x}}{-1} + \frac{e^{2x}}{2} + 3x + 3e^x + C$

$= \boxed{-e^{-x} + \frac{e^{2x}}{2} + 3x + 3e^x + C}$

(Vi)

$$\int \sin(a+b)x\, dx$$

Solution:

$\int \sin(a+b)x\, dx$

$\because \int \sin x\, dx = -\cos x + C$

$= \frac{-\cos(a+b)\,x}{(a+b)\cdot 1} + C$

$= \boxed{\frac{-1}{a+b} \cdot \cos(a+b) + C}$

(Vii)

$$\int \sqrt{1-\cos 2x}\, dx$$

Solution:

$= \int \sqrt{1-\cos 2x}\;\; dx$

$\because \cos 2x = 1-2\sin^2 x$

$\because 2\sin^2 x = 1-\cos 2x$

$= \int \sqrt{2\sin^2 x}\;\; dx$

$= \int \sqrt{2} \cdot \sqrt{\sin^2 x}\; dx$

$= \sqrt{2} \int \sin x\, dx$

$= \sqrt{2}\left(\frac{-\cos x}{1}\right) + c$

$= \boxed{-\sqrt{2}\cos x + C}$

(viii)

$$\int \ln x \cdot \frac{1}{x}\, dx$$

Solution:

$= \int \ln x \cdot \frac{1}{x}\, dx$

$\because \int [f(x)]^n \cdot f'(x) = \frac{[f(x)]^{n+1}}{n+1} + C$

$= \frac{(\ln x)^{1+1}}{1+1} + C$

$= \boxed{\frac{(\ln x)^2}{2} + C}$

(ix)

$$\int \sin^2 x\, dx$$

Solution:

$\int \sin^2 x \,.\, dx$

$\because \cos 2x = 1 - 2\sin^2 x$

$\because 2\sin^2 x = 1 - \cos 2x$

$\because \sin^2 x = \frac{1 - \cos 2x}{2}$

$= \int \left(\frac{1 - \cos 2x}{2}\right) dx$

$= \frac{1}{2} \int (1 - \cos 2x)\, dx$

$= \frac{1}{2} \int 1\, dx - \int \cos 2x\, dx$

$\because \int \cos x\, dx = \sin x + C$

$= \frac{1}{2}\left[x - \frac{\sin 2x}{2}\right] + C$

$= \frac{1}{2}\left[x - \frac{\sin 2x}{2}\right] + C$

$= \boxed{\frac{x}{2} - \frac{\sin 2x}{4} + C}$

(x)

$$\int \frac{1}{1+\cos x}\, dx, \left[\frac{-\pi}{2} < x \frac{\pi}{2}\right]$$

Solution:

$= \int \frac{1}{1+\cos x}$

$\because \cos 2x = 2\cos^2 x - 1$

$\because \cos x = 2\cos^2 \frac{x}{2} - 1$

$\because 2\cos^2 \frac{x}{2} = 1 + \cos x$

$= \int \frac{1}{2\cos^2 \frac{x}{2}}\, dx$

$= \frac{1}{2} \int \frac{1}{\cos^2 \frac{x}{2}}\, dx$

$= \frac{1}{2} \int \sec^2 \frac{x}{2}\, dx$

$\because \int \sec^2 x\, dx = \tan x + C$

$= \frac{1}{2}\left(\frac{\tan \frac{x}{2}}{\frac{1}{2}}\right) + C$

$= \frac{1}{\cancel{2}} \cdot \frac{\cancel{2}}{1} \tan \frac{x}{2} + C$

$= \boxed{\tan \frac{x}{2} + C}$

(xi)

$$\int \frac{ax+b}{ax^2+2bx+C}\,dx$$

Solution:

$$\int \frac{ax+b}{ax^2+2bx+C}\,dx$$

Multiply and divided by "2"

$$= \frac{1}{2}\int \frac{2(ax+b)}{ax^2+2bx+C}\,dx$$

$$= \frac{1}{2}\int \left(\frac{2ax+2b}{ax^2+2bx+C}\right)dx$$

$$\because \int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)|+C$$

$$= \boxed{\frac{1}{2}\ln(ax^2+2bx+C)+C}$$

* * *

(xii)

$$\int \cos 3x \sin 2x\,dx$$

Solution:

$$= \int \cos 3x \sin 2x\,dx$$

Multiply and divided by "2"

$$= \frac{1}{2}\int (2\cos 3x \sin 2x)\,dx$$

$$\because 2\cos\alpha \sin\beta = \sin(\alpha+\beta) - \sin(\alpha-\beta)$$

$$= \frac{1}{2}\int \left\{\sin(3x+2x) - \sin(3x-2x)\right\}dx$$

$$= \frac{1}{2}\int (\sin 5x - \sin x)\,dx$$

$$= \frac{1}{2}\int \sin 5x\,dx - \int \sin x\,dx$$

$$= \frac{1}{2}\left\{-\frac{\cos 5x}{5\cdot(1)} - \frac{-\cos x}{1}\right\}+C$$

$$= \frac{1}{2}\left\{-\frac{\cos 5x}{5} + \cos x\right\}+C$$

$$= \boxed{-\frac{1}{2}\left\{\frac{\cos 5x}{5} - \cos x\right\}+C}$$

* * *

(xiii)

$$\int \frac{\cos 2x-1}{1+\cos 2x}\,dx$$

Solution:

$$\int \frac{\cos 2x-1}{1+\cos 2x}\,dx$$

$$\begin{array}{r|l} & \quad 1 \\ \hline 1+\cos 2x & \cancel{\cos 2x} - 1 \\ & \underline{\cancel{\cos 2x} \pm 1} \\ & \quad -2 \end{array}$$

$$= \int \left(1 - \frac{2}{1+\cos 2x}\right)dx$$

$$\because \cos 2x = 2\cos^2 x - 1$$

$$\because 2\cos^2 x = 1+\cos 2x$$

$$= \int \left(1 - \frac{\cancel{2}}{\cancel{2}\cos^2 x}\right)dx$$

$$= \int \left(1 - \frac{1}{\cos^2 x}\right)dx$$

$$= \int (1 - \sec^2 x)\,dx$$

$$= \int 1\,dx - \int \sec^2 x\,dx$$

$$\because \int \sec^2 dx = \tan x$$

$$= \boxed{x - \tan x + C}$$

=

* * *

(xiv)

$$\int \tan^2 x\,dx$$

Solution:

$$\int \tan^2 x \ dx$$

$\because 1+\tan^2 x = \sec^2 x$

$\therefore \tan^2 x = \sec^2 x - 1$

$$= \int (\sec^2 x - 1)\, dx$$

$$= \int \sec^2 x \ dx - 1\int 1 \ dx$$

$\because \int \sec^2 x\, dx = \tan x$

$\because \int 1 \ dx = x$

$$= \boxed{\tan x - x + C}$$

• ——————— •

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Theory:

Integration by method of substitution:

Sometimes it is possible to convert an integral into a standard form or to an easy integral by a suitable change of a variable. Now we evaluate $\int f(x)\,dx$ by the method of substitution. Let x be a function of a variable t, that is,

If $x = \phi t$, then $dx = \phi'(t)\,dt$

Putting $x = \phi t$, then $dx = \phi'(t)\,dt$ we have

$$\int f(x)\,dx = \int f(\phi(t))\,\phi' t\,dt.$$

Integration by some useful substitution:

We list below suitable substitutions for certain expressions to be integrated.

Expression Involving	Suitable substitution
i) $\sqrt{a^2-x^2}$	$x = a\sin\theta$
ii) $\sqrt{x^2-a^2}$	$x = a\sec\theta$
iii) $\sqrt{a^2+x^2}$	$x = a\tan\theta$.
iv) $\sqrt{x+a}$ (or $\sqrt{x-a}$)	$\sqrt{x+a} = t$ (or $\sqrt{x-a} = t$)
v) $\sqrt{2ax-x^2}$	$x-a = a\sin\theta$
vi) $\sqrt{2ax+x^2}$	$x+a = a\sec\theta$.

Examples:

Example # 1: Evaluate

$$\int \frac{a\,dt}{2\sqrt{at+b}}, (at+b>0)$$

$$= \int \frac{a\,dt}{2\sqrt{at+b}}$$

Let $u = at+b$

$du = a\,dt$

$$= \int \frac{du}{2\sqrt{u}}$$

$$= \frac{1}{2}\int u^{-1/2}\,du$$

$$= \frac{1}{2}\cdot\frac{u^{-1/2+1}}{-1/2+1}$$

$$= \frac{1}{2}\cdot\frac{u^{1/2}}{1/2}$$

$$= \frac{1}{\cancel{2}}\cdot\cancel{2}\;u^{1/2}$$

$$= \sqrt{u}$$

$$= \sqrt{at+b}+c$$

Example # 2: Evaluate

$$\int \frac{x}{\sqrt{4+x^2}}\,dx$$

$$= \int \frac{x}{\sqrt{4+x^2}}\,dx$$

Let $t = 4+x^2$

$dt = 2x\,dx$

$x\,dx = \frac{1}{2}\,dt$

$$= \int \frac{1/2\;dt}{\sqrt{t}}$$

$$= \frac{1}{2}\int \frac{1}{\sqrt{t}}\,dt$$

$$= \frac{1}{2}\int (t)^{-1/2}\,dt$$

$$= \frac{1}{2}\int (t)^{-1/2}\cdot 1\,dt$$

$$= \frac{1}{2}\cdot\frac{(t)^{-1/2+1}}{-1/2+1}$$

$$= \frac{1}{2}\cdot\frac{(t)^{1/2}}{1/2}$$

$$= \frac{1}{2}\cdot 2\,(t)^{1/2}$$

$$= \sqrt{t}$$

$$= \sqrt{4+x^2}+c$$

Example # 3: Evaluate

$$\int x\sqrt{x-a}\,dx \quad (x>a)$$

$$= \int x\sqrt{x-a}\,dx$$

Let $x-a = t \Rightarrow x = a+t$

$dx = dt$

$$= \int (a+t)\sqrt{t}\,dt$$

$$= \int a\sqrt{t} + t\sqrt{t}\,dt$$

$$= \int \left[a(t)^{1/2} + t^{3/2}\right] dt$$

$$= a\int t^{1/2}\,dt + \int t^{3/2}\,dt$$

$$= a\cdot\frac{t^{1/2+1}}{1/2+1} + \frac{t^{3/2+1}}{3/2+1}$$

$$= a\cdot\frac{t^{3/2}}{3/2} + \frac{t^{5/2}}{5/2}$$

$$= \frac{2a}{3}(t)^{3/2} + \frac{2}{5}(t)^{5/2}$$

$$= \frac{2}{3}a(t)^{3/2} + \frac{2}{5}\,t\cdot t^{3/2}$$

$= 2t^{3/2}\left(\frac{a}{3} + \frac{t}{5}\right) + c$

$= 2t^{3/2}\left(\frac{5a+3t}{15}\right)$

$= \frac{2}{15}(x-a)^{3/2}(5a+3(x-a))$

$= \frac{2}{15}(x-a)^{3/2}(5a+3x-3a)$

$= \frac{2}{15}(x-a)^{3/2}(2a+3x) + c$

Example # 4: $\int \frac{\cot\sqrt{x}}{\sqrt{x}}dx$

$= \int \cot\sqrt{x} \cdot \frac{1}{\sqrt{x}}\, dx$

Put $\sqrt{x} = z$

$\frac{1}{2\sqrt{x}}dx = dz$

$\frac{1}{\sqrt{x}}dx = 2dz$

$= \int \cot z \cdot 2dz$

$= 2\int \cot z\, dz$

$= 2\ln|\sin z| + c$

$= 2\ln|\sin\sqrt{x}| + c$

Example # 5: Evaluate

i) $\int \operatorname{cosec} x\, dx$

$= \int \operatorname{cosec} x\, dx$

$= \int \frac{\operatorname{cosec} x(\operatorname{cosec} x - \cot x)}{\operatorname{cosec} x - \cot x}dx$

$= \int \frac{\operatorname{cosec}^2 x - \operatorname{cosec} x \cot x}{\operatorname{cosec} x - \cot x}dx$

Put $t = \operatorname{cosec} x - \cot x$

$dt = -\operatorname{cosec} x \cot x - (-\operatorname{cosec}^2 x)dx$

$dt = -\operatorname{cosec} x \cot x + \operatorname{cosec}^2 x\, dx$

$dt = \operatorname{cosec}^2 x - \operatorname{cosec} x \cot x\, dx$

$= \int \frac{dt}{t}$

$= \ln t$

$= \ln|\operatorname{cosec} x - \cot x| + c$

ii) $\int \sec x\, dx$

$= \int \sec x\, dx$

$= \int \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x}dx$

$= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}dx$

Put $t = \sec x + \tan x$

$dt = \sec x \tan x + \sec^2 x\, dx$

$dt = \sec^2 x + \sec x \tan x\, dx$

$= \int \frac{dt}{t}$

$= \ln t$

$= \ln|\sec x + \tan x| + c$

Example # 6: Evaluate

$\int \cos^3 x \sqrt{\sin x}\, dx$

$= \int \cos^2 x \sqrt{\sin x} \cos x\, dx$

Put $\sin x = t$

$\cos x\, dx = dt$

$= \int (1 - \sin^2 x)\sqrt{\sin x} \cos x\, dx$

$= \int (1 - t^2)\sqrt{t}\, dt$

$= \int (1-t^2)\, t^{1/2}\, dt.$

$= \int t^{1/2} - t^{5/2}\, dt.$

$= \int t^{1/2} dt - \int t^{5/2} dt.$

$= \frac{t^{1/2+1}}{1/2+1} - \frac{t^{5/2+1}}{5/2+1}$

$= \frac{t^{3/2}}{3/2} - \frac{t^{7/2}}{7/2}$

$= \frac{2}{3} t^{3/2} - \frac{7}{2} t^{7/2}$

$= \frac{2}{3} \sin^{3/2} x - \frac{7}{2} \sin^{7/2} x + c.$

Example # 7: Evalute

$$\int \sqrt{1+\sin x}\, dx$$

$= \int \sqrt{1+\sin x}\; dx$

$= \int \sqrt{1+\sin x} \times \frac{\sqrt{1-\sin x}}{\sqrt{1-\sin x}}\, dx$

$= \int \frac{\sqrt{(1)^2-(\sin)^2}}{\sqrt{1-\sin x}}\, dx$

$= \int \frac{\sqrt{\cos^2 x}}{\sqrt{1-\sin x}}\, dx$

$= \int \frac{\cos x}{\sqrt{1-\sin x}}\, dx.$

Put $t = 1 - \sin x$

$dt = -\cos x\, dx$

$-dt = \cos x\, dx$

$= \int \frac{-dt}{\sqrt{t}}$

$= -\int (t)^{-1/2}\, dt.$

$= -\int (t)^{-1/2} \cdot 1\, dt.$

$= -\left[\frac{(t)^{-1/2+1}}{-1/2+1}\right]$

$= -\left[\frac{t^{1/2}}{1/2}\right]$

$= -2\sqrt{t}$

$= -2\sqrt{1-\sin x} + c.$

Example # 8: Find $\int \frac{dx}{x(\ln 2x)^3}.$

$= \int \frac{dx}{x(\ln 2x)^3}$

$= \int (\ln 2x)^{-3} \cdot \frac{1}{x}\, dx$

Put $t = \ln 2x$

$dt = \frac{1(2)dx}{2x}$

$dt = \frac{1}{x}\, dx.$

$= \int (t)^{-3} \cdot\, dt$

$= \frac{t^{-3+1}}{-3+1}$

$= \frac{t^{-2}}{-2}$

$= \frac{-1}{2t^2}.$

$= \frac{-1}{2(\ln 2x)^2} + c$

Example # 9: Find

$$\int a^{x^2} x\, dx$$

Put $x^2 = t$

$2x\, dx = dt.$

$x\, dx = \frac{1}{2} dt.$

$= \int a^t \frac{1}{2}\, dt$

$$= \frac{1}{2}\int a^t \, dt$$

$$= \frac{1}{2} \frac{a^t}{\ln a}$$

$$= \frac{1}{2} \cdot \frac{a^{x^2}}{\ln a} + c$$

Example # 10: Evaluate

$$i) \int \frac{1}{\sqrt{a^2 - x^2}} dx, \; (-a < x < a)$$

Put $x = a\sin\theta$

$dx = a\cos\theta \, d\theta$

$$= \int \frac{1}{\sqrt{a^2 - (a\sin\theta)^2}} a\cos\theta \, d\theta$$

$$= \int \frac{1}{\sqrt{a^2 - a^2\sin^2\theta}} \cdot a\cos\theta \, d\theta$$

$$= \int \frac{1}{\sqrt{a^2(1 - \sin^2\theta)}} \cdot a\cos\theta \, d\theta$$

$$= \int \frac{1}{\sqrt{a^2\cos^2\theta}} a\cos\theta \, d\theta$$

$$= \int \frac{1}{\cancel{a\cos\theta}} \cancel{a\cos\theta} \, d\theta$$

$$= \int 1 \, d\theta$$

$$= \theta + c$$

$$= \sin^{-1}\frac{x}{a} + c$$

$\therefore x = a\sin\theta$

$\sin\theta = \frac{x}{a}$

$\theta = \sin^{-1}\frac{x}{a}$

$$ii) \int \frac{1}{x\sqrt{x^2 - a^2}} dx$$

$$= \int \frac{1}{x\sqrt{x^2 - a^2}} dx$$

Put $x = a\sec\theta$

$dx = a\sec\theta\tan\theta \, d\theta$

$$= \int \frac{\cancel{a\sec\theta}\tan\theta \, d\theta}{\cancel{a\sec\theta}\sqrt{(a\sec\theta)^2 - a^2}}$$

$$= \int \frac{\tan\theta}{\sqrt{a^2\sec^2\theta - a^2}} d\theta$$

$$= \int \frac{\tan\theta}{\sqrt{a^2(\sec^2\theta - 1)}} d\theta$$

$$= \int \frac{\tan\theta}{\sqrt{a^2\tan^2\theta}} d\theta$$

$$= \int \frac{\cancel{\tan\theta}}{a\cancel{\tan\theta}} d\theta$$

$$= \frac{1}{a}\int 1 \, d\theta$$

$$= \frac{1}{a}\theta$$

$$= \frac{1}{a}\sec^{-1}\frac{x}{x} + c$$

$\therefore x = a\sec\theta$

$\sec\theta = \frac{x}{a}$

$\theta = \sec^{-1}\frac{x}{a}$

Example # 1: Evaluate

$$\int \frac{1}{\sqrt{a^2 + x^2}} dx$$

$$= \int \frac{1}{\sqrt{a^2 + x^2}} dx$$

Put $x = a\tan\theta$

$dx = a\sec^2\theta \, d\theta$

$$= \int \frac{a\sec^2\theta \, d\theta}{\sqrt{a^2 + (a\tan\theta)^2}}$$

$$= \int \frac{a\sec^2\theta \, d\theta}{\sqrt{a^2 + a^2\tan^2\theta}}$$

$$= \int \frac{a\sec^2\theta}{\sqrt{a^2(1 + \tan^2\theta)}} d\theta$$

$$= \int \frac{a\sec^2\theta}{\sqrt{a^2\sec^2\theta}} d\theta$$

$$= \int \frac{\cancel{a}\sec^{\cancel{2}}\theta}{\cancel{a}\cancel{\sec\theta}} d\theta$$

$$= \int \sec\theta \, d\theta$$

$$= \int \frac{\sec\theta\,(\sec\theta+\tan\theta)\,d\theta}{\sec\theta+\tan\theta}$$

$$\because \int \frac{f'(x)}{f(x)}\,dx = \ln f(x)+c$$

$$= \ln|\sec\theta+\tan\theta|+c_1$$

Since

$$\sec^2\theta = 1+\tan^2\theta \qquad \because x = a\tan\theta,\quad \tan\theta = \frac{x}{a}$$

$$= 1+\frac{x^2}{a^2}$$

$$\sec^2\theta = \frac{a^2+x^2}{a^2}$$

$$\sec\theta = \frac{\sqrt{a^2+x^2}}{a}$$

So

$$= \ln\left|\frac{\sqrt{a^2+x^2}}{a}+\frac{x}{a}\right|+c_1$$

$$= \ln\left|\frac{\sqrt{a^2+x^2}+x}{a}\right|+c_1$$

$$= \ln|x+\sqrt{a^2+x^2}|-\ln a+c_1$$

$$= \ln|x+\sqrt{a^2+x^2}|+c$$

$$\because c = = c_1-\ln a$$

Example # 2: Evaluate

$$\int \frac{dx}{\sqrt{2x+x^2}}$$

$$= \int \frac{dx}{\sqrt{2x+x^2}}$$

$$= \int \frac{dx}{\sqrt{x^2+2x+1-1}}$$

$$= \int \frac{dx}{\sqrt{(x+1)^2-(1)^2}}$$

Put $x+1 = \sec\theta$

$dx = \sec\theta\tan\theta\,d\theta$

$$= \int \frac{\sec\theta\tan\theta}{\sqrt{(\sec\theta)^2-1}}\,d\theta$$

$$= \int \frac{\sec\theta\tan\theta}{\sqrt{\tan^2\theta}}\,d\theta$$

$$= \int \frac{\sec\theta\,\cancel{\tan\theta}}{\cancel{\tan\theta}}\,d\theta$$

$$= \int \sec\theta\,d\theta$$

$$= \int \frac{\sec\theta\,(\sec\theta+\tan\theta)\,d\theta}{\sec\theta+\tan\theta}$$

$$\because \int \frac{f'(x)}{f(x)}\,dx = \ln f(x)+c$$

$$= \ln|\sec\theta+\tan\theta|+c$$

$$\because \sec\theta = x+1$$

$$\therefore \tan^2\theta = \sec^2\theta-1$$

$$= (x+1)^2-1$$

$$= x^2+2x+1-1$$

$$\tan^2\theta = x^2+2x$$

$$\tan\theta = \sqrt{x^2+2x}$$

$$= \ln|x+1+\sqrt{x^2+2x}|+c$$

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Exercise # 3·3

Question # 1

Evaluate the following integrals:

Evaluate $\int \frac{-2x}{\sqrt{4-x^2}} dx$

Solution:

$$= \int \frac{-2x}{\sqrt{4-x^2}} dx$$

$$= \int \frac{-2x}{(4-x^2)^{1/2}} dx$$

$$= \int (4-x^2)^{-1/2} \cdot -2x \, dx$$

$$\because \int [f(x)]^n \cdot f'(x) = \frac{[f(x)]^{n+1}}{n+1} + C$$

$$= \frac{(4-x^2)^{\frac{-1}{2}+1}}{\frac{-1}{2}+1} + C$$

$$= \frac{(4-x^2)^{\frac{1}{2}}}{\frac{1}{2}} + C$$

$$= \frac{2}{1}(4-x^2)^{1/2} + C$$

$$= \boxed{2\sqrt{4-x^2} + C}$$

Question # 2

Evaluate $\int \frac{dx}{x^2+4x+13}$

Solution:

$$= \int \frac{dx}{x^2+4x+13}$$

$$= \int \frac{dx}{x^2+4x+4+9}$$

$$= \int \frac{dx}{(x+2)^2+(3)^3}$$

put

$x+2 = 3\tan\theta$

$dx = 3\sec^2\theta \, d\theta$

$$= \int \frac{3\sec^2\theta \, d\theta}{(3\tan\theta)^2+(3)^2}$$

$$= \int \frac{3\sec^2\theta \, d\theta}{9\tan^2\theta+9}$$

$$= \int \frac{3\sec^2\theta \, d\theta}{9(\tan^2\theta+1)}$$

$\because 1+\tan^2\theta = \sec^2\theta$

$$= \frac{\cancel{3}}{\cancel{9}_3} \int \frac{\cancel{\sec^2\theta}}{\cancel{\sec^2\theta}} d\theta$$

$$= \frac{1}{3} \int 1 \, d\theta$$

$$= \frac{1}{3} \cdot \theta \, d\theta$$

$\because x+2 = 3\tan\theta$

$\therefore \theta = \tan^{-1}\left(\frac{x+2}{3}\right)$

$$= \boxed{\frac{1}{3} \tan^{-1}\left(\frac{x+2}{3}\right) + C}$$

Question #3

Evaluate $\int \frac{x^2}{4+x^2}\,dx$

Solution:

$= \int \frac{x^2}{4+x^2}\,dx$

$$\begin{array}{r} 1 \\ 4+x^2 \overline{)\,\cancel{x^2}\quad} \\ \underline{\cancel{+x^2}+4} \\ -4 \end{array}$$

$= \int \left(1 - \frac{4}{4+x^2}\right) dx$

$= \int 1\,dx - 4\int \frac{1}{4+x^2}\,dx$

$= x - 4\int \frac{1}{(2)^2+(x)^2}\,dx$

$= x - 4\ I$ —— ①

Where $I = \int \frac{1}{(2)^2+x^2}\,dx$

put

$x = 2\tan\theta$

$dx = 2\sec^2\theta\,d\theta$

$I = \int \frac{2\sec^2\theta\,d\theta}{(2)^2+(2\tan\theta)^2}$

$I = \int \frac{2\sec^2\theta\,d\theta}{4+4\tan^2\theta}$

$I = \int \frac{2\sec^2\theta\,d\theta}{4(1+\tan^2\theta)}$

$\because 1+\tan^2\theta = \sec^2\theta$

$I = \frac{\cancel{2}}{\cancel{4}_2}\int \frac{\cancel{\sec^2\theta}}{\cancel{\sec^2\theta}}\,d\theta$

$= \frac{1}{2}\int 1\,d\theta$

$I = \frac{1}{2}\theta$

$\because \tan\theta = \frac{x}{2}$

$\because \theta = \tan^{-1}\frac{x}{2}$

$I = \frac{1}{2}\tan^{-1}\frac{x}{2} + c$

Equation ① become as

$= x - \overset{2}{\cancel{4}}\left(\frac{1}{\cancel{2}}\tan^{-1}\frac{x}{2}\right) + c$

$= \boxed{x - 2\tan^{-1}\frac{x}{2} + c}$

Question #4

Evaluate $\int \frac{1}{x\ln x}\,dx$

Solution:

$= \int \frac{1}{x\ln x}\,dx$

$= \int \frac{x^{-1}}{\ln x}\,dx$

$= \int \frac{\frac{1}{x}}{\ln x}\,dx$

$\because \int \frac{f'(x)}{f(x)} = \ln|f(x)| + c$

$= \boxed{\ln(\ln x) + c}$

Question #5

Evaluate $\int \frac{e^x}{e^x+3}\,dx$

Solution:

$= \int \frac{e^x}{e^x+3}\,dx$

$\because \int \frac{f'(x)}{f(x)} = \ln|f(x)| + c$

$= \boxed{\ln(e^x+3) + c}$

Question # 6

Evaluate $\int \frac{x+b}{(x^2+2bx+c)^{1/2}} dx$

Solution:

$$= \int \frac{x+b}{(x^2+2bx+c)^{1/2}} dx$$

$$= \int (x^2+2bx+c)^{-1/2} \cdot (x+b)\, dx$$

Multiply and divided by '2'

$$= \frac{1}{2}\int (x^2+2bx+c)^{-1/2} \cdot 2(x+b)\, dx$$

$$= \frac{1}{2}\int (x^2+2bx+c)^{-1/2} \cdot (2x+2b)\, dx$$

$$\because \int [f(x)]^n \cdot f'(x) = \frac{[f(x)]^{n+1}}{n+1} + c$$

$$= \frac{1}{2} \cdot \frac{(x^2+2bx+c)^{-1/2+1}}{-\frac{1}{2}+1} + C$$

$$= \frac{1}{2} \cdot \frac{(x^2+2bx+c)^{1/2}}{\frac{1}{2}} + C$$

$$= \frac{1}{\cancel{2}} \cdot \frac{\cancel{2}}{1} (x^2+2bx+c)^{1/2} + C$$

$$= \boxed{\sqrt{x^2+2bx+c} + C}$$

Question # 7

Evaluate $\int \frac{\sec^2 x}{\sqrt{\tan x}} dx$

Solution:

$$= \int \frac{\sec^2 x}{\sqrt{\tan x}} dx$$

$$= \int \frac{\sec^2 x}{(\tan x)^{1/2}} dx$$

$$= \int (\tan x)^{-1/2} \cdot \sec^2 x\, dx$$

$$\because \int [f(x)]^n \cdot f'(x) = \frac{[f(x)]^{n+1}}{n+1} + c$$

$$= \frac{(\tan x)^{-1/2+1}}{-\frac{1}{2}+1} + c$$

$$= \frac{(\tan x)^{1/2}}{\frac{1}{2}} + c$$

$$= \boxed{2\sqrt{\tan x} + c}$$

Question #8

Show that $\int \frac{dx}{\sqrt{x^2-a^2}} = \ln(x+\sqrt{x^2-a^2}) + c$

Solution:

$$\text{L.H.S.} = \int \frac{dx}{\sqrt{x^2-a^2}}$$

put

$x = a\sec\theta$

$dx = a\sec\theta\tan\theta\, d\theta$

$$= \int \frac{a\sec\theta\tan\theta\, d\theta}{\sqrt{(a\sec\theta)^2 - a^2}}$$

$$= \int \frac{a\sec\theta\tan\theta}{\sqrt{a^2\sec^2\theta - a^2}} d\theta$$

$$= \int \frac{a\sec\theta\tan\theta}{\sqrt{a^2(\sec^2\theta - 1)}} d\theta$$

$\because 1+\tan^2\theta = \sec^2\theta$

$\because \tan^2\theta = \sec^2\theta - 1$

$$= \int \frac{a\sec\theta\tan\theta}{\sqrt{a^2\tan^2\theta}} d\theta$$

$= \int \frac{a \sec\theta \tan\theta}{a \tan\theta} d\theta$

$= \int \sec\theta \, d\theta$

$= \ln|\sec\theta + \tan\theta| + C_1$

$\because x = a\sec\theta$

$\because \sec\theta = \frac{x}{a}$

$= \ln\left| \frac{x}{a} + \sqrt{\tan^2\theta} \right| + C_1$

$\because \tan^2\theta = \sec^2\theta - 1$

$= \ln\left| \frac{x}{a} + \sqrt{\sec^2\theta - 1} \right| + C_1$

$= \ln\left| \frac{x}{a} + \sqrt{\left(\frac{x}{a}\right)^2 - 1} \right| + C_1$

$= \ln\left| \frac{x}{a} + \sqrt{\frac{x^2}{a^2} - 1} \right| + C_1$

$= \ln\left| \frac{x}{a} + \sqrt{\frac{x^2 - a^2}{a^2}} \right| + C_1$

$= \ln\left| \frac{x}{a} + \frac{\sqrt{x^2 - a^2}}{a} \right| + C_1$

$= \ln\left| \frac{x + \sqrt{x^2 - a^2}}{a} \right| + C_1$

$\because \ln\frac{m}{n} = \ln m - \ln n$

$= \ln|x + \sqrt{x^2 - a^2}| - \ln a + C_1$

$= \ln|x + \sqrt{x^2 - a^2}| + C_1 - \ln a$

$\because C_1 - \ln a = C$

$= \ln|x + \sqrt{x^2 - a^2}| + C$ = R.H.S.

L.H.S. = R.H.S.

Hence proved.

(b)

Show that $\int \sqrt{a^2 - x^2}\, dx = \frac{a^2}{2}\sin^{-1}\frac{x}{a} + \frac{x}{2}\sqrt{a^2 - x^2} + C$

Solution:

L.H.S. $= \int \sqrt{a^2 - x^2}\, dx$

put

$x = a\sin\theta$

$dx = a\cos\theta\, d\theta$

$= \int \sqrt{a^2 - (a\sin\theta)} \cdot a\cos\theta\, d\theta$

$= \int \sqrt{a^2 - a^2\sin^2\theta} \cdot a\cos\theta\, d\theta$

$= \int \sqrt{a^2(1 - \sin^2\theta} \cdot a\cos\theta\, d\theta$

$\because 1 - \sin^2\theta = \cos^2\theta$

$= \int \sqrt{a^2\cos^2\theta} \cdot a\cos\theta\, d\theta$

$= \int a\cos\theta \cdot a\cos\theta\, d\theta$

$= \int a^2\cos^2\theta\, d\theta$

$\because \cos 2\theta = 2\cos^2\theta - 1$

$\because 1 + \cos 2\theta = 2\cos^2\theta$

$\because \frac{1 + \cos 2\theta}{2} = \cos^2\theta$

$= a^2 \int \cos^2\theta\, d\theta$

$= a^2 \int \frac{1 + \cos 2\theta}{2} d\theta$

$= \frac{a^2}{2} \int 1 + \cos 2\theta\, d\theta$

$= \frac{a^2}{2} \left\{ \int 1\, d\theta + \int \cos 2\theta\, d\theta \right\}$

$= \frac{a^2}{2} \left\{ \theta + \frac{\sin 2\theta}{2} \right\} + C$

$\because \sin 2\theta = 2\sin\theta\cos\theta$

$= \frac{a^2}{2} \left[\theta + \frac{\cancel{2}\sin\theta\cos\theta}{\cancel{2}} \right] + C$

$= \frac{a^2}{2} \left[\theta + \sin\theta\cos\theta \right] + C$

$= \frac{a^2}{2} \left[\theta + \sin\theta \cdot \sqrt{\cos^2\theta} \right] + C$

$\because \cos^2\theta = 1 - \sin^2\theta$

$= \frac{a^2}{2} \left[\theta + \sin\theta \sqrt{1 - \sin^2\theta} \right] + C$

$\because x = a\sin\theta \Rightarrow \sin\theta = \frac{x}{a}$

$\because \theta = \sin^{-1}\left(\frac{x}{a}\right)$

$$= \frac{a^2}{2}\left\{\sin^{-1}\frac{x}{a} + \frac{x}{a}\sqrt{1-\left(\frac{x}{a}\right)^2}\right\} + C$$

$$= \frac{a^2}{2}\left[\sin^{-1}\frac{x}{a} + \frac{x}{a}\sqrt{1-\frac{x^2}{a^2}}\right] + C$$

$$= \frac{a^2}{2}\left[\sin^{-1}\frac{x}{a} + \frac{x}{a}\sqrt{\frac{a^2-x^2}{a^2}}\right] + C$$

$$= \frac{a^2}{2}\left[\sin^{-1}\frac{x}{a} + \frac{x}{a}\cdot\frac{\sqrt{a^2-x^2}}{a}\right] + C$$

$$= \frac{a^2}{2}\left[\sin^{-1}\frac{x}{a} + \frac{x\cdot\sqrt{a^2-x^2}}{a^2}\right] + C$$

$$= \left\{\frac{a^2}{2}\sin^{-1}\frac{x}{a} + \frac{a^2}{2}\cdot\frac{x}{a^2}\sqrt{a^2-x^2}\right\} + C$$

$$= \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + \frac{x}{2}\sqrt{a^2-x^2} + C = R.H.S.$$

L.H.S. = R.H.S.

Hence proved.

Question #9

Evaluate $\int \frac{dx}{(1+x^2)^{3/2}}$

Solution:

$$= \int \frac{dx}{(1+x^2)^{3/2}}$$

put

$x = \tan\theta$

$dx = \sec^2\theta\, d\theta$

$$= \int \frac{\sec^2\theta\, d\theta}{(1+\tan^2\theta)^{3/2}}$$

$\because 1+\tan^2\theta = \sec^2\theta$

$$= \int \frac{\sec^2\theta\, d\theta}{(\sec^2\theta)^{3/2}}$$

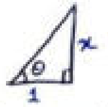
$$= \int \frac{\sec^2\theta}{\sec^3\theta}\, d\theta$$

$$= \int \frac{1}{\sec\theta}\, d\theta$$

$$= \int \cos\theta\, d\theta$$

$$= \sin\theta + C$$

$\because \tan\theta = \frac{x}{1}$



$(H)^2 = B^2 + P^2$

$(H)^2 = (1)^2 + (x)^2$

$(H)^2 = 1 + x^2$

$H = \sqrt{1+x^2}$

$\sin\theta = \frac{x}{\sqrt{1+x^2}}$

$$= \boxed{\frac{x}{\sqrt{1+x^2}} + C}$$

Question #10

Evaluate $\int \frac{1}{(1+x^2)\mathrm{Tan}^{-1}x}\, dx$

Solution:

$$= \int \frac{1}{(1+x^2)\mathrm{Tan}^{-1}x}\, dx$$

$$= \int \frac{(1+x^2)^{-1}}{\mathrm{Tan}^{-1}x}\, dx$$

$$= \int \frac{\frac{1}{(1+x^2)}}{\mathrm{Tan}^{-1}x}\, dx$$

$$\because \int \frac{f'(x)}{f(x)} = \ln|f(x)| + C$$

$$= \boxed{\ln|\mathrm{Tan}^{-1}x| + C}$$

* * *

Question #11

Evaluate $\int \sqrt{\frac{1+x}{1-x}}\, dx$

Solution:

$$= \int \sqrt{\frac{1+x}{1-x}}\, dx$$

$$= \int \frac{\sqrt{1+x}}{\sqrt{1-x}} \times \frac{\sqrt{1+x}}{\sqrt{1+x}}\, dx$$

$$= \int \frac{\sqrt{(1+x)^2}}{\sqrt{(1-x)(1+x)}}\, dx$$

$$= \int \frac{(1+x)}{1-x^2}\, dx$$

put

$x = \sin\theta$

$dx = \cos\theta\, d\theta$

$$= \int \frac{(1+\sin\theta)\cos\theta\, d\theta}{\sqrt{1-\sin^2\theta}}$$

$\because 1-\sin^2\theta = \cos^2\theta$

$$= \int \frac{1+\sin\theta \quad \cos\theta\, d\theta}{\sqrt{\cos^2\theta}}$$

$$= \int \frac{(1+\sin\theta)\cancel{\cos\theta}\, d\theta}{\cancel{\cos\theta}}$$

$$= \int (1+\sin\theta)\, d\theta$$

$$= \int 1\, d\theta + \int \sin\theta\, d\theta$$

$$= \theta - \cos\theta + C$$

$\because x = \sin\theta \Rightarrow \sin^{-1}x = \theta$

$\because \cos\theta = \sqrt{1-\sin^2\theta}$

$\Rightarrow \cos\theta = \sqrt{1-x^2}$

$$= \boxed{\sin^{-1}x - \sqrt{1-x^2} + C}$$

•——•

Question #12

Evaluate $\int \frac{\sin\theta}{1+\cos^2\theta}\, d\theta$

Solution:

$$= \int \frac{\sin\theta}{1+\cos^2\theta}\, d\theta$$

put

$\cos\theta = t$

$-\sin\theta\, d\theta = dt$

$\sin\theta\, d\theta = -dt$

$$= \int \frac{-dt}{1+(t)^2}$$

$$= -\int \frac{1}{1+t^2}\, dt$$

$\because \int \frac{1}{1+t^2}\, dx = \tan^{-1}x + C$

$$= -\tan^{-1}(t) + C$$

$\because \cos\theta = t$

$$= \boxed{-\tan^{-1}(\cos\theta) + C}$$

•——•

Question #13

Evaluate $\int \frac{ax}{\sqrt{a^2-x^4}}\, dx$

Solution:

$$= \int \frac{ax}{\sqrt{(a)^2-(x^2)^2}}\, dx$$

put $x^2 = a\sin\theta$

$2x\, dx = a\cos\theta\, d\theta$

$x\, dx = \frac{1}{2} a\cos\theta\, d\theta$

$$= \int \frac{a \cdot \frac{1}{2} a\cos\theta\, d\theta}{\sqrt{a^2-(a\sin\theta)^2}}$$

$$= \int \frac{\frac{a^2}{2} \cos\theta \, d\theta}{\sqrt{a^2 - (a^2 \sin^2\theta)}}$$

$$= \frac{a^2}{2} \int \frac{\cos\theta \, d\theta}{\sqrt{a^2(1-\sin^2\theta)}}$$

$\because 1 - \sin^2\theta = \cos^2\theta$

$$= \frac{a^2}{2} \int \frac{\cos\theta \, d\theta}{\sqrt{a^2 \cdot \cos^2\theta}}$$

$$= \frac{a^2}{2} \int \frac{\cos\theta}{a \cos\theta} d\theta$$

$$= \frac{a}{2} \int 1 \, d\theta$$

$$= \frac{a}{2} \theta \, d\theta$$

$\because x^2 = a\sin\theta \Rightarrow \sin\theta = \frac{x^2}{a}$

$\because \theta = \sin^{-1} \frac{x^2}{a}$

$$= \boxed{\frac{a}{2} \sin^{-1} \frac{x^2}{2} + C}$$

Question # 14

Evaluate $\int \frac{dx}{\sqrt{7-6x-x^2}}$

Solution:

$$= \int \frac{dx}{\sqrt{7-6x-x^2}}$$

$$= \int \frac{dx}{\sqrt{7-6x^2-x^2-9+9}}$$

$$= \int \frac{dx}{\sqrt{16-(x^2+6x+9)}}$$

$$= \int \frac{dx}{\sqrt{(4)^2-(x+3)^2}}$$

put

$x+3 = 4\sin\theta$

$dx = 4\cos\theta \, d\theta$

$$= \int \frac{4\cos\theta \, d\theta}{\sqrt{(4)^2-(4\sin\theta)^2}}$$

$$= 4 \int \frac{\cos\theta}{\sqrt{16-16\sin^2\theta}} d\theta$$

$$= 4 \int \frac{\cos\theta}{\sqrt{16(1-\sin^2\theta)}} d\theta$$

$\because 1 - \sin^2\theta = \cos^2\theta$

$$= 4 \int \frac{\cos\theta}{\sqrt{16\cos^2\theta}} d\theta$$

$$= \cancel{4} \int \frac{\cancel{\cos\theta}}{\cancel{4}\cancel{\cos\theta}} d\theta$$

$$= \int 1 \, d\theta$$

$$= \theta + C$$

$\because \sin\theta = \frac{x+3}{4}$

$\because \theta = \sin^{-1}\left(\frac{x+3}{4}\right)$

$$= \boxed{\sin^{-1}\left(\frac{x+3}{4}\right) + C}$$

Question # 15

$$\int \frac{\cos x}{\sin x \ln \sin x} dx$$

Solution:

$$= \int \frac{\cos x}{\sin x \ln \sin x} dx$$

$$= \int \frac{\cos x}{\sin x} \cdot \frac{1}{\ln \sin x} dx$$

$$= \int \cot x \cdot \frac{1}{\ln \sin x} dx$$

put

$\ln \sin x = t$

$\frac{1}{\sin x} \cos x \, dx = dt$

$\frac{\cos x}{\sin x} dx = dt$

$\cot x \, dx = dt$

$= \int \frac{dt}{t}$

$\because \int \frac{f'(x)}{f(x)} = \ln|f(x)|+C$

$= \ln|t|+C$

$\because t = \ln(\sin x)$

$= \boxed{\ln|\ln \sin x|+C}$

Question # 16

Evaluate $\int \cos x\left(\frac{\ln \sin x}{\sin x}\right)dx$

Solution:

$= \int \cos x\left(\frac{\ln \sin x}{\sin x}\right)dx$

$= \int \frac{\cos x}{\sin x}\cdot(\ln \sin x)\,dx$

$= \int \cot x\cdot(\ln \sin x)\,dx$

put

$\ln \sin x = t$

$\frac{1}{\sin x}\cos x\,dx = dt$

$\cot x\,dx = dt$

$= \int dt\cdot t$

$= \int t\,dt$

$= \frac{t^{1+1}}{1+1}+C$

$= \frac{t^2}{2}+C$

$\because t = \ln \sin x$

$= \boxed{\frac{(\ln \sin x)^2}{2}+C}$

Question # 17

Evaluate $\int \frac{x\,dx}{4+2x+x^2}$

Solution:

$= \int \frac{x\,dx}{4+2x+x^2}$

Multiply and divided by '2'

$= \frac{1}{2}\int \frac{2x\,dx}{4+2x+x^2}$

Adding and subtracting '2'

$= \frac{1}{2}\int \frac{2x+2-2}{4+2x+x^2}\,dx$

$= \frac{1}{2}\int\left[\frac{2x+2}{x^2+2x+4} - \frac{2}{x^2+2x+4}\right]dx$

$= \frac{1}{2}\int \frac{2x+2}{x^2+2x+4}\,dx - \frac{1}{\cancel{2}}\int \frac{\cancel{2}}{x^2+2x+4}\,dx$

$= \frac{1}{2}\int \frac{2x+2}{x^2+2x+4}\,dx - \int \frac{1}{x^2+2x+4}\,dx$

$\because \int \frac{f'(x)}{f(x)} = \ln|f(x)|+C$

$= \frac{1}{2}\ln|x^2+2x+4| - I$ —— ①

$I = \int \frac{1}{x^2+2x+4}\,dx$

$I = \int \frac{1}{x^2+2x+1+3}\,dx$

$I = \int \frac{1\quad dx}{(x+1)^2+(\sqrt{3})^3}$

put

$x+1 = \sqrt{3}\tan\theta$

$dx = \sqrt{3}\sec^2\theta\,d\theta$

$I = \int \frac{\sqrt{3}\sec^2\theta\,d\theta}{(\sqrt{3}\tan\theta)^2+(\sqrt{3})^3}$

$I = \int \frac{\sqrt{3}\sec^2\theta\,d\theta}{3\tan^2\theta+3}$

$I = \frac{\sqrt{3}}{3}\int \frac{\sec^2\theta\,d\theta}{\tan^2\theta+1}$

$\because 1+\tan^2\theta = \sec^2\theta$

$I = \frac{1}{\sqrt{3}}\int \frac{\cancel{\sec^2\theta}}{\cancel{\sec^2\theta}}\,d\theta$

$$I = \frac{1}{\sqrt{3}} \int 1 \, d\theta$$

$$I = \frac{1}{\sqrt{3}} \theta$$

$\because x+1 = \sqrt{3}\tan\theta \Rightarrow \tan\theta = \frac{x+1}{\sqrt{3}}$

$\because \theta = \tan^{-1}\left(\frac{x+1}{\sqrt{3}}\right)$

$$I = \frac{1}{\sqrt{3}} \tan^{-1} \frac{x+1}{\sqrt{3}} + C$$

so, eq ① become

$$= \boxed{\frac{1}{2} \ln|x^2+2x+4| - \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{x+1}{\sqrt{3}}\right) + C}$$

Question # 18

Evaluate $\int \frac{x}{x^4+2x^2+5} dx$

Solution:

$$= \int \frac{x \, dx}{x^4+2x^2+5}$$

$$= \int \frac{x \, dx}{x^4+2x^2+1+4}$$

$$= \int \frac{x \, dx}{(x^2+1)^2+(2)^2}$$

put

$x^2+1 = 2\tan\theta$

$2x \, dx = 2\sec^2\theta \, d\theta$

$x \, dx = \frac{\cancel{2}\sec^2\theta \, d\theta}{\cancel{2}}$

$x \, dx = \sec^2\theta \, d\theta$

$$= \int \frac{\sec^2\theta \, d\theta}{(2\tan\theta)^2+(2)^2}$$

$$= \int \frac{\sec^2\theta \, d\theta}{4\tan^2\theta+4}$$

$$= \int \frac{\sec^2\theta \, d\theta}{4(\tan^2\theta+1)}$$

$$= \frac{1}{4} \int \frac{\sec^2\theta}{\sec^2\theta} d\theta$$

$$= \frac{1}{4} \int 1 \, d\theta$$

$$= \frac{1}{4} \theta$$

$\because 2\tan\theta = x^2+1 \Rightarrow \tan\theta = \frac{x^2+1}{2}$

$\Rightarrow \theta = \tan^{-1} \frac{x^2+1}{2}$

$$= \boxed{\frac{1}{4} \tan^{-1}\left(\frac{x^2+1}{2}\right) + C}$$

Question # 19

Evaluate $\int \left[\cos\left(\sqrt{x} - \frac{x}{2}\right) x \left(\frac{1}{\sqrt{x}} - 1\right)\right] dx$

Solution:

$$= \int \left[\cos\left(\sqrt{x} - \frac{x}{2}\right) \times \left(\frac{1}{\sqrt{x}} - 1\right)\right] dx$$

put

$\sqrt{x} - \frac{x}{2} = t$

$x^{1/2} - \frac{x}{2} = t \Rightarrow \frac{1}{2}x^{-1/2} - \frac{1}{2} = dt$

$\frac{1}{2\sqrt{x}} - \frac{1}{2} = dt \Rightarrow \frac{1}{2}\left(\frac{1}{\sqrt{x}} - 1\right) = dt$

$\frac{1}{\sqrt{x}} - 1 = 2dt$

$$= \int \cos t \cdot 2dt$$

$$= 2\int \cos t \, dt$$

$$= 2(\sin t) + C$$

$\because t = \sqrt{x} - \frac{x}{2}$

$$= \boxed{2\sin\left(\sqrt{x} - \frac{x}{2}\right) + C}$$

Question # 20

Evaluate $\int \frac{x+2}{\sqrt{x+3}}\,dx$

Solution:

$$= \int \frac{x+2+1-1}{\sqrt{x+3}}\,dx$$

$$= \int \frac{x+3\ -1}{\sqrt{x+3}}\,dx$$

$$= \int \frac{x+3}{\sqrt{x+3}}\,dx - \int \frac{1}{\sqrt{x+3}}\,dx$$

$$= \int \sqrt{x+3}\,dx - \int \frac{1}{\sqrt{x+3}}\,dx$$

$$= \int x+3)^{1/2}dx - \int (x+3)^{-1/2}dx$$

$$\because \int x^n = \frac{x^{n+1}}{n+1} + C$$

$$= \frac{(x+3)^{\frac{1}{2}+1}}{\frac{1}{2}+1} - \frac{(x+3)^{\frac{-1}{2}+1}}{\frac{-1}{2}+1} + C$$

$$= \frac{(x+3)^{3/2}}{3/2} - \frac{(x+3)^{1/2}}{1/2} + C$$

$$= \boxed{\frac{2}{3}(x+3)^{3/2} - 2\sqrt{x+3}\ + C}$$

* * *

Question # 21

Evaluate $\int \frac{\sqrt{2}}{\sin x + \cos x}\,dx$

Solution:

$$= \int \frac{1}{\frac{\sin x + \cos x}{\sqrt{2}}}\,dx$$

$$= \int \frac{1}{\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x}\,dx$$

$$\because \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

$$\because \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

$$= \int \frac{1}{\sin\frac{\pi}{4}\sin x + \cos\frac{\pi}{4}\cos x}\,dx$$

$$= \int \frac{1}{\cos(x-\frac{\pi}{4})}\,dx \qquad \because \sin\alpha\sin\beta + \cos\alpha\cos\beta = \cos(\alpha-\beta)$$

$$= \int \sec\left(x - \frac{\pi}{4}\right)dx$$

$$= \boxed{\ln\left|\sec\left(x-\frac{\pi}{4}\right) + \tan\left(x-\frac{\pi}{4}\right)\right| + C}$$

Question #22

Evaluate $\int \frac{dx}{\frac{1}{2}\sin x + \frac{\sqrt{3}}{2}\cos x}$

Solution:

$$= \int \frac{dx}{\frac{1}{2}\sin x + \frac{\sqrt{3}}{2}\cos x}$$

$$\because \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

$$\because \cos\frac{\pi}{3} = \frac{1}{2}$$

$$= \int \frac{dx}{\sin x\cos\frac{\pi}{3} + \cos x\sin\frac{\pi}{3}}$$

$$\because \sin\alpha\cos\beta + \cos\alpha\sin\beta = \sin(\alpha+\beta)$$

$$= \int \frac{dx}{\sin\left(x+\frac{\pi}{3}\right)}$$

$$= \int \operatorname{cosec}\left(x+\frac{\pi}{3}\right)dx$$

$$\because \int \operatorname{cosec} x = \ln|\operatorname{cosec} x - \cot x| + c$$

$$= \boxed{\ln\left|\operatorname{cosec}\left(x+\frac{\pi}{3}\right) - \cot\left(x+\frac{\pi}{3}\right)\right| + C}$$

* * *

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Theory

Integration By Parts:

$$\int f(x)g'(x)\,dx = f(x)\int g'(x)\,dx - \int g(x)f'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx$$

This is known as the formula for integration by parts

If we put

$$u = f(x) \quad \text{and} \quad dv = g'(x)\,dx$$

Then $du = f'(x)\,dx$ and $v = g(x)$

Then above equation can be written as

$$\int u\,dv = uv - \int v\,du$$

ILATE Rule:

The ILATE rule is a method for selecting the first and second functions when using the integration by parts method to solve integral

- **I** - inverse trig (arc functions)
- **L** - logarithmic functions
- **A** - algebraic (polynomials)
- **T** - trignometric functions
- **E** - exponential functions

ILATE rule

I	inverse trignometric ($\sin^{-1}x$, $\tan^{-1}x$, etc)
L	Logarithmic ($\log_3 x$, $\log x$, $\ln x$. etc)
A	Algebric (x^3, $\sqrt[3]{x}$, etc)
T	Trignometric ($\sin x$, $\cos x$, etc)
E	Exponential (3^x, e^x, etc)

Note:

ILATE in an acronym for inverse, Logarithmic, Algebraic, Trignometric and exponential

Example # 1: Find $\int x\cos x\,dx$

$= \int \underset{I}{x}\,\underset{II}{\cos x}\,dx$ $\quad \therefore \int \sin x\,dx = -\cos x$

$= x \cdot \sin x - \int \sin x \cdot (1)\,dx$

$= x\sin x - (-\cos x) \cdot$ $\quad \therefore \int \cos x\,dx = \sin x$

$= x\sin x + \cos x + C$

Example # 2: Find $\int xe^x dx$

$= \int \underset{I}{x}\,\underset{II}{e^x}\,dx$

$= x\,e^x - \int e^x \cdot 1\,dx$

$= xe^x - e^x + C$

Example # 3: Evaluate $\int x\tan^2 x\,dx$

$= \int x\tan^2 x\,dx$

$\therefore \int f(x) \cdot g'(x)\,dx = f(x)\int g(x) - \int g(x) \cdot f'(x)\,dx + C$

$= \int x(\sec^2 x - 1)\,dx$ $\quad \therefore \tan^2 x = \sec^2 x - 1$

$= \int x\sec^2 x - x\,dx$

$= \int \underset{I}{x}\,\underset{II}{\sec^2 x}\,dx - \int x\,dx$ $\quad \therefore \int \sec^2 dx = \tan x$

$x \cdot \tan x - \int \tan x \cdot 1\,dx - \frac{x^2}{2}$

$x\tan x - (-\ln|\cos x|) - \frac{x^2}{2}$

$x\tan x + \ln|\cos x| - \frac{x^2}{2} + C$

Example # 4: Evaluate $\int x^5 \ln x\,dx$.

$= \int \underset{I}{\ln x} \cdot \underset{II}{x^5}\,dx$

$= \ln x \cdot \frac{x^6}{6} - \int \frac{x^6}{6} \cdot \frac{1}{x}\,dx$

$= \frac{x^6}{6}\ln x - \frac{1}{6}\int x^5\,dx$

$= \frac{x^6}{6}\ln x - \frac{1}{6}\,\frac{x^6}{6}$

$= \frac{x^6}{6}\ln x - \frac{x^6}{36} + C$

Example # 5: Evaluate $\int \ln(x+\sqrt{x^2+1})\,dx$

$= \int \left(\underset{I}{\ln(x+\sqrt{x^2+1})} \cdot \underset{II}{1}\right) dx$

$= \ln(x+\sqrt{x^2+1}) \cdot x - \int x \cdot \frac{d}{dx}\left[\ln(x+\sqrt{x^2+1})\right] dx$

$= \ln(x+\sqrt{x^2+1}) \cdot x - \int x \cdot \frac{1}{x+\sqrt{x^2+1}} \cdot \frac{d}{dx}(x+(x^2+1)^{1/2})\,dx$

$= x\ln(x+\sqrt{x^2+1}) - \int \frac{x}{x+\sqrt{x^2+1}}\left[1 + \frac{1}{2\sqrt{x^2+1}} \cdot 2x\,dx\right]$

$= x\ln(x+\sqrt{x^2+1}) - \int \frac{x}{x+\sqrt{x^2+1}}\left(1 + \frac{x}{\sqrt{x^2+1}}\right) dx$

$= x\ln(x+\sqrt{x^2+1}) - \int \frac{x}{\cancel{x+\sqrt{x^2+1}}}\left(\frac{\cancel{\sqrt{x^2+1}+x}}{\sqrt{x^2+1}}\right) dx$

$= x\ln(x+\sqrt{x^2+1}) - \int \frac{x}{\sqrt{x^2+1}}\,dx.$

$= x\ln(x+\sqrt{x^2+1}) - \frac{1}{2}\int (x^2+1)^{-1/2} \cdot 2x\,dx$

$= x\ln(x+\sqrt{x^2+1}) - \frac{1}{2} \cdot \frac{(x^2+1)^{-1/2+1}}{-1/2+1}$

$= x\ln(x+\sqrt{x^2+1}) - \frac{1}{2} \cdot \frac{(x^2+1)^{1/2}}{1/2}$

$= x\ln(x+\sqrt{x^2+1}) - (x^2+1)^{1/2}$

$= x\ln(x+\sqrt{x^2+1}) - \sqrt{x^2+1} + C.$

Example # 6: Evaluate

$$\int x^2 \cdot ae^{ax} dx$$

$$= \int \underset{I}{x^2} \cdot \underset{II}{ae^{ax}} dx$$

$$= a\left[x^2 \cdot \frac{e^{ax}}{a} - \int \frac{e^{ax}}{a}(2x)\,dx\right]$$

$$= a\left[x^2 \cdot \frac{e^{ax}}{a} - \frac{2}{a}\int \underset{I}{x}\, \underset{II}{e^{ax}} dx\right]$$

$$= a\left[x^2 \cdot \frac{e^{ax}}{a} - \frac{2}{a}\left[x \cdot \frac{e^{ax}}{a} - \int \frac{e^{ax}}{a} \cdot 1 dx\right]\right]$$

$$= x^2 \cdot e^{ax} - 2\left[x \cdot \frac{e^{ax}}{a} - \frac{1}{a}\int e^{ax} dx\right]$$

$$= x^2 \cdot e^{ax} - \frac{2}{a}\left[x \cdot e^{ax} - \frac{1}{a} e^{ax}\right]$$

$$= x^2 \cdot e^{ax} - \frac{2x \cdot e^{ax}}{a} - \frac{2e^{ax}}{a^2} + c$$

Example # 7: Find $\int e^{ax} \cdot \cos bx dx$

$$I = \int \underset{I}{\cos bx} \cdot \underset{II}{e^{ax}} dx$$

$$I = \cos bx \cdot \frac{e^{ax}}{a} - \int \frac{e^{ax}}{a} \cdot (-\sin bx \cdot b)\,dx$$

$$I = \cos bx \cdot \frac{e^{ax}}{a} + \frac{b}{a}\int \underset{I}{\sin bx} \cdot \underset{II}{e^{ax}} dx$$

$$I = \frac{e^{ax}\cos bx}{a} + \frac{b}{a}\left[\sin bx \cdot \frac{e^{ax}}{a} - \int \frac{e^{ax}}{a} \cdot (\cos bx \cdot b)\,dx\right]$$

$$I = \frac{e^{ax}\cos bx}{a} + \frac{b}{a^2}\sin bx\, e^{ax} - \frac{b^2}{a^2}\int e^{ax}\cos bx\, dx$$

$$I = \frac{e^{ax}\cos bx}{a} + \frac{be^{ax}}{a^2}\sin bx - \frac{b^2}{a^2} I + c_1$$

$$I + \frac{b^2 I}{a^2} = \frac{e^{ax}\cos bx}{a} + \frac{b}{a} e^{ax}\sin bx + c_1$$

$$I\left(\frac{a^2+b^2}{a^2}\right) = e^{ax}\left(\frac{\cos bx}{a} + \frac{b\sin bx}{a^2}\right) + c_1$$

$$I = \frac{\cancel{a^2}\, e^{ax}}{a^2+b^2}\left(\frac{a\cos bx + b\sin bx}{\cancel{a^2}}\right) + \frac{a^2 c_1}{a^2+b^2}$$

$$I = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx) + \frac{a^2 c_1}{a^2+b^2}$$

$$I = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx) + C \qquad \therefore C = \frac{a^2 c_1}{a^2+b^2}$$

So Put $a = r\cos\theta$; $b = r\sin\theta$

$$a^2+b^2 = r^2(\cos^2\theta + \sin^2\theta)$$

$$a^2+b^2 = r^2(1)$$

$$a^2+b^2 = r^2 \Rightarrow r = \sqrt{a^2+b^2}$$

$$I = \frac{e^{ax}}{r^2}(r\cos\theta\cos bx + r\sin\theta\sin bx)$$

$$I = \frac{e^{ax} r}{r^2}(\cos bx\cos\theta + \sin bx\sin\theta)$$

$$I = \frac{e^{ax}}{r}(\cos(bx-\theta))$$

$\therefore \cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

$$I = \frac{e^{ax}}{\sqrt{a^2+b^2}}\cos(bx-\theta) + C$$

$\therefore \frac{b}{a} = \frac{r\sin\theta}{r\cos\theta}$

$\frac{b}{a} = \tan\theta$

$\theta = \tan^{-1}\frac{b}{a}$

Thus

$$\int e^{ax}\cos bx\,dx = \frac{1}{\sqrt{a^2+b^2}} e^{ax}\cos\left(bx - \tan^{-1}\frac{b}{a}\right) + c$$

Example # 8: Evaluate $\int \sqrt{a^2+x^2}\, dx$

$$I = \int \underset{I}{\sqrt{a^2+x^2}} \cdot \underset{II}{1}\, dx$$

$$I = \sqrt{a^2+x^2} \cdot x - \int x \cdot \frac{d}{dx}(a^2+x^2)^{1/2} dx$$

$$I = \sqrt{a^2+x^2} \cdot x - \int x \cdot \frac{1}{2(a^2+x^2)^{1/2}} \cdot 2x\, dx$$

$$I = \sqrt{a^2+x^2} \cdot x - \int \frac{x^2}{\sqrt{a^2+x^2}} dx$$

$$I = \sqrt{a^2+x^2} \cdot x - \int \frac{a^2+x^2-a^2}{\sqrt{a^2+x^2}} dx$$

$$I = x\sqrt{a^2+x^2} - \int \frac{a^2+x^2}{\sqrt{a^2+x^2}} - \frac{a^2}{\sqrt{a^2+x^2}}\, dx$$

$$I = x\sqrt{a^2+x^2} - \int \frac{a^2+x^2}{\sqrt{a^2+x^2}} dx + a^2\int \frac{1}{\sqrt{a^2+x^2}} dx$$

$$I = x\sqrt{a^2+x^2} - \int \sqrt{a^2+x^2}\, dx + a^2\int \frac{1}{\sqrt{a^2+x^2}} dx$$

$$I = x\sqrt{a^2+x^2} - I + a^2 \ln|x+\sqrt{a^2+x^2}| + c_1$$

$$2I = x\sqrt{a^2+x^2} + a^2 \ln|x+\sqrt{a^2+x^2}| + c_1$$

$$I = \frac{x}{2}\sqrt{a^2+x^2} + \frac{a^2}{2}\ln|x+\sqrt{a^2+x^2}| + \frac{c_1}{2}$$

Thus

$$\int \sqrt{a^2+x^2}\, dx = \frac{x}{2}\sqrt{a^2+x^2} + \frac{a^2}{2}\ln|x+\sqrt{a^2+x^2}| + c$$

$$\because \frac{c_1}{2} = c$$

$$\because \int \frac{1}{\sqrt{a^2+x^2}} dx = \ln|x+\sqrt{a^2+x^2}| + c$$

Example # 9: Evaluate

$$\int \sin^4 x\, dx$$

$$I = \int \sin^4 x\, dx$$

$$I = \int \sin^2 x \cdot \sin^2 x\, dx$$

$$I = \int \sin^2 x \cdot (1-\cos^2 x)\, dx$$

$$\because \sin^2 x + \cos^2 x = 1$$

$$\sin^2 x = 1 - \cos^2 x$$

$$I = \int \sin^2 x - \sin^2 x \cos^2 x\, dx.$$

$$I = \int \sin^2 x\, dx - \int \sin^2 x \cos^2 x\, dx$$

$$I = \int \frac{1-\cos 2x}{2} dx - \int \underset{\text{I}}{\cos x}\,(\underset{\text{II}}{\sin^2 x \cos x})\, dx$$

$$I = \frac{1}{2}\int 1 - \cos 2x\, dx - \left[\cos x \cdot \frac{\sin^3 x}{x} - \int \frac{\sin^3 x}{3} \cdot (-\sin x)\, dx\right]$$

$$I = \frac{1}{2}\int 1\, dx - \frac{1}{2}\int \cos 2x\, dx - \left[\cos x \cdot \frac{\sin^3 x}{3} + \frac{1}{3}\int \sin^4 dx\right]$$

$$I = \frac{1}{2}x - \frac{1}{2}\cdot\frac{\sin 2x}{2} - \frac{1}{3}\sin^3 x \cos x - \frac{1}{3} I + c_1$$

$$I + \frac{1}{3} I = \frac{1}{2}x - \frac{\sin 2x}{4} - \frac{1}{3}\sin^3 x \cos x + c_1$$

$$\left(1+\frac{1}{3}\right) I = \frac{x}{2} - \frac{\sin 2x}{4} - \frac{1}{3}\sin^3 x \cos x + c_1$$

$$\left(\frac{3+1}{3}\right) I = \frac{x}{2} - \frac{\sin 2x}{4} - \frac{1}{3}\sin^3 x \cos x + c_1$$

$$\frac{4}{3} I = \frac{x}{2} - \frac{\sin 2x}{4} - \frac{1}{3}\sin^3 x \cos x + c_1$$

$$I = \frac{3}{4}\left[\frac{x}{2} - \frac{\sin 2x}{4} - \frac{1}{3}\sin^3 x \cos x + c_1\right]$$

$$I = \frac{3}{8}x - \frac{3}{16}\sin 2x - \frac{1}{4}\sin^3 x \cos x + \frac{3c_1}{4}$$

$$\because \frac{3}{4}c_1 = c$$

Thus

$$\int \sin^4 x\, dx = \frac{3}{8}x - \frac{3}{16}\sin 2x - \frac{1}{4}\sin^3 x \cos x + c.$$

Example # 10:

Evaluate

$$\int \frac{e^x(1+\sin x)}{1+\cos x}\,dx$$

$$= \int \frac{e^x(1+\sin x)}{1+\cos x}\,dx$$

$$= \int \frac{e^x\left(1+2\sin\frac{x}{2}\cos\frac{x}{2}\right)}{2\cos^2\frac{x}{2}}\,dx$$

$\because$ $\sin 2\theta = 2\sin\theta\cos\theta$

$\Rightarrow$ $\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$.

$\because$ $\cos^2\frac{\theta}{2} = \frac{1+\cos\theta}{2}$

$\Rightarrow$ $2\cos^2\frac{\theta}{2} = 1+\cos\theta$.

$$= \int e^x\left(\frac{1}{2\cos^2 x/2} + \frac{\cancel{2}\sin x/2\,\cancel{\cos x/2}}{\cancel{2}\cos^{\cancel{2}} x/2}\right)dx$$

$$= \int e^x\left(\frac{1}{2}\sec^2\frac{x}{2} + \frac{\sin x/2}{\cos x/2}\right)dx$$

$$= \int e^x\left(\frac{1}{2}\sec^2\frac{x}{2} + \tan\frac{x}{2}\right)dx$$

$$= e^x\tan\frac{x}{2} + c$$

$\because$ $\int e^x(f(x)+f'(x))dx = e^x f(x) + c$.

$f(x) = \tan\frac{x}{2}$

$f'(x) = \frac{1}{2}\sec^2\frac{x}{2}$.

Example # 11: Show that

$$\int e^{ax}[a f(x) + f'(x)]dx = e^{ax}f(x) + c$$

$$= \int e^{ax}(a f(x) + f'(x))\,dx$$

$$= \int [e^{ax} a f(x) + e^{ax} f'(x)]\,dx$$

$$= a\int e^{ax} f(x)\,dx + \int \underset{I}{e^{ax}}\,\underset{II}{f'(x)}\,dx$$

$$= a\int e^{ax} f(x)\,dx + e^{ax}f(x) - \int f(x)\cdot e^{ax} a\,dx$$

$\because$ $\int f'(x)\,dx = f(x)$

$$= \cancel{a\int e^{ax} f(x)\,dx} + e^{ax}f(x) - \cancel{a\int e^{ax} f(x)\,dx}$$

$$= e^{ax} f(x) + c$$

Hence proved.

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Exercise # 5.4

Question #1

Evaluate the following integrals by part add a word representing all the function are defined:

(i)

$\int x \cdot \sin x \, dx$

Solution:

$\int x \cdot \sin x \, dx$

1st function = x 2nd function = $\sin x$

Integration by parts:

$= x(-\cos x) - \int (\cos x) \cdot 1 \, dx$

$= -x\cos x + \int \cos x \, dx$

$= -x\cos x + \sin x + C$

$= \boxed{\sin x - x\cos x + C}$

(ii)

$\int \ln x \, dx$

Solution:

$= \int \ln x \cdot 1 \, dx$

1st function = $\ln x$ 2nd function = 1

Integration by parts:

$= \ln x \cdot x - \int \cancel{x} \cdot \frac{1}{\cancel{x}} \, dx$

$= x \ln x - \int 1 \, dx$

$= \boxed{x \ln x - x + C}$

(iii)

$\int x \ln x \, dx$

Solution:

$= \int x \ln x \, dx$

1st function = $\ln x$ 2nd function = x

Integration by parts:

$= \ln x \cdot \frac{x^2}{2} - \int \frac{\cancel{x}^2}{2} \cdot \frac{1}{\cancel{x}} \, dx$

$= \frac{x^2}{2} \ln x - \frac{1}{2} \int x \, dx$

$\because \int x^n dx = \frac{x^{n+1}}{n+1} + C$

$= \frac{x^2}{2} \ln x - \frac{1}{2} \cdot \frac{x^2}{2} + C$

$= \frac{x^2}{2} \ln x - \frac{x^2}{4} + C$

$= \boxed{\frac{x^2}{2}\left[\ln x - \frac{1}{2}\right] + C}$

(iv)

$\int x^2 \ln x \, dx$

Solution:

$= \int x^2 \ln x \, dx$

1st function = $\ln x$ 2nd function = x^2

Integration by parts:

$= \ln x \cdot \frac{x^3}{3} - \int \frac{\cancel{x}^{3^2}}{3} \cdot \frac{1}{\cancel{x}} \, dx$

$= \frac{x^3}{\cancel{3}} \ln x - \frac{1}{3} \int x^2 \, dx$

$\because \int x^n dx = \frac{x^{n+1}}{n+1} + C$

$= \frac{x^3 \ln x}{3} - \frac{1}{3} \cdot \frac{x^3}{3} + C$

$$= \frac{x^3}{3}\ln x - \frac{x^3}{9} + C$$

$$= \boxed{\frac{x^3}{3}\left[\ln x - \frac{1}{3}\right] + C}$$

* * *

(v)

$$\int x^3 \ln x \, dx$$

Solution:

$$= \int x^3 \ln x \, dx$$

1st function= $\ln x$ 2nd function= x^3

Integration by parts:

$$= \ln x \cdot \frac{x^4}{4} - \int \frac{x^{\cancel{4}\,3}}{4} \cdot \frac{1}{\cancel{x}} \, dx$$

$$= \frac{x^4}{4}\ln x - \frac{1}{4}\int x^3 \, dx \qquad \because \int x^n dx = \frac{x^{n+1}}{n+1} + C$$

$$= \frac{x^4}{4}\ln x - \frac{1}{4}\cdot\frac{x^4}{4} + C$$

$$= \frac{x^4}{4}\ln x - \frac{x^4}{16} + C$$

$$= \boxed{\frac{x^4}{4}\left[\ln x - \frac{1}{4}\right] + C}$$

* * *

(vi)

$$\int x^4 \ln x \, dx$$

Solution:

$$\int x^4 \ln x \, dx$$

1st function: $\ln x$ 2nd function= x^4

Integration by parts:

$$= \ln x \cdot \frac{x^5}{5} - \int \frac{x^{\cancel{5}\,4}}{5} \cdot \frac{1}{\cancel{x}} \, dx$$

$$= \frac{x^5}{5}\ln x - \frac{1}{5}\int x^4 \, dx$$

$$\because \int x^n dx = \frac{x^{n+1}}{n+1} + C$$

$$= \frac{x^5}{5}\ln x - \frac{1}{5}\cdot\frac{x^5}{5} + c$$

$$= \frac{x^5}{5}\ln x - \frac{x^5}{25} + C$$

$$= \boxed{\frac{x^5}{5}\left[\ln x - \frac{1}{5}\right] + C}$$

* * *

(vii)

$$\int \operatorname{Tan}^{-1}x \, dx$$

Solution:

$$\int \operatorname{Tan}^{-1}x \cdot 1 \, dx$$

1st function= $\operatorname{Tan}^{-1}x$ 2nd function= 1

Integration by parts:

$$= \operatorname{Tan}^{-1}x \cdot x - \int x \cdot \frac{1}{1+x^2} \, dx$$

$$= x\tan^{-1}x - \int \frac{x}{1+x^2} \, dx$$

Multiply and divided by "2"

$$= x\tan^{-1}x - \frac{1}{2}\int \frac{2x}{1+x^2} \, dx$$

$$\because \int \frac{f'(x)}{f(x)} = \ln|f(x)| + C$$

$$= \boxed{x\tan^{-1}x - \frac{1}{2}\ln|1+x^2| + C}$$

* * *

(viii)

$$\int x^2 \sin x \, dx$$

Solution:

$$\int x^2 \sin x \, dx$$

1st function= x^2 2nd function: $\sin x$

Integration by parts:

$$= x^2(-\cos x) - \int(-\cos x)\ 2x \, dx$$

$$= -x^2\cos x + 2\int x\cos x \, dx$$

Again Integration by parts:

$$= -x^2\cos x + 2\left[x\sin x - \int \sin x \cdot 1 \, dx\right.$$

$$= -x^2\cos x + 2x\sin x - 2\int \sin x \, dx$$

$$= -x^2\cos x + 2x\sin x - 2(-\cos x) + C$$

$$= \boxed{-x^2\cos x + 2x\sin x + 2\cos x + C}$$

(ix)

$$\int x^2 \,\mathrm{Tan}^{-1}x \, dx$$

Solution:

$$= \int x^2 \cdot \mathrm{Tan}^{-1}x \, dx$$

1st function = $\mathrm{Tan}^{-1}x$ 2nd function = x^2

Integration by parts:

$$= \mathrm{Tan}^{-1}x \cdot \frac{x^3}{3} - \int \frac{x^3}{3} \cdot \frac{1}{1+x^2} dx$$

$$= \frac{x^3}{3}\mathrm{Tan}^{-1}x - \frac{1}{3}\int \frac{x^3}{1+x^2} dx$$

$$\begin{array}{r|l} & x \\ \hline 1+x^2 & \cancel{x^3} \\ & \underline{\cancel{\mp x^3} \mp x} \\ & -x \end{array}$$

$$= \frac{x^3}{3}\mathrm{Tan}^{-1}x - \frac{1}{3}\int \left(x - \frac{x}{1+x^2}\right) dx$$

$$= \frac{x^3}{3}\mathrm{Tan}^{-1}x - \frac{1}{3}\int x \, dx + \frac{1}{3}\int \frac{x}{1+x^2} dx$$

$$= \frac{x^3}{3}\mathrm{Tan}^{-1}x - \frac{1}{3} \cdot \frac{x^2}{2} + \frac{1}{6}\int \frac{2x}{1+x^2} dx$$

$$\because \int \frac{f'(x)}{f(x)} = \ln|f(x)| + C$$

$$= \boxed{\frac{x^3}{3}\mathrm{Tan}^{-1}x - \frac{x^3}{6} + \frac{1}{6}\ln|1+x^2| + C}$$

(x)

$$\int x \,\mathrm{Tan}^{-1}x \, dx$$

Solution:

$$\int x \,\mathrm{Tan}^{-1}x \, dx$$

1st function = $\mathrm{Tan}^{-1}x$ 2nd function = x

$$= \mathrm{Tan}^{-1}x \cdot \frac{x^2}{2} - \int \frac{x^2}{2} \cdot \frac{1}{1+x^2} dx$$

$$= \frac{x^2}{2}\mathrm{Tan}^{-1}x - \frac{1}{2}\int \frac{x^2}{1+x^2}$$

$$\begin{array}{r|l} & 1 \\ \hline 1+x^2 & \cancel{x^2} \\ & \underline{\cancel{\mp x^2} \mp 1} \\ & -1 \end{array}$$

$$= \frac{x^2}{2}\mathrm{Tan}^{-1}x - \frac{1}{2}\int \left(1 - \frac{1}{1+x^2}\right) dx$$

$$= \frac{x}{2}\tan x - \frac{1}{2}\int 1 \, dx + \frac{1}{2}\int \frac{1}{1+x^2} dx$$

$$= \frac{x^2}{2}\mathrm{Tan}^{-1}x - \frac{1}{2}x + \frac{1}{2}\mathrm{Tan}^{-1}x + C$$

$$= \frac{x^2}{2}\mathrm{Tan}^{-1}x + \frac{1}{2}\mathrm{Tan}^{-1}x - \frac{x}{2} + C$$

$$= \boxed{\frac{1}{2}\mathrm{Tan}^{-1}x\,(x^2+1) - \frac{x}{2} + C}$$

(xi)

$$\int x^3 \tan^{-1}x \, dx$$

Solution:

$$\int x^3 \tan^{-1}x \, dx$$

1st function = $\tan^{-1}x$ 2nd function = x^3

Integration by parts:

$$= \tan^{-1}x \cdot \frac{x^4}{4} - \int \frac{x^4}{4} \cdot \frac{1}{1+x^2} dx$$

$$= \frac{x^4}{4}\tan^{-1}x - \frac{1}{4}\int \frac{x^4}{1+x^2} dx$$

$$\begin{array}{r|l} & x^2 - 1 \\ \hline 1+x^2 & \cancel{x^4} \\ & \underline{\cancel{\mp x^4} \mp x^2} \\ & \cancel{-x^2} \\ & \underline{\cancel{\mp x^2} \mp 1} \\ & 1 \end{array}$$

$$= \frac{x^4}{4}\tan^{-1}x - \frac{1}{4}\int \left[x^2 - 1 + \frac{1}{1+x^2}\right] dx$$

$$= \frac{x^4}{4}\tan^{-1}x - \frac{1}{4}\int x^2 \, dx + \frac{1}{4}\int 1 \, dx - \frac{1}{4}\int \frac{1}{1+x^2} dx$$

$$= \frac{x^4}{4}\tan^{-1}x - \frac{1}{4} \cdot \frac{x^3}{3} + \frac{1}{4}x - \frac{1}{4}\tan^{-1}x + C$$

$$= \boxed{\frac{1}{4}\left[x^4\tan^{-1}x - \frac{x^3}{3} + x - \tan^{-1}x\right] + C}$$

(xii)

$$\int x^3 \cos x \, dx$$

Solution:

$$= \int x^3 \cos x \, dx$$

1st function = x^3 2nd function = $\cos x$

Integration by parts:

$= x^3 \sin x - \int \sin x \cdot 3x^2 dx$

$= x^3 \sin x - 3\int \underset{II}{\sin x} \cdot \underset{I}{x^2} dx$

Again Integration by parts:

$= x^3 \sin x - 3\left[x^2(-\cos x) - \int(-\cos x)\cdot 2x\, dx\right.$

$= x^3 \sin x + 3x^2 \cos x - 6\int \underset{II}{\cos x}\, \underset{I}{x}\, dx$

Again Integration by parts:

$= x^3 \sin x + 3x^2 \cos x - 6\left[x(\sin x) - \int \sin x \cdot 1\, dx\right.$

$= x^3 \sin x + 3x^2 \cos x - 6x \sin x - 6\int \sin x\, dx$

$= x^3 \sin x + 3x^2 \cos x - 6x \sin x - 6(\cos x) + C$

$= x^3 \sin x - 6x \sin x + 3x^2 \cos x - 6\cos x + C$

$= \boxed{(x^3 - 6x)\sin x + (3x^2 - 6)\cos x + C}$

(xiii)

$$\int \sin^{-1}x\, dx$$

Solution:

$\int \sin^{-1}x \cdot 1\, dx$

1st function = $\sin^{-1}x$ 2nd function = 1

Integration by parts:

$= \sin^{-1}x \cdot x - \int x \cdot \frac{1}{\sqrt{1-x^2}} dx$

$= x\sin^{-1}x - \int \frac{x}{\sqrt{1-x^2}} dx$

$= x \sin^{-1}x - \int (1-x^2)^{-1/2} \cdot x\, dx$

Multiply and divided by "2"

$= x\sin^{-1}x + \frac{1}{2}\int (1-x^2)^{-1/2} \cdot -2x\, dx$

$\because \int f^n x \cdot f'(x) dx = \frac{f^{n+1}(x)}{n+1} + C$

$= x \sin^{-1}x + \frac{1}{2} \cdot \frac{(1-x^2)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + C$

$= x\sin^{-1}x + \frac{1}{2} \cdot \frac{(1-x^2)^{1/2}}{1/2} + C$

$= x\sin^{-1}x + \frac{1}{2}\cdot\frac{2}{1}\sqrt{1-x^2} + C$

$= \boxed{x\sin^{-1}x + \sqrt{1-x^2} + C}$

(xiv)

$$\int x \sin^{-1}x\, dx$$

Solution:

$\int x \sin^{-1}x\, dx$

1st function = $\sin^{-1}x$ 2nd function = x

Integration by parts:

$= \sin^{-1}x \cdot \frac{x^2}{2} - \int \frac{x^2}{2} \cdot \frac{1}{\sqrt{1-x^2}} dx$

$= \frac{x^2}{2}\sin^{-1}x - \frac{1}{2}\int \frac{x^2}{\sqrt{1-x^2}} dx$

$= \frac{x^2}{2}\sin^{-1}x + \frac{1}{2}\int \frac{-x^2}{\sqrt{1-x^2}} dx$

$= \frac{x^2}{2}\sin^{-1}x + \frac{1}{2}\int \frac{1-x^2-1}{\sqrt{1-x^2}} dx$

$= \frac{x^2}{2}\sin^{-1}x + \frac{1}{2}\int\left[\frac{1-x^2}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x^2}}\right]dx$

$= \frac{x^2}{2}\sin^{-1}x + \frac{1}{2}\int\left[\frac{\cancel{\sqrt{1-x^2}}\cdot\sqrt{1-x^2}}{\cancel{\sqrt{1-x^2}}} - \frac{1}{\sqrt{1-x^2}}\right]dx$

$= \frac{x^2}{2}\sin^{-1}x + \frac{1}{2}\int\sqrt{1-x^2}\, dx - \frac{1}{2}\int\frac{1}{\sqrt{1-x^2}} dx$

$\because \int\sqrt{a^2-x^2}\, dx = \frac{a^2}{2}\sin^{-1}\frac{x}{a} + \frac{x}{2}\sqrt{a^2-x^2} + C$

$\Rightarrow \int\sqrt{1-x^2}\, dx = \frac{1}{2}\sin^{-1}x + \frac{x}{2}\sqrt{1-x^2} + C$

$\because \int\frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1}\frac{x}{a} + C$

$\Rightarrow \int\frac{1}{\sqrt{1-x^2}} dx = \sin^{-1}x + C$

$= \frac{x^2}{2}\sin^{-1}x + \frac{1}{2}\left[\frac{1}{2}\sin^{-1}x + \frac{x}{2}\sqrt{1-x^2}\right] - \frac{1}{2}\sin^{-1}x + C$

$= \frac{x^2}{2}\sin^{-1}x + \frac{1}{4}\sin^{-1}x + \frac{x}{4}\sqrt{1-x^2} - \frac{1}{2}\sin^{-1}x + C$

$= \frac{x^2}{2}\sin^{-1}x + \left(\frac{1}{4} - \frac{1}{2}\right)\sin^{-1}x + \frac{x}{4}\sqrt{1-x^2} + C$

$= \boxed{\frac{x^2}{2}\sin^{-1}x - \frac{1}{4}\sin^{-1}x + \frac{x}{4}\sqrt{1-x^2} + C}$

(xv)

$$\int e^x \sin x \cos x \, dx$$

Solution:

$= \int e^x \sin x \cos x \, dx$

Multiply and divided by '2'

$= \frac{1}{2}\int e^x \cdot 2\sin x \cos x \, dx$

$\because 2\sin x \cos x = \sin 2x$

$= \frac{1}{2}\int e^x \sin 2x$ —— ①

Let

$I = \int e^x \sin 2x$

1st function: $\sin 2x$ 2nd function: e^x

Integration by parts:

$= \sin 2x \cdot e^x - \int e^x \cdot (\cos 2x) \cdot 2 \, dx$

$= e^x \sin 2x - 2\int e^x \cos 2x \, dx$

Again Integration by parts:

$= e^x \sin 2x - 2\left[\cos 2x \cdot e^x - \int e^x(-\sin 2x) 2 \, dx\right.$

$= e^x \sin 2x - 2e^x \cos 2x + 4\int e^x (-\sin 2x) \, dx$

$= e^x \sin 2x - 2e^x \cos 2x - 4\int e^x \sin 2x \, dx$

$\because \int e^x \sin 2x \, dx = I$

$= e^x \sin 2x - 2e^x \cos 2x - 4I$

$I + 4I = e^x \sin 2x - 2e^x \cos 2x$

$5I = e^x \sin 2x - 2e^x \cos 2x$

$I = \frac{1}{5}(e^x \sin 2x - 2e^x \cos 2x)$

$I = \frac{1}{5}e^x(\sin 2x - 2\cos 2x)$

Thus eq ① becomes as

$= \frac{1}{2}\left[\frac{1}{5}e^x(\sin 2x - 2\cos 2x)\right)$

$= \frac{1}{5}e^x\left(\frac{1}{2}\sin 2x - \cos 2x\right)$

$\because \cos 2x = 1 - 2\sin^2 x$

$= \frac{1}{5}e^x\left(\frac{1}{2}\sin 2x - (1 - 2\sin^2 x)\right.$

$= \boxed{\frac{1}{5}e^x\left[\frac{1}{2}\sin 2x - 1 + 2\sin^2 x\right] + c}$

(xvi)

$$\int x \sin x \cos x \, dx$$

Solution:

$\int x \sin x \cos x \, dx$

Multiply and divided by '2'

$= \frac{1}{2}\int x \cdot 2\sin x \cos x \, dx$

$\because 2\sin x \cos x = \sin 2x$

$= \frac{1}{2}\int x \sin 2x \, dx$ —— ①

Let

$I = \int x \sin 2x \, dx$

1st function: x 2nd function: $\sin 2x$

Integration by parts:

$= x \cdot \left(-\frac{\cos 2x}{2}\right) - \int\left(-\frac{\cos 2x}{2}\right) \cdot (1) \, dx$

$= -\frac{x\cos 2x}{2} + \frac{1}{2}\int \cos 2x \cdot dx$

$I = -\frac{x\cos 2x}{2} + \frac{1}{2} \cdot \frac{\sin 2x}{2} + c$

Thus eq ① become as

$= \frac{1}{2}\left(\frac{-x\cos 2x}{2} + \frac{1 \sin 2x}{4}\right) + c$

$= -\frac{x\cos 2x}{4} + \frac{1}{8}\sin 2x + c$

$= -\frac{x\cos 2x}{4} + \frac{1 \; \cancel{2}\sin x \cos x}{\cancel{8}\,4} + c$

$= \boxed{\frac{1}{4}(-x\cos 2x + \sin x \cos x) + c}$

* * *

(xvii)

$$\int x \cos^2 x \, dx$$

Solution:

$\int x \cos^2 x \, dx$

$\because \cos^2 x = \frac{1 + \cos 2x}{2}$

$= \int x \left(\frac{1+\cos 2x}{2} \right) dx$

$= \frac{1}{2} \int x (1+\cos 2x)\, dx$

$= \frac{1}{2} \int x + x\cos 2x \, dx$

$= \frac{1}{2} \int x\, dx + \frac{1}{2} \int x \cos 2x \, dx$

1st function: x 2nd function: $\cos 2x$

Integration by parts:

$= \frac{1}{2} \cdot \frac{x^2}{2} + \frac{1}{2} \left[x \cdot \frac{\sin 2x}{2} - \int \frac{\sin 2x}{2} \; (1)\, dx \right.$

$= \frac{x^2}{4} + \frac{1}{2} \left[\frac{x \sin 2x}{2} - \frac{1}{2} \int \sin 2x \, dx \right.$

$= \frac{x^2}{4} + \frac{1}{4} x \sin 2x - \frac{1}{4} \left(\frac{-\cos 2x}{2} \right) + C$

$= \frac{x^2}{4} + \frac{1}{4} x \sin 2x + \frac{\cos 2x}{8} + c$

$= \boxed{\frac{1}{4} \left(x^2 + x\sin 2x + \frac{1}{2} \cos 2x \right) + C}$

* * *

(xviii)

$$\int x \sin^2 x \, dx$$

Solution:

$= \int x \sin^2 x \, dx$

$\because \sin^2\theta = \frac{1-\cos 2\theta}{2}$

$= \int x \left(\frac{1-\cos 2x}{2} \right) dx$

$= \frac{1}{2} \int x (1-\cos 2x)\, dx$

$= \frac{1}{2} \int x - x \cos 2x \, dx$

$= \frac{1}{2} \int x\, dx - \frac{1}{2} \int x \cos 2x \, dx$

1st function: x 2nd function: $\cos x$

Integration by parts:

$= \frac{1}{2} \cdot \frac{x^2}{2} - \frac{1}{2} \left[x \cdot \frac{\sin 2x}{2} - \int \frac{\sin 2x}{2} \; (1)\, dx \right.$

$= \frac{x^2}{4} - \frac{1}{2} \left(\frac{x \sin 2x}{2} - \frac{1}{2} \int \sin 2x \, dx \right.$

$= \frac{x^2}{4} - \frac{1}{4} \sin 2x + \frac{1}{4} \left(\frac{-\cos 2x}{2} \right) + C$

$= \frac{x^4}{4} - \frac{1}{4} \sin 2x - \frac{1}{8} \cos 2x \Big) + C$

$= \boxed{\frac{1}{4} \left(x^4 - \sin 2x - \frac{1}{2} \cos 2x \right) + C}$

* * *

(xix)

$$\int (\ln x)^2 \, dx$$

Solution:

$\int (\ln x)^2 \cdot 1 \, dx$

1st function: $(\ln x)^2$ 2nd function: 1

Integration by parts:

$= (\ln x)^2 \cdot x - \int \cancel{x} \cdot 2 \ln x \cdot \frac{1}{\cancel{x}} \, dx$

$= x (\ln x)^2 - 2 \int \ln x \cdot 1 \, dx$

Again Integration by parts:

$= x (\ln x)^2 - 2 \left[\ln x \cdot x - \int \cancel{x} \cdot \frac{1}{\cancel{x}} \, dx \right]$

$= x (\ln x)^2 - 2x \ln x + 2 \int 1 \, dx$

$= x (\ln x)^2 - 2x \ln x + 2x$

(xx)

$$\int \ln(\tan x) \cdot \sec^2 x \, dx$$

Solution:

$\int \ln(\tan x) \cdot \sec^2 x \, dx$

1st function: $\ln(\tan x)$ 2nd function: $\sec^2 x$

Integration by parts:

$= \ln(\tan x) \cdot \tan x - \int \cancel{\tan x} \cdot \frac{1}{\cancel{\tan x}} \; (\sec^2 x)\, dx$

$= \tan x \cdot \ln(\tan x) - \int \sec^2 x \, dx$

$\because \int \sec^2 x \, dx = \tan x + C$

$= \boxed{\tan x \cdot \ln(\tan x) - \tan x + C}$

* * *

(xxi)

$$\int \frac{x\sin^{-1}x}{\sqrt{1-x^2}}dx$$

Solution:

$$= \int \sin^{-1}x \cdot \frac{x}{\sqrt{1-x^2}}\, dx$$

Multiply and divided by '-2'

$$= \frac{-1}{2}\int \sin^{-1}x \; \frac{-2x}{\sqrt{1-x^2}}\, dx$$

$$= \frac{-1}{2}\int \sin^{-1}x \cdot (1-x^2)^{-1/2} \cdot (-2x)\, dx$$

1st function = $\sin^{-1}x$ 2nd function = $(1-x^2)^{-1/2}\cdot(-2x)$

Integration by parts:

$$= -\frac{1}{2}\int \sin^{-1}x \; \frac{(1-x^2)^{-1/2+1}}{-\frac{1}{2}+1} - \int \frac{(1-x^2)^{-1/2+1}}{-\frac{1}{2}+1} \cdot \frac{1}{\sqrt{1-x^2}}\, dx$$

$$= -\frac{1}{2}\int \sin^{-1}x \cdot \frac{(1-x^2)^{1/2}}{1/2} - \int \frac{(1-x^2)^{1/2}}{1/2} \cdot \frac{1}{\sqrt{1-x^2}}\, dx$$

$$= -\frac{1}{2}\int 2\sin^{-1}x \cdot \sqrt{1-x^2} - 2\int \cancel{\sqrt{1-x^2}} \cdot \frac{1}{\cancel{\sqrt{1-x^2}}}\, dx$$

$$= -[\sin^{-1}x\sqrt{1-x^2} + \int 1\, dx$$

$$= -\sin^{-1}x\sqrt{1-x^2} + x$$

$$= \boxed{x - \sqrt{1-x^2}\,\sin^{-1}x + C}$$

* * *

Question #2

Evaluate the following integrals:

(i)

$$\int \tan^4 x\, dx$$

Solution:

$$= \int \tan^4 x\, dx$$

$$= \int \tan^2 x \cdot \tan^2 x\, dx \qquad \because \tan^2 x = \sec^2 x - 1$$

$$= \int (\sec^2 x - 1) \cdot \tan^2 x\, dx$$

$$= \int \sec^2 x \tan^2 x - \tan^2 x\, dx$$

$$= \int \tan^2 x \sec^2 x - \tan^2 x\, dx$$

$$= \int \tan^2 x \sec^2 x - \int \sec^2 x - 1\, dx$$

$$\because \int f^n(x) \cdot f'(x)\, dx = \frac{f^{n+1}(x)}{n+1} + C$$

$$= \frac{(\tan x)^{2+1}}{2+1} - \int \sec^2 x\, dx + \int 1\, dx$$

$$= \boxed{\frac{(\tan x)^3}{3} - \tan x + x + C}$$

* * *

(ii)

$$\int \sec^4 x\, dx$$

Solution:

$$= \int \sec^4 x\, dx$$

$$= \int \sec^2 x \cdot \sec^2 x\, dx$$

$$\because 1 + \tan^2 x = \sec^2 x$$

$$= \int (1 + \tan^2 x) \cdot \sec^2 x\, dx$$

$$= \int (\sec^2 x + \tan^2 x \sec^2 x)\, dx$$

$$= \int (\sec^2 x\, dx + \int \tan^2 x \sec^2 x\, dx$$

$$\because \int f^n(x) \cdot f'(x)\, dx = \frac{f^{n+1}(x)}{n+1} + C$$

$$= \tan x + \frac{(\tan x)^{2+1}}{2+1} + C$$

$$= \boxed{\tan x + \frac{(\tan x)^3}{3} + C}$$

(iii)

$$\int e^x \sin 2x \cos x \, dx$$

Solution:

$= \int e^x \sin 2x \cos x \, dx$

Multiply and divided by '2'

$= \frac{1}{2}\int e^x (2\sin 2x \cos x)\, dx$

$\because 2\sin\alpha \cos\beta = \sin(\alpha+\beta) + \sin(\alpha-\beta)$

$= \frac{1}{2}\int e^x \left(\sin(2x+x) + \sin(2x-x)\right) dx$

$= \frac{1}{2}\int e^x (\sin 3x + \sin x)\, dx$

$= \frac{1}{2}\int e^x \sin 3x \, dx + \frac{1}{2}\int e^x \sin x \, dx$

$= \frac{1}{2}\{I_1 + I_2\}$ —— Ⓐ

let

$I_1 = \int e^x \sin 3x \, dx$

1st function = $\sin 3x$ 2nd function = e^x

Integration by parts:

$= \sin 3x \cdot e^x - \int e^x \cdot 3\cos 3x \, dx$

$= e^x \sin 3x - 3\int \underset{II}{e^x} \underset{I}{\cos 3x}\, dx$

Again Integration by parts:

$= e^x \sin 3x - 3\left[\cos 3x \cdot e^x - \int e^x \cdot -3\sin 3x \, dx\right] dx$

$I_1 = e^x \sin 3x - 3e^x \cos 3x - 9\int e^x \sin 3x \, dx$

$I_1 = e^x \sin 3x - 3e^x \cos 3x - 9 I_1$

$I_1 + 9I_1 = e^x \sin 3x - 3e^x \cos 3x$

$10 I_1 = e^x \sin 3x - 3e^x \cos 3x$

$I_1 = \frac{1}{10} e^x (\sin 3x - 3\cos 3x)$

Let $I_2 = \int e^x \sin x \, dx$

1st function = $\sin x$ 2nd function = e^x

Integration by parts:

$I_2 = \sin x \; e^x - \int e^x \cos x \, dx$

$I_2 = e^x \sin x - \int \underset{II}{e^x} \underset{I}{\cos x}\, dx$

Again Integration by parts:

$I_2 = e^x \sin x - [\cos x \cdot e^x - \int e^x \cdot -\sin x \, dx$

$I_2 = e^x \sin x - e^x \cos x - \int e^x \sin x \, dx$

$I_2 = e^x \sin x - e^x \cos x - I_2$

$I_2 + I_2 = e^x \sin x - e^x \cos x$

$2I_2 = e^x(\sin x - \cos x)$

$I_2 = \frac{1}{2} e^x (\sin x - \cos x)$

putting the values of I_1 and I_2 in eq Ⓐ

$= \frac{1}{2}\left[\frac{e^x}{10}(\sin 3x - 3\cos 3x) + \frac{e^x}{2}(\sin x - \cos x)\right]$

$= \frac{1}{2} \cdot \frac{e^x}{2}\left[\frac{1}{5}(\sin 3x - 3\cos 3x) + \sin x - \cos x\right]$

$= \boxed{\frac{e^x}{4}\left[\frac{1}{5}\sin 3x - \frac{3}{5}\cos 3x + \sin x - \cos x\right] + C}$

(iv)

$$\int \tan^3 x \cdot \sec x \, dx$$

Solution:

$= \int \tan^3 x \cdot \sec x \, dx$

$= \int \tan^2 x \cdot \tan x \sec x \, dx$

$\because \tan^2 x = \sec^2 x - 1$

$= \int (\sec^2 x - 1) \sec x \cdot \tan x \, dx$

$= \int\left[(\sec^2 x)(\sec x \tan x) - (\sec x \tan x)\right] dx$

$= \int (\sec^2 x)(\sec x \tan x)\, dx - \int \sec x \tan x \, dx$

$\because \int f^n(x) \cdot f'(x)\, dx = \frac{f^{n+1}(x)}{n+1} + C$

$= \frac{(\sec x)^{2+1}}{2+1} - \sec x + C$

$= \boxed{\frac{(\sec x)^3}{3} - \sec x + C}$

(v)

$$\int x^3 \cdot e^{5x} \, dx$$

Solution:

$\int x^3 \cdot e^{5x}$

1st function: x^3 2nd function: e^{5x}

Integration by parts:

$$= x^3 \cdot \frac{e^{5x}}{5} - \int \frac{e^{5x}}{5} \cdot 3x^2 \, dx$$

$$= \frac{1}{5} x^3 e^{5x} - \frac{3}{5} \int \underset{II}{e^{5x}} \cdot \underset{I}{x^2} \, dx$$

Again Integration by parts:

$$= \frac{x^3 e^{5x}}{5} - \frac{3}{5}\left[x^2 \cdot \frac{e^{5x}}{5} - \int \frac{e^{5x}}{5} \cdot 2x \, dx \right]$$

$$= \frac{x^3 e^{5x}}{5} - \frac{3}{25} x^2 e^{5x} + \frac{6}{25} \int \underset{II}{e^{5x}} \cdot \underset{I}{x} \, dx$$

Again Integration by parts:

$$= \frac{x^3 e^{5x}}{5} - \frac{3}{25} x^2 e^{5x} + \frac{6}{25}\left[x \cdot \frac{e^{5x}}{5} - \int \frac{e^{5x}}{5} \cdot (1) \, dx \right]$$

$$= \frac{x^3 e^{5x}}{5} - \frac{3}{25} x^2 e^{5x} + \frac{6}{125} x e^{5x} - \frac{6}{125} \int e^{5x} \, dx$$

$$= \frac{x^3 e^{5x}}{5} - \frac{3x^2 e^{5x}}{25} + \frac{6}{125} x e^{5x} - \frac{6}{125} \cdot \frac{e^{5x}}{5} + C$$

$$= \boxed{\frac{e^{5x}}{5}\left[x^3 - \frac{3x^2}{5} + \frac{6}{25} x - \frac{6}{125} \right] + C}$$

(vi)

$$\int e^{-x} \sin 2x \, dx$$

Solution:

let

$I = \int e^{-x} \sin 2x$

1st function: $\sin 2x$ 2nd function: e^{-x}

Integration by parts:

$$= \sin 2x \cdot \frac{e^{x}}{-1} - \int \frac{e^{-x}}{-1} \cdot \cos x \cdot 2 \, dx$$

$$= -e^{-x} \cdot \sin 2x + \int e^{-x} \cdot 2\cos 2x \, dx$$

$$= -e^{-x} \sin 2x + 2 \int e^{-x} \cos 2x \, dx$$

Again Integration $\overset{II}{}$ by $\overset{I}{}$ parts:

$$= -e^{-x} \sin 2x + 2 \left[\cos 2x \cdot \frac{e^{-x}}{-1} - \int -\frac{e^{-x}}{-1} \cdot -\sin 2x \cdot 2 \, dx \right.$$

$$= -e^{-x} \sin 2x + 2\left[-e^{-x} \cos 2x - 2 \int e^{-x} \sin 2x \, dx \right.$$

$$I = -e^{-x} \sin 2x - 2e^{-x} \cos 2x - 4I$$

$$I + 4I = -e^{-x} \sin 2x - 2e^{-x} \cos 2x + c_1$$

$$5I = -2e^{-x} \left(\frac{1}{2} \sin 2x + \cos 2x \right) + c_1$$

$$I = \frac{-2e^{-x}}{5} \left[\frac{1}{2} \sin 2x + \cos 2x \right] + \frac{c_1}{5}$$

$$\because \frac{c_1}{5} = C$$

$$\boxed{I = \frac{-2}{5} e^{-x} \left(\frac{1}{2} \sin 2x + \cos 2x \right) + C}$$

(vii)

$$\int e^{2x} \cdot \cos 3x \, dx$$

Solution:

$\int e^{2x} \cdot \cos 3x \, dx$

let

$I = \int e^{2x} \cdot \cos 3x \, dx$

1st function: $\cos 3x$ 2nd function: e^{2x}

Integration by parts:

$$I = \cos 3x \cdot \frac{e^{2x}}{2} - \int \frac{e^{2x}}{2} \cdot (-\sin 3x) \cdot 3 \, dx$$

$$I = \frac{e^{2x}}{2} \cos 3x - \frac{1}{2} \int e^{2x} \cdot -3 \sin 3x \, dx$$

$$I = \frac{e^{2x}}{2} \cos 3x + \frac{3}{2} \int \underset{II}{e^{2x}} \underset{I}{\sin 3x} \, dx$$

Again Integration by parts:

$$I = \frac{e^{2x}}{2} \cos 3x + \frac{3}{2} \left[\sin 3x \cdot \frac{e^{2x}}{2} - \int \frac{e^{2x}}{2} \cdot 3\cos 3x \, dx \right.$$

$$I = \frac{e^{2x}}{2} \cos 3x + \frac{3}{2} \left[\frac{e^{2x}}{2} \sin 3x - \frac{3}{2} \int e^{2x} \cdot \cos 3x \, dx \right.$$

$$I = \frac{e^{2x}}{2}\cos 3x + \frac{3}{4}e^{2x}\sin 3x - \frac{9}{4}I$$

$$I + \frac{9}{4}I = \frac{e^{2x}}{2}\cos 3x + \frac{3}{4}e^{x}\sin 3x$$

$$\frac{4+9I}{4} = e^{2x}\left(\frac{1}{2}\cos 3x + \frac{3}{4}\sin 3x\right)$$

$$\frac{13I}{\cancel{4}} = e^{2x}\left(\frac{2\cos 3x + 3\sin 3x}{\cancel{4}}\right)$$

$$13I = e^{2x}(2\cos 3x + 3\sin 3x)$$

$$I = \frac{e^{2x}}{13}(2\cos 3x + 3\sin 3x)$$

$$\boxed{I = \frac{3e^{2x}}{13}\left[\frac{2}{3}\cos 3x + \sin 3x\right] + C}$$

* * *

(viii)

$$\int \operatorname{cosec}^3 x \, dx$$

Solution:

$$\int \operatorname{cosec}^3 x \, dx$$

$$I = \int \operatorname{cosec}^3 x \, dx \Rightarrow I = \int \operatorname{cosec}^2 x \cdot \operatorname{cosec} x \, dx$$

1st function: cosec x 2nd function: $\operatorname{cosec}^2 x$

Integration by parts:

$$I = \operatorname{cosec} x \cdot (-\cot x) - \int -\cot x \cdot \frac{d}{dx}(\operatorname{cosec} x)\,dx$$

$$I = -\operatorname{cosec} x \cot x + \int \cot x \cdot (-\operatorname{cosec} x \cot x)\,dx$$

$$I = -\operatorname{cosec} x \cot x + \int -\cot^2 x \operatorname{cosec} x \, dx$$

$$I = -\operatorname{cosec} x \cot x - \int \cot^2 x \operatorname{cosec} x \, dx$$

$\because 1 + \cot^2 x = \operatorname{cosec}^2 x$

$\because \cot^2 x = \operatorname{cosec}^2 x - 1$

$$I = -\operatorname{cosec} x \cot x - \int (\operatorname{cosec}^2 x - 1)\operatorname{cosec} x \, dx$$

$$I = -\operatorname{cosec} x \cot x - \int \operatorname{cosec}^3 x - \operatorname{cosec} x \, dx$$

$$I = -\operatorname{cosec} x \cot x - \int \operatorname{cose}^3 x \, dx + \int \operatorname{cosec} x \, dx$$

$$I = -\operatorname{cosec} x \cot x - I + \int \operatorname{cosec} x \, dx$$

$$I + I = -\operatorname{cosec} x \cot x + \ln|\operatorname{cosec} x - \cot x|$$

$$2I = -\operatorname{cosec} x \cot x + \ln|\operatorname{cosec} x - \cot x|$$

$$I = \frac{-1}{2}\operatorname{cosec} x \cot x + \frac{1}{2}\ln|\operatorname{cosec} x - \cot x|$$

$$\boxed{I = -\frac{1}{2}\left(\operatorname{cosec} x \cot x - \ln|\operatorname{cosec} x - \cot x|\right) + C}$$

* * *

Question #3

Show that $\int e^{ax}\sin bx \, dx = \frac{1}{\sqrt{a^2+b^2}} e^{ax}\sin\left[bx - \tan^{-1}\frac{b}{a}\right] + C$.

Solution:

Let $I = \int e^{ax}\sin bx$

1st function: $\sin bx$ 2nd function: e^{ax}

Integration by parts:

$$= \sin bx \cdot \frac{e^{ax}}{a} - \int \frac{e^{ax}}{a} \cdot \cos bx \cdot (b)\,dx$$

$$= \frac{e^{ax}}{a}\sin bx - \frac{b}{a}\int \underset{II}{e^{ax}}\,\underset{I}{\cos bx}\,dx$$

Again Integration by parts:

$$= \frac{e^{ax}}{a}\sin bx - \frac{b}{a}\left[\cos bx \cdot \frac{e^{ax}}{a} - \int \frac{e^{ax}}{a} \cdot -\sin bx \cdot (b)\,dx\right.$$

$$= \frac{e^{ax}}{a}\sin bx - \frac{b e^{ax}}{a^2}\cos bx - \frac{b^2}{a^2}\int e^{ax}\sin bx \, dx$$

$$I = \frac{e^{ax}}{a}\sin bx - \frac{b}{a^2}e^{ax}\cos bx - \frac{b^2}{a^2}I$$

$$I + \frac{b^2}{a^2}I = \frac{e^{ax}}{a}\sin bx - \frac{b}{a^2}e^{ax}\cos bx$$

$$\left(\frac{a^2+b^2}{a^2}\right)I = e^{ax}\left[\frac{1}{a}\sin bx - \frac{b}{a^2}\cos bx\right]$$

$$\left(\frac{a^2+b^2}{a^2}\right)I = e^{ax}\left(\frac{a\sin bx - b\cos bx}{a^2}\right)$$

$$I = \frac{e^{ax}}{a^2+b^2}\left(\sin bx \cdot a - \cos bx \cdot b\right)$$

Let

$a = r\cos\theta$ and $b = r\sin\theta$

$$\Rightarrow a^2+b^2 = r^2\cos^2\theta + r^2\sin^2\theta$$

$$a^2+b^2 = r^2(\cos^2\theta + \sin^2\theta)$$

$$\because \cos^2\theta + \sin^2\theta = 1$$

$$a^2+b^2 = r^2 \ (1)$$

$$a^2+b^2 = r^2$$

$$\because \frac{\cancel{r}\sin\theta}{\cancel{r}\cos\theta} = \frac{b}{a}$$

$$\because \tan\theta = \frac{b}{a}$$

$$\because \theta = \tan^{-1}\left(\frac{b}{a}\right)$$

$$I = \frac{e^{ax}}{r^2}\left(\sin bx \ r\cos\theta - \cos bx \cdot r\sin\theta\right)$$

$$I = \frac{e^{ax}}{r^2} \cdot r\left(\sin bx\cos\theta - \cos bx \, \sin\theta\right)$$

$$\because \sin\alpha\cos\beta - \cos\alpha\sin\beta = \sin(\alpha-\beta)$$

$$I = \frac{e^{ax}}{r}\sin(bx-\theta)$$

$$\boxed{I = \frac{e^{ax}}{\sqrt{a^2+b^2}}\sin\left(bx - \tan^{-1}\frac{b}{a}\right) + c}$$

Hence proved.

Question # 4

Evaluate the following indefinite integrals.

(i)

$$\sqrt{a^2-x^2}\,dx$$

Solution:

Let

$$I = \sqrt{a^2-x^2} \cdot 1 \, dx$$

1st function $= \sqrt{a^2-x^2}$ 2nd function: 1

Integration by parts:

$$= \sqrt{a^2-x^2} \cdot x - \int x \, \frac{1}{2\sqrt{a^2-x^2}} \, dx(a^2-x^2)\,dx$$

$$= x\sqrt{a^2-x^2} - \int x \cdot \frac{1}{\cancel{2}\sqrt{a^2-x^2}}(-\cancel{2}x)\,dx$$

$$= x\sqrt{a^2-x^2} - \int \frac{-x^2}{\sqrt{a^2-x^2}}\,dx$$

$$= x\sqrt{a^2-x^2} - \int \frac{a^2-x^2-a^2}{\sqrt{a^2-x^2}}\,dx$$

$$= x\sqrt{a^2-x^2} - \int\left(\frac{a^2-x^2}{\sqrt{a^2-x^2}} - \frac{a^2}{\sqrt{a^2-x^2}}\right)dx$$

$$= x\sqrt{a^2-x^2} - \int\sqrt{a^2-x^2}\,dx + a^2\int\frac{1}{\sqrt{a^2-x^2}}\,dx$$

$$I = x\sqrt{a^2-x^2} - I + a^2\int\frac{1}{\sqrt{a^2-x^2}}\,dx$$

$$\because \int\frac{1}{a^2-x^2}\,dx = \sin^{-1}\left(\frac{x}{a}\right)$$

$$I + I = x\sqrt{a^2-x^2} + a^2\sin^{-1}\left(\frac{x}{a}\right)$$

$$2I = x\sqrt{a^2-x^2} + a^2\sin^{-1}\left(\frac{x}{a}\right)$$

$$\boxed{I = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{g}\sin^{-1}\left(\frac{x}{a}\right) + c}$$

(ii)

$$\int\sqrt{x^2-a^2}\,dx$$

Solution:

Let

$$I = \int\sqrt{x^2-a^2} \cdot 1 \, dx$$

1st function $= \sqrt{x^2-a^2}$ 2nd function $= 1$

Integration by parts:

$$I = \sqrt{x^2-a^2} \cdot x - \int x \cdot \frac{d}{dx}\left(\sqrt{x^2-a^2}\right)dx$$

$$I = x\sqrt{x^2-a^2} - \int x \cdot \frac{1}{\cancel{2}\sqrt{x^2-a^2}}(\cancel{2}x)\,dx$$

$$I = x\sqrt{x^2-a^2} - \int \frac{x^2}{\sqrt{x^2-a^2}}\,dx$$

$$I = x\sqrt{x^2-a^2} - \int \frac{x^2-a^2+a^2}{\sqrt{x^2-a^2}}\,dx$$

$$I = x\sqrt{x^2-a^2} - \int\left(\frac{x^2-a^2}{\sqrt{x^2-a^2}} + \frac{a^2}{\sqrt{x^2-a^2}}\right)dx$$

$$I = x\sqrt{x^2-a^2} - \int \sqrt{x^2-a^2}\,dx - \int \frac{a^2}{\sqrt{x^2-a^2}}\,dx$$

$$I = x\sqrt{x^2-a^2} - I - a^2\int \frac{1}{\sqrt{x^2-a^2}}\,dx$$

$$I + I = x\sqrt{x^2-a^2} - a^2\int \frac{1}{\sqrt{x^2-a^2}}\,dx$$

$$\because \int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln|x+\sqrt{x^2-a^2}|$$

$$2I = x\sqrt{x^2-a^2} - a^2\ln|x+\sqrt{x^2-a^2}|$$

$$\boxed{I = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\ln|x+\sqrt{x^2-a^2}| + C}$$

iii

$$\int \sqrt{4-5x^2}\,dx$$

Solution:

Let $I = \int \sqrt{4-5x^2}\cdot 1\,dx$

1st function: $\sqrt{4-5x^2}$ 2nd function: 1

Integration by parts:

$$I = \sqrt{4-5x^2}\;x - \int x\cdot\frac{1}{\cancel{2}\sqrt{4-5x^2}}(-\overset{5}{\cancel{10}}x)\,dx$$

$$I = x\sqrt{4-5x^2} - \int \frac{-5x^2}{\sqrt{4-5x^2}}\,dx$$

$$I = x\sqrt{4-5x^2} - \int \frac{4-5x^2-4}{\sqrt{4-5x^2}}\,dx$$

$$I = x\sqrt{4-5x^2} - \int\left(\frac{4-5x^2}{\sqrt{4-5x^2}} - \frac{4}{\sqrt{4-5x^2}}\right)dx$$

$$I = x\sqrt{4-5x^2} - \int\sqrt{4-5x^2}\,dx + 4\int\frac{1}{\sqrt{4-5x^2}}\,dx$$

$$I = x\sqrt{4-5x^2} - I + 4\int\frac{1}{\sqrt{5\left(\frac{4}{5}-x^2\right)}}\,dx$$

$$I + I = x\sqrt{4-5x^2} + \frac{4}{\sqrt5}\int\frac{1}{\sqrt{\left(\frac{2}{\sqrt5}\right)^2 - x^2}}\,dx$$

$$\because \int\frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right)$$

$$2I = x\sqrt{4-5x^2} + \frac{4}{\sqrt5}\sin^{-1}\frac{x}{\left(\frac{2}{\sqrt5}\right)}$$

$$I = \frac{x}{2}\sqrt{4-5x^2} + \frac{\overset{2}{\cancel{4}}}{\cancel{2}\sqrt5}\sin^{-1}\frac{\sqrt5\,x}{2} + C$$

$$\boxed{I = \frac{x}{2}\sqrt{4-5x^2} + \frac{2}{\sqrt5}\sin^{-1}\frac{\sqrt5\,x}{2} + C}$$

iv

$$\int\sqrt{3-4x^2}\,dx$$

Solution:

Let

$I = \int\sqrt{3-4x^2}\cdot 1\,dx$

1st function: $\sqrt{3-4x^2}$ 2nd function: 1

Integration by parts:

$$I = \sqrt{3-4x^2}\;x - \int x\,\frac{1}{\cancel{2}\sqrt{3-4x^2}}(-\overset{4}{\cancel{8}}x)\,dx$$

$$I = x\sqrt{3-4x^2} - \int\frac{-4x^2}{\sqrt{3-4x^2}}\,dx$$

$$I = x\sqrt{3-4x^2} - \int\frac{3-4x^2-3}{\sqrt{3-4x^2}}\,dx$$

$$I = x\sqrt{3-4x^2} - \int\left(\frac{3-4x^2}{\sqrt{3-4x^2}} - \frac{3}{\sqrt{3-4x^2}}\right)dx$$

$$I = x\sqrt{3-4x^2} - \int\sqrt{3-4x^2}\,dx + 3\int\frac{1}{\sqrt{3-4x^2}}\,dx$$

$$I = x\sqrt{3-4x^2} - I + 3\int\frac{1}{\sqrt{4\left(\frac{3}{4}-x^2\right)}}\,dx$$

$$I + I = x\sqrt{3-4x^2} + \frac{3}{2}\int\frac{1}{\sqrt{\left(\frac{\sqrt3}{2}\right)^2 - x^2}}\,dx$$

$$\because \int\frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}$$

$$2I = x\sqrt{3-4x^2} + \frac{3}{2}\sin^{-1}\frac{x}{\left(\frac{\sqrt3}{2}\right)}$$

$$\boxed{I = \frac{x}{2}\sqrt{3-4x^2} + \frac{3}{4}\sin^{-1}\frac{2x}{\sqrt{3}} + C}$$

(v)

$$\int \sqrt{x^2+4}\,dx$$

Solution:

Let

$$I = \int \sqrt{x^2+4}\cdot 1\,dx$$

1st function: $\sqrt{x^2+4}$ 2nd function: 1

Integration by parts:

$$I = \sqrt{x^2+4}\cdot x - \int x \frac{1}{\cancel{2}\sqrt{x^2+4}}(\cancel{2}x)\,dx$$

$$I = x\sqrt{x^2+4} - \int \frac{x^2}{\sqrt{x^2+4}}\,dx$$

$$I = x\sqrt{x^2+4} - \int \frac{x^2+4-4}{\sqrt{x^2+4}}\,dx$$

$$I = x\sqrt{x^2+4} - \int\left(\frac{x^2+4}{\sqrt{x^2+4}} - \frac{4}{\sqrt{x^2+4}}\right)dx$$

$$I = x\sqrt{x^2+4} - \int\sqrt{x^2+4}\,dx + 4\int\frac{1}{x^2+4}\,dx$$

$$I = x\sqrt{x^2+4} - I + 4\int\frac{1}{\sqrt{x^2+4}}\,dx$$

$$\because \int\frac{1}{\sqrt{x^2+a^2}}\,dx = \ln|x+\sqrt{x^2+a^2}| + C$$

$$I + I = x\sqrt{x^2+4} + 4\ln|x+\sqrt{x^2+4}|$$

$$2I = x\sqrt{x^2+4} + 4\ln|x+\sqrt{x^2+4}|$$

$$I = \frac{x}{2}\sqrt{x^2+4} + \frac{\overset{2}{\cancel{4}}}{\cancel{2}}\ln|x+\sqrt{x^2+4}|$$

$$\boxed{I = \frac{x}{2}\sqrt{x^2+4} + 2\ln|x+\sqrt{x^2+4}| + C}$$

(vi)

$$\int x^2 e^{ax}\,dx$$

Solution:

$$\int x^2 e^{ax}\,dx$$

Let

$$I = \int x^2 e^{ax}\,dx$$

1st function: x^2 2nd function: e^{ax}

Integration by parts:

$$= x^2\cdot\frac{e^{ax}}{a} - \int\frac{e^{ax}}{a}\cdot 2x\,dx$$

$$= \frac{x^2e^{ax}}{a} - \frac{2}{a}\int \underset{\text{II}}{e^{ax}}\cdot \underset{\text{I}}{x}\,dx$$

Again Integration by parts:

$$= \frac{x^2e^{ax}}{a} - \frac{2}{a}\left[x\cdot\frac{e^{ax}}{a} - \int\frac{e^{ax}}{a}\cdot(1)\,dx\right.$$

$$= \frac{x^2e^{ax}}{a} - \frac{2xe^{ax}}{a^2} + \frac{2}{a^2}\int e^{ax}\,dx$$

$$= \frac{x^2}{\alpha}e^{ax} - \frac{2}{a^2}xe^{ax} + \frac{2}{a^2}\cdot\frac{e^{ax}}{a} + C$$

$$= \frac{x^2}{\alpha}e^{ax} - \frac{2}{a^2}xe^{ax} + \frac{2e^{ax}}{a^3} + C$$

$$= \boxed{\frac{e^{ax}}{\alpha}\left[x^2 - \frac{2}{a} + \frac{2}{a^2}\right] + C}$$

Question #5

Evaluate the following integrals.

(i)

$$\int e^{x}\left(\frac{1}{x}+\ln x\right)dx$$

Solution:

$= \int e^{x}\left(\frac{1}{x}+\ln x\right)dx$

$= \int e^{x}\left(\ln x+\frac{1}{x}\right)dx$

$= \int e^{x}\left(\ln x+\frac{d}{dx}(\ln x)\right)dx$

$\because \int e^{x}[f(x)+f'(x)]dx = e^{x}f(x)+c$

$= \boxed{e^{x}\ln x + c}$

(ii)

$$\int e^{x}(\cos x+\sin x)dx$$

Solution:

$= \int e^{x}(\cos x+\sin x)dx$

$= \int e^{x}(\sin x+\cos x)dx$

$= \int e^{x}\left(\sin x+\frac{d}{dx}(\sin x)\right)dx$

$\because \int e^{x}[f(x)+f'(x)]dx = e^{x}f(x)+c$

$= \boxed{e^{x}\sin x + c}$

(iii)

$$\int e^{ax}\left[a\sec^{-1}x+\frac{1}{x\sqrt{x^2-1}}\right]dx$$

Solution:

$= \int e^{ax}\left[a\sec^{-1}x+\frac{1}{x\sqrt{x^2-1}}\right]dx$

$= \int e^{ax}\left[a\sec^{-1}x+\frac{d}{dx}(\sec^{-1}x)\right]dx$

$\because \int e^{ax}[af(x)+f'(x)]dx = e^{ax}f(x)+c$

$= \boxed{e^{ax}\sec^{-1}x + c}$

(iv)

$$\int e^{3x}\left(\frac{3\sin x-\cos x}{\sin^2 x}\right)dx$$

Solution:

$= \int e^{3x}\left(\frac{3\sin x}{\sin^2 x}-\frac{\cos x}{\sin^2 x}\right)dx$

$= \int e^{3x}\left(3\frac{1}{\sin x}-\frac{1}{\sin x}\cdot\frac{\cos x}{\sin x}\right)dx$

$= \int e^{3x}\left[3\operatorname{cosec}x-\operatorname{cosec}x\cot x\right]dx$

$= \int e^{3x}\left[3\operatorname{cosec}x+\frac{d}{dx}(\operatorname{cosec}x)\right]dx$

$\because \int e^{ax}[af(x)+f'(x)]dx = e^{ax}f(x)+c$

$= \boxed{e^{ax}\operatorname{cosec}x + c}$

(v)

$$\int e^{2x}(-\sin x+2\cos x)dx$$

Solution:

$= \int e^{2x}(-\sin x+2\cos x)dx$

$= \int e^{2x}[2\cos x-\sin x]dx$

$= \int e^{2x}\left[2\cos x+\frac{d}{dx}(\cos x)\right]dx$

$\because \int e^{ax}[af(x)+f'(x)]dx = e^{ax}f(x)+c$

$= \boxed{e^{2x}\cos x + c}$

(vi)

$$\int \frac{xe^x}{(1+x)^2}\,dx$$

Solution:

$$= \int \frac{xe^x}{(1+x)^2}\,dx$$

$$= \int e^x \cdot \frac{1+x-1}{(1+x)^2}\,dx$$

$$= \int e^x \cdot \left[\frac{1+x}{(1+x)^2} - \frac{1}{(1+x)^2}\right] dx$$

$$= \int e^x \left[\frac{1}{1+x} - \frac{1}{(1+x)^2}\right] dx$$

$$\because \frac{d}{dx}\left(\frac{1}{1+x}\right) = \frac{d}{dx}(1+x)^{-1} = -1(1+x)^{-1-1}$$

$$= \frac{-1}{(1+x)^2}$$

$$= \int e^x \left[\frac{1}{1+x} + \frac{d}{dx}\left(\frac{1}{1+x}\right)\right] dx$$

$$\because \int e^x \{f(x) + f'(x)] dx = e^x f(x) + c$$

$$= e^x \frac{1}{1+x} + c$$

$$= \boxed{\frac{e^x}{1+x} + c}$$

* * *

(vii)

$$\int e^{-x} (\cos x - \sin x)\,dx$$

Solution:

$$= \int e^{-x} (\cos x - \sin x)\,dx$$

$$= \int e^{-x} (-\sin x + \cos x)\,dx$$

$$= \int e^{-x} [(-1)\sin x + \cos x]\,dx$$

$$= \int e \left[(-1\sin x) + \frac{d}{dx}(\sin x)\right] dx$$

$$\because \int e^{ax} [a f(x) + f'(x)] dx = e^{ax} f(x) + c$$

$$= \boxed{e^{-x}\sin x + c}$$

* * *

(viii)

$$\int \frac{e^{m\,Tan^{-1}x}}{1+x^2}\,dx$$

Solution:

$$\int e^{m\,Tan^{-1}x} \cdot \frac{1}{1+x^2}\,dx$$

put

$$t = Tan^{-1}x$$

$$dt = \frac{1}{1+x^2}\,dx$$

$$= \int e^{mt}\,dt$$

$$= \frac{e^{mt}}{m} + c \qquad \because t = Tan^{-1}x$$

$$= \frac{e^{m\,Tan^{-1}x}}{m} + c$$

$$= \boxed{\frac{1}{m} e^{m\,Tan^{-1}x} + c}$$

* * *

(ix)

$$\int \frac{2x}{1-\sin x}\,dx$$

Solution:

$$= \int \frac{2x}{1-\sin x}\,dx$$

$$\because \cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$$

$$= \int \frac{2x}{1-\cos\left(\frac{\pi}{2} - x\right)}$$

$\because 1-\cos\theta = 2\sin^2\frac{\theta}{2}$

$$= \int \frac{2x}{2\sin^2\frac{(\frac{\pi}{2}-x)}{2}}dx$$

$$= \int \frac{\cancel{2}x}{\cancel{2}\sin^2\left(\frac{\pi}{4}-\frac{x}{2}\right)}dx$$

$$= \int x\,\text{cosec}^2\left(\frac{\pi}{4}-\frac{x}{2}\right)dx$$

$$= x\left[-\frac{\cot\left(\frac{\pi}{4}-\frac{x}{2}\right)}{\frac{-1}{2}}\right] - \int -\frac{\cot\left(\frac{\pi}{4}-\frac{x}{2}\right)}{\frac{-1}{2}}.(1)\,dx$$

$$= x\;2\cot\left(\frac{\pi}{4}-\frac{x}{2}\right) - 2\int\cot\left(\frac{\pi}{4}-\frac{x}{2}\right)dx$$

$$= 2x\cot\left(\frac{\pi}{4}-\frac{x}{2}\right) - 2\int\frac{\cos\left(\frac{\pi}{4}-\frac{x}{2}\right)}{\sin\left(\frac{\pi}{4}-\frac{x}{2}\right)}dx$$

$$= 2x\cot\left(\frac{\pi}{4}-\frac{x}{2}\right) - 2(-2)\int\frac{\cos\left(\frac{\pi}{4}-\frac{x}{2}\right)\left(\frac{-1}{2}\right)}{\sin\left(\frac{\pi}{4}-\frac{x}{2}\right)}dx$$

$\because \int\frac{f'(x)}{f(x)}dx = \ln|f(x)| + c$

$$= \boxed{2x\cot\left(\frac{\pi}{4}-\frac{x}{2}\right) + 4\ln\left|\sin\left(\frac{\pi}{4}-\frac{x}{2}\right)\right| + c}$$

(x)

$$\int e^x\,\frac{1+x}{(2+x)^2}dx$$

Solution

$$= \int e^x\,\frac{2+x-1}{(2+x)^2}dx$$

$$= \int e^x\left(\frac{2+x}{(2+x)^2} - \frac{1}{(2+x)^2}\right)dx$$

$$= \int e^x\left[\frac{1}{2+x} - \frac{1}{(2+x)^2}\right]dx$$

$$= \int e^x\left[\frac{1}{2+x} + \frac{d}{dx}\left(\frac{1}{2+x}\right)\right]dx$$

$\because \int e^x[f(x)+f'(x)]dx = e^x f(x)+c$

$$= \boxed{e^x\cdot\frac{1}{2+x} + c}$$

(xi)

$$\int\left(\frac{1-\sin x}{1-\cos x}\right)e^x\,dx$$

$$= \int\left(\frac{1-\sin x}{1-\cos x}\right)e^x\,dx$$

$$= \int e^x\cdot\frac{\left(1-2\sin\frac{x}{2}\cos\frac{x}{2}\right)}{2\sin^2\frac{x}{2}}dx$$

$$= \int e^x\left[\frac{1}{2\sin^2\frac{x}{2}} - \frac{\cancel{2}\cancel{\sin\frac{x}{2}}\cos\frac{x}{2}}{\cancel{2}\sin^{\cancel{2}}\frac{x}{2}}\right]dx$$

$$= \int e^x\left[\frac{1}{2}\text{cosec}^2\frac{x}{2} - \cot\frac{x}{2}\right]dx$$

$$= \int e^x\left[-\cot\frac{x}{2} + \frac{1}{2}\text{cosec}^2\frac{x}{2}\right]dx$$

$$= \int e^x\left[-\cot\frac{x}{2} + \frac{d}{dx}\left(-\cot\frac{x}{2}\right)\right]dx$$

$\because \int e^x[f(x)+f'(x)]dx = e^x f(x)+c$

$$= e^x\left(-\cot\frac{x}{2}\right) + c$$

$$= \boxed{-e^x\cot\frac{x}{2} + c}$$

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Theory

Integration involving Partial Fraction:

If $P(x)$, $Q(x)$ are polynomial functions and the denominator ($Q(x) \neq 0$), in the rational function $\frac{P(x)}{Q(x)}$, can be factorized into linear and quadratic (irreducible) factors, then the rational function is written as a sum of simpler rational functions, each of which can be integrated by methods of partial fraction which already known to us. Here we will give examples of the following three cases when the denominator $Q(x)$ contains

Case I. Non-repeated linear factors

$$\frac{x+7}{(x-1)(x+3)} = \frac{A}{x-1} + \frac{B}{x+3}$$

Case II. Repeated and non-repeated linear factors

$$\frac{5x+7}{(x-1)(x+3)^2} = \frac{A}{x-1} + \frac{B}{x+3} + \frac{C}{(x+3)^2}$$

Case III. Linear and non-repeated irreducible quadratic factors or non-repeated irreducible quadratic factors.

$$\frac{11x+3}{(x-3)(x^2+9)} = \frac{A}{x-3} + \frac{Bx+C}{x^2+9}$$

Example # 1: Evaluate

$$\int \frac{-x+6}{2x^2-7x+6}\,dx \quad (x>2)$$

$$= \int \frac{-x+6}{2x^2-4x-3x+6}\,dx$$

$$= \int \frac{-x+6}{2x(x-2)-3(x-2)}\,dx$$

$$= \int \frac{-x+6}{(x-2)(2x-3)}\,dx$$

Let

$$\frac{-x+6}{(x-2)(2x-3)} = \frac{A}{x-2} + \frac{B}{2x-3} \quad \text{— (i)}$$

Multiply by $(x-2)(2x-3)$

$$-x+6 = A(2x-3) + B(x-2) \quad \text{— (ii)}$$

Put $x-2=0 \Rightarrow x=2$ in (ii)

$$-2+6 = A(2(2)-3) + B(2-2)$$

$$4 = A(4-3) + B(0)$$

$$4 = A(1) + 0$$

$$\boxed{A = 4}$$

Put $2x-3=0 \Rightarrow x = 3/2$ in (ii)

$$-\frac{3}{2}+6 = A\left[2\left(\frac{3}{2}\right)-3\right] + B\left(\frac{3}{2}-2\right)$$

$$\frac{-3+12}{2} = A\left[3-3\right] + B\left(\frac{3-4}{2}\right)$$

$$\frac{9}{2} = A(0) + B\left(-\frac{1}{2}\right)$$

$$B = \frac{9}{2} \times -2$$

$$\boxed{B = -9}$$

So (i) become:

$$\frac{-x+6}{(x-2)(x+2)} = \frac{4}{x-2} - \frac{9}{2x-3}$$

$$\int \frac{-x+6}{(x-2)(2x-3)}\,dx = 4\int \frac{1}{x-2}\,dx - 9\int \frac{1}{2x-3}\,dx$$

$$= 4\int \frac{1}{x-2}\,dx - \frac{9}{2}\int \frac{2}{2x-3}\,dx$$

$$= 4\ln|x-2| - \frac{9}{2}\ln|2x-3| + c$$

Example # 2: Evaluate

$$\int \frac{2x^3-9x^2+12x}{2x^2-7x+6}\,dx, \ (x>1)$$

$$= \int \frac{2x^3-9x^2+12x}{2x^2-7x+6}\,dx$$

$$\begin{array}{r|l} & \quad x-1 \\ \hline 2x^2-7x+6 & \cancel{2x^3}-9x^2+12x \\ & \underset{-}{\cancel{2x^3}}\ \underset{+}{-}7x^2\ \underset{-}{+}6x \\ \hline & \quad -\cancel{2x^2}+6x \\ & \quad \underset{+}{-}\cancel{2x^2}\ \underset{-}{+}7x\ \underset{+}{-}6 \\ \hline & \qquad -x+6 \end{array}$$

Thus

$$\int \frac{2x^3-9x^2+12x}{2x^2-7x+6}\,dx = \int (x-1) + \frac{(6-x)}{2x^2-7x+6}\,dx$$

$$\because 2x^2-7x+6 = 2x^2-4x-3x+6 = 2x(x-2)-3(x-2) = (x-2)(2x-3)$$

Now

$$\frac{-x+6}{(x-2)(2x-3)} = \frac{A}{x-2} + \frac{B}{2x-3} \quad \text{— (i)}$$

Multiply by $(x-2)(2x-3)$, we get

$$-x+6 = A(2x-3) + B(x-2) \quad \text{— (ii)}$$

Put $x-2=0 \Rightarrow x=2$ in (ii)

$$-2+6 = A(2(2)-3) + B(2-2)$$

$$4 = A(4-3) + B(0)$$

$$4 = A(1) + 0$$

$$\boxed{A = 4}$$

Put $2x-3=0 \Rightarrow x=\frac{3}{2}$ in (ii)

$$-\frac{3}{2}+6 = A\left[2\left(\frac{3}{2}\right)-3\right] + B\left[\frac{3}{2}-2\right]$$

$$\frac{-3+12}{2} = A(3-3) + B\left(\frac{3-4}{2}\right)$$

$$\frac{9}{2} = A(0) + B\left(-\frac{1}{2}\right)$$

$$\frac{9}{2} = 0 - \frac{1}{2}B$$

$$B = -\cancel{2}\cdot\frac{9}{\cancel{2}}$$

$$\boxed{B = -9}$$

So

$$\frac{-x+6}{(x-2)(2x-3)} = \frac{4}{x-2} - \frac{9}{2x-3}$$

Now

$$\int\frac{2x^3-9x^2+12x}{2x^2-7x+6}dx = \int\left[x-1+\frac{4}{x-2}-\frac{9}{2x-3}\right]dx$$

$$= \int x\,dx - \int 1dx + 4\int\frac{1}{x-2}dx - 9\int\frac{1}{2x-3}dx$$

$$= \frac{x^2}{2} - x + 4\ln|x-2| - \frac{9}{2}\int\frac{2}{2x-3}dx$$

$$= \frac{x^2}{2} - x + 4\ln|x-2| - \frac{9}{2}\ln|2x-3| + c$$

Example # 3: Evaluate

i) $\int\frac{2a}{x^2-a^2}dx$; $(x>a)$

let

$\because x^2-a^2 = (x-a)(x+a)$

$$\frac{2a}{(x-a)(x+a)} = \frac{A}{x-a} + \frac{B}{x+a} \text{ — (i)}$$

Multiply by $(x-a)(x+a)$, we get

$$2a = A(x+a) + B(x-a) \text{ — (ii)}$$

Put $x+a=0 \Rightarrow x=-a$ in (ii)

$$2a = A(-a+a) + B(-a-a)$$

$$2a = 0 + B(-2a)$$

$$\cancel{2a} = -\cancel{2a}B$$

$$\boxed{B = -1}$$

Put $x-a=0 \Rightarrow x=a$ in (ii)

$$2a = A(a+a) + B(a-a)$$

$$2a = 2a\,A + 0$$

$$\boxed{A = 1}$$

So

$$\frac{2a}{(x-a)(x+a)} = \frac{1}{x-a} - \frac{1}{x+a}$$

$$\int\frac{2a}{(x-a)(x+a)}dx = \int\frac{1}{x-a}dx - \int\frac{1}{x+a}dx$$

$$= \ln|x-a| - \ln|x+a| + c$$

$$= \ln\left|\frac{x-a}{x+a}\right| + c$$

(ii) $\int\frac{2a}{a^2-x^2}dx$

let

$$\frac{2a}{(a-x)(a+x)} = \frac{A}{a-x} + \frac{B}{a+x} \text{ — (i)}$$

Multiply by $(a-x)(a+x)$, we get

$$2a = A(a+x) + B(a-x) \text{ — (ii)}$$

Put $a+x=0 \Rightarrow x=-a$ in (ii)

$$2a = A(a+(-a)) + B(a-(-a))$$

$$2a = A(a-a) + B(a+a)$$

$$\cancel{2a} = 0 + \cancel{2a}B$$

$$\boxed{B = 1}$$

Put $a-x=0 \Rightarrow x=a$ in (ii)

$$2a = A(a+a) + B(a-a)$$

$$2a = 2a\,A + B(0)$$

$$\cancel{2a} = \cancel{2a}A$$

$$\boxed{A = 1}$$

So

$$\frac{2a}{(a-x)(a+x)} = \frac{1}{a-x} + \frac{1}{a+x}$$

$$\int\frac{2a}{(a-x)(a+x)}dx = \int\frac{1}{a-x}dx + \int\frac{1}{a+x}dx$$

$$= -\int \frac{-1}{a-x}\,dx + \int \frac{1}{a+x}\,dx$$

$$= -\ln|a-x| + \ln|a+x| + c$$

$$= \ln\left|\frac{a+x}{a-x}\right| + c$$

Example # 4: Evaluate

$$\int \frac{7x-1}{(x-1)^2(x+1)}\,dx \quad (x>1)$$

Let

$$\frac{7x-1}{(x-1)^2(x+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1} \text{ —(i)}$$

Multiply by $(x-1)^2(x+1)$

$$7x-1 = A(x-1)(x+1) + B(x+1) + C(x-1)^2 \text{ —(ii)}$$

Put $x-1=0 \Rightarrow x=1$ in (ii)

$$7(1)-1 = A(1-1)(1+1) + B(1+1) + C(1-1)^2$$

$$7-1 = 0 + 2B + 0$$

$$6 = 2B$$

$$\boxed{B=3}$$

Put $x+1=0 \Rightarrow x=1$ in (ii)

$$7(-1)-1 = A(-1-1)(-1+1) + B(-1+1) + C(-1-1)^2$$

$$-7-1 = 0 + 0 + C(-2)^2$$

$$-8 = 4C$$

$$\boxed{C=-2}$$

From (ii)

$$7x-1 = A(x^2-1) + B(x+1) + C(x^2-2x+1)$$

$$7x-1 = Ax^2 - A + Bx + B + Cx^2 - 2Cx + C$$

Equating coefficient of 'x^2'

$$A + C = 0$$

$$A - 2 = 0$$

$$\boxed{A = 2}$$

So

$$\frac{7x-1}{(x-1)^2(x+1)} = \frac{2}{x-1} + \frac{3}{(x-1)^2} - \frac{2}{x+1}$$

$$\int \frac{7x-1}{(x-1)^2(x+1)}\,dx = 2\int \frac{1}{x-1}\,dx + 3\int \frac{1}{(x-1)^2}\,dx - 2\int \frac{1}{x+1}\,dx$$

$$= 2\ln|x-1| + 3\int (x-1)^{-2}\,dx - 2\ln|x+1|$$

$$= 2\ln|x-1| + 3\cdot\frac{(x-1)^{-1}}{-1} - 2\ln|x+1|$$

$$= 2\ln\left|\frac{x-1}{x+1}\right| - \frac{3}{x-1} + c$$

Example # 5: Evaluate

$$\int \frac{e^x(x^2+1)}{(x+1)^2}\,dx$$

$$= \int \frac{e^x(x^2+1)}{x^2+1+2x}$$

$$\begin{array}{r|l} & \quad 1 \\ \hline x^2+1+2x & x^2+1 \\ & \cancel{x^2} \cancel{\pm 1} \pm 2x \\ \hline & \quad -2x \end{array}$$

$$= \int e^x\left(1 - \frac{2x}{(x+1)^2}\right)dx$$

$$= \int e^x - \frac{e^x\,2x}{(x+1)^2}\,dx$$

$$= \int e^x\,dx - 2e^x\int \frac{x}{(x+1)^2}\,dx \text{ —(I)}$$

Now

$$\frac{x}{(x+1)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2} \text{ —(i)}$$

Multiply by $(x+1)^2$

$$x = A(x+1) + B \text{ — (ii)}$$

Put $x+1=0 \Rightarrow x=-1$ in (ii)

$$-1 = A(-1+1) + B$$

$$-1 = 0 + B$$

$$\boxed{B=-1}$$

From (ii)

$$x = Ax + A + B$$

Equating coefficient of 'x'

$$\boxed{1 = A}$$

So (i)

$$\int \frac{x}{(x+1)^2} dx = \int \frac{1}{x+1} dx - \int \frac{1}{(x+1)^2} dx$$

So Ⓘ

$$= \int e^x dx - 2e^x \left[\int \frac{1}{x+1} dx - \int \frac{1}{(x+1)^2} dx \right]$$

$$= e^x - 2\int \frac{e^x}{x+1} dx + 2\int \frac{e^x}{(x+1)^2} dx$$

$$= e^x - 2\int \frac{e^x}{x+1} dx + 2\int \underset{I}{e^x} \cdot \underset{II}{(x+1)^{-2}} dx$$

$$= e^x - 2\int \frac{e^x}{x+1} dx + 2\left[e^x \cdot \frac{(x+1)^{-2+1}}{-2+1} - \int \frac{(x+1)^{-2+1}}{-2+1} \cdot e^x dx \right]$$

$$= e^x - 2\int \frac{e^x}{x+1} dx + 2\left[e^x \cdot \frac{(x+1)^{-1}}{-1} - \int \frac{(x+1)^{-1}}{-1} \cdot e^x dx \right]$$

$$= e^x - \cancel{2\int \frac{e^x}{x+1} dx} - \frac{2e^x}{x+1} \cancel{+2\int \frac{-e^x}{x+1} dx}$$

$$= e^x - \frac{2e^x}{x+1} + C$$

$$= \frac{e^x x + e^x - 2e^x}{x+1} + C$$

$$= \frac{e^x x - e^x}{x+1} + C$$

$$= \frac{e^x(x-1)}{x+1} + C$$

Example # 6: Evalute

$$\int \frac{1}{x^3-1} dx$$

$$= \int \frac{1}{x^3-1} dx = \int \frac{1}{(x-1)(x^2+1+x)} dx$$

$$\therefore (x)^3 - (1)^3 = (x-1)(x^2+1+2x)$$

Now

$$\frac{1}{(x-1)(x^2+1+x)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1+x} \quad \text{—(i)}$$

Multiply by $(x-1)(x^2+1+x)$, we get

$$1 = A(x^2+1+x) + (Bx+C)(x-1) \quad \text{—(ii)}$$

Put $x-1=0 \Rightarrow x=1$ in (ii)

$$1 = A[(1)^2+(1)+(1)] + (B(1)+C)(1-1)$$

$$1 = A(1+1+1) + (B+C)(0)$$

$$1 = 3A + 0$$

$$\boxed{A = \frac{1}{3}}$$

from (ii)

$$1 = Ax^2 + A + Ax + Bx^2 - Bx + Cx - C$$

Equating coefficient of 'x^2'

$$0 = A + B$$

$$0 = \frac{1}{3} + B$$

$$\boxed{B = -\frac{1}{3}}$$

Equating coefficient of 'x'

$$0 = A - B + C$$

$$0 = \frac{1}{3} - \left(\frac{1}{3}\right) + C$$

$$0 = \frac{1}{3} + \frac{1}{3} + C$$

$$0 = \frac{1+1}{3} + C$$

$$0 = \frac{2}{3} + C$$

$$\boxed{C = -\frac{2}{3}}$$

Thus

$$\frac{1}{x^3-1} = \frac{1}{3(x-1)} + \frac{-1/3 x - 2/3}{x^2+x+1}$$

$$= \frac{1}{3}\cdot\frac{1}{x-1} - \frac{1}{3}\left(\frac{x+2}{x^2+x+1}\right)$$

$$= \frac{1}{3}\cdot\frac{1}{x-1} - \frac{1}{2\times3}\left(\frac{2x+4}{x^2+x+1}\right)$$

$$= \frac{1}{3}\cdot\frac{1}{x-1} - \frac{1}{6}\left(\frac{2x+1+3}{x^2+x+1}\right)$$

$$= \frac{1}{3}\cdot\frac{1}{x-1} - \frac{1}{6}\cdot\frac{2x+1}{x^2+x+1} - \frac{1}{6}\cdot\frac{3}{x^2+x+1}$$

$$= \frac{1}{3}\cdot\frac{1}{x-1} - \frac{1}{6}\cdot\frac{2x+1}{x^2+x+1} - \frac{1}{2}\cdot\frac{1}{x^2+x+1}$$

$$= \frac{1}{3}\cdot\frac{1}{x-1} - \frac{1}{6}\cdot\frac{2x+1}{x^2+x+1} - \frac{1}{2}\cdot\frac{1}{x^2+x+\frac{1}{4}+\frac{3}{4}}$$

$$= \frac{1}{3}\cdot\frac{1}{x-1} - \frac{1}{6}\cdot\frac{2x+1}{x^2+x+1} - \frac{1}{2}\cdot\frac{1}{(x+\frac{1}{2})^2+(\frac{\sqrt3}{2})^2}$$

$$= \frac{1}{3}\int\frac{1\,dx}{x-1} - \frac{1}{6}\int\frac{2x+1\,dx}{x^2+x+1} - \frac{1}{2}\int\frac{1}{(x+\frac{1}{2})^2+(\frac{\sqrt3}{2})^2}dx$$

$$= \frac{1}{3}\ln|x-1| - \frac{1}{6}\ln|x^2+x+1|$$

$$- \frac{1}{2}\cdot\frac{1}{\sqrt3/2}\tan^{-1}\left(\frac{x+1/2}{\sqrt3/2}\right)$$

$$\because \int\frac{1}{x^2+a^2}dx = \frac{1}{a}\tan^{-1}\frac{x}{a}+c$$

$$= \frac{1}{3}\ln|x-1| - \frac{1}{6}\ln|x^2+x+1| - \frac{1}{\sqrt3}\tan^{-1}\left(\frac{2x+1}{\sqrt3}\right)+c$$

Example # 7: Evaluate

$$\int\frac{2x}{x^6-1}dx$$

$$= \int\frac{2x}{x^6-1}dx$$

$$= \int\frac{2x}{(x^2)^3-1}dx$$

Put $x^2 = t$

$2x dx = dt$

$$= \int\frac{dt}{t^3-1} = \int\frac{dt}{(t-1)(t^2+1+t)}$$

Now

$$\frac{1}{(t-1)(t^2+1+t)} = \frac{A}{t-1} + \frac{Bt+C}{t^2+1+t} \quad \text{— (i)}$$

Multiply by $(t-1)(t^2+1+t)$, we get

$$1 = A(t^2+1+t) + (Bt+C)(t-1) \quad \text{—(ii)}$$

Put $t-1=0 \Rightarrow t=1$ in (ii)

$$1 = A[(1)^2+1+1] + (B(1)+C)(1-1)$$

$$1 = A(1+1+1) + (B+C)(0)$$

$$1 = 3A + 0$$

$$\boxed{A = \frac{1}{3}}$$

From (ii)

$$1 = At^2 + A + At + Bt^2 - Bt + Ct - C$$

Equating coefficient of 't^2'

$$0 = A + B$$

$$0 = \frac{1}{3} + B$$

$$\boxed{B = -\frac{1}{3}}$$

Equating coefficient of 't'

$$0 = A - B + C$$

$$0 = \frac{1}{3} - \left(-\frac{1}{3}\right) + C$$

$$0 = \frac{1}{3} + \frac{1}{3} + C$$

$$0 = \frac{2}{3} + C$$

$$\boxed{C = -\frac{2}{3}}$$

Thus

$$\frac{1}{t^3-1} = \frac{1}{3(t-1)} + \frac{-1/3t - 2/3}{t^2+t+1}$$

$$= \frac{1}{3}\cdot\frac{1}{t-1} - \frac{1}{3}\left(\frac{t+2}{t^2+t+1}\right)$$

$$= \frac{1}{3}\cdot\frac{1}{t-1} - \frac{1}{3\times 2}\left(\frac{2t+4}{t^2+t+1}\right)$$

$$= \frac{1}{3}\cdot\frac{1}{t-1} - \frac{1}{6}\left(\frac{2t+1+3}{t^2+t+1}\right)$$

$$= \frac{1}{3}\cdot\frac{1}{t-1} - \frac{1}{6}\left(\frac{2t+1}{t^2+t+1}\right) - \frac{1}{6}\left(\frac{3}{t^2+t+1}\right)$$

$$= \frac{1}{3}\cdot\frac{1}{t-1} - \frac{1}{6}\left(\frac{2t+1}{t^2+t+1}\right) - \frac{1}{2}\left(\frac{1}{t^2+t+1}\right)$$

$$= \frac{1}{3}\cdot\frac{1}{t-1} - \frac{1}{6}\left(\frac{2t+1}{t^2+t+1}\right) - \frac{1}{2}\left(\frac{1}{t^2+t+\frac{1}{4}+\frac{3}{4}}\right)$$

$$= \frac{1}{3}\cdot\frac{1}{t-1} - \frac{1}{6}\left(\frac{2t+1}{t^2+t+1}\right) - \frac{1}{2}\left(\frac{1}{t^2+t+\left(\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right)$$

$$= \frac{1}{3}\cdot\frac{1}{t-1}\cdot\frac{1}{6}\left(\frac{2t+1}{t^2+t+1}\right) - \frac{1}{2}\left(\frac{1}{\left(t+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right)$$

Thus

$$\int\frac{dt}{t^3-1} = \frac{1}{3}\int\frac{1}{t-1}dt - \frac{1}{6}\int\frac{2t+1}{t^2+t+1}dt - \frac{1}{2}\int\frac{1}{\left(t+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}dt$$

$$= \frac{1}{3}\ln|t-1| - \frac{1}{6}\ln|t^2+t+1|$$

$$-\frac{1}{2}\cdot\frac{1}{\sqrt{3}/2}\tan^{-1}\left(\frac{t+1/2}{\sqrt{3}/2}\right)+C$$

$$= \frac{1}{3}\ln|t-1| - \frac{1}{6}\ln|t^2+t+1|$$

$$-\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2t+1}{\sqrt{3}}\right)+C$$

Replacing t by x^2

$$\int\frac{dx}{x^6-1} = \frac{1}{3}\ln|x^2-1| - \frac{1}{6}|x^4+x^2+1|$$

$$-\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x^2+1}{\sqrt{3}}\right)+C$$

Example # 8: Evaluate

$$\int\frac{3}{x(x^3-1)}dx \quad x \neq 0, \quad x \neq -1$$

$$= \int\frac{3}{x(x-1)(x^2+x+1)}dx$$

Now

$$\frac{3}{x(x-1)(x^2+x+1)} = \frac{A}{x} + \frac{B}{x-1} + \frac{Cx+D}{x^2+x+1}$$

Multiply by $x(x-1)(x^2+x+1)$

$$3 = A(x-1)(x^2+x+1) + B(x)(x^2+x+1) + (Cx+D)(x)(x-1) \quad \text{—(i)}$$

Put $x=0$ in (i)

$$3 = A(0-1)[(0)^2+0+1] + B(0)(0^2+0+1) + [C(0)+D](0)(0-1)$$

$$3 = A(-1)(1) + 0 + 0$$

$$-A = 3$$

$$\boxed{A = -3}$$

Put $x-1=0 \Rightarrow x=1$ in (i)

$$3 = A(1-1)(1^2+1+1) + B(1)(1^2+1+1) + [C(1)+D](1)(1-1)$$

$$3 = A(0)(1) + B(1)(3) + (C+D)(0)$$

$$3 = 0 + 3B + 0$$

$$3B = 3$$

$$\boxed{B = 1}$$

From (i)

$$3 = A(x^3-1) + Bx^3 + Bx^2 + Bx + (Cx+D)(x^2-x)$$

$3 = Ax^3 - A + Bx^3 + Bx^2 - Bx + Cx^3 - Cx^2 + Dx^2 - Dx$

Equating coefficient of "x^3"

$0 = A + B + C$

$0 = -3 + 1 + C$

$0 = -2 + C$

$\boxed{C = 2}$

Equating coefficient of 'x^2'

$0 = B - C + D$

$0 = 1 - 2 + D$

$0 = -1 + D$

$\boxed{D = 1}$

Thus

$$\frac{3}{x(x^3-1)} = \frac{-3}{x} + \frac{1}{x-1} + \frac{2x+1}{x^2+x+1}$$

$$\int \frac{3}{x(x^3-1)}dx = -3\int \frac{1}{x}dx + \int \frac{1}{x-1}dx + \int \frac{2x+1}{x^2+x+1}dx$$

$$= -3\ln x + \ln|x-1| + \ln|x^2+x+1| + c$$

$$= -3\ln x + \ln(x-1)(x^2+x+1) + c$$

$$= -3\ln x + \ln(x^3-1) + c$$

$$\int \frac{3}{x(x^3-1)}dx = -3\ln x + \ln(x^3-1) + c$$

Example # 9: Evaluate

$$\int \frac{2x^2+6x}{(x^2+1)(x^2+2x+3)}dx$$

$$= \int \frac{2x^2+6x}{(x^2+1)(x^2+2x+3)}dx$$

Now

$$\frac{2x^2+6x}{(x^2+1)(x^2+2x+3)} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2+2x+3}$$

Multiply by $(x^2+1)(x^2+2x+3)$

$2x^2+6x = (Ax+B)(x^2+2x+3) + (Cx+D)(x^2+1)$

$2x^2+6x = Ax^3 + 2Ax^2 + 3Ax + Bx^2 + 2Bx + 3B$
$\quad + Cx^3 + Cx + Dx^2 + D$

Equating coefficient of "x^3", "x^2", "x" and constant.

For x^3; $0 = A + C$ —(i)

For x^2; $2 = 2A + B + D$ —(ii)

For x; $6 = 3A + 2B + C$ —(iii)

For constant term $0 = 3B + D$ —(iv)

By (iii) - (ii)

$$6 = 3A + 2B + C$$
$$\underset{-}{}2 = \underset{-}{}2A \underset{-}{+} B \quad \underset{-}{+} D$$
$$4 = A + B + C - D$$

$A + C - D = 4 - B$ —(v)

By (i) - (iv)

$$0 = A + C$$
$$0 = \underset{-}{3B} \underset{-}{+} D$$
$$A + C - 3B - D = 0$$

$A + C - D = 3B$ —(vi)

Comparing (v) & (vi)

$$3B = 4 - B$$

$$3B + B = 4$$

$$4B = 4$$

$$\boxed{B = 1}$$

Put in (iv)

$$0 = 3(1) + D$$

$$0 = 3 + D$$

$$\boxed{D = -3}$$

Put $B=1$ & $D=-3$ in (ii)

$$2 = 2A + 1 - 3$$

$$2 = 2A - 2$$

$$2 + 2 = 2A$$

$$4 = 2A$$

$$\boxed{A = 2}$$

Put $A=2$ in (i)

$$0 = 2 + C$$

$$\boxed{C = -2}$$

So

$$\frac{2x^2+6x}{(x^2+1)(x^2+2x+3)} = \frac{2x+1}{x^2+1} + \frac{-2x-3}{x^2+2x+3}$$

$$= \frac{2x}{x^2+1} + \frac{1}{x^2+1} - \frac{2x+3}{x^2+2x+3}$$

$$= \frac{2x}{x^2+1} + \frac{1}{x^2+1} - \frac{2x+2+1}{x^2+2x+3}$$

$$= \frac{2x}{x^2+1} + \frac{1}{x^2+1} - \frac{2x+2}{x^2+2x+3} - \frac{1}{x^2+2x+3}$$

$$\int \frac{2x^2+6x}{(x^2+1)(x^2+2x+1)}dx = \int \frac{2x}{x^2+1}dx + \int \frac{1}{x^2+1}dx - \int \frac{2x+2}{x^2+2x+3}dx - \int \frac{1}{x^2+2x+1+2}dx$$

$$= \ln|x^2+1| + \tan^{-1}x - \ln|x^2+2x+3| - \int \frac{1}{(x+1)^2+(\sqrt{2})^2}dx$$

$$= \ln|x^2+1| + \tan^{-1}x - \ln|x^2+2x+3| - \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x+1}{\sqrt{2}}\right) + C$$

$$\because \int \frac{1}{a^2+x^2}dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c$$

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Exercise # 3·5

Evaluate the following integrals:

Question #1

$$\int \frac{3x+1}{x^2-x-6}\,dx$$

Solution:

$$\int \frac{3x+1}{x^2-x-6}\,dx$$

$\because\ x^2-x-6$

$= x^2-3x+2x-6$

$= x(x-3)+2(x-3)$

$= (x-3)(x+2)$

$\frac{3x+1}{(x-3)(x+2)} = \frac{A}{x-3} + \frac{B}{x+2}$ —— ①

Multiply by $(x-3)(x+2)$ on both sides

$3x+1 = A(x+2)+B(x-3)$ —— ②

$x-3=0 \Rightarrow \boxed{x=3}$ put in eq ②

$3(3)+1 = A(3+2)+B(3-3)$

$9+1 = A(5)+0$

$10 = 5A$

$\frac{{}^{2}\cancel{10}}{\cancel{5}} = A$

$\boxed{A=2}$

$x+2=0 \Rightarrow \boxed{x=-2}$ put in eq ②

$3(g)+1 = A(-2+2)+B(-2-3)$

$-6+1 = 0+B(-5)$

$-5 = -5B$

$\frac{\cancel{-5}}{\cancel{-5}} = B$

$\boxed{B=1}$

$\frac{3x+1}{(x-3)(x+2)} = \frac{2}{x-3} + \frac{1}{x+2}$

$\int \frac{3x+1}{(x-3)(x+2)}dx = \int \frac{2}{x-3}dx + \int \frac{1}{x+2}dx$

$\int \frac{3x+1}{(x-3)(x+2)}dx = 2\int \frac{1}{x-3}dx + \int \frac{1}{x+2}dx$

$\because \int \frac{f'(x)}{f(x)}dx = \ln f(x)+c$

$$\boxed{\int \frac{3x+1}{(x-3)(x+2)}dx = 2\ln|x-3| + \ln|x+2| + C}$$

Question #2

$$\int \frac{5x+8}{(x+3)(2x-1)}\,dx$$

Solution:

$$\int \frac{5x+8}{(x+3)(2x-1)}\,dx$$

$\frac{5x+8}{(x+3)(2x-1)} = \frac{A}{x+3} + \frac{B}{(2x-1)}$ —— ①

Multiply by $(x+3)(2x-1)$ on both sides

$5x+8 = A(2x-1)+B(x+3)$ —— ②

$x+3=0 \Rightarrow \boxed{x=-3}$ put in eq ②

$5(-3)+8 = A(2(-3)-1)+B(-3+3)$

$-15+8 = A(-6-1)+0$

$-7 = -7A$

$\frac{\cancel{-7}}{\cancel{-7}} = A$

$\boxed{A=1}$

$2x-1=0 \Rightarrow x=\frac{1}{2}$ put in eq ②

$5\left(\frac{1}{2}\right)+8 = 0+B\left(\frac{1}{2}+3\right)$

$\frac{5}{2}+8 = B\left(\frac{1+6}{2}\right)$

$\frac{5+16}{2} = B\left(\frac{7}{2}\right)$

${}^{3}\frac{\cancel{21}}{\cancel{2}} \times \frac{\cancel{2}}{\cancel{7}} = B$

$\boxed{B=3}$

$$\frac{5x+8}{(x+3)(2x-1)} = \frac{1}{x+3} + \frac{3}{(2x-1)}\,dx$$

$$\int \frac{5x+8}{(x+3)(2x-1)}\,dx = \int \frac{1}{x+3}\,dx + 3\int \frac{1}{2x-1}\,dx$$

$$\int \frac{5x+8}{(x+3)(2x-1)}\,dx = \int \frac{1}{x+3}\,dx + \frac{3}{2}\int \frac{2}{2x-1}\,dx$$

$$\because \int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C$$

$$\boxed{= \int \frac{5x+8}{(x+3)(2x-1)}\,dx = \ln|x+3| + \frac{3}{2}\ln|2x-1| + C}$$

Question # 3

$$\int \frac{x^2+3x-34}{x^2+2x-15}\,dx$$

Solution:

$$= \int \frac{x^2+3x-34}{x^2+2x-15}\,dx$$

$$\begin{array}{r|l} & 1 \\ \hline x^2+2x-15 & x^2+3x-34 \\ & \underset{-}{x^2} \underset{-}{+2x} \underset{+}{-15} \\ \hline & x-19 \\ \hline \end{array}$$

$$= \int \left[1 + \frac{x-19}{x^2+2x-15}\right] dx$$

$$= \int 1\,dx + \int \frac{x-19}{x^2+2x-15}\,dx \text{ —— (I)}$$

$\because x^2+2x-15$

$= x^2+5x-3x-15$

$= x(x+5)-3(x+5)$

$= (x-3)(x+5)$

$$\frac{x-19}{(x-3)(x+5)} = \frac{A}{x-3} + \frac{B}{x+5} \text{ —— ①}$$

Multiply by $(x-3)(x+5)$ on both sides

$$x-19 = A(x+5)+B(x-3) \text{ —— ②}$$

$x+5=0 \Rightarrow \boxed{x=-5}$ put in eq ②

$$-5-19 = A(-5+5)+B(-5-3)$$

$$-24 = 0(-8)B$$

$$\frac{\cancel{-24}^{3}}{\cancel{-8}} = B$$

$$\boxed{B=3}$$

$x-3=0 \Rightarrow \boxed{x=3}$ put in eq ②

$$3-19 = A(3+5)+B(3-3)$$

$$-16 = A(8)+0$$

$$\frac{\cancel{-16}^{2}}{\cancel{8}} = A$$

$$\boxed{A=-2}$$

$$\frac{x-19}{(x-3)(x+5)} = \frac{-2}{x-3} + \frac{3}{x+5}$$

Thus eq (I) becomes

$$= \int 1\,dx + \int \frac{-2}{x-3}\,dx + 3\int \frac{1}{x+5}\,dx$$

$$= x - 2\int \frac{1}{x-3}\,dx + 3\int \frac{1}{x+5}\,dx$$

$$= \boxed{x-2\ln|x-3|+3\ln|x+5|+C}$$

Question # 4

$$\int \frac{(a-b)x}{(x-a)(x-b)}\,dx, \quad (a>b)$$

Solution:

$$\int \frac{(a-b)x}{(x-a)(x-b)}\,dx$$

$$\frac{(a-b)x}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b} \text{ —— ①}$$

Multiply by $(x-a)(x-b)$ on both sides

$$(a-b)x = A(x-b)+B(x-a) \text{ —— ②}$$

$x-a=0 \Rightarrow \boxed{x=a}$ put in eq ②

$$(a-b)a = A(a-b)+B(a-a)$$

$$\cancel{(a-b)}a = A\cancel{(a-b)}+0$$

$$\boxed{A=a}$$

$x - b = 0 \Rightarrow \boxed{x = b}$ put in eq ②

$(a-b)\,b = A\,(b-b) + B\,(a-b)$

$\cancel{(a-b)}\,b = 0 + B\,\cancel{(a-b)}$

$\boxed{B = b}$

$$\frac{(a-b)x}{(x-a)(x-b)} = \frac{a}{x-a} + \frac{b}{x-b}$$

$$\int \frac{(a-b)x}{(x-a)(x-b)}\,dx = a\int \frac{1}{x-a}\,dx + b\int \frac{1}{x-b}\,dx$$

$$\boxed{\int \frac{(a-b)x}{(x-a)(x-b)}\,dx = a\ln|x-a| + b\ln|x-b| + C}$$

Question #5

$$\int \frac{3-x}{1-x-6x^2}\,dx$$

Solution:

$$= \int \frac{3-x}{1-x-6x^2}\,dx$$

$\because\ 1 - x - 6x^2$

$= -6x^2 - x + 1$

$= -6x^2 - 3x + 2x + 1$

$= -3x(2x+1) + 1(2x+1)$

$= (2x+1)(1-3x)$

$$\frac{3-x}{(2x+1)(1-3x)} = \frac{A}{2x+1} + \frac{B}{1-3x} \quad \text{—— ①}$$

Multiply by $(2x+1)(1-3x)$ on both sides

$3 - x = A\,(1-3x) + B\,(2x+1)$ —— ②

$2x + 1 = 0 \Rightarrow \boxed{x = \frac{-1}{2}}$ put in eq ②

$3 - \left(\frac{-1}{2}\right) + A\left(1 - 3\left(\frac{-1}{2}\right) + B\,(0)\right.$

$3 + \frac{1}{2} = A\left(1 + \frac{3}{2}\right) + 0$

$\frac{6+1}{2} = A\left(\frac{2+3}{2}\right) + 0$

$\frac{7}{\cancel{2}} = A\left(\frac{5}{\cancel{2}}\right)$

$\boxed{\frac{7}{5} = A}$

$1 - 3x = 0 \Rightarrow 1 = 3x \Rightarrow \boxed{x = \frac{1}{3}}$
put in eq ②

$3 - \frac{1}{3} = 0 + B\left(2\left(\frac{1}{3}\right) + 1\right)$

$\frac{9-1}{3} = B\left(\frac{2}{3} + 1\right)$

$\frac{8}{\cancel{3}} = B\left(\frac{2+3}{\cancel{3}}\right)$

$8 = B(5)$

$\boxed{B = \frac{8}{5}}$

$$\frac{3-x}{(2x+1)(1-3x)} = \frac{7}{5(2x+1)} + \frac{8}{5(1-3x)}$$

$$\int \frac{3-x}{(2x+1)(1-3x)}\,dx = \frac{7}{5}\int \frac{1}{2x+1}\,dx + \frac{8}{5}\int \frac{1}{1-3x}\,dx$$

$$= \frac{7}{5 \cdot 2}\int \frac{2}{2x+1}\,dx + \frac{8}{5}\cdot\frac{1}{-3}\int \frac{-3}{1-3x}\,dx$$

$$= \boxed{\frac{7}{10}\ln|2x+1| - \frac{8}{15}\ln|1-3x| + C}$$

Question #6

$$\int \frac{2x}{x^2 - a^2}\,dx$$

Solution:

$$\int \frac{2x}{x^2-a^2}\,dx \Rightarrow \int \frac{2x}{(x-a)(x+a)}\,dx$$

$$\frac{2x}{(x-a)(x+a)} = \frac{A}{x-a} + \frac{B}{x+a} \quad \text{—— ①}$$

Multiply by $(x-a)(x+a)$ on both sides

$2x = A(x+a) + B(x-a)$

$x + a = 0 \Rightarrow \boxed{x = -a}$ put in eq ②

$2(-a) = A(-a+a) + B(-a-a)$

$-2a = 0 + B(-2a)$

$\frac{-2\cancel{a}}{\cancel{-2a}} = B$

$\boxed{B = 1}$

$x - a = 0 \Rightarrow \boxed{x = a}$ put in eq ②

$2(a) = A(a+a) + B(a-a)$

$2a = A(2a) + 0$

$\frac{2\cancel{a}}{\cancel{2a}} = A$

$\boxed{A = 1}$

$\frac{2x}{(x-a)(x+a)} = \frac{1}{x-a} + \frac{1}{x+a}$

$\int \frac{2x}{(x-a)(x+a)} dx = \int \frac{1}{x-a} dx + \int \frac{1}{x+a}$

$\int \frac{2x}{(x-a)(x+a)} dx = \ln|x-a| + \ln|x+a|$

$\because \ln m + \ln n = \ln m \times \ln n$

$\int \frac{2x}{(x-a)(x+a)} dx = \ln|(x-a)(x+a)| + C$

$\because (x-a)(x+a) = x^2 - a^2$

$$\boxed{\int \frac{2x}{(x-a)(x+a)} dx = \ln|x^2 - a^2| + C}$$

Question # 7

$\int \frac{1}{6x^2+5x-4} dx$

Solution:

$\int \frac{1}{6x^2+5x-4} dx$

$\because 6x^2+5x-4$

$= 6x^2 - 3x + 8x - 4$

$= 3x(2x-1) + 4(2x-1)$

$= (2x-1)(3x+4)$

$\frac{1}{(2x-1)(3x+4)} = \frac{A}{2x-1} + \frac{B}{3x+4}$ —— ①

Multiply by $(2x-1)(3x+4)$ on both sides

$1 = A(3x+4) + B(2x-1)$ —— ②

$2x - 1 = 0 \Rightarrow \boxed{x = \frac{1}{2}}$ put in eq ②

$1 = A\left(3\left(\frac{1}{2}\right) + 4\right) + 0$

$1 = A\left(\frac{3}{2} + 4\right) + 0$

$1 = A\left(\frac{3+8}{2}\right)$

$1 = A\left(\frac{11}{2}\right)$

$\boxed{A = \frac{2}{11}}$

$3x + 4 = 0 \Rightarrow \boxed{x = -\frac{4}{3}}$ put in eq ②

$1 = 0 + B\left(2\left(-\frac{4}{3}\right) - 1\right)$

$1 = B\left(\frac{-8-3}{3}\right)$

$1 = B\left(\frac{-11}{3}\right)$

$\boxed{B = \frac{-3}{11}}$

$\frac{1}{(2x-1)(3x+4)} = \frac{2}{11(2x-1)} - \frac{3}{11(3x+4)}$

$\int \frac{1}{6x^2+5x-4} dx = \frac{2}{11} \int \frac{2}{2x-1} dx - \frac{1}{11} \int \frac{3}{3x+4} dx$

$= \frac{1}{11} \ln|2x-1| - \frac{1}{11} \ln|3x+4| + C$

$\because \ln m - \ln n = \ln\left|\frac{m}{n}\right|$

$$= \boxed{\frac{1}{11} \ln\left|\frac{2x-1}{3x+4}\right| + C}$$

Question #8

$$\int \frac{2x^3-3x^2-x-7}{2x^2-3x-2}\,dx$$

Solution:

$$\int \frac{2x^3-3x^2-x-7}{2x^2-3x-2}\,dx$$

$$\begin{array}{r} x \\ 2x^2-3x-2\,\overline{)\,2x^3-3x^2-x-7} \\ \underline{\mp 2x^3 \pm 3x^2 \pm 2x} \\ x-7 \end{array}$$

$$= \int \left(x + \frac{x-7}{2x^2-3x-2}\right) dx$$

$$= \int x\,dx + \int \frac{x-7}{2x^2-3x-2}\,dx \text{ —— (I)}$$

$\because 2x^2-3x-2$

$= 2x^2-4x+x-2$

$= 2x(x-2)+1(x-2)$

$= (x-2)(2x+1)$

$$\frac{x-7}{(x-2)(2x+1)} = \frac{A}{x-2} + \frac{B}{2x+2} \text{ —— ①}$$

Multiply by $(x-2)(2x+1)$ on both sides

$x-7 = A(2x+1) + B(x-2)$ —— ②

$x-2=0 \Rightarrow \boxed{x=2}$ put in eq ②

$2-7 = A(2(2)+1) + B(2-2)$

$-5 = A(4+1) + 0$

$-5 = A(5)$

$\frac{-\cancel{5}}{\cancel{5}} = A$

$\boxed{A=-1}$

$2x+1=0 \Rightarrow \boxed{x=\frac{-1}{2}}$ put in eq ②

$\frac{-1}{2} - 7 = 0 + B\left(\frac{-1}{2} - 2\right)$

$\frac{-1-14}{\cancel{2}} = B\left(\frac{-1-4}{\cancel{2}}\right)$

$-15 = -5B$

$\frac{-\cancel{15}^{3}}{-\cancel{5}} = B$

$\boxed{B=3}$

$$\frac{x-7}{(x-2)(2x+1)} = \frac{-1}{x-2} + \frac{3}{2x+1}$$

$$\int \frac{x-7}{2x^2-3x-2}\,dx = \int \frac{-1}{x-2}\,dx + \int \frac{3}{2x+1}\,dx$$

Thus eq (I) becomes

$$= \int x\,dx - 1\int \frac{1}{x-2}\,dx + \frac{3}{2}\int \frac{2}{2x+1}\,dx$$

$$\boxed{= \frac{x^2}{2} - \ln|x-2| + \frac{3}{2}\ln|2x+1| + C}$$

Question #9

$$\int \frac{3x^2-12x+11}{(x-1)(x-2)(x-3)}\,dx$$

Solution:

$$\int \frac{3x^2-12x+11}{(x-1)(x-2)(x-3)}\,dx$$

$$\frac{3x^2-12x+11}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3} \text{ —— ①}$$

Multiply by $(x-1)(x-2)(x-3)$ on both sides

$3x^2-12x+11 = A(x-2)(x-3)+B(x-1)(x-3)+C(x-1)(x-2)$ —— ②

$x-1=0 \Rightarrow \boxed{x=1}$ put in eq ②

$3(1)^2-12(1)+11 = A(1-2)(1-3) + 0 + 0$

$3-12+11 = A(-1)(-2)$

$2 = 2A$

$\frac{\cancel{2}}{\cancel{2}} = A$

$\boxed{A=1}$

$x-2=0 \Rightarrow \boxed{x=2}$ put in eq ②

$3(2)^2-12(2)+11 = 0 + B(2-1)(2-3) + 0$

$3(4)-24+11 = B(1)(-1)$

$12-24+11 = -B$

$-1 = -B \quad \boxed{B=1}$

$x-3=0 \Rightarrow \boxed{x=3}$ put in eq ②

$3(3)^2-12(3)+11=0+0+C(3-1)(3-2)$

$3(9)-36+11 = C(2)(1)$

$27-36+11 = 2C$

$2 = 2C$

$\frac{\cancel{2}}{\cancel{2}} = C$

$\boxed{C=1}$

$$\frac{3x^2-12x+11}{(x-1)(x-2)(x-3)} = \frac{1}{x-1}+\frac{1}{x-2}+\frac{1}{x-3}$$

$$\int\frac{3x^2-12x+11}{(x-1)(x-2)(x-3)}dx = \int\frac{1}{x-1}dx+\int\frac{1}{x-2}dx+\int\frac{1}{x-3}dx$$

$$=\boxed{\ln|x-1|+\ln|x-2|+\ln|x-3|+C}$$

Question # 10

$$\int\frac{2x-1}{x(x-1)(x-3)}dx$$

Solution:

$$\int\frac{2x-1}{x(x-1)(x-3)}dx$$

$$\frac{2x-1}{x(x-1)(x-3)} = \frac{A}{x}+\frac{B}{x-1}+\frac{C}{x-3} \quad — ①$$

Multiply by $x(x-1)(x-3)$ on both sides

$2x-1 = A(x-1)(x-3)+B(x-3)x+C(x-1)x$ — ②

$\boxed{x=0}$ put in eq ②

$2(0)-1 = A(0-1)(0-3)+0+0$

$0-1 = A(-1)(-3)$

$-1 = 3A$

$\boxed{A=-\frac{1}{3}}$

$x-1=0 \Rightarrow \boxed{x=1}$ put in eq ②

$2(1)-1 = 0+B(1)(1-3)$

$2-1 = B(1)(-2)$

$1 = -2B$

$\boxed{B=-\frac{1}{2}}$

$x-3=0 \Rightarrow \boxed{x=3}$ put in eq ②

$2(3)-1 = 0+0+C(3)(3-1)$

$6-1 = C\,3(2)$

$5 = (6)C$

$\boxed{C=\frac{5}{6}}$

$$\frac{2x-1}{x(x-1)(x-3)} = \frac{-1}{3(x)}-\frac{1}{2(x-1)}+\frac{5}{6(x-3)}$$

$$\int\frac{2x-1}{x(x-1)(x-3)}dx = \frac{-1}{3}\int\frac{1}{x}dx-\frac{1}{2}\int\frac{1}{x-1}dx+\frac{5}{6}\int\frac{1}{x-3}dx$$

$$=\boxed{-\frac{1}{3}\ln|x|-\frac{1}{2}\ln|x-1|+\frac{5}{6}\ln|x-3|+C}$$

Question # 11

$$\int\frac{5x^2+9x+6}{(x^2-1)(2x+3)}dx$$

Solution:

$$\int\frac{5x^2+9x+6}{(x^2-1)(2x+3)}dx$$

$$\frac{5x^2+9x+6}{(x-1)(x+1)(2x+3)} = \frac{A}{x-1}+\frac{B}{x+1}+\frac{C}{2x+3} \quad — ①$$

Multiply by $(x-1)(x+1)(2x+3)$ on both sides

$5x^2+9x+6 = A(x+1)(2x+3)+B(x-1)(2x+3)+C(x-1)(x+1)$ — ②

$x-1=0 \Rightarrow \boxed{x=1}$ put in eq ②

$5(1)^2+9(1)+6 = A(1+1)(2(1)+3)+0+0$

$5+9+6 = A(2)(2+3)$

$20 = A(10)$

$\frac{\cancel{20}^{2}}{\cancel{10}} = A$

$\boxed{A=2}$

$x+1=0 \Rightarrow \boxed{x=-1}$ put in eq ②

$5(-1)^2+9(-1)+6 = 0+B(-1-1)(2(-1)+3)$

$5(1)-9+6 = B(-2)(-2+3)$

$5-9+6 = B(-2)(1)$

$2 = -2B$

$\frac{\cancel{2}}{\cancel{2}} = B$

$\boxed{B = 1}$

$2x+3=0 \Rightarrow \boxed{x = -\frac{3}{2}}$ put in eq ②

$5\left(\frac{-3}{2}\right)^2 + 9\left(\frac{-3}{2}\right) + 6 = 0+0+C\left(\frac{-3}{2}-1\right)\left(\frac{-3}{2}+1\right)$

$5\left(\frac{9}{4}\right) - \frac{27}{2} + 6 = C\left(\frac{-3-2}{2}\right)\left(\frac{-3+2}{2}\right)$

$\frac{45}{4} - \frac{27}{2} + 6 = C\left(-\frac{5}{2}\right)\left(\frac{-1}{2}\right)$

$\frac{45 - 54 + 24}{4} = C\left(\frac{5}{4}\right)$

$15 = = C(5)$

$\frac{{}^{3}\cancel{15}}{\cancel{5}} = C$

$\boxed{C = 3}$

$$\frac{5x^2+9x+6}{(x-1)(x+1)(2x+3)} = \frac{2}{x-1} + \frac{1}{x+1} + \frac{3}{2x+3}$$

$$\int \frac{5x^2+9x+6}{(x^2-1)(2x+3)}dx = 2\int\frac{1}{x-1}dx + \int\frac{1}{x+1}dx + 3\int\frac{1}{2x+3}dx$$

$$= 2\int\frac{1}{x-1}dx + \int\frac{1}{x+1}dx + \frac{3}{2}\int\frac{2}{2x+3}dx$$

$$= \boxed{2\ln|x-1| + \ln|x+1| + \frac{3}{2}\ln|2x+3| + C}$$

Question #12

$$\int\frac{4+7x}{(1+x)^2(2+3x)}dx$$

Solution:

$$\int\frac{4+7x}{(1+x)^2(2+3x)}dx$$

$$\frac{4+7x}{(1+x)^2(2+3x)} = \frac{A}{1+x} + \frac{B}{(1+x)^2} + \frac{C}{2+3x} \quad \text{——①}$$

Multiply by $(1+x)^2(2+3x)$ on bothsides

$4+7x = A(1+x)(2+3x) + B(2+3x) + C(1+x)^2$ —②

$1+x=0 \Rightarrow \boxed{x=-1}$ put in eq ②

$4+7(-1) = 0 + B(2+3(-1)$

$4-7 = B(2-3)$

$-3 = B(-1)$

$\boxed{B = 3}$

$2+3x = 0 \Rightarrow \boxed{x = \frac{-2}{3}}$ put in eq ②

$4+7\left(\frac{-2}{3}\right) = 0+0+C\left(1-\frac{2}{3}\right)^2$

$4 - \frac{14}{3} = C\left(\frac{3-2}{3}\right)^2$

$\frac{12-14}{3} = C\left(\frac{1}{3}\right)^2$

$\frac{-2}{3} = \left(\frac{1}{\cancel{9}^{3}}\right)C$

$\boxed{C = -6}$

From eq ②

$4+7x = A(2+3x+2x+3x^2) + 2B + 3Bx + C(1+x^2+2x)$

$4+7x = 2A + 5Ax + 3Ax^2 + 2B + 3Bx + Cx^2 + C + 2Cx$

Comparing Coefficient of "x^2"

$3A + C = 0$

$3A - 6 = 0$

$3A = 6$

$A = \frac{\cancel{6}^{2}}{\cancel{3}}$

$\boxed{A = 2}$

$$\frac{4+7x}{(1+x)^2(2+3x)} = \frac{2}{1+x} + \frac{3}{(1+x)^2} - \frac{6}{(2+3x)}$$

$$\int\frac{4+7x}{(1+x)^2(2+3x)}dx = 2\int\frac{1}{1+x} + 3\int(1+x)^{-2}dx - 6\int\frac{1}{2+3x}dx$$

$\because \int x^n dx = \frac{x^{n+1}}{n+1} + c$

$$= 2\int\frac{1}{1+x}dx + 3\,\frac{(1+x)^{-2+1}}{-2+1} - \frac{\cancel{6}^{2}}{\cancel{3}}\int\frac{3}{2+3x}dx$$

$\because \int\frac{f'(x)}{f(x)}dx = \ln|f(x)| + c$

$$= 2\ln|1+x| + 3\,\frac{(1+x)^{-1}}{-1} - 2\ln|2+3x|$$

$$= \boxed{2\ln|1+x) - \frac{3}{1+x} - 2\ln|2+3x| + C}$$

Question # 13

$$\int \frac{2x^2}{(x-1)^2(x+1)}dx$$

Solution:

$$\int \frac{2x^2}{(x-1)^2(x+1)}dx$$

$$\frac{2x^2}{(x-1)^2(x+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1} \quad \text{——} ①$$

Multiply by $(x-1)^2(x+1)$ on both sides

$$2x^2 = A(x-1)(x+1) + B(x+1) + C(x-1)^2 \quad \text{——} ②$$

$x-1=0 \Rightarrow \boxed{x=1}$ put in eq ②

$$2(1)^2 = 0 + B(1+1) + 0$$

$$2 = B(2)$$

$$\frac{\cancel{2}}{\cancel{2}} = B$$

$$\boxed{B=1}$$

$x+1=0 \Rightarrow \boxed{x=-1}$ put in eq ②

$$2(-1)^2 = 0 + 0 + C(-1-1)^2$$

$$2(1) = C(-2)^2$$

$$2 = 4C$$

$$\frac{\cancel{2}}{\cancel{4}_2} = C$$

$$\boxed{C = \frac{1}{2}}$$

From eq ②

$$2x^2 = A(x^2-1) + Bx + B + C(x^2+1-2x)$$

$$2x^2 = Ax^2 - A + Bx + B + Cx^2 + C - 2Cx$$

Comparing Coefficient of 'x^2'

$$A + C = 2$$

$$A + \frac{1}{2} = 2$$

$$A = 2 - \frac{1}{2}$$

$$A = \frac{4-1}{2}$$

$$\boxed{A = \frac{3}{2}}$$

$$\frac{2x^2}{(x-1)^2(x+1)} = \frac{3}{2(x-1)} + \frac{1}{(x-1)^2} + \frac{1}{2(x+1)}$$

$$\int \frac{2x^2}{(x-1)^2(x+1)}dx = \int \frac{3}{2(x-1)}dx + \int \frac{1}{(x-1)^2}dx + \int \frac{1}{2(x+1}dx$$

$$\int \frac{2x^2}{(x-1)^2(x+1)}dx = \frac{3}{2}\int \frac{1}{x-1}dx + \int (x-1)^{-2}dx + \frac{1}{2}\int \frac{1}{x+1}dx$$

$$\int \frac{2x^2}{(x-1)^2(x+1)} = \frac{3}{2}\ln|x-1| + \frac{(x-1)^{-1}}{-1} + \frac{1}{2}\ln|x+1| + c$$

$$= \boxed{\frac{3}{2}\ln|x-1| - \frac{1}{x-1} + \frac{1}{2}\ln|x+1| + c}$$

Question # 14

$$\int \frac{1}{(x-1)(x+1)^2}dx$$

Solution:

$$\int \frac{1}{(x-1)(x+1)^2}dx$$

$$\frac{1}{(x-1)(x+1)^2} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{(x+1)^2} \quad \text{——} ①$$

Multiply by $(x-1)(x+1)^2$ on both sides

$$1 = A(x+1)^2 + B(x-1)(x+1) + C(x-1) \quad \text{——} ②$$

$x-1=0 \Rightarrow \boxed{x=1}$ put in eq ②

$$1 = A(1+1)^2 + 0 + 0$$

$$1 = A(2)^2$$

$$1 = 4A$$

$$\boxed{A = \frac{1}{4}}$$

$x+1=0 \Rightarrow \boxed{x=-1}$ put in eq ②

$$1 = A(-1+1)^2 + B(-1-1)(-1+1) + C(-1-1)$$

$$1 = 0 + 0 + C(-2)$$

$$\boxed{C = -\frac{1}{2}}$$

From eq ②

$$1 = A(x^2+1+2x) + B(x^2+\cancel{x}-\cancel{x}-1) + Cx - C$$

$$1 = Ax^2 + A + 2Ax + Bx^2 - B + Cx - C$$

Comparing Coefficient of 'x^2':

$$A + B = 0$$

$$\frac{1}{4} + B = 0$$

$$\boxed{B = -\frac{1}{4}}$$

$$\int \frac{1}{(x-1)(x+1)^2} dx = \int \frac{1}{4(x-1)} dx - \int \frac{1}{4(x+1)} dx - \int \frac{1}{2(x+1)^2} dx$$

$$= \frac{1}{4}\int \frac{1}{x-1} dx - \frac{1}{4}\int \frac{1}{x+1} dx - \frac{1}{2}\int (x+1)^{-2} dx$$

$$= \frac{1}{4}\ln|x-1| - \frac{1}{4}\ln|x+1| - \frac{1}{2}\cdot\frac{(x+1)^{-2+1}}{-2+1} + C$$

$$= \frac{1}{4}\ln|x-1| - \frac{1}{4}\ln|x+1| - \frac{1}{2}\ \frac{(x+1)^{-1}}{-1} + C$$

$$= \boxed{\frac{1}{4}\ln|x-1| - \frac{1}{4}\ln|x+1| + \frac{1}{2(x+1)} + C}$$

Question # 15

$$\int \frac{x+4}{x^3-3x^2+4} dx$$

Solution:

$$\int \frac{x+4}{x^3-3x^2+4} dx$$

$\because x^3-3x^2+4$

$= x^3+x^2-4x^2+4$

$= x^2(x+1)-4(x^2-1)$

$= x^2(x+1)-4(x+1)(x-1)$

$= (x+1)(x^2-4x-4)$

$= (x+1)(x-2)^2$

$$\frac{x+4}{(x+1)(x-2)^2} = \frac{A}{x+1} + \frac{B}{x-2} + \frac{C}{(x-2)^2} \quad \text{—①}$$

Multiply by $(x+1)(x-2)^2$ on both sides

$x+4 = A(x-2)^2 + B(x+1)(x-2) + C(x+1)$ —②

$x+1=0 \Rightarrow \boxed{x=-1}$ put in eq ②

$-1+4 = A(-1-2)^2 + 0 + 0$

$3 = A(-3)^2$

$3 = A(9)$

$\frac{\cancel{3}}{\cancel{9}_3} = A$

$\boxed{A = \frac{1}{3}}$

$x-2=0 \Rightarrow \boxed{x=2}$ put in eq ②

$2+4 = 0+0+C(2+1)$

$6 = C(3)$

$\frac{\overset{2}{\cancel{6}}}{\cancel{3}} = C$

$\boxed{C=2}$

From eq ②

$x+4 = A(x^2-4x+4) + B(x^2-2x+x-2) + Cx + C$

$x+4 = Ax^2-4Ax+4A + Bx^2 - Bx - 2B + Cx + C$

Comparing Coefficient of 'x^2'

$A+B=0$

$\frac{1}{3}+B=0$

$\boxed{B = -\frac{1}{3}}$

$$\frac{x+4}{(x+1)(x-2)^2} = \frac{1}{3(x+1)} - \frac{1}{3(x-2)} + \frac{2}{(x-2)^2}$$

$$\int \frac{x+4}{(x+1)(x-2)^2} dx = \int \frac{1}{3(x+1)} dx - \frac{1}{3}\int \frac{1}{x-2} dx + 2\int (x-2)^{-2} dx$$

$$= \frac{1}{3}\int \frac{1}{x+1} dx - \frac{1}{3}\int \frac{1}{x-2} dx + 2\cdot\frac{(x-2)^{-2+1}}{-2+1} + C$$

$$= \frac{1}{3}\ln|x+1| - \frac{1}{3}\ln|x-2| + 2\ \frac{(x-2)^{-1}}{-1} + C$$

$$= \boxed{\frac{1}{3}\left\{\ln|x+1| - \ln|x-2|\right\} - \frac{2}{x-2} + C}$$

Question # 16

$$\int \frac{x^3-6x^2+25}{(x+1)^2(x-2)^2} dx$$

Solution:

$$\int \frac{x^3-6x^2+25}{(x+1)^2(x-2)^2} dx$$

$$\frac{x^3-6x^2+25}{(x+1)^2(x-2)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x-2} + \frac{D}{(x-2)^2} \quad \text{—①}$$

Multiply by $(x+1)^2(x-2)^2$ on both sides

$x^3-6x^2+25 = A(x+1)(x-2)^2 + B(x-2)^2 + C(x+1)^2(x-2) + D(x+1)^2$ —②

$x+1=0 \Rightarrow \boxed{x=-1}$ put in eq ②

$$(-1)^3 - 6(-1)^2 + 25 = 0 + B(-1-2)^2 + 0 + 0$$

$$-1 - 6(1) + 25 = B(-3)^2$$

$$-1 - 6 + 25 = B(9)$$

$$18 = 9B$$

$$^{2}\frac{\cancel{18}}{\cancel{9}} = B$$

$$\boxed{B = 2}$$

$x - 2 = 0 \Rightarrow \boxed{x = 2}$ put in eq ②

$$(2)^3 - 6(2)^2 + 25 = 0 + 0 + 0 + D(2+1)^2$$

$$8 - 6(4) + 25 = D(3)^2$$

$$8 - 24 + 25 = 9D$$

$$9 = 9D$$

$$\frac{\cancel{9}}{\cancel{9}} = D$$

$$\boxed{D = 1}$$

From eq ②

$$x^3 - 6x^2 + 25 = A(x+1)(x^2 - 4x + 4) + B(x^2 - 4x + 4)$$
$$+ C(x^2 + 1 + 2x)(x - 2) + D(x^2 + 1 + 2x)$$

$$x^3 - 6x^2 + 25 = A(x^3 - 4x^2 + \cancel{4x} + x^2 - \cancel{4x} + 4) + Bx^2 - 4Bx + 4B$$
$$+ C(x^3 + x + 2\cancel{x^2} - 2x^2 - 2 - 4x) + Dx^2 + D + 2Dx$$

$$x^3 - 6x^2 + 25 = Ax^3 - 3Ax^2 + 4A + Bx^2 - 4Bx + 4B$$
$$+ Cx^3 - 3Cx - 2C + Dx^2 + 2Dx + D$$

Comparing Coefficient of "x^3" and "x^2"

"x^3" $A + C = 1$ —— ③

"x^2" $-3A + B + D = -6$

$$-3A + 2 + 1 = -6$$

$$-3A + 3 = -6$$

$$-3A = -6 - 3$$

$$A = \frac{\cancel{-9}\,3}{\cancel{-3}}$$

$\boxed{A = 3}$ put in eq ③

$$3 + C = 1$$

$$C = 1 - 3$$

$$\boxed{C = -2}$$

$$\frac{x^3 - 6x^2 + 25}{(x+1)^2(x-2)^2} = \frac{3}{x+1} + \frac{2}{(x+1)^2} - \frac{2}{x-2} + \frac{1}{(x-2)^2}$$

$$\int \frac{x^3 - 6x^2 + 25}{(x+1)^2(x-2)^2}\,dx = \int \frac{3}{x+1}\,dx + 2\int (x+1)^{-2}\,dx - \int \frac{2}{x-2}\,dx$$
$$+ 1\int (x-2)^{-2}\,dx$$

$$= 3\int \frac{1}{x+1}\,dx + 2\cdot\frac{(x+1)^{-1}}{-1} - 2\int \frac{1}{x-2}\,dx + 1\,\frac{(x-2)^{-1}}{-1}$$

$$= \boxed{3\ln|x+1| - \frac{2}{x+1} - 2\ln|x-2| - \frac{1}{x-2} + C}$$

Question # 17

$$\int \frac{x^3 + 22x^2 + 14x - 17}{(x-3)(x+2)^3}\,dx$$

Solution:

$$\int \frac{x^3 + 22x^2 + 14x - 17}{(x-3)(x+2)^3}\,dx$$

$$\frac{x^3 + 22x^2 + 14x - 17}{(x-3)(x+2)^3} = \frac{A}{x-3} + \frac{B}{x+2} + \frac{C}{(x+2)^2} + \frac{D}{(x+2)^3} \quad \text{—①}$$

Multiply by $(x-3)(x+2)^3$ on both sides

$$x^3 + 22x^2 + 14x - 17 = A(x+2)^3 + B(x-3)(x+2)^2 + C$$
$$(x-3)(x+2) + D(x-3) \quad \text{—— ②}$$

$x - 3 = 0 \Rightarrow \boxed{x = 3}$ put in eq ②

$$(3)^3 + 22(3)^2 + 14(3) - 17 = A(3+2)^3 + 0 + 0 + C$$

$$27 + 22(9) + 42 - 17 = A(5)^3$$

$$27 + 198 + 42 - 17 = A(125)$$

$$250 = (125)A$$

$$\frac{^{2}\cancel{250}}{\cancel{125}_{1}} = A$$

$$\boxed{A = 2}$$

$x + 2 = 0 \Rightarrow \boxed{x = -2}$ put in eq ②

$$(-2)^3 + 22(-2)^2 + 14(-2) - 17 = 0 + 0 + 0 + D(-2-3)$$

$$8 + 22(4) - 28 - 17 = D(-5)$$

$$8 + 88 - 28 - 17 = D(-5)$$

$35 = (-5)D$

$\frac{^{7}\cancel{35}}{-\cancel{5}} = D$

$\boxed{D = -7}$

From eq ②

$x^3+22x^2+14x-17 = A(x^3+6x^2+12x+8) + B(x-3)(x^2+2x+4)$
$+C(x^2-3x+2x-6) + Dx - 3D$

$x^3+22x^2+14x-17 = Ax^3+6Ax^2+12Ax+8A+B(x^3+4x^2+4x$
$-3x^2-12x-12+Cx^2-8x-6C+Dx-3D$

$x^3+22x^2+14x-17 = Ax^3-6Ax^2+12Ax+8A+Bx^3+Bx^2-8B$
$-12B+Cx^2-Cx-6C+Dx-3D$

Comparing Coefficient of "x^3" and "x^2"

"x^3"

$A + B = 1$

$2 + B = 1$

$B = 1-2$

$\boxed{B = -1}$

"x^2"

$+6A + B + C = 22$

$+6(2)+(-1)+C = 22$

$+12 - 1 + C = 22$

$+11 + C = 22$

$C = 22-11$

$\boxed{C = 11}$

$$\frac{x^3+22x^2+14x-17}{(x-3)(x+2)^3} = \frac{2}{x-3} - \frac{1}{x+2} + \frac{11}{(x+2)^2} - \frac{7}{(x+2)^3}$$

$$\int \frac{x^3+22x^2+14x-17}{(x-3)(x+2)^3}dx = 2\int\frac{1}{x-3}dx - 1\int\frac{1}{x+2}dx + 11\int(x+2)^{-2}dx - 7\int(x+2)^{-3}dx$$

$$= 2\ln|x-3| - \ln|x+2| + 11\frac{(x+2)^{-2+1}}{-2+1} - 7\frac{(x+2)^{-3+1}}{-3+1}$$

$$= 2\ln|x-3| - \ln|x+2| + 11\frac{(x+2)^{-1}}{-1} - 7\frac{(x+2)^{-2}}{-2}$$

$$= \boxed{2\ln|x-3| - \ln|x+2| - \frac{11}{x+2} + \frac{7}{2(x+2)^2} + C}$$

Question # 18

$$\int\frac{x-2}{(x+1)(x^2+1)}dx$$

Solution:

$$\int\frac{x-2}{(x+1)(x^2+1)}dx$$

$\frac{x-2}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1}$ —— ①

Multiply by $(x+1)(x^2+1)$ on both sides

$x-2 = A(x^2+1) + Bx+C(x+1)$ —— ②

$x+1 = 0 \Rightarrow \boxed{x=-1}$ put in eq ②

$-1-2 = A((-1)^2+1) + B(-1)+C(-1+1)$

$-3 = A(1+1) + 0$

$-3 = A(2)$

$\frac{-3}{2} = A$

$\boxed{A = \frac{-3}{2}}$

From eq ②

$x-2 = Ax^2+A+Bx^2+Bx+Cx+C$

Comparing Coefficient of x^2 and x

"x^2" $A+B = 0$

$\frac{-3}{2} + B = 0$

$\boxed{B = \frac{3}{2}}$

"x"

$B+C = 1$

$\frac{3}{2} + C = 1$

$C = 1 - \frac{3}{2}$

$C = \frac{2-3}{2}$

$\boxed{C = -\frac{1}{2}}$

$$\frac{x-2}{(x+1)(x^2+1)} = \frac{-3}{2(x+1)} + \frac{3x-1}{2(x^2+1)}$$

$$\int \frac{x-2}{(x+1)(x^2+1)}dx = \int \frac{-3}{2(x+1)}dx + \int \frac{3x-1}{2(x^2+1)}dx$$

$$= \frac{-3}{2}\int \frac{1}{x+1}dx + \frac{1}{2}\int \frac{3x}{x^2+1}dx - \frac{1}{2}\int \frac{1}{x^2+1}dx$$

$$= \frac{-3}{2}\ln|x+1| + \frac{3}{2}\cdot\frac{1}{2}\int \frac{2x}{x^2+1}dx - \frac{1}{2}\int \frac{1}{x^2+1}dx$$

$$= \boxed{\frac{-3}{2}\ln|x+1| + \frac{3}{4}\ln|x^2+1| - \frac{1}{2}\tan^{-1}x + C}$$

Question #19

$$\int \frac{x}{(x-1)(x^2+1)}dx$$

Solution:

$$\int \frac{x}{(x-1)(x^2+1)}dx$$

$$\frac{x}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx+C}{(x^2+1)} \text{ —— } ①$$

Multiply by $(x-1)(x^2+1)$ on both sides

$$x = A(x^2+1) + Bx+C\,(x-1) \text{ —— } ②$$

$x-1=0 \Rightarrow \boxed{x=1}$ put in eq ②

$$1 = A((1)^2+1) + 0$$

$$1 = A\,(1+1)$$

$$1 = 2A$$

$$\boxed{A = \frac{1}{2}}$$

From eq ②

$$x = Ax^2+A+Bx^2-Bx+Cx-C$$

Comparing Coefficient of 'x^2' and 'x'

'x^2' $A+B=0$

$$\frac{1}{2}+B=0$$

$$\boxed{B = \frac{-1}{2}}$$

'x' $-B+C=1$

$$-\left(-\frac{1}{2}\right)+C=1$$

$$\frac{1}{2}+C=1$$

$$C = 1-\frac{1}{2}$$

$$C = \frac{2-1}{2}$$

$$\boxed{C = \frac{1}{2}}$$

$$\frac{x}{(x-1)(x^2+1)} = \frac{1}{2(x-1)} - \frac{x-1}{2(x^2+1)}$$

$$\int \frac{x}{(x-1)(x^2+1)}dx = \int \frac{1}{2(x-1)}dx - \frac{1}{4}\int \frac{2x-2}{x^2+1}dx$$

$$= \frac{1}{2}\int \frac{1}{x-1}dx - \frac{1}{4}\int \frac{2x}{x^2+1}dx + \frac{1}{2}\int \frac{1}{x^2+1}dx$$

$$= \boxed{\frac{1}{2}\ln|x-1| - \frac{1}{4}\ln|x^2+1| + \frac{1}{2}\tan^{-1}x + C}$$

Question #20

$$\int \frac{9x-7}{(x+3)(x^2+1)}dx$$

Solution:

$$\int \frac{9x-7}{(x+3)(x^2+1)}dx$$

$$\frac{9x-7}{(x+3)(x^2+1)} = \frac{A}{x+3} + \frac{Bx+C}{x^2+1} \text{ —— } ①$$

Multiply by $(x+3)(x^2+1)$ on both sides

$$9x-7 = A(x^2+1) + Bx+C\,(x+3) \text{ —— } ②$$

$x+3=0 \Rightarrow \boxed{x=-3}$ put in eq ②

$$9(-3)-7 = A((-3)^2+1)+0$$

$$-27-7 = A\,(9+1)$$

$$-34 = A\,(10)$$

$$\frac{-\cancel{34}^{17}}{{}_{5}\cancel{10}} = A$$

$$\boxed{A = \frac{-17}{5}}$$

From eq ②

$$9x-7 = Ax^2+A+Bx^2+3Bx+Cx+3C$$

"x^2" Comparing Coefficient

$$A+B=0$$

$$\frac{-17}{5}+B=0$$

$$\boxed{B=\frac{17}{5}}$$

"x"

$$3B+C=9$$

$$3\left(\frac{17}{5}\right)+C=9$$

$$\frac{51}{5}+C=9$$

$$C=9-\frac{51}{5}$$

$$C=\frac{45-51}{5}$$

$$\boxed{C=-\frac{6}{5}}$$

$$\frac{9x-7}{(x+3)(x^2+1)} = \frac{-17}{5(x+3)} + \frac{17x-6}{5(x^2+1)}$$

$$\int\frac{9x-7}{(x+3)(x^2+1)}dx = \frac{-17}{5}\int\frac{1}{x+3}dx + \frac{17}{5}\int\frac{x}{x^2+1}dx - \frac{6}{5}\int\frac{1}{x^2+1}dx$$

$$= \frac{-17}{5}\ln|x+3| + \frac{17}{5}\cdot\frac{1}{2}\int\frac{2x}{x^2+1}dx - \frac{6}{5}\tan^{-1}x + C$$

$$= \boxed{\frac{-17}{5}\ln|x+3| + \frac{17}{10}\ln|x^2+1| - \frac{6}{5}\tan^{-1}x + C}$$

Question # 21

$$\int\frac{1+4x}{(x-3)(x^2+4)}dx$$

Solution:

$$\int\frac{1+4x}{(x-3)(x^2+4)}$$

$$\frac{1+4x}{(x-3)(x^2+4)} = \frac{A}{x-3} + \frac{Bx+C}{x^2+4} \quad \text{——} ①$$

$$1+4x = A(x^2+4) + Bx+C\,(x-3) \quad \text{——} ②$$

$$x-3=0 \Rightarrow \boxed{x=3} \text{ put in eq } ②$$

$$1+4(3) = A((3)^2+4) + 0$$

$$1+12 = A(9+4)$$

$$13 = A(13)$$

$$\frac{\cancel{13}}{\cancel{13}} = A$$

$$\boxed{A=1}$$

From eq ②

$$1+4x = Ax^2+4A+Bx^2-3Bx+Cx-3C$$

Comparing Coefficient of "x^2" and "x"

"x^2" $A+B=0$

$$1+B=0$$

$$\boxed{B=-1}$$

"x" $-3B+C=4$

$$-3(-1)+C=4$$

$$3+C=4$$

$$C=4-3$$

$$\boxed{C=1}$$

$$\frac{1+4x}{(x-3)(x^2+4)} = \frac{1}{x-3} - \frac{x-1}{(x^2+4}$$

$$\int\frac{1+4x}{(x-3)(x^2+4)}dx = \int\frac{1}{x-3}dx - \int\frac{x-1}{x^2+4}$$

$$= \int\frac{1}{x-3}dx - \int\frac{x}{x^2+4}dx + \int\frac{1}{x^2+4}dx$$

$= \ln|x-3| - \frac{1}{2}\int \frac{2x}{x^2+4}dx + \int \frac{1}{(x)^2+(2)^2}dx$

$$\boxed{= \ln|x-3| - \frac{1}{2}\ln|x^2+4| + \frac{1}{2}\tan^{-1}\frac{x}{2} + C}$$

Question # 22

$$\int \frac{12}{x^3+8}dx$$

Solution:

$\int \frac{12}{x^3+8}\,dx$ ∵ $a^3+b^3=(a+b)(a^2-ab+b^2)$

$\int \frac{12}{(x)^3+(2)^3}dx \Rightarrow \int \frac{12}{(x+2)(x^2-2x+4)}dx$

$\frac{12}{(x+2)(x^2-2x+4)} = \frac{A}{x+2} + \frac{Bx+C}{x^2-2x+4}$ —— ①

Multiply by $(x+2)(x^2-2x+4)$ on both sides

$12 = A(x^2-2x+4) + Bx+C(x+2)$ —— ②

$x+2=0 \Rightarrow \boxed{x=-2}$ put in eq ②

$12 = A((-2)^2-2(-2)+4)+0$

$12 = A(4+4+4)$

$12 = 12A$

$\frac{\cancel{12}}{\cancel{12}} = A$

$\boxed{A=1}$

From eq ②

$12 = Ax^2-2Ax+4A+Bx^2+2Bx+Cx+2C$

Comparing Coefficient of 'x^2' and 'x'

'x^2'

$A+B=0$

$1+B=0$

$\boxed{B=-1}$

'x'

$-2A+2B+C=0$

$-2(1)+2(1)+C=0$

$-2-2+C=0$

$\boxed{C=4}$

$\frac{12}{(x+2)(x^2-2x+4)} = \frac{1}{x+2} - \frac{x-4}{x^2-2x+4}$

$\int \frac{12}{(x+2)(x^2-2x+4)}dx = \int \frac{1}{x+2}dx - \int \frac{x-4}{x^2-2x+4}dx$

$= \ln|x+2| - \frac{1}{2}\int \frac{2x-8}{x^2-2x+4}dx$

$= \ln|x+2| - \frac{1}{2}\int \frac{2x-2-6}{x^2-2x+4}dx$

$= \ln|x+2| - \frac{1}{2}\int \frac{2x-2}{x^2-2x+4}dx - \frac{6}{2}\int \frac{1}{x^2-2x+4}dx$

$= \ln|x+2| - \frac{1}{2}\ln|x^2-2x+4| - \frac{\overset{3}{\cancel{6}}}{\cancel{2}}\int \frac{1}{x^2-2x+1+3}dx$

$= \ln|x+2| - \frac{1}{2}\ln|x^2-2x+4| - 3\int \frac{1}{(x-1)^2+(\sqrt{3})^2}dx$

$= \ln|x+2| - \frac{1}{2}\ln|x^2-2x+4| - \frac{3}{\sqrt{3}}\cdot\tan^{-1}\left(\frac{x-1}{\sqrt{3}}\right)+c$

$$\boxed{= \ln|x+2| - \frac{1}{2}\ln|x^2-2x+4| - \sqrt{3}\tan^{-1}\left(\frac{x-1}{\sqrt{3}}\right)+C}$$

Question # 23

$$\int \frac{9x+6}{x^3-8}$$

Solution:

$\int \frac{9x+6}{x^3-8}dx$ ∵ $a^3-b^3=(a-b)(a^2+ab+b^2)$

$\int \frac{9x+6}{(x)^3-(2)^3}dx \Rightarrow \int \frac{9x+6}{(x-2)(x^2+2x+4)}dx$

$$\frac{9x+6}{(x-2)(x^2+2x+4)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+2x+4} \text{ ——} ①$$

Multiply by $(x-2)(x^2+2x+4)$ on both sides

$$9x+6 = A(x^2+2x+4)+Bx+C(x-2) \text{ ——} ②$$

$x-2=0 \Rightarrow \boxed{x=2}$ put in eq ②

$$9(2)+6 = A((2)^2+2(2)+4)+0$$

$$18+6 = A(4+4+4)$$

$$24 = A(12)$$

$$\frac{\overset{2}{\cancel{24}}}{\cancel{12}} = A$$

$$\boxed{A=2}$$

From eq ②

$$9x+6 = Ax^2+2Ax+4A+Bx^2-2B+Cx-2C$$

Comparing Coefficient of 'x^2' and 'x'

'x^2' $A+B=0$

$$2+B=0$$

$$\boxed{B=-2}$$

"x" $2A+C-2B=9$

$$2(2)-2(-2)+C=9$$

$$4+4+C=9$$

$$8+C=9$$

$$C=9-8$$

$$\boxed{C=1}$$

$$\frac{9x+6}{(x-2)(x^2+2x+4)} = \frac{2}{x-2} - \frac{2x-1}{x^2+2x+4}$$

$$\int\frac{9x+6}{(x-2)(x^2+2x+4}dx = \int\frac{2}{x-2}dx - \int\frac{2x-1}{x^2+2x+4}dx$$

$$= 2\int\frac{1}{x-2}dx - \int\frac{2x+2-2-1}{x^2+2x+4}dx$$

$$= 2\ln|x-2| - \int\frac{2x+2}{x^2+2x4}dx + 3\int\frac{1}{x^2+2x+4}dx$$

$$= 2\ln|x-2| - \ln|x^2+2x+4| + 3\int\frac{1}{x^2+2x+1+3}dx$$

$$= 2\ln|x-2| - \ln|x^2+2x+4| + 3\int\frac{1}{(x+1)^2+(\sqrt{3})^2}dx$$

$$= 2\ln|x-2| - \ln|x^2+2x+4| + \frac{3}{\sqrt{3}}\tan^{-1}\left(\frac{x+1}{\sqrt{3}}\right)$$

$$= \boxed{2\ln|x-2| - \ln|x^2+2x+4| + \sqrt{3}\tan^{-1}\left(\frac{x+1}{\sqrt{3}}\right) + C}$$

Question #24

$$\int\frac{2x^2+5x+3}{(x-1)^2(x^2+4)}dx$$

Solution:

$$\int\frac{2x^2+5x+3}{(x-1)^2(x^2+4)}dx$$

$$\frac{2x^2+5x+3}{(x-1)^2(x^2+4)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+4} \text{ ——} ①$$

Multiply by $(x-1)^2(x^2+4)$ on both sides

$$2x^2+5x+3 = A(x^2+4)(x-1) + B(x^2+4) + Cx+D(x-1)^2 \text{ ——} ②$$

$x-1=0 \Rightarrow \boxed{x=1}$ put in eq ②

$$2(1)^2+5(1)+3 = 0 + B((1)^2+4) + 0$$

$$2+5+3 = B(1+4) + 0$$

$$10 = B(5)$$

$$\frac{\overset{2}{\cancel{10}}}{\cancel{5}} = B$$

$$\boxed{B=2}$$

From eq ②

$$2x^2+5x+3 = A(x^3+4x-x^2-4)+Bx^2+4B + Cx+D(x^2+1-2x)$$

$$2x^2+5x+3 = Ax^3-Ax^2+4Ax-4A+Bx^2+4B+Cx^3+C$$

$$"x^3" -2Cx^2+Dx^2+D-2Dx$$

$$A+C=0 \Rightarrow -A=+C \text{ —— } ③$$

"x^2"

$$-A+B+D-2C=2$$

$$-A+2+D-2(-A)=2$$

$$-A+2A+D=2-2$$

$$A+D=0$$

$$D=-A \text{ —— } ④$$

"x"

$$5=4A+C-2D$$

$$5=4A-A-2(-A)$$

$$5=4A-A+2A$$

$$5=5A$$

$$\frac{\cancel{5}}{\cancel{5}}=A$$

$$\boxed{A=1}$$

put in eq ③

$$\boxed{C=-1}$$

put the value of A in eq ④

$$\boxed{D=-1}$$

$$\frac{2x^2+5x+3}{(x-1)^2(x^2+4)}=\frac{1}{x-1}+\frac{2}{(x-1)^2}-\frac{x-1}{x^2+4}$$

$$\int\frac{2x^2+5x+3}{(x-1)^2(x^2+4)}dx=\int\frac{1}{x-1}dx+2\int(x-1)^{-2}dx-\int\frac{x+1}{x^2+4}dx$$

$$=\ln|x-1|+2\cdot\frac{(x-1)^{-1}}{-1}-\int\frac{x}{x^2+4}dx-\int\frac{1}{(x)^2+(2)^2}dx$$

$$=\ln|x-1|-\frac{2}{x-1}-\frac{1}{2}\int\frac{2x}{x^2+4}dx-\int\frac{1}{(x)^2+(2)^2}dx$$

$$=\boxed{\ln|x-1|-\frac{2}{x-1}-\frac{1}{2}\ln|x^2+4|-\frac{1}{2}\tan^{-1}\frac{x}{2}+C}$$

Question #25

$$\int\frac{2x^2-x-7}{(x+2)^2(x^2+x+1)}dx$$

Solution:

$$\int\frac{2x^2-x-7}{(x+2)^2(x^2+x+1)}dx$$

$$\frac{2x^2-x-7}{(x+2)^2(x^2+x+1)}=\frac{A}{x+2}+\frac{B}{(x+2)^2}+\frac{Cx+D}{x^2+x+1} \text{ — } ①$$

Multiply by $(x+2)^2(x^2+x+1)$ on both sides

$$2x^2-x-7=A(x+2)(x^2+x+1)+B(x^2+x+1) \text{ — } ②$$
$$+Cx+D(x+2)^2$$

$$x+2=0 \Rightarrow \boxed{x=-2} \text{ put in eq } ②$$

$$2(-2)^2-(-2)-7=0+B((-2)^2+(-2)+1)+0$$

$$2(4)+2-7=B(4-2+1)$$

$$8+2-7=3B$$

$$3=3B$$

$$\frac{\cancel{3}}{\cancel{3}}=B$$

$$\boxed{B=1}$$

From eq ②

$$2x^2-x-7=A(x^3+x^2+x+2x^2+2x+2)+Bx^2+Bx+B$$
$$+Cx+D(x^2+4+4x)$$

$$2x^2-x-7=Ax^3+3Ax^2+3Ax+2A+Bx^2+Bx+B$$
$$+Cx^3+4Cx^2+4Cx+Dx^2+4Dx+4D$$

Comparing Coefficient of "x^3" and "x^2"

"x^3"

$$A+C=0$$

$$C=-A \text{ —— } ③$$

"x^2"

$$2=3A+B+4C+D$$

$$2=3A+1+4(-A)+D$$

$$2=3A+1-4A+D$$

$$2=-A+1+D$$

$$2-1=-A+D$$

$$D=A+1 \text{ —— } ④$$

"x"

$$-1=3A+B+4C+4D$$

$$-1=3A+1+4(-A)+4(A+1)$$

$$-1=3A+1-4A+4A+4$$

$-1 = 3A + 5$

$-1-5 = 3A$

$-6 = 3A$

$\frac{-\cancel{6}^{2}}{\cancel{3}} = A$

$\boxed{A = -2}$

put in eq ③ and ④

$\boxed{C = 2}$

$\boxed{D = -1}$

$$\frac{2x^2-x-7}{(x+2)^2(x^2+x+1)} = \frac{-2}{x+2} + \frac{1}{(x+2)^2} + \frac{2x-1}{x^2+x+1}$$

$$\int \frac{2x^2-x-7}{(x+2)^2(x^2+x+1)}dx = \int \frac{-2}{x+2}dx + \int (x+2)^{-2}dx + \int \frac{2x+1-2}{x^2+x+1}dx$$

$$= -2\int \frac{1}{x+2}dx + \frac{(x+2)^{-1}}{-1} + \int \frac{2x+1}{x^2+x+1}dx - 2\int \frac{1}{x^2+x+1}dx$$

$$= -2\ln|x+2| - \frac{1}{x+2} + \ln|x^2+x+1| - 2\int \frac{1}{x^2+x+\frac{1}{4}+\frac{3}{4}}dx$$

$$= -2\ln|x+2| - \frac{1}{x+2} + \ln|x^2+x+1| - 2\int \frac{1}{(x+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2}dx$$

$$= -2\ln|x+2| - \frac{1}{x+2} + \ln|x^2+x+1| - \frac{2}{\frac{\sqrt{3}}{2}}\tan^{-1}\left(\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right) + C$$

$$= \boxed{-2\ln|x+2| - \frac{1}{x+2} + \ln|x^2+x+1| - \frac{4}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + C}$$

Question #26

$$\int \frac{3x+1}{(4x^2+1)(x^2-x+1)}dx$$

Solution:

$$\int \frac{3x+1}{(4x^2+1)(x^2-x+1)}dx$$

$$\frac{3x+1}{(4x^2+1)(x^2+x+1)} = \frac{Ax+B}{4x^2+1} + \frac{Cx+D}{x^2+x+1} \quad — ①$$

Multiply by $(4x^2+1)(x^2+x+1)$ on both sides

$3x+1 = Ax+B(x^2+x+1) + Cx+D(4x^2+1)$ — ②

$3x+1 = Ax^3+Ax^2+Ax+Bx^2+Bx+B+4Cx^3$
$+ 4Dx^2 + Cx + Dx$

Comparing Coefficient of x^3, x^2, x and constant

"x^3" $A+4C = 0 \Rightarrow A = -4C$ — ③

"x^2" $-A+B+4D = 0$ — ④

"x" $A+B+C = 3$ — ⑤

Constant;

$B+D = 1 \Rightarrow B = 1-D$ — ⑥

put in eq ④

$0 = -(-4C) + (1-D) + 4D$

$0 = 4C + 1 - D + 4D$

$0 = 4C+3D+1$ — ⑦

put eq ③ and ④ in eq ⑤

$3 = -4C - (1-D) + C$

$3 = -4C - 1 + D + C$

$3 = -3C + D - 1$

$D = 3C + 4$ — ⑧

put in eq ⑦

$0 = 4C + 3(3C+4) + 1$

$0 = 4C + 9C + 12 + 1$

$0 = 13C + 13$

$\frac{-\cancel{13}}{\cancel{13}} = C$

$\boxed{C = -1}$ put in eq ③

$A = -4(-1)$

$\boxed{A = 4}$ put in eq ⑥

$= 3C + 4$

$D = 3(-1) + 4$

$D = -3 + 4$

$\boxed{D = 1}$

put in eq ⑥

$B = 1 - 1$

$\boxed{B = 0}$

$$\frac{3x+1}{(4x^2+1)(x^2+x+1)} = \frac{4x+0}{4x^2+1} + \frac{(-1)x+1}{x^2-x+1}$$

$$\frac{3x+1}{(4x^2+1)(x^2+x+1)} = \frac{1}{2}\frac{8x}{4x^2+1} + \frac{x-1}{x^2-x+1}$$

$$\int \frac{3x+1}{(4x^2+1)(x^2+x+1)}dx = \frac{1}{2}\int \frac{8x}{4x^2+1}dx - \frac{1}{2}\int \frac{2x-2}{x^2-x+1}dx$$

$$= \frac{1}{2}\ln|4x^2+1| - \frac{1}{2}\int \frac{2x-1-1}{x^2-x+1}dx$$

$$= \frac{1}{2}\ln|4x^2+1| - \frac{1}{2}\int \frac{2x-1}{x^2-x+1}dx + \frac{1}{2}\int \frac{1}{x^2-x+\frac{1}{4}+\frac{3}{4}}dx$$

$$= \frac{1}{2}\ln|4x^2+1| - \frac{1}{2}\ln|x^2-x+1| + \frac{1}{2}\int \frac{1}{\left(x-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}dx$$

$$= \frac{1}{2}\ln|4x^2+1| - \frac{1}{2}\ln|x^2-x+1| + \frac{\frac{1}{2}}{\sqrt{3}/2}\tan^{-1}\left(\frac{x-\frac{1}{2}}{\sqrt{3}/2}\right) + C$$

$$= \boxed{\frac{1}{2}\ln|4x^2+1| - \frac{1}{2}\ln|x^2-x+1| + \frac{\sqrt{3}}{2}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right) + C}$$

Question # 27

$$\int \frac{4x+1}{(x^2+4)(x^2+4x+5)}dx$$

Solution: $\int \frac{4x+1}{(x^2+4)(x^2+4x+5)}dx$

$$\frac{4x+1}{(x^2+4)(x^2+4x+5)} = \frac{Ax+B}{x^2+4} + \frac{Cx+D}{x^2+4x+5} \quad \text{—①}$$

Multiply by $(x^2+4)(x^2+4x+5)$ on both sides

$4x+1 = Ax+B(x^2+4x+5) + (x+D(x^2+4)$

$4x+1 = Ax^3 + 4Ax^2 + 5Ax + Bx^2 + 4Bx + 5B$
$+4Cx + Cx^3 + Dx^2 + 4D$

Comparing Coefficient of x^3, x^2, x and x^0

"x^3"

$A + C = 0 \Rightarrow A = -C$ —②

"x^2"

$4A + B + D = 0$ —③

"x"

$5A + 4B + 4C = 4$ —④

Constant:

$5B + 4D = 1 \Rightarrow 5B = 1 - 4D$

$B = \frac{1-4D}{5}$ —⑤

put eq ⑤ in ③ and ④

$-4C + \left(\frac{1-4D}{5}\right) + D = 0$

$-20C + 1 - 4D + 5D = 0$

$-20C + D + 1 = 0$

$D = 20C - 1$ —⑥

$5(-C) + 4\left(\frac{1-4D}{5}\right) + 4C = 4$

$-25C + 4 - 16D + 20C = 20$

$D = \frac{-5C-16}{16}$ —⑦

put in eq ⑥

$\frac{-5C-16}{16} = 20C - 1$

$-5C - 16 = 320C - 16$

$-16 + 16 = 320C + 5C$

$0 = 320C + 5C$

$325C = 0$

$\boxed{C = 0}$ put in eq ②

$\boxed{A = 0}$

put in eq ⑥

$20(0) - 1 = D$

$\boxed{D = -1}$

$$B = \frac{1-4(-1)}{5}$$

$$B = \frac{1+4}{5}$$

$$B = \frac{5}{5}$$

$$\boxed{B = 1}$$

$$\frac{4x+1}{(x^2+4)(x^2+4x+5)} = \frac{0x+1}{x^2+4} + \frac{0x-1}{x^2+4x+5}$$

$$\int \frac{4x+1}{(x^2+4)(x^2+4x+5)}dx = \int \frac{1}{x^2+4}dx - \int \frac{1}{x^2+4x+4+1}dx$$

$$= \frac{1}{2}\tan^{-1}\frac{x}{2} - \int \frac{1}{(x+2)^2+(1)^2}dx$$

$$= \boxed{\frac{1}{2}\tan^{-1}\frac{x}{2} - \tan^{-1}(x+2) + C}$$

Question #28

$$\int \frac{6a^2}{(x^2+a^2)(x^2+4a^2)}dx$$

Solution: $\int \frac{6a^2}{(x^2+a^2)(x^2+4a^2)}dx$

$$\frac{6a^2}{(x^2+a^2)(x^2+4a^2)} = \frac{Ax+B}{x^2+a^2} + \frac{Cx+D}{x^2+4a^2} \quad — ①$$

Multiply by $(x^2+a^2)(x^2+4a^2)$ on both sides

$$6a^2 = Ax+B(x^2+4a^2) + Cx+D(x^2+a^2)$$

$$6a^2 = Ax^3 + 4Aa^2x + Bx^2 + 4Ba^2 + Cx^3 + Ca^2x + Dx^2 + Da^2$$

Comparing Coefficient of x^3, x^2, x and x^0

"x^3" $A + C = 0 \Rightarrow A = -C$ — ②

"x^2" $B + D = 0 \Rightarrow B = -D$ — ③

"x" $4a^2A + a^2C = 0 \Rightarrow (4A+C)a^2 = 0$

$4A + C = 0$ — ④

Constant

$$6a^2 = 4a^2B + Da^2$$

$$6\cancel{a^2} = (4B+D)\cancel{a^2}$$

$6 = 4B + D$ — ⑤

put eq ② and ③ in eq ④ and ⑤

$$4(-C)+C=0$$

$$-4C+C=0$$

$$-3C=0$$

$\boxed{C=0}$ put in eq ②

$\boxed{A=0}$

$$4(-D)+D=6$$

$$-4D+D=6$$

$$-3D=6$$

$$D = \frac{\cancel{6}^{2}}{\cancel{-3}}$$

$$\boxed{D=-2}$$

put in eq ③

$\boxed{B=2}$

$$\frac{6a^2}{(x^2+a^2)(x^2+4a^2)} = \frac{0x+2}{x^2+a^2} + \frac{0x-2}{x^2+4a^2}$$

$$\int \frac{6a^2}{(x^2+a^2)(x^2+4a^2)}dx = \int \frac{2}{x^2+a^2}dx + \int \frac{-2}{x^2+4a^2}dx$$

$$= 2\int \frac{1}{x^2+a^2}dx - 2\int \frac{1}{x^2+(2a)^2}dx$$

$$= \frac{2}{a}\tan^{-1}\frac{x}{a} - \frac{\cancel{2}}{\cancel{2}a}\tan^{-1}\frac{x}{2a} + C$$

$$= \boxed{\frac{2}{a}\tan^{-1}\frac{x}{a} - \frac{1}{a}\tan^{-1}\frac{x}{2a} + C}$$

Question # 29

$$\int \frac{2x^2-2}{x^4+x^2+1}\,dx$$

Solution:

$$\int \frac{2x^2-2}{x^4+x^2+1}\,dx$$

$\because x^4+x^2+1$

$=(x^2)^2+(1)^2+2(x^2)(1)-2(x^2)(1)+x^2$

$=(x^2+1)^2-2x^2+x^2$

$=(x^2+1)^2-(x)^2$

$=(x^2+x+1)(x^2-x+1)$

$$\frac{2x^2-2}{x^2+x+1} = \frac{Ax+B}{x^2+x+1}+\frac{Cx+D}{x^2-x+1} \quad \text{—} ①$$

Multiply by $(x^2+x+1)(x^2-x+1)$ on both sides

$2x^2-2 = Ax+B(x^2-x+1)+Cx+D(x^2+x+1)$

$2x^2-2 = Ax^3-Ax^2+Ax+Bx^2-Bx+B+Cx^3+Cx^2+Cx$
$+Dx^2+Dx+D$

Comparing coefficient of x^3, x^2, x and x^0

"x^3"
$A+C=0$ — ②

"x^2"
$A-B+C+D=0$ — ③

"x"
$-A+B+C+D=2$ — ④

"Constant"
$B+D=-2$ — ⑤

put in eq ④

$2=-A+C-2$

$2+2=-A+C$

$-A+C=4$ — ⑥

put eq ② in eq ③

$0=-B+D$ — ⑦

eq ② + ⑥

$$\begin{array}{r} \cancel{A}+C=0 \\ -\cancel{A}+C=4 \\ \hline 2C=4 \\ \hline \end{array}$$

$C=\frac{\cancel{4}\,2}{\cancel{2}}$

$\boxed{C=2}$

put in eq ②

$A+2=0$

$\boxed{A=-2}$

Eq ⑤ + Eq ⑦

$$\begin{array}{r} \cancel{B}+D=-2 \\ -\cancel{B}+D=0 \\ \hline 2D=-2 \\ \hline \end{array}$$

$D=-\frac{\cancel{2}}{\cancel{2}}$

$\boxed{D=-1}$

put in eq ⑤

$B-1=-2$

$B=-2+1$

$\boxed{B=-1}$

$$\frac{2x^2-2}{(x^2+x+1)(x^2-x+1)} = \frac{-2x-1}{x^2+x+1}+\frac{2x-1}{x^2-x+1}$$

$$\int\frac{2x^2-2}{(x^2+x+1)(x^2-x+1)}dx = \int\frac{-2x-1}{x^2+x+1}dx + \int\frac{2x-1}{x^2-x+1}dx$$

$$= -\int\frac{2x+1}{x^2+x+1}dx + \int\frac{2x-1}{x^2-x+1}dx$$

$$= -\ln(x^2+x+1) + \ln|x^2-x+1|$$

$$= \ln|x^2-x+1| - \ln|x^2+x+1|$$

$$= \boxed{\ln\left|\frac{x^2-x+1}{x^2+x+1}\right| + C}$$

Question # 30

$$\int \frac{3x-8}{(x^2-x+2)(x^2+x+2)}\,dx$$

Solution:

$$\int \frac{3x-8}{(x^2-x+2)(x^2+x+2)}\,dx$$

$$\frac{3x-8}{(x^2-x+2)(x^2+x+2)} = \frac{Ax+B}{x^2-x+2} + \frac{Cx+D}{x^2+x+2} \quad \text{——} ①$$

Multiply by $(x^2-x+2)(x^2+x+2)$ on both sides

$$3x-8 = Ax+B(x^2+x+2) + Cx+D(x^2-x+2)$$

$$3x-8 = Ax^3+Ax^2+2Ax+Bx^2+Bx+2B+Cx^3+2Cx$$
$$-Cx^2+Dx^2-Dx+2D$$

Comparing Coefficient of x^3, x^2, x and x'

"x^3"
$$A+C=0 \Rightarrow A=-C \quad \text{——} ②$$

"x^2"
$$A+B-C+D=0 \quad \text{——} ③$$

"x"
$$2A+B+2C=3 \quad \text{——} ④$$

"Constant"
$$2B+2D=-8$$
$$B+D=-4 \quad \text{——} ⑤$$

put eq ② and ⑤ in eq ③ and ④

$$-C+(-4-D)-C+D=0$$
$$-C-4-\cancel{D}-C+\cancel{D}=0$$
$$-2C-4=0$$
$$-2C=4$$
$$\boxed{C=-2}$$

put in eq ②
$$\boxed{A=2}$$

$$2(-C)+(-4-D)+2C-D=3$$
$$-2\cancel{C}-4-D+2\cancel{C}-D=3$$
$$-2D=3+4$$
$$-2D=7$$

$$\boxed{D=-\frac{1}{2}}$$

put in eq ⑤

$$B\cdot\left(-\frac{7}{2}\right)-4$$
$$B=-4+\frac{7}{2}$$
$$B=-\frac{8+7}{2}$$
$$\boxed{B=-\frac{1}{2}}$$

$$\frac{3x-8}{(x^2-x+2)(x^2+x+2)} = \frac{2x-1/2}{x^2-x+2} + \frac{-2x-7/2}{x^2+x+2}$$

$$\int\frac{3x-8}{(x^2-x+2)(x^2+x+2)}\,dx = \int\frac{2x-1+1-7/2}{x^2-x+2}\,dx - \int\frac{2x+1-1+7/2}{x^2+x+1}\,dx$$

$$= \int\frac{2x-1+1/2}{x^2-x+2}\,dx - \int\frac{2x+1+5/2}{x^2+x+1}\,dx$$

$$= \int\frac{2x-1}{x^2-x+2}\,dx + \frac{1}{2}\int\frac{1}{x^2-x+2}\,dx - \int\frac{2x+1}{x^2+x+2}\,dx$$
$$-\int\frac{5/2}{x^2+x+2}\,dx$$

$$= \ln|x^2-x+2| + \frac{1}{2}\int\frac{1}{x^2-x+\frac{1}{4}-\frac{1}{4}+2}\,dx - \ln|x^2+x+2| -$$
$$5/2\int\frac{1}{x^2+x+\frac{1}{4}-\frac{1}{4}+2}\,dx$$

$$= \ln|x^2-x+2| + \frac{1}{2}\int\frac{1}{(x-\frac{1}{2})^2+\frac{7}{4}}\,dx - \ln|x^2+x+2|$$
$$-5/2\int\frac{1}{(x-\frac{1}{2})^2+\frac{7}{4}}\,dx$$

$$= \ln|x^2-x+2| + \frac{1}{2}\int\frac{1}{(x-\frac{1}{2})^2+(\frac{\sqrt{7}}{2})^2}\,dx - \ln|x^2+x+2|$$
$$-5/2\int\frac{1}{(x-\frac{1}{2})^2+(\frac{\sqrt{7}}{2})^2}\,dx$$

$$= \ln|x^2-x+2| + \frac{1}{2}\cdot\frac{1}{\frac{\sqrt{7}}{2}}\tan^{-1}\frac{(x-\frac{1}{2})}{\frac{\sqrt{7}}{2}} - \ln|x^2+x+2|$$
$$-5/2\cdot\frac{1}{\frac{\sqrt{7}}{2}}\tan^{-1}\frac{(x-\frac{1}{2})}{\sqrt{7}/2} + C$$

$$= \boxed{\ln|x^2-x+2| + \frac{1}{\sqrt{7}}\tan^{-1}\left(\frac{2x-1}{\sqrt{7}}\right) - \ln|x^2+x+1| - \frac{5}{\sqrt{7}}\tan^{-1}\left(\frac{2x-1}{\sqrt{7}}\right) + C}$$

Question # 31

$$\int \frac{3x^3+4x^2+9x+5}{(x^2+x+1)(x^2+2x+3)}\,dx$$

Solution:

$$\frac{3x^3+4x^2+9x+5}{(x^2+x+1)(x^2+2x+3)} = \frac{Ax+B}{x^2+x+1} + \frac{Cx+D}{x^2+2x+3} \quad \text{—} ①$$

Multiply by $(x^2+x+1)(x^2+2x+3)$ on both sides

$$3x^3+4x^2+9x+5 = Ax+B(x^2+2x+3)+Cx+D(x^2+x+1)$$

$$3x^3+4x^2+9x+5 = Ax^3+2Ax^2+3Ax+Bx^2+2Bx+3B+Cx^3$$
$$+Cx^2+Cx+Dx^2+Dx+D$$

Comparing coefficient of x^3, x^2 and x, x^0

"x^3" $A+C = 3 \Rightarrow A = 3-C$ — ②

"x^2" $2A+B+C+D = 4$ — ③

"x" $3A+2B+C+D = 9$ — ④

"x^0"

$3B+D = 5$

$D = 5-3B$ — ⑤

put eq ② and ⑤ in eq ③ and ④

$2(3-C)+B+C+5-3B = 4$

$6-2C+B+C+5-3B = 4$

$-C-2B = 4-6-5$

$-(C+2B) = -7$

$C+2B = 7$ — ⑥

$3(3-C)+2B+C+5-3B = 9$

$9-3C+2B+C+5-3B = 9$

$-B-2C = 9-9-5$

$-B = -5+2C$

$5-2C = B$ — ⑦

put in eq ⑥

$C+2(5-2C) = 7$

$C+10-4C = 7$

$-3C = -3$

$\boxed{C = 1}$ put in eq ②

$A = 3-1$

$\boxed{A = 2}$

put in eq ⑦

$B = 5-2(1)$

$B = 5-2$

$\boxed{B = 3}$ put in eq ⑤

$D = 5-3(3)$

$D = 5-9$

$\boxed{D = -4}$

$$\frac{3x^3+4x^2+9x+5}{(x^2+x+1)(x^2+2x+3)} = \frac{2x+3}{x^2+x+1} + \frac{x-4}{x^2+2x+3}$$

$$\int\frac{3x^3+4x^2+9x+5}{(x^2+x+1)(x^2+2x+3)}dx = \int\frac{2x+1+2}{x^2+x+1}dx + \frac{1}{2}\int\frac{2x-8}{x^2+2x+3}dx$$

$$= \int\frac{2x+1}{x^2+x+1}dx + 2\int\frac{1}{x^2+x+\frac{1}{4}+\frac{3}{4}}dx + \frac{1}{2}\int\frac{2x+2-10}{x^2+2x+3}dx$$

$$= \ln|x^2+x+1| + 2\int\frac{1}{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}dx + \frac{1}{2}\int\frac{2x+2}{x^2+2x+3}dx - \frac{10}{2}\int\frac{1}{x^2+2x+3}dx$$

$$= \ln|x^2+x+1| + \frac{2}{\frac{\sqrt{3}}{2}}\cdot\tan^{-1}\frac{\left(x+\frac{1}{2}\right)}{\sqrt{3}/2} + \frac{1}{2}\ln|x^2+2x+3| -$$
$$-5\int\frac{1}{x^2+2x+1+2}dx$$

$$= \ln|x^2+x+1| + \frac{4}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + \frac{1}{2}\ln|x^2+2x+3|$$
$$-5\int\frac{1}{(x+1)^2+(\sqrt{2})^2}dx$$

$$= \ln|x^2+x+1| + \frac{4}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + \frac{1}{2}\ln|x^2+2x+3|$$
$$-\frac{5}{\sqrt{2}}\tan^{-1}\left(\frac{x+1}{\sqrt{2}}\right) + C$$

$$= \boxed{\ln|x^2+x+1| + \frac{4}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + \frac{1}{2}\ln|x^2+2x+3| - \frac{5}{2}\tan^{-1}\left(\frac{x+1}{\sqrt{2}}\right) + C}$$

* Theory

* The Definite Integral

If f(x) is continuos on the interval [a,b] and if f(x) is any indefinite integral, then

$$\int_a^b f(x)\,dx = |F(x)|_a^b = F(b) - F(a)$$

is called definite integral of f(x) between the limit "a" and "b".

- The interval [a, b] is called range of integration.
- The function f(x) is known as the integrand.
- While a and b are known as lower and upper limits of integration respectively.

* Fundamental Theorem of Calculus:

If f(x) is continuous on the interval [a,b] and $\phi'(x) = f(x)$, that is, $\phi(x)$ is any antiderivative of f on [a,b], then

$$\int_a^b f(x)\,dx = \phi(b) - \phi(a)$$

Properties of Definite Integral

If f(x) and g(x) are two continuous functions on the interval [a,b], then

(i) $\int_a^a f(x)\,dx = 0$

(ii) $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$ (Where c is any constant)

(iii) $\int_a^b c\,f(x)\,dx = c\int_a^b f(x)\,dx$

(iv) $\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

(v) $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

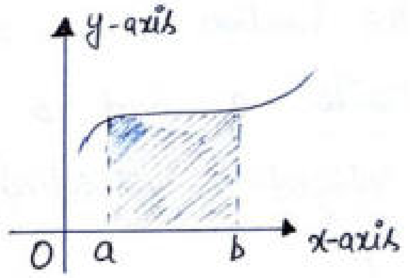
* Application of Definite Integral

Area under the curve

$\int_a^b f(x)\,dx$ gives the area under the curve $y = f(x)$ from $x = a$ to $x = b$ and the x-axis

$$\text{Area} = \int_a^b f(x)\,dx$$

Area under the curve



Example #1

Evaluate: ~~(i)~~

$$\int_{-1}^{3} (x^3 + 3x^2)\,dx$$

Solution:

$= \int_{-1}^{3} (x^3 + 3x^2)\,dx$

$= \int_{-1}^{3} x^3\,dx + \int_{-1}^{3} 3x^2\,dx$

$= \int_{-1}^{3} x^3\,dx + 3\int_{-1}^{3} x^2\,dx$

$\because \int (x)^n\,dx = \frac{x^{n+1}}{n+1}$

$= \left|\frac{x^{3+1}}{3+1}\right|_{-1}^{3} + 3\left|\frac{x^{2+1}}{2+1}\right|_{-1}^{3}$

$= \left|\frac{x^4}{4}\right|_{-1}^{3} + 3\left|\frac{x^3}{3}\right|_{-1}^{3}$

$= \frac{1}{4}\left[(3)^4 - (-1)^4\right] + \frac{\cancel{3}}{\cancel{3}}\left[(3)^3 - (-1)^3\right]$

$= \frac{1}{4}\left[81 - 1\right] + \left[27 - (-1)\right]$

$= \frac{1}{\cancel{4}_1}(\overset{20}{\cancel{80}}) + [27 + 1]$

$= 20 + 28$

$= \boxed{48}$

(ii)

$$\int_1^2 \frac{x^2+1}{x+1}\,dx$$

Solution:

$$\int_1^2 \frac{x^2+1}{x+1}\,dx$$

$$\begin{array}{r|l} & x-1 \\ \hline x+1 & x^2+1 \\ & \underline{-x^2 \pm x} \\ & -x+1 \\ & \underline{\pm x \pm 1} \\ & 2 \end{array}$$

$$= \int_1^2 \left((x-1) + \frac{2}{x+1}\right)dx$$

$$= \int_1^2 (x-1)\,dx + \int_1^2 \frac{2}{x+1}\,dx$$

$$= \int_1^2 x\,dx - \int_1^2 1\,dx + 2\int_1^2 \frac{1}{x+1}\,dx \quad \because \int \frac{f'(x)}{f(x)}dx = \ln f(x)$$

$$= \left|\frac{x^2}{2}\right|_1^2 - |x|_1^2 + \left|2\ln|x+1|\right|_1^2$$

$$= \frac{1}{2}\left[(2)^2-(1)^2\right] - [2-1] + 2\left(\ln|2+1| - \ln|1+1|\right)$$

$$= \frac{1}{2}[4-1] - [1] + 2[\ln 3 - \ln 2] \quad \because \ln m - \ln n = \ln\frac{m}{n}$$

$$= \frac{1}{2}(3) - 1 + 2\ln\frac{3}{2}$$

$$= \frac{3}{2} - 1 + 2\ln\frac{3}{2}$$

$$= \frac{3-2}{2} + 2\ln\frac{3}{2}$$

$$= \boxed{\frac{1}{2} + 2\ln\frac{3}{2}}$$

Example #2

Evaluate:

(i)

$$\int_0^{\sqrt{3}} \frac{x^3+9x+1}{x^2+9}\,dx$$

Solution:

$$\int_0^{\sqrt{3}} \frac{x^3+9x+1}{x^2+9}\,dx$$

$$= \int_0^{\sqrt{3}} \frac{x^3+9x}{x^2+9} + \frac{1}{x^2+9}\,dx$$

$$= \int_0^{\sqrt{3}} \frac{x\cancel{(x^2+9)}}{\cancel{x^2+9}} + \frac{1}{x^2+9}\,dx$$

$$= \int_0^{\sqrt{3}} x + \frac{1}{x^2+9}\,dx$$

$$= \int_0^{\sqrt{3}} x\,dx + \int_0^{\sqrt{3}} \frac{1}{x^2+9}\,dx$$

$$= \left|\frac{x^2}{2}\right|_0^{\sqrt{3}} + \left|\frac{1}{3}\tan^{-1}\frac{x}{3}\right|_0^{\sqrt{3}}$$

$$= \frac{1}{2}\left[(\sqrt{3})^2-(0)^2\right] + \frac{1}{3}\left[\tan^{-1}\frac{\sqrt{3}}{3} - \tan^{-1}\frac{0}{3}\right]$$

$$= \frac{1}{2}(3-0) + \frac{1}{3}\left[\tan^{-1}\frac{1}{\sqrt{3}} - \tan^{-1}0\right]$$

$$= \frac{3}{2} + \frac{1}{3}\left(\frac{\pi}{6} - 0\right)$$

$$= \boxed{\frac{3}{2} + \frac{\pi}{18}}$$

(ii)

$$\int_0^{\pi/4} \sec x(\sec x + \tan x)\,dx$$

Solution:

$$\int_0^{\pi/4} \sec x(\sec x + \tan x)\,dx$$

$$= \int_0^{\pi/4} (\sec^2 x + \sec x \tan x)\, dx$$

$$= \int_0^{\pi/4} \sec^2 x\, dx + \int_0^{\pi/4} \sec x \tan x\, dx$$

$$= \left|\tan x\right|_0^{\pi/4} + \left|\sec x\right|_0^{\pi/4}$$

$$= \left[\tan\frac{\pi}{4} - \tan 0\right] + \left[\sec\frac{\pi}{4} - \sec 0\right]$$

$$= (1-0) + (\sqrt{2} - 1)$$

$$= \cancel{1} + \sqrt{2} - \cancel{1}$$

$$= \boxed{\sqrt{2}}$$

Example #3

Evaluate: $\int_0^{\pi/4} \frac{1}{1-\sin x}\, dx$

Solution:

$$= \int_0^{\pi/4} \left(\frac{1}{1-\sin x} \times \frac{1+\sin x}{1+\sin x}\right) dx$$

$$= \int_0^{\pi/4} \frac{1+\sin x}{(1-\sin x)(1+\sin x)}\, dx$$

$\because (a-b)(a+b) = a^2 - b^2$

$$= \int_0^{\pi/4} \frac{1+\sin x}{1-\sin^2 x}\, dx$$

$\because 1 - \sin^2 x = \cos^2 x$

$$= \int_0^{\pi/4} \frac{1+\sin x}{\cos^2 x}\, dx$$

$$= \int_0^{\pi/4} \left(\frac{1}{\cos^2 x} + \frac{\sin x}{\cos^2 x}\right) dx$$

$$= \int_0^{\pi/4} \left[\sec^2 x + \frac{1}{\cos x} \cdot \frac{\sin x}{\cos x}\right] dx$$

$$= \int_0^{\pi/4} (\sec^2 x + \sec x \tan x)\, dx$$

$$= \int_0^{\pi/4} \sec^2 x\, dx + \int_0^{\pi/4} \sec x \tan x\, dx$$

$$= \left|\tan x\right|_0^{\pi/4} + \left|\sec x\right|_0^{\pi/4}$$

$$= \left(\tan\frac{\pi}{4} - \tan 0\right) + \left(\sec\frac{\pi}{4} - \sec 0\right)$$

$$= (1-0) + (\sqrt{2} - 1)$$

$$= \cancel{1} + \sqrt{2} - \cancel{1}$$

$$= \boxed{\sqrt{2}}$$

Example #4

Evaluate $\int_{-1}^{2} (x + |x|)\, dx$

Solutions:

$$\int_{-1}^{2} (x + |x| dx$$

$$\because \int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx \quad , a<c<b$$

$$= \int_{-1}^{2} (x + |x|)\, dx = \int_{-1}^{0} (x + |x|)\, dx + \int_0^2 (x + |x|)\, dx$$

$\because$ If $x>0$, $|x| = x$

$\because$ If $x<0$, $|x| = -x$

$$= \int_{-1}^{0} (x + (-x))\, dx + \int_0^2 (x + (x))\, dx$$

$$= \int_{-1}^{0} 0\, dx + \int_0^2 2x\, dx$$

$$= 0 + 2 \cdot \frac{x^2}{2}\Big|_0^2$$

$$= \cancel{2} \left|\frac{x^2}{\cancel{2}}\right|_0^2$$

$$\left[(2)^2 - (0)^2\right]$$

$= 4 - 0$

$= \boxed{4}$

Example #5

Evaluate $\displaystyle\int_0^{\sqrt{7}} \frac{3x}{\sqrt{x^2+9}}\,dx$

Solution:

$$= \int_0^{\sqrt{7}} \frac{3x}{\sqrt{x^2+9}}\,dx$$

$$= 3\int_0^{\sqrt{7}} (x^2+9)^{-1/2} \cdot x\,dx$$

Multiply and divided by "2"

$$= \frac{3}{2}\int_0^{\sqrt{7}} (x^2+9)^{-1/2} \cdot 2x\,dx$$

$$\because \int [f(x)]^n \cdot f'(x)\,dx = \frac{[f(x)]^{n+1}}{n+1}$$

$$= \frac{3}{2}\left|\frac{(x^2+9)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}\right|_0^{\sqrt{7}}$$

$$= \frac{3}{2}\left[\frac{(x^2+9)^{1/2}}{1/2}\right]_0^{\sqrt{7}}$$

$$= \frac{3}{\cancel{2}} \cdot \frac{\cancel{2}}{1}\left[((\sqrt{7})^2+9)^{1/2} - ((0)^2+9)^{1/2}\right]$$

$$= 3\left[(7+9)^{1/2} - (9)^{1/2}\right]$$

$$= 3\left[(16)^{1/2} - (9)^{1/2}\right]$$

$$= 3\left[(4^2)^{1/2} - (3^2)^{1/2}\right]$$

$$= 3[4 - 3]$$

$$= 3(1)$$

$= \boxed{3}$

Example #6

Evaluate $\displaystyle\int_{1/2}^{\sqrt{3}/2} \frac{\sin^{-1}x}{\sqrt{1-x^2}}\,dx$

Solution:

$$= \int_{1/2}^{\sqrt{3}/2} \sin^{-1}x \cdot \frac{1}{\sqrt{1-x^2}}\,dx$$

$$\because \int [f(x)]^n \cdot f'(x)\,dx = \frac{[f(x)]^{n+1}}{n+1}$$

$$= \left|\frac{(\sin^{-1}x)^2}{2}\right|_{1/2}^{\sqrt{3}/2}$$

$$= \frac{1}{2}\left[\left(\sin^{-1}\frac{\sqrt{3}}{2}\right)^2 - \left(\sin^{-1}\frac{1}{2}\right)^2\right]$$

$$= \frac{1}{2}\left[\left(\frac{\pi}{3}\right)^2 - \left(\frac{\pi}{6}\right)^2\right]$$

$$= \frac{1}{2}\left[\frac{\pi^2}{9} - \frac{\pi^2}{36}\right]$$

$$= \frac{1}{2}\left[\frac{4\pi^2 - \pi^2}{36}\right]$$

$$= \frac{1}{2}\left(\frac{3\pi^2}{36}\right)$$

$$= \frac{\cancel{3}\pi^2}{\underset{24}{\cancel{72}}}$$

$$= \boxed{\frac{\pi^2}{24}}$$

Example #7

Evaluate $\int_0^{\pi/6} x\cos x\,dx$

Solution:

$$= \int_0^{\pi/6} x \cos x \, dx$$

1st function = x 2nd function = $\cos x$

Integration by parts:

$$= x\sin x - \int \sin x \cdot 1\,dx$$

$$= x\sin x + \cos x + c$$

Taking limits:

$$= |x\sin x + \cos x|_0^{\pi/6}$$

$$= \left[\frac{\pi}{6}\sin\frac{\pi}{6} + \cos\frac{\pi}{6} - (0\sin 0 + \cos 0)\right)$$

$$= \left[\frac{\pi}{6}\left(\frac{1}{2}\right) + \frac{\sqrt{3}}{2} - (0+1)\right)$$

$$= \left[\frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1\right)$$

$$= \boxed{\frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1}$$

Example #8

Evaluate $\int_1^e x\ln x\,dx$

Solution:

$$= \int_1^e x \ln x\,dx$$

1st function = $\ln x$ 2nd function = x

Integration by parts:

$$= \ln x \cdot \int x\,dx - \int\left[\int x\,dx \cdot \frac{d}{dx}(\ln x)\right] dx$$

$$= \ln x \cdot \frac{x^2}{2} - \int \frac{x^2}{2} \cdot \frac{1}{\cancel{x}}\,dx$$

$$= \frac{x^2}{2}\ln x - \frac{1}{2}\int x \cdot dx$$

$$= \frac{x^2}{2}\ln x - \frac{1}{2}\,\frac{x^2}{2} + c$$

$$= \frac{x^2}{2}\ln x - \frac{x^2}{4} + c$$

Taking limit:

$$= \left[\frac{x^2}{2}\ln x - \frac{x^2}{4}\right]_1^e$$

$$= \left[\left(\frac{e^2}{2}\ln e - \frac{e^2}{4}\right) - \left[\frac{(1)^2}{2}\ln 1 - \frac{(1)^2}{4}\right]\right.$$

$\because \ln e = 1$ $\because \ln 1 = 0$

$$= \left[\frac{e^2}{2}(1) - \frac{e^2}{4}\right] - \left[\frac{1}{2}(0) - \frac{1}{4}\right]$$

$$= \left(\frac{e^2}{2} - \frac{e^3}{4}\right) - \left(-\frac{1}{4}\right)$$

$$= \frac{e^2}{2} - \frac{e^3}{4} + \frac{1}{4}$$

$$= \frac{2e^2 - e^2 + 1}{4}$$

$$= \boxed{\frac{e^2+1}{4}}$$

Example #9

If $\int_{-2}^{1} f(x)\,dx = 5$, $\int_{1}^{3} f(x)\,dx = 3$

$\int_{-2}^{1} g(x)\,dx = 4$ evaluate the following definit integrals:

(i)

$$\int_{-2}^{3} f(x)\,dx$$

Solution:

$$= \int_{-2}^{3} f(x)\,dx$$

$$\because \int_{a}^{b} f(x)\,dx = \int_{a}^{c} f(x)\,dx + \int_{c}^{b} f(x)\,dx$$

$$= \int_{-2}^{1} f(x)\,dx + \int_{1}^{3} f(x)\,dx$$

$$= 5 + 3$$

$$= \boxed{8}$$

* * *

(ii)

$$\int_{-2}^{1} \left[2f(x) + 3g(x)\right] dx$$

Solution:

$$\int_{-2}^{1} \left[2f(x) + 3g(x)\right] dx$$

$$= \int_{-2}^{1} 2f(x)\,dx + \int_{-2}^{1} 3g(x)\,dx$$

$$= 2\int_{-2}^{1} f(x)\,dx + 3\int_{-2}^{1} g(x)\,dx$$

$$= 2(5) + 3(4)$$

$$= 10 + 12$$

$$= \boxed{22}$$

* * *

(iii)

$$\int_{-2}^{1} 3f(x)\,dx - \int_{-2}^{1} 2g(x)\,dx$$

Solution:

$$= \int_{-2}^{1} 3f(x)\,dx - \int_{-2}^{1} 2g(x)\,dx$$

$$= 3\int_{-2}^{1} f(x)\,dx - 2\int_{-2}^{1} g(x)\,dx$$

$$= 3(5) - 2(4)$$

$$= 15 - 8$$

$$= \boxed{7}$$

* * *

Exercise # 3·6

Evaluate the following definite integrals:

Question # 1

$$\int_{1}^{2}(x^2+1)dx$$

Solution:

$$= \int_{1}^{2}(x^2+1)\,dx$$

$$= \int_{1}^{2}x^2 dx + \int_{1}^{2}1\,dx$$

$$= \left|\frac{x^3}{3}\right|_1^2 + |x|_1^2$$

$$= \frac{1}{3}|x^3|_1^2 + |x|_1^2$$

$$= \frac{1}{3}\left[(2)^3-(1)^3\right] + [2-1]$$

$$= \frac{1}{3}(8-2) + (1)$$

$$= \frac{1}{3}(7) + 1$$

$$= \frac{7}{3} + 1$$

$$= \frac{7+3}{3}$$

$$= \boxed{\frac{10}{3}}$$

Question # 2

$$\int_{-1}^{1}(x^{\frac{1}{3}}+1)\,dx$$

Solution:

$$\int_{-1}^{1}(x^{\frac{1}{3}}+1)\,dx$$

$$= \int_{-1}^{1}x^{1/3}dx + \int_{-1}^{1}1\,dx$$

$$= \left|\frac{x^{\frac{1}{3}+1}}{\frac{1}{3}+1}\right|_{-1}^{1} + |x|_{-1}^{1}$$

$$= \left|\frac{x^{\frac{4}{3}}}{\frac{4}{3}}\right|_{-1}^{1} + |x|_{-1}^{1}$$

$$= \frac{3}{4}|x^{\frac{4}{3}}|_{-1}^{1} + |x|_{-1}^{1}$$

$$= \frac{3}{4}\left[(1)^{4/3}-(-1)^{4/3}\right] + [1-(-1)]$$

$$= \frac{3}{4}[1-1] + [1+1]$$

$$= \frac{3}{4}(0) + (2)$$

$$= \boxed{2}$$

Question # 3

$$\int_{-2}^{0}\frac{1}{(2x-1)^2}dx$$

Solution:

$$= \int_{-2}^{0}\frac{1}{(2x-1)^2}\,dx$$

$$= \int_{-2}^{0}(2x-1)^{-2}dx$$

Multiply and Divided by '2'

$$= \frac{1}{2}\int(2x-1)^{-2}\cdot 2\,dx$$

$$\because \int [f(x)]^n \cdot f'(x)\,dx = \frac{[f(x)]^{n+1}}{n+1}$$

$$= \frac{1}{2}\left|\frac{(2x-1)^{-2+1}}{-2+1}\right|_{-9}^{0}$$

$$= \frac{1}{2}\left|\frac{(2x-1)^{-1}}{-1}\right|_{-2}^{0}$$

$$= -\frac{1}{2}\left|\frac{1}{2x-1}\right|_{9}^{0}$$

$$= -\frac{1}{2}\left[\frac{1}{2(0)-1} - \frac{1}{2(-2)-1}\right]$$

$$= -\frac{1}{2}\left[\frac{1}{0-1} - \frac{1}{-4-1}\right]$$

$$= -\frac{1}{2}\left[\frac{1}{-1} - \frac{1}{-5}\right]$$

$$= -\frac{1}{2}\left[-1 + \frac{1}{5}\right]$$

$$= -\frac{1}{2}\left[\frac{-5+1}{5}\right]$$

$$= -\frac{1}{\cancel{2}}\left[-\frac{\overset{2}{\cancel{4}}}{5}\right]$$

$$= \boxed{\frac{2}{5}}$$

Question # 4

$$\int_{-6}^{2} \sqrt{3-x}\,dx$$

Solution:

$$= \int_{-6}^{2} \sqrt{3-x}\,dx$$

$$= \int_{-6}^{2} (3-x)^{1/2}\,dx$$

Multiply and Divided by '-1'

$$= -\int_{-6}^{2} (3-x)^{1/2} \cdot (-1)\,dx$$

$$\because \int [f(x)]^n \cdot f'(x)\,dx = \frac{[f(x)]^{n+1}}{n+1}$$

$$=$$

$$= -\left|\frac{(3-x)^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right|_{-6}^{2}$$

$$= -\left|\frac{(3-x)^{3/2}}{3/2}\right|_{-6}^{2}$$

$$= -\frac{2}{3}\left|(3-x)^{3/2}\right|_{-6}^{2}$$

$$= -\frac{2}{3}\left[(3-2)^{3/2} - (3-(-6))^{3/2}\right]$$

$$= -\frac{2}{3}\left[(1)^{3/2} - (3+6)^{3/2}\right]$$

$$= -\frac{2}{3}\left[1 - (9)^{3/2}\right]$$

$$= -\frac{2}{3}\left[1 - (3^2)^{3/2}\right]$$

$$= -\frac{2}{3}\left[1 - (3)^3\right]$$

$$= -\frac{2}{3}\left[1 - 27\right]$$

$$= -\frac{2}{3}(-26)$$

$$= \boxed{\frac{52}{3}}$$

Question # 5

$$\int_{1}^{\sqrt{5}} \sqrt{(2t-1)^3}\,dt$$

Solution:

$$= \int_{1}^{\sqrt{5}} \sqrt{(2t-1)^3}\,dt$$

$$= \int_{1}^{\sqrt{5}} (2t-1)^{3/2}\,dt$$

Multiply and Divided by '2'

$$= \frac{1}{2}\int_{1}^{\sqrt{5}} (2t-1)^{3/2}\,(2)\,dx$$

$$\because \int [f(x)]^n \cdot f'(x)\,dx = \frac{[f(x)]^{n+1}}{n+1}$$

$$= \frac{1}{2}\left|\frac{(2t-1)^{\frac{3}{2}+1}}{\frac{3}{2}+1}\right|_1^{\sqrt{5}}$$

$$= \frac{1}{2}\left|\frac{(2t-1)^{5/2}}{5/2}\right|_1^{\sqrt{5}}$$

$$= \frac{1}{\cancel{2}}\cdot\frac{\cancel{2}}{5}\left[(2\sqrt{5}-1)^{5/2} - (2(1)-1)^{5/2}\right]$$

$$= \frac{1}{5}\left[(2\sqrt{5}-1)^{5/2} - (2-1)^{5/2}\right]$$

$$= \frac{1}{5}\left[(2\sqrt{5}-1)^{5/2} - (1)^{5/2}\right]$$

$$= \boxed{\frac{1}{5}\left[(2\sqrt{5}-1)^{5/2} - 1\right]}$$

Question #6

$$\int_9^{\sqrt{5}} x\sqrt{x^2-1}\;dx$$

Solution:

$$= \int_9^{\sqrt{5}} x\sqrt{x^2-1}\;dx$$

$$= \int_9^{\sqrt{5}} (x^2-1)^{1/2}\cdot x\,dx$$

Multiply and divided by '2'

$$= \frac{1}{2}\int_9^{\sqrt{5}} (x^2-1)^{1/2}\cdot 2x\,dx$$

$$\because \int [f(x)]^n \cdot f'(x)\,dx = \frac{[f(x)]^{n+1}}{n+1}$$

$$= \frac{1}{2}\left|\frac{(x^2-1)^{1/2+1}}{\frac{1}{2}+1}\right|_9^{\sqrt{5}}$$

$$= \frac{1}{2}\left|\frac{(x^2-1)^{3/2}}{3/2}\right|_9^{\sqrt{5}}$$

$$= \frac{1}{\cancel{2}}\cdot\frac{\cancel{2}}{3}\left[(x^2-1)^{3/2}\right]_9^{\sqrt{5}}$$

$$= \frac{1}{3}\left[((\sqrt{5})^2-1)^{3/2} - ((2)^2-1)^{3/2}\right]$$

$$= \frac{1}{3}\left[(5-1)^{3/2} - (4-1)^{3/2}\right]$$

$$= \frac{1}{3}\left[(4)^{3/2} - (3)^{3/2}\right]$$

$$= \frac{1}{3}\left[(2^2)^{3/2} - 3^1\cdot 3^{1/2}\right]$$

$$= \frac{1}{3}\left[(2)^3 - 3\sqrt{3}\right]$$

$$= \frac{1}{3}\left[8 - 3\sqrt{3}\right]$$

$$= \frac{8}{3} - \frac{\cancel{3}\sqrt{3}}{\cancel{3}}$$

$$= \boxed{\frac{8}{3} - \sqrt{3}}$$

Question #7

$$\int_1^2 \frac{x}{x^2+2}\,dx$$

Solution:

$$= \int_1^2 \frac{x}{x^2+2}\,dx$$

Multiply and divided by '2'

$$= \frac{1}{2}\int_1^2 \frac{2x}{x^2+2}\,dx$$

$$\because \int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)|$$

$$= \frac{1}{2}\left|\ln|x^2+2|\right|_1^2$$

$$= \frac{1}{2}\left[\ln|(2)^2+2| - \ln|(1)^2+2|\right]$$

$$= \frac{1}{2}\left[\ln|4+2| - \ln|1+2|\right]$$

$$= \frac{1}{2}\left[\ln 6 - \ln 3\right]$$

$$= \frac{1}{2}\ln \frac{\cancel{6}^{\,2}}{\cancel{3}}$$

$$= \boxed{\frac{1}{2}\ln 2}$$

Question #8

$$\int_2^3 \left(x - \frac{1}{x}\right)^2 dx$$

Solution:

$$= \int_2^3 \left(x - \frac{1}{x}\right)^2 dx$$

$$= \int_2^3 \left(x^2 + \frac{1}{x^2} - 2\right) dx$$

$$= \int_2^3 x^2 dx + \int_2^3 x^{-2} dx - 2\int_2^3 1\, dx$$

$$= \left|\frac{x^3}{3}\right|_2^3 + \left|\frac{x^{-1}}{-1}\right|_2^3 - 2|x|_2^3$$

$$= \frac{1}{3}\left|x^3\right|_2^3 - \left|\frac{1}{x}\right|_2^3 - 2\,|x|_2^3$$

$$= \frac{1}{3}\left[(3)^3 - (2)^3\right] - \left[\frac{1}{3} - \frac{1}{2}\right] - 2\left[3 - 2\right]$$

$$= \frac{1}{3}\left[27 - 8\right] - \left[\frac{2-3}{6}\right] - 2(1)$$

$$= \frac{1}{3}(19) - \left(\frac{-1}{6}\right) - 2$$

$$= \frac{19}{3} + \frac{1}{6} - 2$$

$$= \frac{38 + 1 - 12}{6} = \frac{\cancel{27}^{\,9}}{\cancel{6}_{\,2}}$$

$$= \boxed{\frac{9}{2}}$$

Question #9

$$\int_{-1}^{1} \left(x + \frac{1}{2}\right)\sqrt{x^2+x+1}\; dx$$

Solution:

$$= \int_{-1}^{1} \left(x + \frac{1}{2}\right)\sqrt{x^2+x+1}\; dx$$

$$= \int_{-1}^{1} (x^2+x+1)^{1/2} \left(\frac{2x+1}{2}\right) dx$$

$$= \frac{1}{2}\int_{-1}^{1} (x^2+x+1)^{1/2} \cdot (2x+1)\, dx$$

$$\because \int \left[f(x)\right]^n \cdot f'(x)\, dx = \frac{\left[f(x)\right]^{n+1}}{n+1}$$

$$= \frac{1}{2}\left|\frac{(x^2+x+1)^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right|_{-1}^{1}$$

$$= \frac{1}{2}\left|\frac{(x^2+x+1)^{3/2}}{3/2}\right|_{-1}^{1}$$

$$= \frac{1}{\cancel{2}} \cdot \frac{\cancel{2}}{3}\left|(x^2+x+1)^{3/2}\right|_{-1}^{1}$$

$$= \frac{1}{3}\left[\left((1)^2+1+1\right)^{3/2} - \left((-1)^2 - 1 + 1\right)^{3/2}\right]$$

$$= \frac{1}{3}\left[(1+1+1)^{3/2} - (1-1+1)^{3/2}\right]$$

$$= \frac{1}{4}\left[(3)^{3/2} - (1)^{3/2}\right]$$

$$= \frac{1}{3}\left[3 \cdot 3^{1/2} - 1\right]$$

$$= \frac{1}{3}\left[3\sqrt{3} - 1\right]$$

$$= \frac{\cancel{3}\sqrt{3}}{\cancel{3}} - \frac{1}{3}$$

$$= \boxed{\sqrt{3} - \frac{1}{3}}$$

Question #10

$$\int_0^3 \frac{dx}{x^2+9}$$

Solution:

$$= \int_0^3 \frac{dx}{x^2+9}$$

$$= \int_0^3 \frac{1}{(x)^2+(3)^2}\, dx$$

$$= \quad \because \int \frac{1}{a^2+x^2}\, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}$$

$$= \frac{1}{3} \left| \tan^{-1} \frac{x}{3} \right|_0^3$$

$$= \frac{1}{3}\left[\tan^{-1} \frac{\cancel{3}}{\cancel{3}} - \tan^{-1} \frac{0}{3} \right]$$

$$= \frac{1}{3}\left[\tan^{-1} 1 - \tan^{-1} 0 \right]$$

$$= \frac{1}{3}\left[\frac{\pi}{4} - 0 \right]$$

$$= \boxed{\frac{\pi}{12}}$$

Question #11

$$\int_{\pi/6}^{\pi/3} \cos t \; dt$$

Solution:

$$= \int_{\pi/6}^{\pi/3} \cos t \; dt$$

$$= \left| \sin t \right|_{\pi/6}^{\pi/3}$$

$$= \left[\sin \frac{\pi}{3} - \sin \frac{\pi}{6} \right]$$

$$= \frac{\sqrt{3}}{2} - \frac{1}{2}$$

$$= \boxed{\frac{\sqrt{3}-1}{2}}$$

Question #12

$$\int_1^2 \left[x+\frac{1}{x}\right]^{1/2} \left[1-\frac{1}{x^2}\right] dx$$

Solution:

$$= \int_1^2 \left(x+\frac{1}{x}\right)^{1/2} \cdot \left[1-\frac{1}{x^2}\right] dx$$

$$\because \frac{d}{dx}\left(x+\frac{1}{x}\right)$$

$$= 1+(-1)x^{-2} = 1-x^{-2} \Rightarrow 1-\frac{1}{x^2}$$

$$= \int_1^2 \left(x+\frac{1}{x}\right)^{\frac{1}{2}} \cdot \left[1-\frac{1}{x^2}\right] dx$$

$$\because \int [f(x)]^n \cdot f'(x)\, dx = \frac{[f(x)]^{n+1}}{n+1}$$

$$= \left| \frac{\left(x+\frac{1}{x}\right)^{\frac{1}{2}+1}}{\frac{1}{2}+1} \right|_1^2$$

$$= \left. \frac{\left(x+\frac{1}{x}\right)^{3/2}}{3/2} \right|_1^2$$

$$= \frac{2}{3}\left[\left(2+\frac{1}{2}\right)^{3/2} - \left(1+\frac{1}{1}\right)^{3/2} \right]$$

$$= \frac{2}{3}\left[\left(\frac{4+1}{2}\right)^{3/2} - (2)^{3/2} \right]$$

$$= \frac{2}{3}\left[\left(\frac{5}{2}\right)^{3/2} - (2)^{3/2} \right]$$

$$= \frac{2}{3}\left[\frac{5\cdot\sqrt{5}}{(2)^{3/2}} - \left(2^{3/2}\right) \right]$$

$$= \frac{2}{3}\left[\frac{5\sqrt{5} - \left(2^{3/2}\right)^2}{2^{3/2}} \right]$$

$$= \frac{\cancel{2}}{3}\left[\frac{5\sqrt{5} - 2^3}{\cancel{2}\cdot\sqrt{2}} \right]$$

$$= \boxed{\frac{5\sqrt{5} - 8}{3\sqrt{2}}}$$

Question #13

$$\int_1^2 \ln x \, dx$$

Solution:

$$\int_1^2 \ln x \cdot 1 \, dx$$

1st function = $\ln x$, 2nd function = 1

Integration by parts:

$$= \ln x \cdot x - \int \cancel{x} \cdot \frac{1}{\cancel{x}} \, dx$$

$$= x \ln x - \int 1 \, dx$$

$$= \left| x \ln x - x \right|_1^2$$

$$= (2 \ln 2 - 2) - (1 \ln 1 - 1)$$

$$\because \ln 1 = 0$$

$$= (2 \ln 2 - 2) - (1 \cdot 0 - 1)$$

$$= (2 \ln 2 - 2) + 1$$

$$= 2 \ln 2 - 2 + 1$$

$$= \boxed{2 \ln 2 - 1}$$

Question #14

$$\int_0^2 \left(e^{\frac{x}{2}} - e^{-\frac{x}{2}}\right) dx$$

Solution:

$$\int_0^2 \left(e^{\frac{x}{2}} - e^{-\frac{x}{2}}\right) dx$$

$$= \int_0^2 e^{\frac{x}{2}} \, dx - \int_0^2 e^{-\frac{x}{2}} \, dx$$

$$= \left| \frac{e^{\frac{x}{2}}}{\frac{1}{2}} - \frac{e^{-\frac{x}{2}}}{\frac{-1}{2}} \right|_0^2$$

$$= 2 \left| e^{\frac{x}{2}} \right|_0^2 + 2 \left| e^{-\frac{x}{2}} \right|_0^2$$

$$= 2 \left[e^{\frac{2}{2}} - e^{\frac{0}{2}} \right] + 2 \left[e^{-\frac{2}{2}} - e^{-\frac{0}{2}} \right]$$

$$= 2 \left(e^{1} - e^{0} \right) + 2 \left(e^{-1} - e^{0} \right)$$

$$= 2 [e - 1] + 2 \left(\frac{1}{e} - 1 \right)$$

$$= 2 \left[e - 1 + \frac{1}{e} - 1 \right]$$

$$= 2 \left[e + \frac{1}{e} - 2 \right]$$

$$= 2 \left[\frac{e^2 + 1 - 2}{e} \right]$$

$$= \boxed{\frac{2}{e} (e - 1)^2}$$

Question #15

$$\int_0^{\pi/4} \frac{\cos\theta + \sin\theta}{2 \cos^2\theta} \, d\theta$$

Solution:

$$\int_0^{\pi/4} \frac{\cos\theta + \sin\theta}{2 \cos^2\theta} \, d\theta$$

$$= \frac{1}{2} \int_0^{\pi/4} \left(\frac{\cos\theta}{\cos^2\theta} + \frac{\sin\theta}{\cos^2\theta} \right) d\theta$$

$$= \frac{1}{2} \int_0^{\pi/4} \frac{1}{\cos\theta} + \frac{1}{\cos\theta} \cdot \frac{\sin\theta}{\cos\theta} \, d\theta$$

$$= \frac{1}{2} \int \left(\sec\theta + \sec\theta \tan\theta \right) d\theta$$

$$= \frac{1}{2}\int_0^{\pi/4} \sec\theta\, d\theta + \int_0^{\pi/4} \sec\theta \tan\theta\, d\theta$$

$$= \frac{1}{2}\left| \ln|\sec\theta + \tan\theta| \right|_0^{\pi/4} + |\sec\theta|_0^{\pi/4}$$

$$= \frac{1}{2}\left[\ln\left(\sec\frac{\pi}{4} + \tan\frac{\pi}{4}\right) - \ln(\sec 0 + \tan 0)\right] + \left(\sec\frac{\pi}{4} - \sec 0\right)$$

$$= \frac{1}{2}\left[\ln(\sqrt{2} + 1) - \ln(1 + 0)\right] + (\sqrt{2} - 1)$$

$$= \frac{1}{2}\left[\ln(\sqrt{2}+1) - \ln 1\right] + \sqrt{2} - 1$$

$$\because \ln 1 = 0$$

$$= \frac{1}{2}\left[\ln(\sqrt{2}+1) - 0\right] + \sqrt{2} - 1$$

$$= \boxed{\frac{1}{2}\left[\ln|\sqrt{2}+1|\right] + \sqrt{2} - 1}$$

Question #16

$$\int_0^{\pi/6} \cos^3\theta\, d\theta$$

Solution:

$$= \int_0^{\pi/6} \cos^3\theta\, d\theta$$

$$= \int_0^{\pi/6} \cos^2\theta \cos\theta\, d\theta \qquad \because \cos^2\theta = 1 - \sin^2\theta$$

$$= \int_0^{\pi/6} (1 - \sin^2\theta) \cos\theta\, d\theta$$

$$= \int_0^{\pi/6} (\cos\theta - \sin^2\theta \cos\theta)\, d\theta$$

$$= \int_0^{\pi/6} \cos\theta\, d\theta - \int_0^{\pi/6} \sin^2\theta \cos\theta\, d\theta$$

$$\because \int [f(x)]^n \cdot f'(x)\, dx = \frac{[f(x)]^{n+1}}{n+1}$$

$$= |\sin\theta|_0^{\pi/6} - \left|\frac{\sin^3\theta}{3}\right|_0^{\pi/6}$$

$$= \left[\sin\frac{\pi}{6} - \sin 0\right] - \frac{1}{3}\left[\sin^3\frac{\pi}{6} - \sin^3\theta\, 0\right]$$

$$= \left(\frac{1}{2} - 0\right) - \frac{1}{3}\left[\left(\frac{1}{2}\right)^3 - 0\right]$$

$$= \left(\frac{1}{2}\right) - \frac{1}{3}\left[\frac{1}{8}\right]$$

$$= \frac{1}{2} - \frac{1}{24}$$

$$= \frac{12 - 1}{24}$$

$$= \boxed{\frac{11}{24}}$$

Question #17

$$\int_{\pi/6}^{\pi/4} \cos^2\theta \cot^2\theta\, d\theta$$

Solution:

$$= \int_{\pi/6}^{\pi/4} \cos^2\theta \cdot \cot^2\theta\, d\theta$$

$$\because 1 + \cot^2\theta = \operatorname{cosec}^2\theta$$

$$\because \cot^2\theta = \operatorname{cosec}^2\theta - 1$$

$$= \int_{\pi/6}^{\pi/4} \cos^2\theta\, (\operatorname{cosec}^2\theta - 1)\, d\theta$$

$$= \int_{\pi/6}^{\pi/4} (\cos^2\theta \operatorname{cosec}^2\theta - \cos^2\theta)\, d\theta$$

$$= \int_{\pi/6}^{\pi/4} \cos^2\theta \operatorname{cosec}^2\theta\, d\theta - \int_{\pi/6}^{\pi/4} \cos^2\theta\, d\theta$$

$$= \int_{\pi/6}^{\pi/4} \frac{\cos^2\theta}{\sin^2\theta}\, d\theta - \int_{\pi/6}^{\pi/4} \cos^2\theta\, d\theta$$

$$= \int_{\pi/6}^{\pi/4} \cot^2\theta\, d\theta - \int_{\pi/6}^{\pi/4} \frac{1+\cos 2\theta}{2}\, d\theta$$

$$= \int_{\pi/6}^{\pi/4} (\operatorname{cosec}^2\theta - 1)\, d\theta - \frac{1}{2}\int_{\pi/6} 1 + \cos 2\theta\, d\theta$$

$$= \int_{\pi/6}^{\pi/4} \operatorname{cosec}^2\theta\, d\theta - \int_{\pi/6}^{\pi/4} 1\, d\theta - \frac{1}{2}\int_{\pi/6}^{\pi/4} 1\, d\theta - \frac{1}{2}\int_{\pi/6}^{\pi/4} \cos 2\theta\, d\theta$$

$$= \left|-\cot\theta\right|_{\pi/6}^{\pi/4} - \int_{\pi/6}^{\pi/4}\left(1+\frac{1}{2}\right)d\theta - \frac{1}{2}\left|\frac{\sin 2\theta}{2}\right|_{\pi/6}^{\pi/4}$$

$$= \left|-\cot\theta\right|_{\pi/6}^{\pi/4} - \frac{3}{2}\left|\theta\right|_{\pi/6}^{\pi/4} - \frac{1}{4}\left|\sin 2\theta\right|_{\pi/6}^{\pi/4}$$

$$= -\left[\cot\frac{\pi}{4} - \cot\frac{\pi}{6}\right] - \frac{3}{2}\left[\frac{\pi}{4} - \frac{\pi}{6}\right] - \frac{1}{4}\left[\sin \cancel{2}\left(\frac{\pi}{\underset{2}{\cancel{4}}}\right) - \sin \cancel{2}\left(\frac{\pi}{\underset{3}{\cancel{6}}}\right)\right]$$

$$= -(1-\sqrt{3}) - \frac{3}{2}\left(\frac{3\pi-2\pi}{12}\right) - \frac{1}{4}\left(\sin\frac{\pi}{2} - \sin\frac{\pi}{3}\right]$$

$$= -1+\sqrt{3} - \frac{3}{2}\left(\frac{\pi}{12}\right) - \frac{1}{4}\left(1 - \frac{\sqrt{3}}{2}\right)$$

$$= -1+\sqrt{3} - \frac{\cancel{3}\pi}{\underset{8}{\cancel{24}}} - \frac{1}{4}\left(1 - \frac{\sqrt{3}}{2}\right)$$

$$= -1+\sqrt{3} - \frac{\pi}{8} - \frac{1}{4} + \frac{\sqrt{3}}{8}$$

$$= \frac{-8+8\sqrt{3}-\pi-2+\sqrt{3}}{8}$$

$$= \boxed{\frac{9\sqrt{3}-10-\pi}{8}}$$

Question #18

$$\int_0^{\pi/4} \cos^4 t\, dt$$

Solution:

$$= \int_0^{\pi/4} \cos^4 t\, dt$$

$$= \int_0^{\pi/4} (\cos^2 t)^2\, dt$$

$$\because \cos^2\theta = \frac{1+\cos 2\theta}{2}$$

$$= \int_0^{\pi/4} \left(\frac{1+\cos 2t}{2}\right)^2 dt$$

$$= \int_0^{\pi/4} \frac{1+2\cos 2t+\cos^2 2t}{4}\, dt$$

$$= \frac{1}{4}\int_0^{\pi/4} \left(1 + 2\cos 2t + \frac{1\cos 4t}{2}\right)dt$$

$$= \frac{1}{4}\left[\int_0^{\pi/4} 1\, dt + 2\int_0^{\pi/4} \cos 2t\, dt + \frac{1}{2}\int_0^{\pi/4} \cos^2 2t\, dt\right]$$

$$= \frac{1}{4}\left[|t|_0^{\pi/4} + 2\left|\frac{\sin 2t}{2}\right|_0^{\pi/4} + \frac{1}{2}\int_0^{\pi/4} 1+\cos 4t\, dt\right]$$

$$= \frac{1}{4}\left[\left[\frac{\pi}{4} - 0\right] + \frac{\cancel{2}}{\cancel{2}}\left|\sin 2t\right|_0^{\pi/4} + \frac{1}{2}\int_0^{\pi/4} 1\, dt + \frac{1}{2}\int_0^{\pi/4} \cos 4t\, dt\right.$$

$$= \frac{1}{4}\left[\frac{\pi}{4} + 1\left[\sin \cancel{2}\frac{\pi}{\underset{2}{\cancel{4}}} - \sin 2(0)\right] + \frac{1}{2}|t|_0^{\pi/4} + \frac{1}{2}\left|\frac{\sin 4t}{4}\right|_0^{\pi/4}\right.$$

$$= \frac{1}{4}\left[\frac{\pi}{4} + 1\left[\sin\frac{\pi}{2} - 0\right] + \frac{1}{2}\left(\frac{\pi}{4} - 0\right) + \frac{1}{8}\left[\sin \cancel{4}\frac{\pi}{\cancel{4}} - \sin 4(0)\right]\right.$$

$$= \frac{1}{4}\left[\frac{\pi}{4} + 1\left[(1) - 0\right] + \frac{\pi}{8} + \frac{1}{8}(\sin\pi - 0)\right.$$

$$= \frac{1}{4}\left[\frac{\pi}{4} + 1 + \frac{\pi}{8} + \frac{1}{8}(0)\right]$$

$$= \frac{1}{4}\left[\frac{\pi}{4} + 1 + \frac{\pi}{8}\right]$$

$$= \frac{1}{4}\left(\frac{2\pi+8+\pi}{8}\right)$$

$= \frac{1}{4}\left(\frac{3\pi+8}{8}\right)$

$= \boxed{\frac{3\pi+8}{32}}$

Question #19

$$\int_0^{\pi/3} \cos^2\theta \ \sin\theta \ d\theta$$

Solution:

$= \int_0^{\pi/3} \cos^2\theta \sin\theta \, d\theta$

$= -\int_0^{\pi/3} \cos^2\theta \, (-\sin\theta) \, d\theta$

$\because \int [f(x)]^n \cdot f'(x) dx = \frac{[f(x)]^{n+1}}{n+1}$

$= -\left|\frac{\cos^3\theta}{3}\right|_0^{\pi/3}$

$= -\frac{1}{3}\left|\cos^3\theta\right|_0^{\pi/3}$

$= -\frac{1}{3}\left[\cos^3\frac{\pi}{3} - \cos^3(0)\right]$

$= -\frac{1}{3}\left[\left(\frac{1}{2}\right)^3 - (1)^3\right]$

$= -\frac{1}{3}\left[\frac{1}{8} - 1\right]$

$= -\frac{1}{3}\left[\frac{1-8}{8}\right]$

$= -\frac{1}{3}\left[\frac{-7}{8}\right]$

$= \boxed{\frac{7}{24}}$

Question #20

$$\int_0^{\pi/4} (1+\cos^2\theta) \tan^2\theta \ d\theta$$

Solution:

$= \int_0^{\pi/4} \tan^2\theta + \cos^2\theta \tan^2\theta \ d\theta$

$= \int_0^{\pi/4} \tan^2\theta + \cancel{\cos^2\theta} \cdot \frac{\sin^2\theta}{\cancel{\cos^2\theta}} \ d\theta$

$= \int_0^{\pi/4} \tan^2\theta \ d\theta + \int_0^{\pi/4} \sin^2\theta \ d\theta$

$\because \tan^2\theta = \sec^2\theta - 1$

$\because \sin^2\theta = \frac{1-\cos2\theta}{2}$

$= \int_0^{\pi/4} \sec^2\theta - 1 \ d\theta + \int_0^{\pi/4} \frac{1-\cos2\theta}{2} \ d\theta$

$= \int_0^{\pi/4} \sec^2\theta \ d\theta - \int_0^{\pi/4} 1 \ d\theta + \frac{1}{2}\int_0^{\pi/4} 1 \ d\theta - \frac{1}{2}\int_0^{\pi/4} \cos2\theta d\theta$

$= |\tan\theta|_0^{\pi/4} - |\theta|_0^{\pi/4} + \frac{1}{2}|\theta|_0^{\pi/4} - \frac{1}{2}\left|\frac{\sin2\theta}{2}\right|_0^{\pi/4}$

$= \left(\tan\frac{\pi}{4} - \tan0\right) - \left(\frac{\pi}{4} - 0\right) + \frac{1}{2}\left(\frac{\pi}{4} - 0\right) - \frac{1}{4}\left[\sin \cancel{2} \frac{\pi}{\cancel{4}_2} - \sin 2(0)\right]$

$= (1-0) - \frac{\pi}{4} + \frac{\pi}{8} - \frac{1}{4}\left(\sin\frac{\pi}{2} - \sin0\right)$

$= 1 - \frac{\pi}{4} + \frac{\pi}{8} - \frac{1}{4} \ (1-0)$

$= 1 - \frac{\pi}{4} + \frac{\pi}{8} - \frac{1}{4}$

$= \frac{8 - 2\pi + \pi - 2}{8} = \boxed{\frac{6-\pi}{8}}$

Question #21

$$\int_0^{\pi/4} \frac{\sec\theta}{\sin\theta + \cos\theta}\, d\theta$$

Solution:

$$= \int_0^{\pi/4} \frac{\sec\theta}{\sin\theta+\cos\theta}\, d\theta$$

$$= \int_0^{\pi/4} \frac{\sec\theta}{\cos\theta\left(\frac{\sin\theta}{\cos\theta}+1\right)}\, d\theta$$

$$= \int_0^{\pi/4} \frac{\sec\theta \sec\theta}{\tan\theta+1}\, d\theta$$

$$= \int_0^{\pi/4} \frac{\sec^2\theta}{\tan\theta+1}\, d\theta$$

$$\because \int \frac{f'(x)}{f(x)}\, dx = \ln|f(x)$$

$$= \left|\ln|\tan\theta+1|\right|_0^{\pi/4}$$

$$= \ln|\tan\tfrac{\pi}{4}+1| - \ln|\tan 0+1)$$

$$= \ln(1+1) - \ln|0+1)$$

$$= \ln 2 - \ln 1$$

$$= \ln 2 - 0 \quad \because \ln 1 = 0$$

$$= \boxed{\ln 2}$$

Question #22

$$\int_{-1}^{5} |x-3|\, dx$$

Solution:

$|x-3| = -(x-3)$ if $-1 < x < 3$

$|x-3| = (x-3)$ if $3 < x < 5$

$$= \int_{-1}^{3} -(x-3)\, dx + \int_{3}^{5} (x-3)\, dx$$

$$= -\left|\frac{(x-3)^2}{2}\right|_{-1}^{3} + \left|\frac{(x-3)^2}{2}\right|_{3}^{5}$$

$$= -\frac{1}{2}\left[(3-3)^2 - (-1-3)^2\right] + \frac{1}{2}\left[(5-3)^2 - (3-3)^2\right]$$

$$= -\frac{1}{2}\left[(0)^2 - (-4)^2\right] + \frac{1}{2}\left[(2)^2 - (0)^2\right]$$

$$= -\frac{1}{2}\left[0-(16)\right] + \frac{1}{2}(4-0)$$

$$= -\frac{1}{2}(-16) + \frac{\cancel{4}^{\,2}}{\cancel{2}}$$

$$= \frac{\overset{8}{\cancel{16}}}{\cancel{2}} + 2$$

$$= 8+2$$

$$= \boxed{10}$$

Question #23

$$\int_{1/8}^{1} \frac{(x^{1/3}+2)^2}{x^{2/3}}\, dx$$

Solution:

$$\int_{1/8}^{1} \frac{(x^{1/3}+2)^2}{x^{2/3}}\, dx$$

$$= \int_{1/8}^{1} (x^{1/3}+2)^2 \cdot x^{-2/3}\, dx$$

Multiply and divided by "3"

$$= 3\int_{1/8}^{1} (x^{1/3}+2)^2 \cdot \frac{1}{3} x^{-2/3}\, dx$$

$$\because \int [f(x)]^n \cdot f'(x)\, dx = \frac{[f(x)]^{n+1}}{n+1}$$

$$= 3 \left| \frac{(x^{1/3}+2)^3}{3} \right|_{1/8}^{1}$$

$$= \frac{\cancel{3}}{\cancel{3}}\left[((1)^{1/3}+2)^3 - \left[\left(\frac{1}{8}\right)^{1/3}+2\right]^3\right.$$

$$= \left[(1+2)^3 - \left[\left(\frac{1}{2^3}\right)^{1/3}+2\right]^3\right.$$

$$= \left[\ (3)^3 - \left[\ \frac{1}{2}+2\right]^3\right.$$

$$= \left[\ 27 - \left[\ \frac{1+4}{2}\right]^3\right.$$

$$= \left[\ 27 - \left(\frac{5}{2}\right)^3\right]$$

$$= 27 - \frac{125}{8}$$

$$= \frac{216-125}{8}$$

$$= \boxed{\frac{91}{8}}$$

Question #24

$$\int_1^3 \frac{x^2-2}{x+1}\, dx$$

Solution:

$$= \int_1^3 \frac{x^2-2}{x+1}\, dx$$

$$= \begin{array}{r|l} & x-1 \\ \hline x+1 & \cancel{x^2}-2 \\ & -\cancel{x^2} \mp x \\ \hline & -\cancel{x}-2 \\ & \mp\cancel{x} \mp 1 \\ \hline & -1 \end{array}$$

$$= \int_1^3 \left((x-1) - \frac{1}{x+1}\right) dx$$

$$= \int_1^3 (x-1)\, dx - \int_1^3 \frac{1}{x+1}\, dx$$

$$= \frac{(x-1)^2}{2}\Big|_1^3 - \ln|x+1|\Big|_1^3$$

$$= \frac{1}{2}\left[(3-1)^2 - (1-1)^2\right] - \left[\ln|3+1| - \ln|1+1|\right]$$

$$= \frac{1}{2}\left[(2)^2 - (0)^2\right] - \left[\ln 4 - \ln 2\right]$$

$$= \frac{1}{2}(4-0) - \left[\ln 4 - \ln 2\right]$$

$$= \frac{\cancel{4}^{2}}{\cancel{2}} - \ln\frac{\cancel{4}^{2}}{\cancel{2}}$$

$$= \boxed{2-\ln 2}$$

Question #25

$$\int_2^3 \frac{3x^2-2x+1}{(x-1)(x^2+1)}\, dx$$

Solution:

$$= \int_2^3 \frac{3x^2-2x+1}{(x-1)(x^2+1)}\, dx$$

$$= \int_{2}^{3} \frac{3x^2-2x+1}{x^3-x^2+x-1}\, dx$$

$$\because \int \frac{f'(x)}{f(x)}\, dx = \ln|f(x)|$$

$$= \left[\ln|x^3-x^2+x-1|\right]_2^3$$

$$= \ln|(3)^3-(3)^2+3-1| - \ln|(2)^3-(2)^2+2-1|$$

$$= \ln|27-9+3-1| - \ln|8-4+1|$$

$$= \ln 20 - \ln 5$$

$$\because \ln m - \ln n = \ln\frac{m}{n}$$

$$= \ln \frac{\overset{4}{\cancel{20}}}{\cancel{5}}$$

$$= \boxed{\ln 4}$$

Question #26

$$\int_{0}^{\pi/4} \frac{\sin x - 1}{\cos^2 x}\, dx$$

Solution:

$$= \int_{0}^{\pi/4} \frac{\sin x - 1}{\cos^2 x}\, dx$$

$$= \int_{0}^{\pi/4} \left(\frac{\sin x}{\cos^2 x} - \frac{1}{\cos^2 x}\right) dx$$

$$= \int_{0}^{\pi/4} \left(\frac{1}{\cos x}\cdot\frac{\sin x}{\cos x} - \frac{1}{\cos^2 x}\right) dx$$

$$= \int_{0}^{\pi/4} \sec x \tan x - \sec^2 x\, dx$$

$$= \int_{0}^{\pi/4} \sec x \tan x\, dx - \int_{0}^{\pi/4} \sec^2 x\, dx$$

$$= \left|\sec x\right|_0^{\pi/4} - \left|\tan x\right|_0^{\pi/4}$$

$$= \left[\sec\frac{\pi}{4} - \sec 0\right] - \left[\tan\frac{\pi}{4} - \tan 0\right]$$

$$= (\sqrt{2} - 1) - (1 - 0)$$

$$= \sqrt{2} - 1 - 1$$

$$= \boxed{\sqrt{2} - 2}$$

Question #27

$$\int_{0}^{\pi/4} \frac{1}{1+\sin x}\, dx$$

Solution:

$$= \int_{0}^{\pi/4} \frac{1}{1+\sin x} \times \frac{1-\sin x}{1-\sin x}\, dx$$

$$= \int_{0}^{\pi/4} \frac{1-\sin x}{(1+\sin x)(1-\sin x)}\, dx$$

$$\because (a+b)(a-b) = a^2 - b^2$$

$$= \int_{0}^{\pi/4} \frac{1-\sin x}{1-\sin^2 x}\, dx$$

$$\because 1 - \sin^2 x = \cos^2 x$$

$$= \int_{0}^{\pi/4} \frac{1-\sin x}{\cos^2 x}\, dx$$

$$= \int_{0}^{\pi/4} \frac{1}{\cos^2 x} - \frac{\sin x}{\cos^2 x}\, dx$$

$$= \int_{0}^{\pi/4} \frac{1}{\cos^2 x} - \frac{1}{\cos x}\cdot\frac{\sin x}{\cos x}\, dx$$

$$= \int_{0}^{\pi/4} \sec^2 x\, dx - \int_{0}^{\pi/4} \sec x \tan x\, dx$$

$= \left|\tan x\right|_0^{\pi/4} - \left|\sec x\right|_0^{\pi/4}$

$= \left[\tan\frac{\pi}{4} - \tan 0\right] - \left[\sec\frac{\pi}{4} - \sec 0\right]$

$= (1-0) - [\sqrt{2} - 1)$

$= 1 - \sqrt{2} + 1$

$= \boxed{2-\sqrt{2}}$

Question #28

$$\int_0^1 \frac{3x}{\sqrt{4-3x}}\,dx$$

Solution:

$= \int_0^1 \frac{3x}{\sqrt{4-3x}}\,dx$

$= -\int_0^1 \frac{-3x}{\sqrt{4-3x}}\,dx$

By Adding and subtracting "4"

$= -\int_0^1 \left(\frac{4-3x-4}{\sqrt{4-3x}}\right)dx$

$= -\int_0^1 \left(\frac{4-3x}{\sqrt{4-3x}} - \frac{4}{\sqrt{4-3x}}\right)dx$

$= -\int_0^1 \sqrt{4-3x} - 4(4-3x)^{-1/2}\Big)dx$

$= -\int_0^1 (4-3x)^{1/2}dx + 4\int_0^1 (4-3x)^{-1/2}dx$

Multiply and divided by (-3)

$= -\left(\frac{-1}{3}\right)\int_0^1 (4-3x)^{1/2}\cdot -3\,dx + 4\left(\frac{-1}{3}\right)\int_0^1 (4-3x)^{-1/2}\cdot -3\,dx$

$\because \int [f(x)]^n \cdot f'(x)\,dx = \frac{[f(x)]^{n+1}}{n+1}$

$= \frac{1}{3}\left|\frac{(4-3x)^{1/2+1}}{\frac{1}{2}+1}\right|_0^1 - \frac{4}{3}\left|\frac{(4-3x)^{-1/2+1}}{\frac{-1}{2}+1}\right|_0^1$

$= \frac{1}{3}\left[\frac{(4-3x)^{3/2}}{3/2}\right]_0^1 - \frac{4}{3}\left|\frac{(4-3x)^{1/2}}{1/2}\right|_0^1$

$= \frac{1}{3}\cdot\frac{2}{3}\left[(4-3(1))^{3/2} - (4-3(0))^{3/2}\right] - \frac{4}{3}\cdot 2\left[(4-3(1))^{1/2} - (4-3(0))^{3/2}\right.$

$= \frac{2}{9}\left[(4-3)^{3/2} - (4-0)^{3/2}\right] - \frac{8}{3}\left[(4-3)^{1/2} - (4-0)^{1/2}\right]$

$= \frac{2}{9}\left[(1)^{3/2} - (4)^{3/2}\right] - \frac{8}{3}\left[(1)^{1/2} - (4)^{1/2}\right]$

$= \frac{2}{9}\left[1 - (2^2)^{3/2}\right] - \frac{8}{3}\left[1 - (2^2)^{1/2}\right]$

$= \frac{2}{9}\left[1 - (2)^3\right] - \frac{8}{3}\left[1 - 2\right]$

$= \frac{2}{9}[1-8) - \frac{8}{3}[-1)$

$= \frac{2}{9}(-7) + \frac{8}{3}$

$= \frac{-14}{9} + \frac{8}{3}$

$= \frac{-14+24}{9}$

$= \boxed{\frac{10}{9}}$

Question #29

$$\int_{\pi/6}^{\pi/2} \frac{\cos x}{\sin x(2+\sin x)}\,dx$$

Solution: $\int_{\pi/6}^{\pi/2} \frac{\cos x}{\sin x\cdot\sin x\left(\frac{2}{\sin x}+1\right)}\,dx$

$= \int_{\pi/6}^{\pi/2} \frac{1}{\sin x}\cdot\frac{\cos x}{\sin x}\cdot\frac{1}{(2\operatorname{cosec} x+1)}\,dx$

$= \int_{\pi/6}^{\pi/2} \frac{\operatorname{cosec} x\cot x}{2\operatorname{cosec} x+1}\,dx$

Multiply and divided by "-2"

$$-\frac{1}{2}\int_{\pi/6}^{\pi/2}\left(\frac{-2\,\text{cosec}\,x\cot x}{2\,\text{cosec}\,x+1}\right)dx$$

$$\because \int\frac{f'(x)}{f(x)}=\ln|f(x)|$$

$$=-\frac{1}{2}\left|\ln|2\,\text{cosec}\,x+1|\right|_{\pi/6}^{\pi/2}$$

$$=-\frac{1}{2}\left[\ln\left|2\,\text{cosec}\frac{\pi}{2}+1\right|-\ln\left|2\,\text{cosec}\frac{\pi}{6}+1\right|\right]$$

$$=-\frac{1}{2}\left[\ln|2(1)+1|-\ln|2(2)+1|\right]$$

$$=-\frac{1}{2}\left[\ln|2+1|-\ln|4+1|\right]$$

$$=-\frac{1}{2}\left[\ln 3-\ln 5\right]$$

$$\because \ln m-\ln n=\ln\frac{m}{n}$$

$$=\frac{1}{2}\left[\ln 5-\ln 3\right]$$

$$=\boxed{\frac{1}{2}\ln\frac{5}{3}}$$

Question #30

$$\int_0^{\pi/2}\frac{\sin x}{(1+\cos x)(2+\cos x)}\,dx$$

Solution:

$$\int_0^{\pi/2}\frac{\sin x}{(1+\cos x)(2+\cos x)}\,dx$$

put

$t=\cos x$

$dt=-\sin x\,dx$

$-dt=\sin x\,dx$

As

$x\to 0$ then $t\to 1$

$x\to\frac{\pi}{2}$ then $t\to 0$

$$=\int_1^0\frac{-dt}{(1+t)(2+t)}\qquad \because \cos 0=1 \quad \because \cos\frac{\pi}{2}=0$$

$$=-\int_0^1\frac{-dt}{(1+t)(2+t)}$$

$$=\int_0^1\frac{1}{(1+t)(2+t)}\,dt$$

$$\frac{1}{(1+t)(2+t)}=\frac{A}{1+t}+\frac{B}{2+t}\quad\text{——}\ ①$$

Multiply by $(1+t)(2+t)$ on both sides

$$1=A(2+t)+B(1+t)\quad\text{——}\ ②$$

$1+t=0 \Rightarrow \boxed{t=-1}$ put in eq ②

$1=A(2-1)+B(1-1)$

$1=A(1)+0$

$1=A$

$\boxed{A=1}$

$2+t=0 \Rightarrow \boxed{t=-2}$ put in eq ②

$1=A(2-2)+B(1-2)$

$1=0+B(-1)$

$\boxed{B=-1}$

$$\frac{1}{(1+t)(2+t)}=\frac{1}{1+t}-\frac{1}{2+t}$$

$$\int_0^1\frac{1}{(1+t)(2+t)}\,dt=\int_0^1\frac{1}{1+t}\,dt-\int_0^1\frac{1}{2+t}\,dt$$

$$=\left|\ln|1+t|\right|_0^1-\left|\ln|2+t|\right|_0^1$$

$$=\left[\ln|1+1|-\ln|1+0|\right]-\left[\ln|2+1|-\ln|2+0|\right]$$

$$=\left[\ln 2-\ln 1\right]-\left[\ln 3-\ln 2\right]$$

$$=\ln 2-\ln 1-\ln 3+\ln 2$$

$$\because \ln 1=0$$

$$=2\ln 2-0-\ln 3$$

$$\because \ln m^n=n\ln m$$

$$=\ln 2^2-\ln 3$$

$$=\ln 4-\ln 3\qquad \because \ln m-\ln n=\ln\left(\frac{m}{n}\right)$$

$$=\boxed{\ln\frac{4}{3}}$$

* Theory

Example #1

Find the area bounded by the curve $y = 4 - x^2$ and the x-axis.

Solution:

$y = 4 - x^2$

For x-intercep, put $y = 0$

$0 = 4 - x^2$

$x^2 = 4$

$x = \pm 2$

x	$y = 4 - x^2$
-2	0
-1	3
0	4
1	3
2	0

$y = 4 - x^2 \geqslant 0$ in $[-2, 2]$

$$\text{Area} = \int_{-2}^{2} (4 - x^2)\,dx$$

$$\text{Area} = 4\int_{-2}^{2} 1\,dx - \int_{-2}^{2} x^2\,dx$$

$$\text{Area} = 4x\Big|_{-2}^{2} - \frac{x^3}{3}\Big|_{-2}^{2}$$

$$\text{Area} = 4[(2) + 2] - \frac{1}{3}[(2)^3 - (-2)^3]$$

$$\text{Area} = 4\ (4) - \frac{1}{3}[8 - (-8)]$$

$$\text{Area} = 16 - \frac{1}{3}(8 + 8)$$

$$\text{Area} = 16 - \frac{16}{3}$$

$$\text{Area} = \frac{48 - 16}{3}$$

$$\boxed{\text{Area} = \frac{32}{3}\ \text{sq}\cdot\text{unit}}$$

Example #2

Find the area bounded by the curve $y = x^3 + 3x^2$ and the x-axis

Solution

$y = x^3 + 3x^2$

For x-intercept, put $y = 0$

$0 = x^3 + 3x^2$

$0 = x^2(x + 3)$

$x^2 = 0$, $x + 3 = 0$

$\boxed{x = 0}$, $\boxed{x = -3}$

x	$y = x^3 + 3x^2$
-3	0
-2	4
-1	2
0	0

$y = x^3 + 3x^2 \geqslant 0$ in $[-3, 0]$

$$\text{Area} = \int_{-3}^{0} x^3 + 3x^2\,dx$$

$$\text{Area} = \int_{-3}^{0} x^3\,dx + 3\int_{-3}^{0} x^2\,dx$$

$$\text{Area} = \frac{x^4}{4}\Big|_{-3}^{0} + \cancel{3}\cdot\frac{x^3}{\cancel{3}}\Big|_{-3}^{0}$$

$$\text{Area} = \frac{1}{4}|x^4|_{-3}^{0} + |x^3|_{-3}^{0}$$

$$\text{Area} = \frac{1}{4}[(0)^4 - (-3)^4] + [(0)^3 - (-3)^3]$$

$$\text{Area} = \frac{1}{4}[0 - 81] + [0 - (-27)]$$

$$\text{Area} = \frac{1}{4}(-81) + 27$$

$$\text{Area} = \frac{-81}{4} + 27$$

$$\text{Area} = \frac{-81 + 108}{4}$$

$$\boxed{\text{Area} = \frac{27}{4}\ \text{sq}\cdot\text{unit}}$$

Example #3

Find the area bounded by the $y = x(x^2-4)$ and the x-axis.

Solution:

$y = x(x^2-4)$

$y = x^3-4x$

For x-intercept
put $y=0$

$0 = x^3-4x$

$0 = x(x^2-4)$

$x=0 \quad , \quad x^2-4=0$

$x^2=4$

$x=\pm 2$

$\boxed{x=0}$, $\boxed{x=+2}$, $\boxed{x=-2}$

x	$y=x^3-4x$
0	0
1	-3
2	8

$y= x^3-4x \leq 0$ in $[0,2]$

$Area_1 = -\int_0^2 (x^3-4x)dx$

$Area_1 = -\left[\int_0^2 x^3dx - 4\int_0^2 x\,dx\right]$

$Area_1 = -\left[\frac{x^4}{4}\Big|_0^2 - \cancel{4}^{2}\cdot\frac{x^2}{\cancel{2}}\Big|_0^2\right]$

$Area_1 = -\left[\frac{1}{4}|x^4|_0^2 - 2x^2|_0^2\right]$

$Area_1 = -\left[\frac{1}{4}[(2)^4-(0)^4] - 2[(2)^2-(0)^2]\right]$

$Area_1 = -\left[\frac{1}{4}(16) - 2[4)\right]$

$Area_1 = -[4-8]$

$Area_1 = -4+8$

$\boxed{A_1 = 4}$

$y = x^3-4x \geqslant 0$ in $[-2,0]$

x	$y=x^3-4x$
-2	0
-1	3
0	0

$Area_2 = \int_{-2}^{0}(x^3-4x)dx$

$A_2 = \int_{-2}^{0} x^3dx - 4\int_{-2}^{0} x\,dx$

$A_2 = \frac{x^4}{4}\Big|_{-2}^{0} - \cancel{4}^{2}\cdot\frac{x^2}{\cancel{2}}\Big|_{-2}^{0}$

$A_2 = \frac{1}{4}[x^4]_{-2}^{0} - 2x^2|_{-2}^{0}$

$A_2 = \frac{1}{4}[(0)^4-(-2)^4] - 2[(0)^2-(-2)^2]$

$A_2 = \frac{1}{4}[0-16] - 2[0-4]$

$A_2 = \frac{1}{4}(-16) - 2(-4)$

$A_2 = -4+8$

$\boxed{A_2 = 4}$

$Area = A_1 + A_2$

$Area = 4+4$

$\boxed{Area = 8 \text{ sq.unit}}$

Example #4

Find the area bounded by the curve $f(x) = x^3-2x^2+1$ and the x-axis in the 1st quadrant.

Solution:

$f(x) = x^3-2x^2+1$

$y = x^3-2x^2+1$

For x-intercept

put $y=0$

$0=x^3-2x^2+1$

$0=x^3-x^2-x^2+1$

$x^2(x-1)-1(x^2-1)=0$

$x^2(x-1)-1(x-1)(x+1)=0$

$(x-1)\{x^2-1(x+1)]=0$

$(x-1)(x^2-x-1)=0$

$x-1=0 \quad , \quad x^2-x-1=0$

$\boxed{x=1}$

$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

$x=\frac{-(-1)\pm\sqrt{(-1)^2-4(1)(-1)}}{2(1)}$

$x=\frac{1\pm\sqrt{1+4}}{2}$

$\boxed{x=\frac{1\pm\sqrt{5}}{2}}$

For 1st quadrant

$y \geqslant 0$ in $[0,1]$

$\text{Area}=\int_0^1 (x^3-2x^2+1)\,dx$

$\text{Area}=\int_0^1 x^3dx-2\int_0^1 x^2dx+\int_0^1 1\,dx$

$\text{Area}=\frac{x^4}{4}\Big|_0^1-2\cdot\frac{x^3}{3}\Big|_0^1+x\Big|_0^1$

$\text{Area}=\frac{1}{4}[(1)^4-(0)^4]-\frac{2}{3}[(1)^3-(0)^3]+(1-0))$

$\text{Area}=\frac{1}{4}(1)-\frac{2}{3}(1)+(1)$

$\text{Area}=\frac{1}{4}-\frac{2}{3}+1$

$\text{Area}=\frac{3-8+12}{12}$

$\boxed{\text{Area}=\frac{7}{12}\text{ sq.unit}}$

Example #5

Find the area between the x-axis and the curve $y^2=4-x$ in the first quadrant from $x=0$ to $x=3$

Solution:

$y^2=4-x$

$y=\pm\sqrt{4-x}$

$y=\sqrt{4-x} \quad , \quad y=-\sqrt{4-x}$

$y>0$ in Ist quadrant so, we reject

$y=-\sqrt{4-x}$

Thus,

$\text{Area}=\int_0^3 \sqrt{4-x}\,dx$

$\text{Area}=-1\int_0^3 \sqrt{4-x}\cdot(-1)\,dx$

$\text{Area}=-\int_0^3 (4-x)^{1/2}\cdot(-1)\,dx$

$\text{Area}=-\left[\frac{(4-x)^{1/2+1}}{1/2+1}\right]_0^3$

$\text{Area}=-\left[\frac{(4-x)^{3/2}}{3/2}\right]_0^3$

$\text{Area}=-\frac{2}{3}[(4-3)^{3/2}-(4-0)^{3/2}]$

$\text{Area}=-\frac{2}{3}[(1)^{3/2}-(4)^{3/2}]$

$\text{Area}=-\frac{2}{3}[1-(2^2)^{3/2}]$

$\text{Area}=-\frac{2}{3}[1-(2)^3]$

$\text{Area}=-\frac{2}{3}[1-8]$

$\text{Area}=-\frac{2}{3}(-7)$

$\boxed{\text{Area}=\frac{14}{3}\text{ sq.unit}}$

Exercise # 3·7

Question # 1

Find the area between the x-axis and the curve $y = x^2+1$ from $x=1$ to $x=2$.

Solution:

$$y = x^2+1$$

$$y = x^2+1 > 0 \text{ in } [a, b]$$

$$\text{Area} = \int_1^2 y\,dx$$

$$\text{Area} = \int_1^2 x^2+1\,dx$$

$$\text{Area} = \int_1^2 x^2\,dx + \int_1^2 1\,dx$$

$$\text{Area} = \left|\frac{x^3}{3}\right|_2^2 + |x|_1^2$$

$$\text{Area} = \frac{1}{3}\left[(2)^3-(1)^3\right] + (2-1)$$

$$\text{Area} = \frac{1}{3}[8-1] + (1)$$

$$\text{Area} = \frac{1}{3}(7)+1$$

$$\text{Area} = \frac{7}{3}+1$$

$$\text{Area} = \frac{7+3}{3}$$

$$\boxed{\text{Area} = \frac{10}{3} \text{ sq. units}}$$

•———•

Question # 2

Find the area above the x-axis under the curve $y = 5-x^2$ from $x=-1$ to $x=2$.

Solution:

$$y = 5-x^2$$

Area above the x-axis

$$\text{Area} = \int_{-1}^{2} 5-x^2\,dx$$

$$\text{Area} = \int_{-1}^{2} 5\,dx - \int_{-1}^{2} x^2\,dx$$

$$\text{Area} = 5|x|_{-1}^{2} - \left|\frac{x^3}{3}\right|_{-1}^{2}$$

$$\text{Area} = 5[2-(-1)] - \frac{1}{3}\left[(2)^3-(-1)^3\right]$$

$$\text{Area} = 5(2+1) - \frac{1}{3}[8-(-1)]$$

$$\text{Area} = 5(3) - \frac{1}{3}[8+1]$$

$$\text{Area} = 15 - \frac{1}{\cancel{3}}(\overset{3}{\cancel{9}})$$

$$\text{Area} = 15-3$$

$$\boxed{\text{Area} = 12 \text{ sq. units}}$$

•———•

Question # 3

Find the area below the curve $y = 3\sqrt{x}$ and above the x-axis between $x=1$ and $x=4$

Solution:

$y = 3\sqrt{x}$

Area is above the x-axis

Area $= \int_1^4 y\,dx$

Area $= \int_1^4 3\sqrt{x}\,dx$

Area $= 3\int_1^4 x^{1/2}\,dx$

Area $= 3\left|\frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right|_1^4$

Area $= 3\left[\frac{x^{3/2}}{3/2}\right]_1^4$

Area $= \cancel{3}\cdot\frac{2}{3}\left[(4)^{3/2} - (1)^{3/2}\right]$

Area $= 2\left[(2^2)^{3/2} - 1\right]$

Area $= 2\left[(2)^3 - 1\right]$

Area $= 2(8-1)$

Area $= 2(7)$

$\boxed{\text{Area} = 14 \text{ sq. unit}}$

Question # 4

Find the area bounded by cos function from $x = -\frac{\pi}{2}$ to $x = \frac{\pi}{2}$

Solution:

$y = \cos x$

$y = \cos x \geq 0$ in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Area $= \int_{-\pi/2}^{\pi/2} \cos x\,dx$

Area $= \left|\sin x\right|_{-\pi/2}^{\pi/2}$

Area $= \left[\sin\frac{\pi}{2} - \sin\frac{\pi}{2}\right]$

Area $= \left[(1) - (-1)\right]$

Area $= 1+1$

$\boxed{\text{Area} = 2 \text{ sq. unit}}$

Question # 5

Find the area between the x-axis and the curve $y = 4x - x^2$

Solution:

$y = 4x - x^2$

For x-intercept

put $y = 0$

$0 = 4x - x^2$

$0 = x(4-x)$

$\boxed{x = 0}$, $4 - x = 0$

$\boxed{4 = x}$

x	$y = 4x - x^2$
0	0
1	3
2	4
3	3
4	0

$y = 4x - x^2 \geq 0$ in $[0, 4]$

Area $= \int_0^4 4x - x^2\,dx$

Area $= 4\int_0^4 x\,dx - \int_0^4 x^2\,dx$

Area $= \overset{2}{\cancel{4}}\cdot\frac{x^2}{\cancel{2}}\Big|_0^4 - \frac{x^3}{3}\Big|_0^4$

Area $= 2x^2\Big|_0^4 - \frac{1}{3}\left|x^3\right|_0^4$

Area $= 2\left[(4)^2 - (0)^2\right] - \frac{1}{3}\left[(4)^3 - (0)^3\right]$

Area $= 2\left[16 - 0\right] - \frac{1}{3}\left[64 - 0\right]$

Area $= 32 - \frac{64}{3}$

$$\text{Area} = \frac{96-64}{3}$$

$$\boxed{\text{Area} = \frac{32}{3} \text{ sq. unit}}$$

Question #6

Determine the area bounded by the parabola $y = x^2+2x-3$ and x-axis:

Solution:

$$y = x^2+2x-3$$

For x-intercept put $y=0$

$$0 = x^2+2x-3$$

$$0 = x^2+3x-x-3$$

$$0 = x(x+3)-1(x+3)$$

$$0 = (x+3)(x-1)$$

x	$y = x^2+2x-3$
-3	0
-2	-3
-1	-4
0	-3
1	0

$x+3=0$, $x-1=0$

$\boxed{x=-3}$, $\boxed{x=1}$

$$y = x^2+2x-3 \leqslant 0 \text{ in } [-3,1]$$

$$\text{Area} = -\int_{-3}^{1}(x^2+2x-3)\,dx$$

$$\text{Area} = -\left[\int_{-3}^{1} x^2\,dx + 2\int_{-3}^{1} x\,dx - 3\int_{-3}^{1} dx\right]$$

$$\text{Area} = -\left[\frac{x^3}{3}\Big|_{-3}^{1} + \cancel{2}\cdot\frac{x^2}{\cancel{2}}\Big|_{-3}^{1} - 3x\Big|_{-3}^{1}\right]$$

$$\text{Area} = -\left[\frac{1}{3}\left[x^3\right]_{-3}^{1} + |x^2|_{-3}^{1} - 3|x|_{-3}^{1}\right]$$

$$\text{Area} = -\left[\frac{1}{3}\left[(1)^3-(-3)^3\right] + \left[(1)^2-(-3)^2\right] - 3\left[1-(-3)\right]\right]$$

$$\text{Area} = -\left[\frac{1}{3}\left[1-(-27)\right] + (1-9) - 3\left[1+3\right]\right]$$

$$\text{Area} = -\left[\frac{1}{3}(1+28) - 8 - 3(4)\right]$$

$$\text{Area} = -\left[\frac{28}{3} - 8 - 12\right]$$

$$\text{Area} = -\left[\frac{28}{3} - 20\right]$$

$$\text{Area} = -\left[\frac{28-60}{3}\right]$$

$$\text{Area} = -\left[-\frac{32}{3}\right]$$

$$\boxed{\text{Area} = \frac{32}{3} \text{ sq. unit}}$$

Question #7

Find the area bounded by the curve $y = x^3+1$, the x-axis and line $x=2$.

Solution:

$$y = x^3+1 \qquad \because a^3+b^3 = (a+b)(a^2-ab+b^2)$$

$$y = (x+1)(x^2-x+1)$$

For x-intercept put $y=0$

$$0 = (x+1)(x^2-x+1)$$

$x+1=0$, $x^2-x+1=0$

$\boxed{x=-1}$, (Imaginary roots) Not possible

x	$y = x^3+1$
-1	0
0	1
2	9

$$y = x^3+1 > 0 \text{ in } [-1,2]$$

$$\text{Area} = \int_{-1}^{2} x^3+1\,dx$$

$$\text{Area} = \int_{-1}^{2} x^3\,dx + \int_{-1}^{2} 1\,dx$$

$$\text{Area} = \left|\frac{x^4}{4}\right|_{-1}^{2} + |x|_{-1}^{2}$$

$$\text{Area} = \frac{1}{4}\left[(2)^4 - (-1)^4\right] + \left[2-(-1)\right]$$

$$\text{Area} = \frac{1}{4}\left[16-1\right] + \left[2+1\right]$$

$$\text{Area} = \frac{1}{4}(15) + (3)$$

$$\text{Area} = \frac{15}{4} + 3$$

$$\text{Area} = \frac{15+12}{4}$$

$$\boxed{\text{Area} = \frac{27}{4} \text{ sq-unit}}$$

Question #8

Find the area bounded by the curve $y = x^3 - 4x$ and the x-axis.

Solution:

$$y = x^3 - 4x$$

For x-intercept
put $y=0$

$$0 = x^3 - 4x$$

$$0 = (x^2-4)x$$

$$x = 0, \quad x^2 - 4 = 0$$

$$x^2 = 4$$

$$x = \pm 2$$

$\boxed{x=0}$, $\boxed{x=2}$, $\boxed{x=-2}$

x	$y = x^3-4x$
0	0
1	-3
2	0

$y = x^3 - 4x \leq 0$ in $[0,2]$

$$\text{Area}_1 = -\int_0^2 x^3 - 4x\,dx$$

$$A_1 = -\left[\int_0^2 x^3 dx - 4\int_0^2 x\,dx\right]$$

$$A_1 = -\left[\frac{x^4}{4}\Big|_0^2 - 4\cdot\frac{x^2}{2}\Big|_0^2\right]$$

$$A_1 = -\left[\frac{1}{4}\left[x^4\right]_0^2 - 2\left[x^2\right]_0^2\right]$$

$$A_1 = -\left[\frac{1}{4}\left[(2)^4 - (0)^4\right] - 2\left[(2)^2 - (0)^2\right]\right]$$

$$A_1 = -\left[\frac{1}{4}\left[16-0\right] - 2\left[4-0\right]\right]$$

$$A_1 = -\left[\frac{\overset{4}{\cancel{16}}}{\cancel{4}} - 8\right]$$

$$A_1 = -(4-8)$$

$$A_1 = -(-4)$$

$$\boxed{A_1 = 4}$$

x	$y = x^3-4x$
-2	0
-1	3
0	0

$y = x^3 - 4x \geq 0$ in $[-2,0]$

$$\text{Area}_2 = \int_{-2}^{0} x^3 - 4x\,dx$$

$$\text{Area}_2 = \int_{-2}^{0} x^3 dx - 4\int_{-2}^{0} x\,dx$$

$$A_2 = \int \frac{x^4}{4}\Big|_{-2}^{0} - 4\cdot\frac{x^2}{2}\Big|_{-2}^{0}$$

$$A_2 = \frac{1}{4}\left[x^4\right]_{-2}^{0} - 2\,x^2\Big]_{-2}^{0}$$

$$A_2 = \frac{1}{4}\left[(0)^4 - (-2)^4\right] - 2\left[(0)^2 - (-2)^2\right]$$

$$A_2 = \frac{1}{4}\left[0-16\right] - 2\left[0-4\right]$$

$$A_2 = \frac{1}{4}(-16) + 8$$

$$A_2 = \frac{-\cancel{16}}{\cancel{4}} + 8$$

$$A_2 = -4 + 8$$

$$\boxed{A_2 = 4}$$

Area = $A_1 + A_2$

Area = $4+4$

$\boxed{\text{Area} = 8 \text{ sq unit}}$

Question #9

Find the area between the curve $y = x(x-1)(x+1)$ and the x-axis.

Solution:

$y = x(x-1)(x+1)$

$y = x(x^2-1)$

$y = x^3 - x$

For x-intercept put $y=0$

$x(x-1)(x+1) = 0$

$x=0$, $x-1=0$, $x+1=0$

$\boxed{x=0}$, $\boxed{x=1}$, $\boxed{x=-1}$

$y = x^3 - x \leq 0$ in $[0,1]$

x	$y=x^3-x$
0	0
$\frac{1}{2}$	$-\frac{3}{8}$
1	0

$$\text{Area} = -\int_0^1 x^3 - x\,dx$$

$$\text{Area}_1 = -\left[\int_0^1 x^3\,dx - \int_0^1 x\,dx\right]$$

$$A_1 = -\left[\frac{x^4}{4}\Big|_0^1 - \frac{x^2}{2}\Big|_0^1\right]$$

$$A_1 = -\left[\frac{1}{4}\left[(1)^4-(0)^4\right] - \frac{1}{2}\left[(1)^2-(0)^2\right]\right]$$

$$A_1 = -\left[\frac{1}{4}(1-0) - \frac{1}{2}\left[1-0\right]\right]$$

$$A_1 = -\left[\frac{1}{4} - \frac{1}{2}\right]$$

$$A_1 = -\frac{1}{4} + \frac{1}{2}$$

$$A_1 = \frac{-1+2}{4}$$

$$\boxed{A_1 = \frac{1}{4}}$$

x	$y=x^3-x$
-1	0
$-\frac{1}{2}$	$\frac{3}{8}$
0	0

$y = x^3 - x \geq 0$ in $[-1,0]$

$$\text{Area}_2 = \int_{-1}^0 x^3 - x\,dx$$

$$A_2 = \int_{-1}^0 x^3\,dx - \int_{-1}^0 x\,dx$$

$$A_2 = \left|\frac{x^4}{4}\right|_{-1}^0 - \left|\frac{x^2}{2}\right|_{-1}^0$$

$$A_2 = \frac{1}{4}\left[(0)^4-(-1)^4\right] - \frac{1}{2}\left[(0)^2-(-1)^2\right]$$

$$A_2 = \frac{1}{4}\left[-1\right] - \frac{1}{2}\left[-1\right]$$

$$A_2 = -\frac{1}{4} + \frac{1}{2}$$

$$A_2 = \frac{-1+2}{4}$$

$$\boxed{A_2 = \frac{1}{4}}$$

Area = $A_1 + A_2$

Area = $\frac{1}{4} + \frac{1}{4}$

Area = $\frac{1+1}{4}$

Area = $\frac{\cancel{2}^{1}}{\cancel{4}_{2}}$

$\boxed{\text{Area} = \frac{1}{2} \text{ sq.unit}}$

Question#10

Find the area above the x-axis bounded by the curve $y^2 = 3 - x$ from $x = -1$ to $x = 2$

Solution:

$$y^2 = 3-x$$

$$y = \sqrt{3-x}$$

$$y = \sqrt{3-x} > 0 \text{ in } [-1, 2]$$

$$\text{Area} = \int_{-1}^{2} (3-x)^{1/2}\, dx$$

$$\text{Area} = -\int_{-1}^{2} (3-x)^{1/2} -1\, dx$$

$$\because \int [f(x)]^n \cdot f'(x)\, dx = \frac{[f(x)]^{n+1}}{n+1}$$

$$\text{Area} = -\left|\frac{(3-x)^{1/2+1}}{\frac{1}{2}+1}\right|_{-1}^{2}$$

$$\text{Area} = -\left[\frac{(3-x)^{3/2}}{3/2}\right]_{-1}^{2}$$

$$\text{Area} = -\frac{2}{3}\left[(3-2)^{3/2} - (3+1)^{3/2}\right]$$

$$\text{Area} = -\frac{2}{3}\left[(1)^{3/2} - (4)^{3/2}\right]$$

$$\text{Area} = -\frac{2}{3}\left[1 - (2^2)^{3/2}\right]$$

$$\text{Area} = -\frac{2}{3}\left[1 - (2)^3\right]$$

$$\text{Area} = -\frac{2}{3}\left[1-8\right]$$

$$\text{Area} = -\frac{2}{3}(-7)$$

$$\boxed{\text{Area} = \frac{14}{3} \text{ sq unit}}$$

Question#11

Find the area between the x-axis and the curve $y = \cos\frac{1}{2}x$ from $x = -\pi$ to π.

Solution:

$$y = \cos\frac{1}{2}x$$

$$y = \cos\frac{x}{2} \geq 0 \text{ in } [-\pi, \pi]$$

$$\text{Area} = \int_{-\pi}^{\pi} \cos\frac{x}{2}\, dx$$

$$\text{Area} = \left|\frac{\sin\frac{x}{2}}{\frac{1}{2}}\right|_{-\pi}^{\pi}$$

$$\text{Area} = 2\left[\sin\frac{\pi}{2} - \sin -\frac{\pi}{2}\right]$$

$$\text{Area} = 2\left[1 - (-1)\right]$$

$$\text{Area} = 2(1+1)$$

$$\text{Area} = 2(2)$$

$$\boxed{\text{Area} = 4 \text{ sq unit}}$$

Question #12

Find the area between the x-axis and the curve $y = \sin 2x$ from $x = 0$ to $x = \frac{\pi}{3}$

Solution:

$$y = \sin 2x$$

$$y = \sin 2x \geq 0 \text{ in } \left[0, \frac{\pi}{3}\right]$$

$$\text{Area} = \int_0^{\pi/3} \sin 2x \, dx$$

$$\text{Area} = \left| -\frac{\cos 2x}{2} \right|_0^{\pi/3}$$

$$\text{Area} = -\frac{1}{2}\left[\cos 2\frac{\pi}{3} - \cos 2(0)\right]$$

$$\text{Area} = -\frac{1}{2}\left[\cos 120^\circ - \cos 0^\circ\right]$$

$$\text{Area} = -\frac{1}{2}\left[-\frac{1}{2} - 1\right]$$

$$\text{Area} = -\frac{1}{2}\left[\frac{-1-2}{2}\right]$$

$$\text{Area} = -\frac{1}{2}\left[\frac{-3}{2}\right]$$

$$\boxed{\text{Area} = \frac{3}{4} \text{ sq. units}}$$

Question #13

Find the area between the x-axis and the curve $y = \sqrt{2ax - x^2}$ when $a > 0$.

Solution:

$$y = \sqrt{2ax - x^2}$$

For x-intercept put $y = 0$

$$0 = \sqrt{2ax - x^2}$$

$$0 = 2ax - x^2$$

$$0 = x(2a - x)$$

$\boxed{x = 0}$, $2a - x = 0$

$\boxed{2a = x}$

x	$y = 2ax - x^2$
0	0
a	a
2a	0

$y = 2ax - x^2 \geq 0$ in $[0, 2a]$

$$\text{Area} = \int_0^{2a} \sqrt{2ax - x^2} \, dx$$

$$\text{Area} = \int_0^{2a} (2ax - x^2)^{1/2} \, dx$$

By Adding and subtracting "a^2"

$$\text{Area} = \int_0^{2a} \left(\sqrt{2ax - x^2 + a^2 - a^2}\right) dx$$

$$\text{Area} = \int_0^{2a} \sqrt{-(x^2 - 2ax + a^2 - a^2)} \, dx$$

$$\text{Area} = \int_0^{2a} \sqrt{-((x-a)^2 - a^2)} \, dx$$

$$\text{Area} = \int_0^{2a} \sqrt{a^2 - (x-a)^2} \, dx$$

put

$$x - a = a\sin\theta$$

$$dx = a\cos\theta \, d\theta$$

$\boxed{x = 0}$

$$0 - \cancel{a} = \cancel{a}\sin\theta$$

$$-1 = \sin\theta$$

$$\theta = \sin^{-1}(-1)$$

$$\boxed{\theta = -\frac{\pi}{2}}$$

$\boxed{x = 2a}$

$$2a - a = a\sin\theta$$

$$\cancel{a} = \cancel{a}\sin\theta$$

$$\theta = \sin^{-1}(1)$$

$$\boxed{\theta = \frac{\pi}{2}}$$

$$\text{Area} = \int_{-\pi/2}^{\pi/2} \sqrt{a^2 - (a\sin\theta)^2} \; a\cos\theta \, d\theta$$

$$\text{Area} = \int_{-\pi/2}^{\pi/2} \sqrt{a^2 - a^2\sin^2\theta} \; a\cos\theta \, d\theta$$

$$\text{Area} = \int_{-\pi/2}^{\pi/2} \sqrt{a^2(1-\sin^2\theta)}\;\; a\cos\theta\, d\theta$$

$$\because\; 1-\sin^2\theta = \cos^2\theta$$

$$\text{Area} = \int_{-\pi/2}^{\pi/2} \sqrt{a^2\cos^2\theta}\;\; \cdot a\cos\theta\, d\theta$$

$$\text{Area} = \int_{\pi/2}^{\pi/2} a\cos\theta \cdot a\cos\theta\, d\theta$$

$$\text{Area} = \int_{-\pi/2}^{\pi/2} a^2\cos^2\theta\, d\theta$$

$$\text{Area} = a^2\int_{-\pi/2}^{\pi/2} \cos^2\theta\, d\theta$$

$$\because\; \cos^2\theta = \frac{1+\cos 2\theta}{2}$$

$$\text{Area} = a^2\int_{-\pi/2}^{\pi/2} \left(\frac{1+\cos 2\theta}{2}\right) d\theta$$

$$\text{Area} = \frac{a^2}{2}\int_{-\pi/2}^{\pi/2} (1+\cos 2\theta)\, d\theta$$

$$\text{Area} = \frac{a^2}{2}\left[\int_{-\pi/2}^{\pi/2} 1\, d\theta + \int_{-\pi/2}^{\pi/2} \cos 2\theta\, d\theta\right]$$

$$\text{Area} = \frac{a^2}{2}\left[\left|\theta\right|_{-\pi/2}^{\pi/2} + \left|\frac{\sin 2\theta}{2}\right|_{-\pi/2}^{\pi/2}\right]$$

$$\text{Area} = \frac{a^2}{2}\left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right] + \frac{1}{2}\left|\sin 2\theta\right|_{-\pi/2}^{\pi/2}$$

$$\text{Area} = \frac{a^2}{2}\left[\frac{\pi}{2} + \frac{\pi}{2}\right] + \frac{1}{2}\left[\sin 2\left(\frac{\pi}{2}\right) - \sin 2\left(-\frac{\pi}{2}\right)\right]$$

$$\text{Area} = \frac{a^2}{2}\left[\frac{\pi+\pi}{2}\right] + \frac{1}{2}\left(\sin\pi - \sin(-\pi)\right)$$

$$\because\; \sin(-\theta) = -\sin\theta$$

$$\text{Area} = \frac{a^2}{2}\left[\left(\frac{2\pi}{2}\right) + \frac{1}{2}\left[\sin\pi + \sin\pi\right]\right]$$

$$\text{Area} = \frac{a^2}{2}\left[\pi + \frac{1}{2}(0+0)\right]$$

$$\text{Area} = \frac{a^2}{2}\left[\pi + \frac{1}{2}(0)\right]$$

$$\text{Area} = \frac{a^2}{2}\left[\pi + 0\right]$$

$$\text{Area} = \frac{a^2}{2}\pi$$

$$\boxed{\text{Area} = \frac{1}{2}\pi a^2 \;\; \text{sq}\cdot\text{unit}}$$

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* Theory

* Differential Equation:

An equation containing at least one derivative of a dependent, variable with respect to an independant variable is called differential Equation:

Example:

(i) $y\frac{dy}{dx} + 2x = 0$

(ii) $x\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2x = 0$

* Order of Differential Equation:

The order of differential equation is the order of highest derivative it contains.

(i) $y\frac{dy}{dx} + 2x = 0$ (order is 1)

(ii) $x\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2x = 0$ (Order is 2)

* Degree of differential Equation:

The degree of differential equation is the highest power of the differential coefficient present in the equation.

(i) $x\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2x = 0$ (Degree 1)

(ii) $y\left(\frac{dy}{dx}\right)^2 + 2x = 0$ (Degree 2)

Example #1

Solve the differential equation $(x-1)dx + ydy = 0$

Solution:

$$(x-1)dx + ydy = 0$$

$$y\,dy = -(x-1)\,dx$$

$$y\,dy = (1-x)\,dx$$

Integration both sides

$$\int y\,dy = \int (1-x)\,dx$$

$$\int y\,dy = \int 1\,dx - \int x\,dx$$

$$\frac{y^2}{2} = x - \frac{x^2}{2} + c_1$$

Multiply by '2' both sides

$$y^2 = 2x - x^2 + 2c_1$$

$$\because c = 2c_1$$

$$y^2 = 2x - x^2 + c$$

$$\boxed{x^2 - 2x + y^2 = c}$$

Example #2

Solve the differential equation

$x^2(2y+1)\frac{dy}{dx} - 1 = 0$

Solution:

$$x^2(2y+1)\frac{dy}{dx} - 1 = 0$$

Separating the variables

$$x^2(2y+1)\frac{dy}{dx} = 1$$

$$(2y+1)\frac{dy}{dx} = \frac{1}{x^2}$$

$$(2y+1)\,dy = \frac{1}{x^2}\,dx$$

Integration both sides

$$\int (2y+1)\,dy = \int \frac{1}{x^2}\,dx$$

$$\int 2y\,dy + \int 1\,dy = \int x^{-2}\,dx$$

$$\cancel{2}\cdot\frac{y^2}{\cancel{2}} + y = \frac{x^{-2+1}}{-2+1} + c$$

$$y^2 + y = \frac{x^{-1}}{-1} + c$$

$$\boxed{y^2 + y = -\frac{1}{x} + c}$$

Example #3

Solve the differential equation

$\frac{1}{x}\frac{dy}{dx} - 2y = 0 \quad , x \neq 0 , y > 0$

Solution:

$$\frac{1}{x}\frac{dy}{dx} - 2y = 0$$

Separating the variables:

$$\frac{dy}{x\,dx} = 2y$$

$$\frac{1}{y}\,dy = 2x\,dx$$

Integration both sides

$$\int \frac{1}{y}\,dy = \int 2x\,dx$$

$$\int \frac{1}{y}\,dy = 2\int x\,dx$$

$$\ln y = \cancel{2}\cdot\frac{x^2}{\cancel{2}} + c_1$$

$$\ln y = x^2 + c_1$$

$$e^{\ln y} = e^{x^2 + c_1}$$

$$y = e^{x^2}\cdot e^{c_1}$$

$\because c = e^{c_1}$

$y = cx^2$

$$\boxed{y = ce^{x^2}}$$

•———•

Example # 4

Solve $\frac{dy}{dx} = \frac{y^2+1}{e^{-x}}$

Solution:

$$\frac{dy}{dx} = \frac{y^2+1}{e^{-x}}$$

Separating Variables:

$$e^{-x}dy = (y^2+1)\,dx$$

$$\frac{1}{y^2+1}\,dy = \frac{1}{e^{-x}}\,dx$$

Integration both sides:

$$\int \frac{1}{y^2+1}\,dy = \int e^x\,dx$$

$$\tan^{-1}y = e^x + c$$

$$\boxed{y = \tan(e^x+c)}$$

•———•

Example #5

Solve $2e^x\tan y\,dx + (1-e^x)\sec^2 y\,dy = 0$

Solution:

$$2e^x\tan y\,dx + (1-e^x)\sec^2 y\,dy = 0$$

Separating Variables:

$$(1-e^x)\sec^2 y\,dy = -2e^x\tan y\,dx$$

$$\frac{\sec^2 y}{\tan y}\,dy = \frac{-2e^x}{1-e^x}\,dx$$

Integration both sides

$$\int \frac{\sec^2 y}{\tan y}\,dy = 2\int \frac{e^{-x}}{1-e^{x}}\,dx$$

$$\ln|\tan y| = 2\ln|1-e^x| + \ln c$$

$$\ln|\tan y| = \ln(1-e^x)^2 + \ln c$$

$$\cancel{\ln}|\tan y| = \cancel{\ln}|c(1-e^x)^2|$$

$$\boxed{\tan y = c(1-e^x)^2}$$

•———•

Example #6

Solve $(\sin y + y\cos y)\,dy = [x(2\ln x+1)]\,dx.$

Solution:

$$(\sin y + y\cos y)\,dy = x(2\ln x+1)\,dx$$

Integration both sides:

$$\int \sin y + y\cos y\,dy = \int x(2\ln x+1)\,dx$$

$$\int \sin y\,dy + \int y\cdot\cos y\,dy = 2\int x\ln x\,dx + \int x\,dx$$

Integration by parts:

$$-\cos y + y\cdot\sin y - \int \sin y\cdot 1\,dy = 2\left\{\ln x\cdot\frac{x^2}{x} - \int \frac{x^2}{2}\cdot\frac{1}{x}\,dx\right\} + \frac{x^2}{2}$$

$$-\cancel{\cos y} + y\sin y + \cancel{\cos y} = x^2\ln x - \frac{x^2}{2} + \frac{x^2}{2} + c$$

$$\boxed{y\sin y = x^2\ln x + c}$$

•———•

✗ Initial Conditions:

The arbitrary constants involving in the solution of different equations can be determined by the given conditions. Such conditions are called Initial Value Conditions.

Example #7

The slope of the tangent at any point of the curve is given by $\frac{dy}{dx} = 2x-2$, find the equation of the curve if $y=0$ when $x=-1$

Solution:

$$\frac{dy}{dx} = 2x-2$$

Separating the variables:

$$dy = (2x-2)dx$$

Integration both sides

$$\int 1\, dy = \int (2x-2)\, dx$$

$$\int 1\, dy = \int 2x\, dx - 2\int 1\, dx$$

$$y = \cancel{2}\cdot\frac{x^2}{\cancel{2}} - 2\cdot x + c$$

$$y = x^2 - 2x + c \text{ —— } ①$$

At $\boxed{x=-1}$ and $\boxed{y=0}$

$$0 = (-1)^2 - 2(-1) + c$$

$$0 = 1 + 2 + c$$

$$0 = 3 + c \quad \boxed{c=-3}$$

From eq ①

$$\boxed{y = x^2 - 2x - 3}$$

Example #8

Solve $\frac{dy}{dx} = \frac{3}{4}x^2 + x - 3$, if $y=0$ when $x=2$.

Solution:

$$\frac{dy}{dx} = \frac{3}{4}x^2 + x - 3$$

Separating the variables:

$$dy = \left(\frac{3}{4}x^2 + x - 3\right) dx$$

Integration both sides

$$\int 1\, dy = \int \left(\frac{3}{4}x^2 + x - 3\right) dx$$

$$\int 1\, dy = \frac{3}{4}\int x^2\, dx + \int x\, dx - 3\int 1\, dx$$

$$y = \frac{\cancel{3}}{4}\cdot\frac{x^3}{\cancel{3}} + \frac{x^2}{2} - 3x$$

$$y = \frac{x^3}{4} + \frac{x^2}{2} - 3x + C \text{ —— } ①$$

At $y=0$ when $x=2$

$$0 = \frac{(2)^3}{4} + \frac{(2)^2}{2} - 3(2) + C$$

$0 = \frac{\overset{2}{\cancel{8}}}{\cancel{4}} + \frac{4^2}{\cancel{2}} - 6 + c$

$0 = 2 + 8 - 6 + c$

$0 = 4 - 6 + c$

$0 = -2 + c$

$\boxed{c = 2}$

From eq ①

$y = \frac{x^3}{4} + \frac{x^2}{2} - 3x + 2$

Multiply by "4"

$\boxed{4y = x^3 + x^2 - 12x + 8}$

Example #9

A particle is moving in a straight line and its acceleration is given by $a = 2t - 7$,

(i)

find v (velocity) in term of t if $v = 10 m/sec$, when $t = 0$

Solution:

$a = 2t - 7$ $\quad \because a = \frac{dv}{dt}$

$\frac{dv}{dt} = 2t - 7$

Separating the variables:

$dv = 2t - 7\,dt$

Integration both sides

$\int dv = \int 2t - 7\,dt$

$\int dv = 2\int t\,dt - 7\int 1\,dt$

$v = \cancel{2}\cdot\frac{t^2}{\cancel{2}} - 7t + c_1$

$v = t^2 - 7t + c_1$

At $\boxed{v = 10}$ when $\boxed{t = 0}$

$10 = (0)^2 - 7(0) + c_1$

$10 = 0 - 0 + c_1$

$\boxed{c_1 = 10}$

So, eq ① becomes

$\boxed{v = t^2 - 7t + 10}$

(ii)

find s (distance) in term of t if $s = 0$, when $t = 0$

Solution:

$v = t^2 - 7t + 10$

$\frac{ds}{dt} = t^2 - 7t + 10$

$ds = dt(t^2 - 7t + 10)$

$ds = t^2 dt - 7t dt + 10 dt$

Integration both sides:

$\int ds = \int t^2 dt - 7\int t\,dt + 10\int 1\,dt$

$S = \frac{t^3}{3} - 7\frac{t^2}{2} + 10t + c_2$ —— ②

As $\boxed{S = 0}$ when $\boxed{t = 0}$

$0 = \frac{(0)^3}{3} - 7\frac{(0)^2}{2} + 10(0) + c_2$

$0 = 0 - 0 + 0 + c_2$

$\boxed{c_2 = 0}$

So eq ②

$S = \frac{t^3}{3} - \frac{7}{2}t^2 + 10t + 0$

$\boxed{S = \frac{t^3}{3} - \frac{7}{2}t^2 + 10t}$

Example # 10

In a culture, becteria increases at the rate proportional to the number of becteria present. If becteria are 100 initially and are doubled in 2 hours. Find the number of becteria present from four hours later.

Solutions:

Let p be the number of bacteria present at time t, then

$$\frac{dP}{dt} \propto P$$

$$\frac{dP}{dt} = KP$$

$$dp = Kp\,dt$$

$$\frac{1}{P}dp = K\,dt$$

Integration both sides

$$\int \frac{1}{P}dp = K\int 1\,dt$$

$$\ln p = Kt + c_1$$

$$e^{\ln p} = e^{kt+c_1}$$

$$P = e^{kt} \cdot e^{c_1} \qquad \because e^{c_1} = C$$

$$P = ce^{kt} \text{ ——— } ①$$

put $t = 0$, $P = 100$ in ①

$$100 = ce^{k(0)}$$

$$100 = C$$

$$P = 100e^{kt} \text{ ——— } ②$$

put $t = 2$, $t = 2(100) = 200$

$$200 = 100e^{k(2)}$$

$$2 = e^{2k}$$

$$\ln 2 = \ln e^{2K}$$

$$\ln 2 = 2K \ln e$$

$$\ln 2 - 2 = K$$

$$\frac{\ln 2}{2} = K$$

From eq ②

$$P = 100e^{(\ln 2/2)t}$$

put $t = 4$

$$P = 100e^{(\frac{\ln 2}{2})4^2}$$

$$P = 100e^{\ln 4}$$

$$P = 100\,4$$

$$P = 400$$

Required number of bacteria present 4 hours later

Example #11

A ball is thrown vartically upward with the Velocity of 1470cm/sec, Neglecting air resistance, find.

(i) Velocity of ball at any time t.

(ii) distance traveled in any time t.

(iii) maximum height attained by the ball.

i) Solution:

Let v be the velocity of ball at any time t.

so,

$$\frac{dv}{dt} = -g \quad (\text{for upward motion})$$

$$dv = -g\,dt$$

Integration both sides

$$\int dv = -g\int 1\,dt$$

$$v = -gt + c_1 \text{ --- } ⓘ$$

put $t=0$ then $v = 1470$ cm/sec in eq ⓘ

$$1470 = -g(0) + c_1$$

$$1470 = 0 + c_1$$

$$c_1 = 1470$$

Thus eq ⓘ

$$v = -gt + 1470$$

∵ In c.g.s system
∵ $g = 980$ cm/sec

$$v = -980t + 1470 \text{ --- } (ii)$$

ii) Let h be the height of ball then

$$v = \frac{dh}{dt}$$

$$\frac{dh}{dt} = -980t + 1470$$

Integration

$$\int dh = \int(-980t + 1470)\,dt$$

$$h = -980\int t\,dt + 1470\int 1\,dt$$

$$h = -980\cdot\frac{t^2}{2} + 1470t + c_2 \text{ --- } (iii)$$

when $t=0$ then $h=0$ so, eq (iii)

$$0 = -980\cdot\underline{(0)^2} + 1470(0) + c_2$$

$$\boxed{c_2 = 0}$$

so, eq (iii) becomes

$$h = -490t^2 + 1470t \text{ --- } (iv)$$

iii) When ball will be at maximum height then $v=0$ so put in (ii)

$$0 = -980t + 1470$$

$$980t = 1470$$

$$t = \frac{1470}{980}$$

$$\boxed{t = \frac{3}{2}}$$

put in eq (iv)

$$h = -490\left(\frac{3}{2}\right)^2 + 1470\left(\frac{3}{2}\right)$$

$$h = -490\left(\frac{9}{4}\right) + 1470\left(\frac{3}{2}\right)$$

$$h = -1102.5 + 2205$$

$$\boxed{h = 1102.5 \text{ cm}}$$

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Exercise # 3·8

Question #1

Check that each of the following equations, written against the differential equation is its Solution.

(i)

$x\frac{dy}{dx} = 1+y$, $y = cx-1$

Solution:

$$x\frac{dy}{dx} = 1+y$$

Separating the variables

$$xdy = (1+y)dx$$

$$\frac{1}{1+y}dy = \frac{1}{x}dx$$

Integration both sides

$$\int\frac{1}{1+y}dy = \int\frac{1}{x}dx$$

$$\ln|1+y| = \ln|x| + \ln|c|$$

$\because \ln m + \ln n = \ln(mn)$

$$\cancel{\ln}|1+y| = \cancel{\ln}|cx|$$

$$1+y = cx$$

$$\boxed{y = cx-1}$$

(ii)

$x^2(2y+1)\frac{dy}{dx} - 1 = 0$, $y^2+y = c-\frac{1}{x}$

Solution:

$$x^2(2y+1)\frac{dy}{dx} - 1 = 0$$

Separating the Variables:

$$x^2(2y+1)\frac{dy}{dx} = 1$$

$$(2y+1)dy = \frac{1}{x^2}dx$$

Integration both sides

$$\int(2y+1)dy = \int\frac{1}{x^2}dx$$

$$\int 2y\,dy + \int 1\,dy = \int x^{-2}dx$$

$$\cancel{2}\cdot\frac{y^2}{\cancel{2}} + y = \frac{x^{-2+1}}{-2+1} + c$$

$$y^2+y = \frac{x^{-1}}{-1} + c$$

$$y^2+y = -\frac{1}{x} + c$$

$$\boxed{y^2+y = c-\frac{1}{x}}$$

(iii)

$y\frac{dy}{dx} - e^{2x} = 1$, $y^2 = e^{2x} + 2x + C$

Solution:

$$y\frac{dy}{dx} - e^{2x} = 1$$

Separating the variables

$$y\frac{dy}{dx} = 1+e^{2x}$$

$$y\,dy = (1+e^{2x})\,dx$$

Integration both sides

$$\int y\,dy \cdot \int(1+e^{2x})dx$$

$$\int y\,dy = \int 1\,dx + \int e^{2x}dx$$

$$\frac{y^2}{2} = x + \frac{e^{2x}}{2} + c_1$$

Multiply by '2'

$$y^2 = 2x + e^{2x} + 2c_1$$

$$\because 2c_1 = C$$

$$\boxed{y^2 = e^{2x} + 2x + C}$$

(iv)

$\frac{1}{x}\frac{dy}{dx} - 2y = 0$, $y = ce^{x^2}$

Solution:

$$\frac{1}{x}\frac{dy}{dx} - 2y = 0$$

Separating the variables:

$$\frac{1}{x}\frac{dy}{dx} = 2y$$

$$\frac{1}{y}dy = 2x\,dx$$

Integration both sides

$$\int\frac{1}{y}dy = 2\int x\,dx$$

$$\ln y = \cancel{2}\cdot\frac{x^2}{\cancel{2}} + c_1$$

$$\ln y = x^2 + c_1$$

$$e^{\ln y} = e^{x^2+c_1}$$

$$y = e^{x^2}\cdot e^{c_1}$$

$$\because e^{c_1} = c$$

$$\boxed{y = ce^{x^2}}$$

(v)

$\frac{dy}{dx} = \frac{y^2+1}{e^{-x}}$, $y = \tan(e^x + c)$

Solution:

$$\frac{dy}{dx} = \frac{y^2+1}{e^{-x}}$$

Separating the variables

$$\frac{1}{y^2+1}dy = \frac{1}{e^{-x}}dx$$

Integration both sides

$$\int\frac{1}{y^2+1}dy = \int e^x dx$$

$$\tan^{-1}y = e^x + c$$

$$\boxed{y = \tan(e^x + c)}$$

Question #2

$$\frac{dy}{dx} = -y$$

Solution:

$$\frac{dy}{dx} = -y$$

$dy = -y\,dx$

$\frac{1}{y}dy = -1\,dx$

Integration both sides

$\int \frac{1}{y}dy = -\int 1\,dx$

$\ln y = -x + c_1$

$e^{\ln y} = e^{-x+c_1}$

$y = e^{-x}\cdot e^{c_1}$ $\because e^{c_1} = c$

$\boxed{y = ce^{-x}}$

Question #3

$ydx + xdy = 0$

Solution

$ydx + xdy = 0$

Separation the variable

$xdy = -ydx$

$\frac{1}{y}dy = -\frac{1}{x}dx$

Integration both sides

$\int \frac{1}{y}dy = -\int \frac{1}{x}dx$

$\ln y = -\ln x + \ln c$

$\ln y + \ln x = \ln c$

$\because \ln m + \ln n = \ln(mn)$

$\cancel{\ln} xy = \cancel{\ln} c$

$\boxed{xy = c}$

Question #4

$\frac{dy}{dx} = \frac{1-x}{y}$

Solution:

$\frac{dy}{dx} = \frac{1-x}{y}$

Separating the variables:

$ydy = 1-x\,dx$

Integration both sides

$\int ydy = \int(1-x)\,dx$

$\int ydy = \int 1dx - \int xdx$

$\frac{y^2}{2} = x - \frac{x^2}{2} + c_1$

Multiply by '2'

$y^2 = 2x - x^2 + 2c_1$

$\because 2c_1 = c$

$y^2 = 2x - x^2 + c$

$\boxed{y^2 = x(2-x) + c}$

Question #5

$\frac{dy}{dx} = \frac{y}{x^2}$, $(y > 0)$

Solution:

$\frac{dy}{dx} = \frac{y}{x^2}$

Separating the variables

$\frac{1}{y}dy = \frac{1}{x^2}dx$

Integration both sides

$\int \frac{1}{y}dy = \int \frac{1}{x^2}dx$

$$\ln y = \frac{x^{-2+1}}{-2+1} + c_1$$

$$\ln y = \frac{x^{-1}}{-1} + c_1$$

$$\ln y = -\frac{1}{x} + c_1$$

$$e^{\ln y} = e^{-\frac{1}{x}+c_1}$$

$$y = e^{c_1} \cdot e^{-\frac{1}{x}}$$

$$\because e^{c_1} = c$$

$$\boxed{y = ce^{-\frac{1}{x}}}$$

•——•

Question #6

$$\sin y \ \operatorname{cosec} x \ \frac{dy}{dx} = 1$$

Solution:

$$\sin y \cdot \operatorname{cosec} x \ \frac{dy}{dx} = 1$$

Separating the variables

$$\sin y \ \operatorname{cosec} x \ dy = dx$$

$$\sin y \ dy = \frac{1}{\operatorname{cosec} x} dx$$

$$\sin y \ dy = \sin x \ dx$$

Integration both sides

$$\int \sin y \ dy = \int \sin x \ dx$$

$$-\cos y = -\cos x + c_1$$

$$\cos y = \cos x - c_1$$

$$\because -c_1 = C$$

$$\boxed{\cos y = \cos x + C}$$

•——•

Question #7

$$x \ dy + y(x-1)dx = 0$$

Solution:

$$x \ dy + y(x-1) dx = 0$$

Separating the variables:

$$x \ dy = -y(x-1) \ dx$$

$$\frac{1}{y} dy = -\left(\frac{x-1}{x}\right)dx$$

$$\frac{1}{y} dy = -\left[\frac{x}{x} - \frac{1}{x}\right] dx$$

$$\frac{1}{y} dy = -\left[1 - \frac{1}{x}\right] dx$$

Integration both sides

$$\int \frac{1}{y} dy = \int -1 \ dx + \int \frac{1}{x} dx$$

$$\ln y = -x + \ln x + \ln c$$

$$\ln y - \ln x = -x + \ln c$$

$$\because \ln m - \ln n = \ln\left(\frac{m}{n}\right)$$

$$\ln \frac{y}{x} = -x + \ln c$$

$$e^{\ln \frac{y}{x}} = e^{-x+\ln c}$$

$$\frac{y}{x} = e^{-x} \cdot e^{\ln c}$$

$$\frac{y}{x} = ce^{-x}$$

$$\boxed{y = cxe^{-x}}$$

•——•

Question #8

$$\frac{x^2+1}{y+1} = \frac{x}{y} \frac{dy}{dx}, \quad (x, y > 0)$$

Solution:

$$\frac{x^2+1}{y+1} = \frac{x}{y}\frac{dy}{dx}$$

Separating the variables:

$$\frac{x^2+1}{x}dx = \frac{y+1}{y}dy$$

$$\left(\frac{x^2}{x}+\frac{1}{x}\right)dx = \left(\frac{y}{y}+\frac{1}{y}\right)dy$$

$$\left(x+\frac{1}{x}\right)dx = \left(1+\frac{1}{y}\right)dy$$

$$\left(1+\frac{1}{y}\right)dy = \left(x+\frac{1}{x}\right)dx$$

Integration both sides

$$\int 1\,dy + \int\frac{1}{y}dy = \int x\,dx + \int\frac{1}{x}dx$$

$$y+\ln y = \frac{x^2}{2}+\ln x+\ln c$$

$$y+\ln y = \frac{x^2}{2}+\ln cx$$

$$\ln y-\ln cx = \frac{x^2}{2}-y$$

$$\ln\frac{y}{cx} = \frac{x^2}{2}-y$$

$$e^{\ln\left(\frac{y}{cx}\right)} = e^{\frac{x^2}{2}-y}$$

$$\frac{y}{cx} = e^{\frac{x^2}{2}-y}$$

$$y = cx\cdot e^{\frac{x^2}{2}}\cdot e^{-y}$$

$$\frac{y}{e^{-y}} = cxe^{\frac{x^2}{2}}$$

$$\boxed{ye^{y} = cxe^{\frac{x^2}{2}}}$$

• —— •

Question #9

$$\frac{1}{x}\frac{dy}{dx} = \frac{1}{2}(1+y^2)$$

Solution:

$$\frac{1}{x}\frac{dy}{dx} = \frac{1}{2}(1+y^2)$$

Separating the variables

$$dy = \frac{1}{2}x(1+y^2)dx$$

$$\frac{1}{1+y^2}dy = \frac{1}{9}x\,dx$$

Integration both sides

$$\int\frac{1}{1+y^2}dy = \frac{1}{2}\int x\,dx$$

$$\tan^{-1}y = \frac{1}{2}\cdot\frac{x^2}{2}+c$$

$$\tan^{-1}y = \frac{x^2}{4}+c$$

$$\boxed{y = \tan\left(\frac{x^2}{4}+c\right)}$$

• —— •

Question #10

$$2x^2y\frac{dy}{dx} = x^2-1$$

Solution:

$$2x^2y\frac{dy}{dx} = x^2-1$$

Separating the variable

$$2y\,dy = \frac{x^2-1}{x^2}dx$$

$$2y\,dy = \left(\frac{x^2}{x^2}-\frac{1}{x^2}\right)dx$$

$$2y\,dy = (1-x^{-2})dx$$

Integration both sides

$$2\int y\,dy = \int 1\,dx - \int x^{-2}dx$$

$$\cancel{2}\cdot\frac{y^2}{\cancel{2}} = x - \frac{x^{-2+1}}{-2+1}+c$$

$$\boxed{y^2 = x + \frac{1}{x} + c}$$

Question #11

$$\frac{dy}{dx} + \frac{2xy}{2y+1} = x$$

Solution:

$$\frac{dy}{dx} + \frac{2xy}{2y+1} = x$$

Separating the variables:

$$\frac{dy}{dx} = x - \frac{2xy}{2y+1}$$

$$\frac{dy}{dx} = \frac{\cancel{2xy} + x - \cancel{2xy}}{2y+1}$$

$$\frac{dy}{dx} = \frac{x}{2y+1}$$

$$(2y+1)dy = x\,dx$$

Integration both sides

$$2\int y\,dy + \int 1\,dy = \int x\,dx$$

$$\cancel{2}\frac{y^2}{\cancel{2}} + y = \frac{x^2}{2} + c$$

$$y^2 + y = \frac{x^2}{2} + c$$

$$\boxed{y(y+1) = \frac{x^2}{2} + c}$$

Question #12

$$(x^2 + yx^2)\frac{dy}{dx} + y^2 + xy^2 = 0$$

Solution:

$$(x^2 - yx^2)\frac{dy}{dx} + y^2 + xy^2 = 0$$

$$x^2(1-y)\frac{dy}{dx} + y^2(1+x) = 0$$

$$x^2(1-y)\frac{dy}{dx} = -y^2(1+x)$$

$$\frac{1-y}{y^2}\,dy = -\left(\frac{1+x}{x^2}\right)dx$$

$$\left(\frac{1}{y^2} - \frac{y}{y^2}\right)dy = -\left(\frac{1}{x^2} + \frac{x}{x^2}\right)dx$$

$$\left(\frac{1}{y^2} - \frac{1}{y}\right)dy = -\left(\frac{1}{x^2} + \frac{1}{x}\right)dx$$

Integration both sides.

$$\int\frac{1}{y^2}dy - \int\frac{1}{y}dy = -\left[\int\frac{1}{x^2}dx + \int\frac{1}{x}dx\right]$$

$$\int y^{-2}dy - \int\frac{1}{y}dy = -\left[\int x^{-2}dx + \int\frac{1}{x}dx\right]$$

$$\frac{y^{-2+1}}{-2+1} - \ln y = -\left[\frac{x^{-2+1}}{-2+1} + \ln x\right] + c_1$$

$$\frac{y^{-1}}{-1} - \ln y = -\left[\frac{x^{-1}}{-1} + \ln x\right] + c_1$$

$$-\frac{1}{y} - \ln y = -\left[-\frac{1}{x} + \ln x\right] + c_1$$

$$\ln y + \frac{1}{y} = \ln x - \frac{1}{x} - c_1$$

$$\because -c_1 = c$$

$$\boxed{\ln y + \frac{1}{y} = \ln x - \frac{1}{x} + c}$$

Question #13

$$\sec^2 x \tan y\,dx + \sec^2 y \tan x\,dy = 0$$

Solution:

$$\sec^2 x \tan y\,dy + \sec^2 y \tan x\,dy = 0$$

$$\sec^2 y \tan x \, dy = -\sec^2 x \tan y \, dx$$

$$\frac{\sec^2 y}{\tan y} dy = -\frac{\sec^2 x}{\tan x} dx$$

Integration both sides

$$\int \frac{\sec^2 y}{\tan y} dy = -\int \frac{\sec^2 x}{\tan x} dx$$

$$\ln(\tan y) = -\ln(\tan x) + \ln c$$

$$\ln(\tan y) + \ln(\tan x) = \ln c$$

$$\cancel{\ln}(\tan y . \tan x) = \cancel{\ln} c$$

$$\boxed{\tan y . \tan x = C}$$

Question #14

$$\left[y - x\frac{dy}{dx}\right] = 2\left[y^2 + \frac{dy}{dx}\right]$$

Solution:

$$\left(y - x\frac{dy}{dx}\right) = 2\left(y^2 + \frac{dy}{dx}\right)$$

$$y - x\frac{dy}{dx} = 2y^2 + 2\frac{dy}{dx}$$

$$y - 2y^2 = 2\frac{dy}{dx} + x\frac{dy}{dx}$$

$$y(1-2y) = (x+2)\frac{dy}{dx}$$

$$y(1-2y)dx = (x+2)dy$$

$$\frac{1}{x+2}dx = \frac{1}{y(1-2y)}dy$$

$$\frac{dy}{y(1-2y)} = \frac{1}{x+2}dx \quad \text{—— (I)}$$

Using partial fraction to solve $\frac{1}{y(1-2y)}$

$$\frac{1}{y(1-2y)} = \frac{A}{y} + \frac{B}{1-2y} \quad \text{——} ①$$

Multiply by 'y' (1-2y) both sides.

$$1 = A(1-2y) + B(y) \quad \text{——} ②$$

$\boxed{y=0}$ put in eq②

$$1 = A[1-2(0)] + B(0)$$

$$1 = A(1-0) + 0$$

$$1 = A(1)$$

$$\boxed{A=1}$$

$1-2y=0 \quad 1=2y \Rightarrow \boxed{y=\frac{1}{2}}$ put in②

$$1 = A \; 1-2\left(\frac{1}{2}\right) + B\left(\frac{1}{2}\right)$$

$$1 = 0 + B\left(\frac{1}{2}\right)$$

$$\boxed{B=2}$$

$$\frac{1}{y(1-2y)} = \frac{1}{y} + \frac{2}{1-2y}$$

Eq (I) becomes

$$\left(\frac{1}{y} + \frac{2}{1-2y}\right)dy = \frac{dx}{x+2}$$

Taking Integration both sides

$$\int \frac{1}{y}dy - \int \frac{2}{2y-1}dy = \int \frac{1}{x+2}dx$$

$$\ln y - \ln|2y-1| = \ln|x+2| + \ln c$$

$$\cancel{\ln} \frac{y}{2y-1} = \cancel{\ln}[c(x+2)]$$

$$\boxed{\frac{y}{2y-1} = c(x+2)}$$

Question #15

$$1+\cos x\tan y\frac{dy}{dx}=0$$

Solution:

$$1+\cos x\tan y\frac{dy}{dx}=0$$

Separating the variables

$$\cos x\tan y\frac{dy}{dx}=-1$$

$$\cos x\tan y\,dy=-dx$$

$$\tan y\,dy=-\frac{1}{\cos x}dx$$

Integration both sides

$$\int\tan y\,dy=\int-\frac{1}{\cos x}dx$$

$$\int\frac{\sin y}{\cos y}dy=-\int\sec x\,dx$$

Multiply by '-1'

$$\int\frac{-\sin y}{\cos y}dy=+\int\sec x\,dx$$

$$\ln|\cos y|=\ln|\sec x+\tan x|+\ln c$$

$$\cancel{\ln}(\cos y)=\cancel{\ln}\left[c(\sec x+\tan x)\right]$$

$$\boxed{\cos y=c(\sec x+\tan x)}$$

* * *

Question #16

$$y-x\frac{dy}{dx}=3\left[1+x\frac{dy}{dx}\right]$$

Solution:

$$y-x\frac{dy}{dx}=3\left[1+x\frac{dy}{dx}\right]$$

$$y-x\frac{dy}{dx}=3+3x\frac{dy}{dx}$$

$$y-3=x\frac{dy}{dx}+3x\frac{dy}{dx}$$

$$y-3=4x\frac{dy}{dx}$$

$$\frac{1}{4x}dx=\frac{1}{y-3}dy$$

Integration both sides

$$\frac{1}{4}\int\frac{1}{x}dx=\int\frac{1}{y-3}dy$$

$$\frac{1}{4}\ln|x|+\ln c=\ln|y-3|$$

$$\because n\ln m=\ln m^n$$

$$\ln x^{1/4}+\ln c=\ln(y-3)$$

$$\cancel{\ln}cx^{1/4}=\cancel{\ln}(y-3)$$

$$y-3=cx^{1/4}$$

$$\boxed{y=3+cx^{1/4}}$$

* * *

Question #17

$$\sec x+\tan y\frac{dy}{dx}=0$$

Solution:

$$\sec x+\tan y\frac{dy}{dx}=0$$

Separating the variables

$$\tan y\,dy=-\sec x\,dx$$

Integration both sides

$$\int \tan y\,dy = -\int \sec x\,dx$$

$$\int \frac{\sin y}{\cos y}\,dy = -\int \sec x\,dx$$

Mutiply by '-1'

$$\int \frac{-\sin y}{\cos y}\,dy = \int \sec x\,dx$$

$$\ln(\cos y) = \ln(\sec x+\tan x)+\ln c$$

$$\cancel{\ln}(\cos y) = \cancel{\ln}[c(\sec x+\tan x)]$$

$$\boxed{\cos y = c(\sec x+\tan x)}$$

Question #18

$$(e^x+e^{-x})\frac{dy}{dx} = e^x-e^{-x}$$

Solution:

$$(e^x+e^{-x})\frac{dy}{dx} = e^x-e^{-x}$$

Separating the variables

$$(e^x+e^{-x})\,dy = e^x-e^{-x}\,dx$$

$$dy = \left(\frac{e^x-e^{-x}}{e^x+e^{-x}}\right)dx$$

Integration both sides

$$\int 1\,dy = \int \frac{e^x-e^{-x}}{e^x+e^{-x}}\,dx$$

$$\boxed{y = \ln|e^x+e^{-x}|+c}$$

Question # 19

Find the general solution of equation $\frac{dy}{dx} - x = xy^2$ Also find the particular solution if $y=1$ when $x=0$

Solution:

$$\frac{dy}{dx} - x = xy^2$$

Separating the variables

$$\frac{dy}{dx} = x + xy^2$$

$$\frac{dy}{dx} = x(1+y^2)$$

$$\frac{dy}{1+y^2} = x\,dx$$

Integration both sides

$$\int \frac{1}{1+y^2}\,dy = \int x\,dx$$

$$\tan^{-1}y = \frac{x^2}{2}+C \quad \text{——} ①$$

put $\boxed{y=1}$, $\boxed{x=0}$

$$\tan^{-1}(1) = \frac{(0)^2}{2}+C$$

$$\frac{\pi}{4} = 0+C$$

$$\boxed{C = \frac{\pi}{4}}$$

put in eq ①

$$\boxed{\tan^{-1}y = \frac{x^2}{2}+\frac{\pi}{4}}$$

Question #20

Solve the differential equation $\frac{dx}{dt} = 2x$ given that $x=4$, when $t=0$.

Solution:

$$\frac{dx}{dt} = 2x$$

Separating the variables

$$dx = 2x\,dt$$

$$\frac{1}{x}dx = 2\,dt$$

Integration both sides

$$\int \frac{1}{x}dx = 2\int 1\,dt$$

$$\ln x = 2t + c_1$$

$$e^{\ln x} = e^{2t+c_1}$$

$$x = e^{2t}\cdot e^{c_1}$$

$\because e^{c_1} = C$

$$x = e^{2t}\cdot c \quad —— ①$$

put $\boxed{x=4}$ and $\boxed{t=0}$

$$4 = e^{2(0)}\cdot c$$

$$4 = e^{0}\cdot c$$

$$\boxed{4 = c}$$

put in eq ①

$$x = e^{2t}4$$

$$\boxed{x = 4e^{2t}}$$

Question #21

Solve the differential equation $\frac{ds}{dt} + 2st = 0$. Also find the particular equation if $s = 4e$, when $t=0$.

Solution:

$$\frac{ds}{dt} + 2st = 0$$

Separating the valiables

$$\frac{ds}{dt} = -2st$$

$$ds = -2st\,dt$$

$$\frac{1}{s}ds = -2t\,dt$$

Integration both sides

$$\int \frac{1}{s}ds = -2\int t\,dt$$

$$\ln s = -\cancel{2}\cdot\frac{t^2}{\cancel{2}} + c_1$$

$$\ln s = -t^2 + c_1$$

$$e^{\ln s} = e^{-t^2+c_1}$$

$$s = e^{c_1}\cdot e^{-t^2}$$

$\because e^{c_1} = c$

$$s = ce^{-t^2} \quad —— ①$$

when $s = 4e$, $t=0$

$$4e = c\cdot e^{-(0)^2}$$

$$4e = ce^{0}$$

$\boxed{4e = c}$ put in eq ①

$$s = 4e\cdot e^{-t^2}$$

$$\boxed{s = 4e^{1-t^2}}$$

Question # 22

In a culture bacteria increases at the rate proportional to the number of bacteria present. If bacteria are 200 initially and are doubled in 2 hours. Find the number of bacteria present four hours later.

Solution:

let p be the number of bacteria then

$$\frac{dP}{dt} = KP$$

$$\frac{1}{P}\,dp = K\,dt$$

Integration both sides

$$\int \frac{1}{P}\,dp = K\int 1\,dt$$

$$\ln p = Kt + \ln c$$

$$\ln p - \ln c = Kt$$

$$\ln \frac{P}{c} = Kt$$

$$e^{\ln \frac{P}{c}} = e^{kt}$$

$$\frac{P}{c} = e^{kt}$$

$$p = ce^{Kt} \text{ —— } ⓘ$$

put $p = 200$, $t = 0$

$$200 = ce^{K(0)}$$

$$200 = ce^{0} \quad \because e^{0} = 1$$

$$\boxed{c = 200}$$

so eq ⓘ

$$P = 200e^{Kt} \text{ —— } ⓘⓘ$$

put $p = 400$ when $t = 2$ in eq ⓘⓘ

so eq ⓘ

$$400 = 200e^{8K}$$

$$2 = e^{2K}$$

$$\ln 2 = \ln e^{2K}$$

$$2K = \ln 2$$

$$K = \frac{1}{2}\ln 2$$

so eq ⓘⓘ $p = 200e^{Kt}$ $\because t = 4$

$$P = 200e^{\left(\frac{\ln 2}{2}\right)(\overset{2}{4})}$$

$$P = 200e^{2\ln 2}$$

$$P = 200e^{\ln 2^{2}}$$

$$P = 200e\ln^{4}$$

$$P = 200(4)$$

$$\boxed{P = 800}$$

Which is the required number of bacteria present four hours later.

• —— •

Question # 23

A ball is thrown vertically upward with the velocity of 2450cm/sec. Neglecting air resistance find,

i) Velocity of ball at any time t.

ii) distance traveled in any time t.

iii) maximum height attained by the ball.

Solution:

Let v is the velocity and g is acceleration, so

(i) $\frac{dv}{dt} = -g$ (For upward direction)

$dv = -g\,dt$

Integration both sides

$\int dv = -g \int dt$

$v = -gt + c_1$ —— ①

put $v = 2450$, $t = 0$

$2450 = -g(0) + c_1$

$\boxed{2450 = c_1}$

Thus eq ①

$v = -gt + 2450$

$\because g = 980$

$v = -980t + 2450$

ii) Let h be height so

$v = \frac{dh}{dt}$

$\frac{dh}{dt} = v$

$\frac{dh}{dt} = -980t + 2450$

$dh = -980t\,dt + 2450\,dt$

Integration both sides

$\int dh = -980 \int t\,d + \int 2450\,dt$

$h = -980\frac{t^2}{2} + 2450t + c_2$ —— ②

put $h = 0$, $t = 0$

$0 = -490(0)^2 + 2450(0) + c_2$

$\boxed{c_2 = 0}$

Thus eq ②

$h = 0490\ t^2 + 2450$ —— ③

iii) For maximum height $v = 0$

$0 = -980t + 2450$

From eq ①

$980t = 2450$

$t = \frac{2450}{980}$

$t = 5/2$

$h = 2450\left(\frac{5}{2}\right) - 490\left(\frac{5}{2}\right)^2$

$h = 6125 - 30625$

$h = 3062.5$

maximum height = 3062.5 cm

Divided by 100

$\boxed{\text{Maximum height} = 30.6\ m}$

UNIT
4

Introduction to Analytic Geometry

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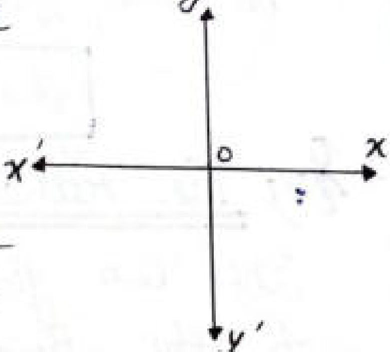
Unit no 4

(Introduction to Analytic Geometry)

Theory:

i) Coordinate System:

Draw in a plane two mutually perpendicular number lines x'x and y'y, one horizontal and other vertical. Let their point of intersection be O which we call the origin and the real number 0 of both the lines is represented by O. The two lines are called the coordinate axes. The horizontal line x'Ox is called x-axis and the vertical line y'Oy is called y-axis.



Note:

If (x,y) are the coordinates of a point P, then the first number (component) of the ordered pair is called the x-coordinate or abscissa of P and the second member of the ordered pair is called the y-coordinate or ordinate of P

ii) Quadrants:

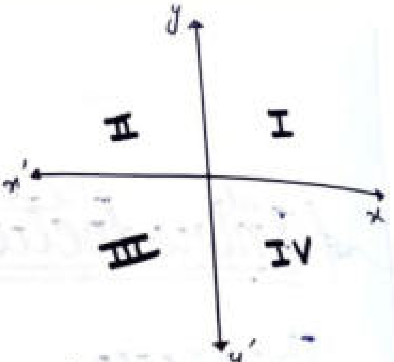
The coordinate axes divide the plane into four equal Parts called quadrants. They are defined as fallows:

Quadrant I: All points with (x,y) $x>0; y>0$

Quadrant II: All points (x,y) with $x<0; y>0$

Quadrant III: All points (x,y) with $x<0; y<0$

Quadrant IV: All points (x,y) with $x>0; y<0$



iii) **The Distance Formula:** Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be two points in the plane

$$\boxed{d = |AB| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}}$$

iv) **The ratio formula:** Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be the two given points in a plane. The coordinates of the point dividing the line segment AB in ratio $k_1 : k_2$ are

$$\boxed{\left(\frac{k_1 x_2 + k_2 x_1}{k_1 + k_2} ; \frac{k_1 y_2 + k_2 y_1}{k_1 + k_2}\right)}$$

v) **Mid point formula:** If $k_1 : k_2 = 1:1$, then P becomes midpoint of $\overline{AB}$ and coordinates of P are:

$$\boxed{x = \frac{x_1 + x_2}{2} ; y = \frac{y_1 + y_2}{2}}$$

Note:

The centroid of a ΔABC is a point that divides each median in the ratio 2:1
Medians of a triangle are concurrent

Note:

Bisectors of angles of a triangle are concurrent.

Example no 1: Show that the points A(-1,2), B(7,5) and C(2,-6) are vertices of a right triangle.

If these points are the vertices of a right triangle than. By P.Theorem

$|BC|^2 = |AB|^2 + |AC|^2$

$\therefore d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

By Distance formula

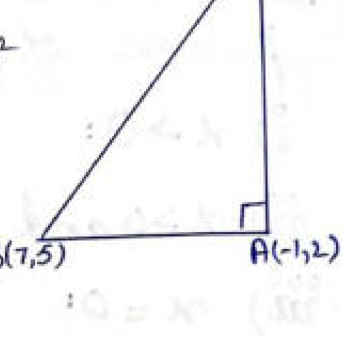
$\sqrt{(7-2)^2+(5+6)^2} = \sqrt{(7+1)^2+(5-2)^2} + \sqrt{(-1-2)^2+(2+6)^2}$

$\sqrt{(5)^2+(11)^2} = \sqrt{(8)^2+(3)^2} + \sqrt{(-3)^2+(8)^2}$

$\sqrt{25+121} = \sqrt{64+9} + \sqrt{9+64}$

$(\sqrt{146})^2 = (\sqrt{73})^2 + (\sqrt{73})^2$

$146 = 73+73 \Rightarrow 146 = 146$ Hence proved.



Example no 2: The point C(-5,3) is the centre of a circle and P(7,-2) lies on the circle. What is the radius of the circle?

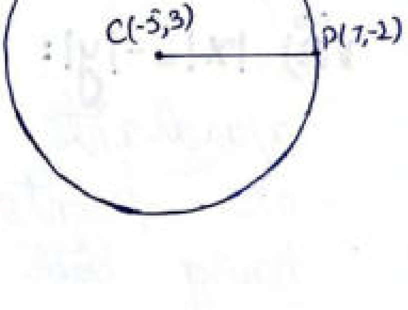
The radius of the circle is the distance from C to P. C(-5,3) and P(7,-2)

By Distance formula

$\text{Radius} = |CP| = \sqrt{7-(-5)^2+(-2-3)^2} \Rightarrow \sqrt{(7+5)^2+(-5)^2}$

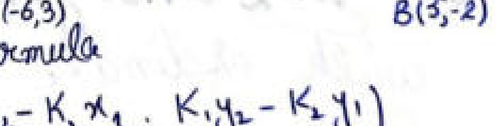
$\Rightarrow \sqrt{(12)^2+(-5)^2} \Rightarrow \sqrt{144+25} \Rightarrow \sqrt{169}$

$\boxed{r = 13}$



Example no 3: Find the coordinates of the point that divides the join of A(-6,3) and B(5,-2) in the ratio 2:3

(i) internally (ii) externally



Here $K_1 = 2$; $K_2 = 3$; $x_1 = -6$; $x_2 = 5$; $y_1 = 3$; $y_2 = -2$

(i) By the formula

$P(x,y) = \left(\frac{K_1x_2 + K_2x_1}{K_1 + K_2} ; \frac{K_1y_2 + K_2y_1}{K_1+K_2}\right)$

$'' = \left(\frac{(2)(5)+(3)(-6)}{2+3} ; \frac{(2)(-2)+(3)(3)}{2+3}\right)$

$'' = \left(\frac{10-18}{5} ; \frac{-4+9}{5}\right)$

$'' = \left(\frac{-8}{5} ; \frac{5}{5}\right)$

$P(x,y) = \left(-\frac{8}{5} ; 1\right)$

(ii) By the formula

$P(x,y) = \left(\frac{K_1x_2 - K_2x_1}{K_1 - K_2} ; \frac{K_1y_2 - K_2y_1}{K_1-K_2}\right)$

$'' = \left(\frac{(2)(5)-(3)(-6)}{2-3} ; \frac{(2)(-2)-(3)(3)}{2-3}\right)$

$'' = \left(\frac{10+18}{-1} ; \frac{-4-9}{-1}\right)$

$'' = \left(\frac{28}{-1} ; -\frac{13}{-1}\right)$

$P(x,y) = (-28 ; 13)$

Exersise no 4.1

Question no 1: Describe the location in the plane of the point P(x, y) for which

i) $x > 0$: The right-half plane

ii) $x > 0$ and $y > 0$: The first quadrant.

iii) $x = 0$: The y-axis

iv) $y = 0$: The x-axis

v) $x < 0$ and $y \geq 0$: The second quadrant

vi) $x = y$: Points in the first and third quadrants having equal abscissae and ordinates

vii) $|x| = -|y|$: Points in the first and third quadrants having both the coordinates equal or points in the second and fourth quadrants having both the coordinates equall but opposite signs.

viii) $|x| \geq 3$: Point on the x-axis less than or equal to -3 or greater than or equal to 3

ix) $x > 2$ and $y = 2$: Points in the first quadrants with ordinate 2 and abscissa greater than 2

x) x and y have opposite signs: The second and fourth quadrants.

Question no 28 Find in each of the fallowing

i) The distance between the two given points

ii) Mid point of the line segment joining the two points

a) $A(3,1)$; $B(-2,-4)$

Distance:

By Using distance formula

$$|AB| = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

$$'' = \sqrt{(-2-3)^2+(-4-1)^2}$$

$$'' = \sqrt{(-5)^2+(-5)^2}$$

$$'' = \sqrt{25+25}$$

$$'' = \sqrt{50}$$

$$\boxed{|AB| = 5\sqrt{2}}$$

Mid point:

By Using Mid point formula

$$\text{Mid point} = \left(\frac{x_1+x_2}{2} ; \frac{y_1+y_2}{2}\right)$$

$$'' = \left(\frac{3-2}{2} ; \frac{1-4}{2}\right)$$

$$\boxed{M.P = \left(\frac{1}{2} ; -\frac{3}{2}\right)}$$

b) $A(-8,3)$, $B(2,-1)$

Distance:

By Using distance formula

$$|AB| = \sqrt{(2+8)^2+(-1-3)^2}$$

$$'' = \sqrt{(10)^2+(-4)^2}$$

$$'' = \sqrt{100+16}$$

$$'' = \sqrt{116}$$

$$\boxed{|AB| = 2\sqrt{29}}$$

Mid point:

By using Mid point formula

$$\text{Mid point} = \left(\frac{x_1+x_2}{2} ; \frac{y_1+y_2}{2}\right)$$

$$'' = \left(\frac{-8+2}{2} ; \frac{3-1}{2}\right)$$

$$'' = \left(\frac{-6}{2} ; \frac{2}{2}\right)$$

$$\boxed{M.P = (-3, 1)}$$

c) $A(-\sqrt{5}; -\frac{1}{3})$; $B(-3\sqrt{5}, 5)$

Distance:

$$|AB| = \sqrt{(-3\sqrt{5}+\sqrt{5})^2+\left(5+\frac{1}{3}\right)^2}$$

$$'' = \sqrt{(-2\sqrt{5})^2+\left(\frac{15+1}{3}\right)^2}$$

$$'' = \sqrt{4(5)+\left(\frac{16}{3}\right)^2}$$

$$'' = \sqrt{20+\frac{256}{9}} \Rightarrow \sqrt{\frac{180+256}{9}}$$

$$'' = \sqrt{\frac{436}{9}} \Rightarrow \boxed{|AB| = \frac{2\sqrt{109}}{3}}$$

Mid point:

$$\text{Mid point} = \left(\frac{-\sqrt{5}-3\sqrt{5}}{2} ; \frac{-\frac{1}{3}+5}{2}\right)$$

$$'' = \left(\frac{-4\sqrt{5}}{2} ; \frac{\frac{-1+15}{3}}{2}\right)$$

$$'' = \left(-2\sqrt{5} ; \frac{14/3}{2}\right)$$

$$'' = \left(-2\sqrt{5} ; \frac{14}{6}\right)$$

$$\boxed{M.P = \left(-2\sqrt{5} ; \frac{7}{3}\right)}$$

Question no 3: Which of the fallowing points are at a distance of 15 units from the origin?

a) $(\sqrt{176}, 7)$

Let $A(0;0); B(\sqrt{176};7)$

By Using distance formula

$|AB| = \sqrt{(\sqrt{176}-0)^2+(7-0)^2}$

$'' = \sqrt{(\sqrt{176})^2+(7)^2}$

$'' = \sqrt{176+49}$

$'' = \sqrt{225}$

$|AB| = 15$

b) $(10,-10)$

Let $A(0,0), B(10,-10)$

By Using distance formula

$|AB| = \sqrt{(10-0)^2+(-10-0)^2}$

$'' = \sqrt{(10)^2+(10)^2}$

$'' = \sqrt{100+100}$

$'' = \sqrt{200}$

$|AB| = 10\sqrt{2}$

c) $(1,15)$

Let $A(0,0), B(1,15)$

By Using distance formula

$|AB| = \sqrt{(1-0)^2+(15-0)^2}$

$'' = \sqrt{(1)^2+(15)^2}$

$'' = \sqrt{1+225}$

$|AB| = \sqrt{226}$

d) $(\frac{15}{2}, \frac{15}{2})$

Let $A(0,0), B(\frac{15}{2}; \frac{15}{2})$

By Using distance formula

$|AB| = \sqrt{(\frac{15}{2}-0)^2+(\frac{15}{2}-0)^2}$

$'' = \sqrt{(\frac{15}{2})^2+(\frac{15}{2})^2}$

$'' = \sqrt{\frac{225}{4}+\frac{225}{4}} \Rightarrow \sqrt{\frac{225+225}{4}}$

$'' = \sqrt{\frac{450}{4}} \Rightarrow |AB| = \frac{15\sqrt{2}}{2}$

Question no 4: Show ~~that~~:

i) The points $A(0;2); B(\sqrt{3};-1)$ and $C(0,-2)$ are the vertices of a right triangle

If these point are the vertices of a right triangle than by P. Theorem

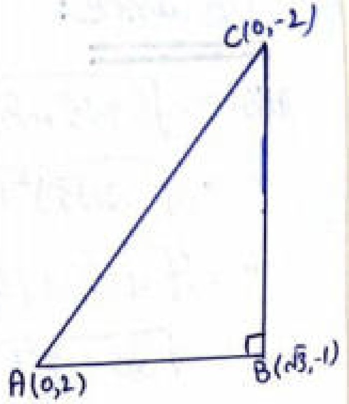
$|AC|^2 = |AB|^2 + |BC|^2$

$\sqrt{(0-0)^2+(-2-2)^2}^2 = \sqrt{(0-\sqrt{3})^2+(2-(-1))^2}^2 + \sqrt{(\sqrt{3}-0)^2+(-1-(-2))^2}^2$

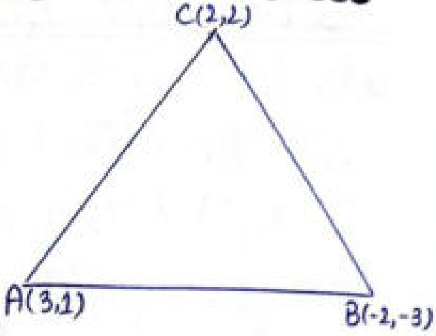
$\sqrt{0+(-4)^2} = \sqrt{(-\sqrt{3})^2+(3)^2} + \sqrt{(\sqrt{3})^2+(1)^2}$

$\sqrt{16} = \sqrt{3+9} + \sqrt{3+1}$

$(\sqrt{16})^2 = (\sqrt{12})^2+(\sqrt{4})^2 \Rightarrow 16 = 12+4 \Rightarrow 16 = 16$ Hence proved



ii) The points A(3,1), B(-2,-3) and C(2,2) are ~~two~~ vertices of an isosceles triangle.



These points are the vertices of an isosceles triangle when

$|AB| = |BC|$ or $|BC| = |AC|$ or $|AC| = |AB|$

By using distance formula

$|AB| = \sqrt{(-2-3)^2+(-3-1)^2}$

$\quad '' = \sqrt{(-5)^2+(-4)^2}$

$\quad '' = \sqrt{25+16}$

$|AB| = \sqrt{41}$

$|BC| = \sqrt{(-2-2)^2+(-3-2)^2}$

$\quad '' = \sqrt{(-4)^2+(-5)^2}$

$\quad '' = \sqrt{16+25}$

$|BC| = \sqrt{41}$

$|AC| = \sqrt{(2-3)^2+(2-1)^2}$

$\quad '' = \sqrt{(-1)^2+(1)^2}$

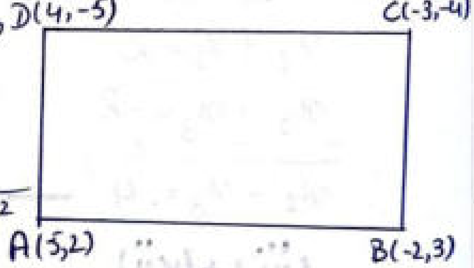
$\quad '' = \sqrt{1+1}$

$|AC| = \sqrt{2}$

So $|AB| = |BC|$

iii) The points A(5,2), B(-2,3), C(-3,-4) and D(4,-5) are vertices of a Parallelogram. Is the parallelogram a square?

These points be the points of parallelogram



if $|AB| = |CD|$ or $|AD| = |BC|$

A(5,2); B(-2,3); C(-3,-4), D(4,-5)

$\sqrt{(-2-5)^2+(3-2)^2} = \sqrt{(-3-4)^2+(-4+5)^2}$

$\sqrt{(-7)^2+(1)^2} = \sqrt{(-7)^2+(1)^2}$

$\sqrt{49+1} = \sqrt{49+1}$

$\sqrt{50} = \sqrt{50}$

So $|AB| = |DC|$

$\sqrt{(5-4)^2+(2+5)^2} = \sqrt{(-2+3)^2+(3+4)^2}$

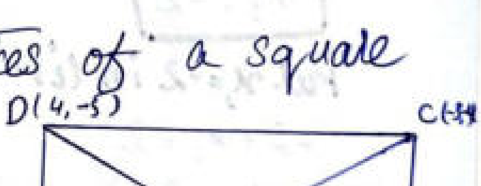
$\sqrt{(1)^2+(7)^2} = \sqrt{(1)^2+(7)^2}$

$\sqrt{1+49} = \sqrt{1+49}$

$\sqrt{50} = \sqrt{50}$

So $|AD| = |BC|$

These points will also be the vertices of a square



if $|AC| = |BD|$

$\sqrt{(-3-5)^2+(-4-2)^2} = \sqrt{(-2-4)^2+(3+5)^2}$

$\sqrt{(-8)^2+(-6)^2} = \sqrt{(-6)^2+(8)^2}$

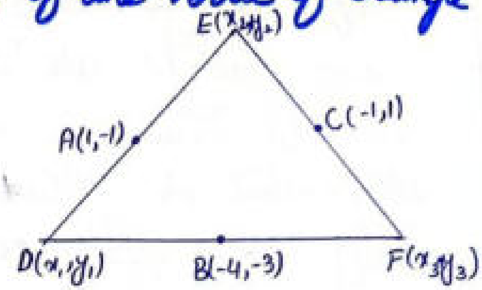
$\sqrt{64+36} = \sqrt{36+64}$

$\sqrt{100} = \sqrt{100}$

$10 = 10$

So these points also be the vertices of a square.

Question no 58 The midpoints of the sides of a triangle are (1,-1), (-4,-3) and (-1,1). Find coordinates of the vertice of triangle



Let the vertices of the triangle are

$D(x_1,y_1)$, $E(x_2,y_2)$ & $F(x_3,y_3)$

The given point $A(1,-1)$, $B(-4,-3)$ & $C(-1,1)$

∴ A is the midpoint of DE

$(1,-1) = \left(\frac{x_1+x_2}{2}; \frac{y_1+y_2}{2}\right) \Rightarrow 1 = \frac{x_1+x_2}{2}; -1 = \frac{y_1+y_2}{2} \Rightarrow x_1+x_2 = 2$ —(i); $y_1+y_2 = -2$ —(ii)

∴ B is the midpoint of DF

$(-4,-3) = \left(\frac{x_1+x_3}{2}; \frac{y_1+y_3}{2}\right) \Rightarrow -4 = \frac{x_1+x_3}{2}; -3 = \frac{y_1+y_3}{2} \Rightarrow x_1+x_3 = -8$ —(iii), $y_1+y_3 = -6$ —(iv)

∴ C is the mid point of EF

$(-1,1) = \left(\frac{x_2+x_3}{2}; \frac{y_2+y_3}{2}\right) \Rightarrow -1 = \frac{x_2+x_3}{2}; 1 = \frac{y_2+y_3}{2} \Rightarrow x_2+x_3 = -2$ —(v); $y_2+y_3 = 2$ —(vi)

(i) − (v)

$x_1 + x_2 = 2$

$\underline{\overset{-}{}x_2 \overset{-}{+} x_3 = \overset{+}{-}2}$

$x_1 - x_3 = 4$ ——— (vii)

(iii) + (vii)

$x_1 + \cancel{x_3} = -8$

$\underline{x_1 - \cancel{x_3} = 4}$

$2x_1 = -4$

$\boxed{x_1 = -2}$

Put $x_1 = -2$ in (i)

$-2 + x_2 = 2$

$x_2 = 2 + 2$

$\boxed{x_2 = 4}$

Put $x_2 = 4$ in (v)

$4 + x_3 = -2$

$x_3 = -2 - 4$

$\boxed{x_3 = -6}$

(ii) − (vi)

$y_1 + y_2 = -2$

$\underline{\overset{-}{}y_2 \overset{-}{+} y_3 = \overset{-}{}2}$

$y_1 - y_3 = -4$ ——— (viii)

(iv) + (viii)

$y_1 + \cancel{y_3} = -6$

$\underline{y_1 - \cancel{y_3} = -4}$

$2y_1 = -10$

$\boxed{y_1 = -5}$

Put $y_1 = -5$ in (ii)

$-5 + y_2 = -2$

$y_2 = -2 + 5$

$\boxed{y_2 = 3}$

Put $y_2 = 3$ in (vi)

$3 + y_3 = 2$

$y_3 = 2 - 3$

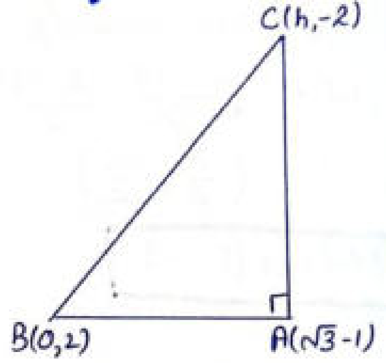
$\boxed{y_3 = -1}$

$D(x_1, y_1) = D(-2,-5)$

$E(x_2, y_2) = E(4,3)$

$F(x_3, y_3) = F(-6,-1)$

Question no 6: Find h such that the points $A(\sqrt{3},-1)$, $B(0,2)$ and $C(h,-2)$ are vertices of a right triangle with right angle at the vertex A.



If these points are the vertices of a right triangle than. By P.Theorem

$$|BC|^2 = |AB|^2 + |AC|^2$$

By Using distance formula:

$A(\sqrt{3},-1), B(0,2), C(h,-2)$

$|BC| = \sqrt{(h-0)^2 + (-2-2)^2}$

$|BC| = \sqrt{(h)^2 + (-4)^2}$

$|BC| = \sqrt{h^2+16}$

$|BC|^2 = h^2+16$

$|AB| = \sqrt{(0-\sqrt{3})^2 + (2-(-1))^2}$

$|AB| = \sqrt{(-\sqrt{3})^2 + (3)^2}$

$|AB| = \sqrt{3+9}$

$|AB| = \sqrt{12}$

$|AB|^2 = 12$

$|AC| = \sqrt{(h-\sqrt{3})^2 + (2-(-1))^2}$

$|AC| = \sqrt{h^2-2h\sqrt{3}+3+(-1)^2}$

$|AC| = \sqrt{h^2-2h\sqrt{3}+3+1}$

$|AC| = \sqrt{h^2-2h\sqrt{3}+4}$

$|AC|^2 = h^2-2h\sqrt{3}+4$

By Putting the values:

$h^2+16 = 12+h^2-2h\sqrt{3}+4 \Rightarrow h^2+16 = h^2-2h\sqrt{3}+16$

$\cancel{h^2}+\cancel{16}-\cancel{h^2}+2h\sqrt{3}-\cancel{16} = 0 \Rightarrow 2h\sqrt{3} = 0 \Rightarrow h = \frac{0}{2\sqrt{3}}$

$$\boxed{h=0}$$

Question no 7: Find h such that $A(-1,h)$, $B(3,2)$ and $C(7,3)$ are collinear.

$A(-1,h), B(3,2), C(7,3)$

Since the points are collinear

$$\begin{vmatrix} -1 & h & 1 \\ 3 & 2 & 1 \\ 7 & 3 & 1 \end{vmatrix} = 0$$

$-1(2-3) - h(3-7) + 1(9-14) = 0$

$-1(-1) - h(-4) + 1(-5) = 0$

$1 + 4h - 5 = 0$

$4h - 4 = 0$

$4h = 4 \Rightarrow \boxed{h=1}$

Note:

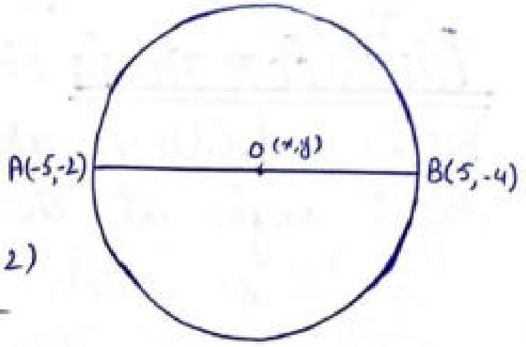
⇒ The points $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ are said to collinear if

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

⇒ The points are said to be collinear if they lie on the same line.

Question no 8: The points A(-5,-2) and B(5,-4) are ends of a diameter of a circle. Find the centre and radius of the circle

A(-5,-2) ; B(5,-4)



For Centre:

Using M.P formula

$M.P = \left(\frac{-5+5}{2} ; \frac{-2-4}{2}\right)$

$\text{"} = \left(\frac{0}{2} ; \frac{-6}{2}\right)$

$\boxed{M.P = O = (0,-3)}$

For radius:

$r = |OA| = |OB|$

So we find $|OA|$. O(0,-3), A(-5,-2)

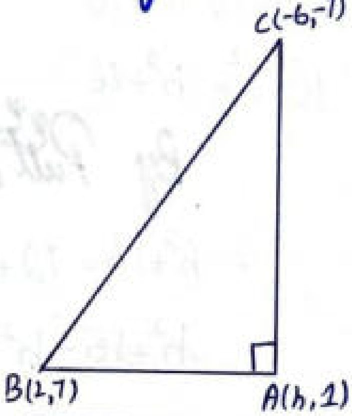
$|OA| = r = \sqrt{(-5-0)^2+(-2-(-3))^2}$

$\text{"} = \sqrt{(-5)^2+(1)^2}$

$\text{"} = \sqrt{25+1}$

$\boxed{r = \sqrt{26}}$

Question no 9: Find h such that the points A(h,1) B(2,7) and C(-6,-7) are vertices of a right triangle with right angle at the vertex A.



If these points are the vertices of a right triangle than. By P.Theorem

$|BC|^2 = |AB|^2 + |AC|^2$

By Using distance formula:

A(h,1), B(2,7), C(-6,-7)

$|BC| = \sqrt{(-6-2)^2+(-7-7)^2}$

$\text{"} = \sqrt{(-8)^2+(-14)^2}$

$\text{"} = \sqrt{64+196}$

$\text{"} = \sqrt{260}$

$|BC|^2 = 260$

$|AB| = \sqrt{(h-2)^2+(7-1)^2}$

$\text{"} = \sqrt{h^2-4h+4+(6)^2}$

$\text{"} = \sqrt{h^2-4h+4+36}$

$\text{"} = \sqrt{h^2-4h+40}$

$|AB|^2 = h^2-4h+40$

$|AC| = \sqrt{(h-(-6))^2+(1-(-7))^2}$

$\text{"} = \sqrt{(h+6)^2+(1+7)^2}$

$\text{"} = \sqrt{h^2+12h+36+64}$

$\text{"} = \sqrt{h^2+12h+100}$

$|AC|^2 = h^2+12h+100$

By putting the values

$260 = h^2-4h+40+h^2+12h+100$

$260 = 2h^2+8h+140$

$2h^2+8h+140-260 = 0$

$2h^2+8h-120 = 0$

÷ing by 2

$\frac{2h^2}{2}+\frac{8h}{2}-\frac{120}{2} = 0$

$h^2+4h-60 = 0$

$h^2+10h-6h-60 = 0$

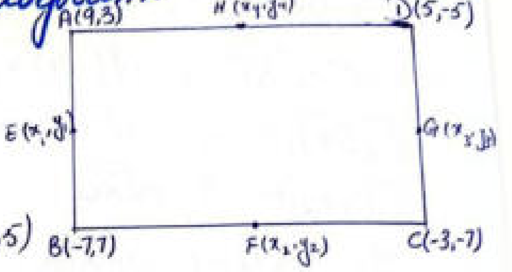
$h(h+10)-6(h+10) = 0$

$(h+10)(h-6) = 0$

$h+10 = 0 ; h-6 = 0 \Rightarrow \boxed{h = -10 ; 6}$

$h = -10 ; h = 6$

Question no 10: A quadrilateral has the points A(9,3), B(-7,7), C(-3,-7) and D(5,-5) as its vertices. Find the midpoints of its sides. Show that the figure formed by joining the midpoints consecutively is a parallelogram.



Let E, F, G and H be the midpoints of AB; BC; CD and AD respectively

∴ E is the midpoint of AB

$E(x_1,y_1) = \left(\frac{9-7}{2}; \frac{3+7}{2}\right) \Rightarrow E(x_1,y_1) = \left(\frac{2}{2}; \frac{10}{2}\right) \Rightarrow E(x_1,y_1) = (1,5)$

∴ F is the midpoint of BC

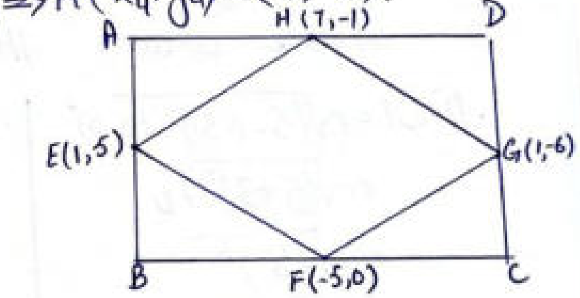
$F(x_2,y_2) = \left(\frac{-7-3}{2}; \frac{7-7}{2}\right) \Rightarrow E(x_2,y_2) = \left(\frac{-10}{2}; \frac{0}{2}\right) \Rightarrow F(x_2,y_2) = (-5,0)$

∴ G is the midpoint of CD

$G(x_3,y_3) = \left(\frac{-3+5}{2}; \frac{-7-5}{2}\right) \Rightarrow G(x_3,y_3) = \left(\frac{2}{2}; \frac{-12}{2}\right) \Rightarrow G(x_3,y_3) = (1,-6)$

∴ H is the mid point of AD

$H(x_4,y_4) = \left(\frac{9+5}{2}; \frac{3-5}{2}\right) \Rightarrow H(x_4,y_4) = \left(\frac{14}{2}; \frac{-2}{2}\right) \Rightarrow H(x_4,y_4) = (7,-1).$



According to the condition

If these midpoints are the vertices of a parallelogram than

|EF| = |GH| or |EH| = |FG|

E(1,5), F(-5,0), G(1,-6), H(7,-1).

$\sqrt{(-5-1)^2+(0-5)^2} = \sqrt{(7-1)^2+(-1-(-6))^2}$

$\sqrt{(-6)^2+(-5)^2} = \sqrt{(6)^2+(5)^2}$

$\sqrt{36+25} = \sqrt{36+25}$

$\sqrt{61} = \sqrt{61}$

|EF| = |GH|

$\sqrt{(7-1)^2+(-1-5)^2} = \sqrt{(1-(-5))^2+(-6-0)^2}$

$\sqrt{(6)^2+(-6)^2} = \sqrt{(6)^2+(-6)^2}$

$\sqrt{36+36} = \sqrt{36+36}$

$\sqrt{72} = \sqrt{72}$

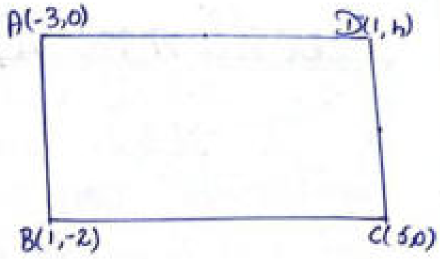
|EH|=|FG|

Hence proved these points are the vertices of an parallelogram.

Question no 11: Find h such that the quadrilateral with vertices A(-3,0), B(1,-2), C(5,0) and D(1,h) is parallelogram. Is it square?

A(-3,0), B(1,-2), C(5,0), D(1,h)

Since these points are the vertices of a parallelogram so



|AD| = |BC|

$\sqrt{(-3-1)^2+(0-h)^2} = \sqrt{(1-5)^2+(-2-0)^2}$

$\sqrt{(-4)^2+(-h)^2} = \sqrt{(-4)^2+(-2)^2}$

$\sqrt{16+h^2} = \sqrt{16+4}$

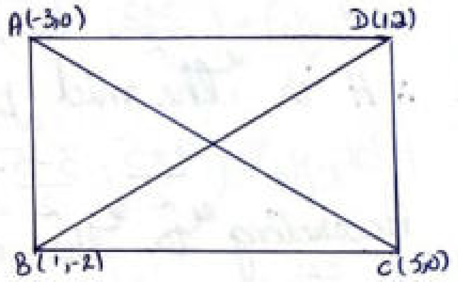
$\sqrt{16+h^2} = \sqrt{20}$

By taking square on Both sides

$(\sqrt{16+h^2})^2 = (\sqrt{20})^2 \Rightarrow 16+h^2 = 20 \Rightarrow h^2 = 20-16 \Rightarrow h^2 = 4$

$\boxed{h = \pm 2}$

We take h=2 because If we take h=-2 so the vertices B and D become same which is not possible



If these point also be the points of a square than |AC| = |BD|

$|AC| = \sqrt{(5-(-3))^2+(0-0)^2}$

$= \sqrt{(5+3)^2+0}$

$= \sqrt{(8)^2}$

$= 8$

$|BD| = \sqrt{(1-1)^2+(2-(-2))^2}$

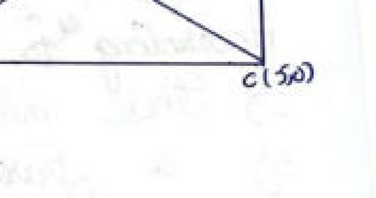
$= \sqrt{(0)^2+(2+2)^2}$

$= \sqrt{(4)^2}$

$= 4$

So 8 ≠ 4 So these points are not the vertices of a square.

Question no 12: If two vertices of an equilateral triangle are A(-3,0), B(3,0) find the third vertex. How many of these triangles are possible?



Let C(x,y) is the third vertex

Since ABC is equilateral Δ

So |AB| = |BC| = |AC| or |AB| = |AC| ; |BC| = |AC|

|AB| = |AC|

$\sqrt{(3-(-3))^2+(0-0)^2} = \sqrt{(x+3)^2+(y-0)^2}$

$\sqrt{(6)^2+(0)^2} = \sqrt{x^2+6x+9+y^2}$

|BC| = |AC|

$\sqrt{(x-3)^2+(y-0)^2} = \sqrt{(x+3)^2+(y-0)^2}$

$\sqrt{(x-3)^2+y^2} = \sqrt{(x+3)^2+y^2}$

$\sqrt{x^2-6x+9+y^2} = \sqrt{x^2+6x+9+y^2}$

$\sqrt{36+0} = \sqrt{x^2+y^2+6x+9}$

By taking square on B·sides

$(\sqrt{36})^2 = (\sqrt{x^2+y^2+6x+9})^2$

$36 = x^2+y^2+6x+9$

$x^2+y^2+6x+9-36=0$

$x^2+y^2+6x-27=0$ —(i)

By taking square on Both sides.

$(\sqrt{x^2-6x+9+y^2})^2 = (\sqrt{x^2+6x+9+y^2})^2$

$x^2-6x+9+y^2 = x^2+6x+9+y^2$

$\cancel{x^2}-6x+\cancel{9}+\cancel{y^2}-\cancel{x^2}-6x-\cancel{9}-\cancel{y^2}=0$

$-12x=0$

$\boxed{x=0}$

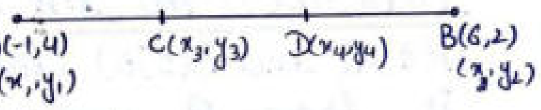
Put $x=0$ in (i)

$(0)^2+y^2+6(0)-27=0 \Rightarrow y^2-27=0 \Rightarrow y^2=27 \Rightarrow \sqrt{y^2}=\sqrt{27} \Rightarrow \boxed{y \pm 3\sqrt{3}}$

So $C(x,y)=C(0,\pm 3\sqrt{3})$. Hence two triangles are possible
$(0,3\sqrt{3}), (0,-3\sqrt{3})$.

Question no 13: Find the points trisecting the join of A(-1,4) and B(6,2).

Suppose points $C(x_3,y_3)$ and $D(x_4,y_4)$ trisect line AB.



∴ C divides AB in ratio 1:2 where $K_1=1$; $K_2=2$

So the coordinates are

By the formula

$$C(x_3,y_3) = \left(\frac{K_1x_2+K_2x_1}{K_1+K_2} ; \frac{K_1y_2+K_2y_1}{K_1+K_2}\right)$$

$$" = \left(\frac{(1)(6)+(2)(-1)}{1+2} ; \frac{(1)(2)+(2)(4)}{1+2}\right)$$

$$" = \left(\frac{6-2}{3} ; \frac{2+8}{3}\right)$$

$$C(x_3,y_3) = \left(\frac{4}{3} ; \frac{10}{3}\right)$$

∴ D divides AB in ratio 2:1 where $K_1=2$; $K_2=1$

So coordinates are

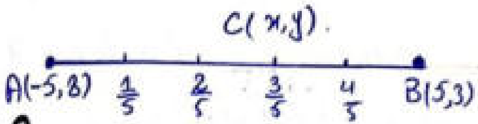
$$D(x_4,y_4) = \left(\frac{(2)(6)+(1)(-1)}{2+1} ; \frac{(2)(2)+(1)(4)}{2+1}\right)$$

$$" = \left(\frac{12-1}{3} ; \frac{4+4}{3}\right)$$

$$D(x_4,y_4) = \left(\frac{11}{3} ; \frac{8}{3}\right)$$

Question no 14: Find the point three-fifth of the way along the line segment from A(-5,8) to B(5,3)

Let C(x,y) be a required point



A(-5,8) ; B(5,3).

∴ C divides the line AB in ratio 3:2

So the coordinates are

$$C(x,y) = \left(\frac{(3)(5)+(2)(-5)}{3+2} ; \frac{(3)(3)+(2)(8)}{3+2}\right)$$

$$" = \left(\frac{15-10}{5} ; \frac{9+16}{5}\right)$$

$$" = \left(\frac{5}{5} , \frac{25}{5}\right)$$

$C(x,y) = (1 , 5)$ So **(1,5)** is the required point.

Question no 15: Find the point P on the joint of A(1,4) and B(5,6) that is twice as far from A as B is from A and lies

i) on the same side of A as B

∴ B become midpoint of AP

So $(5,6) = \left(\frac{1+x}{2} ; \frac{4+y}{2}\right)$

$5 = \frac{1+x}{2}$; $6 = \frac{4+y}{2}$

$10 = 1+x$; $12 = 4+y$

$x = 10-1$; $y = 12-4$

$x = 9$; $y = 8$

So **P(9,8)** is required point

ii) on the opposite side of A as B does

∴ A divides PB in ratio 2:1

So $(1,4) = \left(\frac{(2)(5)+(1)(x)}{2+1} ; \frac{(2)(6)+(1)(y)}{2+1}\right)$

$1 = \frac{10+x}{3}$; $4 = \frac{12+y}{3}$

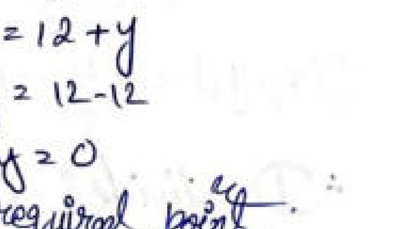
$3 = 10+x$; $12 = 12+y$

$x = 3-10$; $y = 12-12$

$x = -7$; $y = 0$

So **P(-7,0)** is required point.

Question no 16: Find the point which is equidistant from the points A(5,3), B(-2,2) and C(4,2). What is the radius of the circumcircle of the ΔABC?



Let P(x,y) be the point which is equidistant from given points.

|AP| = |BP| = |CP| or |AP| = |BP| ; |BP| = CP|

A(5,3), B(-2,2), C(4,2)

|AP| = |BP|

$\sqrt{(x-5)^2+(y-3)^2} = \sqrt{(x+2)^2+(y-2)^2}$

By taking square on B·sides

$(x-5)^2+(y-3)^2 = (x+2)^2+(y-2)^2$ —— (i)

Now **|BP| = |CP|**

$\sqrt{(x+2)^2+(y-2)^2} = \sqrt{(x-4)^2+(y-2)^2}$

By taking square on B·sides

$(x+2)^2+\cancel{(y-2)^2} = (x-4)^2+\cancel{(y-2)^2}$

$x^2+4x+4 = x^2-8x+16$

$\cancel{x^2}+4x+4-\cancel{x^2}+8x-16=0$

$12x-12=0$

$12x=12$

$\boxed{x=1}$

Put x = 1 in (i)

$(1-5)^2+(y-3)^2=(1+2)^2+(y-2)^2$

$(-4)^2+y^2-6y+9=(3)^2+y^2-4y+4$

$16+y^2-6y+9=9+y^2-4y+4$

$y^2-6y+25=y^2-4y+13$

$\cancel{y^2}-6y+25-\cancel{y^2}+4y-13=0$

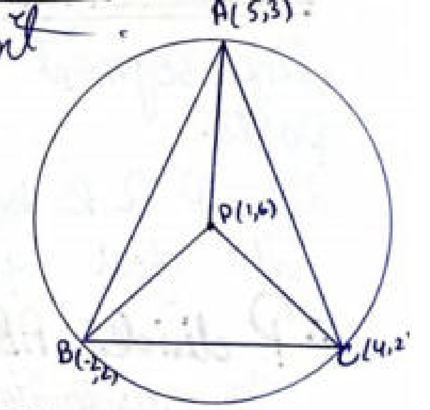
$-2y+12=0$

$-2y=-12$

$y=\frac{-12}{-2}$

$\boxed{y=6}$

Hence **P(1,6)** is the required point.



Radius of circumcircle:

Radius of circumcircle of ΔABC

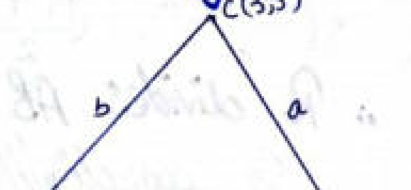
$r = |AP| = |BP| = |CP|$

So we find |AP|

$|AP| = \sqrt{(1-5)^2+(6-3)^2}$

" $= \sqrt{(-4)^2+(3)^2} \Rightarrow \sqrt{16+9} \Rightarrow \sqrt{25} \Rightarrow \boxed{|AP| = r = 5}$

Question no 17: The points (4,-2), (-2,4) and (5,5) are the vertices of a triangle. Find in-centre of the triangle.



Here |BC| = a ; |AC| = b ; |AB| = c

$A(x_1, y_1) = A(4,-2)$

$B(x_2, y_2) = B(-2,4)$

$C(x_3, y_3) = C(5,5)$

$a = |BC| = \sqrt{(5-(-2))^2+(5-4)^2} \Rightarrow \sqrt{(5+2)^2+(1)^2} \Rightarrow \sqrt{(7)^2+(1)^2} \Rightarrow \sqrt{49+1} \Rightarrow \sqrt{50} \Rightarrow 5\sqrt{2}$

$b = |AC| = \sqrt{(5-4)^2+(5+2)^2} \Rightarrow \sqrt{(1)^2+(7)^2} \Rightarrow \sqrt{1+49} \Rightarrow \sqrt{50} \Rightarrow 5\sqrt{2}$

$c = |AB| = \sqrt{(-2-4)^2+(4-(-2))^2} \Rightarrow \sqrt{(-6)^2+(6)^2} \Rightarrow \sqrt{36+36} \Rightarrow \sqrt{72} \Rightarrow 6\sqrt{2}$

∴ Incentre formula

$$= \left(\frac{ax_1+bx_2+cx_3}{a+b+c} ; \frac{ay_1+by_2+cy_3}{a+b+c}\right)$$

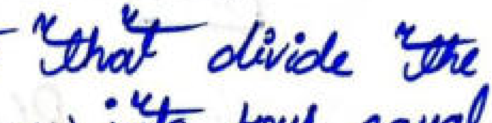
By putting the values:

$$= \left(\frac{(5\sqrt{2})(4)+(5\sqrt{2})(-2)+(6\sqrt{2})(5)}{5\sqrt{2}+5\sqrt{2}+6\sqrt{2}} ; \frac{(5\sqrt{2})(-2)+(5\sqrt{2})(4)+(6\sqrt{2})(5)}{5\sqrt{2}+5\sqrt{2}+6\sqrt{2}}\right)$$

$$= \left(\frac{20\sqrt{2}-10\sqrt{2}+30\sqrt{2}}{16\sqrt{2}} ; \frac{-10\sqrt{2}+20\sqrt{2}+30\sqrt{2}}{16\sqrt{2}}\right)$$

$$= \left(\frac{40\sqrt{2}}{16\sqrt{2}} ; \frac{40\sqrt{2}}{16\sqrt{2}}\right) \Rightarrow \boxed{= \left(\frac{5}{2} ; \frac{5}{2}\right)}$$

Question no 18: Find the points that divide the line segment joining $A(x_1, y_1)$ and $B(x_2, y_2)$ into four equal parts.



Let P, Q, R be the required points into four equal parts

∴ P divides AB in ratio 1:3 where $K_1 = 1 ; K_2 = 3$

So coordinates are

$$= \left(\frac{(1)(x_2)+(3)(x_1)}{1+3} ; \frac{(1)(y_2)+(3)(y_1)}{1+3}\right) \Rightarrow \left(\frac{x_2+3x_1}{4} ; \frac{y_2+3y_1}{4}\right)$$

∴ Q is the mid point of AB.

So coordinates are

$$= \left(\frac{x_1+x_2}{2} ; \frac{y_1+y_2}{2}\right)$$

∴ R divides AB in ratio 3:1 where $K_1 = 3 ; K_2 = 1$

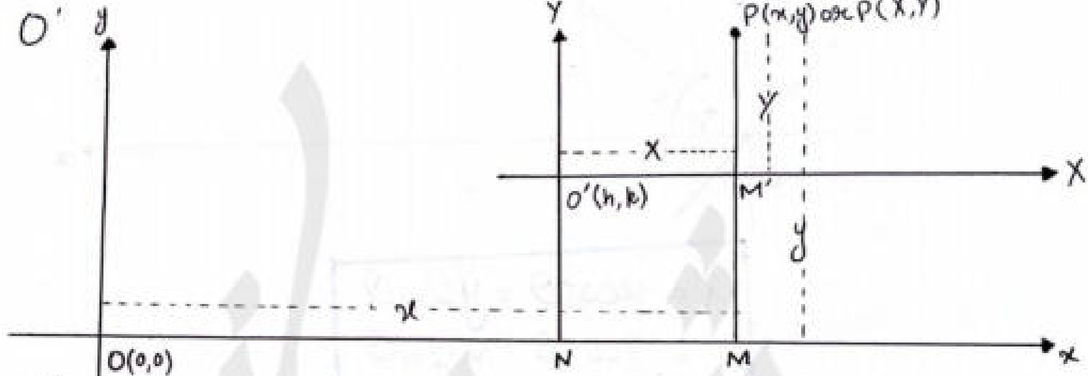
So coordinates are

$$= \left(\frac{(3)(x_2)+(1)(x_1)}{3+1} ; \frac{(3)(y_2)+(1)(y_1)}{3+1}\right) \Rightarrow \left(\frac{3x_2+x_1}{4} ; \frac{3y_2+y_1}{4}\right)$$

Theory:

i) Translation of Axes:

Let xy-coordinates system be given and O'(h,k) be any point in the plane. Through O' draw two mutually perpendicular lines O'X, O'Y such that O'X is parallel to Ox. The new axes O'X and O'Y are called the translation of the Ox- and Oy-axes through the point O'



Note:

In translation of axes, origin is shifted to another point in the plane but the axes remains parallel to the old axes.

If P be a point with coordinates (x,y) reffered to xy-coordinates system and the axes be translated through the point O'(h,k) and O'X, O'Y be the new axes.

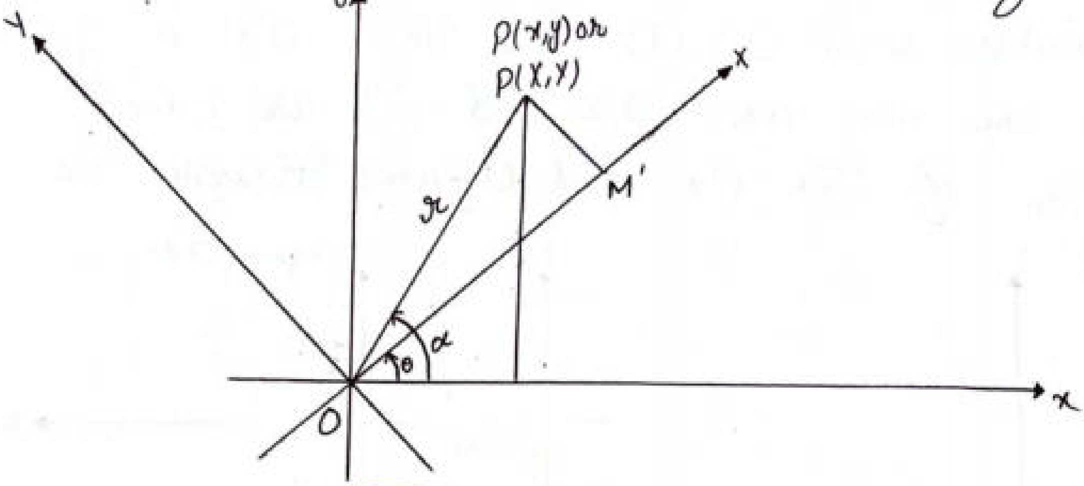
Thus the coordinates of P refferred to XY-system are

$$X = x-h \quad \text{or} \quad P(X,Y) = (x-h,\, y-k)$$
$$Y = y-k$$

Moreover, $x = X+h \; ; \; y = Y+k$

ii) Rotation of Axes:

Let xy-coordinates system be given. We rotate Ox and Oy about the origin through an angle $\theta\,(0<\theta<90^\circ)$ so that the new axes are OX and OY. This process is called the rotation of the axes.



$$\boxed{\begin{array}{l} X = x\cos\theta + y\sin\theta \\ Y = y\cos\theta - x\sin\theta \end{array}}$$

•—•

Example # 1: The coordinates of a point P are (-6,9). The axes are tranlated through the point O'(-3,2) Find the coordinates of P reffered to the ~~x~~ new axes.

$P(x,y) = P(-6,9)$

$O'(h,k) = O'(-3,2)$

$P(X,Y) = (x-h,\ y-k)$

$\quad " = (-6-(-3),\ 9-2)$

$\quad " = (-6+3\ ,\ 7)$

$\boxed{P(X,Y) = (-3,7)}$

•—•

Example # 2: Given Data $P(X,Y) = P(2,-3)$
$O'(h,k) = O'(4,6)$

$P(x,y) = (X+h,\ Y+k)$

$\quad " = (2+4\ ;\ -3+6)$

$\boxed{P(x,y) = (6,\ 3)}$

•—•

Example no 3: Given Data $P(x,y) = P(5,7)$ and $\theta = 30^\circ$

$P(X,Y) = ?$

$X = x\cos\theta + y\sin\theta$

$X = 5\cos 30^\circ + 7\sin 30^\circ$

$X = 5\dfrac{\sqrt{3}}{2} + 7\cdot\dfrac{1}{2}$

$X = \dfrac{5\sqrt{3}+7}{2}$

$Y = y\cos\theta - x\sin\theta$

$Y = 7\cos 30^\circ - 5\sin 30^\circ$

$Y = 7\dfrac{\sqrt{3}}{2} - \dfrac{5}{2}$

$Y = \dfrac{7\sqrt{3}-5}{2}$

$$\boxed{P(X,Y) = \left(\frac{5\sqrt{3}+7}{2} ; \frac{7\sqrt{3}-5}{2}\right)}$$

Example no 4: Given Data $P(X,Y) = P(-1,-7)$; $\theta = \arctan\dfrac{4}{3}$

$\theta = \arctan\dfrac{4}{3} \Rightarrow \theta = \tan^{-1}\dfrac{4}{3} \Rightarrow \tan\theta = \dfrac{4}{3} = \dfrac{P}{B}$

$Hyp^2 = Perp^2 + Base^2 \Rightarrow H^2 = 4^2 + 3^2 \Rightarrow H^2 = 16+9 \Rightarrow H^2 = 25 \Rightarrow H = 5$

$\Rightarrow \sin\theta = \dfrac{P}{H} = \dfrac{4}{5}$ and $\cos\theta = \dfrac{B}{H} = \dfrac{3}{5}$.

$P(x,y) = ?$

$X = x\cos\theta + y\sin\theta$

$-1 = x\cdot\dfrac{3}{5} + y\cdot\dfrac{4}{5}$

$-5 = 3x + 4y$

$3x + 4y + 5 = 0$ —— (i)

$Y = y\cos\theta - x\sin\theta$

$-7 = y\cdot\dfrac{3}{5} - x\cdot\dfrac{4}{5}$

$-35 = 3y - 4x$

$3y - 4x + 35 = 0$ —— (ii)

By Solving (i) & (ii)

$4(i) + 3(ii).$

$\cancel{12x} + 16y + 20 = 0$

$-\cancel{12x} + 9y + 105 = 0$

$25y + 125 = 0$

$25y = -125$

$\boxed{y = -5}$

Put in (i)

$3x + 4(-5) + 5 = 0$

$3x - 20 + 5 = 0$

$3x - 15 = 0$

$3x = 15 \Rightarrow \boxed{x = 5}$

$P(x,y) = P(5,-5)$

Exercise no 4.2

Question no 1: The two points P and O' are given in xy-coordinate system. Find the XY-coordinates of P reffered to the translated axes O'X and O'Y

i) $P(3,2), O'(1,3)$

$P(x,y) = P(3,2)$

$O'(h,k) = O'(1,3)$

$P(X,Y) = ?$

$P(X,Y) = (x-h, y-k)$

$\text{"} = (3-1; 2-3)$

$\boxed{P(X,Y) = (2,-1)}$

iii) $P(-6,-8), O'(-4,-6)$

$P(x,y) = P(-6,-8)$

$O'(h,k) = O'(-4,-6)$

$P(X,Y) = (x-h, y-k).$

$\text{"} = (-6-(-4), -8-(-6))$

$\text{"} = (-6+4, -8+6)$

$\boxed{P(X,Y) = (-2,-2)}$

ii) $P(-2,6), O'(-3,2)$

$P(x,y) = P(-2,6)$

$O'(h,k) = O'(-3,2)$

$P(X;Y) = (x-h, y-k)$

$\text{"} = (-2-(-3), 6-2)$

$\boxed{P(X,Y) = (1,4)}$

iv) $P\left(\frac{3}{2}, \frac{5}{2}\right), O'\left(\frac{-1}{2}, \frac{7}{2}\right)$

$P(X,Y) = (x-h, y-k)$

$\text{"} = \left(\frac{3}{2}+\frac{1}{2}, \frac{5}{2}-\frac{7}{2}\right)$

$\text{"} = \left(\frac{3+1}{2}; \frac{5-7}{2}\right)$

$\text{"} = \left(\frac{4}{2}; -\frac{2}{2}\right)$

$\boxed{P(X,Y) = (2,-1)}.$

Question no 2: The coordinates of P are given in the XY-coordinates system. Find the coordinates of P in xy-coordinate system.

i) $P(8,10), O'(3,4)$

$P(X,Y) = P(8,10)$

$O'(h,k) = O'(3,4)$

$P(x,y) = (X+h, Y+k)$

$\text{"} = (8+3, 10+4)$

$\boxed{P(x,y) = (11,14)}$

ii) $P(-5-3), O'(-2,-6)$

$P(X,Y) = P(-5,-3)$

$O'(h,k) = O'(-2,-6)$

$P(x,y) = (X+h, Y+k)$

$\text{"} = (-5-2; -3-6)$

$\boxed{P(x,y) = (-7,-9)}$

iii) $P\left(\frac{-3}{4}, \frac{-7}{6}\right), O'\left(\frac{1}{4}, \frac{-1}{6}\right)$

$$P(x,y) = (X+h, Y+k)$$
$$" = \left(\frac{-3}{4} + \frac{1}{4}, \frac{-7}{6} - \frac{1}{6}\right)$$
$$" = \left(\frac{-3+1}{4} ; \frac{-7-1}{6}\right)$$
$$" = \left(\frac{-2}{4}, \frac{-8}{6}\right)$$
$$\boxed{P(x,y) = \left(\frac{-1}{2}, \frac{-4}{3}\right)}$$

iv) $P(4,-3); O'(-2,3)$

$$P(X,Y) = P(4,-3)$$
$$O'(h,k) = O'(-2,3)$$
$$P(x,y) = (X+h, Y+k)$$
$$" = (4-2 ; -3+3)$$
$$\boxed{P(x,y) = (2, 0)}$$

Question no 3: The xy-coordinate axes are rotated about the origin through the indicated angle. The new axes are OX and OY. Find the XY-coordinates of the point P with the given xy-coordinates.

i) $P(5,3) ; \theta = 45^\circ$

$$P(X,Y) = (x\cos\theta + y\sin\theta ; y\cos\theta - x\sin\theta)$$
$$" = (5\cos 45^\circ + 3\sin 45^\circ ; 3\cos 45^\circ - 5\sin 45^\circ)$$
$$" = \left(5\cdot\frac{1}{\sqrt{2}} + 3\cdot\frac{1}{\sqrt{2}} ; 3\cdot\frac{1}{\sqrt{2}} - 5\cdot\frac{1}{\sqrt{2}}\right)$$
$$" = \left(\frac{5+3}{\sqrt{2}} ; \frac{3-5}{\sqrt{2}}\right) \Rightarrow \left(\frac{8}{\sqrt{2}} ; \frac{-2}{\sqrt{2}}\right) \Rightarrow \left(\frac{8}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} ; \frac{-2}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}\right)$$
$$\Rightarrow \left(\frac{\overset{4}{8}\sqrt{2}}{2} ; -\frac{2\sqrt{2}}{2}\right) \Rightarrow \boxed{P(X;Y) = (4\sqrt{2} ; -\sqrt{2})}$$

ii) $P(3,-7). \theta = 30^\circ$

$$P(X,Y) = (x\cos\theta + y\sin\theta ; y\cos\theta - x\sin\theta).$$
$$" = (3\cos 30^\circ - 7\sin 30^\circ, -7\cos 30^\circ - 3\sin 30^\circ)$$
$$" = \left(3\cdot\frac{\sqrt{3}}{2} - 7\cdot\frac{1}{2} ; -7\cdot\frac{\sqrt{3}}{2} - 3\cdot\frac{1}{2}\right)$$
$$\boxed{P(X,Y) = \left(\frac{3\sqrt{3}-7}{2} ; \frac{-7\sqrt{3}-3}{2}\right)}$$

iii) $P(11,-15). \theta = 60^\circ$

$$P(X,Y) = (x\cos\theta + y\sin\theta , y\cos\theta - x\sin\theta)$$

$$P(X,Y) = (11\cos 60^\circ + (-15)\sin 60^\circ \;;\; -15\cos 60^\circ \mp 11\sin 60^\circ)$$

$$" = \left(11 \cdot \frac{1}{2} - 15 \cdot \frac{\sqrt{3}}{2} \;;\; -15 \cdot \frac{1}{2} - 11 \cdot \frac{\sqrt{3}}{2}\right)$$

$$\boxed{P(X,Y) = \left(\frac{11-15\sqrt{3}}{2} \;;\; \frac{-15-11\sqrt{3}}{2} \right)}$$

iv) $P(15,10)$, $\theta = \arctan \frac{1}{3}$

$$P(x,y) = P(15,10)$$

$$\theta = \arctan \frac{1}{3} \Rightarrow \theta = \tan^{-1}\frac{1}{3} \Rightarrow \tan\theta = \frac{1}{3} = \frac{P}{B}$$

$$H^2 = P^2 + B^2 \Rightarrow H^2 = (1)^2 + (3)^2 \Rightarrow H^2 = 1+9 \Rightarrow H^2 = 10 \Rightarrow H = \sqrt{10}$$

$$\Rightarrow \sin\theta = \frac{1}{\sqrt{10}} = \frac{P}{H} \quad \text{and} \quad \cos\theta = \frac{3}{\sqrt{10}} = \frac{B}{H}$$

$$P(X,Y) = (x\cos\theta + y\sin\theta \;;\; y\cos\theta - x\sin\theta)$$

$$" = (15\cos\theta + 10\sin\theta,\; 10\cos\theta - 15\sin\theta)$$

$$" = \left(15 \cdot \frac{3}{\sqrt{10}} + 10 \cdot \frac{1}{\sqrt{10}} \;;\; 10 \cdot \frac{3}{\sqrt{10}} - 15 \cdot \frac{1}{\sqrt{10}}\right)$$

$$" = \left(\frac{45+10}{\sqrt{10}} \;;\; \frac{30-15}{\sqrt{10}} \right)$$

$$\boxed{P(X,Y) = \left(\frac{55}{\sqrt{10}} \;;\; \frac{15}{\sqrt{10}} \right)}$$

Question no 4: The xy-coordinate axes are rotated about the origin through the indicated angle and the new axes are OX and OY. Find the xy-coordinates of P with the given XY-coordinates.

i) $P(-5,3)$; $\theta = 30^\circ$

$$P(X,Y) = P(-5,3) \;;\; \theta = 30^\circ$$

$$X = x\cos\theta + y\sin\theta$$

$$-5 = x\cos 30^\circ + y\sin 30^\circ$$

$$-5 = x \cdot \frac{\sqrt{3}}{2} + y \cdot \frac{1}{2}$$

$$-10 = \sqrt{3}x + y$$

$$\sqrt{3}x + y = -10 \text{ --- (i)}$$

$$Y = y\cos\theta - x\sin\theta$$

$$3 = y\cos 30^\circ - x\sin 30^\circ$$

$$3 = y \cdot \frac{\sqrt{3}}{2} - x \cdot \frac{1}{2}$$

$$6 = \sqrt{3}y - x$$

$$-x + \sqrt{3}y = 6 \text{ --- (ii)}$$

By Solving (i) & (ii)

(i)+√3(ii)

$$\cancel{\sqrt{3}x}+y=-10$$

$$-\cancel{\sqrt{3}x}+3y=6\sqrt{3}$$

$$4y=6\sqrt{3}-10$$

$$y=\frac{6\sqrt{3}-10}{4}$$

$$y=\frac{2(3\sqrt{3}-5)}{4}$$

$$\boxed{y=\frac{3\sqrt{3}-5}{2}}$$

Put in (ii)

$$-x+\sqrt{3}\left(\frac{3\sqrt{3}-5}{2}\right)=6$$

$$-x+\frac{3(\sqrt{3})^2-5\sqrt{3}}{2}=6$$

$$-x+\frac{3(3)-5\sqrt{3}}{2}=6$$

$$\frac{9-5\sqrt{3}}{2}-6=x$$

$$\frac{9-5\sqrt{3}-12}{2}=x$$

$$\boxed{x=\frac{-5\sqrt{3}-3}{2}}$$

$$\boxed{P(x,y)=\left(\frac{-5\sqrt{3}-3}{2}\,;\,\frac{3\sqrt{3}-5}{2}\right)}$$

iii) $P(-7\sqrt{2};5\sqrt{2}):\theta=45°$

$$P(X,Y)=P(-7\sqrt{2};5\sqrt{2});\ \theta=45°$$

$$X=x\cos\theta+y\sin\theta$$

$$-7\sqrt{2}=x\cos45°+y\sin45°$$

$$-7\sqrt{2}=x\cdot\frac{1}{\sqrt{2}}+y\cdot\frac{1}{\sqrt{2}}$$

$$-7\sqrt{2}=\frac{1}{\sqrt{2}}(x+y)$$

$$-7(2)=x+y$$

$$x+y=-14 \text{ —— (i)}$$

$$Y=y\cos\theta-x\sin\theta.$$

$$5\sqrt{2}=y\cos45°-x\sin45°$$

$$5\sqrt{2}=y\cdot\frac{1}{\sqrt{2}}-x\cdot\frac{1}{\sqrt{2}}$$

$$5\sqrt{2}=\frac{1}{\sqrt{2}}(y-x)$$

$$5(2)=y-x$$

$$-x+y=10 \text{ —— (ii)}$$

By Solving (i) and (ii)

(i)+(ii)

$$\cancel{x}+y=-14$$

$$-\cancel{x}+y=10$$

$$2y=-4$$

$$\boxed{y=-2}$$

Put in (i)

$$x-2=-14$$

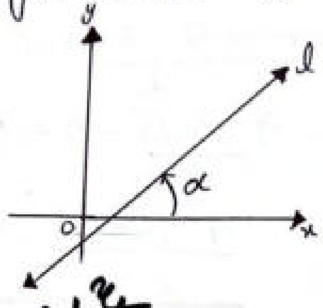
$$x=-14+2$$

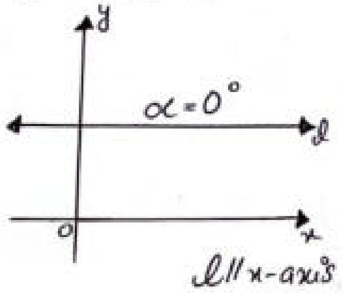
$$\boxed{x=-12}$$

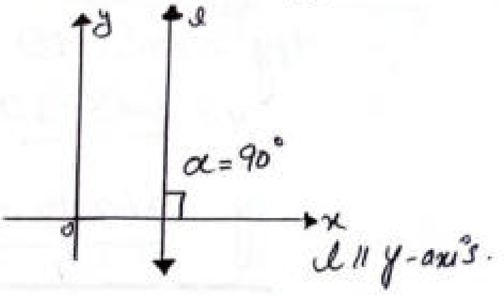
$$\boxed{P(x,y)=(-12,-2)}$$

Theory:

i) Inclination of a line: The angle $\alpha (0° < \alpha < 180°)$ measured counterclockwise from positive x-axis to a non-horizontal straight line l is called the inclination of l.







Note:

- If l is parallel to x-axis, then $\alpha = 0°$
- If l is parallel to y-axis, then $\alpha = 90°$

ii) Slope or gradient of a line: When we walk on an inclined plane, we cover horizontal distance (run) as well as vertical distance (rise) at the same time. The measure of steepness (ratio of rise to run) is termed as slope or gradient of the inclined path and is denoted by m.

$$m = \frac{rise}{run} = \frac{y}{x} = \tan\alpha \quad \Rightarrow \quad m = \tan\alpha.$$

Note:

- If l is horizontal its slope is zero; $m = 0$
- If l is vertical its slope is undefined; $m = \infty$
- If $0° < \alpha < 90°$, m is positive
- If $90° < \alpha < 180°$, then m is negative

- **Slope or gradient of a Straight line joining two points:**

If a non-vertical line l with inclination a passes through two points $P(x_1, y_1)$ and $Q(x_2, y_2)$, then the slope or gradient m of l is given by

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \tan\alpha$$

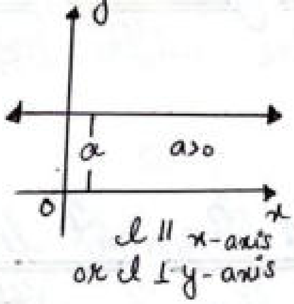
Note:

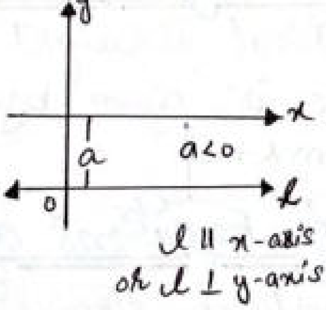
- $m \neq \frac{y_2 - y_1}{x_1 - x_2}$ and $m \neq \frac{y_1 - y_2}{x_2 - x_1}$
- l is horizontal, iff $m = 0$ $(\because \alpha = 0^\circ)$
- l is vertical, iff m is not defined $(\because \alpha = 90^\circ)$
- If Slope of AB = Slope of BC, Then the points A, B and C are callinear.

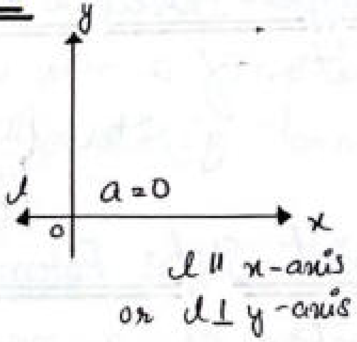
⇒ The two lines l_1 and l_2 with respective slopes m_1 and m_2 are

- Parallel iff $m_1 = m_2$
- Perpendicular iff $m_1 \cdot m_2 = -1$ or $m_1 = \frac{-1}{m_2}$ or $m_1 \cdot m_2 + 1 = 0$

• Equation of a Straight Line Parallel to the x-axis (or perpendicular to the y-axis)







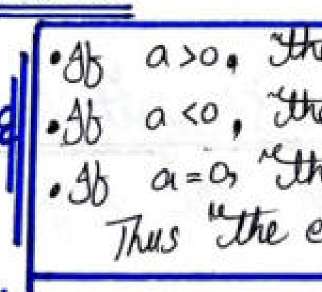
$$\boxed{y = a}$$

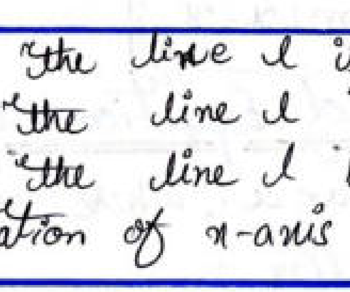
Note:

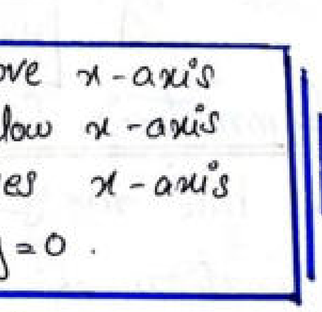
- If $a > 0$, then the line l is above x-axis
- If $a < 0$, then the line l is below x-axis
- If $a = 0$, then the line l becomes x-axis

Thus the equation of x-axis is $y = 0$.

• Equation of a Straight line Parallel to the y-axis (or perpendicular to the x-axis)







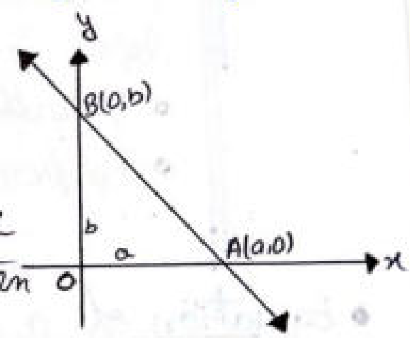
$\boxed{x = b}$

Note:

- If b>0, then the line is on right of the y-axis
- If b<0, then the line is on left of the y-axis
- If b=0, then the line becomes the y-axis. Thus the equation of y-axis is x=0.

Derivation of Stantard Forms of Equations of Straight line:

Intercepts:



⇒ If a line intersects x-axis at (a,0), then a is called x-intercept of the line

⇒ If a line intersects y-axis at (0,b), then b is called y-intercept of the line

Slope-Intercept form of Equations of a Straight lines:

Equation of a non-vertical straight line with slop m and y-intercept c is given by: $\boxed{y = mx + c}$

Point-Slope Form of Equations of a Straight line:

Equation of a non-vertical straight line l with Slope m and passing through a point $Q(x_1, y_1)$ is given by: $\boxed{y - y_1 = m(x - x_1)}$

Symmetric Form of Equations of a Straight line:

We have $m = \frac{y-y_1}{x-x_1} = \tan\alpha$. where α is the angle of inclination of the line

$m = \tan\alpha \Rightarrow m = \frac{\sin\alpha}{\cos\alpha} \Rightarrow \frac{y-y_1}{x-x_1} = \frac{\sin\alpha}{\cos\alpha} \Rightarrow (y-y_1)\cos\alpha = (x-x_1)\sin\alpha$

$\frac{y-y_1}{\sin\alpha} = \frac{x-x_1}{\cos\alpha}$ or $\boxed{\frac{x-x_1}{\cos\alpha} = \frac{y-y_1}{\sin\alpha} = r}$

Two-Points Form of Equations of a Straight lines:

Equation of a non-vertical straight line passing through two points $Q(x_1, y_1)$ and $R(x_2, y_2)$ is

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) \quad \text{or} \quad y - y_2 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_2)$$

Two-Intercept Form of Equations of a Straight Line:

Equation of a line whose non-zero x and y-intercept are a and b respectively

$$\boxed{\frac{x}{a} + \frac{y}{b} = 1}$$

Normal Form of Equation of a Straight line:

An equation of a non-vertical straight line l, such that length of the perpendicular form the origin to l is P and α is the inclination of this perpendicular, is

$$\boxed{x\cos\alpha + y\sin\alpha = P}$$

• A Linear Equation in two Variables Represents a Straight line:

The linear equation $ax+by+c=0$ in two variables x and y represents a straight line. A linear equation in two variables x and y is $\boxed{ax+by+c=0}$ ①

Proof: Here a and b cannot be both zero. So the following cases arise:

Case I: $a \neq$; $b=0$

In the case eq ① takes the form:

$$ax + c = 0$$
$$x = \frac{-c}{a}$$

Which is an equation of the straight line parallel to the y-axis at a directed distance $\frac{-c}{a}$ from the y-axis.

Case II: $a=0$; $b \neq 0$

In the case eq ① takes the form:

$$by + c = 0$$
$$y = -\frac{c}{b}$$

Which is an equation of straight line parallel to the x-axis at a distance $\frac{-c}{b}$ from the x-axis.

Case III: $a \neq 0$; $b \neq 0$

In the case eq ① takes the form:

$$by = -ax - c$$

or $y = \frac{-a}{b}x - \frac{c}{b} = mx + c$

Which is the slope-intercept form of the straight line with slope equals $-\frac{a}{b}$ and y-intercept $-\frac{c}{b}$

Thus the equation $ax+by+c=0$, always represnt a Straight line.

- ## To Transform the General Linear Equation to Standard Forms

To transform the equation $ax+by+c=0$ in the standard forms:

Slope-Intercept Form:

We have

$by = -ax - c$ or $y = \frac{-ax}{b} - \frac{c}{b} = mx + c$; where $m = \frac{-a}{b}$; $c = \frac{-c}{b}$

Point-Slope Form:

The slope of the line $ax+by+C=0$ is $-\frac{a}{b}$. A point on the line is $\left(\frac{-c}{a}, 0\right)$

$\therefore y - y_1 = m(x - x_1)$.

Equation of the line becomes $y - 0 = \frac{-a}{b}\left(x - \left(-\frac{c}{a}\right)\right)$

$$y = -\frac{a}{b}\left(x + \frac{c}{a}\right)$$

Which is the point-slope form.

Symmetric Form:

$m = \tan\alpha = \frac{-a}{b} = \frac{\text{Perpendicular}}{\text{Base}}$

$H^2 = P^2 + B^2 \Rightarrow H^2 = (-a)^2 + (b)^2 \Rightarrow H^2 = a^2 + b^2 \Rightarrow H = \pm\sqrt{a^2+b^2}$

so $\sin\alpha = \frac{a}{\pm\sqrt{a^2+b^2}}$; $\cos\alpha = \frac{b}{\pm\sqrt{a^2+b^2}}$

A point of $ax+by+c=0$ is $\left(\frac{-c}{a}, 0\right)$

Equation of the symmetric form become $\frac{x - \left(-\frac{c}{a}\right)}{b/\pm\sqrt{a^2+b^2}} = \frac{y - 0}{a/\pm\sqrt{a^2+b^2}} = r$

Two Points Form:

We chose two arbitrary points on $ax+by+c=0$. Two such points are $\left(\frac{-c}{a}, 0\right)$ and $\left(0, \frac{-c}{b}\right)$

$$\frac{y-0}{0+\frac{c}{b}} = \frac{x+\frac{c}{a}}{\frac{-c}{a}-0} \qquad \therefore \frac{y-y_1}{y_1-y_2} = \frac{x-x_1}{x_1-x_2}$$

Two Intercept Form:

$ax+by=-c$ or $\frac{ax}{-c} + \frac{by}{-c} = 1 \Rightarrow \frac{x}{-c/a} + \frac{y}{-c/b} = 1$

which is an equation in two intercepts form.

Normal Form:

The equation $ax+by+c=0$ can be written in the normal form as

$$\frac{ax+by}{\pm\sqrt{a^2+b^2}} = \frac{-c}{\pm\sqrt{a^2+b^2}} \quad \text{— ①}$$

Note:

The sign of the radical to be such that the right side of ① is positive.

- ## Position of a Point with respect to a line

Let $P(x_1, y_1)$ be a point in the plane not lying on l $\quad l: ax+by+c=0$ — ①

then P lies

- above the line ① if $ax_1+by_1+c>0$
- below the line ① if $ax_1+by_1+c<0$

- ## The point of Intersection of two Straight lines.

$$\frac{x_1}{b_1c_2-b_2c_1} = \frac{y_1}{a_2c_1-a_1c_2} = \frac{1}{a_1b_2-a_2b_1}$$

Note

$a_1b_1-a_2b_1\neq 0$, otherwise $l_1 \parallel l_2$

- ## Condition of Concurrency of Three Straight Lines

Three non-parallel lines

$l_1: a_1x+b_1y+c_1=0$
$l_2: a_2x+b_2y+c_2=0$ are concurrent iff $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$
$l_3: a_3x+b_3y+c_3=0$

Note:

=> Altitudes of a triangle are concurrent
=> Right bisectors of a triangle are concurrent

- ## Distance of a Point Form a Line

The distance d from the points $P(x_1, y_1)$ to the line $l_1, l: ax+by+c=0$

is given by: $d = \dfrac{|ax_1+by_1+c|}{\sqrt{a^2+b^2}}$

- **Area of a Triangular Region Whose Vertices are Given**

The area of a triangle Δ is given by

$$\Delta = \frac{1}{2}\left[x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)\right] \text{ or } \Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Example # 1: Show that the points A(-3,6), B(3,2) and C(6,0) are collinear

The points A, B, C are collinear if: Slope of AB = Slope of BC

Slope of AB $= \frac{2-6}{3-(-3)} \Rightarrow \frac{-4}{3+3} \Rightarrow \frac{-4}{6} \Rightarrow -\frac{2}{3}$ $\quad \because m = \frac{y_2-y_1}{x_2-x_1}$

Slope of BC $= \frac{0-2}{6-3} \Rightarrow \frac{-2}{3}$

$\therefore$ Slope of AB = Slope of BC

Thus A, B, C are collinear.

Example no 2: Show that the triangle with vertices A(1,1), B(4,5) and C(12,-1) is a right triangle.

Slope of AB $= \frac{5-1}{4-1} \Rightarrow \frac{4}{3} = m_1$; Slope of BC $= \frac{-1-5}{12-4} \Rightarrow \frac{-6}{8} \Rightarrow \frac{-3}{4} = m_2$

Since $m_1 \cdot m_2 = \frac{4}{3} \cdot \frac{-3}{4} = -1$

Therefore $AB \perp BC$

So ΔABC is ~~of~~ a right triangle.

Example no 3: Find the equation of the straight line if **(a)** its slope is 2 and y-intercept is 5
(b) it is perpendicular to a line with slope -6 and c is $\frac{4}{3}$

(a) $m = 2$; $c = 5$

By PoiSlope intercept form

$\therefore y = mx + c$

$$\boxed{y = 2x + 5}$$

is the required equation.

(b) Slope of given line $= -6$

Slope of required line $= \frac{1}{6}$

($\because$ given line is $\perp$ to required line)

$m = \frac{1}{6}$; $c = \frac{4}{3}$

By Slope-intercept form

$\therefore y = mx + c$

$y = \frac{1}{6}x + \frac{4}{3}$

Multiplying by 6 on B.S

$$\boxed{6y = x + 8}$$

is the required equation.

Example no 4: Write down an equation of the straight line passing through (5,1) and parallel to a line passing through the points (0,-1), (7,-15)

Slope $= m = \frac{-15-(-1)}{7-0} \Rightarrow \frac{-15+1}{7} \Rightarrow \frac{-14}{7} \Rightarrow \boxed{-2 = m}$

$m = -2$; $P(5,1)$.

By point-slope form:

$y - 1 = -2(x-5)$ $\quad\quad \because y - y_1 = m(x - x_1)$.

$y - 1 = -2x + 10$

$y - 1 + 2x - 10 = 0$

$\boxed{2x + y - 11 = 0}$

Example no 5: Find an equation of line through the points (-2,1) and (6,-4)

By Two-Points form:

$\frac{y-1}{-4-1} = \frac{x-(-2)}{6-(-2)} \Rightarrow \frac{y-1}{-5} = \frac{x+2}{8}$ $\quad\quad \because \frac{y-y_1}{y_2-y_1} = \frac{x-x_1}{x_2-x_1}$

$8y - 8 = -5x - 10$

$8y - 8 + 5x + 10 = 0$

$\boxed{5x + 8y + 2 = 0}$

Example no 6: Write down an equation of the line which cuts x-axis at (2,0) and y-axix at (0,-4).

By Two-Intercepts form:

$\frac{x}{2} + \frac{y}{-4} = 1$ $\quad\quad \because \frac{x}{a} + \frac{y}{b} = 1$.

Multiply by -4 on B.S

$-2x + y = -4$

$\boxed{2x - y - 4 = 0}$

Example no 7: Find an equation of the line through the Point P(2,3) which forms an isoceles triangle with the coordinate axes in the I quadrant.

Let $A(a,0), B(0,a)$ then slope $= m = \frac{a-0}{0-a} \Rightarrow \frac{a}{-a} \Rightarrow -1$.

$m = -1$; $P(2,3)$

By point-slope form:

$y - 3 = -1(x-2)$ $\quad\quad \because y - y_1 = m(x - x_1)$.

$y - 3 = -x + 2$

$y - 3 + x - 2 = 0 \Rightarrow \boxed{x + y - 5 = 0}$

Example no 8: The length of Perpendicular from the origin to a line is 5units and the inclination of this perpendicular is 120°. Find the slope and y-intercept of the line.

$P = 5; \quad \theta = 120°.$

By Normal form:

$x\cos 120° + y\sin 120° = 5$ ∴ $x\cos\theta + y\sin\theta = P$.

$-\frac{1}{2}x + \frac{\sqrt{3}}{2}y = 5$ ∴ Multiply by -2 on B.S

$x - \sqrt{3}y = -10$

$x - \sqrt{3}y + 10 = 0$

To find the slope and y-intercept

$x + 10 = \sqrt{3}y$

$\frac{x}{\sqrt{3}} + \frac{10}{\sqrt{3}} = y$

where $m = \frac{1}{\sqrt{3}}$; $c = \frac{10}{\sqrt{3}}$.

Example no 9: Transform the equation $5x-12y+39=0$ into

i) Slope-intercept form:

$12y = 5x+39 \Rightarrow y = \frac{5}{12}x + \frac{39}{12}$ where $m = \frac{5}{12}$; $c = \frac{39}{12}$.

ii) Two-intercept form:

$5x-12y = -39 \Rightarrow \frac{5x}{-39} + \frac{12y}{13} = 1 \Rightarrow \frac{x}{-39/5} + \frac{y}{13/12} = 1$

iii) Normal form:

$5x - 12y = -39$. Divide both sides by $\pm\sqrt{5^2+12^2} \Rightarrow \pm 13$. We take = -13

Hence $\frac{-5x}{13} + \frac{12y}{13} = \frac{-39}{-13} \Rightarrow \frac{-5x}{13} + \frac{12y}{13} = 3$

iv) Point-slope form:

A point on the line is $(-\frac{39}{5}, 0)$ and its slope is $\frac{5}{12}$

$y - 0 = \frac{5}{12}\left(x + \frac{39}{5}\right)$

v) Two-Points form: Let $(-\frac{39}{5}, 0)$ & $(0, \frac{39}{12})$ be two points

$$\frac{y-0}{0-\frac{39}{12}} = \frac{x-(-\frac{39}{5})}{-\frac{39}{5}-0}$$

vi) Symmetric form: As $\tan\alpha = \frac{5}{12}$, So $\sin\alpha = \frac{5}{13}$; $\cos\alpha = \frac{12}{13}$; $P(-\frac{39}{5}, 0)$

$$\frac{x-(-\frac{39}{5})}{12/13} = \frac{y-0}{5/13} = r \text{ (Say)}$$

Example no 10: Sketch the line $3x+2y+6=0$.

$3x+2y+6=0$ —(i)

Put $x=0$ in (i)

$3(0)+2y+6=0$

$2y=-6$

$y=-3$

$(0,-3)$

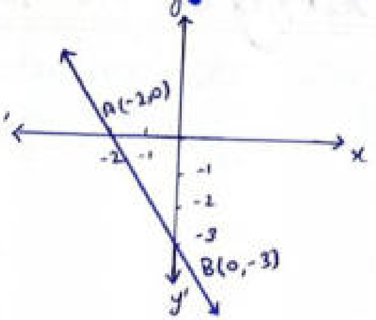
Put $y=0$ in (i)

$3x+2(0)+6=0$

$3x=-6$

$x=-2$

$(-2,0)$



The two points A(-2,0), B(0,3) are on ①

Example no 12: Check whether the points (-2,4) lies above or below the line. $4x+5y-3=0$ —①

$b=5$ (+ive)

$4(-2)+5(4)-3$

$-8+20-3$

$9>0$ (+ive) —②

The coefficient of y in ① and the expression ② have same sign so point (-2,4) lies above ①

Example no 13: Check whether the origin and the point P(5,-8) lie on the same side or on the opposite sides of the line: $3x+7y+15=0$ —(i)

$(0,0)$; $P(5,-8)$

Put (0,0) in (i)

$3(0)+7(0)+15$

$15>0$ (+ive) —(ii)

Put (5,-8) in (i)

$3(5)+7(-8)+15$

$15-56+15$

$-26<0$ (-ive) (iii)

The expression (ii) & (iii) have opposite sign. So they lie on the opposite sides.

Example no 1: Find the point of intersection of the lines $5x+7y=35$ —(i); $3x-7y=21$ —(ii)

By adding (i) & (ii)

$$\begin{array}{r} 5x+7y=35 \\ 3x-7y=21 \\ \hline 8x\quad=56 \end{array}$$

$\boxed{x=7}$

Put $x=7$ in (i)

$5(7)+7y=35$

$35+7y=35$

$7y=35-35$

$7y=0$

$\boxed{y=0}$

Thus (7,0) is the point of intersection.

Example no 2: Check whether the following lines are concurrent or not. If concurrent find the point of concurrency.

$$3x - 4y - 3 = 0 \text{ ——— } ①$$

$$5x + 12y + 1 = 0 \text{ ——— } ②$$

$$32x + 4y - 17 = 0 \text{ ——— } ③$$

$$= \begin{vmatrix} 3 & -4 & -3 \\ 5 & 12 & 1 \\ 32 & 4 & -17 \end{vmatrix}$$

Expand R_1

$$= 3\begin{vmatrix} 12 & 1 \\ 4 & -17 \end{vmatrix} - (-4)\begin{vmatrix} 5 & 1 \\ 32 & -17 \end{vmatrix} - 3\begin{vmatrix} 5 & 12 \\ 32 & 4 \end{vmatrix}$$

$$= 3[-204-4] + 4[-85-32] - 3[20-384]$$

$$= 3(-208) + 4(-117) - 3(-364)$$

$$= -624 - 468 + 1092$$

$$= 0$$

Thus the lines are concurrent.

The point of intersection of any two lines is the required point of concurrency.

From ① and ② we have

$$\therefore \frac{x}{b_1c_2 - b_2c_1} = \frac{y}{a_2c_1 - a_1c_2} = \frac{1}{a_1b_2 - a_2b_1}$$

$$\frac{x}{-4+36} = \frac{y}{-15-3} = \frac{1}{36+20}$$

$$\frac{x}{32} = \frac{y}{-18} = \frac{1}{56}$$

$$\frac{x}{32} = \frac{1}{56} \quad ; \quad \frac{y}{-18} = \frac{1}{56}$$

$$x = \frac{32}{56} \quad ; \quad y = \frac{-18}{56}$$

$$x = \frac{4}{7} \quad ; \quad y = \frac{-9}{28}$$

$\left(\frac{4}{7}, \frac{-9}{28}\right)$ is the point of intersection

Example no 4: Find the distance between the parallel lines $l_1 : 2x-5y+13=0$

$l_2 : 2x-5y+6=0$

$$\therefore d = \frac{|c_2-c_1|}{\sqrt{a^2+b^2}}$$

$$2x-5y+13=0$$
$$2x-5y+6=0$$

$c_1 = 13$; $c_2 = 6$; $a = 2$; $b = -5$

$$d = \frac{|6-13|}{\sqrt{(2)^2+(-5)^2}}$$

$$= \frac{|-7|}{\sqrt{4+25}}$$

$$= \frac{7}{\sqrt{29}}.$$

Hence distance between given parallel lines is $\frac{7}{\sqrt{29}}$.

Example no 5: Find the area of the region bounded by the triangle with vertices $(a, b+c)$, $(a, b-c)$ and $(-a, c)$

$$\therefore \Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

$$\Delta = \frac{1}{2}\begin{vmatrix} a & b+c & 1 \\ a & b-c & 1 \\ -a & c & 1 \end{vmatrix}$$

Expand by R_1

$$\Delta = \frac{1}{2}\left[a(b-c-c) - (b+c)(a+a) + 1(ac + a(b-c))\right]$$

$$\Delta = \frac{1}{2}\left[a(b-2c) - (b+c)(2a) + 1(\cancel{ac} + ab - \cancel{ac})\right]$$

$$\Delta = \frac{1}{2}\left[ab - 2ac - 2ab - 2ac + ab\right]$$

$$\Delta = \frac{1}{2}\left[\cancel{2ab} - 2ac - 2ac - \cancel{2ab}\right]$$

$$\Delta = \frac{1}{2}\left[-4ac\right]$$

$$\boxed{\Delta = -2ac}$$

Example no 6: By considering the area of the region bounded by the triangle with vertices A(1,4), B(2,-3), C(3,-10). Check whether the three points are collinear or not

$$\therefore\ \Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

$$\Delta = \frac{1}{2}\begin{vmatrix} 1 & 4 & 1 \\ 2 & -3 & 1 \\ 3 & -10 & 1 \end{vmatrix}$$

Expand by R_1

$$\Delta = \frac{1}{2}\left[1(-3+10) - 4(2-3) + 1(-20+9)\right]$$

$$\Delta = \frac{1}{2}\left[1(7) - 4(-1) + 1(-11)\right]$$

$$\Delta = \frac{1}{2}\left[7 + 4 - 11\right]$$

$$\Delta = \frac{1}{2}(11 - 11)$$

$$\Delta = \frac{1}{2}(0)$$

$$\Delta = 0$$

Thus the points are collinear.

Exercise no 4.3

Question no 1: Find the slope and inclination of the line joining the points:

i) (-2,4); (5,11)

Let A(-2,4), B(5,11)

Slope:

$\therefore$ Slope $= m = \frac{y_2 - y_1}{x_2 - x_1}$

$m =$ Slope of AB $= \frac{11-4}{5-(-2)}$

$= \frac{7}{5+2} \Rightarrow \frac{7}{7} \Rightarrow \boxed{m = 1}$

Inclination:

Angle of inclination:

$\therefore m = \tan \alpha$

$\alpha = \tan^{-1}(m)$

$\alpha = \tan^{-1}(1)$

$\boxed{\alpha = 45^\circ}$

ii) (3,-2):(2,7)

Let A(3,-2); B(2,7)

Slope:

$\therefore m = \frac{y_2 - y_1}{x_2 - x_1}$

$m =$ Slope of AB $= \frac{7-(-2)}{2-3}$

$= \frac{7+2}{-1} \Rightarrow \frac{9}{-1} \Rightarrow \boxed{m = -9}$

Inclination:

Angle of inclination:

$\therefore m = \tan \alpha$

$\alpha = \tan^{-1}(m)$

$\alpha = \tan^{-1}(-9)$

$\boxed{\alpha = 96.34^\circ}$

iii) (4,6).(4,8)

Let A(4,6), B(4,8)

Slope:

$\therefore m = \frac{y_2 - y_1}{x_2 - x_2}$

Slope of AB $= \frac{8-6}{4-4}$

$= \frac{2}{0}$

$\boxed{m = \text{undefined}}$

Inclination:

Angle of inclination:

$\therefore m = \tan \alpha$

$\alpha = \tan^{-1}(m)$

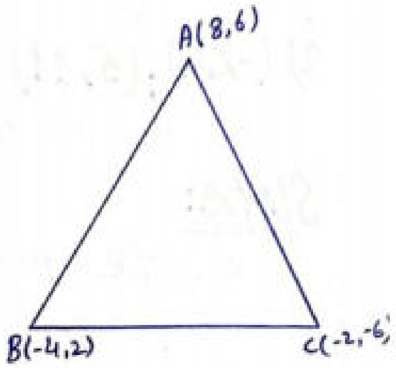
$\alpha = \tan^{-1}(\infty)$

$\boxed{\alpha = 90^\circ}$

Question no 28 In the triangle A(8,6), B(-4,2) and C(-2,-6), find the Slope of

A(8,6), B(-4,2), C(-2,-6)

i) each side of the triangle.

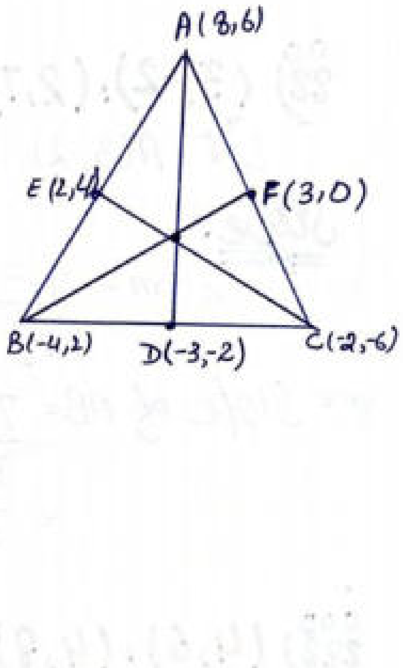


Slope of AB $= \frac{2-6}{-4-8} \Rightarrow \frac{-4}{-12} \Rightarrow \frac{1}{3}$

Slope of BC $= \frac{-6-2}{-2+4} \Rightarrow \frac{-8}{2} \Rightarrow -4$

Slope of AC $= \frac{-6-6}{-2-8} \Rightarrow \frac{-12}{-10} \Rightarrow \frac{6}{5}$.

ii) each median of the triangle



∴ D is the midpoint of BC

So coordinates are: $\left(\text{Midpoint formula } \therefore \left(\frac{x_1+x_2}{2} ; \frac{y_1+y_2}{2}\right) \right)$

$= \left(\frac{-4-2}{2} ; \frac{2-6}{2}\right) \Rightarrow \left(\frac{-6}{2} ; \frac{-4}{2}\right) \Rightarrow (-3,-2)$.

Slope of median AD

$m_1 = \frac{-2-6}{-3-8} \Rightarrow m_1 = \frac{-8}{-11} \Rightarrow \boxed{m_1 = \frac{8}{11}}$

∴ F is the midpoint of AC

So coordinates are:

$= \left(\frac{8-2}{2} ; \frac{6-6}{2}\right) \Rightarrow \left(\frac{6}{2} ; \frac{0}{2}\right) \Rightarrow (3,0)$

Slope of median BF

$m_2 = \frac{0-2}{3-(-4)} \Rightarrow m_2 = \frac{-2}{3+4} \Rightarrow \boxed{m_2 = -\frac{2}{7}}$

∴ E is the midpoint of AB

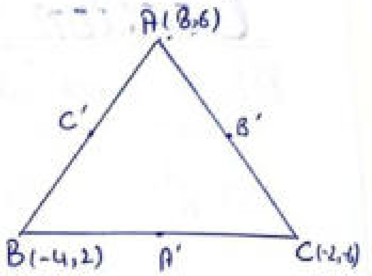
So coordinates are:

$= \left(\frac{8-4}{2}, \frac{6+2}{2}\right) \Rightarrow \left(\frac{4}{2}, \frac{8}{2}\right) \Rightarrow (2,4)$

Slope of median CE

$m_3 = \frac{4-(-6)}{2-(-2)} \Rightarrow m_3 = \frac{4+6}{2+2} \Rightarrow m_3 = \frac{10}{4} \Rightarrow \boxed{m_3 = \frac{5}{2}}$

iii) each altitude of the Triangle



∴ AA' be altitude from point A on BC

Slope of BC = $\frac{-6-2}{-2+4} \Rightarrow \frac{-8}{2} \Rightarrow -4$

Slope of altitude AA' = $\frac{1}{4}$ (∴ $AA' \perp BC$)

∴ BB' be altitude from point B on AC

Slope of AC = $\frac{-6-6}{-2-8} \Rightarrow \frac{-12}{-10} \Rightarrow \frac{6}{5}$.

Slope of altitude BB' = $-\frac{5}{6}$ (∴ $BB' \perp AC$)

∴ CC' be altitude from point C on AB

Slope of AB = $\frac{2-6}{-4-8} \Rightarrow \frac{-4}{-12} \Rightarrow \frac{1}{3}$.

Slope of altitude CC' = -3. (∴ $CC' \perp AB$)

Question no 3: By means of slopes, show that the following points lie on the same line:

(a) (-1,-3);(1,5);(2,9)

Let A(-1,-3),B(1,5),C(2,9).

These points will be lie on the same line if Slope of AB = Slope of BC

m_1 = Slope of AB = $\frac{5+3}{1+1} \Rightarrow \frac{8}{2} \Rightarrow 4$

m_2 = Slope of BC = $\frac{9-5}{2-1} \Rightarrow \frac{4}{1} \Rightarrow 4$

$m_1 = m_2$

(b) (4,-5);(7,5);(10,15)

Let A(4,-5), B(7,5), C(10,15)

These points will be lie on the same line if $m_1 = m_2$

Slope of AB = $m_1 = \frac{5+5}{7-4} \Rightarrow \frac{10}{3}$

Slope of BC = $m_2 = \frac{15-5}{10-7} \Rightarrow \frac{10}{3}$

So $m_1 = m_2$.

(c) (-4,6);(3,8);(10,10)

Let A(-4,6), B(3,8), C(10,10).

Slope of AB = $m_1 = \frac{8-6}{3+4} \Rightarrow \frac{2}{7}$

Slope of BC = $m_2 = \frac{10-8}{10-3} \Rightarrow \frac{2}{7}$

So $m_1 = m_2$.

(d) (a,2b);(c,a+b); (2c-a ; 2a)

Let A(a,2b); B(c,a+b); C(2c-a, 2a).

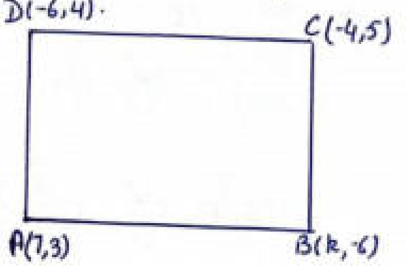
Slope of AB = $\frac{a+b-2b}{c-a} \Rightarrow \frac{a-b}{c-a} = m_1$

Slope of BC = $\frac{2a-(a+b)}{2c-a-c} \Rightarrow \frac{2a-a-b}{2c-a-c}$

$\Rightarrow \frac{a-b}{c-a} = m_2$

$m_1 = m_2$

Question no 4: Find k so that line joining A(7,3), B(k,-6) and the line joining C(-4,5), D(-6,4) are (i) Parallel (ii) Perpendicular



A(7,3), B(k,-6), C(-4,5), D(-6,4).

Slope of $AB = \frac{-6-3}{k-7} \Rightarrow \frac{-9}{k-7} = m_1$

Slope of $CD = \frac{4-5}{-6+4} \Rightarrow \frac{-1}{-2} \Rightarrow m_2 = \frac{1}{2}$

i) Parallel: If these points are parallel then $m_1 = m_2$

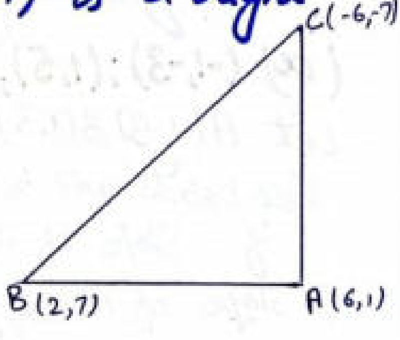
$\frac{-9}{k-7} = \frac{1}{2} \Rightarrow -18 = k-7 \Rightarrow k = -18+7 \Rightarrow \boxed{k = -11}$

ii) Perpendicular: If these points are perpendicular than $m_1 m_2 = -1$

$\frac{-9}{k-7} \cdot \frac{1}{2} = -1 \Rightarrow \frac{-9}{2(k-7)} = -1 \Rightarrow -9 = -2(k-7) \Rightarrow -9 = -2k+14 \Rightarrow -9-14 = -2k$

$\Rightarrow -23 = -2k \Rightarrow k = \frac{-23}{-2} \Rightarrow \boxed{k = \frac{23}{2}}$

Question no 5: Using slopes, show that the triangle with its vertices A(6,1), B(2,7) and C(-6,-7) is a right triangle.



A(6,1), B(2,7), C(-6,-7).

Slope of $AB = \frac{1-7}{6-2} \Rightarrow \frac{-6}{4} \Rightarrow \frac{-3}{2} = m_1$

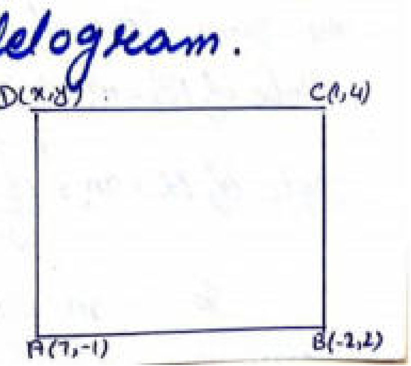
Slope of $AC = \frac{-7-1}{-6-6} \Rightarrow \frac{-8}{-12} \Rightarrow \frac{2}{3} = m_2$.

$\left(\frac{-3}{2}\right)\left(\frac{2}{3}\right) = -1 \Rightarrow -1 = -1$

Therefore $AB \perp AC$ or $AC \perp AB$.

So ΔABC is a right triangle.

Question no 6: The three points A(7,-1), B(-2,2) and C(1,4) are consective vertices of a parallelogram. Find the fourth vertex.



As these points are the vertices of Parallelogram

Slope of AB = Slope of DC.

Slope of AB $= \frac{2-(-1)}{-2-7} \Rightarrow \frac{2+1}{-9} \Rightarrow \frac{3}{-9} \Rightarrow \frac{-1}{3}$

Slope of DC $= \frac{4-y}{1-x}$

$$-\frac{1}{3} = \frac{4-y}{1-x} \Rightarrow -1(1-x) = 3(4-y) \Rightarrow -1+x = 12-3y \Rightarrow -1+x-12+3y=0$$

$$x+3y-13=0 \text{ ——— (i)}$$

Now Slope of AD = Slope of BC

Slope of AD $= \frac{y+1}{x-7}$

Slope of BC $= \frac{4-2}{1-(-2)} \Rightarrow \frac{2}{1+2} \Rightarrow \frac{2}{3}$

$$\frac{y+1}{x-7} = \frac{2}{3} \Rightarrow 3(y+1) = 2(x-7) \Rightarrow 3y+3 = 2x-14 \Rightarrow 0 = 2x-14-3y-3$$

$$2x-3y-17=0 \text{ ——— (ii)}$$

(i)+(ii)

$$\begin{array}{l} x+3y-13=0 \\ 2x-3y-17=0 \\ \hline 3x \quad -30=0 \end{array}$$

$3x=30$

$\boxed{x=10}$

Put in (i)

$10+3y-13=0$

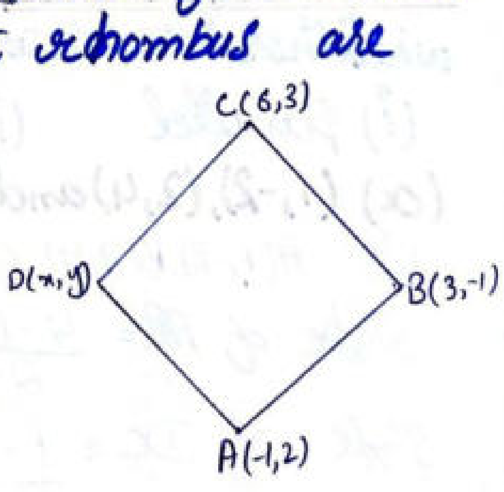
$3y-3=0$

$3y=3$

$\boxed{y=1}$

$D(x,y)=(10,1)$

Question no 7: The points A(-1,2), B(3,-1), C(6,3) are consecutive vertices of a rhombus. Find the fourth vertex and show that the diagonals of the rhombus are perpendicular to each other.



A(-1,2), B(3,-1), C(6,3), D(x,y)

Let D is the required fourth vertex

Slope of AB = Slope of DC

$$\frac{-1-2}{3+1} = \frac{y-3}{x-6} \Rightarrow \frac{-3}{4} = \frac{y-3}{x-6}$$

$$\Rightarrow -3(x-6) = 4(y-3) \Rightarrow -3x+18 = 4y-12$$

$$\Rightarrow 0 = 4y-12+3x-18 \Rightarrow 3x+4y-30=0 \text{ ——— (i)}$$

Slope of AD = Slope of BC

$\frac{y-2}{x+1} = \frac{3+1}{6-3} \Rightarrow \frac{y-2}{x+1} = \frac{4}{3} \Rightarrow 3(y-2) = 4(x+1).$

$\Rightarrow 3y-6 = 4x+4 \Rightarrow 0 = 4x+4-3y+6 \Rightarrow 4x-3y+10=0$ —— (ii)

3(i) + 4(ii)

$9x + \cancel{12y} - 90 = 0$

$16x - \cancel{12y} + 40 = 0$

$25x - 50 = 0$

$25x = 50$

$\boxed{x = 2}$

Put in (i)

$3(2) + 4y - 30 = 0$

$6 + 4y - 30 = 0$

$4y - 24 = 0$

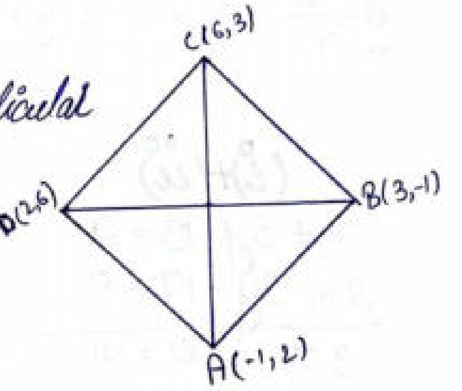
$4y = 24$

$\boxed{y = 6}$

$D(x,y) = (2,6)$

Prove:

The diagonals of rhombus are perpendicular if Slope of AC × slope of BD = -1.



Slope of AC $= \frac{3-2}{6+1} \Rightarrow \frac{1}{7}$

Slope of BD $= \frac{6+1}{2-3} \Rightarrow \frac{7}{-1} \Rightarrow -7.$

$\frac{1}{\cancel{7}} \cdot -\cancel{7} = -1 \Rightarrow -1 = -1$

$\therefore AC \perp BD.$ Hence proved

Question no 8: Two pairs of points are given. Find wheather the two lines determined by these points are: (i) parallel (ii) perpendicular (iii) none.

(a) (1,-2), (2,4) and (4,1), (-8,2)

Let $A(1,-2), B(2,4), C(4,1), D(-8,2).$

Slope of AB $= \frac{4-(-2)}{2-1} \Rightarrow \frac{4+2}{1} \Rightarrow 6 = m. \quad \because m = \frac{y_2-y_1}{x_2-x_1}$

Slope of DC $= \frac{1-2}{4-(-8)} \Rightarrow \frac{-1}{4+8} \Rightarrow \frac{-1}{12} = m_2$

i) Parallel:

If these points are parallel then.

$m_1 = m_2$

$6 \neq -\frac{1}{12}$

So these points are not parallel

ii) Perpendicular:

If these points are perpendicular than

$m_1 \cdot m_2 = -1$

$\cancel{6} \cdot -\frac{1}{\cancel{12}_{6}} = -1$

$-\frac{1}{6} \neq -1$

So these points are not perpendicular also

(b) (-3,4);(6,2) and (4,5);(-2,-7)

Let A(-3,4), B(6,2), C(4,5), D(-2,-7)

$m_1 =$ Slope of AB $= \frac{2-4}{6-(-3)} \Rightarrow \frac{-2}{6+3} \Rightarrow -\frac{2}{9}$ $\quad \therefore m = \frac{y_2 - y_1}{x_2 - x_1}$

$m_2 =$ Slope of DC $= \frac{5-(-7)}{4-(-2)} \Rightarrow \frac{5+7}{4+2} \Rightarrow \frac{12}{6} \Rightarrow 2.$

i) Parallel:

If these points are parallel then

$m_1 = m_2$

$-\frac{2}{9} \neq 2$

So these points are not parallel.

ii) Perpendicular:

If these points are perpendicular then

$m_1 \cdot m_2 = -1$

$-\frac{2}{9} \cdot 2 = -1$

$-\frac{4}{9} \neq -1$

So these point are not perpendicular also.

Question no 9: Find an equation of

a) the horizontal line through (7,-9)

The slope of horizontal line $m = 0$.

By Point-slope form of equation:

∴ $y - y_1 = m(x - x_1)$.

$y-(-9) = 0\,(x-7)$

$\boxed{y+9=0}$

b) the vertical line through (-5,3)

The slope of vertical line $m =$ undefined or $\frac{1}{0}$

By Point-slope form of equation:

∴ $y - y_1 = m(x - x_1)$

$y-3 = \frac{1}{0}(x-(-5))$

$0(y-3) = x+5$

$\boxed{0=x+5}$

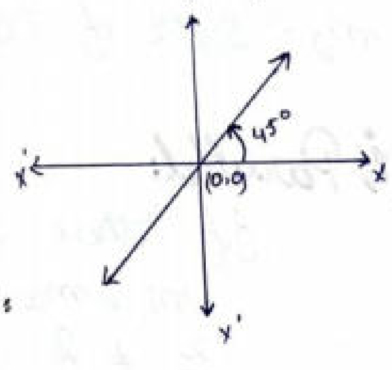
c) the line bisecting the first and third quadrants.

$\theta = 45^\circ$; $P(0,0)$.

∴ $m = \tan\theta$

$m = \tan(45^\circ)$

$\boxed{m=1}$



By point-slop form of equation.

$y-0 = 1(x-0)$

$\boxed{y = x}$

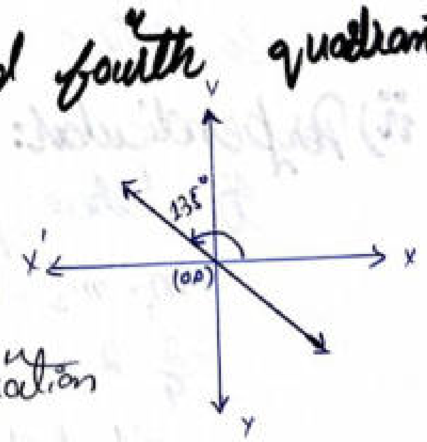
d) the line bisecting the second and fouth quadrants.

$\theta = 135^\circ$; $P(0,0)$

∴ $m = \tan\theta$

$m = \tan(135^\circ)$

$\boxed{m=-1}$



By Point-slope form of equation

$y-0 = -1(x-0)$

$y = -x$

or $\boxed{x = -y}$.

Question no 10: Find an equation of the line

a) Through A(-6,5) having Slope 7

By point-Slope form of equation:

$$\therefore y - y_1 = m(x - x_1).$$

$$y - 5 = 7(x - (-6))$$

$$y - 5 = 7(x + 6)$$

$$y - 5 = 7x + 42$$

$$0 = 7x + 42 - y + 5$$

$$\boxed{7x - y + 47 = 0}$$

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b) Through (8,-3) having slope 0

By point-Slope form of equation:

$$\therefore y - y_1 = m(x - x_1)$$

$$y - (-3) = 0(x - 8) \Rightarrow \boxed{y + 3 = 0}$$

c) Through (-8,5) having Slope undefined

By Point-Slope form of equation

$$\therefore y - y_1 = m(x - x_1)$$

$$y - 5 = \frac{1}{0}(x - (-8).$$

$$0(y - 5) = (x + 8)$$

$$\boxed{0 = x + 8}$$

d) Through (-5,-3) and (9,-1)

By two-Point form of equation:

$$\therefore \frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$$

$$\frac{y - (-3)}{-1 - (-3)} = \frac{x - (-5)}{9 - (-5)} \Rightarrow \frac{y + 3}{-1 + 3} = \frac{x + 5}{9 + 5} \Rightarrow \frac{y + 3}{2} = \frac{x + 5}{14}$$

$$14(y + 3) = 2(x + 5) \Rightarrow 14y + 42 = 2x + 10$$

$$0 = 2x + 10 - 14y - 42$$

$$0 = 2x - 14y - 32$$

$$0 = x - 7y - 16$$

$$\boxed{x - 7y - 16 = 0}$$

e) y-intercept -7 and slope -5

By Slope-intercept form of equations:

$$\therefore y = mx + c$$
$$y = -5x - 7$$
$$y + 5x + 7 = 0$$
$$\boxed{5x + y + 7 = 0}$$

f) x-intercept -3 and y-intercept 4

By two-intercept form of equation:

$$\therefore \frac{x}{a} + \frac{y}{b} = 1$$

$$-\frac{x}{3} + \frac{y}{4} = 1 \Rightarrow \frac{4x - 3y}{-12} = 1 \Rightarrow 4x - 3y = -12$$

$$\boxed{4x - 3y + 12 = 0}$$

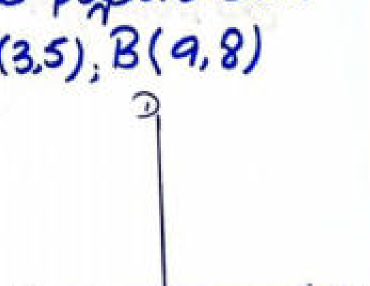
g) x-intercept -9 and slope -4

$(-9,0)$; $m = -4$

By Point-slope form of equation:

$$\therefore y - y_1 = m(x - x_1).$$
$$y - 0 = -4(x - (-9))$$
$$y = -4(x + 9)$$
$$y = -4x - 36$$
$$\boxed{4x + y + 36 = 0}$$

Question no 11: Find an equation of the perpendicular bisector of the segment joining the points $A(3,5)$; $B(9,8)$



$\therefore$ C is the midpoint of AB

So coordinates are:

$\therefore \left(\text{Midpoint formula } \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)\right)$

$$\left(\frac{3+9}{2}; \frac{5+8}{2}\right) \Rightarrow \left(\frac{12}{2}; \frac{13}{2}\right) \Rightarrow \left(6, \frac{13}{2}\right).$$

Slope of AB $= \frac{8-5}{9-3} \Rightarrow \frac{3}{6} \Rightarrow \frac{1}{2}$ $\quad \therefore \left(m = \frac{y_2 - y_1}{x_2 - x_1}\right)$

Slope of Perpendicular bisector CD $= -2$ $\quad (\therefore CD \perp AB)$.

Now eq of $\perp$ bisector CD through $C(6, \frac{13}{2})$ and slope -2 is

By Point slope form of equation:

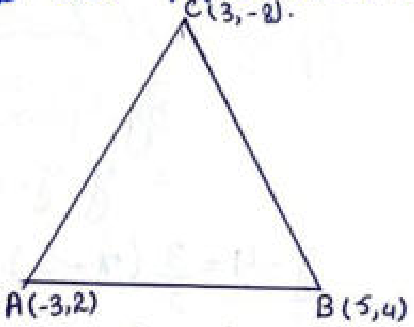
$$\therefore y-y_1 = m(x-x_1).$$

$$y-\frac{13}{2} = -2(x-6)$$

$$\frac{2y-13}{2} = -2(x-6) \Rightarrow 2y-13=-4(x-6) \Rightarrow 2y-13=-4x+24$$

$$\Rightarrow 2y-13+4x-24=0 \Rightarrow \boxed{4x+2y-37=0}$$

Question no 12: Find eq of the sides, altitudes and medians of the triangle whose vertices are A(-3,2), B(5,4) and C(3,-8).



Equation of Sides:

A(-3,2), B(5,4), C(3,-8)

∴ AB, BC, AC be the sides of ΔABC

Eq. of Side AB is

By two-point form:

$$\therefore y-y_1 = \frac{y_2-y_1}{x_2-x_1}(x-x_1)$$

$$y-2=\frac{4-2}{5-(-3)}(x-(-3))$$

$$\frac{y-2}{2}=\frac{x+3}{8}$$

$$8(y-2)=2(x+3).$$

$$8y-16=2x+6$$

$$0=2x+6-8y+16$$

$$2x-8y+22=0$$

$$\boxed{x-4y+11=0}$$

Eq of side BC

By two-point form

$$\therefore y-y_1 = \frac{y_2-y_1}{x_2-x_1}(x-x_1)$$

$$y-4=\frac{-8-4}{3-5}(x-5)$$

$$y-4=\frac{-12}{-2}(x-5)$$

$$y-4=+6(x-5)$$

$$y-4=6x-30$$

$$0=6x-30-y+4$$

$$\boxed{6x-y-26=0}$$

Eq of side AC

By two-point form

$$\therefore y-y_1 = \frac{y_2-y_1}{x_2-x_1}(x-x_1)$$

$$y-2=\frac{-8-2}{3-(-3)}(x-(-3))$$

$$y-2=\frac{-10}{6}(x+3)$$

$$y-2=-\frac{5}{3}(x+3)$$

$$3y-6=-5x-15$$

$$3y-6+5x+15=0$$

$$\boxed{5x+3y+9=0}$$

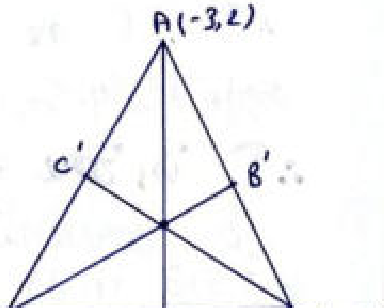
Equation of altitudes:

∴ AA' be altitude from point A on side BC

Slope of BC = $\frac{-8-4}{3-5} \Rightarrow -\frac{12}{-2} \Rightarrow 6$ $\quad \therefore m=\frac{y_2-y_1}{x_2-x_1}$

Slope of Altitude AA' = $-\frac{1}{6}$ (∵ AA'⊥BC)

Eq. of altitude AA' from point A(-3,2) having slope $-\frac{1}{6}$ is



By Point Slope form of equation:

$\therefore y-y_1=m(x-x_1)$

$(y-2)=-\frac{1}{6}(x-(-3)) \Rightarrow 6(y-2)=-1(x+3) \Rightarrow 6y-12=-x-3$

$\Rightarrow 6y-12+x+3=0 \Rightarrow \boxed{x+6y-9=0}$

∴ BB' be altitude from point B on side AC

Slope of $AC=\frac{-8-2}{3-(-3)} \Rightarrow \frac{-10}{3+3} \Rightarrow -\frac{10}{6} \Rightarrow -\frac{5}{3}$ $\quad \therefore m=\frac{y_2-y_1}{x_2-x_1}$

Slope of altitude $BB'=\frac{3}{5}$ $(\because BB' \perp AC)$

Eq of altitude BB' from Point B(5,4) having Slope $\frac{3}{5}$ is

By Point-Slope form equation:

$\therefore y-y_1=m(x-x_1)$.

$y-4=\frac{3}{5}(x-5) \Rightarrow 5y-20=3x-15 \Rightarrow 0=3x-15-5y+20$

$\boxed{3x-5y+5=0}$

∴ CC' be altitude from point C on side AB

Slope of $AB=\frac{4-2}{5-(-3)} \Rightarrow \frac{2}{5+3} \Rightarrow \frac{2}{8} \Rightarrow \frac{1}{4}$ $\quad \therefore m=\frac{y_2-y_1}{x_2-x_1}$

Slope of altitude $CC'=-4$ $\quad (\because CC' \perp AB)$.

Eq. of altitude CC' from ~~fo~~point C(3,-8) having Slope -4 is

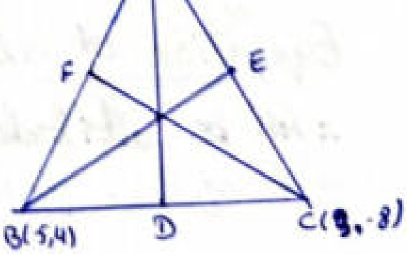
By point-slope form of equation:

$y-(-8)=-4(x-3) \Rightarrow y+8=-4x+12 \Rightarrow y+8+4x-12=0$

$\Rightarrow \boxed{4x+y-4=0}$ $\quad \therefore y-y_1=m(x-x_1)$.

Equation of Medians:

∴ D, E, F are midpoints of sides BC, AC, AB respectively. So, AD, BE and CF are medians.



∴ D is the mid point of BC

So coordinates are

$D\left(\frac{5+3}{2}; \frac{4-8}{2}\right) \Rightarrow \left(\frac{8}{2}; \frac{-4}{2}\right) \Rightarrow (4,-2)$

Eq of median AD is

$\frac{y-2}{-2-2}=\frac{x-(-3)}{4-(-3)} \Rightarrow \frac{y-2}{-4}=\frac{x+3}{7}$ $\quad \therefore \frac{y-y_1}{y_2-y_1}=\frac{x-x_1}{x_2-x_1}$

$7y - 14 = -4x - 12 \Rightarrow 7y - 14 + 4x + 12 = 0 \Rightarrow \boxed{4x + 7y - 2 = 0}$

∴ **E is the midpoint of side AC**

So coordinates of E are:

$E\left(\frac{-3+3}{2}; \frac{2-8}{2}\right) \Rightarrow \left(\frac{0}{2}, \frac{-6}{2}\right) \Rightarrow E(0,-3)$ Midpoint formula ∴ $\left(\frac{x_1+x_2}{2}; \frac{y_1+y_2}{2}\right)$

Eq of median BE is

$\frac{y-4}{-3-4} = \frac{x-5}{0-5} \Rightarrow \frac{y-4}{-7} = \frac{x-5}{-5}$ ∴ $\frac{y-y_1}{y_2-y_1} = \frac{x-x_1}{x_2-x_1}$

$-5y + 20 = -7x + 35 \Rightarrow -5y + 20 + 7x - 35 = 0 \Rightarrow \boxed{7x - 5y - 15 = 0}$

∴ **F is the midpoint of side AB**

So coordinates of F are:

$F\left(\frac{-3+5}{2}; \frac{2+4}{2}\right) \Rightarrow \left(\frac{2}{2}; \frac{6}{2}\right) \Rightarrow F(1,3)$ ∴ Midpoint formula $\left(\frac{x_1+x_2}{2}; \frac{y_1+y_2}{2}\right)$

Eq of median CF is

$\frac{y+8}{3-(-8)} = \frac{x-3}{1-3} \Rightarrow \frac{y+8}{11} = \frac{x-3}{-2} \Rightarrow -2y - 16 = 11x - 33$

$\Rightarrow -2y - 16 = 11x - 33 \Rightarrow 0 = 11x - 33 + 2y + 16 \Rightarrow \boxed{11x + 2y - 17 = 0}$

Question no 13: Find an equation of line through (-4,-6) and perpendicular to a line having slope $-\frac{3}{2}$.

P(-4,-6)

Slope of given line = $\frac{-3}{2}$

Required line is ⊥ to given line so

slope of required line = $\frac{2}{3}$

So equation of lin is (P(-4,-6) having slope $\frac{2}{3}$)

$y - y_1 = m(x - x_1)$

$y - (-6) = \frac{2}{3}(x - (-4))$

$3(y+6) = 2(x+4)$

$3y + 18 = 2x + 8$

$0 = 2x + 8 - 3y - 18$

$\boxed{2x - 3y - 10 = 0}$

Question no 14: Find an equation of the line through (11,-5) and parallel to a line with slope -24.

P(11,-5)

Slope of given line = -24

Slope of required line = -24 (∵ required line is parallel to given line)

Eq of required line is (P(11,-5) having slope -24)

$$y - y_1 = m(x - x_1)$$

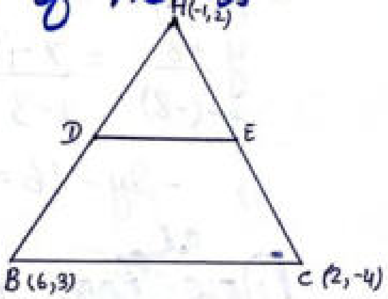
$$y - (-5) = -24(x - 11)$$

$$y + 5 = -24x + 264$$

$$y + 5 + 24x - 264 = 0$$

$$\boxed{24x + y - 259 = 0}$$

Question no 15: The points A(-1,2), B(6,3) and C(2,-4) are vertices of a triangle. Show that the line joining the midpoint D of AB and the midpoint E of AC is parallel to BC and |DE| = $\frac{1}{2}$|BC|



∵ D is the midpoint of AB

So coordinates are

$D\left(\frac{-1+6}{2} ; \frac{2+3}{2}\right) \Rightarrow D\left(\frac{5}{2} ; \frac{5}{2}\right)$ (∵ Midpoint formula $\frac{x_1+x_2}{2} ; \frac{y_1+y_2}{2}$)

∵ E is the midpoint of AC

So coordinates are

$E\left(\frac{-1+2}{2} ; \frac{2-4}{2}\right) \Rightarrow E\left(\frac{1}{2} ; -\frac{2}{2}\right) \Rightarrow E\left(\frac{1}{2} ; -1\right)$

Slope of BC $= \frac{-4-3}{2-6} \Rightarrow \frac{-7}{-4} \Rightarrow \frac{7}{4}$

Slope of DE $= \frac{-1-5/2}{1/2-5/2} \Rightarrow \frac{\frac{-2-5}{2}}{\frac{1-5}{2}} \Rightarrow \frac{\frac{-7}{2}}{\frac{-4}{2}} \Rightarrow \frac{-7}{-4} \Rightarrow \frac{7}{4}$

∴ Slope of BC = Slope of DE

So DE ∥ BC.

Prove:

$|BC| = \sqrt{(2-6)^2 + (-4-3)^2} \Rightarrow \sqrt{(-4)^2 + (-7)^2} \Rightarrow \sqrt{36 + 49} \Rightarrow \sqrt{65}$

$|DE| = \sqrt{\left(\frac{1}{2} - \frac{5}{2}\right)^2 + \left(-1 - \frac{5}{2}\right)^2} \Rightarrow \sqrt{\left(\frac{1-5}{2}\right)^2 + \left(\frac{-2-5}{2}\right)^2} \Rightarrow \sqrt{\left(\frac{-4}{2}\right)^2 + \left(\frac{-7}{2}\right)^2} \Rightarrow \sqrt{\frac{16}{4} + \frac{49}{4}} \Rightarrow \sqrt{\frac{16+49}{4}}$

$\Rightarrow \sqrt{\frac{65}{4}} \Rightarrow \frac{\sqrt{65}}{2}$

$\Rightarrow |DE| = \frac{1}{2}|BC|$

Hence proved

Question no 21: Convert each of the following equation into:

i) Slope intercept form ii) two intercept form iii) Normal form

a) $2x-4y+11=0$

Slope-intercept form

$$2x-4y+11=0$$
$$4y = 2x+11$$
$$y = \frac{2x}{4} + \frac{11}{4}$$
$$y = \frac{1}{2}x + \frac{11}{4}$$

where

$m = \frac{1}{2}$; $c = \frac{11}{4}$

two-intercept form

$$2x-4y+11=1$$
$$2x-4y=-11$$
$$-\frac{2x}{11} - \frac{4y}{-11} = 1$$
$$\frac{-2x}{11} + \frac{4y}{11} = 1$$
$$\frac{x}{-\frac{11}{2}} + \frac{y}{\frac{11}{4}} = 1$$

Normal form

$$2x-4y=-11$$

Divide By $\sqrt{2^2+4^2} \Rightarrow \pm\sqrt{20}$ or $\pm 2\sqrt{5}$

We take $-2\sqrt{5}$

$$\frac{2x}{-2\sqrt{5}} + \frac{\overset{2}{\cancel{4}}y}{2\sqrt{5}} = \frac{11}{2\sqrt{5}}$$
$$\frac{-1x}{\sqrt{5}} + \frac{2}{\sqrt{5}}y = \frac{11}{2\sqrt{5}}$$
$$\tan a = \frac{\sin a}{\cos a} \Rightarrow \frac{2/\sqrt{5}}{-1/\sqrt{5}} \Rightarrow -2$$
$$\alpha = 180^\circ - 63.43^\circ \Rightarrow 116^\circ 34^\circ$$
$$x\cos 116.57^\circ + y\sin 116.57^\circ = \frac{11}{2\sqrt{5}}$$

b) $4x+7y-2=0$

Slope-intercept form

$$4x+7y-2=0$$
$$7y = -4x+2$$
$$y = \frac{-4x}{7} + \frac{2}{7}$$

where

$m = \frac{-4}{7}$; $c = \frac{2}{7}$

two-intercept form

$$4x+7y-2=0$$
$$4x+7y=2$$
$$\frac{4x}{2} + \frac{7y}{2} = 1$$
$$2x + \frac{7}{2}y = 1$$
$$\frac{x}{1/2} + \frac{y}{2/7} = 1$$

Normal form

$$4x+7y=2$$

Divide by $\sqrt{4^2+7^2} \Rightarrow \sqrt{16+49} \Rightarrow \pm\sqrt{65}$

We take $\sqrt{65}$

$$\frac{4x}{\sqrt{65}} + \frac{7y}{\sqrt{65}} = \frac{2}{\sqrt{65}}$$
$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha} \Rightarrow \frac{7/\sqrt{65}}{4/\sqrt{65}} \Rightarrow \frac{7}{4}$$
$$\alpha = \tan^{-1}\left(\frac{7}{4}\right) \Rightarrow \alpha = 60.26^\circ$$
$$x\cos 60.26^\circ + y\sin 60.26^\circ = \frac{2}{\sqrt{65}}$$

c) $15y-8x+3=0$

Slope-intercept form

$$15y-8x+3=0$$
$$15y = 8x-3$$
$$y = \frac{8x}{15} - \frac{3}{15}$$
$$y = \frac{8x}{15} - \frac{1}{5}$$

where

$m = \frac{8}{15}$; $c = \frac{-1}{5}$

Two-intercept form

$$15y-8x+3=0$$
$$15y-8x=-3$$
$$\frac{15y}{-3} - \frac{8x}{-3} = 1$$
$$-5y + \frac{8x}{3} = 1$$
$$\frac{y}{-1/5} + \frac{x}{3/8} = 1$$
$$\frac{x}{3/8} + \frac{y}{-1/5} = 1$$

Normal form

$$-8x+15y = -3$$

Divide by $\sqrt{(-8)^2+(15)^2}$

$$\Rightarrow \sqrt{289} \Rightarrow \pm 17$$

We take -17

$$+\frac{8x}{17} - \frac{15y}{17} = \frac{3}{17}$$
$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha} = \frac{-15/17}{8/17} \Rightarrow \frac{-15}{8}$$
$$\alpha = \tan^{-1}(-1.875)$$
$$\alpha = 360^\circ - 61.93^\circ$$
$$\alpha = 298.07^\circ$$
$$x\cos 298.07^\circ + y\sin 298.07^\circ = \frac{3}{17}$$

Question no 22: In each of the following check whether the two lines are:
i) Parallel ii) Perpendicular iii) neither parallel nor perpendicular

a) $2x+y-3=0$ — (i)
$4x+2y+5=0$ — (ii)

Slope of (i)	Slope of (ii)
$m_1 = \frac{-a}{b}$	$m_2 = \frac{-a}{b}$
$m_1 = \frac{-2}{1}$	$m_2 = \frac{-4}{2}$
$m_1 = -2$	$m_2 = -2$

$\therefore m_1 = m_2$

$\Rightarrow$ Given lines are parallel

b) $3y = 2x+5$
$3x+2y-8=0$

$2x-3y+5$ — (i)	$3x+2y-8=0$ — (ii)
Slope of (i)	Slope of (ii)
$m_1 = -\frac{a}{b}$	$m_2 = -\frac{a}{b}$
$m_1 = \frac{-2}{-3}$	$m_2 = \frac{-3}{2}$
$m_1 = \frac{2}{3}$	

$m_1 \cdot m_2 = -1$

$\left(\frac{2}{3}\right)\left(\frac{-3}{2}\right) = -1$

$\Rightarrow$ Given lines are Perpendicular

c) $4y+2x-1=0$ — i
$x-2y-7=0$ — ii

Slope of (i)	Slope of (ii)
$m_1 = \frac{-a}{b}$	$m_2 = \frac{-a}{b}$
$m_1 = \frac{-2}{4}$	$m_2 = \frac{-1}{-2}$
$m_2 = -\frac{1}{2}$	$m_2 = \frac{1}{2}$

$m_1 \neq m_2$ and $m_1 \cdot m_2 \neq -1$

$\Rightarrow$ So given lines are neither $\parallel$ nor $\perp$

d) $4x-y+2=0$ — (i)
$12x-3y+1=0$ — (ii)

Slope of (i)	Slope of (ii)
$m_1 = \frac{-a}{b}$	$m_2 = \frac{-a}{b}$
$= \frac{-4}{-1}$	$= \frac{-12}{-3}$
$m_1 = 4$	$m_2 = 4$

$m_1 = m_2$

$\Rightarrow$ So, Given lines are parallel.

e) $12x+35y-7=0$ — i
$105x-36y+11=0$ — (ii)

Slope of (i)	Slope of (ii)
$m_1 = \frac{-a}{b}$	$m_2 = -\frac{a}{b}$
$m_1 = \frac{-12}{35}$	$m_2 = \frac{-105}{-36}$
	$= \frac{35}{12}$

$m_1 \cdot m_2 = -1$

$\left(\frac{-12}{35}\right)\left(\frac{35}{12}\right) = -1$

$\Rightarrow$ Given lines are perpendicular

Question no 23: Find the distance between the given parallel lines. sketch the lines. Also find an equation of the parallel line lying mid way between them.

a) $3x-4y+3=0$

$3x-4y+7=0$

$3x-4y+3=0 \; —(l_1)$, $3x-4y+7=0 \; —(l_2)$

For (l_1)

Put $x=0$

$3(0)-4y+3=0$

$-4y=-3$

$y=\frac{3}{4}$

Put $y=0$

$3x-4(0)+3=0$

$3x=-3$

$x=-1$

So $(0,\frac{3}{4})$ & $(-1,0)$ on l_1

For (l_2)

Put $x=0$

$3(0)-4y+7=0$

$-4y=-7$

$y=\frac{7}{4}$

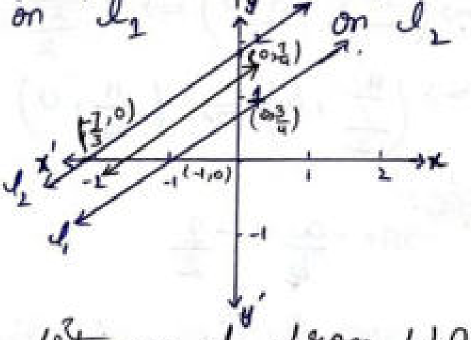
Put $y=0$

$3x-4(0)+7=0$

$3x=-7$

$x=\frac{-7}{3}$

So $(0,\frac{7}{4})$ & $(\frac{-7}{3},0)$ on l_2



Now distance d from $(-1,0)$ to l_2 is

$$\therefore d=\frac{|ax_1+by_1+c|}{\sqrt{a^2+b^2}}$$

$$=\frac{|3(-1)-4(0)+7|}{\sqrt{(3)^2+(-4)^2}} \Rightarrow \frac{|-3-0+7|}{\sqrt{9+16}}$$

$$\Rightarrow \frac{|-3+7|}{\sqrt{25}} \Rightarrow \frac{|4|}{5} \Rightarrow \frac{4}{5}=d$$

Thus distance between the Parallel lines is $\frac{4}{5}$

Now midpoint of $(-1,0)$ and $(\frac{-7}{3},0)$ is

$$=\left(\frac{-1-\frac{7}{3}}{2},\frac{0+0}{2}\right) \Rightarrow \left(\frac{\frac{-3-7}{3}}{2};\frac{0}{2}\right)$$

$$\Rightarrow \left(\frac{\frac{-10}{3}}{2};0\right) \Rightarrow \left(\frac{-10}{6};0\right) \Rightarrow \left(\frac{-5}{3};0\right)$$

Slope $m=\frac{-a}{b}=-\frac{3}{-4} \Rightarrow \frac{3}{4}$

Now required eq. of line passing through points $(\frac{-5}{3},0)$ and slope $\frac{3}{4}$ is $\therefore y-y_1=m(x-x_1)$

$$y-0=\frac{3}{4}\left(x+\frac{5}{3}\right)$$

$$4y=3x+5$$

$$\Rightarrow \boxed{3x-4y+5=0}$$

b) $12x+5y-6=0$

$12x+5y+13=0$

$12x+5y-6=0 \; —(l_1)$

$12x+5y+13=0 \; —(l_2)$

For (l_1)

Put $x=0$

$12(0)+5y-6=0$

$y=\frac{6}{5}$

Put $y=0$

$12x+5(0)-6=0$

$12x=6$

$x=\frac{6}{12} \Rightarrow \frac{1}{2}$

So $(0,\frac{6}{5})$ & $(\frac{1}{2},0)$ on l_1

For (l_2)

Put $x=0$

$12(0)+5y+13=0$

$y=-\frac{13}{5}$

Put $y=0$

$12x+5(0)+13=0$

$x=\frac{-13}{12}$

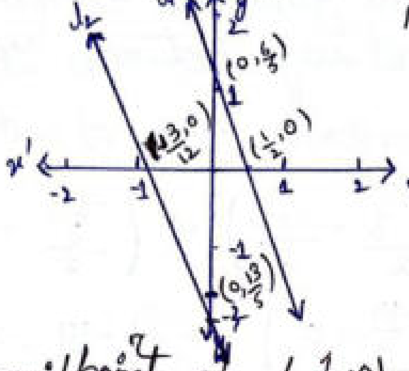
So $(0,\frac{-13}{5})$ & $(\frac{-13}{12},0)$ on l_2

Now distance d from $(\frac{1}{2},0)$ to l_2 is $\therefore d=\frac{|ax_1+by_1+c|}{\sqrt{a^2+b^2}}$

$$d=\frac{|12(\frac{1}{2})+5(0)+13|}{\sqrt{(12)^2+(5)^2}} \Rightarrow \frac{|6+0+13|}{\sqrt{144+25}}$$

$$d=\frac{|19|}{\sqrt{169}} \Rightarrow \frac{19}{13}=d$$

$\Rightarrow d = \frac{19}{13}$ Thus distance b/w the Parallel lines is $\frac{19}{13}$.



Now midpoint of $(\frac{1}{2};0)$ & $(\frac{-13}{12};0)$ is

$$= \left(\frac{\frac{1}{2}-\frac{13}{12}}{2} ; \frac{0+0}{2}\right) \Rightarrow \left(\frac{\frac{6-13}{12}}{2} ; \frac{0}{2}\right)$$

$$\Rightarrow \left(\frac{\frac{-7}{12}}{2} ; 0\right) \Rightarrow \left(\frac{-7}{24} ; 0\right)$$

Slope:

$$m = -\frac{a}{b} \Rightarrow -\frac{12}{5}$$

Now required eq. of line passing through point $(\frac{-7}{24};0)$ and $m = \frac{-12}{5}$ is

$$y - 0 = \frac{-12}{5}\left(x + \frac{7}{24}\right) \qquad \because y - y_1 = m(x - x_1)$$

$$5y = -12x - \frac{7}{2}$$

$$\boxed{12x + 5y + \frac{7}{2} = 0}$$

c) $x + 2y - 5 = 0$

$2x + 4y = 1$

$x + 2y - 5 = 0$ — (l_1) $\qquad 2x + 4y - 1 = 0$ (l_2)

For (l_1)	For (l_2)
Put $x = 0$	Put $x = 0$
$0 + 2y - 5 = 0$	$2(0) + 4y - 1 = 0$
$y = \frac{5}{2}$	$y = \frac{1}{4}$
Put $y = 0$	Put $y = 0$
$x + 2(0) - 5 = 0$	$2x + 4(0) - 1 = 0$
$x = 5$	$x = \frac{1}{2}$
$\therefore (0,\frac{5}{2})$ & $(5,0)$ on l_1	$\therefore (0,\frac{1}{4})$ & $(\frac{1}{2},0)$ on l_2.

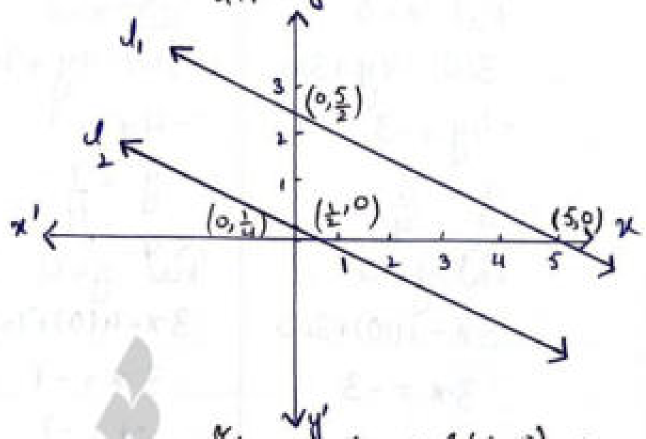
Now distance d from $(5,0)$ to l_2 is

$$\because d = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2+b^2}}$$

$$d = \frac{|2(5) + 4(0) - 1|}{\sqrt{(2)^2 + (4)^2}} \Rightarrow d = \frac{|10 + 0 - 1|}{\sqrt{4+16}}$$

$$d = \frac{|9|}{\sqrt{20}} \Rightarrow \frac{9}{2\sqrt{5}} = d$$

Thus distance b/w the parallel lines is $\frac{9}{2\sqrt{5}}$.



Now midpoint of $(5,0)$ & $(\frac{1}{2},0)$ is

$$\Rightarrow \left(\frac{5+\frac{1}{2}}{2} ; \frac{0+0}{2}\right) \Rightarrow \left(\frac{\frac{10+1}{2}}{2} ; \frac{0}{2}\right)$$

$$\Rightarrow \left(\frac{\frac{11}{2}}{2}, 0\right) \Rightarrow \left(\frac{11}{4}, 0\right)$$

Slope:

$$m = -\frac{a}{b} = -\frac{1}{2}$$

Now required eq. of line passing through point $(\frac{11}{4},0)$ and slope $-\frac{1}{2}$ is

$$\because y - y_1 = m(x - x_1)$$

$$y - 0 = -\frac{1}{2}\left(x - \frac{11}{4}\right)$$

$$2y = -x + \frac{11}{4}$$

$$2y + x - \frac{11}{4} = 0$$

$$\boxed{x + 2y - \frac{11}{4} = 0}$$

Question no 24: Find an equation of the line through $(-4,7)$ and parallel to the line $2x-7y+4=0$

Given line $2x-7y+4=0$

Slope of given line $= m = -\frac{a}{b} = \frac{-2}{-7} \Rightarrow \frac{2}{7}$

Slope of required line $= \frac{2}{7}$ $\therefore$ (required line is ∥ to given line)

Thus equation of required line through $(-4,7)$ having slope $\frac{2}{7}$ is

$y-7 = \frac{2}{7}(x-(-4))$ $\therefore y-y_1 = m(x-x_1)$

$$7(y-7) = 2(x+4)$$
$$7y-49 = 2x+8$$
$$7y-49-2x-8=0$$
$$-2x+7y-57=0$$
$$\boxed{2x-7y+57=0}$$

Question no 25: Find an equation of the line through $(5,-8)$ and perpendicular to the join of $A(-15,-8)$, $B(10,7)$.

Slope of $AB = \frac{7-(-8)}{10-(-15)} \Rightarrow \frac{7+8}{10+15} \Rightarrow \frac{15}{25} \Rightarrow \frac{3}{5}$ $\therefore m = \frac{y_2-y_1}{x_2-x_1}$

Slope of required line $= \frac{-5}{3}$ $\therefore$ (required line is ⊥ to given line)

Thus equation of required line through $(5,-8)$ having slope $\frac{-5}{2}$ is

$$y-(-8) = -\frac{5}{3}(x-5)$$
$$3(y+8) = -5(x-5)$$
$$3y+24 = -5x+25$$
$$3y+24+5x-25=0$$
$$\boxed{5x+3y-1=0}$$

Question no 26: Find equations of two parallel lines perpendicular to $2x-y+3=0$ such that the product of the x and y-intercepts of each is 3.

Give line $2x-y+3=0$

Slope of given line $= \frac{-a}{b} = \frac{-2}{-1} \Rightarrow 2$

Slope of required line $= \frac{-1}{2}$ $\therefore$ (required line is ⊥ to given line)

Let y-intercept of required line $= C$

So eq. of required line having slope $-\frac{1}{2}$ and y-intercept c is

$y = mx+c \Rightarrow y = -\frac{1}{2}x + c$

$$\frac{1}{2}x + y = c \quad — ①$$

$$\frac{x}{2c} + \frac{y}{c} = 1 \qquad \left(\frac{x}{a} + \frac{y}{b} = 1\right)$$

This is the two intercepts form of equation of line with x-intercept=2c and y-intercept= c

∴ Product of intercepts = 3

$$(c)(2c) = 3 \Rightarrow 2c^2 = 3 \Rightarrow c^2 = \frac{3}{2} \Rightarrow c = \pm\sqrt{\frac{3}{2}}$$

By putting the value of c in ①

$$\frac{1}{2}x + y = \pm\sqrt{\frac{3}{2}} \Rightarrow \frac{1}{2}x + y = \pm\sqrt{\frac{3\times 2}{2\times 2}} \Rightarrow \frac{1}{2}x + y = \frac{\sqrt{6}}{2}$$

$$\Rightarrow x + 2y = \pm\sqrt{6}$$

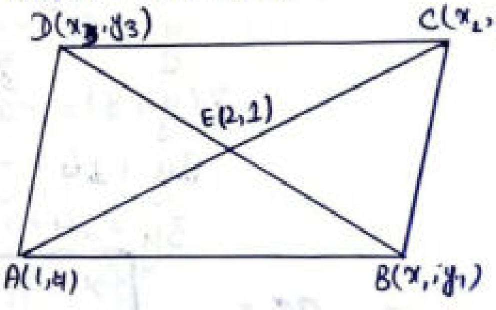
Thus required two parallel lines are

$$x + 2y = \sqrt{6} \qquad \text{and} \qquad x + 2y = -\sqrt{6}.$$

$$\frac{x}{\sqrt{6}} + \frac{2y}{\sqrt{6}} = \frac{\sqrt{6}}{\sqrt{6}} \qquad\qquad \frac{x}{-\sqrt{6}} + \frac{2y}{-\sqrt{6}} = \frac{-\sqrt{6}}{-\sqrt{6}}$$

$$\boxed{\frac{x}{\sqrt{6}} + \frac{y}{\sqrt{6}/2} = 1} \qquad\qquad \boxed{\frac{x}{-\sqrt{6}} + \frac{y}{-\sqrt{6}/2} = 1}.$$

Question no 27: One vertex of a parallelogram is (1,4). the diagonal intersect at (2,1) and the sides have slopes 1 and $\frac{-1}{7}$. Find the other three vertices.



Let (1,4) be given vertex and B(x_1,y_1), C(x_2,y_2) and D(x_3,y_3) be required vertices.

∴ E is the midpoint of AC

Its coordinates are:

$$(2,1) = \left(\frac{1+x_2}{2}; \frac{4+y_2}{2}\right) \Rightarrow 2 = \frac{1+x_2}{2}\ ;\ 1 = \frac{4+y_2}{2} \Rightarrow 4 = 1 + x_2\ ;\ 2 = 4 + y_2$$

$$x_2 = 4-1\ ;\ y_2 = 2-4 \Rightarrow x_2 = 3\ ;\ y_2 = -2$$

$$C(x_2, y_2) = (3,-2)$$

Now:

$$\text{Slope of } AD = \frac{y_3 - 4}{x_3 - 1} \Rightarrow \frac{y_3 - 4}{x_3 - 1} = 1 \Rightarrow y_3 - 4 = x_3 - 1$$

$$\Rightarrow x_3 - 1 - y_3 + 4 = 0 \Rightarrow x_3 - y_3 + 3 = 0 \quad —(i)$$

Slope of BC $= \frac{-2-y_1}{3-x_1} \Rightarrow \frac{-2-y_1}{3-x_1} = 1 \Rightarrow -2-y_1 = 3-x_1$

$\Rightarrow -2-y_1-3+x_1=0 \Rightarrow x_1-y_1-5=0$ —— (ii)

Slope of AB $= \frac{y_1-4}{x_1-1} \Rightarrow \frac{y_1-4}{x_1-1} = -\frac{1}{7} \Rightarrow 7(y_1-4) = -1(x_1-1)$

$\Rightarrow 7y_1-28 = -x_1+1 \Rightarrow 7y_1-28+x_1-1=0 \Rightarrow x_1+7y_1-29=0$ —(iii)

Slope of DC $= \frac{-2-y_3}{3-x_3} \Rightarrow \frac{-2-y_3}{3-x_3} = -\frac{1}{7} \Rightarrow 7(-2-y_3) = -1(3-x_3)$

$-14-7y_3 = -3+x_3 \Rightarrow 0 = x_3-3+14+7y_3 \Rightarrow x_3+7y_3+11=0$ —(iv)

(iv) - (i)

$$\begin{array}{r} \cancel{x_3}+7y_3+11=0 \\ \cancel{x_3} - y_3 + 3 = 0 \\ - \quad + \quad - \\ \hline 8y_3+8=0 \end{array}$$

$8y_3 = -8$

$\boxed{y_3 = -1}$

Put in (i)

$x_3-(-1)+3=0$

$x_3+1+3=0$

$x_3+4=0$

$\boxed{x_3=-4}$

(iii) - (ii)

$$\begin{array}{r} \cancel{x_1}+7y_1-29=0 \\ \cancel{x_1} - y_1 - 5 = 0 \\ - \quad + \quad + \\ \hline 8y_1-24=0 \end{array}$$

$8y_1 = 24$

$\boxed{y_1 = 3}$

Put in (ii)

$x_1-3-5=0$

$x_1-8=0$

$\boxed{x_1 = 8}$

Hence required vertices are

$B(x_1, y_1) = B(8,3)$

$C(x_2, y_2) = C(3,-2)$

$D(x_3, y_3) = D(-4,-1)$

Question no 28: Find whether the given point lies above or below the given line

a) $(5,8)$; $2x-3y+6=0$

$2x-3y+6=0$ — (i)

Here $b=-3<0$

$2(5)-3(8)+6$

$10-24+6$

$-8<0$ — (ii)

As the coefficient of y in (i) and expression in (ii) have same sign so it lie above (i)

b) $(-7,6)$; $4x+3y-9=0$

$4x+3y-9=0$ — (i)

Here $b=3>0$

$4(-7)+3(6)-9$

$-28+18-9$

$-19<0$ — (ii)

As the coefficient of y in (i) and expression in (ii) have opposite sign so it lie below (i) re

Question no 29: Check whether the given points are on the same or opposite sides of the given line

a) (0,0) and (-4,7) ; $6x-7y+70=0$

$6x-7y+70=0$ —— (i)

Put (0,0) in (i)

$6(0)-7(0)+70$

$0-0+70$

$70>0$ (+ive) —— (ii)

Put (-4,7) in (i)

$6(-4)-7(7)+70$

$-24-49+70$

$-3<0$ (-ive) —— (iii)

As expression (ii) & (iii) have opposite sign. So the points lie on the opposite sides of (i)

b) (2,3) and (-2,3) ; $3x-5y+8=0$

$3x-5y+8=0$ —— (i)

Put (2,3) in (i)

$3(2)-5(3)+8$

$6-15+8$

$-1<0$ (-ive) —— (ii)

Put (-2,3) in (i)

$3(-2)-5(3)+8$

$-6-15+8$

$-13<0$ (-ive) —— (iii)

As expression (ii) & (iii) have same sign. so the points lie on the same sidedes of (i).

re to

Question no 30: Find the distance from the point P(6,-1) to the line $6x-4y+9=0$.

$$d=\frac{|6(6)-4(-1)+9|}{\sqrt{(6)^2+(-4)^2}}$$

$$\therefore d=\frac{|ax_1+by_1+c|}{\sqrt{a^2+b^2}}$$

$$d=\frac{|36+4+9|}{\sqrt{36+16}}$$

$$d=\frac{|49|}{\sqrt{52}} \Rightarrow d=\frac{49}{\sqrt{52}}$$

re to

Question no 31: Find the area of the triangle whose vertices are A(5,3), B(-2,2), C(4,2)

A(5,3), B(-2,2), C(4,2)

$$\therefore \Delta=\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

$$\Delta=\frac{1}{2}\begin{vmatrix} 5 & 3 & 1 \\ -2 & 2 & 1 \\ 4 & 2 & 1 \end{vmatrix}$$

Expand R_1

$$\Delta = \frac{1}{2}\left[5(2-2)-3(-2-4)+1(-4-8)\right]$$

$$\Delta = \frac{1}{2}\left[5(0)-3(-6)+1(-12)\right]$$

$$\Delta = \frac{1}{2}\left[0+18-12\right]$$

$$\Delta = \frac{1}{2}(6)$$

$\Delta = 3$ square units

•————•

Question no 32: The coordinates of three points are A(2,3), B(-1,1) and C(4,-5). By computing the area bounded by ABC check wheter the points are collinear.

A(2,3), B(-1,1), C(4,-5)

$$\therefore \quad \Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

$$\Delta = \frac{1}{2}\begin{vmatrix} 2 & 3 & 1 \\ -1 & 1 & 1 \\ 4 & -5 & 1 \end{vmatrix}$$

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Expand by R_1

$$\Delta = \frac{1}{2}\left[2(1+5)-3(-1-4)+1(5-4)\right]$$

$$\Delta = \frac{1}{2}\left[2(6)-3(-5)+1(1)\right]$$

$$\Delta = \frac{1}{2}\left[12+15+1\right]$$

$$\Delta = \frac{1}{2}(28)$$

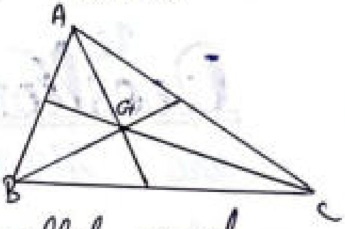
$\Delta = 14$ square units

The points are not collinear.

•————•

Theory:

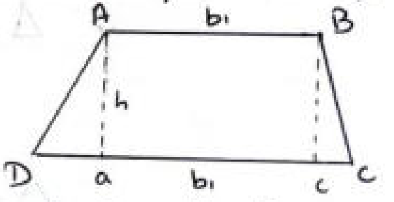
- **Median of a triangle:** The median of a triangle is a line segment from the vertex to the midpoint of the opposite side. Because a triangle has three vertexes, it has also three medians.
- **Centroid of a triangle:** The point at which three median of a triangle intersects is called centroid of a triangle. In figure the Point G is centroid.



- **Trapezium:** A quadrilateral having two parallel and two non-parallel sides is called trapezium

Area of trapezoidal region = $\frac{1}{2}$(sum of ∥ sides)(distance b/w ∥ sides).

In figure Δ of trapezium = $\frac{1}{2}(AB+DC)(h)$



- **Angle between two lines:**

Let l_1 and l_2 be two non-vertical lines such that they are not perpendicular to each axes. If m_1 and m_2 are the slopes of l_1 and l_2 respectively, then the θ from l_1 to l_2 is given by:

$$\tan\theta = \frac{m_2 - m_1}{1 + m_1 m_2}$$

Corallary 1. $l_1 \parallel l_2$ if and only if

$m_1 = m_2 \Rightarrow \theta = 0°$

Corallary 2. $l_1 \perp l_2$ if and only if

$1 + m_1 m_2 = 0 \Rightarrow \theta = \infty$

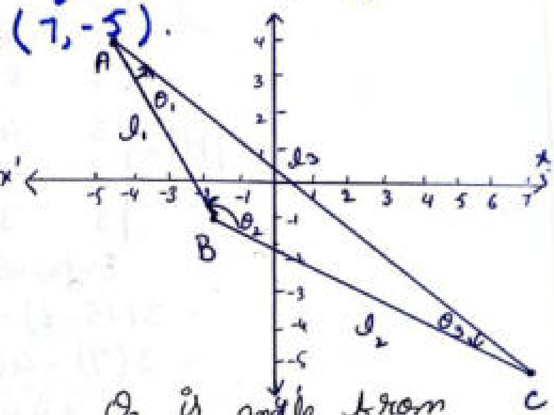
Example no 1: Find the angle from the line with slope $-\frac{7}{3}$ to the line with slope $\frac{5}{2}$

$$\therefore \tan\theta = \frac{m_2 - m_1}{1 + m_1 m_2}$$

$$\tan\theta = \frac{\left(\frac{5}{2}\right) - \left(\frac{-7}{3}\right)}{1 + \left(\frac{-7}{3}\right)\left(\frac{5}{2}\right)} \Rightarrow \frac{\frac{5}{2} + \frac{7}{3}}{1 - \frac{35}{6}} \Rightarrow \frac{\frac{15+14}{6}}{\frac{6-35}{6}} \Rightarrow \frac{29}{-29} \Rightarrow -1$$

$$\tan\theta = -1 \Rightarrow \theta = \tan^{-1}(-1) \Rightarrow \boxed{\theta = 135^\circ}$$

Example no 2: Find the angles of the triangle whose vertices are A(-5,4), B(-2,-1), C(7,-5).



$$\text{Slope of } AB(l_1) = m_1 = \frac{-1-4}{-2+5} \Rightarrow \frac{-5}{3}$$

$$\text{Slope of } BC(l_2) = m_2 = \frac{-5+1}{7+2} \Rightarrow \frac{-4}{9}$$

$$\text{Slope of } AC(l_3) = m_3 = \frac{4+5}{-5-7} \Rightarrow \frac{9}{-12} \Rightarrow -\frac{3}{4}.$$

θ_1 is angle from l_1 to l_3

$$\therefore \tan\theta_1 = \frac{m_3 - m_1}{1 + m_3 m_1}$$

$$\tan\theta_1 = \frac{-\frac{3}{4} - \left(-\frac{5}{3}\right)}{1 + \left(-\frac{3}{4}\right)\left(-\frac{5}{3}\right)}$$

$$\tan\theta_1 = \frac{-\frac{3}{4} + \frac{5}{3}}{1 + \frac{5}{4}}$$

$$\tan\theta_1 = \frac{\frac{-9+20}{12}}{\frac{4+5}{4}}$$

$$\tan\theta_1 = \frac{11}{12} \times \frac{4}{9}$$

$$\tan\theta_1 = \frac{11}{27}$$

$$\theta_1 = \tan^{-1}\left(\frac{11}{27}\right)$$

$$\theta_1 = \tan^{-1}(0.4074)$$

$$\boxed{\theta_1 = 22.2^\circ}$$

θ_2 is angle from l_2 to l_1

$$\therefore \tan\theta_2 = \frac{m_1 - m_2}{1 + m_1 m_2}$$

$$\tan\theta_2 = \frac{-\frac{5}{3} - \left(-\frac{4}{9}\right)}{1 + \left(-\frac{5}{3}\right)\left(-\frac{4}{9}\right)}$$

$$\tan\theta_2 = \frac{-\frac{5}{3} + \frac{4}{9}}{1 + \frac{20}{27}}$$

$$\tan\theta_2 = \frac{\frac{-15+4}{9}}{\frac{27+20}{27}}$$

$$\tan\theta_2 = -\frac{11}{9} \times \frac{27}{47}$$

$$\tan\theta_2 = \frac{-33}{47}$$

$$\theta_2 = \tan^{-1}\left(-\frac{33}{47}\right)$$

$$\theta_2 = \tan^{-1}(-0.7021)$$

$$\theta_2 = -35.1^\circ$$

$$\theta_2 = 180^\circ - 35.1^\circ$$

$$\boxed{\theta_2 = 144.9^\circ}$$

θ_3 is angle from l_3 to l_2

$$\tan\theta_3 = \frac{m_2 - m_3}{1 + m_2 m_3}$$

$$\tan\theta_3 = \frac{-\frac{4}{9} - \left(-\frac{3}{4}\right)}{1 + \left(-\frac{4}{9}\right)\left(-\frac{3}{4}\right)}$$

$$\tan\theta_3 = \frac{-\frac{4}{9} + \frac{3}{4}}{1 + \frac{1}{3}}$$

$$\tan\theta_3 = \frac{\frac{-16+27}{36}}{\frac{3+1}{3}}$$

$$\tan\theta_3 = \frac{11}{36} \times \frac{3}{4}$$

$$\tan\theta_3 = \frac{11}{48}$$

$$\theta_3 = \tan^{-1}\left(\frac{11}{48}\right)$$

$$\theta_3 = \tan^{-1}(0.2292)$$

$$\boxed{\theta_3 = 12.9^\circ}$$

Example no 3: Express the system $\left.\begin{matrix} 3x+4y-7=0 \\ 2x-5y+8=0 \\ x+y-3=0 \end{matrix}\right\}$ in matrix form and check whether the three lines are concurrent.

The matrix form of the system is

$$\begin{bmatrix} 3 & 4 & -7 \\ 2 & -5 & 8 \\ 1 & 1 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Coefficient matrix of the system is

$$A = \begin{bmatrix} 3 & 4 & -7 \\ 2 & -5 & 8 \\ 1 & 1 & -3 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 3 & 4 & -7 \\ 2 & -5 & 8 \\ 1 & 1 & -3 \end{vmatrix}$$

Expand by R_1

$$= 3(15-8)-4(-6-8)-7(2+5)$$
$$= 3(7)-4(-14)-7(7)$$
$$= 21+56-49$$
$$= 28 \neq 0$$ The lines are not concurrent.

Example no 4: Find a system of equations corresponding to the matrix form $\begin{bmatrix} 1 & 2 & 5 \\ 3 & 5 & 1 \\ 4 & 7 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. Are the lines represented by a system concurrent?

$$\begin{bmatrix} x+2y+5 \\ 3x+5y+1 \\ 4x+7y+6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By Using the defination of equality of two matrices

$$\left.\begin{matrix} x+2y+5=0 \\ 3x+5y+1=0 \\ 4x+7y+6=0 \end{matrix}\right]$$

as the required system of equation. The coefficient matrix A of the system is such that

$$|A| = \begin{vmatrix} 1 & 2 & 5 \\ 3 & 5 & 1 \\ 4 & 7 & 6 \end{vmatrix}$$

Expand by R_1

$$= 1(30-7)-2(18-4)+5(21-20)$$
$$= 23-2(14)+5(1)$$
$$= 23-28+5$$
$$= 28-28$$
$$= 0$$

The lines are concurrent.

Exercise no 4.4

Question no 1:

Find the point of intersection of the lines

i) $x - 2y + 1 = 0$ —① and $2x - y + 2 = 0$ —②

①$-2$②

$$x - \cancel{2y} + 1 = 0$$
$$-\,4x \mp \cancel{2y} \pm 4 = 0$$
$$-3x \quad -3 = 0$$
$$-3x = 3$$
$$\boxed{x = -1}$$

Put in ①

$$-1 - 2y + 1 = 0$$
$$-2y = 0$$
$$\boxed{y = 0}$$

(-1,0) is the point of intersection.

ii) $3x + y + 12 = 0$ —① and $x + 2y - 1 = 0$ —②

2①$-$②

$$6x + \cancel{2y} + 24 = 0$$
$$-\,x + \cancel{2y} \mp 1 = 0$$
$$5x \quad + 25 = 0$$
$$5x = -25$$
$$\boxed{x = -5}$$

Put in ①

$$3(-5) + y + 12 = 0$$
$$-15 + y + 12 = 0$$
$$y - 3 = 0$$
$$\boxed{y = 3}$$

(-5,3) is the point of intersection

iii) $x + 4y - 12 = 0$ —① and $x - 3y + 3 = 0$ —②

①$-$②

$$\cancel{x} + 4y - 12 = 0$$
$$\cancel{x} \mp 3y \pm 3 = 0$$
$$7y - 15 = 0$$
$$7y = 15$$
$$\boxed{y = \frac{15}{7}}$$

Put in ①

$$x + 4\left(\frac{15}{7}\right) - 12 = 0$$
$$x + \frac{60}{7} - 12 = 0$$
$$x = -\frac{60}{7} + 12$$
$$x = \frac{-60 + 84}{7}$$
$$\boxed{x = \frac{24}{7}}$$

$\left(\frac{24}{7}; \frac{15}{7}\right)$ is the point of intersection

Question no 2:

Find an equation of the line through

i) the point (2,-9) and the intersection of lines

$2x + 5y - 8 = 0$ & $3x - 4y - 6 = 0$

$2x + 5y - 8 = 0$ —① ; $3x - 4y - 6 = 0$ —②

3①$-$2②

$$6\cancel{x} + 15y - 24 = 0$$
$$-\,\cancel{6x} \mp 8y \mp 12 = 0$$
$$23y - 12 = 0$$
$$23y = 12$$
$$\boxed{y = \frac{12}{23}}$$

Put y in ①

$$2x + 5\left(\frac{12}{23}\right) - 8 = 0$$
$$2x + \frac{60}{23} - 8 = 0$$
$$2x = -\frac{60}{23} + 8$$
$$2x = \frac{-60 + 184}{23}$$
$$2x = \frac{124}{23} \Rightarrow \boxed{x = \frac{62}{23}}$$

So point of intersection is $\left(\frac{-62}{23}, \frac{12}{23}\right)$

Now slope of $\left(\frac{62}{23}, \frac{12}{23}\right)$ and $(2,-9)$ is

$$m = \frac{-9-\frac{12}{23}}{2-\frac{62}{23}} \Rightarrow \frac{\frac{-207-12}{23}}{\frac{46-62}{23}} \Rightarrow \frac{-219}{-16} \Rightarrow \frac{219}{16}$$

Now equation of line through $(2,-9)$ and $m = \frac{219}{16}$

$$y - y_1 = m(x - x_1)$$

$$y-(-9) = \frac{219}{16}(x-2)$$

$$16(y+9) = 219(x-2)$$

$$16y+144 = 219x-438$$

$$0 = 219x-438-16y-144$$

$$219x-16y-582 = 0.$$

is the required equation

ii) the intersection of the lines $x-y-4=0$ and $7x+y+20=0$ and ⓐ parallel ⓑ perpendicular to the line $6x+y-14=0$

$x-y-4=0$ —① ; $7x+y+20=0$ —②

① + ②

$$\begin{array}{r} x-y-4=0 \\ 7x+y+20=0 \\ \hline 8x+16=0 \end{array}$$

$$8x = -16$$

$$\boxed{x=-2}$$

Put x in ①

$$-2-y-4=0$$

$$-y-6=0$$

$$-y=6$$

$$\boxed{y=-6}$$

Point of intersection is $(-2,-6)$

Given line is $6x+y-14=0$

ⓐ Slope of Given line $= \frac{-a}{b} \Rightarrow \frac{-6}{1} \Rightarrow -6$

Slope of required line $= -6$

($\because$ required line ∥ give line)

Equation of line through $(-2,-6)$ & $m=-6$

$$y-(-6) = -6(x-(-2))$$

$$y+6 = -6(x+2)$$

$$y+6 = -6x-12$$

$$y+6+6x+12 = 0$$

$$6x+y+18 = 0$$

is a required equation

ⓑ Slope of given line $= -6$

Slope of required line $= \frac{1}{6}$

($\because$ required line ⊥ given line)

Equation of line through $(-2,-6)$ & $m = \frac{1}{6}$

$$y-(-6) = \frac{1}{6}(x-(-2))$$

$$6(y+6) = 1(x+2)$$

$$6y+36 = x+2$$

$$0 = x+2-6y-36$$

$$x-6y-34 = 0$$

is a required equation

iii) through the intersection of the lines $x+2y+3=0$; $3x+4y+7=0$ and making equal intercepts on the axes.

Given lines $x+2y+3=0$; $3x+4y+7=0$ is

$$(x+2y+3)+k(3x+4y+7) = 0 \quad ①$$

$$x+2y+3+3kx+4ky+7k = 0$$

$$x+3kx+2y+4ky+3+7k = 0$$

$$(1+3k)x+(2+4k)y+(3+7k) = 0$$

$$(1+3k)x+(2+4k)y = -(3+7k)$$

$$\frac{(1+3k)x}{-(3+7k)} + \frac{(2+4k)y}{-(3+7k)} = 1$$

$$\frac{x}{\frac{-(3+7k)}{(1+3k)}} + \frac{y}{\frac{-(3+7k)}{(2+4k)}} = 1$$

Which is two intercept form of equation of line with

$x\text{-intercept} = \frac{-(3+7k)}{(1+3k)}$; $y\text{-intercept} = \frac{-(3+7k)}{(2+4k)}$

We have given

$x\text{-intercept} = y\text{-intercept}$

$$\frac{-(3+7k)}{(1+3k)} = -\frac{(3+7k)}{(2+4k)}$$

$$\frac{1}{1+3k} = \frac{1}{2+4k}$$

$$2+4k = 1+3k$$

$$4k-3k = 1-2$$

$$\boxed{k = -1}$$

Put k in (i)

$$x+2y+3-1(3x+4y+7)=0$$

$$x+2y+3-3x-4y-7=0$$

$$-2x-2y-4=0$$

$$-2(x+y+2)=0$$

$$x+y+2=0$$

is the required equation

Question no 3:

Find an equation of the line through the intersection of $16x-10y-33=0$; $12x+14y+29=0$ and the intersection of $x-y+4=0$, $x-7y+2=0$

First we find intersection of

$16x-10y-33=0$ —① ; $12x+14y+29=0$ —②

14① + 10②

$$224x-\cancel{140y}-462=0$$

$$120x+\cancel{140y}+290=0$$

$$344x-172=0$$

$$344x=172$$

$$x = 172/344$$

$$\boxed{x = \frac{1}{2}}$$

Put in ①

$$16\left(\frac{1}{2}\right)-10y-33=0$$

$$8-10y-33=0$$

$$-10y-25=0$$

$$-10y=25$$

$$y=-\frac{25}{10}$$

$$\boxed{y=\frac{-5}{2}}$$

So point of intersection is $\left(\frac{1}{2}; \frac{-5}{2}\right)$

Now we find intersection of

$x-y+4=0$ —③ ; $x-7y+2=0$ —④

③-④

$$\cancel{x}-y+4=0$$

$$\cancel{-x}\mp 7y \mp 2=0$$

$$6y+2=0$$

$$6y=-2$$

$$y=\frac{-2}{6}$$

$$\boxed{y=\frac{-1}{3}}$$

Put in ③

$$x-\left(-\frac{1}{3}\right)+4=0$$

$$x+\frac{1}{3}+4=0$$

$$x=\frac{-1}{3}-4$$

$$x=\frac{-1-12}{3}$$

$$\boxed{x=-\frac{13}{3}}$$

So point of intersection is $\left(\frac{-13}{3}; \frac{-1}{3}\right)$

Slope through $\left(\frac{1}{2}, \frac{-5}{2}\right)$ and $\left(\frac{-13}{3}, \frac{-1}{3}\right)$ is

$$m = \frac{-\frac{1}{3}-\left(\frac{-5}{2}\right)}{-\frac{13}{3}-\frac{1}{2}} \Rightarrow \frac{-\frac{1}{3}+\frac{5}{2}}{\frac{-13}{3}-\frac{1}{2}} \Rightarrow \frac{\frac{-2+15}{6}}{\frac{-26-3}{6}}$$

$$m = \frac{\frac{13}{6}}{\frac{-29}{6}} \Rightarrow m = \frac{-13}{29}$$

So equation of required line through $(\frac{1}{2}, \frac{-5}{2})$ and slope $-\frac{13}{29}$ is

$$y-(\frac{-5}{2}) = \frac{-13}{29}(x-\frac{1}{2})$$

$$29(y+\frac{5}{2}) = -13(x-\frac{1}{2})$$

$$29y + \frac{145}{2} = -13x + \frac{13}{2}$$

$$29y + \frac{145}{2} + 13x - \frac{13}{2} = 0$$

$$13x + 29y + \frac{145}{2} - \frac{13}{2} = 0$$

$$13x + 29y + \frac{145-13}{2} = 0$$

$$13x + 29y + \frac{132}{2} = 0$$

$$\Rightarrow 13x + 29y + 66 = 0$$

Required line.

Question no 4:

Find the condition that the lines $y = m_1x + c_1$; $y = m_2x + c_2$ and $y = m_3x + c_3$ are concurrent.

Arranging given lines

$$m_1x - y + c_1 = 0$$

$$m_2x - y + c_2 = 0$$

$$m_3x - y + c_3 = 0$$

$\therefore$ given lines are concurrent

$$\begin{vmatrix} m_1 & -1 & c_1 \\ m_2 & -1 & c_2 \\ m_3 & -1 & c_3 \end{vmatrix} = 0$$

$$\begin{vmatrix} m_1 & -1 & c_1 \\ m_2-m_1 & 0 & c_2-c_1 \\ m_3-m_1 & 0 & c_3-c_1 \end{vmatrix} = 0 \quad \text{by } R_2-R_1,\ R_3-R_1$$

Expanding by C_2

$$-(-1)[(m_2-m_1)(c_3-c_1)-(m_3-m_1)(c_2-c_1)]+0-0=0$$

$$(m_2-m_1)(c_3-c_1)-(m_3-m_1)(c_2-c_1) = 0$$

$$(m_2-m_1)(c_3-c_1) = (m_3-m_1)(c_2-c_1)$$

Required condition.

Question no 5:

Determine the value of P such that the lines $2x-3y-1=0$; $3x-y-5=0$ and $3x+Py+8=0$ meet at a point.

Given lines
$2x-3y-1=0$
$3x-y-5=0$
$3x+Py+8=0$
are concurrent.

$$\begin{vmatrix} 2 & -3 & -1 \\ 3 & -1 & -5 \\ 3 & P & 8 \end{vmatrix} = 0$$

Expand by R_1

$$2(-8+5P)+3(24+15)-1(3P+3)=0$$

$$2(-8+5P)+3(39)-1(3P+3)=0$$

$$-16+10P+117-3P-3=0$$

$$7P+98=0$$

$$7P=-98$$

$$\boxed{P=-14}$$

Question no 6:

Show that the lines $4x-3y-8=0$; $3x-4y-6=0$ and $x-y-2=0$ are concurrent and the third-line bisects the angle formed by the first two lines

Given lines are
$l_1;\ 4x-3y-8=0$
$l_2;\ 3x-4y-6=0$
$l_3;\ x-y-2=0$

$$\begin{vmatrix} 4 & -3 & -8 \\ 3 & -4 & -6 \\ 1 & -1 & -2 \end{vmatrix}$$

Expand R_1

$$= 4(8-6)-(-3)(-6+6)-8(-3+4)$$

$= 4(2)+3(0)-8(1)$

$= 8+0-8$

$= 0$

So given lines are concurrent

Slope of $l_1 = m_1 = \frac{-a}{b} = \frac{4}{3}$

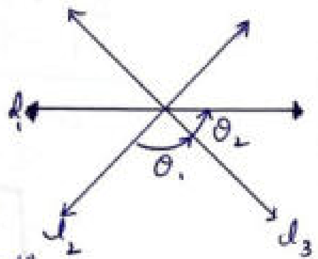
Slope of $l_2 = m_2 = \frac{-a}{b} = \frac{3}{4}$

Slope of $l_3 = m_3 = -\frac{a}{b} = 1$

Now the condition to be proved

$\theta_1 = \theta_2$

(i.e. third line bisects the angle made by first two lines)



$\tan\theta_1 = \frac{m_3 - m_2}{1 + m_3 m_2}$ ∴ θ_1 is angle from l_2 to l_3

$\tan\theta_1 = \frac{1-\frac{3}{4}}{1+(1)(\frac{3}{4})} \Rightarrow \frac{\frac{4-3}{4}}{\frac{4+3}{4}} \Rightarrow \frac{\frac{1}{4}}{7/4}$

$\tan\theta_1 = \frac{1}{7} \Rightarrow \theta_1 = \tan^{-1}(\frac{1}{7})$.

$\tan\theta_2 = \frac{m_1 - m_3}{1 + m_1 m_3}$ ∴ θ_2 is the angle from l_3 to l_1

$\tan\theta_2 = \frac{\frac{4}{3}-1}{1+(\frac{4}{3})(1)} \Rightarrow \frac{\frac{4-3}{3}}{\frac{3+4}{3}} \Rightarrow \frac{\frac{1}{3}}{\frac{7}{3}}$

$\tan\theta_2 = \frac{1}{7} \Rightarrow \theta_2 = \tan^{-1}(\frac{1}{7})$.

$\theta_1 = \theta_2$.

Question no 7:

The vertices of a triangle are A(-2,3), B(-4,1), C(3,5)
Find coordinates of the
i) Centroid
ii) Orthocentre
iii) circumcentre of the triangle

Are these three points collinear?

A(-2,3), B(-4,1), C(3,5)

i) Centroid:

$x_1 = -2;\ y_1 = 3;\ x_2 = -4,\ y_2 = 1,\ x_3 = 3,\ y_3 = 5$

∴ $\text{Centroid} = \left(\frac{x_1+x_2+x_3}{3}; \frac{y_1+y_2+y_3}{3}\right)$

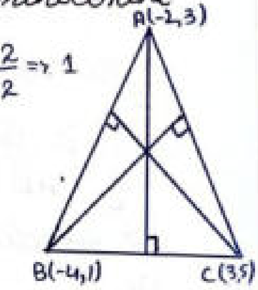
$= \left(\frac{-2-4+3}{3}; \frac{3+1+5}{3}\right)$

$= \left(\frac{-3}{3}; \frac{9}{3}\right)$

Centroid $= (-1, 3)$

ii) Orthocentre:

The point of intersection of altituteds is called orthocentre



Slope of $AB = \frac{1-3}{-4+2} \Rightarrow \frac{-2}{-2} \Rightarrow 1$

Slope of $BC = \frac{5-1}{3+4} \Rightarrow \frac{4}{7}$

∴ altitudes are ⊥ to sides

Slope of altitude on $AB = -\frac{1}{1} \Rightarrow -1$

Slope of altitude on $BC = -\frac{7}{4}$

Now eq. of altitude on AB with slope -1 from C(3,5)

$y-5 = -1(x-3)$

$y-5 = -x+3$

$y-5+x-3=0$

$x+y-8=0$ — ①

Now eq. of altitude on BC with slope $-\frac{7}{4}$ from A(-2,3)

$y-3 = -\frac{7}{4}(x-(-2))$

$4y-12 = -7(x+2)$

$4y-12 = -7x-14$

$4y-12+7x+14=0$

$7x+4y+2=0$ — ②

7① - ②

$$\begin{array}{r} \cancel{7x}+7y-56=0 \\ \cancel{7x}+4y+2=0 \\ \hline 3y-58=0 \end{array}$$

$3y = 58$

$$\boxed{y = \frac{58}{3}}$$

Put in ①

$x + \frac{58}{3} - 8 = 0$

$x + \frac{58-24}{3} = 0$

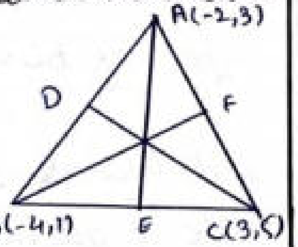
$x + \frac{34}{3} = 0$

$$\boxed{x = -\frac{34}{3}}$$

So orthocentre is $\left(\frac{-34}{3}; \frac{58}{3}\right)$

iii) Circumcentre:

The point of intersection of right bisectors is called circumcentre



∴ D is the midpoint of AB

So coordinates of D are

$= \left(\frac{-2-4}{2}; \frac{3+1}{2}\right) \Rightarrow \left(\frac{-6}{2}; \frac{4}{2}\right) \Rightarrow D(-3,2)$

∴ E is the midpoint of BC

So coordinates of E are

$= \left(\frac{-4+3}{2}; \frac{1+5}{2}\right) \Rightarrow \left(\frac{-1}{2}; \frac{6}{2}\right) \Rightarrow E\left(\frac{-1}{2}; 3\right)$

Slope of AB $= m_1 = \frac{1-3}{-4+2} \Rightarrow \frac{-2}{-2} = 1$

Slope of BC $= m_2 = \frac{5-1}{3+4} \Rightarrow \frac{4}{7}$

Slope of ⊥ bisector on AB $= -\frac{1}{1} \Rightarrow -1$

Slope of ⊥ bisector on BC $= \frac{-7}{4}$

Now eq. of ⊥ bisector with slope -1 and $D(-3,2)$

$y - 2 = -1(x+3)$

$y - 2 = -x - 3$

$x + y - 2 + 3 = 0$

$x + y + 1 = 0$ — ①

Now eq. of ⊥ bisector with slope $-\frac{7}{4}$ and $E\left(-\frac{1}{2}, 3\right)$

$y - 3 = -\frac{7}{4}\left(x + \frac{1}{2}\right)$

$4y - 12 = -7x - \frac{7}{2}$

$7x + 4y - 12 + \frac{7}{2} = 0$

$14x + 8y - 24 + 7 = 0$

$14x + 8y - 17 = 0$ — ②

8① - ②

$8x + \cancel{8y} + 8 = 0$

$\underset{-}{14} \underset{-}{+} \cancel{8y} \underset{+}{-} 17 = 0$

$-6x + 25 = 0$

$-6x = -25$

$$\boxed{x = \frac{25}{6}}$$

Put in ①

$\frac{25}{6} + y + 1 = 0$

$y + \frac{25+6}{6} = 0$

$y + \frac{31}{6} = 0$

$$\boxed{y = -\frac{31}{6}}$$

So circumcentre is $\left(\frac{25}{6}; \frac{-31}{6}\right)$

Now we check $(-1,3)$, $\left(-\frac{34}{3}, \frac{58}{3}\right)$ and $\left(\frac{25}{6}, \frac{-31}{6}\right)$ are collinear.

$$\begin{vmatrix} -1 & 3 & 1 \\ -34/3 & 58/3 & 1 \\ 25/6 & -31/6 & 1 \end{vmatrix}$$

Expand by R_1

$= -1\left(\frac{58}{3} + \frac{31}{6}\right) - 3\left(-\frac{34}{3} - \frac{25}{6}\right) + 1\left[\left(-\frac{34}{3}\right)\left(-\frac{31}{6}\right) - \left(\frac{25}{6}\right)\left(\frac{58}{3}\right)\right]$

$= -\frac{58}{3} - \frac{31}{6} + 34 + \frac{25}{2} + \frac{1054}{18} - \frac{1450}{18}$

$= \frac{-348 - 93 + 612 + 225 + 1054 - 1450}{18}$

$= \frac{1891 - 1891}{18}$

$= \frac{0}{18}$

$= 0$ Hence points are collinear.

Question no 8:

Check whether the lines $4x-3y-8=0; 3x-4y-6=0; x-y-2=0$ are concurrent. If so, find the point where they meet

Given lines

$4x-3y-8=0$ — ①

$3x-4y-6=0$ — ②

$x-y-2=0$ — ③

$$= \begin{vmatrix} 4 & -3 & -8 \\ 3 & -4 & -6 \\ 1 & -1 & -2 \end{vmatrix}$$

Expand by R_1

$= 4(8-6)+3(-6+6)-8(-3+4)$

$= 4(2)+3(0)-8(1)$

$= 8+0-8$

$= 0$

So given lines are concurrent

So the point of intersection is

② − 4③

$$\begin{array}{r} 3x-4y-6=0 \\ 4x-4y-8=0 \\ \underline{-\quad+\quad+\quad} \\ -x+2=0 \end{array}$$

$-x=-2$

$\boxed{x=2}$

Put x in ③

$2-y-2=0$

$-y=0$

$\boxed{y=0}$

So the point of intersection is $(2,0)$

Question no 9:

Find the coordinates of the vertices of the triangle formed by the lines $x-2y-6=0; 3x-y+3=0; 2x+y-4=0$. Also find measures of the angles of the triangle.

Given lines

$x-2y-6=0$ — ①

$3x-y+3=0$ — ②

$2x+y-4=0$ — ③

By solving ①, ② & ③

By solving ① & ②

3① − ②

$$\begin{array}{r} 3x-6y-18=0 \\ 3x-y+3=0 \\ \underline{-\quad+\quad-} \\ -5y-21=0 \end{array}$$

$-5y=21$

$y=\frac{-21}{5}$

Put $y=-\frac{21}{5}$ in ①

$x-2\left(-\frac{21}{5}\right)-6=0$

$x+\frac{42}{5}-6=0$

$x=\frac{-42}{5}+6$

$x=\frac{-42+30}{5}$

$x=\frac{-12}{5}$

$A\left(\frac{-12}{5},\frac{-21}{5}\right)$

By solving ② & ③

② + ③

$$\begin{array}{r} 3x-y+3=0 \\ \underline{2x+y-4=0} \\ 5x-1=0 \end{array}$$

$5x=1$

$x=\frac{1}{5}$

Put $x=\frac{1}{5}$ in ②

$3\left(\frac{1}{5}\right)-y+3=0$

$\frac{3}{5}-y+3=0$

$\frac{3}{5}+3=y$

$\frac{3+15}{5}=y$

$\frac{18}{5}=y$

$y=\frac{18}{5}$

$B\left(\frac{1}{5},\frac{18}{5}\right)$

By solving ① & ③

① + 2③

$$\begin{array}{r} x-2y-6=0 \\ \underline{4x+2y-8=0} \\ 5x-14=0 \end{array}$$

$5x=14$

$x=\frac{14}{5}$

Put $x=\frac{14}{5}$ in ①

$\frac{14}{5}-2y-6=0$

$\frac{14}{5}-6=2y$

$\frac{14-30}{5}=2y$

$-\frac{16}{5}=2y$

$y=-\frac{16}{5\times2}$

$y=-\frac{8}{5}$

$C\left(\frac{14}{5},-\frac{8}{3}\right)$

Slope of $AB = l_1 = m_1 = \dfrac{\frac{18}{5}+\frac{21}{5}}{\frac{1}{5}+\frac{12}{5}} \Rightarrow \dfrac{\frac{18+21}{5}}{\frac{1+12}{5}}$

$\Rightarrow \frac{39}{13} \Rightarrow m_1=3$

Slope of $BC = l_2 = m_2 = \dfrac{-\frac{8}{5}-\frac{18}{5}}{\frac{14}{5}-\frac{1}{5}} \Rightarrow \dfrac{\frac{-8-18}{5}}{\frac{14-1}{5}}$

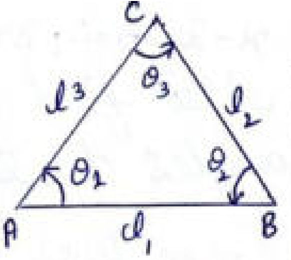
$\Rightarrow \frac{-26}{13} \Rightarrow m_2=-2$

Slope of $AC = l_3 = m_3 = \dfrac{-\frac{8}{5}+\frac{21}{5}}{\frac{14}{5}+\frac{12}{5}} \Rightarrow \dfrac{\frac{-8+21}{5}}{\frac{14+12}{5}}$

$\Rightarrow \frac{13}{26} \Rightarrow \frac{1}{2}=m_3$

$$\tan\theta_1 = \frac{m_3 - m_1}{1 + m_3 m_1}$$

$$\Rightarrow \frac{\frac{1}{2} - 3}{1 + (\frac{1}{2})(3)} \Rightarrow \frac{\frac{1-6}{2}}{\frac{2+3}{2}} \Rightarrow \frac{\frac{-5}{2}}{\frac{5}{2}}$$



$\tan\theta_1 = -1 \Rightarrow \theta_1 = \tan^{-1}(-1) \Rightarrow \theta_1 = 135^\circ$

Acute Angle : $180^\circ - 135^\circ$

$$\boxed{\theta_1 = 45^\circ}$$

$$\tan\theta_2 = \frac{m_1 - m_2}{1 + m_1 m_2}$$

$$\Rightarrow \frac{3-(-2)}{1+(3)(-2)} \Rightarrow \frac{3+2}{1-6} \Rightarrow \frac{5}{-5} \Rightarrow -1$$

$\tan\theta_2 = -1 \Rightarrow \theta_2 = \tan^{-1}(-1) \Rightarrow \theta_2 = 135^\circ$

Acute Angle : $= 180^\circ - 135^\circ$

$$\boxed{\theta_2 = 45^\circ}$$

$$\tan\theta_3 = \frac{m_2 - m_3}{1 + m_2 m_3}$$

$$\Rightarrow \frac{-2 - \frac{1}{2}}{1 + (-2)(\frac{1}{2})} \Rightarrow \frac{\frac{-4-1}{2}}{1-1} \Rightarrow \frac{-5/2}{0} = \infty$$

$\tan\theta_3 = \infty \Rightarrow \theta_3 = \tan^{-1}(\infty) \Rightarrow \boxed{\theta_3 = 90^\circ}$

Question no10:

Find the angle measured from the line l_1 to the line l_2 where

a) l_1 : Joining (2,7) and (7,10)
l_2 : Joining (1, 1) and (-5,3)

Slope of $l_2 = m_1 = \frac{10-7}{7-2} \Rightarrow \frac{3}{5}$

Slope of $l_2 = m_2 = \frac{3-1}{-5-1} \Rightarrow \frac{2}{-6} \Rightarrow \frac{-1}{3}$

If θ is the angle from l_1 to l_2

$$\tan\theta = \frac{m_2 - m_1}{1 + m_2 m_1}$$

$$\Rightarrow \frac{-\frac{1}{3} - \frac{3}{5}}{1 + (\frac{-1}{3})(\frac{3}{5})} \Rightarrow \frac{\frac{-5-9}{15}}{1 - \frac{1}{5}} \Rightarrow \frac{-\frac{14}{15}}{\frac{4}{5}}$$

$$\Rightarrow -\frac{14}{15} \times \frac{5}{4} \Rightarrow \frac{-7}{6} = \tan\theta$$

$\theta = \tan^{-1}(-\frac{7}{6})$

$\theta = -49^\circ 23'$

$\theta = 180^\circ - 49^\circ 23' \Rightarrow \boxed{\theta = 130^\circ 37'}$

Acute Angle:

$\theta = 180^\circ - 130^\circ 37'$

$$\boxed{\theta = 49^\circ 23'}$$

b) l_1 : Joining (3,-1) and (5,7)
l_2 : Joining (2,4) and (-8,2)

Slop of $l_1 = m_1 = \frac{7+1}{5-3} \Rightarrow \frac{8}{2} \Rightarrow 4$

Slop of $l_2 = m_2 = \frac{2-4}{-8-2} \Rightarrow \frac{-2}{-10} \Rightarrow \frac{1}{5}$

$$\tan\theta = \frac{m_2 - m_1}{1 + m_2 m_1}$$

$$\Rightarrow \frac{\frac{1}{5} - 4}{1 + (\frac{1}{5})(4)} \Rightarrow \frac{\frac{1-20}{5}}{1 + \frac{4}{5}} \Rightarrow \frac{\frac{-19}{5}}{\frac{5+4}{5}} \Rightarrow \frac{\frac{-19}{5}}{\frac{9}{5}}$$

$\tan\theta = \frac{-19}{9} \Rightarrow \theta = \tan^{-1}(\frac{-19}{9})$

$\theta = -64^\circ 39'$

$\theta = 180^\circ - 64^\circ 39' \Rightarrow \boxed{\theta = 115^\circ 20'}$

Acute angle

$\theta = 180^\circ - 115^\circ 20'$

$$\boxed{\theta = 64^\circ 39'}$$

c) l_1 : Joining (1,-7) and (6,-4)
l_2 : Joining (-1,2) and (-6,-1)

Slope of $l_1 = m_1 = \frac{-4+7}{6-1} \Rightarrow \frac{3}{5}$

Slope of $l_2 = m_2 = \frac{-1-2}{-6+1} \Rightarrow \frac{-3}{-5} \Rightarrow \frac{3}{5}$

$$\tan\theta = \frac{m_2 - m_1}{1 + m_2 m_1}$$

$$\Rightarrow \frac{\frac{3}{5} - \frac{3}{5}}{1 + (\frac{3}{5})(\frac{3}{5})} \Rightarrow \frac{\frac{3-3}{5}}{1 + \frac{9}{25}} \Rightarrow \frac{\frac{0}{5}}{\frac{25+9}{25}} \Rightarrow \frac{0}{34/25}$$

$\tan\theta = 0 \Rightarrow \theta = \tan^{-1} 0$

$\theta = 0^\circ$

This is also an acute angle

d) l_1 : Joining (-9,-1) and (3,-5)

l_2 : Joining (2,7) and (-6,-7)

Slope of $l_1 = m_1 = \frac{-5+1}{3+9} \Rightarrow \frac{-4}{12} \Rightarrow -\frac{1}{3}$

Slope of $l_2 = m_2 = \frac{-7-7}{-6-2} \Rightarrow \frac{-14}{-8} \Rightarrow \frac{7}{4}$

$$\tan\theta = \frac{m_2-m_1}{1+m_2m_1}$$

$$\Rightarrow \frac{\frac{7}{4}-\left(-\frac{1}{3}\right)}{1+\left(\frac{7}{4}\right)\left(-\frac{1}{3}\right)} \Rightarrow \frac{\frac{7}{4}+\frac{1}{3}}{1-\frac{7}{12}} \Rightarrow \frac{\frac{21+4}{12}}{\frac{12-7}{12}}$$

$\tan\theta = \frac{25}{5} \Rightarrow \tan\theta = 5$

$\theta = \tan^{-1}(5) \Rightarrow \theta = 78°41'$

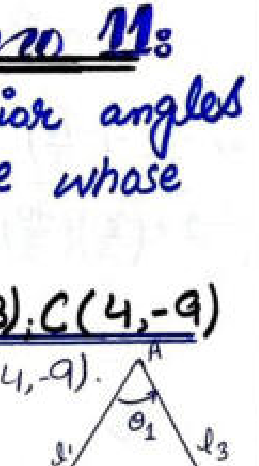
This is also acute angle.

Question no 11:

Find the interior angles of the triangle whose vertices are

(a) A(-2,11), B(-6,-3), C(4,-9)

A(-2,11), B(-6,-3), C(4,-9).



Slope of AB $= m_1 = \frac{-3-11}{-6+2}$

$m_1 \Rightarrow \frac{-14}{-4} \Rightarrow \boxed{\frac{7}{2} = m_1}$

Slope of BC $= m_2 = \frac{-9+3}{4+6}$

$m_2 = \frac{-6}{10} \Rightarrow \boxed{\frac{-3}{5} = m_2}$

Slope of AC $= \frac{-9-11}{4+2} \Rightarrow \frac{-20}{6} \Rightarrow \boxed{m_3 = \frac{-10}{3}}$

∴ θ_1 is the angle from l_1 to l_3

$$\tan\theta_1 = \frac{m_3-m_1}{1+m_3m_1}$$

$$\Rightarrow \frac{-\frac{10}{3}-\frac{7}{2}}{1+\left(\frac{-10}{3}\right)\left(\frac{7}{2}\right)} \Rightarrow \frac{\frac{-20-21}{6}}{1-\frac{70}{6}} \Rightarrow \frac{\frac{-41}{6}}{\frac{-64}{6}} \Rightarrow \frac{41}{64}$$

$\tan\theta_1 = \frac{41}{64} \Rightarrow \theta_1 = \tan^{-1}\left(\frac{41}{64}\right)$

$\boxed{\theta_1 = 32.64°}$

∴ θ_2 is the angle from l_2 to l_1

$$\tan\theta_2 = \frac{m_1-m_2}{1+m_1m_2}$$

$$\Rightarrow \frac{\frac{7}{2}-\left(\frac{-3}{5}\right)}{1+\left(\frac{7}{2}\right)\left(\frac{-3}{5}\right)} \Rightarrow \frac{\frac{7}{2}+\frac{3}{5}}{1-\frac{21}{10}} \Rightarrow \frac{\frac{35+6}{10}}{\frac{10-21}{10}} \Rightarrow \frac{\frac{41}{10}}{\frac{-11}{10}}$$

$\tan\theta_2 = -\frac{41}{11} \Rightarrow \theta_2 = \tan^{-1}\left(\frac{-41}{11}\right)$

$\Rightarrow \theta_2 = -74.98°$

$\Rightarrow \theta_2 = 180° - 74.98° \Rightarrow \boxed{\theta_2 = 105°}$

∴ θ_3 is the angle from l_3 to l_2

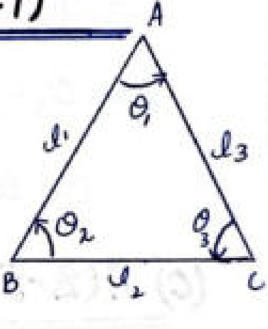
$$\tan\theta_3 = \frac{m_2-m_3}{1+m_2m_3}$$

$$\Rightarrow \frac{\frac{-3}{5}-\left(-\frac{10}{3}\right)}{1+\left(\frac{-3}{5}\right)\left(\frac{-10}{3}\right)} \Rightarrow \frac{\frac{-3}{5}+\frac{10}{3}}{1+2} \Rightarrow \frac{\frac{-9+50}{15}}{3}$$

$$\Rightarrow \frac{\frac{41}{15}}{3} \Rightarrow \frac{41}{15} \times \frac{1}{3} \Rightarrow \frac{41}{45} = \tan\theta_3$$

$\theta_3 = \tan^{-1}\left(\frac{41}{45}\right) \Rightarrow \boxed{\theta_3 = 42.34°}$

(b) A(6,1); B(2,7); C(-6,-7)



Slope of AB $= m_1 = \frac{7-1}{2-6}$

$\Rightarrow \frac{6}{-4} \Rightarrow \boxed{m_1 = \frac{-3}{2}}$

Slope of BC $= m_2 = \frac{-7-7}{-6-2}$

$\Rightarrow \frac{-14}{-8} \Rightarrow \boxed{m_2 = \frac{7}{4}}$

Slope of AC $= m_3 = \frac{-7-1}{-6-6} \Rightarrow \frac{-8}{-12} \Rightarrow \boxed{\frac{2}{3} = m_3}$

∴ θ_1 is the angle from l_1 to l_3

$$\therefore \tan\theta_1 = \frac{m_3-m_1}{1+m_3m_1}$$

$$\Rightarrow \frac{\frac{2}{3}-\left(\frac{-3}{2}\right)}{1+\left(\frac{2}{3}\right)\left(\frac{-3}{2}\right)} \Rightarrow \frac{\frac{2}{3}+\frac{3}{2}}{1-1} \Rightarrow \frac{\frac{4+9}{6}}{0}$$

$\tan\theta_1 = \infty \Rightarrow \theta_1 = \tan^{-1}(\infty)$.

$\boxed{\theta_1 = 90°}$

∴ θ_2 is the angle from l_2 to l

$$\tan\theta_2 = \frac{m_1 - m_2}{1 + m_1 m_2}$$

$$\Rightarrow \frac{\frac{-3}{2} - \frac{7}{4}}{1 + \left(\frac{-3}{2}\right)\left(\frac{7}{4}\right)} \Rightarrow \frac{\frac{-6-7}{4}}{1 - \frac{21}{8}} \Rightarrow \frac{\frac{-13}{4}}{\frac{8-21}{8}}$$

$$\Rightarrow \frac{\frac{-13}{4}}{\frac{-13}{8}} \Rightarrow -\frac{13}{4} \times -\frac{8}{13} \Rightarrow -2$$

$$\tan\theta_2 = -2 \Rightarrow \theta_2 = \tan^{-1}(-2)$$

$$\theta_2 = -63{\cdot}43^\circ$$

$$\theta_2 = 180^\circ - 63{\cdot}43^\circ \Rightarrow \theta_2 = 116{\cdot}6^\circ$$

<u>Acute Angles</u>: $\theta_2 = 180^\circ - 116{\cdot}6^\circ$

$$\boxed{\theta_2 = 63{\cdot}44^\circ}$$

∴ θ_3 is the angle from l_3 to l_2

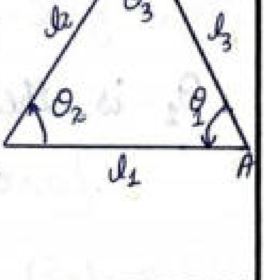
$$\tan\theta_3 = \frac{m_2 - m_3}{1 + m_2 m_3}$$

$$\Rightarrow \frac{\frac{7}{4} - \frac{2}{3}}{1 + \left(\frac{7}{4}\right)\left(\frac{2}{3}\right)} \Rightarrow \frac{\frac{21-8}{12}}{1 + \frac{7}{6}} \Rightarrow \frac{\frac{13}{12}}{\frac{13}{6}} \Rightarrow$$

$$\tan\theta_3 = \frac{13}{12} \times \frac{6}{13} \Rightarrow \tan\theta_3 = \frac{1}{2}$$

$$\theta_3 = \tan^{-1}\left(\frac{1}{2}\right) \Rightarrow \boxed{\theta_3 = 26{\cdot}57^\circ}$$

(c) A(2,-5); B(-4,-3); C(-1,5)



Slope of AB $= \frac{-3-(-5)}{-4-2}$

$$\Rightarrow \frac{-3+5}{-6} \Rightarrow \frac{2}{-6} \Rightarrow \boxed{m_1 = -\frac{1}{3}}$$

Slope of BC $= \frac{5-(-3)}{-1-(-4)}$

$$\Rightarrow \frac{5+3}{-1+4} \Rightarrow \boxed{m_2 = \frac{8}{3}}$$

Slope of AC $= \frac{5-(-5)}{-1-2} \Rightarrow \frac{5+5}{-3} \Rightarrow \boxed{m_3 = \frac{10}{-3}}$

∴ θ_1 is the angle from l_3 to l_1

$$\tan\theta_1 = \frac{m_1 - m_3}{1 + m_1 m_3}$$

$$\Rightarrow \frac{-\frac{1}{3} - \left(-\frac{10}{3}\right)}{1 + \left(-\frac{1}{3}\right)\left(-\frac{10}{3}\right)} \Rightarrow \frac{-\frac{1}{3} + \frac{10}{3}}{1 + \frac{10}{9}} \Rightarrow \frac{\frac{-1+10}{3}}{\frac{9+10}{9}}$$

$$\tan\theta_1 = \frac{\frac{9}{3}}{\frac{19}{9}} = \frac{9}{3} \times \frac{9}{19}^{3} = \frac{27}{19}$$

$$\tan\theta_1 = \frac{27}{19} \Rightarrow \theta_1 = \tan^{-1}\left(\frac{27}{19}\right) \Rightarrow \boxed{\theta_1 = 54{\cdot}87^\circ}$$

∴ θ_2 is the angle from l_1 to l_3

$$\tan\theta_2 = \frac{m_3 - m_1}{1 + m_3 m_1}$$

$$\Rightarrow \frac{\frac{8}{3} - \left(\frac{-1}{3}\right)}{1 + \left(\frac{8}{3}\right)\left(\frac{-1}{3}\right)} \Rightarrow \frac{\frac{8}{3} + \frac{1}{3}}{1 - \frac{8}{9}} \Rightarrow \frac{\frac{8+1}{3}}{\frac{9-8}{9}} = \frac{\frac{9}{3}}{\frac{1}{9}}$$

$$\tan\theta_2 = \frac{9}{3} \times 9 \Rightarrow \tan\theta_2 = 27$$

$$\theta_2 = \tan^{-1}(27) = \boxed{\theta_2 = 87{\cdot}9^\circ}$$

∴ θ_3 is the angle from l_2 to l_3

$$\tan\theta_3 = \frac{m_3 - m_2}{1 + m_3 m_2}$$

$$\Rightarrow \frac{\frac{-10}{3} - \frac{8}{3}}{1 + \left(\frac{-10}{3}\right)\left(\frac{8}{3}\right)} \Rightarrow \frac{\frac{-10}{3} - \frac{8}{3}}{1 - \frac{80}{9}} \Rightarrow \frac{\frac{-10-8}{3}}{\frac{9-80}{9}}$$

$$\Rightarrow \frac{\frac{-18}{3}}{\frac{-71}{9}} \Rightarrow \frac{-18}{3} \times \frac{9}{-71}^{3} = \frac{54}{71}$$

$$\tan\theta_3 = \frac{54}{71} \Rightarrow \theta_3 = \tan^{-1}\left(\frac{54}{71}\right)$$

$$\boxed{\theta_3 = 37{\cdot}2^\circ}$$

(d) A(2,8); B(-5,4); C(4,-9)

Slope of AB $= m_1 = \frac{4-8}{-5-2}$

$$m_1 = \frac{-4}{-7} \Rightarrow \boxed{m_1 = \frac{4}{7}}$$

Slope of BC $= m_2 = \frac{-9-4}{4-(-5)}$

$$m_2 = \frac{-13}{4+5} \Rightarrow \boxed{m_2 = \frac{-13}{9}}$$

Slope of AC $= m_3 = \frac{-9-8}{4-2}$

$$\boxed{m_3 = -\frac{17}{2}}$$

∴ θ_1 is the angle from l_1 to l_2

$$\tan\theta_1 = \frac{m_2 - m_1}{1 + m_2 m_1}$$

$$\Rightarrow \frac{\frac{-13}{9} - \frac{4}{7}}{1 + \left(\frac{-13}{9}\right)\left(\frac{4}{7}\right)} \Rightarrow \frac{\frac{-91-36}{63}}{1 - \frac{54}{63}} \Rightarrow \frac{\frac{-127}{63}}{\frac{9}{63}}$$

$$\tan\theta_1 = \frac{-127}{9} \Rightarrow \theta_1 = \tan^{-1}\left(\frac{-127}{9}\right)$$

$$\theta_1 = -85.95^\circ$$

$$\theta_1 = 180^\circ - 85.95^\circ \Rightarrow \theta_1 = 94.05^\circ$$

Acute Angle:

$$\theta_1 = 180^\circ - 94.05^\circ$$

$$\boxed{\theta_1 = 85.95^\circ}$$

∴ θ_2 is the angle from l_3 to l_1

$$\tan\theta_2 = \frac{m_1 - m_3}{1 + m_1 m_3}$$

$$\Rightarrow \frac{\frac{4}{7} - \left(\frac{-17}{2}\right)}{1 + \left(\frac{4}{7}\right)\left(\frac{-17}{2}\right)} \Rightarrow \frac{\frac{4}{7} + \frac{17}{2}}{1 - \frac{68}{14}} \Rightarrow \frac{\frac{8+119}{14}}{\frac{14-68}{14}}$$

$$\tan\theta_2 = \frac{-127}{54} \Rightarrow \theta_2 = -66.57^\circ$$

$$\Rightarrow \theta_2 = 180^\circ - 66.57^\circ \Rightarrow \theta_2 = 113.04^\circ$$

Acute Angle:

$$\theta_2 = 180^\circ - 113.04^\circ$$

$$\boxed{\theta_2 = 66.96^\circ}$$

∴ θ_3 is the angle from l_2 to l_3

$$\tan\theta_3 = \frac{m_3 - m_2}{1 + m_3 m_2}$$

$$\Rightarrow \frac{\frac{-17}{2} - \left(\frac{-13}{9}\right)}{1 + \left(\frac{-17}{2}\right)\left(\frac{-13}{9}\right)} \Rightarrow \frac{\frac{-17}{2} + \frac{13}{9}}{1 + \frac{221}{18}} \Rightarrow \frac{\frac{-156+26}{18}}{\frac{18+221}{18}}$$

$$\tan\theta_3 = \frac{-127}{239} \Rightarrow \theta_3 = -27.99^\circ$$

$$\theta_3 = 180^\circ - 27.99^\circ \Rightarrow \theta_3 = 152^\circ$$

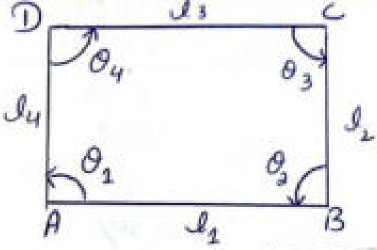
Acute Angle:

$$\theta_3 = 180^\circ - 152^\circ$$

$$\boxed{\theta_3 = 28^\circ}$$

Question no 12:

Find the interior angles of the quadrilateral whose vertices are A(5,2); B(-2,3); C(-3;-4); D(4;-5)



Slope of AB = $m_1 = \frac{3-2}{-2-5} \Rightarrow \boxed{\frac{-1}{7} = m_1}$

Slope of BC = $m_2 = \frac{-4-3}{-3-(-2)} \Rightarrow \frac{-7}{-3+2} \Rightarrow \frac{-7}{-1} \Rightarrow \boxed{7 = m_2}$

Slope of CD = $m_3 = \frac{-5-(-4)}{4-(-3)} \Rightarrow \frac{-5+4}{4+3} \Rightarrow \boxed{\frac{-1}{7} = m_3}$

Slope of DA = $m_4 = \frac{2-(-5)}{5-4} \Rightarrow \frac{2+5}{1} \Rightarrow \boxed{7 = m_4}$

∴ θ_1 is angle from l_1 to l_4

$$\tan\theta_1 = \frac{m_4 - m_1}{1 + m_4 m_1}$$

$$\tan\theta_1 = \frac{7 - \left(-\frac{1}{7}\right)}{1 + (7)\left(-\frac{1}{7}\right)}$$

$$\tan\theta_1 = \frac{1 + \frac{1}{7}}{1 - 1}$$

$$\tan\theta_1 = \frac{\frac{7+1}{7}}{0}$$

$$\tan\theta_1 = \infty$$

$$\boxed{\theta_1 = 90^\circ}$$

∴ θ_2 is angle from l_2 to l_1

$$\tan\theta_2 = \frac{m_1 - m_2}{1 + m_1 m_2}$$

$$\tan\theta_2 = \frac{\frac{-1}{7} - 7}{1 + \left(-\frac{1}{7}\right)(7)}$$

$$\tan\theta_2 = \frac{\frac{-1-49}{7}}{1-1}$$

$$\tan\theta_2 = \frac{-\frac{50}{7}}{0}$$

$$\tan\theta_2 = \infty$$

$$\boxed{\theta_2 = 90^\circ}$$

∴ θ_3 is angle from l_3 to l_2

$$\tan\theta_3 = \frac{m_2 - m_3}{1 + m_2 m_3}$$

$$\tan\theta_3 = \frac{7 - (-\frac{1}{7})}{1 + (7)(-\frac{1}{7})}$$

$$\tan\theta_3 = \frac{7 + 1/7}{1 - 1}$$

$$\tan\theta_3 = \frac{\frac{49+1}{7}}{0}$$

$$\tan\theta_3 = \infty$$

$$\boxed{\theta_3 = 90^\circ}$$

∴ θ_4 is angle from l_4 to l_3

$$\tan\theta_4 = \frac{m_3 - m_4}{1 + m_3 m_4}$$

$$\tan\theta_4 = \frac{-\frac{1}{7} - 7}{1 + (-\frac{1}{7})(7)}$$

$$\tan\theta_4 = \frac{\frac{-1-49}{7}}{1-1}$$

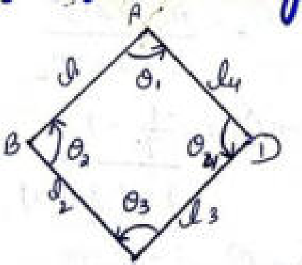
$$\tan\theta_4 = \frac{-50/7}{0}$$

$$\tan\theta_4 = \infty$$

$$\boxed{\theta_4 = 90^\circ}$$

Question no 13:

Show that the points $A(0,0)$; $B(2,1)$; $C(3,3)$; $D(1,2)$ are the vertices of rhombus and find its interior angles



$A(0,0)$; $B(2,1)$; $C(3,3)$; $D(1,2)$

Slope of $AB(l_1) = m_1 = \frac{1-0}{2-0} \Rightarrow \boxed{m_1 = \frac{1}{2}}$

Slope of $BC(l_2) = m_2 = \frac{3-1}{3-2} \Rightarrow \boxed{m_2 = 2}$

Slope of $CD(l_3) = m_3 = \frac{2-3}{1-3} \Rightarrow \boxed{m_3 = \frac{1}{2}}$

Slope of $AD(l_4) = m_4 = \frac{2-0}{1-0} \Rightarrow \boxed{m_4 = 2}$

∴ θ_1 is the angle from l_1 to l_4

$$\tan\theta_1 = \frac{m_4 - m_1}{1 + m_4 m_1}$$

$$\Rightarrow \frac{2 - \frac{1}{2}}{1 + (2)(\frac{1}{2})} \Rightarrow \frac{\frac{4-1}{2}}{1+1} \Rightarrow \frac{3/2}{2} \Rightarrow \frac{3}{2} \times \frac{1}{2}$$

$$\tan\theta_1 = \frac{3}{4} \Rightarrow \theta_1 = \tan^{-1}\left(\frac{3}{4}\right)$$

$$\boxed{\theta_1 = 36^\circ 52'}$$

∴ θ_2 is the angle from l_2 to l_1

$$\tan\theta_2 = \frac{m_1 - m_2}{1 + m_1 m_2}$$

$$\Rightarrow \frac{\frac{1}{2} - 2}{1 + (\frac{1}{2})(2)} \Rightarrow \frac{\frac{1-4}{2}}{1+1} \Rightarrow \frac{-3/2}{2} \Rightarrow \frac{-3}{2} \times \frac{1}{2}$$

$$\tan\theta_2 = \frac{-3}{4} \Rightarrow \theta_2 = \tan^{-1}\left(-\frac{3}{4}\right)$$

$$\Rightarrow \theta_2 = -36^\circ 52' \Rightarrow \theta_2 = 180^\circ - 36^\circ 52'$$

$$\boxed{\theta_2 = 143^\circ 8'}$$

∴ θ_3 is the angle from l_3 to l_2

$$\tan\theta_3 = \frac{m_2 - m_3}{1 + m_2 m_3}$$

$$\Rightarrow \frac{2 - \frac{1}{2}}{1 + (2)(\frac{1}{2})} \Rightarrow \frac{\frac{4-1}{2}}{1+1} \Rightarrow \frac{\frac{3}{2}}{2} \Rightarrow \frac{3}{2} \times \frac{1}{2}$$

$$\tan\theta_3 = \frac{3}{4} \Rightarrow \theta_3 = \tan^{-1}\left(\frac{3}{4}\right)$$

$$\boxed{\theta_3 = 36^\circ 52'}$$

∴ θ_4 is the angle from l_4 to l_3

$$\tan\theta_4 = \frac{m_3 - m_4}{1 + m_3 m_4}$$

$$\Rightarrow \frac{\frac{1}{2} - 2}{1 + (\frac{1}{2})(2)} \Rightarrow \frac{\frac{1-4}{2}}{1+1} \Rightarrow \frac{-3/2}{2} \Rightarrow \frac{-3}{2} \times \frac{1}{2}$$

$$\tan\theta_4 = \frac{-3}{4} \Rightarrow \theta_4 = \tan^{-1}\left(-\frac{3}{4}\right)$$

$$\Rightarrow \theta_4 = -36^\circ 52'$$

$$\Rightarrow \theta_4 = 180^\circ - 36^\circ 52'$$

$$\boxed{\theta_4 = 143^\circ 8'}$$

refo

Question no 14:

Find the area of the region bounded by the triangle whose sides are

$7x - y - 10 = 0 ; 10x + y - 41 = 0 ; 3x + 2y + 3 = 0$

The three lines are given

$7x - y - 10 = 0$ —①

$10x + y - 41 = 0$ —②

$3x + 2y + 3 = 0$ —③

By Solving these equations

By solving ① and ②

① + ②

$7x - \cancel{y} - 10 = 0$

$10x \cancel{+y} - 41 = 0$

$17x - 51 = 0$

$17x = 51$

$\boxed{x_1 = 3}$

Put x=3 in ①

$7(3) - y - 10 = 0$

$21 - y - 10 = 0$

$-y + 11 = 0$

$\boxed{y_1 = 11}$

(3, 11)

By solving ② and ③

2② - ③

$20x + 2\cancel{y} - 82 = 0$

$\underline{-}3x \underline{+}\cancel{2y} \underline{+} 3 = 0$

$17x - 85 = 0$

$17x = 85$

$\boxed{x_2 = 5}$

Put x=5 in ③

$3(5) + 2y + 3 = 0$

$15 + 2y + 3 = 0$

$2y + 18 = 0$

$2y = -18$

$\boxed{y_2 = -9}$

(5, -9)

By solving ③ and ①

2① + ③

$14x - \cancel{2y} - 20 = 0$

$3x + \cancel{2y} + 3 = 0$

$17x - 17 = 0$

$17x = 17$

$\boxed{x_3 = 1}$

Put x=1 in ①

$7(1) - y - 10 = 0$

$7 - y - 10 = 0$

$-y - 3 = 0$

$-y = 3$

$\boxed{y_3 = -3}$

(1, -3)

Area:

$$\therefore \Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

$$\Delta = \frac{1}{2}\begin{vmatrix} 3 & 11 & 1 \\ 5 & -9 & 1 \\ 1 & -3 & 1 \end{vmatrix}$$

Expand by R_1

$$\Delta = \frac{1}{2}\left[3(-9+3) - 11(5-1) + 1(-15+9)\right]$$

$$\Delta = \frac{1}{2}\left[3(-6) - 11(4) + 1(-6)\right]$$

$$\Delta = \frac{1}{2}\left[-18 - 44 - 6\right]$$

$$\Delta = \frac{1}{2}(-68)$$

$$\Delta = -34$$

$\boxed{\Delta = 34 \text{ square unit}}$

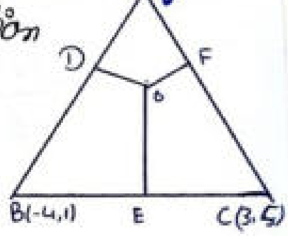
(∴ Area is always positive)

refo

Question no 15:

The vertices of the triangle are A(-2;3), B(-4;1) and C(3;5). Find the centre of the circum centre of the triangle.

=> The point of intersection of right bisector is called circumcentre.



A(-2;3), B(-4,1), C(3,5)

∴ D is the midpoint of AB

So coordinates of D are

$$= \left(\frac{-2-4}{2} , \frac{3+1}{2}\right) \Rightarrow \left(\frac{-6}{2} ; \frac{4}{2}\right) \Rightarrow D(-3;2)$$

$$\text{Slope of } AB = \frac{1-3}{-4+2} \Rightarrow \frac{-2}{-2} \Rightarrow 1$$

$$\text{Slope of right bisector } OD = -\frac{1}{1} \Rightarrow -1 = m$$

Eq of OD is

By point-slope form:

$y - 2 = -1(x - (-3))$ ∴ $y - y_1 = m(x - x_1)$

$y - 2 = -1(x + 3)$

$y - 2 = -x - 3$

$y - 2 + x + 3 = 0$

$x + y + 1 = 0$ —①

∴ E is the midpoint of BC
so coordinates of E are

$$= \left(\frac{-4+3}{2}; \frac{1+5}{2}\right) \Rightarrow \left(\frac{-1}{2}; \frac{6}{2}\right) \Rightarrow E\left(\frac{-1}{2}; 3\right)$$

Slope of BC = $\frac{5-1}{3+4} \Rightarrow \frac{4}{7}$

Slope of bisector OE = $\frac{-7}{4}$

Eq of OE is

By point slope forms

$$y-3 = \frac{-7}{4}\left(x-\left(-\frac{1}{2}\right)\right)$$

$$4(y-3) = -7\left(x+\frac{1}{2}\right)$$

$$4y-12 = -7x-\frac{7}{2}$$

$$4y-12+7x+\frac{7}{2} = 0$$

Multiply by 2

$$8y-24+14x+7=0$$

$$14x+8y-17=0 \text{ — } ②$$

By solving these equations

$$8① - ②$$

$$8x+\cancel{8y}+8=0$$

$$\underline{\underset{-}{}14x \underset{-}{+}\cancel{8y} \underset{+}{-} 17=0}$$

$$-6x \quad +25=0$$

$$-6x=-25$$

$$x=\frac{25}{6}$$

Put $x=\frac{25}{6}$ in ①

$$\frac{25}{6}+y+1=0$$

$$y=-\frac{25}{6}-1$$

$$y=\frac{-25-6}{6}$$

$$y=\frac{-31}{6}$$

$$\left(\frac{25}{6}; -\frac{31}{6}\right)$$

So $\left(\frac{25}{6}; -\frac{31}{6}\right)$ is the circum centre of the triangle

Question no 16:

Express the given system of equations in matrix form. Find in each case whether the lines are concurrent.

(a) $x+3y-2=0; 2x-y+4=0; x-11y+14=0$

The matrix form of the system

$$\begin{bmatrix} 1 & 3 & -2 \\ 2 & -1 & 4 \\ 1 & -11 & 14 \end{bmatrix}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Coefficient matrix of the system

$$A = \begin{bmatrix} 1 & 3 & -2 \\ 2 & -1 & 4 \\ 1 & -11 & 14 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & 3 & -2 \\ 2 & -1 & 4 \\ 1 & -11 & 14 \end{vmatrix}$$

Expand by R_1

$$= 1\begin{vmatrix} -1 & 4 \\ -11 & 14 \end{vmatrix} - 3\begin{vmatrix} 2 & 4 \\ 1 & 14 \end{vmatrix} - 2\begin{vmatrix} 2 & -1 \\ 1 & -11 \end{vmatrix}$$

$$= 1(-14+44) - 3(28-4) - 2(-22+1)$$

$$= 1(30) - 3(24) - 2(-21)$$

$$= 30-72+42$$

$$= 0$$

So the lines are concurrent.

(b) $2x+3y+4=0; x-2y-3=0; 3x+y-8=0$

Given lines

$$2x+3y+4=0$$

$$x-2y-3=0$$

$$3x+y-8=0$$

The matrix form of the system

$$\begin{bmatrix} 2 & 3 & 4 \\ 1 & -2 & -3 \\ 3 & 1 & -8 \end{bmatrix}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Coefficient matrix of the system

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & -2 & -3 \\ 3 & 1 & -8 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 2 & 3 & 4 \\ 1 & -2 & -3 \\ 3 & 1 & -8 \end{vmatrix}$$

Expand by R_1

$$= 2\begin{vmatrix} -2 & -3 \\ 1 & -8 \end{vmatrix} - 3\begin{vmatrix} 1 & -3 \\ 3 & -8 \end{vmatrix} + 4\begin{vmatrix} 1 & -2 \\ 3 & 1 \end{vmatrix}$$

$= 2(16+3) - 3(-8+9) + 4(1+6)$

$= 2(19) - 3(1) + 4(7)$

$= 38 - 3 + 28$

$= 63 \neq 0$

So the lines are not concurrent.

(C) $3x-4y-2=0;\ x+2y-4=0;\ 3x-2y+5=0$

Given lines

$3x - 4y - 2 = 0$

$x + 2y - 4 = 0$

$3x - 2y + 5 = 0$

The matrix form of the system

$$\begin{bmatrix} 3 & -4 & -2 \\ 1 & 2 & -4 \\ 3 & -2 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Coefficient matrix of the system

$$A = \begin{bmatrix} 3 & -4 & -2 \\ 1 & 2 & -4 \\ 3 & -2 & 5 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 3 & -4 & -2 \\ 1 & 2 & -4 \\ 3 & -2 & 5 \end{vmatrix}$$

Expand by R_1

$$= 3\begin{vmatrix} 2 & -4 \\ -2 & 5 \end{vmatrix} - (-4)\begin{vmatrix} 1 & -4 \\ 3 & 5 \end{vmatrix} - 2\begin{vmatrix} 1 & 2 \\ 3 & -2 \end{vmatrix}$$

$= 3(10-8) + 4(5+12) - 2(-2-6)$

$= 3(2) + 4(17) - 2(-8)$

$= 6 + 68 + 16$

$= 90 \neq 0$

So the lines are not concurrent.

Question no 17:

Find a system of linear equations corresponding to the given matrix form. Check whether the lines represented by the system are concurrent

(a) $$\begin{bmatrix} 1 & 0 & -1 \\ 2 & 0 & 1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -1 \\ 2 & 0 & 1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ — ①}$$

Multiply the matrixes on the L.H.S of ①, we have

$$\begin{bmatrix} x + 0y - 1 \\ 2x + 0y + 1 \\ 0x - y + 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By using the defination of equality of two matrices

$$x + 0y - 1 = 0$$
$$2x + 0y + 1 = 0$$
$$0x - y + 2 = 0$$

The coefficient matrix A of the system is such that

$$|A| = \begin{vmatrix} 1 & 0 & -1 \\ 2 & 0 & 1 \\ 0 & -1 & 2 \end{vmatrix}$$

Expand By C_2

$$= 0 + 0 - (-1)\begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix}$$

$$= 1(1+2)$$

$$= 1(3)$$

$$= 3 \neq 0$$

So the lines are not concurrent

(b) $$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 4 & -3 \\ 3 & 6 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 4 & -3 \\ 3 & 6 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ — ①}$$

Multiply the matrices on the L.H.S of ①; we have

$$\begin{bmatrix} x + y + 2 \\ 2x + 4y - 3 \\ 3x + 6y - 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By Using the defination of equality of two matrices

$$x + y + 2 = 0$$
$$2x + 4y - 3 = 0$$
$$3x + 6y - 5 = 0$$

The coefficient matrix A of the system is such that

$$|A| = \begin{vmatrix} 1 & 1 & 2 \\ 2 & 4 & -3 \\ 3 & 6 & -5 \end{vmatrix}$$

Expand by R_1

$$= 1\begin{vmatrix} 4 & -3 \\ 6 & -5 \end{vmatrix} - 1\begin{vmatrix} 2 & -3 \\ 3 & -5 \end{vmatrix} + 2\begin{vmatrix} 2 & 4 \\ 3 & 6 \end{vmatrix}$$

$$= 1(-20 + 18) - 1(-10 + 9) + 2(12 - 12)$$

$$= 1(-2) - 1(-1) + 2(0)$$

$$= -2 + 1 + 0$$

$$= -1 \neq 0$$

So the lines are not concurrent.

Theory:

- **Homogeneous Equation:** let $f(x,y)$ be any equation in the variables x and y, equation $f(x,y)=0$ is called a homogeneous equation of degree n (a positive integer) if

$$f(kx,ky)=k^n f(x,y) \text{ for some real number } k$$

Every homogeneous second degree equation $ax^2+2hxy+by^2=0$ represents a pair of lines through the origin.

The lines are

i) Real and distinct, if $h^2>ab$

ii) Real and coincident, if $h^2=ab$

iii) Imaginary, if $h^2<ab$

- **Measure of angle between the lines Represented by $ax^2+2hxy+by^2=0$:**

$$\tan\theta=\frac{2\sqrt{h^2-ab}}{a+b}$$

- Two lines are parallel, if $\theta=0°$ so that $\tan\theta=0$ which implies $h^2-ab=0$ which is condition for the lines to be coincident.
- Two lines are orthogonal ($\perp$), if $\theta=90°$ so that $\tan\theta=\infty$ (undefined) which implies $a+b=0$. Hence the condition for $ax^2+2hxy+by^2=0$ to represent a pair of orthogonal (perpendicular) lines is the sum of the coefficients of x^2 and y^2 is 0.

Example no 1: Find the equation of each of the lines represented by $20x^2+17xy-24y^2=0$

$$20x^2+17xy-24y^2=0$$
$$20x^2-15xy+32xy-24y^2=0$$
$$5x(4x-3y)+8y(4x-3y)=0$$
$$(4x-3y)(5x+8y)=0$$
$$4x-3y=0 \quad ; \quad 5x+8y=0$$

Required equations.

Example no 2: Find measure of the angle between the lines represented by $x^2 - xy - 6y^2 = 0$

$$x^2 - xy - 6y^2 = 0 \text{ — } ①$$

Compare ① with $ax^2 + 2hxy + by^2 = 0$

$a = 1$; $2h = -1$; $b = -6$

$h = \frac{-1}{2}$

$\therefore \tan\theta = \frac{2\sqrt{h^2 - ab}}{a+b}$

$$\tan\theta = \frac{2\sqrt{(-\frac{1}{2})^2 - (1)(-6)}}{1+(-6)} \Rightarrow \frac{2\sqrt{\frac{1}{4}+6}}{1-6} \Rightarrow \frac{2\sqrt{\frac{1+24}{4}}}{-5} \Rightarrow \frac{2\sqrt{\frac{25}{4}}}{-5}$$

$$\tan\theta = \frac{2\left(\frac{5}{2}\right)}{-5} \Rightarrow \tan\theta = \frac{5}{-5} \Rightarrow \tan\theta = -1 \Rightarrow \theta = \tan^{-1}(-1)$$

$$\Rightarrow \boxed{\theta = 135^\circ} \quad \text{Acute Angle} = 180^\circ - 135^\circ \Rightarrow \boxed{\theta = 45^\circ}$$

Example no 3: Find a joint equation of the straight lines through the origin perpendicular to the lines represented by $x^2 + xy - 6y^2 = 0$

Let $y = m_1 x$ and $y = m_2 x$ be two lines passing through the origin.

Now slope of lines ⊥ to given lines are $-\frac{1}{m_2}$ and $\frac{-1}{m_2}$.

then their equations are

$y = -\frac{1}{m_1}x$; $y = \frac{-1}{m_2}x$

$m_1 y = -x$; $m_2 y = -x$

$m_1 y + x = 0$; $m_2 y + x = 0$

$x + m_1 y = 0$; $x + m_2 y = 0$

Their joint equation

$(x + m_1 y)(x + m_2 y) = 0$

$x^2 + x m_2 y + x m_1 y + m_1 m_2 y^2 = 0$

$x^2 + m_2 xy + m_1 xy + m_1 m_2 y^2 = 0$

$x^2 + (m_2 + m_1) xy + m_1 m_2 y^2 = 0$

$x^2 + (m_1 + m_2) xy + m_1 m_2 y^2 = 0$ — ①

$x^2 + xy - 6y^2 = 0$

$a = 1$; $2h = 1$; $b = -6$

$m_1 + m_2 = -\frac{2h}{b} \Rightarrow \frac{-1}{-6} \Rightarrow \frac{1}{6}$

$m_1 m_2 = \frac{a}{b} \Rightarrow \frac{1}{-6}$

So ① becomes

$x^2 + \left(\frac{1}{6}\right)xy + \left(\frac{-1}{6}\right)y^2 = 0$

Multiply by "6"

$6x^2 + xy - y^2 = 0$

Exercise no 4.5

Question no 1:

$$10x^2-23xy-5y^2=0$$

$$10x^2-23xy-5y^2=0 \quad — ①$$

$$10x^2-25xy+2xy-5y^2=0$$

$$5x(2x-5y)+y(2x-5y)=0$$

$$(2x-5y)(5x+y)=0$$

$$2x-5y=0 \ ; \ 5x+y=0$$

Required pair of lines

Now Compare ① with

$$ax^2+2hxy+by^2=0$$

$$a=10 ; 2h=-23 ; b=-5$$

$$h=-\frac{23}{2}$$

$$\therefore \tan\theta = \frac{2\sqrt{h^2-ab}}{a+b}$$

$$\tan\theta = \frac{2\sqrt{\left(-\frac{23}{2}\right)^2-(10)(-5)}}{10+(-5)}$$

$$\tan\theta = \frac{2\sqrt{\frac{529}{4}+50}}{10-5}$$

$$\tan\theta = \frac{2}{5}\sqrt{\frac{529+200}{4}}$$

$$\tan\theta = \frac{2}{5}\sqrt{\frac{729}{4}}$$

$$\tan\theta = \frac{2}{5}\left(\frac{27}{2}\right)$$

$$\tan\theta = \frac{27}{5}$$

$$\theta = \tan^{-1}\left(\frac{27}{5}\right)$$

$$\theta = \tan^{-1}(5.4)$$

$$\boxed{\theta = 79.51^\circ}$$

Question no 2:

$$3x^2+7xy+2y^2=0$$

$$3x^2+7xy+2y^2=0 \quad — ①$$

$$3x^2+6xy+xy+2y^2=0$$

$$3x(x+2y)+y(x+2y)=0$$

$$(x+2y)(3x+y)=0$$

$$x+2y=0 \ ; \ 3x+y=0$$

Required pair of lines

Now compare ① with

$$ax^2+2hxy+by^2=0$$

$$a=3 ; 2h=7 ; b=2$$

$$h=\frac{7}{2}$$

$$\therefore \tan\theta = \frac{2\sqrt{h^2-ab}}{a+b}$$

$$\tan\theta = \frac{2\sqrt{\left(\frac{7}{2}\right)^2-(3)(2)}}{3+2}$$

$$\tan\theta = \frac{2\sqrt{\frac{49}{4}-6}}{5}$$

$$\tan\theta = \frac{2}{5}\sqrt{\frac{49-24}{4}}$$

$$\tan\theta = \frac{2}{5}\sqrt{\frac{25}{4}}$$

$$\tan\theta = \frac{2}{5}\left(\frac{5}{2}\right)$$

$$\tan\theta = 1$$

$$\theta = \tan^{-1}(1)$$

$$\boxed{\theta = 45^\circ}$$

Question no 3:

$$9x^2+24xy+16y^2=0$$

$$9x^2+24xy+16y^2=0$$

$$9x^2+12xy+12xy+16y^2=0$$

$$3x(3x+4y)+4y(3x+4y)=0$$

$$(3x+4y)(3x+4y)=0$$

$$3x+4y=0 \;;\; 3x+4y=0$$

Required pair of lines

The lines are real and coincident. So, for coincident lines $\boxed{\theta=0^\circ}$

Question no 4:

$$2x^2+3xy-5y^2=0$$

$$2x^2+3xy-5y^2=0 \quad \text{—} ①$$

$$2x^2+5xy-2xy-5y^2=0$$

$$x(2x+5y)-y(2x+5y)=0$$

$$(2x+5y)(x-y)=0$$

$$2x+5y=0 \;;\; x-y=0.$$

Required pair of lines.

Now compare ① with

$$ax^2+2hxy+by^2=0$$

$$a=2 \;;\; 2h=3 \;;\; b=-5$$

$$h=\frac{3}{2}$$

$$\therefore \tan\theta=\frac{2\sqrt{h^2-ab}}{a+b}$$

$$\tan\theta=\frac{2\sqrt{\left(\frac{3}{2}\right)^2-(2)(-5)}}{2+(-5)}$$

$$\tan\theta=\frac{2\sqrt{\frac{9}{4}+10}}{2-5}$$

$$\tan\theta=\frac{2\sqrt{\frac{9+40}{4}}}{-3}$$

$$\tan\theta=-\frac{2}{3}\sqrt{\frac{49}{4}}$$

$$\tan\theta=-\frac{2}{3}\left(\frac{7}{2}\right)$$

$$\tan\theta=-\frac{7}{3}$$

$$\theta=\tan^{-1}\left(-\frac{7}{3}\right)$$

$$\theta=-66.80^\circ$$

$$\theta=180^\circ-66.80^\circ$$

$$\boxed{\theta=113.2^\circ}$$

Question no 5:

$$6x^2-19xy+15y^2=0$$

$$6x^2-19xy+15y^2=0 \quad \text{—} ①$$

$$6x^2-10xy-9xy+15y^2=0$$

$$2x(3x-5y)-3y(3x-5y)=0$$

$$(3x-5y)(2x-3y)=0$$

$$3x-5y=0 \;;\; 2x-3y=0$$

Required pair of lines.

Now compare ① with

$$ax^2+2hxy+by^2=0$$

$$a=6 \;;\; 2h=-19 \;;\; b=15$$

$$h=-\frac{19}{2}$$

$$\therefore \tan\theta=\frac{2\sqrt{h^2-ab}}{a+b}$$

$$\tan\theta=\frac{2\sqrt{\left(-\frac{19}{2}\right)^2-(6)(15)}}{6+15}$$

$$\tan\theta=\frac{2\sqrt{\frac{361}{4}-90}}{21}$$

$$\tan\theta=\frac{2}{21}\sqrt{\frac{361-360}{4}}$$

$$\tan\theta=\frac{2}{21}\sqrt{\frac{1}{4}}$$

$$\tan\theta=\frac{2}{21}\left(\frac{1}{2}\right)$$

$$\boxed{\tan\theta=\frac{1}{21}}$$

$\theta = \tan^{-1}\left(\frac{1}{21}\right)$

$\boxed{\theta = 2.73^\circ}$

Question no 6:

$$x^2 + 2xy\sec\alpha + y^2 = 0$$

$$x^2 + 2xy\sec\alpha + y^2 = 0 \quad \text{—①}$$

$$y^2 + (2x\sec\alpha)y + x^2 = 0$$

By Using quodraatic formula

$a = 1;\ b = 2x\sec\alpha;\ c = x^2$

$$y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$y = \frac{-2x\sec\alpha \pm \sqrt{(2x\sec\alpha)^2 - 4(1)(x^2)}}{2(1)}$$

$$y = \frac{-2x\sec\alpha \pm \sqrt{4x^2\sec^2\alpha - 4x^2}}{2}$$

$$y = \frac{-2x\sec\alpha \pm \sqrt{4x^2(\sec^2\alpha - 1)}}{2}$$

$$y = \frac{-2x\sec\alpha \pm 2x\sqrt{\sec^2\alpha - 1}}{2}$$

$$y = \frac{-2x\sec\alpha \pm 2x\sqrt{\tan^2\alpha}}{2}$$

$$y = \frac{-2x\sec\alpha \pm 2x\tan\alpha}{2}$$

$$y = \frac{-2x(\sec\alpha \mp \tan\alpha)}{2}$$

$$y = -x(\sec\alpha \pm \tan\alpha)$$

$$y = -x\left(\frac{1}{\cos\alpha} \pm \frac{\sin\alpha}{\cos\alpha}\right)$$

$$y = -x\left(\frac{1 \pm \sin\alpha}{\cos\alpha}\right)$$

$y = -x\left(\frac{1+\sin\alpha}{\cos\alpha}\right);\ y = -x\left(\frac{1-\sin\alpha}{\cos\alpha}\right)$

$y\cos\alpha = -x(1+\sin\alpha);\ y\cos\alpha = -x(1-\sin\alpha)$

$y\cos\alpha + x(1+\sin\alpha) = 0;\ y\cos\alpha + x(1-\sin\alpha) = 0$

$x(1+\sin\alpha) + y\cos\alpha = 0;\ x(1-\sin\alpha) + y\cos\alpha = 0$

Required pair of lines

Now compare ① with

$$ax^2 + 2hxy + by^2 = 0$$

$a = 1;\ 2h = 2\sec\alpha;\ b = 1$

$h = \sec\alpha$

$$\therefore \tan\theta = \frac{2\sqrt{h^2 - ab}}{a+b}$$

$$\tan\theta = \frac{2\sqrt{(\sec\alpha)^2 - (1)(1)}}{1+1}$$

$$\tan\theta = \frac{2\sqrt{\sec^2\alpha - 1}}{2}$$

$$\tan\theta = \frac{\cancel{2}\sqrt{\tan^2\alpha}}{\cancel{2}}$$

$$\tan\theta = \tan\alpha$$

$\boxed{\theta = \alpha}$

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Question no 7:

Find a point equation of the lines through the origin and perpendicular to the lines: $x^2-2xy\tan\alpha-y^2=0$

Let $y=m_1x$ and $y=m_2x$ be two lines passing through the origin.

Now slope of lines ⊥ to given lines are $-\frac{1}{m_1}$ and $\frac{-1}{m_2}$, then their eq. are

$y=-\frac{1}{m_1}x$; $y=-\frac{1}{m_2}x$

$m_1y=-x$; $m_2y=-x$

$m_1y+x=0$; $m_2y+x=0$

$x+m_1y=0$; $x+m_2y=0$

Their joint equation

$(x+m_1y)(x+m_2y)=0$

$x^2+xm_2y+xm_1y+m_1m_2y^2=0$

$x^2+(m_1+m_2)xy+m_1m_2y^2=0$ —①

$x^2-2xy\tan\alpha-y^2=0$ —②

Compare ② with $ax^2+2hxy+by^2=0$

$a=1;\ 2h=-2\tan\alpha,\ b=-1$

$m_1+m_2=\frac{-2h}{b}$

$=-\frac{(-2\tan\alpha)}{-1}$

$=-2\tan\alpha$

$m_1m_2=\frac{a}{b}$

$=\frac{1}{-1}\Rightarrow -1$

Now ① becomes

$x^2+(2\tan\alpha)xy+(-1)y^2=0$

$\boxed{x^2-2\tan\alpha\, xy-y^2=0}$

Question no 8:

Find a joint equation of the lines through the origin and perpendicular - to the lines: $ax^2+2hxy+by^2=0$

Let $y=m_2x$ and $y=m_1x$ be two lines passing through the origin.

Now slope of lines ⊥ to given lines are $\frac{-1}{m_1}$ and $-\frac{1}{m_2}$, then their eq. are

$y=-\frac{1}{m_1}x$; $y=\frac{1}{m_2}x$

$m_1y=-x$; $m_2y=-x$

$m_1y+x=0$; $m_2y+x=0$

$x+m_1y=0$; $x+m_2y=0$

Their joint equation

$(x+m_1y)(x+m_2y)=0$

$x^2+xm_2y+xm_1y+m_1m_2y^2=0$

$x^2+(m_1+m_2)xy+m_1m_2y^2=0$ —①

$ax^2+2hxy+by^2=0$ —②

Compare ② with $ax^2+2hxy+by^2=0$

$a=a;\ 2h=2h;\ b=b$

$m_1+m_2=-\frac{2h}{b}$

$m_1m_2=\frac{a}{b}$

So ① becomes

$x^2-\frac{2h}{b}xy+\frac{a}{b}y^2=0$

Multiply by "b" on B.S

$\boxed{bx^2-2hxy+ay^2=0}$

Question no 9: Find the area of the region bounded by: $10x^2 - xy - 21y^2 = 0$ and $x + y + 1 = 0$

$10x^2 - xy - 21y^2 = 0$

$10x^2 - 15xy + 14xy - 21y^2 = 0$

$5x(2x - 3y) + 7y(2x - 3y) = 0$

$(2x - 3y)(5x + 7y) = 0$

$5x + 7y = 0$ — ①

$2x - 3y = 0$ — ②

$x + y + 1 = 0$ — ③

By Solving ① & ②

2① - 5②

$$\begin{array}{r} \cancel{10x} + 14y = 0 \\ \underline{\overset{-}{\cancel{10x}} \overset{+}{-} 15y = 0} \\ 29y = 0 \end{array}$$

$y = 0$

Put in ①

$5x + 7(0) = 0$

$5x = 0$

$x = 0$

Point of intersection of ①, ② is $(0, 0)$

By Solving ② & ③

② + 3③

$$\begin{array}{r} 2x - \cancel{3y} = 0 \\ \underline{3x + \cancel{3y} + 3 = 0} \\ 5x + 3 = 0 \end{array}$$

$5x = -3$

$x = \frac{-3}{5}$

Put in ③

$\frac{-3}{5} + y + 1 = 0$

$y = \frac{3}{5} - 1$

$y = \frac{3 - 5}{5}$

$y = \frac{-2}{5}$

Point of intersection of ② & ③ is $(\frac{-3}{5}, \frac{-2}{5})$.

By Solving ① & ③

① - 5③

$$\begin{array}{r} \cancel{5x} + 7y = 0 \\ \underline{\overset{-}{\cancel{5x}} \overset{-}{+} 5y \overset{-}{+} 5 = 0} \\ 2y - 5 = 0 \end{array}$$

$2y = 5$

$y = \frac{5}{2}$

Put in ①

$5x + 7\left(\frac{5}{2}\right) = 0$

$5x + \frac{35}{2} = 0$

$5x = -\frac{35}{2}$

$x = \frac{-35}{2 \times 5}$

$x = \frac{-7}{2}$

Point of intersection of ① & ③ is $\left(\frac{-7}{2}; \frac{5}{2}\right)$

Area:

$$\therefore \Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

$$\Delta = \frac{1}{2}\begin{vmatrix} 0 & 0 & 1 \\ -\frac{3}{5} & -\frac{2}{5} & 1 \\ -\frac{7}{2} & \frac{5}{2} & 1 \end{vmatrix}$$

Expand by R_1

$$\Delta = \frac{1}{2}\left[0 - 0 + 1\begin{vmatrix} -\frac{3}{5} & -\frac{2}{5} \\ -\frac{7}{2} & \frac{5}{2} \end{vmatrix}\right]$$

$$= \frac{1}{2}\left[\left(-\frac{3}{5}\right)\left(\frac{5}{2}\right) - \left(\frac{-2}{5}\right)\left(\frac{-7}{2}\right)\right]$$

$$\Delta = \frac{1}{2}\left[-\frac{3}{2} - \frac{7}{5}\right]$$

$$\Delta = \frac{1}{2}\left[\frac{-15-14}{10}\right]$$

$$\Delta = \frac{1}{2}\left[-\frac{29}{10}\right]$$

$$\Delta = \frac{-29}{20}$$

$$\Delta = \frac{29}{20} \text{ square units}$$

(∴ Area is always positive)

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UNIT
5

Linear Inequalities
and
Linear Programming

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Unit no 5

Linear Inequatities and linear Programming

Theory:-

i) **Inequality:** Inequalities are the expressions invalving four symbols: greater than ($>$) ; greater then or equal to ($\geq$)
less than ($<$) ; less then or equal to ($\leq$)

For example: (i) $ax<b$ (ii) $2x+3y\leq 0$ (iii) $5x-2y>0$ (iv) $2x-4y\geq 0$

The fallowing operations will not offect the order or sense of inequality

(i) Adding or Subtracting a constant to each of it side
(ii) Multiplying or dividing each side of it by a positive constant

Note:

> Order or sense of equality is changed by Multiplying or dividing its each side by a negative constant.

ii) **Linear Inequality:** A linear inequality in two variables x and y can be one of the fallowing forms:
$ax+by<c$; $ax+by>c$; $ax+by\leq c$; $ax+by\geq c$
Where a, b & c are constants and a,b are not both zero

iii) **Salution of Linear Inequality:** A solution of linear inequality in x and y in an order pair of the numbers which satisfies the inequality

For Example: The order pair $(1,1)$ is a salution of the inequality $x+2y<6$

Because $(1)+2(1)\leq 6$
$1+2<6$
$3<6$ (True)

Note:

> There are infinite many order pairs that satisfy the linear inequality.

Graph of linear inequality is the half planes.

Associated or corresponding equation: The linear equation $ax+by=0$ is called associated or corresponding equation of linear inequalities

$ax+by<c$; $ax+by>c$; $ax+by\leq c$; $ax+by\geq c$.

Corner point or vertex: A point of the solution region where two of its boundary lines intersect, is called corner points or vertex.

* * *

Example 1: Graph the inequality $x+2y<6$

$$x+2y<6 \quad \text{——} ①$$

(Step I)

The associated of ① is

$$x+2y=6 \quad \text{——} ②$$

(Step II)

For x-intercept	For y-intercept
Put $y=0$ in ②	Put $x=0$ in ②
$x+2(0)=6$	$0+2y=6$
$x+0=6$	$2y=6$
$x=6$	$y=3$
(6,0)	(0,3)

(Step III)

Put test point (0,0) in ①

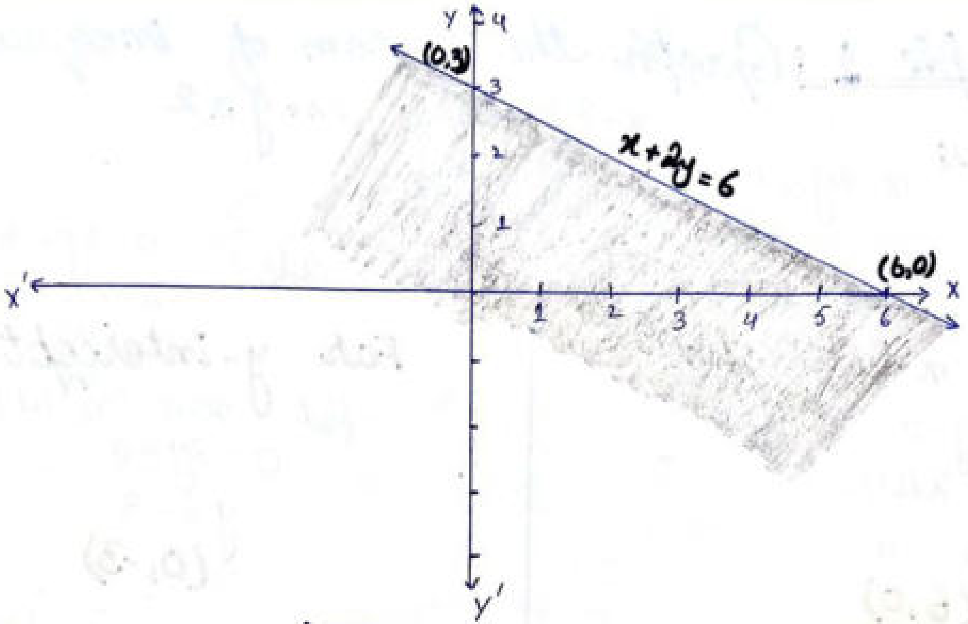
$$x+2y<6$$

$$0+2(0)<6$$

$$0+0<6$$

$$0<6$$

(True)



Example 2: Graph the fallowing linear inequalities in xy-planes;

(i) $2x \geq -3$ (ii) $y \leq 2$

Salution:

$2x \geq -3$ ——(i)

$y \leq 2$ ——(ii)

The associated of (i) & (ii)

$2x = -3$ $y = 2$

$x = -\frac{3}{2}$

$(-\frac{3}{2}, 0)$ $(0, 2)$

put test point (0,0) in (i) & (ii)

$2(0) \geq -3$ $0 \leq 2$

$0 \geq -3$

(True) (True)

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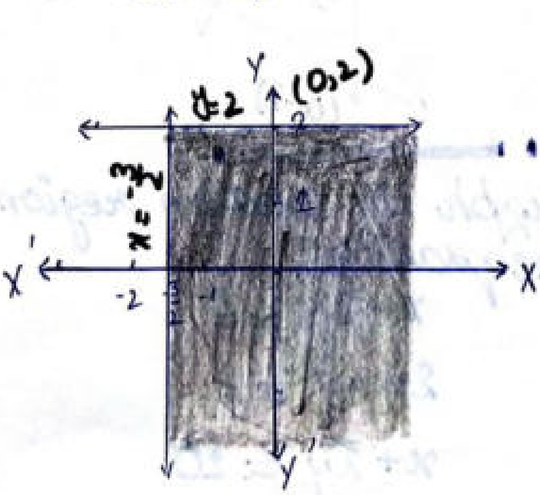


Example 1: Graph the system of inequalities

$x-2y \le 6$; $2x+y \ge 2$

Solution:

$x-2y \le 6$ —(i)

$2x+y \ge 2$ —(ii)

The associated of (i) & (ii) are: $x-2y=6$ —(a)

$2x+y=2$ —(b)

For x-intercept

Put $y=0$ in (a)

$x-2(0)=6$

$x=6$

(6,0)

Put $y=0$ in (b)

$2x+0=2$

$x=1$

(1,0)

For y-intercept

Put $x=0$ in (a)

$0-2y=6$

$y=-3$

(0,-3)

Put $x=0$ in (b)

$2(0)+y=2$

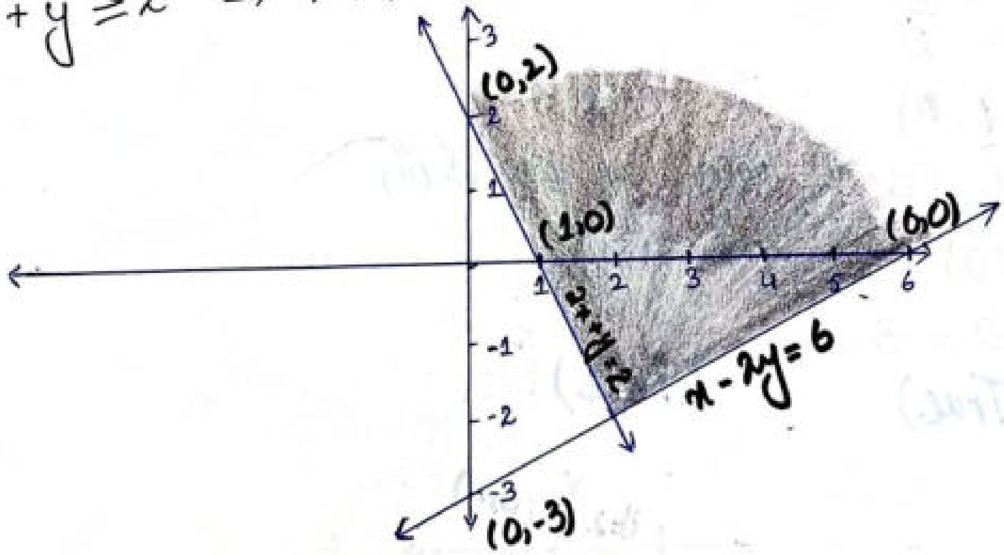
$y=2$

(0,2)

Put test point in (i) & (ii)

$x-2y \le 6 \Rightarrow 0-2(0) \le 6 \Rightarrow 0 \le 6$ (True)

$2x+y \ge 2 \Rightarrow 2(0)+0 \ge 2 \Rightarrow 0 \ge 2$ (False)



Example 2: Graph the solution region for the following system of inequalities.

$x-2y \le 6$

$2x+y \ge 2$

$x+2y \le 10$

Solution:

$x - 2y \leq 6$ —— ①

$2x + y \geq 2$ —— ②

$x + 2y \leq 10$ —— ③

The associated of ①, ② & ③ are

$x - 2y = 6$ —— (a)

$2x + y = 2$ —— (b)

$x + 2y = 10$ —— (c)

For x-intercept

Put $y = 0$ in (a)

$x - 2(0) = 6$

$x = 6$

(6,0)

Put $y=0$ in (b)

$2x + 0 = 2$

$x = 1$

(1,0)

Put $y = 0$ in (c)

$x + 2(0) = 10$

$x = 10$

(10,0)

For y-intercept

Put $x = 0$ in (a)

$0 - 2y = 6$

$y = -3$

(0,-3)

Put $x=0$ in (b)

$2(0) + y = 2$

$y = 2$

(0,2)

Put $x=0$ in (c)

$0 + 2y = 10$

$y = 5$

(0,5)

Put test point (0,0) in ①, ② & ③

$x - 2y \leq 6$

$0 - 2(0) \leq 6$

$0 \leq 6$

(True)

$2x + y \geq 2$

$2(0) + 0 \geq 2$

$0 \geq 2$

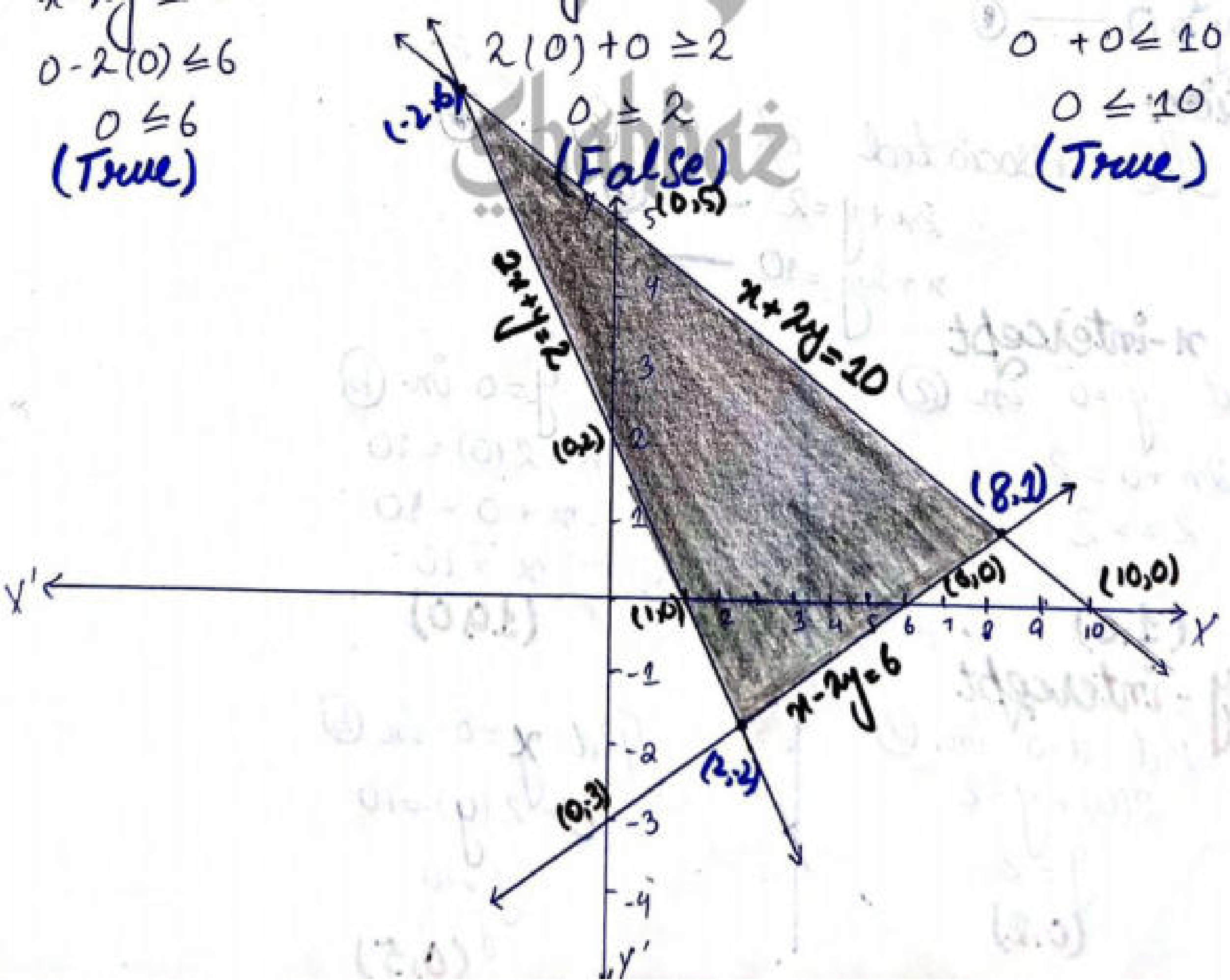
(False)

$0 + 2(0) \leq 10$

$0 + 0 \leq 10$

$0 \leq 10$

(True)



Corner points: By Solving equation ⓐ, ⓑ & ⓒ

By Solving ⓐ & ⓑ

$x-2y=6$ — ⓐ

$2x+y=2$ — ⓑ

2ⓐ - ⓑ

$$\cancel{2x}-4y=12$$
$$\cancel{2x}+y=2$$
$$-5y=10$$
$$\boxed{y=-2}$$

Put $y=-2$ in ⓐ

$x-2(-2)=6$

$x+4=6$

$x=6-4$

$\boxed{x=2}$ $(2,-2)$

By Solving ⓑ & ⓒ

$2x+y=2$ — ⓑ

$x+2y=10$ — ⓒ

ⓑ - 2ⓒ

$$\cancel{2x}+y=2$$
$$\cancel{2x}+4y=20$$
$$-3y=-18$$
$$\boxed{y=6}$$

Put $y=6$ in ⓑ

$2x+6=2$

$2x=2-6$

$2x=-4$

$\boxed{x=-2}$ $(-2,6)$

By Solving ⓐ & ⓒ

$x-2y=6$ — ⓐ

$x+2y=10$ — ⓒ

ⓐ + ⓒ

$$x-\cancel{2y}=6$$
$$x+\cancel{2y}=10$$
$$2x=16$$
$$\boxed{x=8}$$

Put $x=8$ in ⓐ

$8-2y=6$

$-2y=6-8$

$-2y=-2$

$(8,1)$ $\boxed{y=1}$

Example 3: Graph the fallowing systems of inequalities

i) $2x+y\geq 2$ — ①

$x+2y\leq 10$ — ②

$y\geq 0$ — ③

Solution:

The associated of ①, ② & ③

$2x+y=2$ — ⓐ

$x+2y=10$ — ⓑ

For x-intercept

Put $y=0$ in ⓐ

$2x+0=2$

$2x=2$

$x=1$

$(1,0)$

put $y=0$ in ⓑ

$x+2(0)=10$

$x+0=10$

$x=10$

$(10,0)$

For y-intercept

Put $x=0$ in ⓐ

$2(0)+y=2$

$y=2$

$(0,2)$

Put $y=0$ in ⓑ

$0+2(y)=10$

$2y=10$

$y=5$

$(0,5)$

Put test point (0,0). in ① & ②

$2x + y \geq 2$

$2(0) + 0 \geq 2$

$0 \geq 2$

(False)

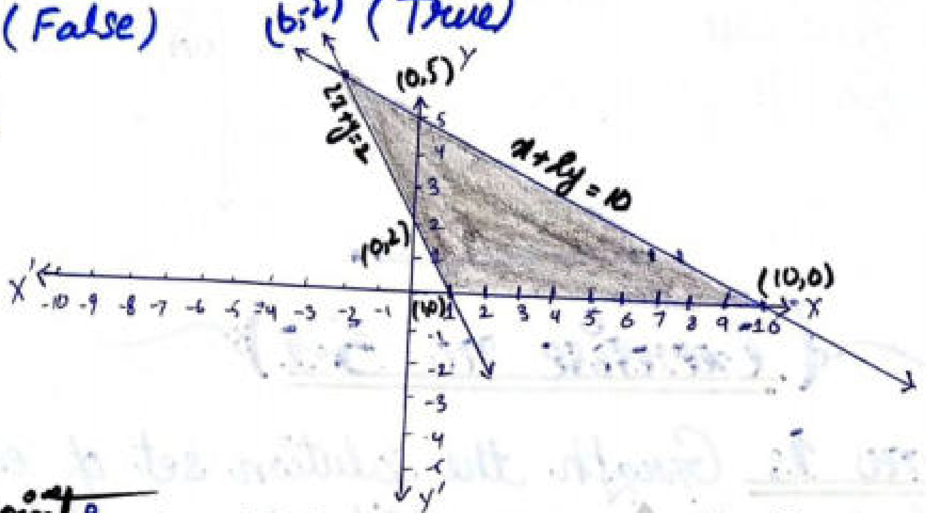
$x + 2y \leq 10$

$0 + 2(0) \leq 10$

$0 \leq 10$

(True)

Graph



Corner point: By Solving equation ⓐ & ⓑ

2ⓐ − ⓑ

$4x + \cancel{2y} = 4$

$x + \cancel{2y} = 10$

$- \qquad - \qquad -$

$3x = -6$

$\boxed{x = -2}$

Put $x = -2$ in ⓑ

$-2 + 2y = 10$

$2y = 10 + 2$

$2y = 12$

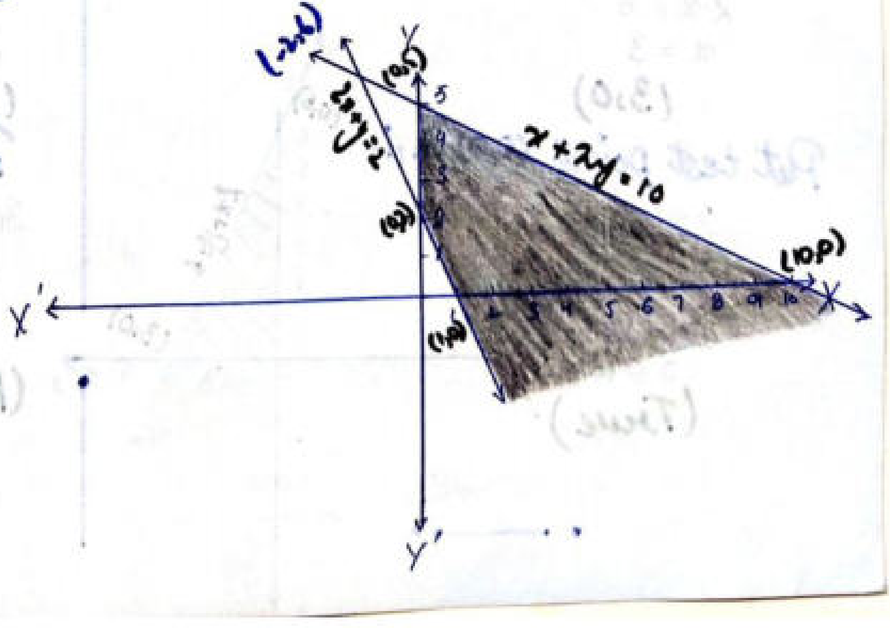
$\boxed{y = 6}$

ii) $2x + y \geq 2$

$x + 2y \leq 10$

$x \geq 0$

Same answer as given in the above question and the Graph is

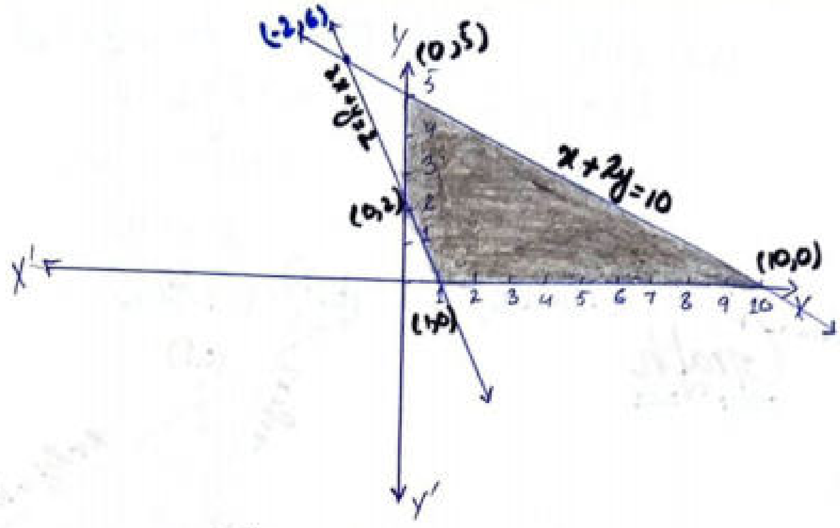


iii) $2x+y \geq 2$

$x+2y \leq 10$

$x \geq 0;\ y \geq 0$

Same answel as given in the first palt and the graph is



(Exercise no 5.1)

Question no 1: Graph the solution set of each of the fallowing linear inequality in xy-plane:

i) $2x+y \leq 6$

$2x+y \leq 6$ —(i)

The associated of (i) is

$2x+y=6$ —(ii)

Put $x=0$ in (ii)

$2(0)+y=6$

$0+y=6$

$y=6$

(0,6)

Put $y=0$ in (ii)

$2x+0=6$

$2x=6$

$x=3$

(3,0)

Put test point (0,0) in (i)

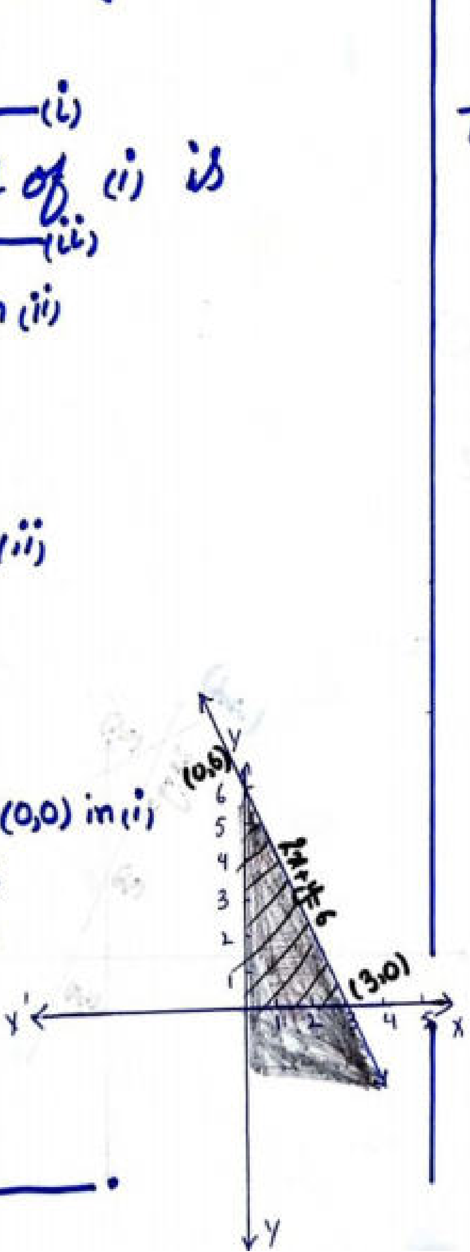
$2x+y \leq 6$

$2(0)+0 \leq 6$

$0+0 \leq 6$

$0 \leq 6$

(True)



ii) $3x+7y \geq 21$

$3x+7y \geq 21$ —(i)

The associate of (i) is

$3x+7y=21$ —(ii)

Put $x=0$ in (ii)

$3(0)+7y=21$

$7y=21$

$y=3$

(0,3)

Put $y=0$ in (ii)

$3x+7(0)=21$

$3x=21$

$x=7$

(7,0)

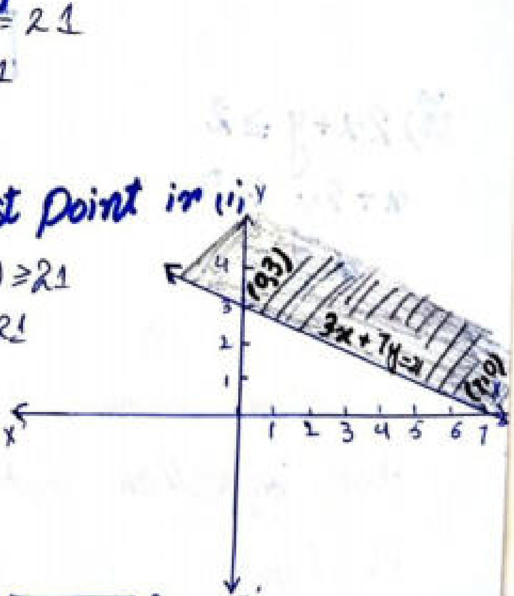
Put test point in (i)

$3(0)+7(0) \geq 21$

$0+0 \geq 21$

$0 \geq 21$

(False)



iii) $3x - 2y \geq 6$

$3x - 2y \geq 6$ —(i)

The associated of (i) is

$3x - 2y = 6$ —(ii)

Put $x=0$ in (ii)

$3(0) - 2y = 6$

$-2y = 6$

$y = -3$

$(0, -3)$

Put $y=0$ in (ii)

$3x - 2(0) = 6$

$3x = 6$

$x = 2$

$(2, 0)$

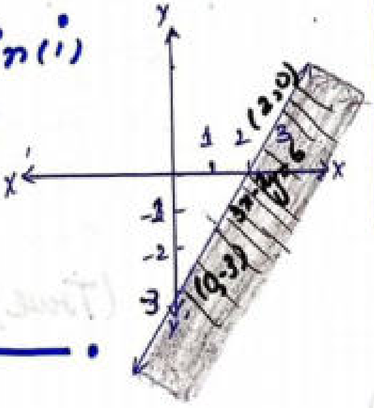
Put test point in (i)

$3x - 2y \geq 6$

$3(0) - 2(0) \geq 6$

$0 \ngeq 6$

(False)



iv) $5x - 4y \leq 20$

$5x - 4y \leq 120$ —(i)

The associated of (i) is

$5x - 4y = 20$ —(ii)

Put $x=0$ in (ii)

$5(0) - 4y = 20$

$-4y = 20$

$y = -5$

$(0, -5)$

Put $y=0$ in (ii)

$5x - 4(0) = 20$

$5x = 20$

$x = 4$

$(4, 0)$

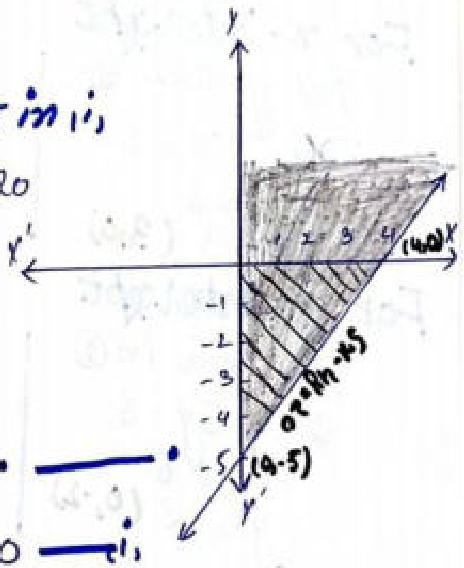
Put test point in (i)

$5(0) - 4(0) \leq 20$

$0 - 0 \leq 20$

$0 \leq 20$

(True)



V) $2x + 1 \geq 0$

$2x + 1 \geq 0$ —(i)

The associated of (i) is

$2x + 1 = 0$

$2x = -1$

$x = -\frac{1}{2}$

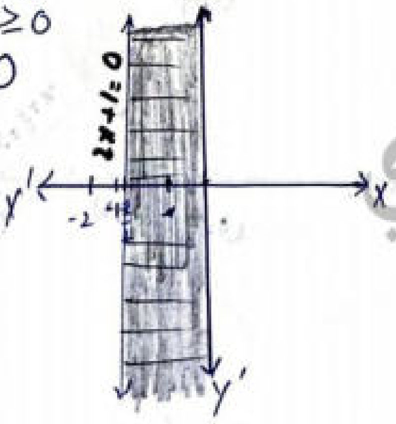
Put test point in (i)

$2(0) + 1 \geq 0$

$0 + 1 \geq 0$

$1 \geq 0$

(True)



vi) $3y - 4 \leq 0$

$3y - 4 \leq 0$ —(i)

The associated of (i) is

$3y - 4 = 0$

$3y = 4$

$y = \frac{4}{3}$

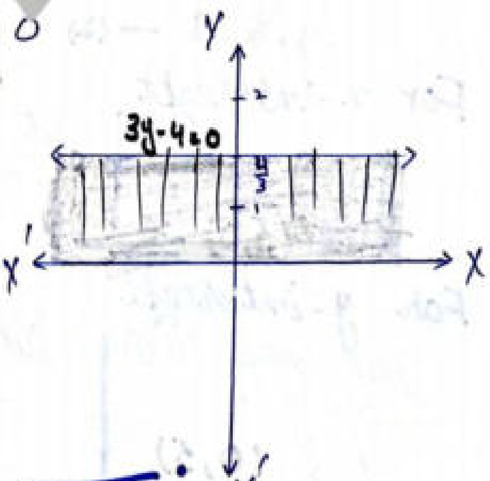
Put test point in (i)

$3(0) - 4 \leq 0$

$0 - 4 \leq 0$

$-4 \leq 0$

(True)



Question no 2: Graph the solution sets of the following systems of linear inequalites by shading:

i) $2x - 3y \le 6$

$2x + 3y \le 12$

$2x - 3y \le 6$ —— (i)

$2x + 3y \le 12$ —— (ii)

The associated of (i) & (ii)

$2x - 3y = 6$ —— ①

$2x + 3y = 12$ —— ②

For x-intercept

Put $y=0$ in ①

$2(x) - 3(0) = 6$

$2x = 6$

$x = 3$ (3,0)

Put $y=0$ in ②

$2x + 3(0) = 12$

$2x = 12$

$x = 6$ (6,0)

For y-intercept

Put $x=0$ in ①

$2(0) - 3y = 6$

$-3y = 6$

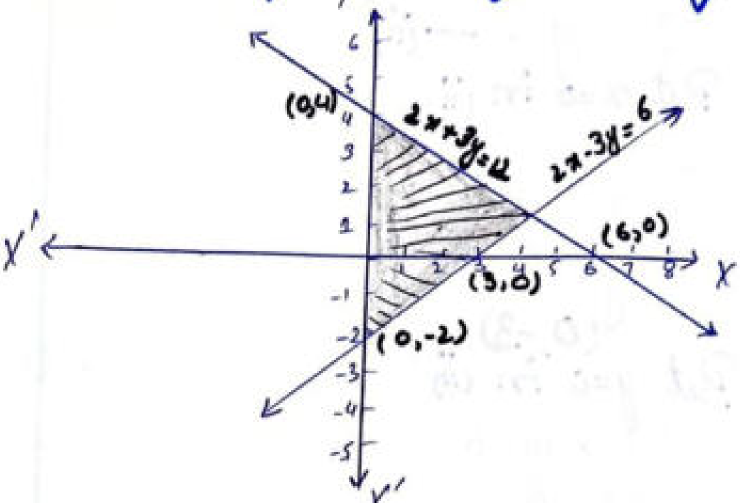
$y = -2$ (0,-2)

Put $x=0$ in ②

$2(0) + 3y = 12$

$3y = 12$

$y = 4$ (0,4)



Put test point (0,0) in (i) & (ii)

$2x - 3y \le 6$	$2x + 3y \le 12$
$2(0) - 3(0) \le 6$	$2(0) + 3(0) \le 12$
$0 - 0 \le 6$	$0 + 0 \le 12$
$0 \le 6$	$0 \le 12$
(True)	(True)

ii) $x + y \ge 5$

$-y + x \le 1$

$x + y \ge 5$ —— (i)

$-y + x \le 1$ —— (ii)

The associated of (i) & (ii)

$x + y = 5$ — (1)

$-y + x = 1$ — (2)

For x-intercept

Put $y=0$ in ①

$x + 0 = 5$

$x = 5$ (5,0)

Put $y=0$ in ②

$-0 + x = 1$

$x = 1$ (1,0)

For y-intercept

Put $x=0$ in ①

$0 + y = 5$

$y = 5$ (0,5)

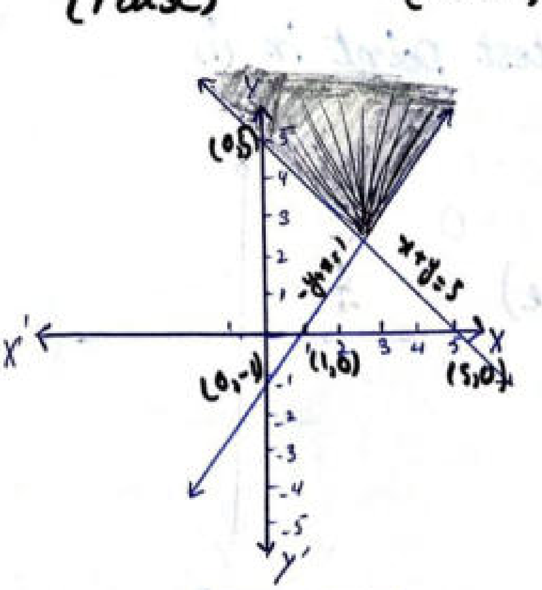
Put $x=0$ in ②

$-y + 0 = 1$

$y = -1$ (0,-1)

Put test point in (i) & (ii)

$x + y \ge 5$	$-y + x \le 1$
$0 + 0 \ge 5$	$-0 + 0 \le 1$
$0 \ge 5$	$0 \le 1$
(False)	(True)



iii) $3x+7y \geq 21$

$x-y \leq 2$

$3x+7y \geq 21$ — (i)

$x-y \leq 21$ — (ii)

The associated of (i) & (ii)

$3x+7y=21$ — (1

$x-y=2$ — (2

For x-intercept

Put $y=0$ in ①
$3x+7(0)=21$
$3x=21$
$x=7$ (7,0)

Put $y=0$ in ②
$x-0=2$
$x=2$
(2,0)

For y-intercept

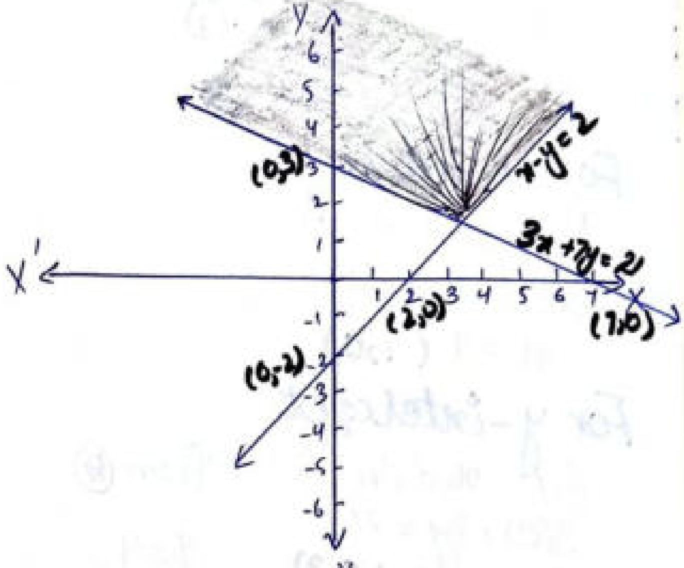
Put $x=0$ in ①
$3(0)+7y=21$
$7y=21$
$y=3$ (0,3)

Put $x\infty$ in ②
$0-y=2$
$-y=2$
$y=-2$ (0,-2)

Put test point in (i) & (ii)

$3x+7y \geq 21$
$3(0)+7(0) \geq 21$
$0 \geq 21$
(False)

$x-y \leq 2$
$0-0 \leq 2$
$0 \leq 2$
(True)



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iv) $4x-3y \leq 12$

$x \geq -\frac{3}{2}$

$4x-3y \leq 12$ — (i)

$x \geq -\frac{3}{2}$ — (ii)

The associated of (i) & (ii)

$4x-3y=12$ — ①

$x=-\frac{3}{2}$ — ②

For x-intercept

Put $y=0$ in ①
$4x-3(0)=12$
$x=3$ (3,0)

from ②
$x=-\frac{3}{2}$

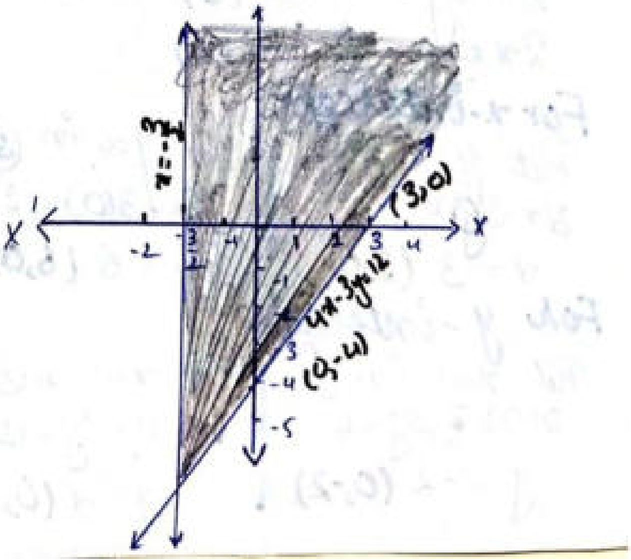
For y-intercept

Put $x=0$ in ①
$4(0)-3y=12$
$y=-4$ (0,-4)

Put test point (0,0) in (i) → & (ii)

$4x-3y \leq 12$
$4(0)-3(0) \leq 12$
$0 \leq 12$
(True)

$x \geq -\frac{3}{2}$
$0 \geq -\frac{3}{2}$
(True)



v) $3x+7y \geq 21$

$y \leq 4$

$3x+7y \geq 21$ — (i)

$y \leq 4$ — (ii)

The associated of (i) & (ii)

$3x+7y = 21$ — ①

$y = 4$ — ②

For x-intercept

Put $y=0$ in ①

$3x+7(0)=21$

$x=7$ **(7,0)**

For y-intercept

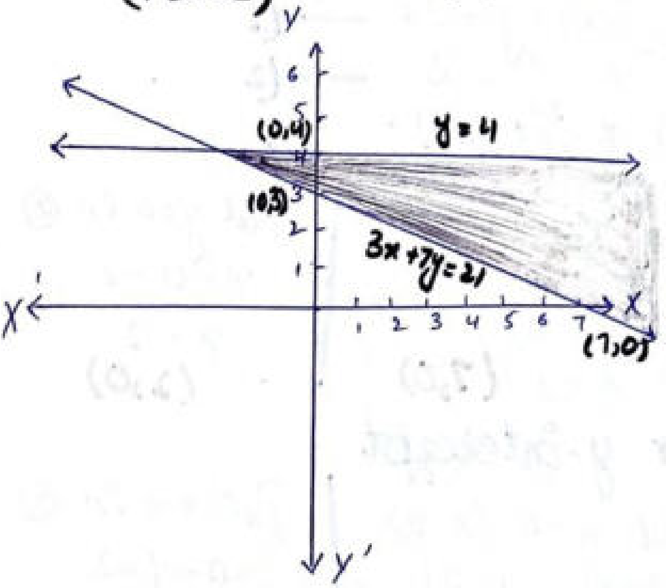
Put $x=0$ in ① | from ②

$3(0)+7y=21$ | $y=4$

$y=3$ **(0,3)**

Put test point (0,0) in (i) & (ii)

$3x+7y \geq 21$	$y \leq 4$
$3(0)+7(0) \geq 21$	$0 \leq 4$
$0 \geq 21$	
(False)	(True)



Question no 3: Indicate the solution region of the fallowing systems of inequalities by Shading:

i) $2x-3y \leq 6$

$2x+3y \leq 12$

$y \geq 0$

$2x-3y \leq 6$ — (i)

$2x+3y \leq 12$ — (ii)

Put test point in (i) & (ii)

(i)	(ii)
$2(0)-3(0) \leq 6$	$2(0)+3(0) \leq 12$
$0 \leq 6$	$0 \leq 12$
(True)	(True)

The associated of (i) & (ii)

$2x-3y = 6$ — (1)

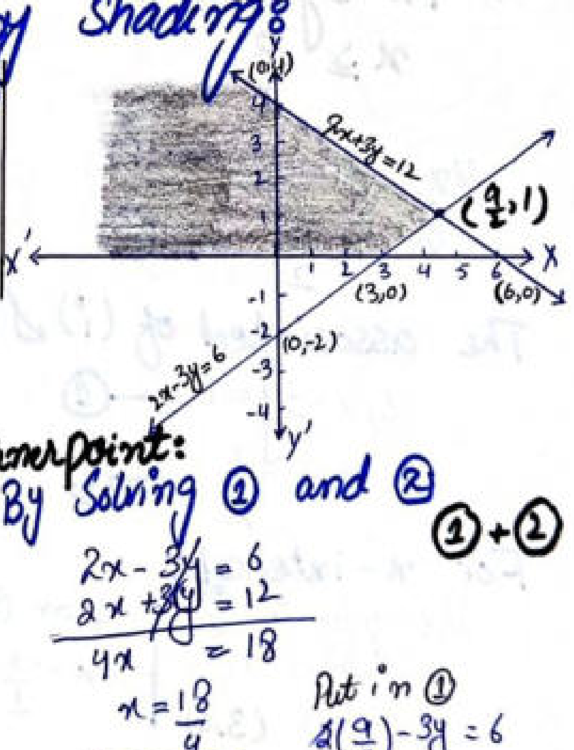
$2x+3y = 12$ — (2)

For x-intercept

Put $y=0$ in ①	Put $y=0$ in ②
$2x-3(0)=6$	$2x+3(0)=12$
$x=3$ **(3,0)**	$x=6$ **(6,0)**

For y-intercept

Put $x=0$ in ①	Put $x=0$ in ②
$2(0)-3y=6$	$2(0)+3y=12$
$y=-2$ **(0,-2)**	$y=4$ **(0,4)**



Cornerpoint:

By Solving ① and ②

①+②

$$\begin{array}{r} 2x-\cancel{3y}=6 \\ 2x+\cancel{3y}=12 \\ \hline 4x\quad=18 \end{array}$$

$x=\frac{18}{4}$

$\boxed{x=\frac{9}{2}}$

Put in ①

$2(\frac{9}{2})-3y=6$

$-3y=6-9$

$-3y=-3$

$\boxed{y=1}$

ii) $x+y \le 5$

$y-2x \le 2$

$x \ge 0$

$x+y \le 5$ —(i)

$y-2x \le 2$ —(ii)

The associated of (i) and (ii)

$x+y=5$ —(1

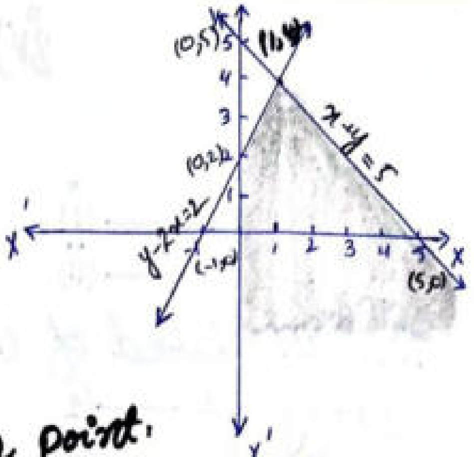
$y-2x=2$ —(2

For x-intercept

Put $y=0$ in ①	Put $y=0$ in ②
$x+0=5$	$0-2x=2$
$x=5$ (5,0)	$x=-1$ (-1,0)

For y-intercept

Put $x=0$ in ①	Put ~~y~~ $x=0$ in ②
$0+y=5$	$y-2(0)=2$
$y=5$ (0,5)	$y=2$ (0,2)



Corner point.

By Solving ① & ②

①-②

$$\begin{array}{r} x+y=5 \\ -2x+y=2 \\ + \quad - \quad - \\ \hline 3x \quad =3 \end{array}$$

$\boxed{x=1}$

Put $x=1$ in ①

$1+y=5$

$y=5-1$

$\boxed{y=4}$

Put test point in (i) & (ii)

(i)	(ii)
$0+0 \le 5$	$0-2(0) \le 2$
$0 \le 5$ (True)	$0 \le 2$ (True

iii) $x+y \ge 5$

$x-y \ge 1$

$y \ge 0$

$x+y \ge 5$ —(i)

$x-y \ge 1$ —(ii)

The associated of (i) & (ii)

$x+y=5$ —①

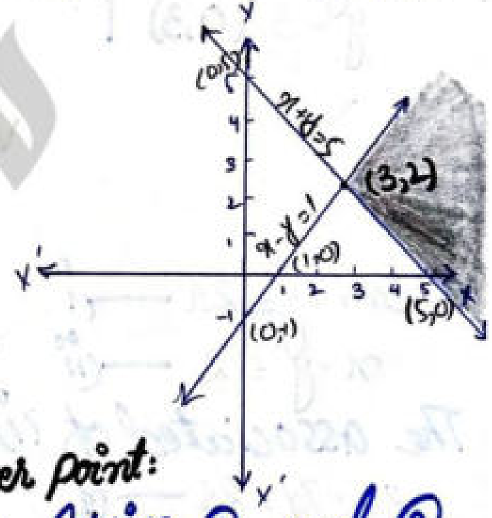
$x-y=1$ —②

For x-intercept

Put $y=0$ in ①	Put $y=0$ in ②
$x+0=5$	$x-0=1$
$x=5$ (5,0)	$x=1$ (1,0)

For y-intercept

Put $x=0$ in ①	Put $x=0$ in ②
$0+y=5$	$0-y=1$
$y=5$	$-y=1$
(0,5)	$y=-1$ (0,-1)



Corner point:

By Solving ① and ②

①+②

$$\begin{array}{r} x+y=5 \\ x-y=1 \\ \hline 2x \quad =6 \end{array}$$

$\boxed{x=3}$

Put in ①

$3+y=5$

$y=5-3$

$\boxed{y=2}$

Put test point in (i) & (ii)

(i)	(ii)
$0+0 \ge 5$	$0-0 \ge 1$
$0 \ge 5$	$0 \ge 1$
(False)	(False)

iv) $3x + 7y \leq 21$
$x - y \leq 2$
$x \geq 0$

$3x + 7y \leq 21$ —— (i)
$x - y \leq 2$ —— (ii)

The associated of (i) & (ii)

$3x + 7y = 21$ —— (1
$x - y = 2$ —— (2

For x-intercept

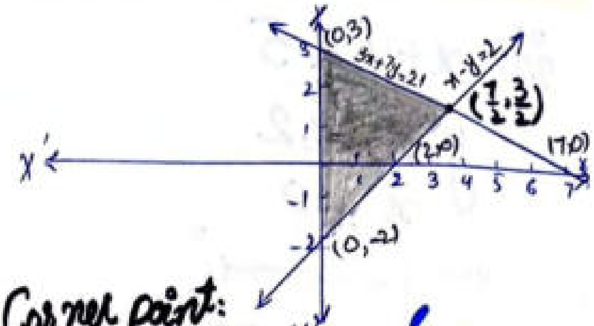
Put $y = 0$ in ①
$3x + 7(0) = 21$
$3x = 21$
$x = 7$ **(7,0)**

Put $y = 0$ in ②
$x - 0 = 2$
$x = 2$
(2,0)

For y-intercept

Put $x = 0$ in ①
$3(0) + 7y = 21$
$7y = 21$
$y = 3$ **(0,3)**

Put $x = 0$ in ②
$0 - y = 2$
$-y = 2$
$y = -2$ **(0,-2)**



Corner point:

By Solving ① & ②

① + 7②

$3x + 7y = 21$
$7x - 7y = 14$
$10x = 35$
$x = \frac{35}{10}$
$\boxed{x = \frac{7}{2}}$

Put x in ②
$\frac{7}{2} - y = 2$
$-y = 2 - \frac{7}{2}$
$-y = \frac{4-7}{2}$
$-y = -\frac{3}{2}$
$\boxed{y = \frac{3}{2}}$

Put test point in (i) & (ii)

(i)
$3(0) + 7(0) \leq 21$
$0 \leq 21$ (True)

(ii)
$0 - 0 \leq 2$
$0 \leq 2$ (True)

---

v) $3x + 7y \leq 21$
$x - y \leq 2$
$y \geq 0$

$3x + 7y \leq 21$ —— (i)
$x - y \leq 2$ —— (ii)

The associated of (i) & (ii)

$3x + 7y = 21$ —— (1
$x - y = 2$ —— (2

For x-intercept

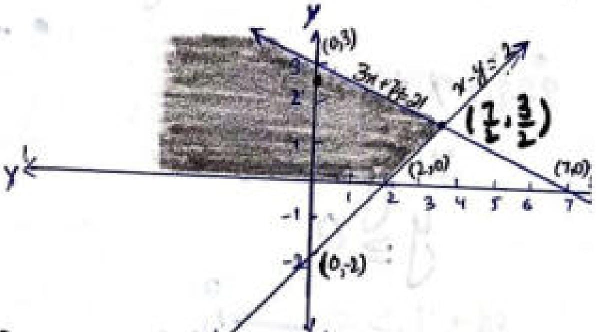
Put $y = 0$ in ①
$3x + 7(0) = 21$
$3x = 21$
$x = 7$ **(7,0)**

Put $y = 0$ in ②
$x - 0 = 2$
$x = 2$
(2,0)

For y-intercept

Put $x = 0$ in ①
$3(0) + 7y = 21$
$7y = 21$
$y = 3$ **(0,3)**

Put $x = 0$ in ②
$0 - y = 2$
$-y = 2$
$y = -2$ **(0,-2)**



Corner point:

By Solving ① & ②

① + 7②

$3x + 7y = 21$
$7x - 7y = 14$
$10x = 35$
$x = \frac{35}{10}$
$\boxed{x = \frac{7}{2}}$

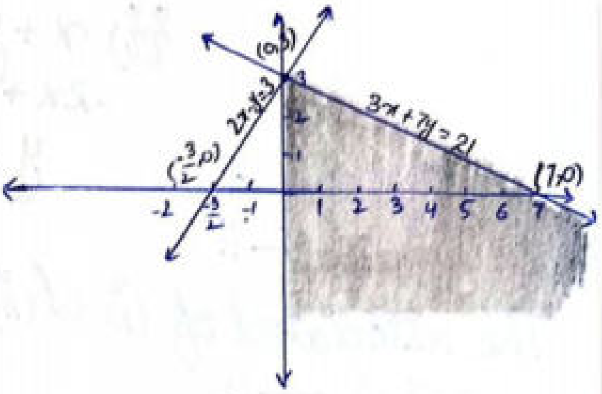
Put $x = \frac{7}{2}$ in ②
$\frac{7}{2} - y = 2$
$-y = 2 - \frac{7}{2}$
$-y = \frac{4-7}{2}$
$-y = -\frac{3}{2}$
$\boxed{y = \frac{3}{2}}$

Put test point in (i) & (ii)

(i)
$3(0) + 7(0) \leq 21$
$0 \leq 21$ (True)

(ii)
$0 - 0 \leq 2$
$0 \leq 2$ (True)

vi) $3x+7y \leq 21$
$2x-y \geq -3$
$x \geq 0$



$3x+7y \leq 21$ — (i)
$2x-y \geq -3$ — (ii)

The associated of (i) & (ii)
$3x+7y=21$ — (1
$2x-y=-3$ — (2

For x-intercept

Put y=0 in ①	Put y=0 in ②
$3x+7(0)=21$	$2x-0=-3$
$3x=21$	$2x=-3$
$x=7$ **(7,0)**	$x=-\frac{3}{2}$ **$(-\frac{3}{2},0)$**

For y-intercept

Put x=0 in ①	Put x=0 in ②
$3(0)+7y=21$	$2(0)-y=-3$
$7y=21$	$-y=-3$
$y=3$ **(0,3)**	$y=3$ **(0,3)**

Put test point (0,0) in (i) & (ii)

(i)
$3(0)+7(0) \leq 21$
$0+0 \leq 21$
$0 \leq 21$
(T)

(ii)
$2(0)-0 \geq -3$
$0 \geq -3$
(T)

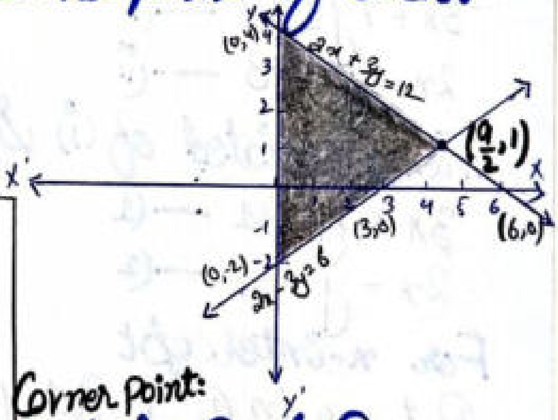
Question no 4: Graph the solution region of the following system of linear inequalities and find corner point of each:

i) $2x-3y \leq 6$
$2x+3y \leq 12$
$x \geq 0$

$2x-3y \leq 6$ — (i
$2x+3y \leq 12$ — (ii

The associated of (i) & (ii)
$2x-3y=6$ — (1
$2x+3y=12$ — (2

Put test point in (i) & (ii)
(i) $2(0)-3(0) \leq 6$
$0 \leq 6$ **(T)**
(ii) $2(0)+3(0) \leq 12$
$0 \leq 12$ **(T)**



For x-intercept

Put y=0 in ①	Put y=0 in ②
$2x-3(0)=6$	$2x+3(0)=12$
$2x=6$	$2x=12$
$x=3$ **(3,0)**	$x=6$ **(6,0)**

For y-intercept

Put x=0 in ①	Put x=0 in ②
$2(0)-3y=6$	$2(0)+3y=12$
$y=-2$ **(0,-2)**	$y=4$ **(0,4)**

Corner point:
By Solving ① & ②
① + ②

$$\begin{array}{r} 2x-3y=6 \\ 2x+3y=12 \\ \hline 4x\quad=18 \end{array}$$
$x=\frac{18}{4}$
$\boxed{x=\frac{9}{2}}$

Put x in ②
$2(\frac{9}{2})+3y=12$
$3y=12-9$
$3y=3$
$\boxed{y=1}$

ii) $x+y \leq 5$
$-2x+y \leq 2$
$y \geq 0$

$x+2y \geq 5$ — i
$-2x+y \leq 2$ — ii

The associated of (i) & (ii)

$x+y=5$ — (1
$-2x+y=2$ — (2

For x-intercept

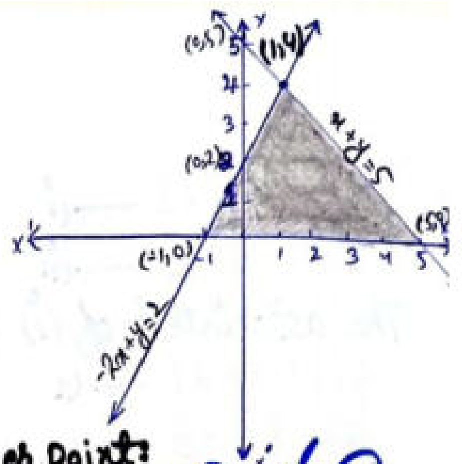
Put $y=0$ in ①
$x+0=5$
$x=5$ **(5,0)**

Put $y=0$ in ②
$-2x+0=2$
$x=-1$ **(-1,0)**

For y-intercept

Put $x=0$ in ①
$0+y=5$
$y=5$ **(0,5)**

Put $x=0$ in ②
$-2(0)+y=2$
$y=2$ **(0,2)**



Cornerpoints:
By Solving ① & ②
① - ②

$x+y=5$
$\underset{+}{-}2x \underset{-}{+} y = \underset{-}{2}$
$3x = 3$
$\boxed{x=1}$

Put in ①
$1+y=5$
$y=5-1$
$\boxed{y=4}$

Put test point in (i) & (ii)

(i)
$0+0 \leq 5$
$0 \leq 5$ **(T)**

(ii)
$-2(0)+0 \leq 2$
$0 \leq 2$ **(T)**

---

iii) $3x+7y \leq 21$
$2x-y \leq -3$
$y \geq 0$

$3x+7y \leq 21$ — (i
$2x-y \leq -3$ — (ii

The associated of (i) & (ii)

$3x+7y=21$ — (1
$2x-y=-3$ — (2

For x-intercept

Put $y=0$ in ①
$3x+7(0)=21$
$3x=21$
$x=7$ **(7,0)**

Put $y=0$ in ②
$2x-0=-3$
$2x=-3$
$x=-\frac{3}{2}$ $(-\frac{3}{2},0)$

For y-intercept

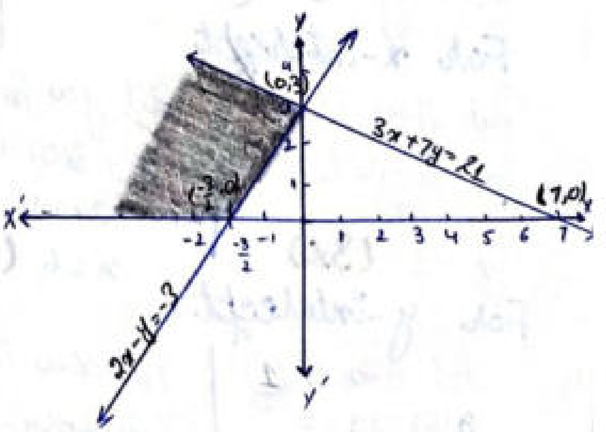
Put $x=0$ in ①
$3(0)+7y=21$
$7y=21$
$y=3$ **(0,3)**

Put $x=0$ in ②
$2(0)-y=-3$
$-y=-3$
$y=3$ **(0,3)**

Put test point (0,0) in (i) & (ii)

(i)
$3(0)+7(0) \leq 21$
$0+0 \leq 21$
$0 \leq 21$ **(T)**

(ii)
$2(0)-0 \leq -3$
$0-0 \leq -3$
$0 \leq -3$ **(F)**



(iv) $3x+2y \geq 6$

$x+3y \leq 6$

$y \geq 0$

$3x+2y \geq 6$ — (i)

$x+3y \leq 6$ — (ii)

The associated of (i) & (ii)

$3x+2y=6$ — (1)

$x+3y=6$ — (2)

Put test point in (i) & (ii)

(i)	(ii)
$3(0)+2(0) \geq 6$	$0+3(0) \leq 6$
$0+0 \geq 6$	$0+0 \leq 6$
$0 \geq 6$	$0 \leq 6$
(F)	(T)

For x-intercept

Put $y=0$ in ①

$3x+2(0)=6$

$3x=6$

$x=2$ (2,0)

Put $y=0$ in ②

$x+3(0)=6$

$x+0=6$

$x=6$ (6,0)

For y-intercept

Put $x=0$ in ①

$3(0)+2y=6$

$2y=6$

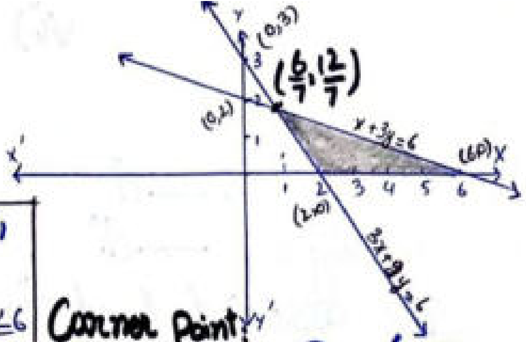
$y=3$ (0,3)

Put $x=0$ in ②

$0+3y=6$

$3y=6$

$y=2$ (0,2)



Corner Point:

By Solving ① & ②

① − 3②

$3x+2y=6$

$\underline{-\cancel{3x} \underset{-}{+}9y \underset{-}{=} 18}$

$-7y=-12$

$\boxed{y=\frac{12}{7}}$

Put y in ②

$x+3(\frac{12}{7})=6$

$x+\frac{36}{7}=6 \Rightarrow x=6-\frac{36}{7}$

$x=\frac{42-36}{7} \Rightarrow \boxed{x=\frac{6}{7}}$

V) $5x+7y \leq 35$

$-x+3y \leq 3$

$x \geq 0$

$5x+7y \leq 35$ — (i)

$-x+3y \leq 3$ — (ii)

The associated of (i) & (ii)

$5x+7y=35$ — (1)

$-x+3y=3$ — (2)

Put test point in (i), (ii)

(i)	(ii)
$5(0)+7(0) \leq 35$	$-0+3(0) \leq 3$
$0+0 \leq 35$	$-0+0 \leq 3$
$0 \leq 35$	$0 \leq 3$
(T)	(T)

For x-intercept

Put $y=0$ in ①

$5x+7(0)=35$

$5x=35$

$x=7$ (7,0)

Put $y=0$ in ②

$-x+3(0)=3$

$-x=3$

$x=-3$ (-3,0)

For y-intercept

Put $x=0$ in ①

$5(0)+7y=35$

$7y=35$

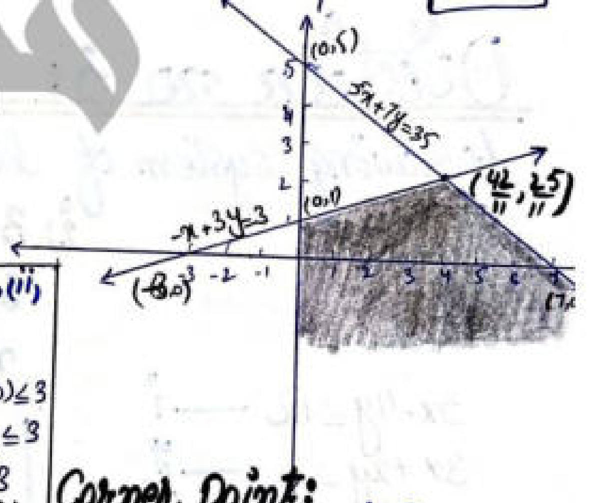
$y=5$ (0,5)

Put $x=0$ in ②

$-0+3y=3$

$3y=3$

$y=1$ (0,1)



Corner Point:

By Solving ① & ②

① + 5②

$5x+7y=35$

$\underline{-5x+15y=15}$

$22y=50$

$y=\frac{50}{22}$

$\boxed{y=\frac{25}{11}}$

Put y in ②

$-x+3(\frac{25}{11})=3$

$-x+\frac{75}{11}=3$

$-x=3-\frac{75}{11}$

$-x=\frac{33-75}{11}$

$-x=-\frac{42}{11}$

$\boxed{x=\frac{42}{11}}$

vi) $5x+7y \leq 35$

$x-2y \leq 2$

$x \geq 0$

$5x+7y \leq 35$ —— (i

$x-2y \leq 2$ —— (ii

The associated of (i) & (ii)

$5x+7y = 35$ — (1

$x-2y = 2$ — (2

Put test point (0,0) in (i) & (ii)

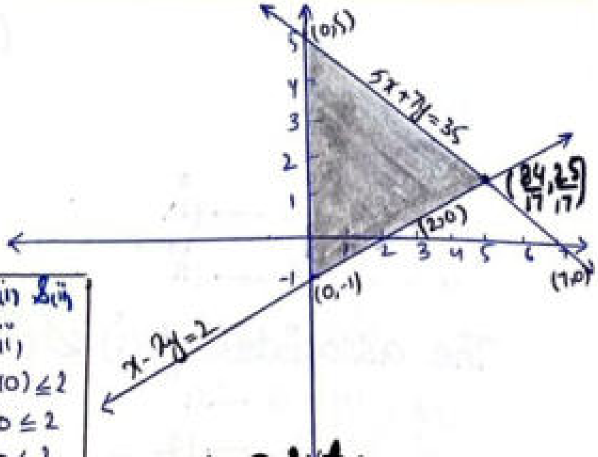
(i)	(ii)
$5(0)+7(0) \leq 35$	$0-2(0) \leq 2$
$0+0 \leq 35$	$0-0 \leq 2$
$0 \leq 35$	$0 \leq 2$
(T)	(T)

For x-intercept

Put y=0 in ①	Put y=0 in ②
$5x+7(0)=35$	$x-2(0)=2$
$5x=35$	$x-0=2$
$x=7$ (7,0)	$x=2$ (2,0)

For y-intercept

Put x=0 in ①	Put x=0 in ②
$5(0)+7y=35$	$0-2y=2$
$0+7y=35$	$-2y=2$
$y=5$ (0,5)	$y=-1$ (0,-1)



Corner point:

By Solving ① & ②

①-5②

$\cancel{5x}+7y=35$

$\cancel{5x}-10y=10$

$-\quad+\quad-$

$17y=25$

$\boxed{y=\frac{25}{17}}$

Put y in ②

$x-2\left(\frac{25}{17}\right)=2$

$x-\frac{50}{17}=2$

$x=2+\frac{50}{17}$

$\boxed{x=\frac{84}{17}}$

Question no 5: Graph the solution region of the following system of linear inequalities by shading:

i) $3x-4y \leq 12$

$3x+2y \geq 3$

$x+2y \leq 9$

$3x-4y \leq 12$ —— i

$3x+2y \geq 3$ —— ii

$x+2y \leq 9$ —— iii

The associated of (i), (ii) & (iii) are

$3x-4y=12$ — (a

$3x+2y=3$ — (b

$x+2y=9$ — (c

For x-intercept

Put y=0 in ⓐ	Put y=0 in ⓑ	Put y=0 in ⓒ
$3x-4(0)=12$	$3x+2(0)=3$	$x+2(0)=9$
$3x=12$	$3x=3$	$x+0=9$
$x=4$ (4,0)	$x=1$ (1,0)	$x=9$ (9,0)

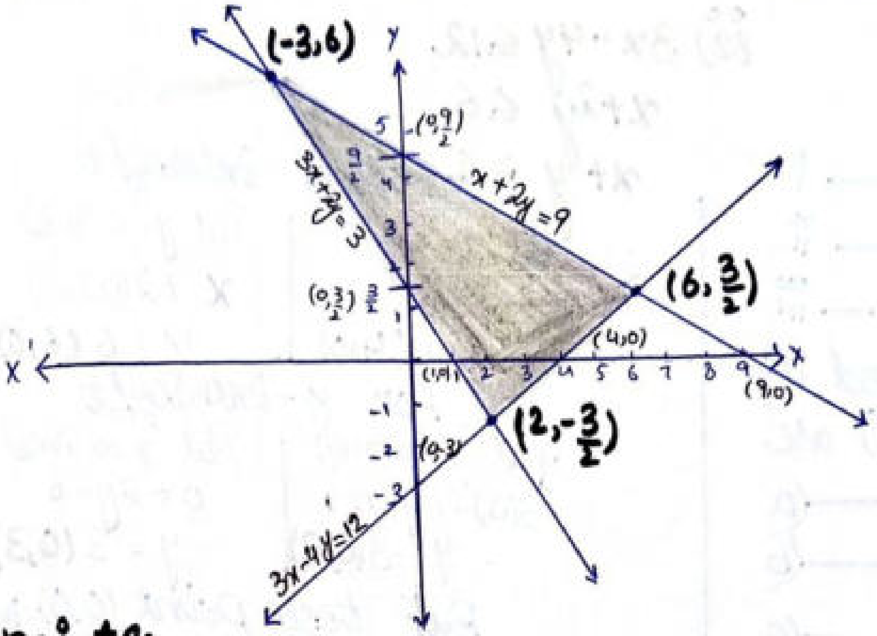
For y-intercept

Put x=0 in ⓐ	Put x=0 in ⓑ	Put x=0 in ⓒ
$3(0)-4y=12$	$3(0)+2y=3$	$0+2y=9$
$-4y=12$	$2y=3$	$2y=9$
$y=-3$ (0,-3)	$y=\frac{3}{2}$ $(0,\frac{3}{2})$	$y=\frac{9}{2}$ $(0,\frac{9}{2})$

Put test point in (i), (ii) & (iii)

(i)	(ii)	(iii)
$3(0)-4(0) \leq 12$	$3(0)+2(0) \geq 3$	$0+2(0) \leq 9$
$0 \leq 12$	$0 \geq 3$	$0 \leq 9$
(T)	(F)	(T)

Graph:



Corner points:

By Solving ⓐ, ⓑ & ⓒ

ⓑ - ⓒ

$$3x+\cancel{2y} = 3$$
$$x+\cancel{2y} = 9$$
$$- \quad - \quad -$$
$$2x = -6$$
$$x = \frac{-6}{2}$$
$$\boxed{x = -3}$$

Put $x=-3$ in ⓒ

$$-3+2y = 9$$
$$2y = 9+3$$
$$2y = 12$$
$$y = \frac{12}{2}$$
$$\boxed{y = 6}$$

ⓐ+2ⓒ

$$3x-\cancel{4y} = 12$$
$$2x+\cancel{4y} = 18$$
$$5x = 30$$
$$x = \frac{30}{5}$$
$$\boxed{x = 6}$$

Put $x=6$ in ⓒ

$$6+2y = 9$$
$$2y = 9-6$$
$$2y = 3$$
$$\boxed{y = \frac{3}{2}}$$

ⓐ - ⓑ

$$\cancel{3x}-4y = 12$$
$$\cancel{3x}+2y = 3$$
$$- \quad -$$
$$-6y = 9$$
$$y = \frac{-9}{6}$$
$$\boxed{y = \frac{-3}{2}}$$

Put in ⓐ

$$3x-4\left(\frac{-3}{2}\right) = 12$$
$$3x+\frac{12}{2} = 12$$
$$3x+6 = 12$$
$$3x = 12-6$$
$$3x = 6$$
$$x = \frac{6}{3}$$
$$\boxed{x = 2}$$

ii) $3x-4y\leq 12$
$x+2y\leq 6$
$x+y\geq 1$

$3x-4y\leq 12$ — i
$x+2y\leq 6$ — ii
$x+y\geq 1$ — iii

The associated of (i), (ii), (iii) are
$3x-4y=12$ — (a)
$x+2y=6$ — (b)
$x+y=1$ — (c)

For x-intercept

Put $y=0$ in (a)	Put $y=0$ in (b)	Put $y=0$ in (c)
$3x-4(0)=12$	$x+2(0)=6$	$x+0=1$
$x=4$ **(4,0)**	$x=6$ **(6,0)**	$x=1$ **(1,0)**

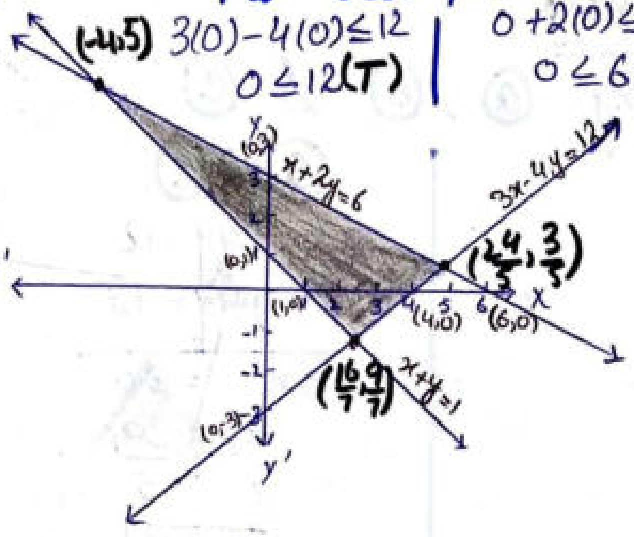
For y-intercept

Put $x=0$ in (a)	Put $x=0$ in (b)	Put $x=0$ in (c)
$3(0)-4y=12$	$0+2y=6$	$0+y=1$
$y=-3$ **(0,-3)**	$y=3$ **(0,3)**	$y=1$ **(0,1)**

Put test point (0,0) in (i), (ii) & (iii)

$3(0)-4(0)\leq 12$	$0+2(0)\leq 6$	$0+0\geq 1$
$0\leq 12$ (T)	$0\leq 6$ (T)	$0\geq 1$ (F)

Graph:



Corner points:

By Solving (a), (b) & (c)

By Solving b & c

(b) - (c)

$x+2y=6$
$\underline{\cancel{x}+y=1}$
$\boxed{y=5}$

Put in (c)

$x+5=1$

$x=1-5$

$\boxed{x=-4}$

By Solving a & b

(a) + 2(b)

$3x-4y=12$
$\underline{2x+4y=12}$
$5x=24$

$\boxed{x=\frac{24}{5}}$

Put in (b)

$\frac{24}{5}+2y=6$

$2y=6-\frac{24}{5}$

$2y=\frac{30-24}{5}$

$2y=\frac{6}{5}$

$\boxed{y=\frac{3}{5}}$

By Solving a & c

(a) + 4(c)

$3x-4y=12$
$\underline{4x+4y=4}$
$7x=16$

$\boxed{x=\frac{16}{7}}$

put in c

$\frac{16}{7}+y=1$

$y=1-\frac{16}{7}$

$y=\frac{7-16}{7}$

$\boxed{y=-\frac{9}{7}}$

iii) $2x + y \leq 4$

$2x - 3y \geq 12$

$x + 2y \leq 6$

$2x + y \leq 4$ —— (i)

$2x - 3y \geq 12$ —— (ii)

$x + 2y \leq 6$ —— (iii)

The associated of (i), (ii) & (iii)

$2x + y = 4$ —— (a

$2x - 3y = 12$ —— (b

$x + 2y = 6$ —— (c

For x-intercept

Put $y=0$ in ⓐ	Put $y=0$ in ⓑ	Put $y=0$ in ⓒ
$2x + 0 = 4$	$2x - 3(0) = 12$	$x + 2(0) = 6$
$x = 2$	$x = 6$	$x = 6$
(2,0)	(6,0)	(6,0)

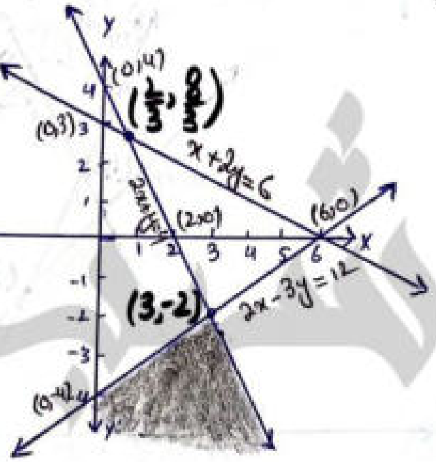
For y-intercept

Put $x=0$ in ⓐ	Put $x=0$ in ⓑ	Put $x=0$ in ⓒ
$2(0) + y = 4$	$2(0) - 3y = 12$	$0 + 2y = 6$
$y = 4$	$y = -4$	$y = 3$
(0,4)	(0,-4)	(0,3)

Put test point (0,0) in (i), (ii) & (iii)

$2(0) + 0 \leq 4$	$2(0) - 3(0) \geq 12$	$0 + 2(0) \leq 6$
$0 \leq 4$	$0 \geq 12$	$0 \leq 6$
(T)	(F)	(T)

Graph:



Corner points:

By Solving ⓐ, ⓑ & ⓒ

ⓐ - 2ⓒ

$\cancel{2x} + y = 4$

$\cancel{2x} + 4y = 12$

$- \quad - \quad -$

$-3y = -8$

$\boxed{y = \frac{8}{3}}$

Put in ⓒ

$x + 2\left(\frac{8}{3}\right) = 6$

$x + \frac{16}{3} = 6$

$x = 6 - \frac{16}{3} \Rightarrow \frac{18 - 16}{3}$

$\boxed{x = \frac{2}{3}}$

ⓐ - ⓑ

$\cancel{2x} + y = 4$

$\cancel{2x} - 3y = 12$

$- \quad + \quad -$

$4y = -8$

$\boxed{y = -2}$

Put in ⓐ

$2x + (-2) = 4$

$2x = 4 + 2$

$2x = 6$

$\boxed{x = 3}$

iv) $2x+y\le 10$
$x+y\le 7$
$-2x+y\le 4$

$2x+y\le 10$ —(i)

$x+y\le 7$ —(ii)

$-2x+y\le 4$ —(iii)

The associated of (i), (ii) & (iii)

$2x+y=10$ —(a)

$x+y=7$ —(b)

$-2x+y=4$ —(c)

For x-intercept

Put $y=0$ in ⓐ
$2x+0=10$
$x=5$
(5,0)

Put $y=0$ in ⓑ
$x+0=7$
$x=7$
(7,0)

Put $y=0$ in ⓒ
$-2x+0=4$
$x=-2$
(-2,0)

For y-intercept

Put $x=0$ in ⓐ
$2(0)+y=10$
$y=10$
(0,10)

Put $x=0$ in ⓑ
$0+y=7$
$y=7$
(0,7)

Put $x=0$ in ⓒ
$-2(0)+y=4$
$y=4$
(0,4)

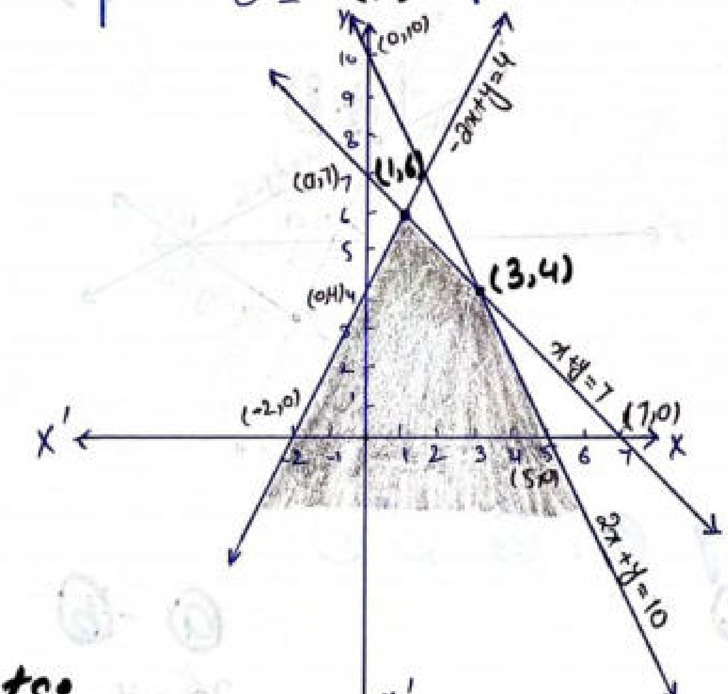
Put test point (0,0) in (i), (ii) & (iii)

$2(0)+0\le 10$
$0\le 10$ **(T)**

$0+0\le 7$
$0\le 7$ **(T)**

$-2(0)+0\le 4$
$0\le 4$ **(T)**

Graph:



Corner Points:

By solving ⓐ, ⓑ & ⓒ

ⓐ - ⓑ

$$2x+\cancel{y}=10$$
$$\underline{\mathop{-}x\mathop{+}\cancel{y}=\mathop{7}_{-}}$$
$$\boxed{x=3}$$

Put in ⓑ

$3+y=7$

$y=7-3$

$\boxed{y=4}$

ⓑ - ⓒ

$$x+\cancel{y}=7$$
$$\underline{\mathop{-2x}_{+}+\cancel{y}=\mathop{4}_{-}}$$
$$3x=3$$
$$\boxed{x=1}$$

Put in ⓑ

$1+y=7$

$y=7-1$

$\boxed{y=6}$

v) $2x+3y \leq 18$
$2x+y \leq 10$
$-2x+y \leq 2$

$2x+3y \leq 18$ —(i)
$2x+y \leq 10$ —(ii)
$-2x+y \leq 2$ —(iii)

The associated of (i), (ii) & (iii)

$2x+3y = 18$ —(a)
$2x+y = 10$ —(b)
$-2x+y = 2$ —(c)

For x-intercept

Put y=0 in ⓐ
$2x+3(0) = 18$
$x = 9$
$(9,0)$

Put y=0 in ⓑ
$2x+0 = 10$
$x = 5$
$(5,0)$

Put y=0 in ⓒ
$-2x+0 = 2$
$x = -1$
$(-1,0)$

For y-intercept

Put x=0 in ⓐ
$2(0)+3y = 18$
$y = 6$
$(0,6)$

Put x=0 in ⓑ
$2(0)+y = 10$
$y = 10$
$(0,10)$

Put x=0 in ⓒ
$-2(0)+y = 2$
$y = 2$
$(0,2)$

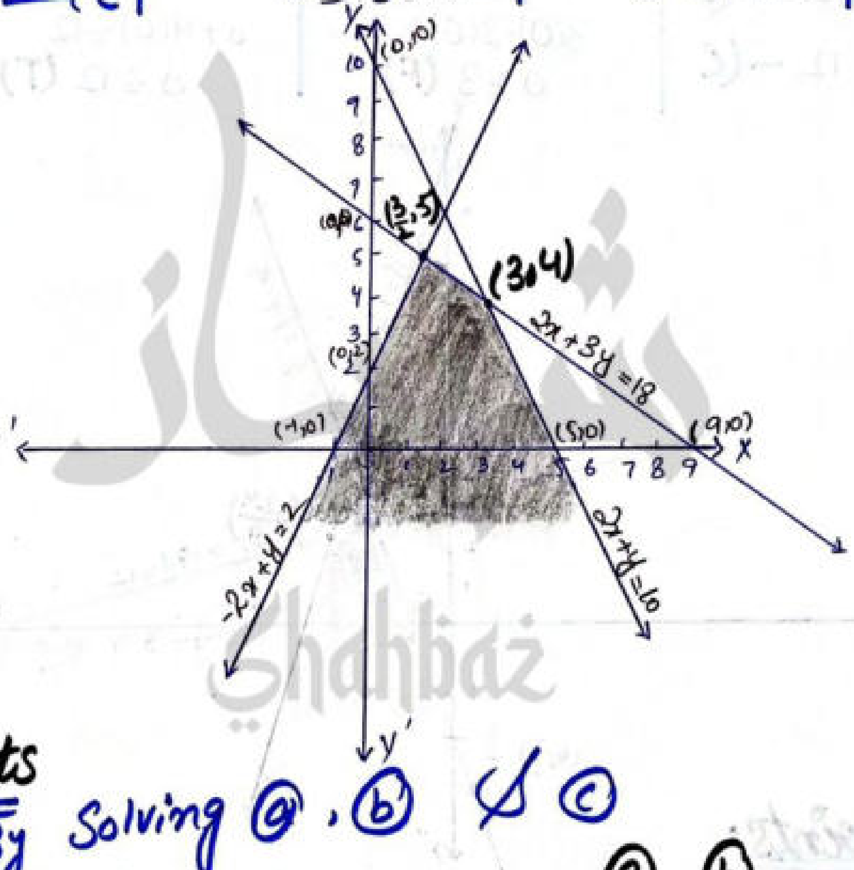
Put test Point (0,0) in (i), (ii) & (iii)

$2(0)+3(0) \leq 18$
$0 \leq 18$ (T)

$2(0)+0 \leq 10$
$0 \leq 10$ (T)

$-2(0)+0 \leq 2$
$0 \leq 2$ (T)

Graph:



Corner points

By Solving ⓐ, ⓑ & ⓒ

ⓐ + ⓒ

$2x+3y = 18$
$-2x+y = 2$
$4y = 20$
$\boxed{y = 5}$

Put in ⓒ
$-2x+5 = 2$
$-2x = 2-5$
$-2x = -3$
$\boxed{x = \frac{3}{2}}$

ⓐ - ⓑ

$2x+3y = 18$
$\underline{\cancel{2x}+y = 10}$
$\underline{- \quad - \quad -}$
$2y = 8$
$\boxed{y = 4}$

Put in ⓑ
$2x+4 = 10$
$2x = 10-4$
$2x = 6$
$\boxed{x = 3}$

vi) $3x-2y\geq 3$

$x+4y\leq 12$

$3x+y\leq 12$

$3x-2y\geq 3$ — i

$x+4y\leq 12$ — ii

$3x+y\leq 12$ — iii

The associated of (i), (ii) & (iii)

$3x-2y=3$ — (a

$x+4y=12$ — (b

$3x+y=12$ — (c

For x-intercept

Put $y=0$ in ⓐ

$3x-2(0)=3$

$x=1$

$(1,0)$

Put $y=0$ in ⓑ

$x+4(0)=12$

$x=12$

$(12,0)$

Put $y=0$ in ⓒ

$3x+0=12$

$x=4$

$(4,0)$

For y-intercept

Put $x=0$ in ⓐ

$3(0)-2y=3$

$y=-\frac{3}{2}$

$(0,-\frac{3}{2})$

Put $x=0$ in ⓑ

$0+4y=12$

$y=3$

$(0,3)$

Put $x=0$ in ⓒ

$3(0)+y=12$

$y=12$

$(0,12)$

Put test point (0,0) in (i), (ii) & (iii)

$3(0)-2(0)\geq 3$

$0\geq 3$ (F)

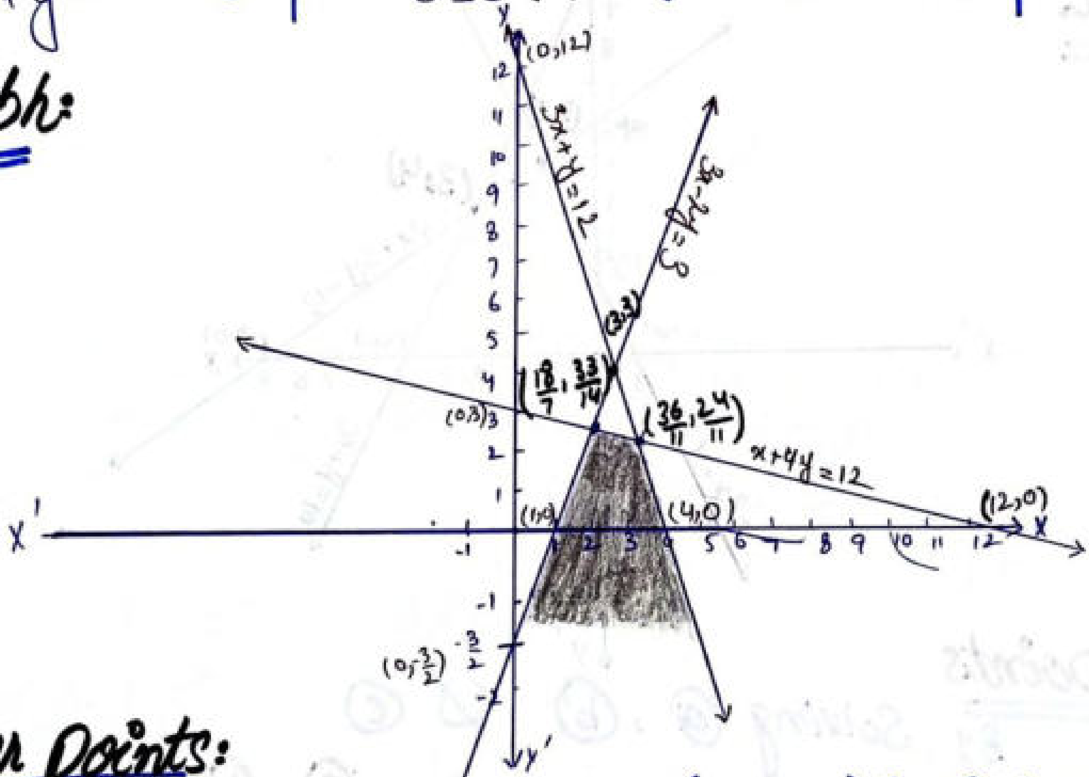
$0+4(0)\leq 12$

$0\leq 12$ (T)

$3(0)+0\leq 12$

$0\leq 12$ (T)

Graph:



Corner Points:

By Solving a & b

$2ⓐ+ⓑ$

$6x-4y=6$

$x+4y=12$

$7x=18$

$\boxed{x=\frac{18}{7}}$

Put in ⓑ

$\frac{18}{7}+4y=12$

$4y=12-\frac{18}{7}$

$4y=\frac{84-18}{7}$

$4y=\frac{66}{7}$

$y=\frac{\cancel{66}^{33}}{7\times \cancel{4}_2}$

$\boxed{y=\frac{33}{14}}$

By Solving b & c

$3ⓑ-ⓒ$

$\cancel{3x}+12y=36$

$\cancel{3x}+y=12$

$-\quad -\quad -$

$11y=24$

$\boxed{y=\frac{24}{11}}$

Put in ⓒ

$3x+\frac{24}{11}=12$

$3x=12-\frac{24}{11}$

$3x=\frac{132-24}{11}$

$3x=\frac{108}{11}$

$x=\frac{108}{11\times 3}$

$\boxed{x=\frac{36}{11}}$

By Solving a & c

$ⓐ-ⓒ$

$\cancel{3x}-2y=3$

$\cancel{3x}+y=12$

$-\quad -\quad -$

$-3y=-9$

$\boxed{y=3}$

Put in ⓒ

$3x+3=12$

$3x=12-3$

$3x=9$

$\boxed{x=3}$

Theory:-

i) **Prablem Constrain:** Tackling a certain problem from every day life each linear inequality concerning the prablem which is called problem constraint

ii) **Problem Constrains:** The system of linear inequality involed in the problem concerned are called Problem Constraints.

iii) **Non-negative Constrains:** The variable used in the system of linear inequalities relating to the problems of every day are non-negative and called non-negative constraints

iv) **Decision variables:** The non-negative constraints which are used to take a decision are called decision variables.

v) **Feasible Region:** A region which is restricted to the first quadrant is referred to as a feasible region for the set of given constraints.

vi) **Feasilble Salution:** Each point of the feasible region is called a feasible salution.

vii) **Feasible salution set:** A set consisting of all the feasible solutions of the system of linear inequalities is called feasiable salution set.

Example # 1: Graph the feasible region and find the corner points for the fallowing system of inequalities

$x - y \leq 3$

$x + 2y \leq 6$

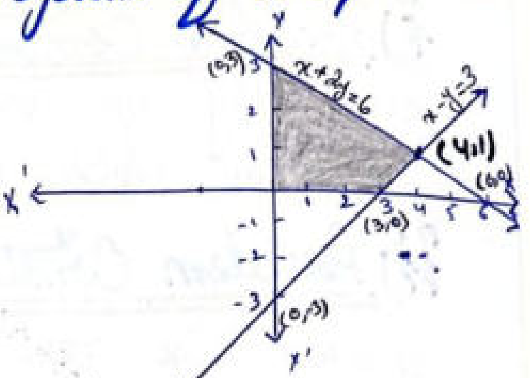
$x \geq 0 ; y \geq 0$

$x - y \leq 3$ — i

$x + 2y \leq 6$ — ii

Put testpoint in (i) & (ii)

(i)	(ii)
$0 - 0 \leq 3$	$0 + 2(0) \leq 6$
$0 \leq 3$	$0 \leq 6$
(T)	(T)



The associated of (i) & (ii)

$x - y = 3$ — 1

$x + 2y = 6$ — 2

For x-intercept

Put $y=0$ in ①	Put $y=0$ in ②
$x - 0 = 3$	$x + 2(0) = 6$
$x = 3$ **(3,0)**	$x = 6$ **(6,0)**

For y-intercept

Put $x=0$ in ①	Put $x=0$ in ②
$0 - y = 3$	$0 + 2y = 6$
$y = -3$ **(0,-3)**	$y = 3$ **(0,3)**

Cornerpoint:

By Solving ① & ②

① - ②

$$\cancel{x} - y = 3$$
$$\cancel{x} + 2y = 6$$
$$- \quad - \quad -$$
$$-3y = -3$$
$$\boxed{y = 1}$$

Put in ①

$$x - 1 = 3$$
$$x = 3 + 1 \Rightarrow \boxed{x = 4}$$

Cornerpoints (0,0),(3,0)(4,1),(0,3)

Example # 3: Graph the feasible regions subject to the fallowing constraints

a) $2x - 3y \leq 6$

$2x + y \geq 2$

$x \geq 0 ; y \geq 0$

$2x - 3y \leq 6$ — i

$2x + y \geq 2$ — ii

Put test point in (i) & (ii)

$2(0) - 3(0) \leq 6$	$2(0) + 0 \geq 2$
$0 \leq 6$ (T)	$0 \geq 2$ (F)

The associated of (i) & (ii)

$2x - 3y = 6$ — 1

$2x + y = 2$ — 2

For x-intercept

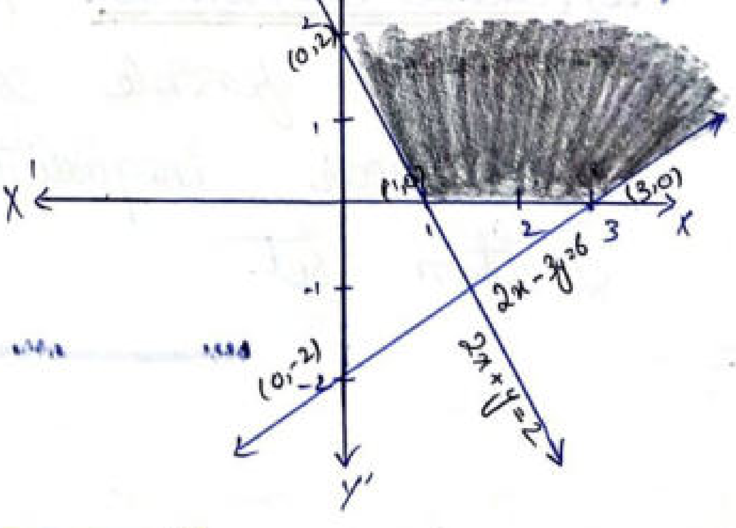
Put $y=0$ in ① & ②

$2x - 3(0) = 6$	$2x + 0 = 2$
$x = 3$ **(3,0)**	$x = 1$ **(1,0)**

For y-intercept

Put $x=0$ in ① & ②

$2(0) - 3y = 6$	$2(0) + y = 2$
$y = -2$ **(0,-2)**	$y = 2$ **(0,2)**



b) $2x-3y \le 6$

$2x+y \ge 2$

$x+2y \le 8$

$x \ge 0;\ y \ge 0$

$2x-3y \le 6$ —— (i

$2x+y \ge 2$ —— (ii

$x+2y \le 8$ —— (iii

The associated of (i), (ii) & (iii)

$2x-3y=6$ —— (a

$2x+y=2$ —— (b

$x+2y=8$ —— (c

For x-intercept

Put y=0 in ⓐ

$2x-3(0)=6$

$2x=6$

$x=3$ (3,0)

Put y=0 in ⓑ

$2x+0=2$

$2x=2$

$x=1$ (1,0)

Put y=0 in ⓒ

$x+2(0)=8$

$x+0=8$

$x=8$ (8,0)

For y-intercept

Put x=0 in ⓐ

$2(0)-3y=6$

$-3y=6$

$y=-2$ (0,-2)

Put x=0 in ⓑ

$2(0)+y=2$

$0+y=2$

$y=2$ (0,2)

Put x=0 in ⓒ

$0+2y=8$

$2y=8$

$y=4$ (0,4)

Put test point (0,0) in (i), (ii) & (iii)

$2x-3y \le 6$

$2(0)-3(0) \le 6$

$0 \le 6$ (T)

$2x+y \ge 2$

$2(0)+0 \ge 2$

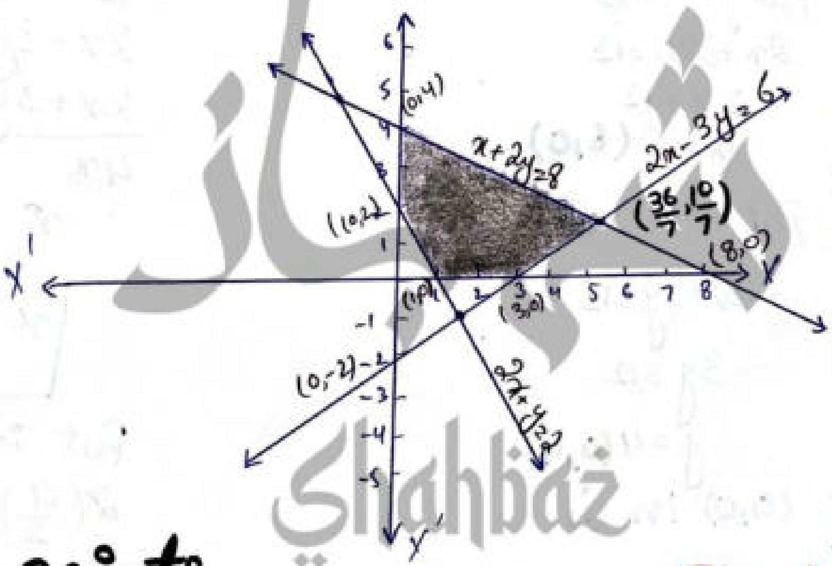
$0 \ge 2$ (F)

$x+2y \le 8$

$0+2(0) \le 8$

$0 \le 8$ (T)

Graph:



Corner point:

By Solving ⓐ & ⓒ

ⓐ - 2ⓒ

$\cancel{2x}-3y=6$

$\cancel{2x}+4y=16$

$\underline{-\quad -\quad\ \ -}$

$-7y=-10$

$\boxed{y=\frac{10}{7}}$

Put in ⓒ

$x+2\left(\frac{10}{7}\right)=8$

$x+\frac{20}{7}=8$

$x=8-\frac{20}{7}$

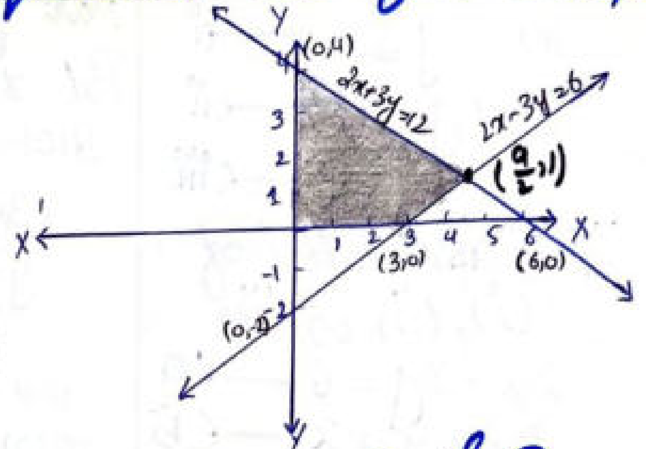
$x=\frac{56-20}{7}$

$\boxed{x=\frac{36}{7}}$

Corner points: (1,0), (3,0), $(\frac{36}{7},\frac{10}{7})$, (0,4), (0,2).

Exercise no 5.2

Question no 1: Graph the feasible region of the following system of linear inequalities and find corner points.



i) $2x-3y\le 6$

$2x+3y\le 12$

$x\ge 0;\ y\ge 0$

$2x-3y\le 6$ — (i)

$2x+3y\le 12$ — (ii)

The associated of (i) & (ii)

$2x-3y=6$ — ①

$2x+3y=12$ — ②

For x-intercept

Put $y=0$ in ①	Put $y=0$ in ②
$2x-3(0)=6$	$2x+3(0)=12$
$2x=6$	$2x=12$
$x=3$ (3,0)	$x=6$ (6,0)

For y-intercept

Put $x=0$ in ①	Put $x=0$ in ②
$2(0)-3y=6$	$2(0)+3y=12$
$-3y=6$	$3y=12$
$y=-2$ (0,-2)	$y=4$ (0,4)

Put test point (0,0) in (i) & (ii)

$2x-3y\le 6$	$2x+3y\le 12$
$2(0)-3(0)\le 6$	$2(0)+3(0)\le 12$
$0-0\le 6$	$0+0\le 12$
$0\le 6$	$0\le 12$
(True)	(True)

By solving ① & ②

① + ②

$$\begin{aligned} 2x-\cancel{3y}&=6\\ 2x+\cancel{3y}&=12\\ \hline 4x&=18 \end{aligned}$$

$x=\frac{18}{4}$

$\boxed{x=\frac{9}{2}}$

Put in ①

$2(\frac{9}{2})-3y=6$

$-3y=6-9$

$-3y=-3$

$\boxed{y=1}$

Corner point $(0,0),(3,0),(\frac{9}{2},1),(0,4)$

ii) $x+y\le 5$

$-2x+y\le 2$

$x\ge 0;\ y\ge 0$

$x+y \le 5$ —— i

$-2x+y \le 2$ —— ii

The associated of (i) & (ii)

$x+y=5$ —— (1

$-2x+y=2$ —— (2

For x-intercept

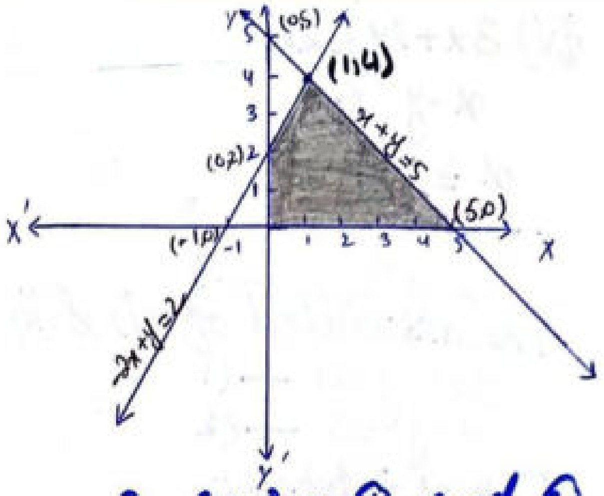
Put $y=0$ in ①	Put $y=0$ in ②
$x+0=5$	$-2x+0=2$
$x=5$ **(5,0)**	$x=-1$ **(-1,0)**

For y-intercept

Put $x=0$ in ①	Put $x=0$ in ②
$0+y=5$	$-2(0)+y=2$
$y=5$ **(0,5)**	$y=2$ **(0,2)**

Put test point in (i) & (ii)

$0+0 \le 5$	$-2(0)+0 \le 2$
$0 \le 5$ **(T)**	$0 \le 2$ **(T)**



By Solving ① and ②

①-②

$$\begin{array}{r} x+y=5 \\ -2x+y=2 \\ + \quad - \quad - \\ \hline 3x \quad =3 \end{array}$$

$$\boxed{x=1}$$

Put in ①

$1+y=5$

$$\boxed{y=4}$$

Corner point (0,0), (5,0)(1,4), (0,2)

iii) $x+y \le 5$

$-2x+y \ge 2$

$x \ge 0, y \ge 0$

$x+y \le 5$ —— i

$-2x+y \ge 2$ —— ii

The associated of (i) & (ii)

$x+y=5$ —— (1

$-2x+y=2$ —— (2

For x-intercept

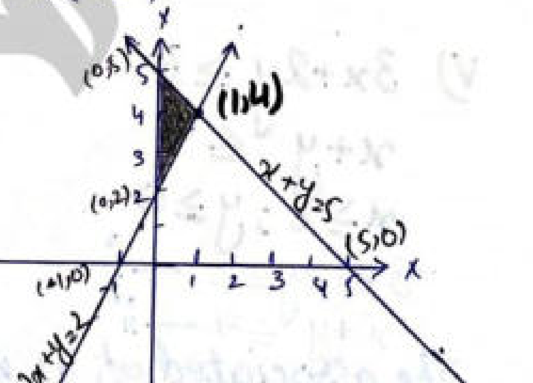
Put $y=0$ in ①	Put $y=0$ in ②
$x+0=5$	$-2x+0=2$
$x=5$ **(5,0)**	$x=-1$ **(-1,0)**

For y-intercept

Put $x=0$ in ①	Put $x=0$ in ②
$0+y=5$	$-2(0)+y=2$
$y=5$ **(0,5)**	$y=2$ **(0,2)**

Put test point (0,0) in (i) & (ii)

$0+0 \le 5$	$-2(0)+0 \ge 2$
$0 \le 5$ **(T)**	$0 \ge 2$ **(F)**



By Solving ① & ②

①-②

$$\begin{array}{r} x+y=5 \\ -2x+y=2 \\ + \quad - \quad - \\ \hline 3x \quad =3 \end{array}$$

$$\boxed{x=1}$$

Put in ①

$1+y=5$

$$\boxed{y=4}$$

Corner points (0,2), (1,4), (0,5)

iv) $3x+7y \le 21$

$x-y \le 3$

$x \ge 0 ; y \ge 0$

$3x+7y \le 21$ — i

$x-y \le 3$ — ii

The associated of (i) & (ii)

$3x+7y=21$ — (1

$x-y=3$ — (2

For x-intercept

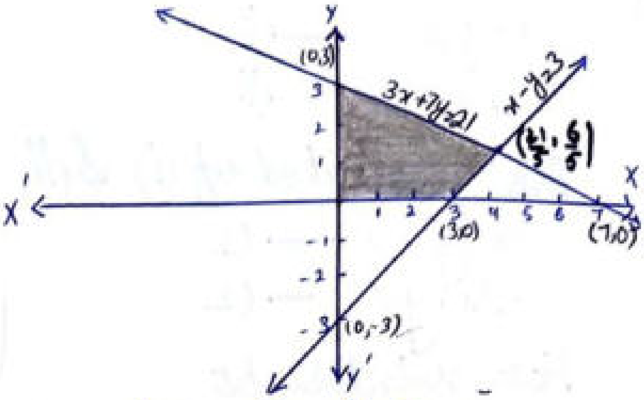
Put $y=0$ in ①	Put $y=0$ in ②
$3x+7(0)=21$	$x-0=3$
$3x=21$	$x=3$ **(3,0)**
$x=7$ **(7,0)**	

For y-intercept

Put $x=0$ in ①	Put $x=0$ in ②
$3(0)+7y=21$	$0-y=3$
$7y=21$	$y=-3$ **(0,-3)**
$y=3$ **(0,3)**	

Put test point in (i) & (ii)

$3(0)+7(0) \le 21$	$0-0 \le 3$
$0 \le 21$ **(T)**	$0 \le 3$ **(T)**



By Solving ① & ②

① + 7②

$3x+7y=21$	Put in ②
$7x-7y=21$	$\frac{21}{5}-y=3$
$10x \quad =42$	$-y=3-\frac{21}{5}$
$x=\frac{42}{10}$	$-y=\frac{15-21}{5}$
$\boxed{x=\frac{21}{5}}$	$-y=\frac{-6}{5}$
	$\boxed{y=\frac{6}{5}}$

Corner point $(0,0),(3,0)\left(\frac{21}{5},\frac{6}{5}\right),(0,3)$.

v) $3x+2y \ge 6$

$x+y \le 4$

$x \ge 0 ; y \ge 0$

$3x+2y \ge 6$ — i

$x+y \le 4$ — ii

The associated of (i) & (ii)

$3x+2y=6$ — (1

$x+y=4$ — (2

For x-intercept

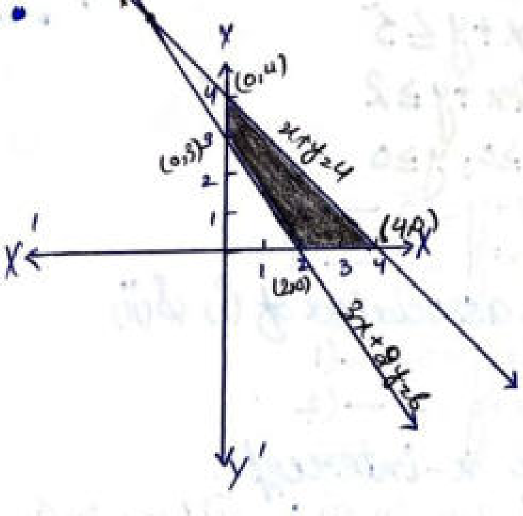
Put $y=0$ in ①	Put $y=0$ in ②
$3x+2(0)=6$	$x+0=4$
$x=2$ **(2,0)**	$x=4$ **(4,0)**

For y-intercept

Put $x=0$ in ①	Put $x=0$ in ②
$3(0)+2y=6$	$0+y=4$
$y=3$ **(0,3)**	$y=4$ **(0,4)**

Put test point in (i) & (ii)

$3(0)+2(0) \ge 6$	$0+0 \le 4$
$0 \ge 6$ **(F)**	$0 \le 4$ **(T)**



Corner points:

(2,0),(4,0),(0,4),(0,3)

vi) $5x+7y \leq 35$

$x-2y \leq 4$

$x \geq 0 ; y \geq 0$

$5x+7y \leq 35$ — i

$x-2y \leq 4$ — ii

The associated of (i) & (ii)

$5x+7y=35$ —(1

$x-2y=4$ —(2

For x-intercept

Put $y=0$ in ①

$5x+7(0)=35$

$5x=35$

$x=7$ **(7,0)**

Put $y=0$ in ②

$x-2(0)=4$

$x-0=4$

$x=4$ **(4,0)**

For y-intercept

Put $x=0$ in ①

$5(0)+7y=35$

$y=5$ **(0,5)**

Put $x=0$ in ②

$0-2y=4$

$y=-2$ **(0,-2)**

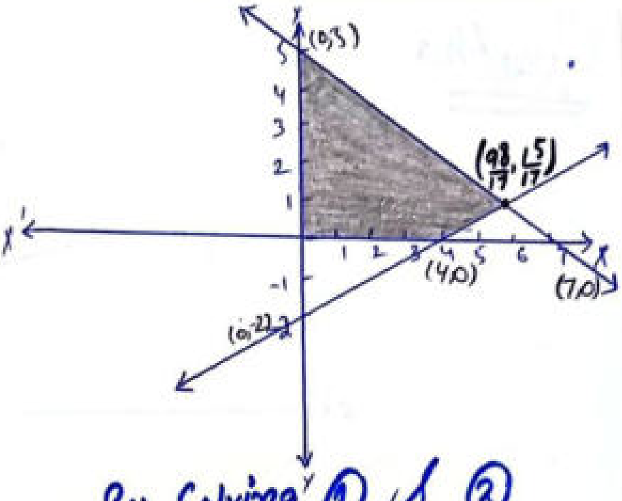
Put test point (0,0) in(i) & (ii)

$5(0)+7(0) \leq 35$

$0 \leq 35$ **(T)**

$0-2(0) \leq 4$

$0 \leq 4$ **(T)**



By Solving ① & ②

① - 5②

$\cancel{5x}+7y=35$

$\cancel{5x}-10y=20$

$- \quad + \quad -$

$17y=15$

$$\boxed{y=\frac{15}{17}}$$

Put in ②

$x-2(\frac{15}{17})=4$

$x - \frac{30}{17} = 4$

$x = 4 + \frac{30}{17}$

$x \pm \frac{68+30}{17}$

$$\boxed{x=\frac{98}{17}}$$

Corner point $(0,0),(4,0),(\frac{98}{17},\frac{15}{17}),(0,5)$

Question no 28: Graph the feasible region of the following system of linear inequalities and find the corner point:

i) $2x+y \leq 20$

$x+4y \leq 12$

$x+2y \leq 10$

$x \geq 0 ; y \geq 0$

$2x+y \leq 10$ — i

$x+4y \leq 12$ — ii

$x+2y \leq 10$ — iii

The associated of (i),(ii) & (iii)

$2x+y=10$ —(1

$x+4y=12$ —(2

$x+2y=10$ —(3

For x-intercept

Put $y=0$ in (1)

$2x+0=10$

$x=5$ **(5,0)**

Put $y=0$ in ②

$x+4(0)=12$

$x=12$ **(12,0)**

Put $y=0$ in ③

$x+2(0)=10$

$x=10$ **(10,0)**

For y-intercept

Put $x=0$ in ①

$2(0)+y=10$

$y=10$ **(0,10)**

Put $x=0$ in ②

$0+4y=12$

$y=3$ **(0,3)**

Put $x=0$ in ③

$0+2y=10$

$y=5$ **(0,5)**

Put test point in (i), (ii) & (iii)

$2x+y \leq 10$

$2(0)+0 \leq 10$

$0 \leq 10$

(T)

$x+4y \leq 12$

$0+4(0) \leq 12$

$0 \leq 12$

(T)

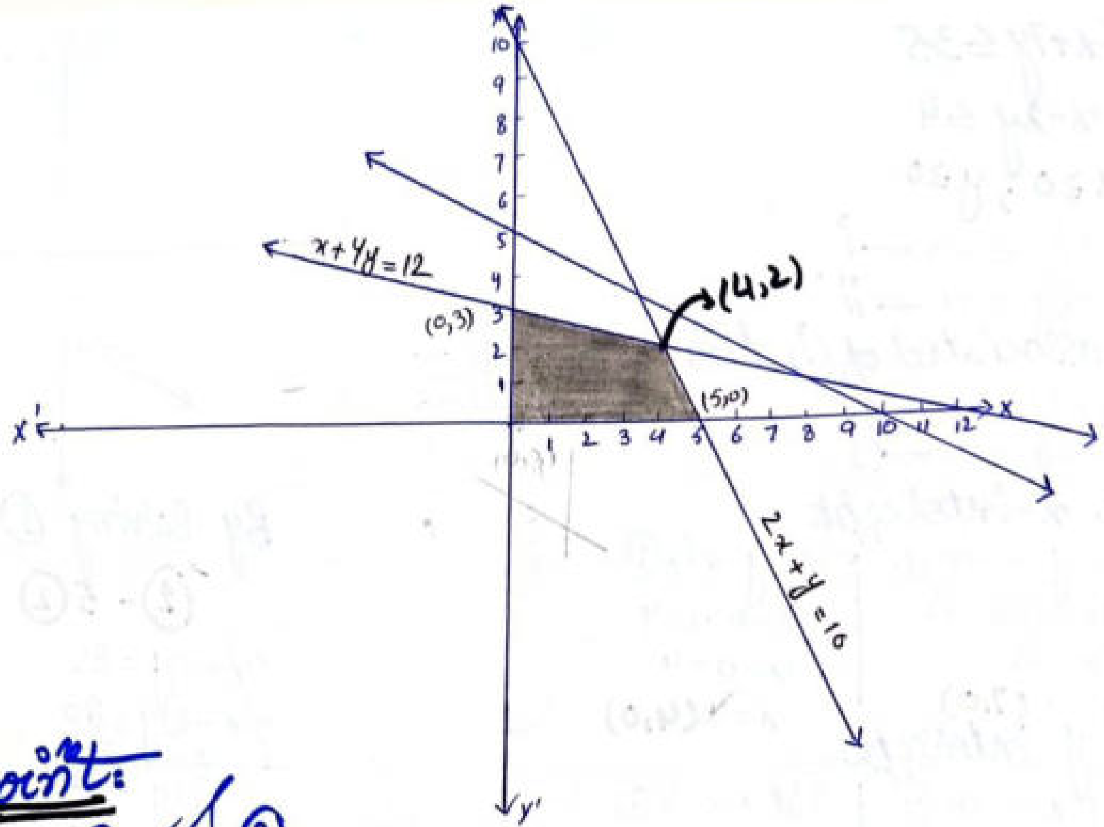
$x+2y \leq 10$

$0+2(0) \leq 10$

$0 \leq 10$

(T)

Graph:



Corner point:

By solving ① & ②

$8x+4y=40$

$\underline{-x \pm 4y = -12}$

$7x = 28$

$\boxed{x = 4}$

4② - ②

Put $x=4$ in ②

$4+4y=12$

$4y=12-4$

$4y=8$

$\boxed{y=2}$

Corner point: $(0,0),(5,0),(4,2),(0,3)$

ii) $2x+3y \leq 18$

$2x+y \leq 10$

$x+4y \leq 12$

$x \geq 0$; $y \geq 0$

$2x+3y \leq 18$ — i

$2x+y \leq 10$ — ii

$x+4y \leq 12$ — iii

The associated of (i), (ii) & (iii)

$2x+3y=18$ — (1

$2x+y=10$ — (2

$x+4y=12$ — (3

For x-intercept:

Put $y=0$ in ①

$2x+3(0)=18$

$x=9$

(9,0)

Put $y=0$ in ②

$2x+0=10$

$x=5$

(5,0)

Put $y=0$ in ③

$x+4(0)=12$

$x=12$

(12,0)

For y-intercept

Put $x=0$ in ①

$2(0)+3y=18$

$y=6$

(0,6)

Put $x=0$ in ②

$2(0)+y=10$

$y=10$

(0,10)

Put $x=0$ in ③

$0+4y=12$

$y=3$

(0,3)

Put test point (0,0) in (i), (ii) & (iii)

$2x+3y \leq 18$

$2(0)+3(0) \leq 18$

$0 \leq 18$

(T)

$2x+y \leq 10$

$2(0)+0 \leq 10$

$0 \leq 10$

(T)

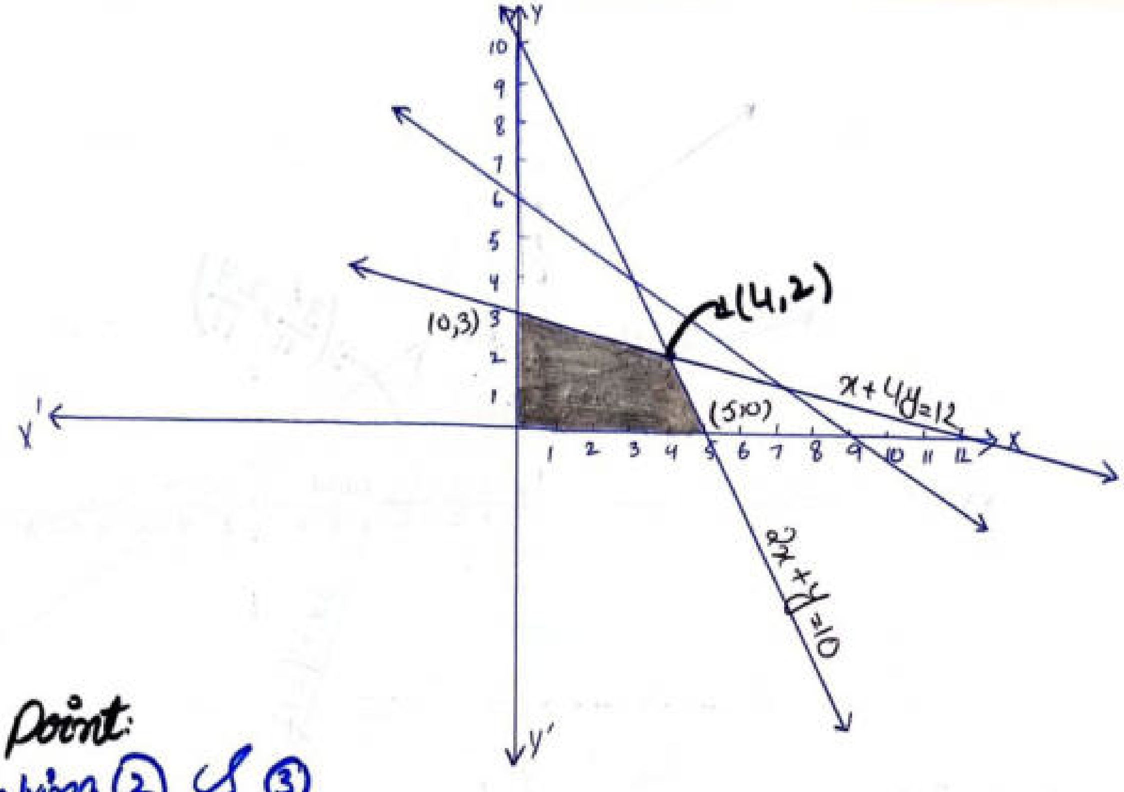
$x+4y \leq 12$

$0+4(0) \leq 12$

$0 \leq 12$

(T)

Graph:



Corner point:

By Solving ② & ③

4[illegible] - ③

$8x + 4y = 40$

$\cancel{x} + \cancel{4y} = 12$

$\underline{-\quad -\quad\quad -}$

$7x = 28$

$\boxed{x = 4}$

Put in ②

$2(4) + y = 10$

$8 + y = 10$

$y = 10 - 8$

$\boxed{y = 2}$

Corner points: (0,0), (5,0), (4,2), (0,3).

iii) $2x + 3y \leq 18$

$x + 4y \leq 12$

$3x + y \leq 12$

$x \geq 0\,;\ y \geq 0$

$2x + 3y \leq 18$ — i

$x + 4y \leq 12$ — ii

$3x + y \leq 12$ — iii

The associated of (i), (ii) & (iii)

$2x + 3y = 18$ — (1

$x + 4y = 12$ — (2

$3x + y = 12$ — (3

For x-intercept:

Put y=0 in ①

$2x + 3(0) = 18$

$2x = 18$

$x = 9$ **(9,0)**

Put y=0 in ②

$x + 4(0) = 12$

$x + 0 = 12$

$x = 12$ **(12,0)**

Put y=0 in ③

$3x + 0 = 12$

$3x = 12$

$x = 4$ **(4,0)**

For y-intercept

Put x=0 in ①

$2(0) + 3y = 18$

$3y = 18$

$y = 6$ **(0,6)**

Put x=0 in ②

$0 + 4y = 12$

$4y = 12$

$y = 3$ **(0,3)**

Put x=0 in ③

$3(0) + y = 12$

$0 + y = 12$

$y = 12$ **(0,12)**

Put test point in (i), (ii) & (iii)

$2x + 3y \leq 18$

$2(0) + 3(0) \leq 18$

$0 \leq 18$

(T)

$x + 4y \leq 12$

$0 + 4(0) \leq 12$

$0 \leq 12$

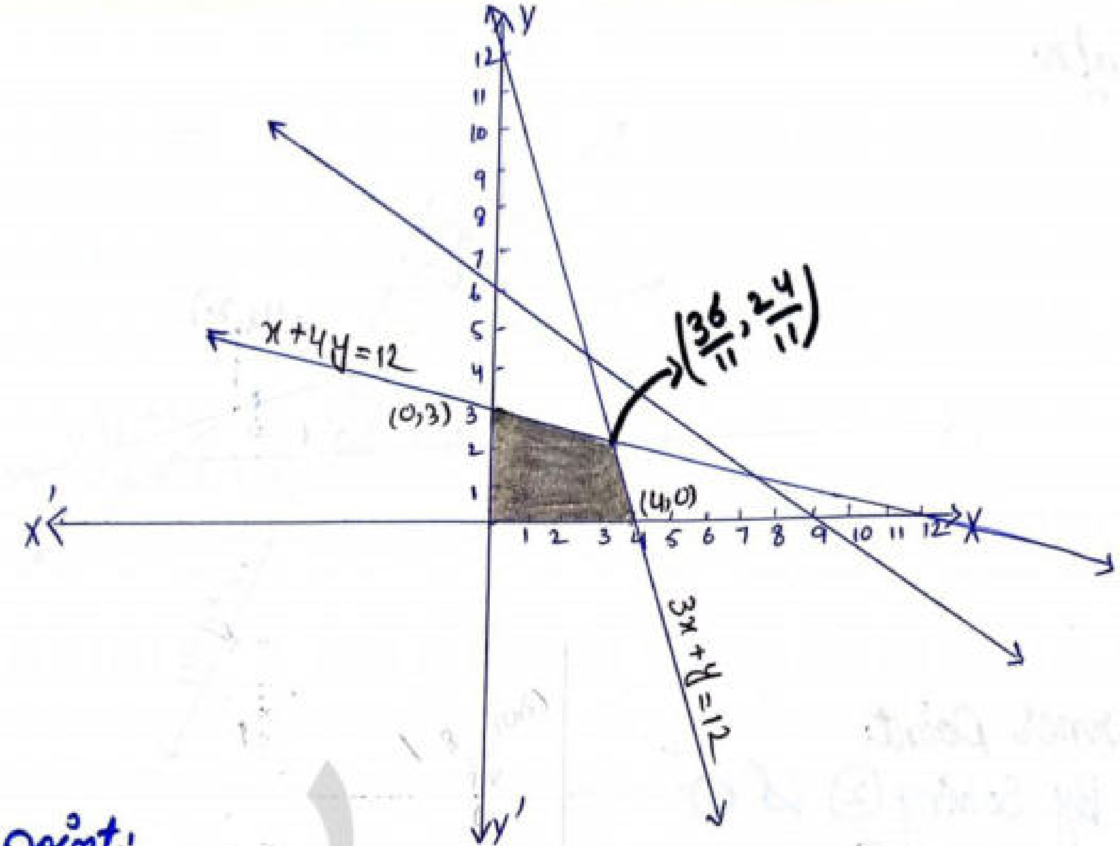
(T)

$3x + y \leq 12$

$3(0) + 0 \leq 12$

$0 \leq 12$

(T)



Corner point:

By Solving ② & ③

② - 4③

$$x+4y=12$$
$$12x+4y=48$$
$$-\quad -\quad -$$
$$-11x=-36$$
$$\boxed{x=\frac{36}{11}}$$

Put in ②

$$\frac{36}{11}+4y=12 \Rightarrow 4y=12-\frac{36}{11}$$
$$4y=\frac{132-36}{11} \Rightarrow 4y=\frac{96}{11}$$
$$y=\frac{\cancel{96}^{\,24}}{11\times 4} \Rightarrow \boxed{y=\frac{24}{11}}$$

Corner points $(0,0),(4,0),\left(\frac{36}{11},\frac{24}{11}\right),(0,3)$

iv) $x+2y\le 14$
$3x+4y\le 36$
$2x+y\le 10$
$x\ge 0;\ y\ge 0$

$x+2y\le 14$ — i
$3x+4y\le 36$ — ii
$2x+y\le 10$ — iii

The associated of (i) (ii) & (iii)

$x+2y=14$ —(1
$3x+4y=36$ —(2
$2x+y=10$ —(3

For x-intercept

Put $y=0$ in ①
$x+2(0)=14$
$x=14$ **(14,0)**

Put $y=0$ in ②
$3x+4(0)=36$
$x=12$ **(12,0)**

Put $y=0$ in ③
$2x+0=10$
$x=5$ **(5,0)**

For y-intercept

Put $x=0$ in ①
$0+2y=14$
$y=7$ **(0,7)**

Put $x=0$ in ②
$3(0)+4y=36$
$y=9$ **(0,9)**

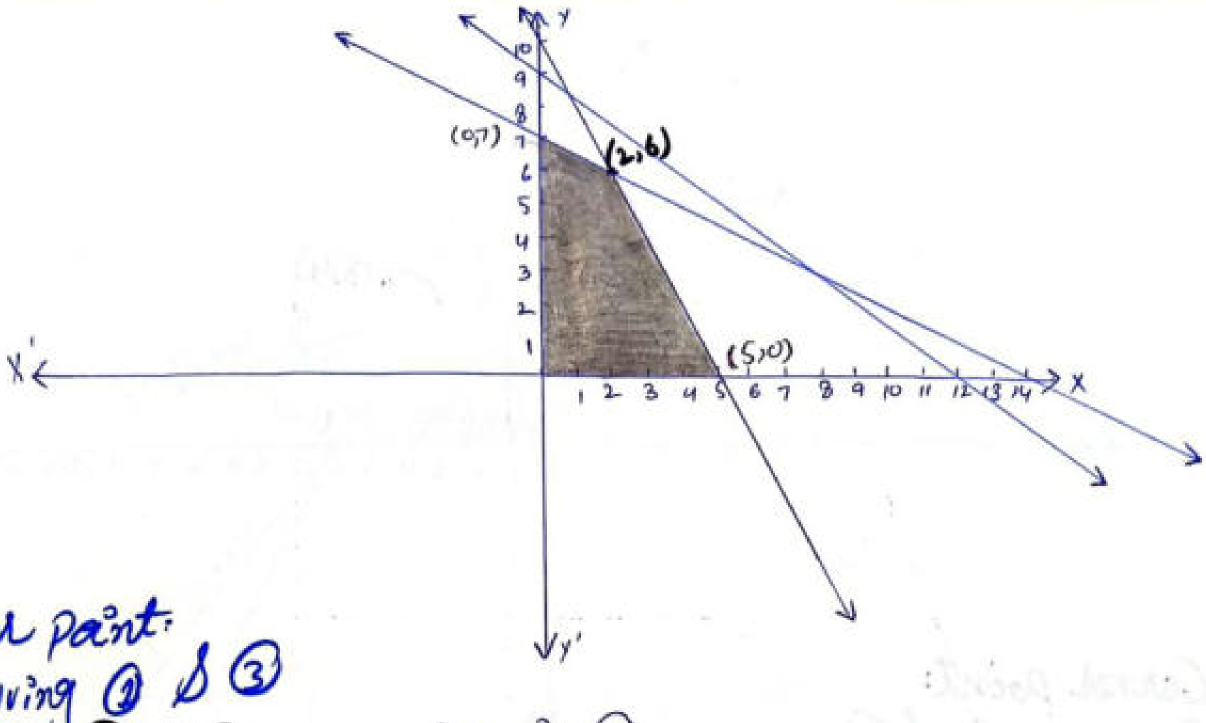
Put $x=0$ in ③
$2(0)+y=10$
$y=10$ **(0,10)**

Put test point in (i),(ii) & (iii)

$x+2y\le 14$
$0+2(0)\le 14$
$0\le 14$
(T)

$3x+4y\le 36$
$3(0)+4(0)\le 36$
$0\le 36$
(T)

$2x+y\le 10$
$2(0)+0\le 10$
$0\le 10$
(T)



Corner point:
By solving ② & ③

②-2③

$$x+2y=14$$
$$4x+2y=20$$
$$-\quad -\quad -$$
$$-3x = -6$$
$$\boxed{x=2}$$

Put in ①

$$2+2y=14$$
$$2y=14-2$$
$$2y=12$$
$$\boxed{y=6}$$

Corner points (0,0), (5,0), (2,6), (0,7)

V) $x+3y \le 15$
$2x+y \le 12$
$4x+3y \le 24$
$x \ge 0;\ y \ge 0$

$x+3y \le 15$ — i
$2x+y \le 12$ — ii
$4x+3y \le 24$ — iii

The associated of (i), (ii) & (iii)

$x+3y=15$ — 1
$2x+y=12$ — 2
$4x+3y=24$ — 3

For - x-intercept

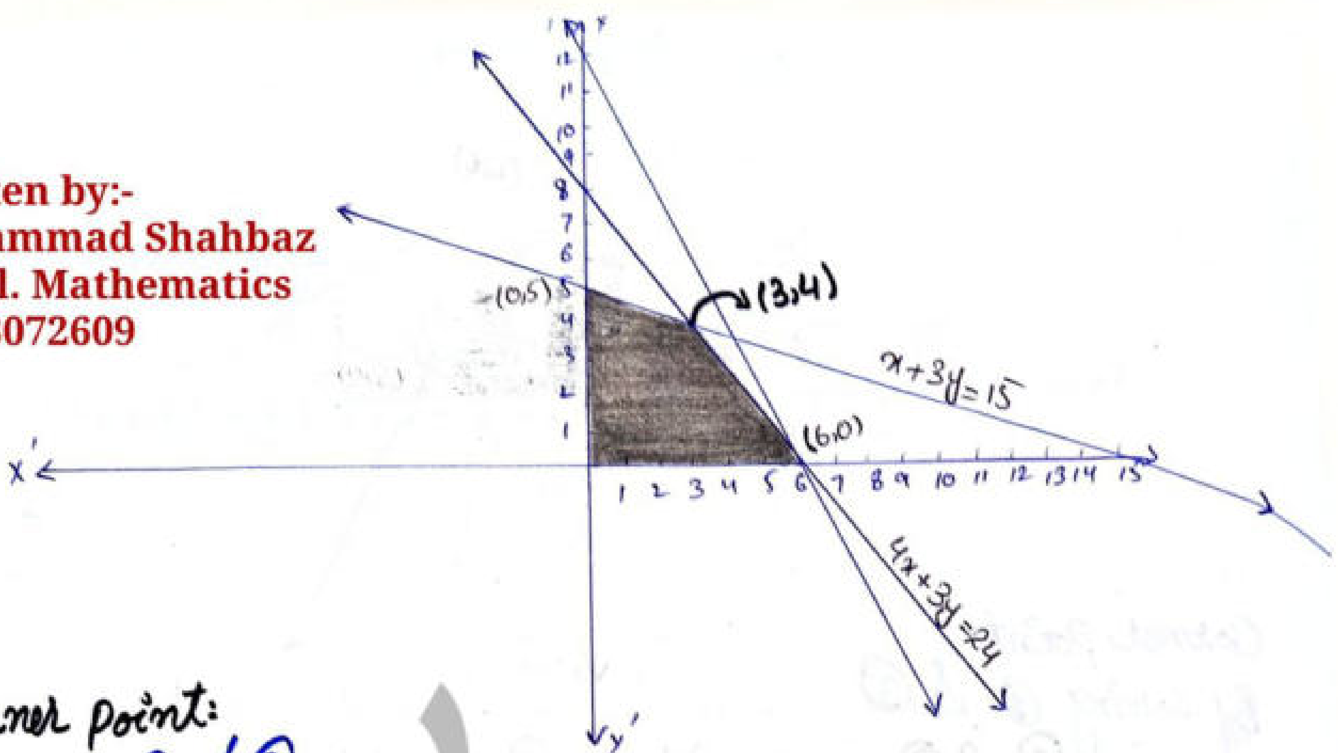
Put $y=0$ in ①	Put $y=0$ in ②	Put $y=0$ in ③
$x+3(0)=15$	$2x+0=12$	$4x+3(0)=24$
$x=15$ **(15,0)**	$x=6$ **(6,0)**	$x=6$ **(6,0)**

For y-intercept

Put $x=0$ in ①	Put $x=0$ in ②	Put $x=0$ in ③
$0+3y=15$	$2(0)+y=12$	$4(0)+3y=24$
$y=5$ **(0,5)**	$y=12$ **(0,12)**	$y=8$ **(0,8)**

Put test Point (0,0) in (i), (ii) & (iii)

$0+3(0) \le 15$	$2(0)+0 \le 12$	$4(0)+3(0) \le 24$
$0 \le 15$	$0 \le 12$	$0 \le 24$
(T)	(T)	(T)



Corner point:

By Solving ① & ③

②-③

$$\begin{aligned} x+3y&=15\\ \underline{\overset{-}{4x}+\overset{-}{3y}}&=\overset{-}{24}\\ -3x&=-9 \end{aligned}$$

$$\boxed{x=3}$$

Put in ①

$3+3y=15$

$3y=15-3$

$3y=12$

$$\boxed{y=4}$$

Corner point (0,0), (6,0), (3,4), (0,5)

vi) $2x+y\le 20$

$8x+15y\le 120$

$x+y\le 11$

$x\ge 0\ ;\ y\ge 0$

$2x+y\le 20$ — i

$8x+15y\le 120$ — ii

$x+y\le 11$ — iii

The associated of (i), (ii) & (iii)

$2x+y=20$ —(1

$8x+15y=120$ —(2

$x+y=11$ —(3

For x-intercept

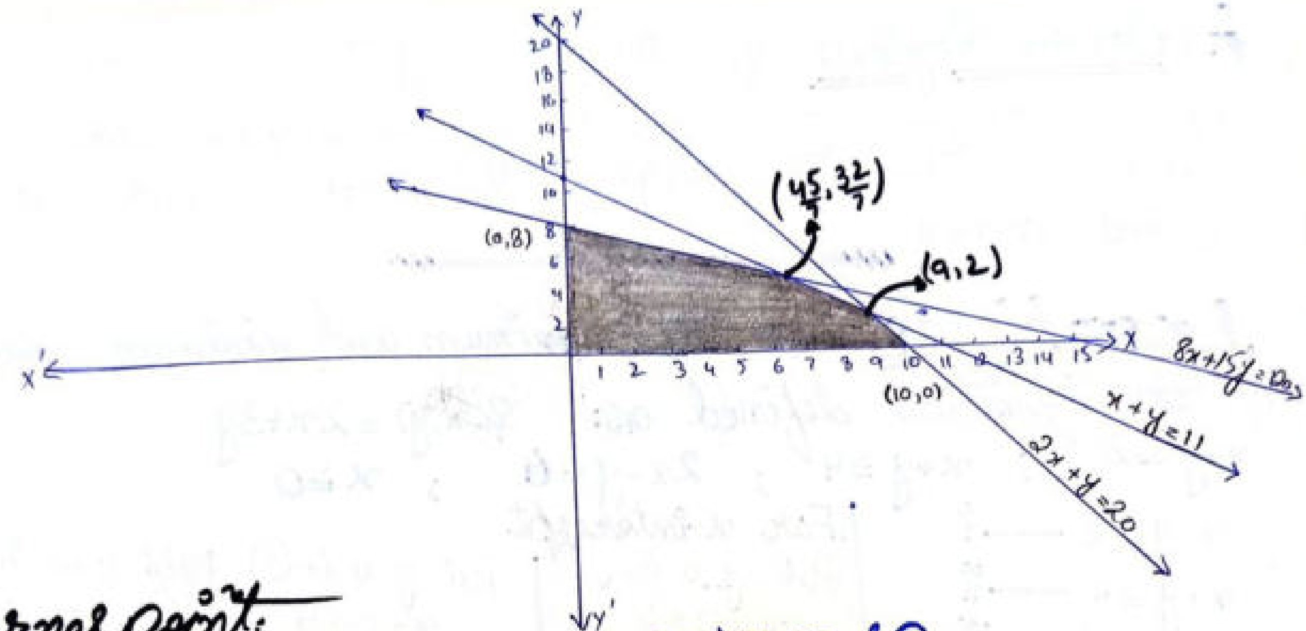
Put $y=0$ in ①	Put $y=0$ in ②	Put $y=0$ in ③
$2x+0=20$	$8x+15(0)=120$	$x+0=11$
$2x=20$	$8x=120$	$x=11$
$x=10$ (10,0)	$x=15$ (15,0)	(11,0)

For y-intercept

Put $x=0$ in ①	Put $x=0$ in ②	Put $x=0$ in ③
$2(0)+y=20$	$8(0)+15y=120$	$0+y=11$
$y=20$ (0,20)	$y=8$ (0,8)	$y=11$ (0,11)

Put test point in (i), (ii) & (iii)

$2(0)+0\le 20$	$8(0)+15(0)\le 120$	$0+0\le 11$
$0\le 20$	$0\le 120$	$0\le 11$
(T)	(T)	(T)



Corner point:

By Solving ① & ③

① - ③

$$\begin{aligned} 2x+y&=20 \\ x+y&=11 \\ - \quad - \quad & \quad - \\ \hline x &= 9 \end{aligned}$$

$$\boxed{x=9}$$

Put in ③

$$9+y=11$$

$$y=11-9$$

$$\boxed{y=2}$$

By Solving ② & ③

② - 15③

$$\begin{aligned} 8x+15y&=120 \\ 15x+15y&=165 \\ - \quad - \quad & \quad - \\ \hline -7x &= -45 \end{aligned}$$

$$\boxed{x=\frac{45}{7}}$$

Put in ③

$$\frac{45}{7}+y=11 \Rightarrow y=11-\frac{45}{7}$$

$$y=\frac{77-45}{7}$$

$$\boxed{y=\frac{32}{7}}$$

Corner points $(0,0), (10,0), (9,2), (\frac{45}{7}, \frac{32}{7}), (0,8)$.

Theory:

i) **Objective function:** A function which is to be maximized or minimized is called an objective function.

ii) **Optimal solution:** The feasible region which maximizes or minimizes the objective function is called the optimal solution.

iii) **Theorem of linear programming:** The theorem of linear programming states that the maximum and minimum values of the objective function occur at corner points of the feasible region

iv) Convex Region: If the line segment obtained by joining any two points of a region lies entirely within the region, then the region is called convex

Example 18 Find the maximum and minimum values of the function defined as: $f(x,y)=2x+3y$

$x-y\le 2$; $x+y\ge 4$; $2x-y\le 6$; $x\ge 0$

$x-y\le 2$ — i

$x+y\ge 4$ — ii

$2x-y\le 6$ — iii

The associated of (i), (ii) & (iii)

$x-y=2$ —(1

$x+y=4$ —(2

$2x-y=6$ —(3

For x-intercept

Put $y=0$ in ①
$x-0=2$
$x=2$ (2,0)

Put $y=0$ in ②
$x+0=4$
$x=4$ (4,0)

Put $y=0$ in ③
$2x-0=6$
$x=3$ (3,0)

For y-intercept

Put $x=0$ in ①
$0-y=2$
$y=-2$ (0,-2)

Put $x=0$ in ②
$0+y=4$
$y=4$ (0,4)

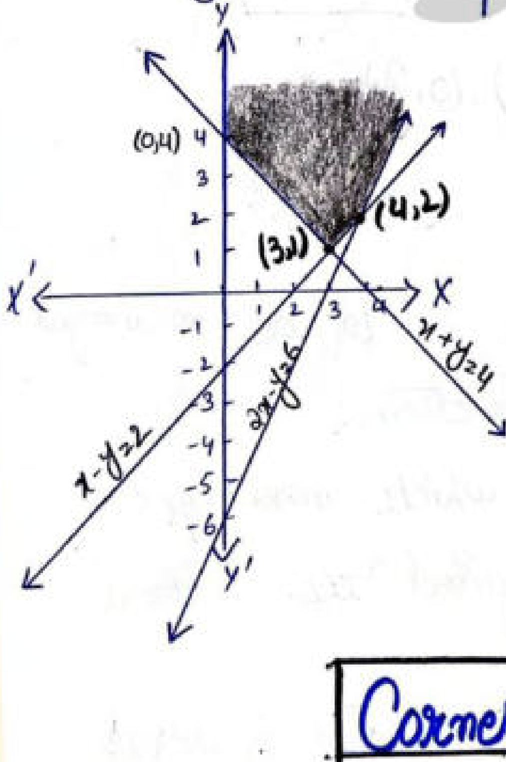
Put $x=0$ in ③
$2(0)-y=6$
$y=-6$ (0,-6)

Put test point in (i), (ii) & (iii)

$0-0\le 2$
$0\le 2$ (T)

$0+0\ge 4$
$0\ge 4$ (F)

$2(0)+0\le 6$
$0\le 6$ (T)



By Solving ① , ② & ③

①+②

$$\begin{aligned}x-y&=2\\ x+y&=4\\ \hline 2x&=6\end{aligned}$$

$\boxed{x=3}$

Put in ①

$3-y=2$

$-y=2-3$

$\boxed{y=1}$

②-③

$$\begin{aligned}x-y&=2\\ 2x-y&=6\\ -\quad+\quad&\quad -\\ \hline -x&=-4\end{aligned}$$

$\boxed{x=4}$

Put in ①

$4-y=2$

$-y=2-4$

$-y=-2 \Rightarrow \boxed{y=2}$

Corner point	$f(x,y)=2x+3y$
(0,4)	$=2(0)+3(4)\Rightarrow 0+12\Rightarrow 12$
(3,1)	$=2(3)+3(1)\Rightarrow 6+3\Rightarrow 9$
(4,2)	$=2(4)+3(2)\Rightarrow 8+6\Rightarrow 14$

The function is minimize at the corner point (3,1) and maximize at the corner point (4,2)

Example 2: Find the minimum and maximum values of f and ϕ defind as: $f(x,y)=4x+5y$

$\phi(x,y)=4x+6y$

Under the constraints:

$2x-3y\le 6$; $2x+y\ge 2$; $2x+3y\le 12$; $x\ge 0$; $y\ge 0$.

$2x-3y\le 6$ —(i)

$2x+y\ge 2$ —(ii)

$2x+3y\le 12$ —(iii)

The associated of (i), (ii) & (iii)

$2x-3y=6$ —(1

$2x+y=2$ —(2

$2x+3y=12$ —(3

For x-intercept

Put y=0 in ①

$2x-3(0)=6$

$2x=6$

$x=3$ (3,0)

Put y=0 in ②

$2x+0=2$

$2x=2$

$x=1$ (1,0)

Put y=0 in ③

$2x+3(0)=12$

$2x=12$

$x=6$ (6,0)

For y-intercept

Put x=0 in ①

$2(0)-3y=6$

$-3y=6$

$y=-2$ (0,-2)

Put x=0 in ②

$2(0)+y=2$

$0+y=2$

$y=2$ (0,2)

Put x=0 in ③

$2(0)+3y=12$

$3y=12$

$y=4$-(0,4)

Put test point in (i), (ii) & (iii)

$2(0)-3(0)\le 6$

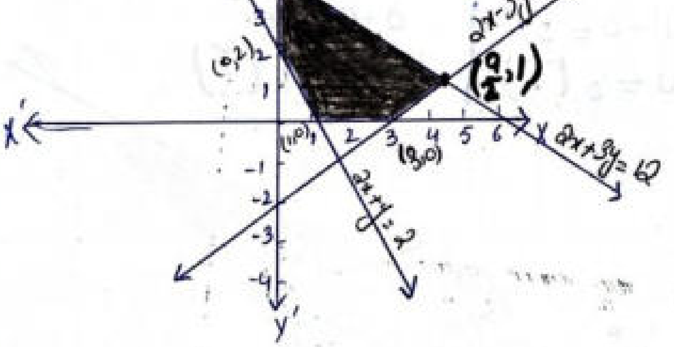
$0\le 6$ (T)

$2(0)+0\ge 2$

$0\ge 2$ (F)

$2(0)+3(0)\le 12$

$0\le 12$ (T)



By Solving ① & ③

①+③

$2x-3y=6$

$2x+3y=12$

$4x=18$

$x=\frac{18}{4}\Rightarrow \boxed{x=\frac{9}{2}}$

Put in (1)

$2(\frac{9}{2})-3y=6$

$9-3y=6$

$-3y=6-9$

$-3y=-3$

$\boxed{y=1}$

Corner Point	$f(x,y)=4x+5y$
(0,4)	$=4(0)+5(4)\Rightarrow 0+20\Rightarrow 20$
(0,2)	$=4(0)+5(2)\Rightarrow 0+10\Rightarrow 10$
(1,0)	$=4(1)+5(0)\Rightarrow 4+0\Rightarrow 4$
(3,0)	$=4(3)+5(0)\Rightarrow 12+0\Rightarrow 12$
$(\frac{9}{2},1)$	$=4(\frac{9}{2})+5(1)\Rightarrow 18+5\Rightarrow 23$

The function is maximize at the corner point $(\frac{9}{2},1)$ and minimize at the cornel point (1,0)

Cornerpoint	$f(x,y)=4x+6y$
(0,4)	$=4(0)+6(4) \Rightarrow 0+24 \Rightarrow 24$
(0,2)	$=4(0)+6(2) \Rightarrow 0+12 \Rightarrow 12$
(1,0)	$=4(1)+6(0) \Rightarrow 4+0 \Rightarrow 4$
(3,0)	$=4(3)+6(0) \Rightarrow 12+0 \Rightarrow 12$
$(\frac{9}{2},1)$	$=4(\frac{9}{2})+6(1) \Rightarrow 18+6 \Rightarrow 24$

The function is maximize at the cornel points (0,4), $(\frac{9}{2},1)$ and, minimize at the cornel point (1,0)

◆

{ Exercise no 5.3 }

Question no 1: Maximize $f(x,y) = 2x+5y$ subject to the constraints $2y-x \leq 8$; $x-y \leq 4$; $x \geq 0$; $y \geq 0$

$2y-x \leq 8$ — i

$x-y \leq 4$ — ii

Put test point (0,0) in (i) & (ii)

$2(0)-0 \leq 8$

$0 \leq 8$ (T)

$0-0 \leq 4$

$0 \leq 4$ (T)

The associated of (i) & (ii)

$2y-x = 8$ — (a

$x-y = 4$ — (b

For x-intercept

Put $y=0$ in ⓐ

$2(0)-x = 8$

$x = -8$ (-8,0)

Put $y=0$ in ⓑ

$x-0=4$

$x=4$ (4,0)

For y-intercept

Put $x=0$ in ⓐ

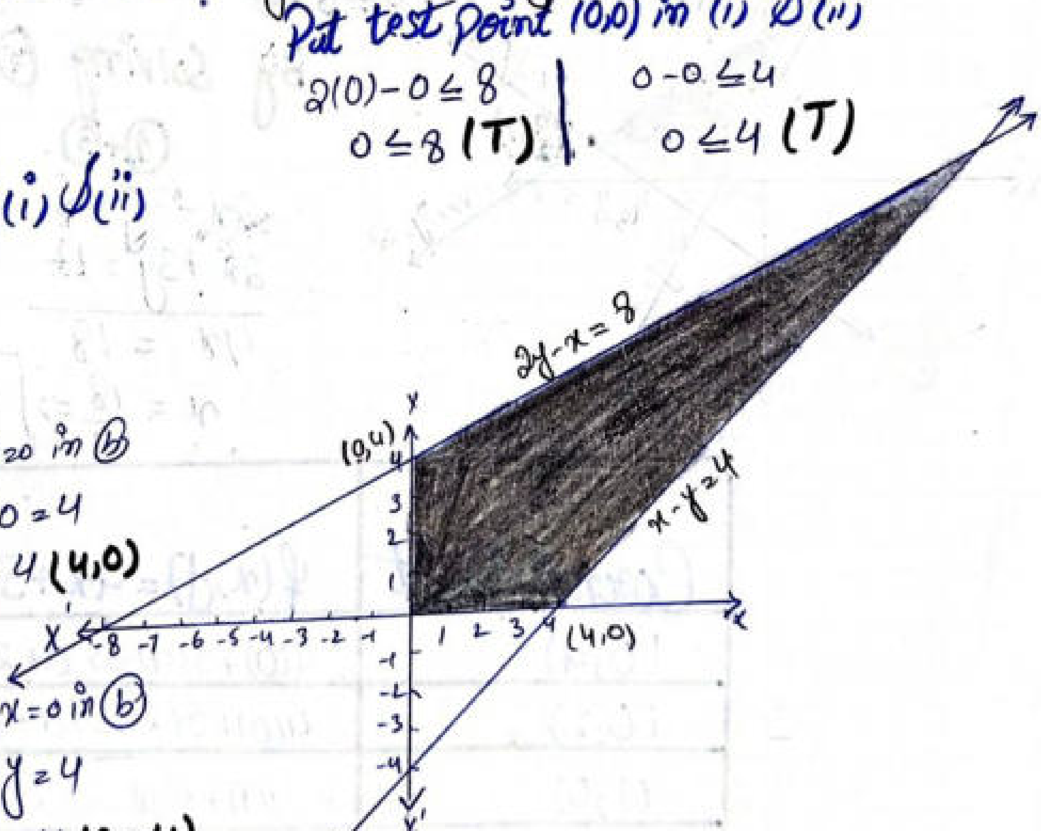
$2y-0=8$

$y=4$ (0,4)

Put $x=0$ in ⓑ

$0-y=4$

$y=-4$ (0,-4)



By Solving ⓐ & ⓑ

ⓐ + ⓑ

$\cancel{-x}+2y=8$

$\cancel{x}-y=4$

$\boxed{y=12}$

Put in ⓑ

$x-12=4$

$x=4+12$

$\boxed{x=16}$

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Corner point	$f(x,y)=2x+5y$
(0,0)	$=2(0)+5(0)\Rightarrow 0+0\Rightarrow 0$
(4,0)	$=2(4)+5(0)\Rightarrow 8+0\Rightarrow 8$
(16,12)	$=2(16)+5(12)\Rightarrow 32+60\Rightarrow 92$
(0,4)	$=2(0)+5(4)\Rightarrow 0+20\Rightarrow 20$

Maximize at the corner point **(16,12)**

Question no 28 Maximize $f(x,y)=x+3y$

subject to the constraints $2x+5y\le 30;\ 5x+4y\le 20;\ x\ge 0;\ y\ge 0.$

$2x+5y\le 30$ — i

$5x+4y\le 20$ — ii

Put test point (0,0) in (i) & (ii)

$2(0)+5(0)\le 30$

$0\le 30$ **(T)**

$5(0)+4(0)\le 20$

$0\le 20$ **(T)**

The associated of (i) & (ii)

$2x+5y=30$ — ⓐ

$5x+4y=20$ — ⓑ

For x-intercept

Put y=0 in ⓐ

$2x+5(0)=30$

$2x=30$

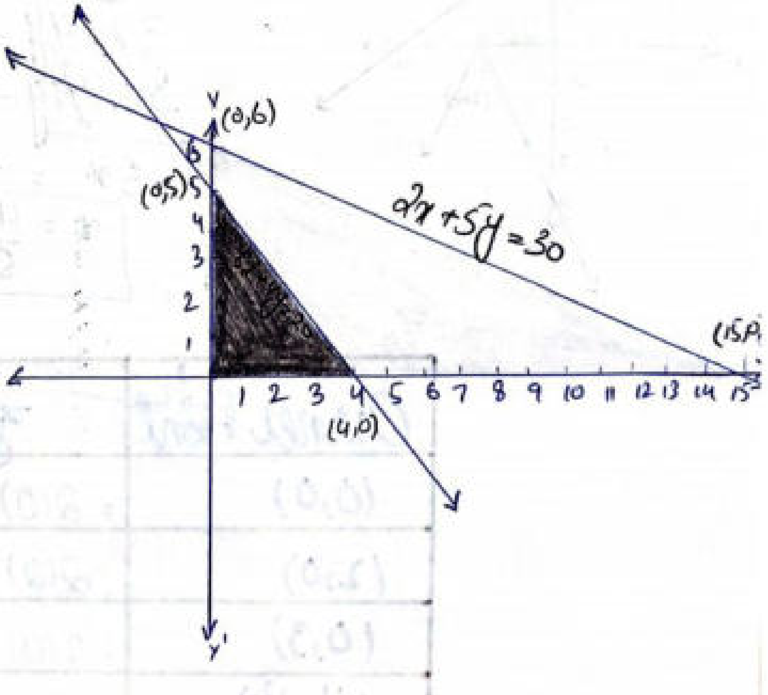
$x=15$ **(15,0)**

Put y=0 in ⓑ

$5x+4(0)=20$

$5x=20$

$x=4$ **(4,0)**



For y-intercept

Put x=0 in ⓐ

$2(0)+5y=30$

$5y=30$

$y=6$ **(0,6)**

Put x=0 in ⓑ

$5(0)+4y=20$

$4y=20$

$y=5$ **(0,5)**

Corner Point	$f(x,y)=x+3y$
$(0,0)$	$=0+3(0)\Rightarrow 0+0\Rightarrow 0$
$(4,0)$	$=4+3(0)\Rightarrow 4+0\Rightarrow 4$
$(0,5)$	$=0+3(5)\Rightarrow 0+15\Rightarrow 15$

Maximize at the corner point $(0,5)$

Question no 3: Maximize $z=2x+3y$ subject to the constraints $3x+4y\le 12;\ 2x+y\le 4;\ 2x-y\le 4;\ x\ge 0;\ y\ge 0$.

$3x+4y\le 12$ — i
$2x+y\le 4$ — ii
$2x-y\le 4$ — iii

The associated of (i), (ii) & (iii)

$3x+4y=12$ — ⓐ
$2x+y=4$ — ⓑ
$2x-y=4$ — ⓒ

For x-intercept

Put $y=0$ in ⓐ
$3x+4(0)=12$
$x=4$ $(4,0)$

Put $y=0$ in ⓑ
$2x+0=4$
$x=2$ $(2,0)$

Put $y=0$ in ⓒ
$2x-0=4$
$x=2$ $(2,0)$

For y-intercept

Put $x=0$ in ⓐ
$3(0)+4y=12$
$y=3$ $(0,3)$

Put $x=0$ in ⓑ
$2(0)+y=4$
$y=4$ $(0,4)$

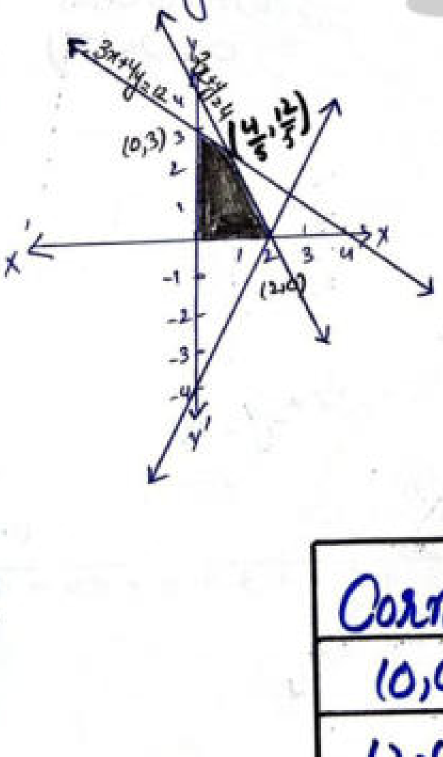
Put $x=0$ in ⓒ
$2(0)-y=4$
$y=-4$ $(0,-4)$

Put test Point in (i), (ii) & (iii)

$3(0)+4(0)\le 12$
$0\le 12$ (T)

$2(0)+0\le 4$
$0\le 4$ (T)

$2(0)-0\le 4$
$0\le 4$ (T)



By Solving ⓐ & ⓑ

ⓐ − 4ⓑ

$$\begin{array}{r} 3x+4y=12 \\ \underset{-}{8x}+\underset{-}{4y}=\underset{-}{16} \\ \hline -5x=-4 \end{array}$$

$$\boxed{x=\frac{4}{5}}$$

Put in ⓑ

$2(\frac{4}{5})+y=4$

$\frac{8}{5}+y=4\Rightarrow y=4-\frac{8}{5}$

$y=\frac{20-8}{5}\Rightarrow \boxed{y=\frac{12}{5}}$

Corner Point	$z=2x+3y$
$(0,0)$	$=2(0)+3(0)\Rightarrow 0+0\Rightarrow 0$
$(2,0)$	$=2(2)+3(0)\Rightarrow 4+0\Rightarrow 4$
$(0,3)$	$=2(0)+3(3)\Rightarrow 0+9\Rightarrow 9$
$(\frac{4}{5},\frac{12}{5})$	$=2(\frac{4}{5})+3(\frac{12}{5})\Rightarrow \frac{8}{5}+\frac{36}{5}\Rightarrow \frac{44}{5}\Rightarrow 8.8$

Maximize at the corner point — **(0,3)**

Question no 4: Minimize $z = 2x + y$

subject to the constraints $x + y \geq 3$; $7x + 5y \leq 35$; $x \geq 0$; $y \geq 0$.

$x + y \geq 3$ — (i)

$7x + 5y \leq 35$ — (ii)

The associated of (i) & (ii)

$x + y = 3$ — ⓐ

$7x + 5y = 35$ — ⓑ

Put test point in (i) & (ii)

$0 + 0 \geq 3$

$0 \geq 3$ **(F)**

$7(0) + 5(0) \leq 35$

$0 \leq 35$ **(T)**

For x-intercept

Put $y = 0$ in ⓐ

$x + 0 = 3$

$x = 3$ **(3,0)**

Put $y = 0$ in ⓑ

$7x + 5(0) = 35$

$x = 5$ **(5,0)**

For y-intercept

Put $x = 0$ in ⓐ

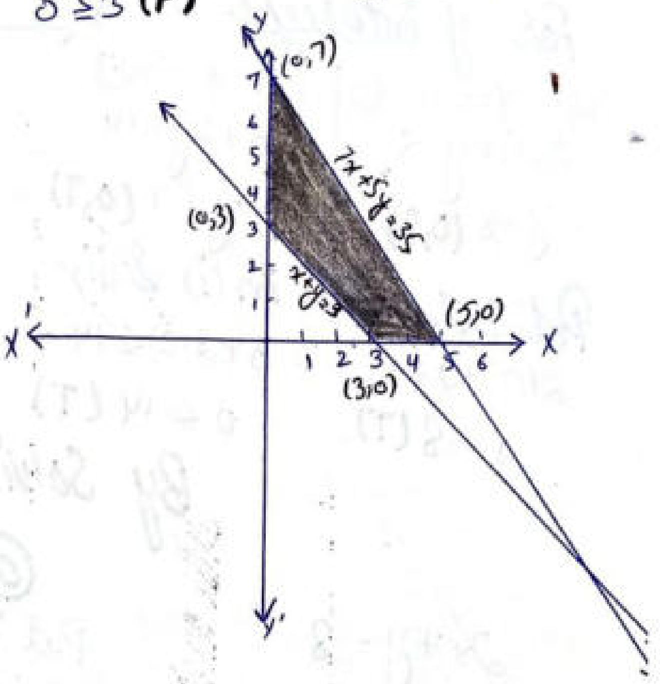
$0 + y = 3$

$y = 3$ **(0,3)**

Put $x = 0$ in ⓑ

$7(0) + 5y = 35$

$y = 7$ **(0,7)**



Corner Point	$z = 2x + y$
(3,0)	$= 2(3) + 0 \Rightarrow 6 + 0 \Rightarrow 6$
(5,0)	$= 2(5) + 0 \Rightarrow 10 + 0 \Rightarrow 10$
(0,7)	$= 2(0) + 7 \Rightarrow 0 + 7 \Rightarrow 7$
(0,3)	$= 2(0) + 3 \Rightarrow 0 + 3 \Rightarrow 3$

Minimize at the corner point **(0,3)**

Question no 5: Maximize $f(x,y) = 2x + 3y$

Subject to the constraints $2x + y \leq 8$; $x + 2y \leq 14$; $x \geq 0$; $y \geq 0$

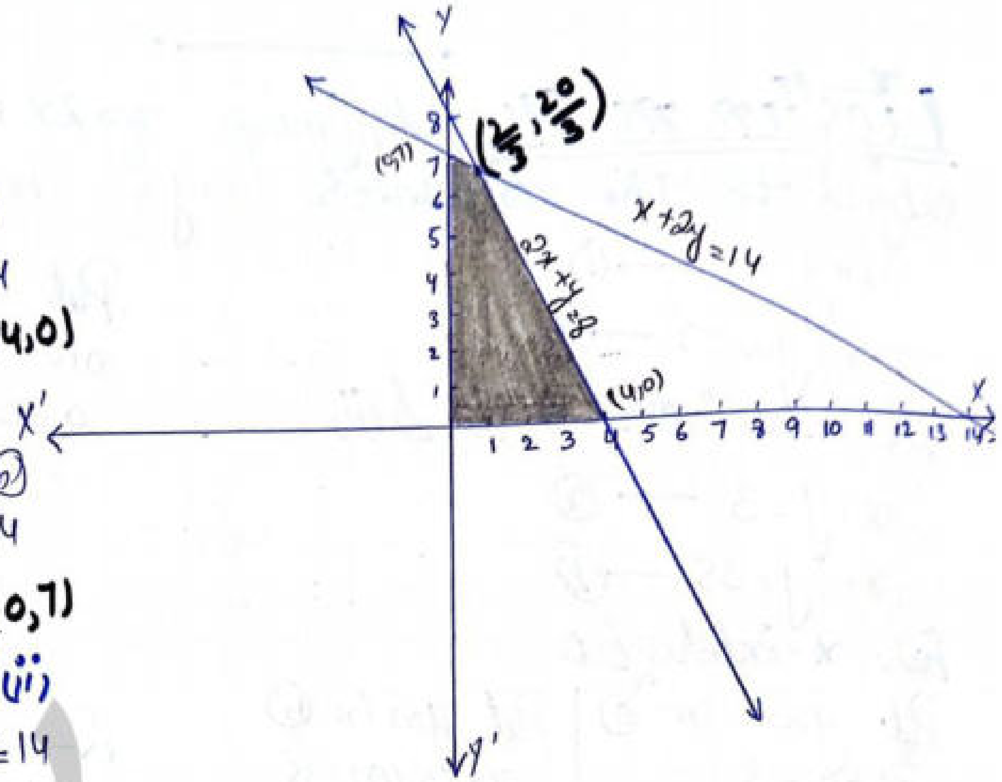
$2x + y \leq 8$ — (i)

$x + 2y \leq 14$ — (ii)

The associated of (i) & (ii)

$2x+y=8$ — ⓐ

$x+2y=14$ — ⓑ



For x-intercept

Put $y=0$ in ⓐ	Put $y=0$ in ⓑ
$2x+0=8$	$x+2(0)=14$
$x=4$ (4,0)	$x=14$ (14,0)

For y-intercept

Put $x=0$ in ⓐ	Put $x=0$ in ⓑ
$2(0)+y=8$	$0+2y=14$
$y=8$ (0,8)	$y=7$ (0,7)

Put test point in (i) & (ii)

$2(0)+0 \leq 8$

$0 \leq 8$ (T)

$0+2(0) \leq 14$

$0 \leq 14$ (T)

By Solving ⓐ & ⓑ

ⓐ - 2ⓑ

$$\cancel{2x}+y=8$$
$$\cancel{2x}+4y=28$$
$$-3y=-20$$
$$\boxed{y=\frac{20}{3}}$$

Put in ⓐ

$$2x+\frac{20}{3}=8$$
$$2x=8-\frac{20}{3} \Rightarrow 2x=\frac{24-20}{3}$$
$$2x=\frac{4}{3} \Rightarrow x=\frac{4}{3\times2} \Rightarrow \boxed{x=\frac{2}{3}}$$

Corner Point	$f(x,y)=2x+3y$
(0,0)	$=2(0)+3(0) \Rightarrow 0+0 \Rightarrow 0$
(4,0)	$=2(4)+3(0) \Rightarrow 8+0 \Rightarrow 8$
$(\frac{2}{3}, \frac{20}{3})$	$=2(\frac{2}{3})+3(\frac{20}{3}) \Rightarrow \frac{4}{3}+20 \Rightarrow \frac{64}{3} \Rightarrow 21.3$
(0,7)	$=2(0)+3(7) \Rightarrow 0+21 \Rightarrow 21$

Maximize at the corner point $(\frac{2}{3}, \frac{20}{3})$

Question no 6: Minimize $z = 3x + y$

Subject to the constraints $3x + 5y \geq 15$; $x + 3y \geq 9$; $x \geq 0$; $y \geq 0$

$3x + 5y \geq 15$ —(i)

$x + 3y \geq 9$ —(ii)

The associated of (i) & (ii)

$3x + 5y = 15$ —ⓐ

$x + 3y = 9$ —ⓑ

For x-intercept

Put $y = 0$ in ⓐ

$3x + 5(0) = 15$

$x = 5$ **(5,0)**

For y-intercept

Put $x = 0$ in ⓐ

$3(0) + 5y = 15$

$y = 3$ **(0,3)**

Put $y = 0$ in ⓑ

$x + 3(0) = 9$

$x = 9$ **(9,0)**

Put $x = 0$ in ⓑ

$0 + 3y = 9$

$y = 3$ **(0,3)**

Put test point in (i) & (ii)

$3x + 5y \geq 15$

$3(0) + 5(0) \geq 15$

$0 \geq 15$ **(F)**

$x + 3y \geq 9$

$0 + 3(0) \geq 9$

$0 \geq 9$ **(F)**

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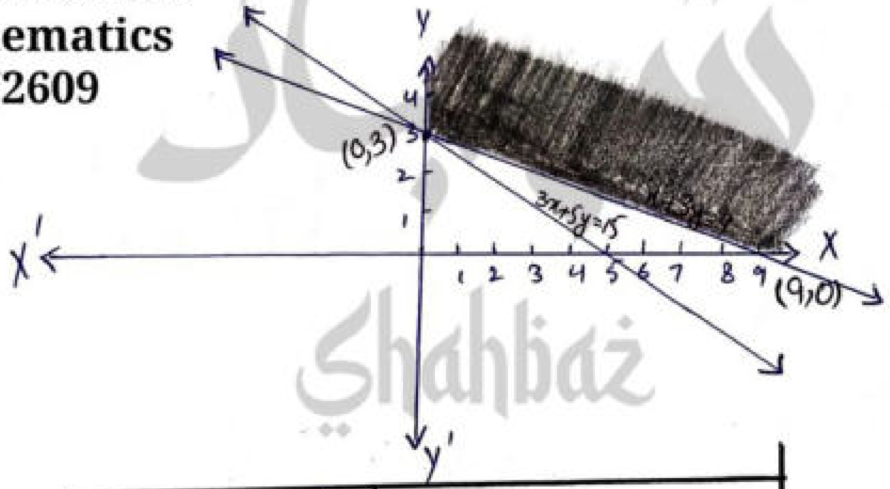

Corner Point	$z = 3x + y$
(0,3)	$= 3(0) + 3 \Rightarrow 0 + 3 \Rightarrow 3$
(9,0)	$= 3(9) + 0 \Rightarrow 27 + 0 \Rightarrow 27$

Minimize at the corner point (0,3)

The End

UNIT
6

Conic Section

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Unit no 6

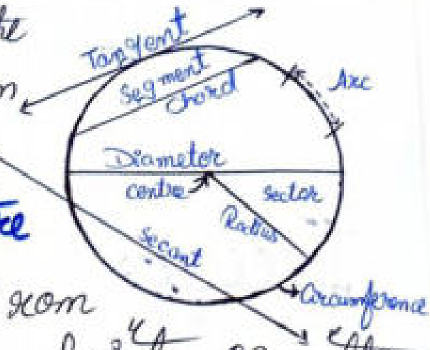
Conic Section

Theory:

Conic Section:

Conic Section or simply **conics**, are the curves obtained by cutting a (double) right circular cone by a plane. Let RS be the line through the centre C of a given circle and perpendicular to its plane. Let A be a fixed point on RS. All lines through A and points on a circle generate a **right circular cone**. The lines are called rulings or generators of the cone. The surface generated by cone consists of two parts, called **nappes**, meeting at the fixed point A, called the **vertex** or **apex** of the cone. The line RS is called **axis** of the cone.

- If the cone is cut by a plane perpendicular to the axis of the cone, then the section is a **circle**
- If the cutting plane is slightly tilted and cuts only one nappe of the cone, the resulting section is an **ellipse**
- If the intersecting plane is parallel to a generator of the cone, but intersects its one nappe only, the curve of intersection is a **parabola**
- If the cutting plane is parallel to the axis of the cone and intersects both of its nappes. then the curve of intersection is a **hyperbola**.



Circle: The set of all points in the plane that are equally distant from a fixed point is called **circle**. The fixed point is called the **centre** of the circle and the distance of from the centre of the circle to any point on the circle is called the **radius** of the circle.

A line segment whose end points lie on a circle is called a **chord** of the circle. A **diameter** of a circle is a chord containing the centre of the circle.

Equation of Circle in Standard form:

If $C(h,k)$ is centre of a circle, r its radius and $P(x,y)$ any point on the circle then equation of circle is given as

$$(x-h)^2+(y-k)^2=r^2 \quad \text{——(i)}$$

- If the centre of the circle is origin, then equation (i) reduces to $x^2+y^2=r^2$
- If $r=0$, the circle is called a **point circle** which consists of the centre only.
- $x=r\cos\theta$, $y=r\sin\theta$ are called **parametric equation** of the circle.

General Form of an Equation of a Circle: The equation $x^2+y^2+2gx+2fy+C=0$ is called general form of an equation of a circle with centre $(-g,-f)$ and radius $\sqrt{g^2+f^2-c}$

Note:

Every second-degree equation in two variables "x" and "y" in which coefficient of x^2 and y^2 is same and contains no term involving the product xy, represents the circle.

Example # 1: Write an equation of the circle with centre (-3,5) and radius 7

$C(h,k) = C(-3,5)$

radius $= r = 7$.

$\therefore (x-h)^2+(y-k)^2 = r^2$

$(x+3)^2+(y-5)^2 = (7)^2$

$x^2+9+6x+y^2+25-10y = 49$

$x^2+y^2+6x-10y+34-49=0$

$x^2+y^2+6x-10y-15=0$

Required Equation.

Example # 2: Show that the equations: $5x^2+5y^2+24x+36y+10=0$ represents a circle. Also find its centre and radius.

$5x^2+5y^2+24x+36y+10=0$

Dividing by '5'

$x^2+y^2+\frac{24}{5}x+\frac{36}{5}y+2=0$

Compare with

$x^2+y^2+2gx+2fy+c=0$

$2g = \frac{24}{5}$; $2f = \frac{36}{5}$; $c=2$

$g = \frac{12}{5}$; $f = \frac{18}{5}$

Centre $(-g,-f) = \left(-\frac{12}{5}, -\frac{18}{5}\right)$

Radius $= \sqrt{g^2+f^2-c}$

$= \sqrt{\left(\frac{12}{5}\right)^2+\left(\frac{18}{5}\right)^2-2}$

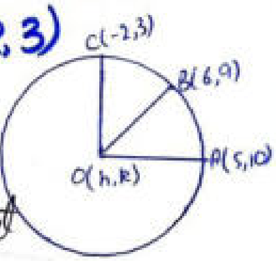
$r = \sqrt{\frac{144}{25}+\frac{324}{25}-2}$

$r = \sqrt{\frac{144+324-50}{25}}$

$r = \sqrt{\frac{418}{25}}$

$r = \frac{\sqrt{418}}{5}$

Example # 3: Find an equation of the circle which passes through the points A(5,10); B(6,9); C(-2,3)



We know that

$|AO| = |BO| \wedge |AO| = |CO|$

We have to find that

$|AO|^2 = |BO|^2$

$(h-5)^2+(k-10)^2 = (h-6)^2+(k-9)^2$

$h^2+25-10h+k^2+100-20k = h^2+36-12h+k^2+81-18k$

$h^2-10h+k^2-20k+125 = h^2-12h+k^2-18k+117$

$\cancel{h^2}-10h+\cancel{k^2}-20k+125-\cancel{h^2}+12h-\cancel{k^2}+18k-117=0$

$2h-2k+8=0$

$h-k+4=0$ — ①

Now

$|AO|^2 = |CO|^2$

$(h-5)^2+(k-10)^2 = (h+2)^2+(k-3)^2$

$h^2+25-10h+k^2+100-20k = h^2+4+4h+k^2-6k+9$

$h^2-10h+k^2-20k+125 = h^2+4h+k^2-6h+13$

$\cancel{h^2}-10h+\cancel{k^2}-20k+125-\cancel{h^2}-4h-\cancel{k^2}+6h-13=0$

$-14h-14k+112=0$

$h+k-8=0$ — ②

① + ②

$h - \cancel{k} + 4 = 0$

$h + \cancel{k} - 8 = 0$

$2h \quad - 4 = 0$

$2h = 4$

$\boxed{h = 2}$

Put $h=2$ in ②

$2 - k + 4 = 0$

$-k + 6 = 0$

$-k = -6$

$\boxed{k = 6}$

$O(h,k) = (2,6)$

$r = |AO| = \sqrt{(2-5)^2+(6-10)^2}$

$= \sqrt{(-3)^2+(-4)^2}$

$= \sqrt{9+16} = \sqrt{25}$

$\boxed{r = 5}$

Equation of a Circle:

$\therefore (x-h)^2+(y-k)^2 = r^2$

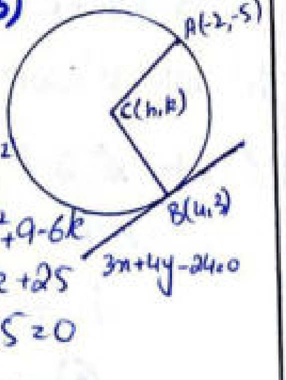
$(x-2)^2+(y-6)^2 = (5)^2$

$x^2+4-4x+y^2+36-12y = 25$

$x^2+y^2-4x-12y+40-25=0$

$x^2+y^2-4x-12y+15=0.$

Example # 5: Find an equation of a circle passing through the point (-2,-5) and touching the line $3x+4y-24=0$ at the point (4,3)



from figure:

$|AC|^2 = |BC|^2$

$(h+2)^2+(k+5)^2 = (h-4)^2+(k-3)^2$

$h^2+4+4h+k^2+25+10k = h^2+16-8h+k^2+9-6k$

$\cancel{h^2}+\cancel{k^2}+4h+10k+29 = \cancel{h^2}+\cancel{k^2}-8h-6k+25$

$4h+10k+29+8h+6k-25=0$

$12h+16k+4=0$

$3h+4k+1=0 \quad$ —①

Slope of $3x+4y-24=0 = -\frac{3}{4}$

Slope of $BC = \frac{k-3}{h-4}$

$\left(\frac{-3}{4}\right)\left(\frac{k-3}{h-4}\right) = -1 \qquad \because$ both are $\perp$

$-3(k-3) = -4(h-4) \Rightarrow -3k+9 = -4h+16$

$-3k+9+4h-16=0 \Rightarrow 4h-3k-7=0$ —②

3① + 4②

$9h+\cancel{12k}+3=0$

$16h-\cancel{12k}-28=0$

$25h \quad -25=0$

$25h = 25$

$\boxed{h = 1}$

Put $h=1$ in ①

$3(1)+4k+1=0$

$3+4k+1=0$

$4k+4=0$

$4k=-4$

$\boxed{k=-1}$

Centre $(h,k) = (1,-1)$.

radius $= |AC| = \sqrt{(-1+5)^2+(1+2)^2}$

$= \sqrt{(4)^2+(3)^2}$

$= \sqrt{16+9} = \sqrt{25}$

$r = 5$

Equation of a Circle:

$\therefore (x-h)^2+(y-k)^2 = r^2$

$(x-1)^2+(y+1)^2 = (5)^2$

$x^2+1-2x+y^2+1+2y = 25$

$x^2+y^2-2x+2y+2-25=0$

$x^2+y^2-2x+2y-23=0$

Required Equation

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Exercise # 6.1

Question no 1: In each of the fallowing, find an equation of the circle with

a) centre at (5,-2) and radius 4

$$C(h,k) = C(5,-2)$$
$$r = 4$$
$$\therefore (x-h)^2+(y-k)^2 = r^2$$
$$(x-5)^2+(y+2)^2 = (4)^2$$
$$x^2+25-10x+y^2+4y+4 = 16.$$
$$x^2+y^2-10x+4y+29-16=0$$
$$x^2+y^2-10x+4y+13=0$$

Required Equation.

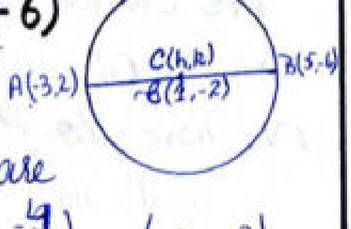
b) centre at $(\sqrt{2}, -3\sqrt{3})$ and radius $2\sqrt{2}$

$$C(h,k) = C(\sqrt{2}, -3\sqrt{3})$$
$$r = 2\sqrt{2}$$
$$\therefore (x-h)^2+(y-k)^2 = r^2$$
$$(x-\sqrt{2})^2+(y+3\sqrt{3})^2 = (2\sqrt{2})^2$$
$$x^2-2(x)(\sqrt{2})+(\sqrt{2})^2+y^2+2(y)(3\sqrt{3})+(3\sqrt{3})^2 = 4(2).$$
$$x^2-2\sqrt{2}\,x+2+y^2+6\sqrt{3}\,y+27 = 8.$$
$$x^2+y^2-2\sqrt{2}\,x+6\sqrt{3}\,y+29-8=0.$$
$$x^2+y^2-2\sqrt{2}\,x+6\sqrt{3}\,y+21=0.$$

Required Equation.

c) ends of a diameter at (-3,2) and (5,-6)

∴ C is the midpoint of AB



So coordinates are

$$= \left(\frac{-3+5}{2}, \frac{2-6}{2}\right) = \left(\frac{2}{2}, \frac{-4}{2}\right) = (1,-2)$$

$$r = |AC| = |BC|$$
$$r = |AC| = \sqrt{(1-(-3))^2+(-2-2)^2}$$
$$r = \sqrt{(1+3)^2+(-4)^2}$$
$$r = \sqrt{16+16}$$
$$r = \sqrt{32}$$
$$\therefore (x-h)^2+(y-k)^2 = r^2$$
$$(x-1)^2+(y+2)^2 = (\sqrt{32})^2$$
$$x^2+1-2x+y^2+4+4y = 32$$
$$x^2+y^2-2x+4y+5-32=0$$
$$x^2+y^2-2x+4y-27=0$$

Required Equation.

Question no 2: Find the centre and radius of the circle with the given equation.

a) $x^2+y^2+12x-10y=0$

$$x^2+y^2+12x-10y=0 \quad \text{—①}$$

Compare ① with

$$x^2+y^2+2gx+2fy+c=0$$
$$2g=12 \; ; \; 2f=-10 \; ; \; c=0$$
$$g=6 \; ; \; f=-5$$
$$\text{Centre} = (-g,-f) = (-6,5)$$
$$\text{Radius} = \sqrt{g^2+f^2-c}$$
$$= \sqrt{(6)^2+(-5)^2-0}$$
$$= \sqrt{36+25}$$
$$r = \sqrt{61}$$

b) $5x^2+5y^2+14x+12y-10=0$

$$5x^2+5y^2+14x+12y-10=0$$

Dividing by '5'

$x^2+y^2+\frac{14}{5}x+\frac{12}{5}y-2=0$ —①

Compare ① with

$x^2+y^2+2gx+2fy+c=0$

$2g=\frac{14}{5}$; $2f=\frac{12}{5}$; $c=-2$

$g=\frac{14}{5\times2}$; $f=\frac{12}{5\times2}$

$g=\frac{7}{5}$; $f=\frac{6}{5}$

Centre $=(-g,-f)=(-\frac{7}{5},\frac{-6}{5})$

Radius $=\sqrt{g^2+f^2-c}$

$=\sqrt{(\frac{7}{5})^2+(\frac{6}{5})^2-(-2)}$

$=\sqrt{\frac{49}{25}+\frac{36}{25}+2}$

$=\sqrt{\frac{49+36+50}{25}}$

$=\sqrt{\frac{135}{25}}=\sqrt{\frac{27}{5}}$

c) $x^2+y^2-6x+4y+13=0$

$x^2+y^2-6x+4y+13=0$ —①

Compare ① with

$x^2+y^2+2gx+2fy+c=0$

$2g=-6$; $2f=4$; $c=13$

$g=-3$; $f=2$

Centre $=(-g,-f)=(3,-2)$

Radius $=\sqrt{g^2+f^2-c}$

$=\sqrt{(-3)^2+(2)^2-13}$

$=\sqrt{9+4-13}$

$=\sqrt{13-13}$

$=\sqrt{0}$

$=0$

d) $4x^2+4y^2-8x+12y-25=0$

$4x^2+4y^2-8x+12y-25=0$

Dividing by '4'

$x^2+y^2-2x+3y-\frac{25}{4}=0$ —①

Compare ① with

$x^2+y^2+2gx+2fy+c=0$

$2g=-2$; $2f=3$; $c=-\frac{25}{4}$

$g=-1$; $f=\frac{3}{2}$

Centre $=(-g,-f)=(1,-\frac{3}{2})$

Radius $=\sqrt{g^2+f^2-c}$

$=\sqrt{(-1)^2+(\frac{3}{2})^2-(-\frac{25}{4})}$

$=\sqrt{1+\frac{9}{4}+\frac{25}{4}}$

$=\sqrt{\frac{4+9+25}{4}}$

$=\sqrt{\frac{38}{4}}$

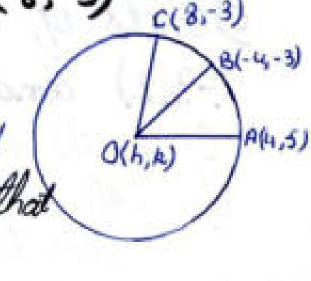
$=\sqrt{\frac{19}{2}}$

Question no 3: Write an equation of the circle that passes through the given points:

a) A(4,5), B(-4,-3); C(8,-3)

We know that

|AO|=|BO| ; |AO|=|CO|



We have to find that

$|AO|^2=|BO|^2$

$(h-4)^2+(k-5)^2=(h+4)^2+(k+3)^2$

$$h^2+16-8h+k^2+25-10k=h^2+16+8h+k^2+9+6k$$

$$h^2-8h+k^2-10k+41=h^2+8h+k^2+6k+25$$

$$\cancel{h^2}-8h+\cancel{k^2}-10k+41-\cancel{h^2}-8h-\cancel{k^2}-6k-25=0$$

$$-16h-16k+16=0$$

$$-16(h+k-1)=0$$

$$h+k-1=0 \text{ —— } ①$$

Now

$$|AO|^2=|CO|^2$$

$$(h-4)^2+(k-5)^2=(h-8)^2+(k+3)^2$$

$$h^2+16-8h+k^2+25-10k=h^2+64-16h+k^2+9+6k$$

$$h^2-8h+k^2-10k+41=h^2-16h+k^2+6k+73$$

$$\cancel{h^2}-8h+\cancel{k^2}-10k+41-\cancel{h^2}+16h-\cancel{k^2}-6k-73=0$$

$$8h-16k-32=0$$

$$8(h-2k-4)=0$$

$$h-2k-4=0 \text{ —— } ②$$

① - ②

$$\cancel{h}+k-1=0$$

$$\cancel{h}\underset{+}{-}2k\underset{+}{-}4=0$$

$$3k+3=0$$

$$3k=-3$$

$$\boxed{k=-1}$$

Put $k=-1$ in ①

$$h+(-1)-1=0$$

$$h-1-1=0$$

$$h-2=0$$

$$\boxed{h=2}$$

$$O(h,k)=O(2,-1)$$

$$r=|AO|=|BO|=|CO|$$

$$r=|AO|=\sqrt{(2-4)^2+(-1-5)^2}$$

$$=\sqrt{(-2)^2+(-6)^2}$$

$$r=\sqrt{4+36}$$

$$r=\sqrt{40}$$

Equation of a circle:

$$\therefore (x-h)^2+(y-k)^2=r^2$$

$$(x-2)^2+(y+1)^2=(\sqrt{40})^2$$

$$x^2+4-4x+y^2+1+2y=40$$

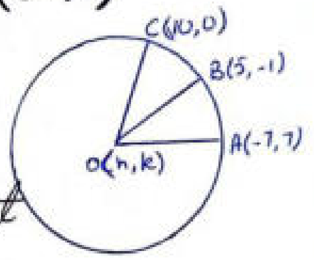
$$x^2+y^2-4x+2y+5-40=0$$

$$x^2+y^2-4x+2y-35=0$$

Required Equation

b) $A(-7,7)$; $B(5,-1)$; $C(10,0)$

We know that



$$|AO|=|BO| \ ; \ |AO|=|CO|$$

We have to find that

$$|AO|^2=|BO|^2$$

$$(h+7)^2+(k-7)^2=(h-5)^2+(k+1)^2$$

$$h^2+49+14h+k^2+49-14k=h^2+25-10h+k^2+1+2k$$

$$h^2+14h+k^2-14k+98=h^2-10h+k^2+2k+26$$

$$\cancel{h^2}+14h+\cancel{k^2}-14k+98-\cancel{h^2}+10h-\cancel{k^2}-2k-26=0$$

$$24h-16k+72=0$$

$$8(3h-2k+9)=0$$

$$3h-2k+9=0 \text{ —— } ①$$

Now

$$|AO|^2=|CO|^2$$

$$(h+7)^2+(k-7)^2=(h-10)^2+(k-0)^2$$

$$h^2+49+14k+k^2+49-14k=h^2+100-20h+k^2$$

$$\cancel{h^2}+14k+\cancel{k^2}-14k+98-\cancel{h^2}-100+20h-\cancel{k^2}=0$$

$$34h-14k-2=0$$

$$2(17h-7k-1)=0$$

$$17h-7k-1=0 \text{ —— } ②$$

17① - 3②

$$51\cancel{h} - 34k + 153 = 0$$
$$-\cancel{51h} \mp 21k \mp 3 = 0$$
$$-13k + 156 = 0$$
$$-13k = -156$$
$$\boxed{k = 12}$$

Put $k = 12$ in ①

$$3h - 2(12) + 9 = 0$$
$$3h - 24 + 9 = 0$$
$$3h - 15 = 0$$
$$3h = 15$$
$$\boxed{h = 5}$$

Centre $O(h,k) = O(5,12)$

Radius $= r = |AO| = |BO| = |CO|$

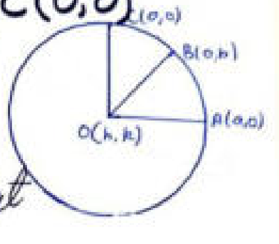
$$r = |AO| = \sqrt{(5-(-7))^2 + (12-7)^2}$$
$$= \sqrt{(5+7)^2 + (12-7)^2}$$
$$= \sqrt{(12)^2 + (5)^2}$$
$$= \sqrt{144 + 25}$$
$$= \sqrt{169}$$
$$r = 13$$

Equation of a circle:

$$(x-h)^2 + (y-k)^2 = r^2$$
$$(x-5)^2 + (y-12)^2 = (13)^2$$
$$x^2 + 25 - 10x + y^2 + 144 - 24y = 169$$
$$x^2 + y^2 - 10x - 24y + 169 - 169 = 0$$
$$x^2 + y^2 - 10x - 24y = 0$$

Required Equation

c) $A(a,0)$; $B(0,b)$; $C(0,0)$



We know that

$|AO| = |BO|$; $|AO| = |CO|$

We have to find that

$$|AO|^2 = |BO|^2$$
$$(h-a)^2 + (k-0)^2 = (h-0)^2 + (k-b)^2$$
$$h^2 + a^2 - 2ah + k^2 = h^2 + k^2 + b^2 - 2kb$$
$$\cancel{h^2} + a^2 - 2ah + \cancel{k^2} - \cancel{h^2} - \cancel{k^2} - b^2 + 2bk = 0$$
$$a^2 - b^2 - 2ah + 2bk = 0 \quad \text{—①}$$

Now

$$|AO|^2 = |CO|^2$$
$$(h-a)^2 + (k-0)^2 = (h-0)^2 + (k-0)^2$$
$$h^2 + a^2 - 2ah + k^2 = h^2 + k^2$$
$$\cancel{h^2} + a^2 - 2ah + \cancel{k^2} - \cancel{h^2} - \cancel{k^2} = 0$$
$$a^2 - 2ah = 0 \quad \text{—②}$$

① - ②

$$\cancel{a^2} - b^2 - \cancel{2ah} + 2bk = 0$$
$$-\cancel{a^2} \qquad \mp\cancel{2ah}$$
$$-b^2 + 2bk = 0$$
$$2bk - b^2 = 0$$
$$b(2k - b) = 0$$
$$2k - b = 0$$
$$2k = b$$
$$\boxed{k = \frac{b}{2}}$$

Put in ①

$$a^2 - b^2 - 2ah + \cancel{2}b\left(\frac{b}{\cancel{2}}\right) = 0$$
$$a^2 - \cancel{b^2} - 2ah + \cancel{b^2} = 0$$
$$a^2 - 2ah = 0$$
$$a(a - 2h) = 0$$
$$a - 2h = 0$$
$$a = 2h$$
$$\boxed{h = a/2}$$

Centre $O(h,k) = O\left(\frac{a}{2}, \frac{b}{2}\right)$

Radius $= r = |AO| = |BO| = |CO|$

$$r = |CO| = \sqrt{\left(\frac{a}{2}-0\right)^2 + \left(\frac{b}{2}-0\right)^2}$$

$$r = \sqrt{\frac{a^2}{4} + \frac{b^2}{4}} = \sqrt{\frac{a^2+b^2}{4}}$$

Equation of a Circle

$$(x-h)^2 + (y-k)^2 = r^2$$

$$\left(x-\frac{a}{2}\right)^2 + \left(y-\frac{b}{2}\right)^2 = \left(\frac{\sqrt{a^2+b^2}}{2}\right)^2$$

$$x^2 + \frac{a^2}{4} - ax + y^2 + \frac{b^2}{4} - by = \frac{a^2+b^2}{4}$$

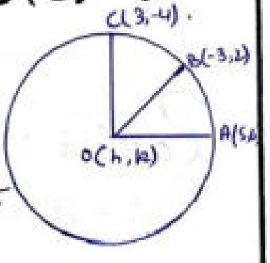
$$x^2 + y^2 - ax - by + \cancel{\frac{a^2+b^2}{4}} - \cancel{\frac{a^2+b^2}{4}} = 0$$

$$x^2 + y^2 - ax - by = 0.$$

Required Equation.

d) $A(5,6); B(-3,2); C(3,-4)$

We know that



$|AO| = |BO|$; $|AO| = |CO|$

We have to find that

$$|AO|^2 = |BO|^2$$

$$(h-5)^2 + (k-6)^2 = (h+3)^2 + (k-2)^2$$

$$h^2 + 25 - 10h + k^2 + 36 - 12k = h^2 + 9 + 6h + k^2 + 4 - 4k$$

$$h^2 - 10h + k^2 - 12k + 61 = h^2 + 6h + k^2 - 4k + 13$$

$$\cancel{h^2} - 10h + \cancel{k^2} - 12k + 61 - \cancel{h^2} - 6h - \cancel{k^2} + 4k - 13 = 0$$

$$-16h - 8k + 48 = 0$$

$$-8(2h + k - 6) = 0$$

$$2h + k - 6 = 0 \quad — ①$$

Now

$$|AO|^2 = |CO|^2$$

$$(h-5)^2 + (k-6)^2 = (h-3)^2 + (k+4)^2$$

$$h^2 + 25 - 10h + k^2 + 36 - 12k = h^2 + 9 - 6h + k^2 + 16 + 8k$$

$$h^2 - 10h + k^2 - 12k + 61 = h^2 - 6h + k^2 + 8k + 25$$

$$\cancel{h^2} - 10h + \cancel{k^2} - 12k + 61 - \cancel{h^2} + 6h - \cancel{k^2} - 8k - 25 = 0$$

$$-4h - 20k + 36 = 0$$

$$-4(h + 5k - 9) = 0$$

$$h + 5k - 9 = 0 \quad — ②$$

$5① - ②$

$$10h + 5\cancel{k} - 30 = 0$$

$$\underset{-}{}h \underset{-}{+} \cancel{5k} \underset{+}{-} 9 = 0$$

$$9h \quad - 21 = 0$$

$$9h = 21$$

$$h = \frac{21}{9}$$

$$\boxed{h = \frac{7}{3}}$$

Put in ②

$$\frac{7}{3} + 5k - 9 = 0.$$

$$5k = 9 - \frac{7}{3}$$

$$5k = \frac{27-7}{3}$$

$$5k = \frac{20}{3}$$

$$k = \frac{20}{3 \times 5}$$

$$\boxed{k = \frac{4}{3}}$$

Centre $O(h,k) = O\left(\frac{7}{3}, \frac{4}{3}\right)$

Radius $r = |AO| = |BO| = |CO|$

$$r = |AO| = \sqrt{\left(\frac{7}{3}-5\right)^2 + \left(\frac{4}{3}-6\right)^2}$$

$$= \sqrt{\left(\frac{7-15}{3}\right)^2 + \left(\frac{4-18}{3}\right)^2}$$

$$r=\sqrt{\left(-\frac{8}{3}\right)^2+\left(-\frac{14}{3}\right)^2}$$

$$r=\sqrt{\frac{64}{9}+\frac{196}{9}}=\sqrt{\frac{64+196}{9}}$$

$$r=\sqrt{\frac{260}{9}}$$

Equation of a Circle:

$$\therefore (x-h)^2+(y-k)^2=r^2$$

$$\left(x-\frac{7}{3}\right)^2+\left(y-\frac{4}{3}\right)^2=\left(\sqrt{\frac{260}{9}}\right)^2$$

$$x^2+\frac{49}{9}-\frac{14}{3}x+y^2+\frac{16}{9}-\frac{8}{3}y=\frac{260}{9}$$

$$x^2+y^2-\frac{14}{3}x-\frac{8}{3}y+\frac{49}{9}+\frac{16}{9}-\frac{260}{9}=0$$

$$\frac{9x^2+9y^2-42x-24y+49+16-260}{9}=0$$

$$9x^2+9y^2-42x-24y-195=0$$

$$9(x^2+y^2)-42x-24y-195=0$$

Dividing by '3'

$$3(x^2+y^2)-14x-8y-65=0.$$

Required Equation.

Question no 4: In each of the following, find an equation of the circle passing through.

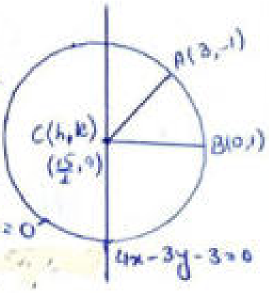
a) A(3,-1); B(0,1) and having centre at $4x-3y-3=0$.

Given:

$$4h-3k-3=0 \text{ — } ①$$

We know that

$$|AC|^2=|BC|^2$$



$$(h-3)^2+(k+1)^2=(h-0)^2+(k-1)^2$$

$$h^2+9-6h+k^2+1+2k=h^2+k^2+1-2k$$

$$\cancel{h^2}-6h+\cancel{k^2}+2k+10-\cancel{h^2}-\cancel{k^2}-1+2k=0$$

$$-6h+4k+9=0 \text{ — } ②$$

$$4① + 3②$$

$$16h-\cancel{12k}-12=0$$

$$-18h+\cancel{12k}+27=0$$

$$-2h \quad +15=0$$

$$-2h=-15$$

$$\boxed{h=\frac{15}{2}}$$

Put in ①

$$4\left(\frac{15}{\cancel{2}}\right)-3k-3=0$$

$$30-3k-3=0$$

$$-3k+27=0$$

$$-3k=-27$$

$$\boxed{k=9}$$

$$C(h,k)=\left(\frac{15}{2},9\right)$$

Radius $r=|AC|=|BC|$

$$r=|AC|=\sqrt{\left(\frac{15}{2}-3\right)^2+\left(9-(-1)\right)^2}$$

$$r=\sqrt{\left(\frac{15-6}{2}\right)^2+(9+1)^2}$$

$$r=\sqrt{\left(\frac{9}{2}\right)^2+(10)^2}$$

$$r=\sqrt{\frac{81}{4}+100}$$

$$r=\sqrt{\frac{81+400}{4}}$$

$$r=\sqrt{\frac{481}{4}}$$

Equation of a Circle:

$$\therefore (x-h)^2+(y-k)^2=r^2$$

$$\left(x-\frac{15}{2}\right)^2+(y-9)^2=\left(\sqrt{\frac{481}{4}}\right)^2$$

$$x^2+\frac{225}{4}-15x+y^2+81-18y=\frac{481}{4}$$

$$x^2+y^2-15x-18y+\frac{225}{4}+81-\frac{481}{4}=0$$

$$\frac{4x^2+4y^2-60x-72y+225+324-481=0}{4}$$

$$4x^2+4y^2-60x-72y+225+324-481=0$$

$$4x^2+4y^2-60x-72y+68=0$$

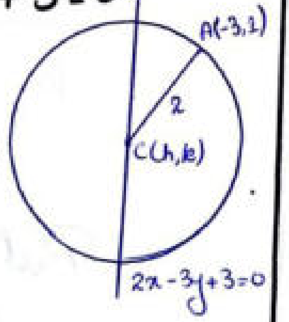
$$4(x^2+y^2)-60x-72y+68=0$$

Dividing by '4'

$$x^2+y^2-15x-18y+17=0.$$

Required Equation

b) A(-3,1) with radius 2 and centre at $2x-3y+3=0$.



Given:

$$2h-3k+3=0 \quad — ①$$

From figure:

$$|AC|^2=4$$

$$(h-(-3))^2+(k-1)^2=4$$

$$(h+3)^2+(k-1)^2=4$$

$$h^2+9+6h+k^2+1-2k=4$$

$$h^2+6h+k^2-2k+10-4=0$$

$$h^2+6h+k^2-2k+6=0 \quad — ②$$

from ①

$$2h-3k+3=0$$

$$2h=3k-3$$

$$h=\frac{3k-3}{2} \quad — ③$$

Put ③ in ②

$$\left(\frac{3k-3}{2}\right)^2+\overset{3}{\cancel{6}}\left(\frac{3k-3}{\cancel{2}}\right)+k^2-2k+6=0$$

$$\frac{9k^2+9-18k}{4}+9k-9+k^2-2k+6=0$$

$$\frac{9k^2+9-18k+36k-36+4k^2-8k+24}{4}=0$$

$$13k^2+10k-3=0$$

$$13k^2+13k-3k-3=0$$

$$13k(k+1)-3(k+1)=0$$

$$(k+1)(13k-3)=0$$

$$k+1=0 \quad ; \quad 13k-3=0$$

$$\boxed{k=-1} \quad ; \quad 13k=3$$

$$\boxed{k=\frac{3}{13}}$$

Put in ③ ; Put in ③

$$h=\frac{3(-1)-3}{2} \quad ; \quad h=\frac{3\left(\frac{3}{13}\right)-3}{2}$$

$$h=\frac{-3-3}{2} \quad ; \quad h=\frac{\frac{9}{13}-3}{2}$$

$$h=\frac{-6}{2} \quad ; \quad h=\frac{\frac{9-39}{13}}{2}$$

$$\boxed{h=-3} \quad ; \quad h=\frac{-30}{26}$$

$$\boxed{h=\frac{-15}{13}}$$

$C(-3,-1)$; $C\left(-\frac{15}{13},\frac{3}{13}\right)$

Equation of a circle when $h=-3$ and $k=-1$.

$$\therefore (x-h)^2+(y-k)^2=r^2$$

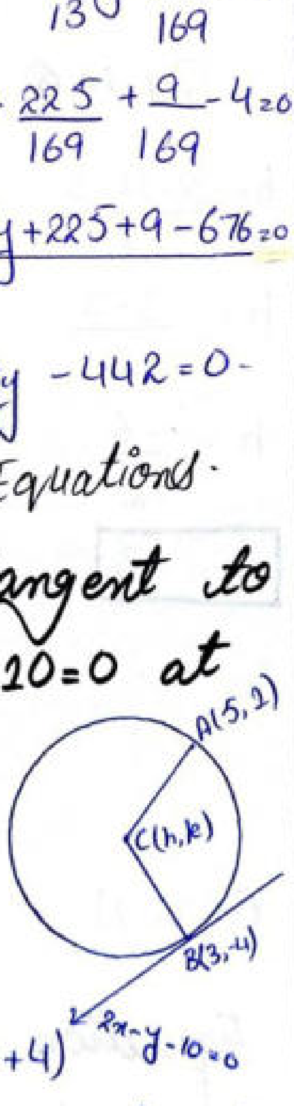
$$(x-(-3))^2+(y-(-1))^2=(2)^2$$
$$(x+3)^2+(y+1)^2=4.$$
$$x^2+9+6x+y^2+1+2y-4=0.$$
$$x^2+y^2+6x+2y+6=0-$$

Equation of a circle when $h=-\frac{15}{13}$ and $k=\frac{3}{13}$

$$\therefore (x-h)^2+(y-k)^2=r^2.$$
$$\left(x-\left(-\frac{15}{13}\right)\right)^2+\left(y-\frac{3}{13}\right)^2=(2)^2$$
$$\left(x+\frac{15}{13}\right)^2+\left(y-\frac{3}{13}\right)^2=4.$$
$$x^2+\frac{30}{13}x+\frac{225}{169}+y^2-\frac{6}{13}y+\frac{9}{169}=4$$
$$x^2+y^2+\frac{30}{13}x+\frac{6}{13}y+\frac{225}{169}+\frac{9}{169}-4=0$$
$$\frac{169x^2+169y^2+390x-78y+225+9-676=0}{169}$$
$$169(x^2+y^2)+390x-78y-442=0-$$

Required Equations.

C) A(5,1) and tangent to the line $2x-y-10=0$ at B(3,-4)



From figure

$$|AC|^2=|BC|^2$$
$$(h-5)^2+(k-1)^2=(h-3)^2+(k+4)^2$$
$$h^2+25-10h+k^2+1-2k=h^2+9-6h+k^2+16+8k$$
$$h^2-10h+k^2-2k+26=h^2-6h+k^2+8k+25$$
$$\cancel{h^2}-10h+\cancel{k^2}-2k+26-\cancel{h^2}+6h-\cancel{k^2}-8k-25=0$$
$$-4h-10k+1=0 \text{ —— ①}$$

Slope of $2x-y-10=\frac{-a}{b}=\frac{-2}{-1}=2$

Slope of $BC=\frac{k+4}{h-3}$ $\therefore \frac{y_2-y_1}{x_2-x_1}$

∴ both lines are ⊥ so,

$$(2)\left(\frac{k+4}{h-3}\right)=-1.$$
$$2(k+4)=-1(h-3).$$
$$2k+8=-h+3$$
$$h+2k+8-3=0$$
$$h+2k+5=0 \text{ —— ②}$$

① + 5②

$$-4h-\cancel{10k}+1=0$$
$$5h+\cancel{10k}+25=0$$
$$h \quad +26=0$$
$$\boxed{h=-26}$$

Put in ②

$$-26+2k+5=0$$
$$2k-21=0$$
$$2k=21$$
$$\boxed{k=\frac{21}{2}}$$

Centre $(h,k)=\left(-26,\frac{21}{2}\right)$

$$r=|AC|=|BC|$$
$$r=|AC|=\sqrt{(-26-5)^2+\left(\frac{21}{2}-1\right)^2}$$
$$r=\sqrt{(-31)^2+\left(\frac{21-2}{2}\right)^2}$$

$$r = \sqrt{(-31)^2 + \left(\frac{19}{2}\right)^2}$$

$$r = \sqrt{961 + \frac{361}{4}}$$

$$r = \sqrt{\frac{3844 + 361}{4}}$$

$$r = \sqrt{\frac{4205}{4}}$$

Equation of a circle:

$$\therefore (x-h)^2 + (y-k)^2 = r^2$$

$$(x-(-26))^2 + \left(y - \frac{21}{2}\right)^2 = \left(\sqrt{\frac{4205}{4}}\right)^2$$

$$(x+26)^2 + \left(y - \frac{21}{2}\right)^2 = \frac{4205}{4}$$

$$x^2 + 676 + 52x + y^2 + \frac{441}{4} - 21y - \frac{4205}{4} = 0$$

$$x^2 + y^2 + 52x - 21y + 676 + \frac{441}{4} - \frac{4205}{4} = 0$$

$$\frac{4x^2 + 4y^2 + 208x - 84y + 2704 + 44 - 4205 = 0}{4}$$

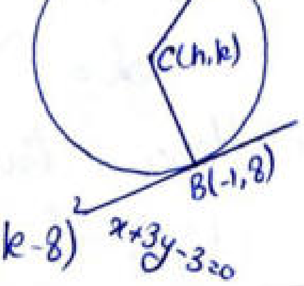
$$4(x^2+y^2) + 208x - 84y - 1060 = 0$$

Dividing by '4'

$$x^2 + y^2 + 52x - 21y - 265 = 0$$

Required Equation.

d) A(1,4), B(-1,8) and tangent to the line $x + 3y - 3 = 0$.



From figure

$$|AC|^2 = |BC|^2$$

$$(h-1)^2 + (k-4)^2 = (h+1)^2 + (k-8)^2$$

$$h^2 + 1 - 2h + k^2 + 16 - 8k = h^2 + 1 + 2h + k^2 + 64 - 16k$$

$$h^2 - 2h + k^2 - 8k + 17 = h^2 + 2h + k^2 - 16k + 65$$

$$\cancel{h^2} - 2h + \cancel{k^2} - 8k + 17 - \cancel{h^2} - 2h - \cancel{k^2} + 16k - 65 = 0$$

$$-4h + 8k - 48 = 0$$

$$-4(h - 2k + 12) = 0$$

$$h - 2k + 12 = 0 \quad \text{—①}$$

Slope of $x + 3y - 3 = 0 = \frac{-a}{b} = \frac{-1}{3}$

Slope of $BC = \frac{k-8}{h+1}$ $\quad \because \frac{y_2 - y_1}{x_2 - x_1}$

$\because$ both lines are $\perp$ so,

$$\left(\frac{-1}{3}\right)\left(\frac{k-8}{h+1}\right) = -1.$$

$$-1(k-8) = -3(h+1)$$

$$-k + 8 = -3h - 3$$

$$3h - k + 8 + 3 = 0$$

$$3h - k + 11 = 0 \quad \text{—②}$$

① - 2②

$$\begin{array}{l} h - \cancel{2k} + 12 = 0 \\ \underset{-}{}6h \underset{+}{-} \cancel{2k} \underset{-}{+} 22 = 0 \\ \hline -5h - 10 = 0 \end{array}$$

$$-5h = 10$$

$$\boxed{h = -2}$$

Put in ①

$$-2 - 2k + 12 = 0$$

$$-2k + 10 = 0$$

$$-2k = -10$$

$$\boxed{k = 5}$$

Centre $(h,k) = C(-2,5)$

$r = |AC| = |BC|$

$r = |AC| = \sqrt{(-2-1)^2+(5-4)^2}$

$r = \sqrt{(-3)^2+(1)^2}$

$r = \sqrt{9+1}$

$r = \sqrt{10}$

Equation of a circle:

$\therefore (x-h)^2+(y-k)^2 = r^2$

$(x+2)^2+(y-5)^2 = (\sqrt{10})^2$

$x^2+4+4x+y^2+25-10y = 10$

$x^2+y^2+4x-10y+29-10=0$

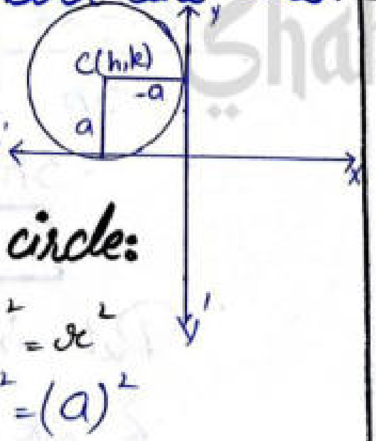
$x^2+y^2+4x-10y+19=0$

Required Equation.

Question no 5: Find an equation of a circle of radius a and lying in the second quardant such that it is tangent to both the axes.

Centre$(h,k) = (-a,a)$

radius $= a$



Equation of a circle:

$\therefore (x-h)^2+(y-k)^2 = r^2$

$(x-(-a))^2+(y-a)^2=(a)^2$

$(x+a)^2+(y-a)^2 = a^2$

$x^2+a^2+2ax+y^2+\cancel{a^2}-2ay-\cancel{a^2}=0$

$x^2+y^2+2ax-2ay+a^2=0$

Required Equation.

Question no 6: Show that the lines $3x-2y=0$ and $2x+3y-13=0$ are tangents to the circle $x^2+y^2+6x-4y=0$

Given

$x^2+y^2+6x-4y=0$

Compare with

$x^2+y^2+2gx+2fy+c=0$

$2g=6$; $2f=-4$; $c=0$

$g=3$; $f=-2$

Centre $(-g,-f) = (-3,2)$

radius $= r = \sqrt{g^2+f^2-c}$

$r = \sqrt{(3)^2+(-2)^2-0}$

$r = \sqrt{9+4}$

$r = \sqrt{13}$

Distance of Centre from line $3x-2y=0$

$d = \frac{|3(-3)-2(2)+0|}{\sqrt{(3)^2+(-2)^2}}$ $\quad \therefore \frac{|ax_1+by_1+c|}{\sqrt{a^2+b^2}}$

$d = \frac{|-9-4|}{\sqrt{9+4}} = \frac{|-13|}{\sqrt{13}}$

$d = \frac{13}{\sqrt{13}} \Rightarrow d = \frac{\sqrt{13}\sqrt{13}}{\sqrt{13}}$

$d = \sqrt{13}$

Hence line $3x-2y=0$ is tangent.

Now Distance of centre from line $2x+3y-13=0$.

$$d = \frac{|2(-3)+3(2)-13|}{\sqrt{(2)^2+(3)^2}} \quad \therefore \frac{|ax_1+by_1+c|}{\sqrt{a^2+b^2}}$$

$$d = \frac{|-6+6-13|}{\sqrt{4+9}} = \frac{|-13|}{\sqrt{13}}$$

$$d = \frac{13}{\sqrt{13}} \Rightarrow d = \frac{\cancel{\sqrt{13}}\sqrt{13}}{\cancel{\sqrt{13}}}$$

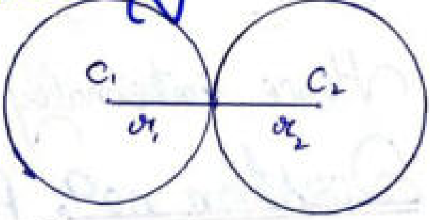
$$d = \sqrt{13}$$

Hence line $2x+3y-13=0$ is tangent.

Question no7: Show that the circles

$$x^2+y^2+2x-2y-7=0 \quad \text{and}$$
$$x^2+y^2-6x+4y+9=0$$

touch externally.



For Circle C_1

$$x^2+y^2+2x-2y-7=0$$

Compare with

$$x^2+y^2+2gx+2fy+c=0$$

$2g=2x$; $2f=-2$; $c=-7$

$g=1$; $f=-1$

Centre, $(-g,-f) = C_1(-1,1)$

$$r_1 = \sqrt{g^2+f^2-c}$$
$$= \sqrt{(1)^2+(-1)^2-(-7)}$$
$$r_1 = \sqrt{1+1+7}$$
$$r_1 = \sqrt{9}$$
$$r_1 = 3$$

For Circle C_2

$$x^2+y^2-6x+4y+9=0$$

Compare with

$$x^2+y^2+2gx+2fy+c=0$$

$2g=-6$; $2f=4$; $c=9$

$g=-3$; $f=2$

Centre $(-g,-f) = C_2(3,-2)$

$$r_2 = \sqrt{g^2+f^2-c}$$
$$r_2 = \sqrt{(-3)^2+(2)^2-9}$$
$$r_2 = \sqrt{9+4-9}$$
$$r_2 = \sqrt{4}$$
$$r_2 = 2$$

$$|C_1C_2| = \sqrt{(3+1)^2+(-2-1)^2}$$
$$= \sqrt{(4)^2+(-3)^2}$$
$$= \sqrt{16+9}$$
$$= \sqrt{25}$$
$$= 5.$$

$$|C_1C_2| = r_1+r_2$$
$$= 3+2$$
$$= 5$$

Hence externally touched

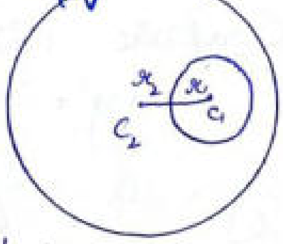
Question no 8: Show that the circles

$x^2+y^2+2x-8=0$ and

$x^2+y^2-6x+6y-46=0$

touche internally.

For Circle C_1



$x^2+y^2+2x-8=0$

Compare with

$x^2+y^2+2gx+2fy+c=0$

$2g=2$; $2f=0$; $c=-8$

$g=1$; $f=0$

Centre $(-g,-f) = C_1(-1,0)$. .

$r_1 = \sqrt{g^2+f^2-c}$

$r_1 = \sqrt{(1)^2+(0)^2-(-8)}$

$r_1 = \sqrt{1+0+8}$

$r_1 = \sqrt{9}$

$r_1 = 3$

For Circle C_2

$x^2+y^2-6x+6y-46=0$

Compare with

$x^2+y^2+2gx+2fy+c=0$

$2g=-6$; $2f=6$; $c=-46$

$g=-3$; $f=3$

Centre $(-g,-f) = C_2(3,-3)$

$r_2 = \sqrt{g^2+f^2-c}$

$r_2 = \sqrt{(-3)^2+(3)^2-(-46)}$

$r_2 = \sqrt{9+9+46}$

$r_2 = \sqrt{64}$

$r_2 = 8$

$|C_1C_2| = \sqrt{(3+1)^2+(-3-0)^2}$

$= \sqrt{(4)^2+(-3)^2}$

$= \sqrt{16+9}$

$= \sqrt{25}$

$= 5$

$|C_1C_2| = r_2 - r_1$

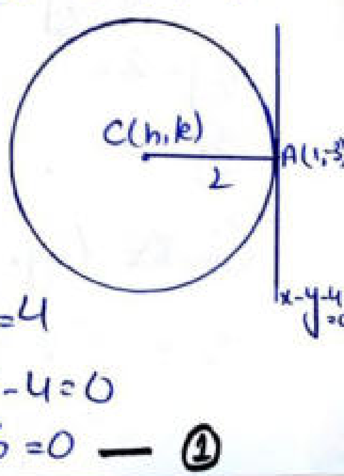
$= 8-3$

$= 5$

Hence internally touched.

Question no 9: Find equations of the circle of radius 2 and tangent to the line $x-y-4=0$ at $A(1,-3)$.

From figure



$|AC|^2 = 4$

$(h-1)^2+(k+3)^2=4$

$h^2+1-2h+k^2+9+6k=4$

$h^2-2h+k^2+6k+10-4=0$

$h^2-2h+k^2+6k+6=0$ — ①

Slope of $x-y-4=0 = -\frac{a}{b} = \frac{-1}{-1} = 1$

Slope of $AC = \frac{k+3}{h-1}$ $\quad \because \frac{y_2-y_1}{x_2-x_1}$

∴ both lines are ⊥ so,

$$(1)\left(\frac{k+3}{h-1}\right) = -1.$$

$$1(k+3) = -1(h-1)$$

$$k+3 = -h+1.$$

$$h+k+3-1=0$$

$$h+k+2=0.$$

$$h=-k-2 \quad — ②$$

Put $h=-k-2$ in ①

$$(-k-2)^2-2(-k-2)+k^2+6k+6=0$$

$$k^2+4+4k+2k+4+k^2+6k+6=0$$

$$2k^2+12k+14=0$$

$$2(k^2+6k+7)=0$$

$$k^2+6k+7=0.$$

$a=1$; $b=6$; $c=7$

By Using quadratic formula

$$k = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$k = \frac{-6\pm\sqrt{(6)^2-4(1)(7)}}{2(1)}$$

$$k = \frac{-6\pm\sqrt{36-28}}{2}$$

$$k = \frac{-6\pm\sqrt{8}}{2}$$

$$k = \frac{-6\pm2\sqrt{2}}{2}$$

$$k = \frac{\cancel{2}(-3\pm\sqrt{2})}{\cancel{2}}$$

$$k = -3\pm\sqrt{2}$$

$\boxed{k=-3+\sqrt{2}}$; $\boxed{k=-3-\sqrt{2}}$

Put in ② $\qquad$ Put in ②

$h=-(-3+\sqrt{2})-2 \qquad h=-(-3-\sqrt{2})-2$

$h=3-\sqrt{2}-2 \qquad h=3+\sqrt{2}-2$

$\boxed{h=1-\sqrt{2}} \qquad \boxed{h=1+\sqrt{2}}$

Equation of a Circle when $h=1-\sqrt{2}$ and $k=-3+\sqrt{2}$

$$\because (x-h)^2+(y-k)^2=r^2$$

$$(x-(1-\sqrt{2}))^2+(y-(-3+\sqrt{2}))^2=(2)^2$$

$$(x-1+\sqrt{2})^2+(y+3-\sqrt{2})^2=4$$

Equation of a Circle when $h=1+\sqrt{2}$ and $k=-3-\sqrt{2}$.

$$\because (x-h)^2+(y-k)^2=r^2$$

$$(x-(1+\sqrt{2}))^2+(y-(-3-\sqrt{2}))^2=(2)^2$$

$$(x-1-\sqrt{2})^2+(y+3+\sqrt{2})^2=4.$$

Required Equations.

Theory:

• Equation of tangent line to the circle:

A tangent to a curve is a line that touches the curve without cutting through it.
The equation of tangent to the circle $x^2+y^2+2gx+2fy+c=0$ at the point $P(x_1, y_1)$ is given by

$$y - y_1 = \frac{x_1+g}{y_1+f}(x - x_1) \quad \text{or} \quad xx_1 + yy_1 + g(x+x_1) + f(y+y_1) + c = 0$$

• Equation of Normal line to the circle:

The normal to the curve at $P(x_1, y_1)$ is the line through $P(x_1, y_1)$ perpendicular to the tangent
The equation of Normal to the circle $x^2+y^2+2gx+2fy+c=0$ at the point $P(x_1, y_1)$ is given by

$$y - y_1 = \frac{y_1+f}{x_1+g}(x - x_1) \quad \text{or} \quad (y - y_1)(x+g) = (x - x_1)(y_1+f)$$

Note:

> The line $y = mx + c$ intersects the circle $x^2+y^2 = r^2$ at the most two points.

• The position of the point with respect to the circle:

- The point $P(x_1, y_1)$ lies outside the circle $x^2+y^2+2gx+2fy+c=0$ if $x_1^2+y_1^2+2gx_1+2fy_1+c > 0$
- The point $P(x_1, y_1)$ lies inside the circle $x^2+y^2+2gx+2fy+c=0$ if $x_1^2+y_1^2+2gx_1+2fy_1+c < 0$
- The point $P(x_1, y_1)$ lies on the circle $x^2+y^2+2gx+2fy+c=0$

if $x_1^2+y_1^2+2gx_1+2fy_1+c=0$

• Length of the tangent to a circle:

Let $P(x_1, y_1)$ be a point outside the circle $x^2+y^2+2gx+2fy+c=0$ then length of point $P(x_1,y_1)$ to the circle $x^2+y^2+2gx+2fy+c=0$ is given by

$$\text{Length of the tangent} = \sqrt{x_1^2+y_1^2+2gx_1+2fy_1+c}$$

Example # 1: Determine whether the point $P(-5,6)$ lies outside or inside the circle $x^2+y^2+4x-6y-12=0$

$$x^2+y^2+4x-6y-12=0.$$
$$(-5)^2+(6)^2+4(-5)-6(6)-12$$
$$25+36-20-36-12$$
$$-7<0$$

Thus the point $(-5,6)$ lies inside the circle.

Example # 2: Find the coordinates of the points of intersection of the line $2x+y=5$ and the circle $x^2+y^2+2x-9=0$. Also find the length of the intercepted chord

Given $2x+y=5$

$$y=5-2x \quad \text{——} ①$$

Given Circle:

$$x^2+y^2+2x-9=0$$

By putting the value of y

$$x^2+(5-2x)^2+2x-9=0$$
$$x^2+25+4x^2-20x+2x-9=0$$
$$5x^2-18x+16=0$$
$$5x^2-10x-8x+16=0$$
$$5x(x-2)-8(x-2)=0$$
$$(x-2)(5x-8)=0$$
$$x-2=0 \; ; \; 5x-8=0$$
$$\boxed{x=2} \; ; \; \boxed{x=\frac{8}{5}}$$

Put in ① ; Put in ①

$$y=5-2(2) \; ; \; y=5-2\left(\frac{8}{5}\right)$$
$$y=5-4 \; ; \; y=5-\frac{16}{5}$$
$$\boxed{y=1} \; ; \; y=\frac{25-16}{5} \Rightarrow \boxed{\frac{9}{5}=y}$$

Thus the point of intersection are $P_1(2,1)$ and $P_2\left(\frac{8}{5},\frac{9}{5}\right)$

Length of the Chord

$$|\overline{P_1P_2}| = \sqrt{\left(\frac{8}{5}-2\right)^2+\left(\frac{9}{5}-1\right)^2}$$
$$=\sqrt{\left(\frac{8-10}{5}\right)^2+\left(\frac{9-5}{5}\right)^2}$$
$$=\sqrt{\left(\frac{-2}{5}\right)^2+\left(\frac{4}{5}\right)^2} = \sqrt{\frac{4}{25}+\frac{16}{25}}$$
$$=\sqrt{\frac{4+16}{25}} = \sqrt{\frac{20}{25}} = \sqrt{\frac{4}{5}}$$

Example # 3: Write equations of two tangents from (2,3) to the circle $x^2+y^2=9$.

Equation of tangent:

$$y = mx + a\sqrt{1+m^2} \quad \text{——} ①$$

if it passes through (2,3)

$$3 = 2m + 3\sqrt{1+m^2}$$

$$3-2m = 3\sqrt{1+m^2}$$

Taking square on Both sides

$$(3-2m)^2 = 9(1+m^2)$$

$$9+4m^2-12m = 9+9m^2$$

$$0 = \cancel{9}+9m^2-\cancel{9}-4m^2+12m$$

$$5m^2+12m=0$$

$$m(5m+12)=0$$

$$\boxed{m=0} \ ; \ 5m+12=0$$

$$\boxed{m=-\frac{12}{5}}$$

Put $m=0$ in ①

$$y=(0)x+3\sqrt{1+0^2}$$

$$y=0+3\sqrt{1}$$

$$y=3$$

$$y-3=0.$$

Put $m=-\frac{12}{5}$ in ①

$$y=\frac{-12}{5}x+3\sqrt{1+\left(\frac{-12}{5}\right)^2}$$

$$y=-\frac{12}{5}x+3\sqrt{1+\frac{144}{25}}$$

$$y=\frac{-12x}{5}+3\sqrt{\frac{25+144}{25}}$$

$$y=\frac{-12x}{5}+3\sqrt{\frac{169}{25}}$$

$$y=-\frac{12}{5}x+3\left(\frac{13}{5}\right)$$

$$y=-\frac{12}{5}x+\frac{39}{5}$$

$$5y=-12x+39$$

$$12x+5y-39=0$$

Required tangents

Example # 4: Write equations of the tangents to the circle $x^2+y^2-4x+6y+9=0$ at the points on the circle whose ordinate is -2

Given that:

$$x^2+y^2-4x+6y+9=0 \quad \text{——} ①$$

Put ordinate $= y = -2$

$$x^2+(-2)^2-4x+6(-2)+9=0$$

$$x^2+4-4x-12+9=0$$

$$x^2-4x+1=0$$

$a=1$; $b=-4$; $c=1$

$$\therefore x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$x=\frac{-(-4)\pm\sqrt{(-4)^2-4(1)(1)}}{2(1)}$$

$$x=\frac{4\pm\sqrt{16-4}}{2}$$

$$x=\frac{4\pm\sqrt{12}}{2}$$

$$x=\frac{4\pm\sqrt{2^2\times3}}{2}$$

$$x=\frac{4\pm2\sqrt{3}}{2}$$

$$x = \frac{2(2 \pm \sqrt{3})}{2}$$

$$x = 2 \pm \sqrt{3}$$

$P_1(2+\sqrt{3}, -2)$; $P_2(2-\sqrt{3}, -2)$.

Diff eq ① w.r.t 'x'

$$2x + 2y\frac{dy}{dx} - 4 + 6\frac{dy}{dx} = 0$$

$$2y\frac{dy}{dx} + 6\frac{dy}{dx} + 2x - 4 = 0$$

$$(2y+6)\frac{dy}{dx} = -2x + 4$$

$$\frac{dy}{dx} = \frac{2(-x+2)}{2(y+3)}$$

$$\frac{dy}{dx} = \frac{-x+2}{y+3}.$$

Slope of Tangent: $\frac{dy}{dx}\Big/_{P_1(2+\sqrt{3},-2)}$

$$= \frac{-(2+\sqrt{3})+2}{-2+3} = \frac{-2-\sqrt{3}+2}{1} = -\sqrt{3}$$

Equation of Tangents

$$y+2 = -\sqrt{3}(x-2-\sqrt{3})$$

$$y+2 = -\sqrt{3}x + 2\sqrt{3} + (\sqrt{3})^2$$

$$y+2+\sqrt{3}x - 2\sqrt{3} - 3 = 0$$

$$\sqrt{3}x + y - 2\sqrt{3} - 1 = 0$$

Now at $P_2(2-\sqrt{3}, -2)$

Slope of Tangent $= \frac{dy}{dx}\Big/_{P_2(2-\sqrt{3},-2)}$

$$= \frac{-(2-\sqrt{3})+2}{-2+3} = \frac{-2+\sqrt{3}+2}{1} = \sqrt{3}$$

Equation of Tangents

$$y+2 = \sqrt{3}(x-2+\sqrt{3})$$

$$y+2 = \sqrt{3}x - 2\sqrt{3} + (\sqrt{3})^2$$

$$\sqrt{3}x - 2\sqrt{3} + 3 - y - 2 = 0$$

$$\sqrt{3}x - y - 2\sqrt{3} + 1 = 0.$$

Required Tangents.

Example # 6: Find the length of the tangent from the point $P(-5, 10)$ to the circle $5x^2+5y^2+14x+12y-10=0$.

$$5x^2+5y^2+14x+12y-10=0$$

Dividing by '5'

$$x^2+y^2+\frac{14}{5}x+\frac{12}{5}y-2=0$$

Compare with

$$x^2+y^2+2gx+2fy+c=0$$

$$2g = \frac{14}{5}\ ;\ 2f = \frac{12}{5}\ ;\ c = -2$$

$$x_1 = -5\ ;\ y_1 = 10$$

$$\therefore \text{length of tangent} = \sqrt{x_1^2+y_1^2+2gx_1+2fy_1+c}$$

$$= \sqrt{(-5)^2+(10)^2+\left(\tfrac{14}{5}\right)(-5)+\left(\tfrac{12}{5}\right)(10)+(-2)}$$

$$= \sqrt{25+100-14+24-2}$$

$$\boxed{\sqrt{133} = \text{length of Tangent}}$$

Exercise # 6.2

Question no 2: Write down equations of tangent and normal to the circle

i) $x^2+y^2=25$ at $(4,3)$ and at $(5\cos\theta, 5\sin\theta)$

$$x^2+y^2-25=0$$

Diff w.r.t 'x'

$$\frac{d}{dx}(x^2+y^2-25)=0$$

$$2x+2y\frac{dy}{dx}=0$$

$$2y\frac{dy}{dx}=-2x$$

$$\frac{dy}{dx}=\frac{-2x}{2y}$$

$$\frac{dy}{dx}=\frac{-x}{y}$$

Slope of tangent $= \frac{dy}{dx}\Big/_{(4,3)} = \frac{-4}{3}$

Equation of tangent:

$$y-3=\frac{-4}{3}(x-4)$$

$$3(y-3)=-4(x-4)$$

$$3y-9=-4x+16$$

$$4x+3y-9-16=0$$

$$4x+3y-25=0$$

Slope of Normal $= \frac{3}{4}$

Equation of Normal:

$$y-3=\frac{3}{4}(x-4)$$

$$4(y-3)=3(x-4)$$

$$4y-12=3x-12$$

$$0=3x-\cancel{12}-4y+\cancel{12}$$

$$3x-4y=0$$

Now at $(5\cos\theta, 5\sin\theta)$

Slope of tangents: $\frac{dy}{dx} = -\frac{\cancel{5}\cos\theta}{\cancel{5}\sin\theta} = \frac{-\cos\theta}{\sin\theta}$

Equation of tangent:

$$y-5\sin\theta=\frac{-\cos\theta}{\sin\theta}(x-5\cos\theta)$$

$$\sin\theta(y-5\sin\theta)=-\cos\theta(x-5\cos\theta)$$

$$y\sin\theta-5\sin^2\theta=-x\cos\theta+5\cos^2\theta$$

$$y\sin\theta+x\cos\theta=5\cos^2\theta+5\sin^2\theta$$

$$x\cos\theta+y\sin\theta=5(\sin^2\theta+\cos^2\theta)$$

$$x\cos\theta+y\sin\theta=5(1)$$

$$x\cos\theta+y\sin\theta=5$$

Slope of Normal $= \frac{\sin\theta}{\cos\theta}$

Equation of Normal:

$$y-5\sin\theta=\frac{\sin\theta}{\cos\theta}(x-5\cos\theta)$$

$$\cos\theta(y-5\sin\theta)=\sin\theta(x-5\cos\theta)$$

$$y\cos\theta-5\sin\theta\cos\theta=x\sin\theta-5\sin\theta\cos\theta$$

$$0=x\sin\theta\underline{-5\sin\theta\cos\theta}-y\cos\theta\underline{+5\sin\theta\cos\theta}$$

$$x\sin\theta-y\cos\theta=0$$

ii) $3x^2+3y^2+5x-13y+2=0$ at $\left(1, \frac{10}{3}\right)$

$3x^2+3y^2+5x-13y+2=0$

Diff w.r.t 'x'

$6x+6y\frac{dy}{dx}+5-13\frac{dy}{dx}=0$

$(6y-13)\frac{dy}{dx}+6x+5=0.$

$(6y-13)\frac{dy}{dx}=-6x-5.$

$\frac{dy}{dx}=\frac{-6x-5}{6y-13}$

Slope of tangent $=\frac{dy}{dx}\Big|_{(1,\frac{10}{3})}$

$=\frac{-6(1)-5}{6(\frac{10}{3})-13}=\frac{-6-5}{20-13}=\frac{-11}{7}$

Equation of tangent:

$y-\frac{10}{3}=-\frac{11}{7}(x-1)$ $\quad\because y-y_1=m(x-x_1)$

$7y-\frac{70}{3}=-11x+11$

$21y-70=-33x+33$

$21y-70+33x-33=0$

$33x+21y-103=0$

Slope of Normal $=\frac{7}{11}$

Equation of Normal:

$y-\frac{10}{3}=\frac{7}{11}(x-1)$

$11y-\frac{110}{3}=7x-7$

$33y-110=21x-21$

$0=21x-21-33y+110$

$21x-33y+89=0$

Question # 2: Write down equation of tangent and normal to the circle $4x^2+4y^2-16x+24y-117=0$ at the points on the circle whose abscissa is -4.

Given that

$4x^2+4y^2-16x+24y-117=0$ —(i

Put abscissa $=x=-4$

$4(-4)^2+4y^2-16(-4)+24y-117=0$

$4(16)+4y^2+64+24y-117=0$

$64+4y^2+64+24y-117=0$

$4y^2+24y+11=0$

$4y^2+22y+2y+11=0.$

$2y(2y+11)+1(2y+11)=0$

$(2y+11)(2y+1)=0$

$2y+11=0$; $2y+1=0$

$2y=-11$; $2y=-1$

$y=-\frac{11}{2}$; $y=-\frac{1}{2}$

$P_1\left(-4,-\frac{11}{2}\right)$; $P_2\left(-4,-\frac{1}{2}\right)$

Diff eq (i) with respect to 'x'

$\frac{d}{dx}(4x^2+4y^2-16x+24y-117)=\frac{d}{dx}(0).$

$8x+8y\frac{dy}{dx}-16+24\frac{dy}{dx}=0$

$8x-16+(8y+24)\frac{dy}{dx}=0$

$(8y+24)\frac{dy}{dx}=-8x+16$

$\frac{dy}{dx}=\frac{8(-x+2)}{\cancel{8}(y+3)}$

$$\frac{dy}{dx} = \frac{-x+2}{y+3}$$

Slope of tangent $= \frac{dy}{dx}\Big|_{P_1(-4,\,-\frac{11}{2})}$

$$= \frac{-(-4)+2}{-\frac{11}{2}+3} = \frac{4+2}{\frac{-11+6}{2}} = \frac{6}{\frac{-5}{2}} = \frac{-12}{5}$$

Equation of tangent:

$$y-\left(-\frac{11}{2}\right) = -\frac{12}{5}(x-(-4))$$

$$y+\frac{11}{2} = \frac{-12}{5}(x+4)$$

$$5y+\frac{55}{2} = -12x-48$$

$$10y+55 = -24x-96$$

$$10y+55+24x+96 = 0$$

$$\boxed{24x+10y+151=0}$$

Slope of Normal $= \frac{5}{12}$

Equation of Normal:

$$y+\frac{11}{2} = \frac{5}{12}(x+4)$$

$$12y+\frac{132}{2} = 5x+20$$

$$24y+132 = 10x+40$$

$$0 = 10x+40-24y-132$$

$$10x-24y-92 = 0$$

$\because$ Dividing by 2

$$\boxed{5x-12y-46=0}$$

Now at $P_2(-4, -\frac{1}{2})$

Slope of tangent $= \frac{dy}{dx}\Big|_{P_2(-4,\,-\frac{1}{2})}$

$$= \frac{-(-4)+2}{-\frac{1}{2}+3} = \frac{4+2}{\frac{-1+6}{2}} = \frac{6}{\frac{5}{2}} = \frac{12}{5}$$

Equation of tangent:

$$y+\frac{1}{2} = \frac{12}{5}(x+4)$$

$$5y+\frac{5}{2} = 12x+48$$

$$10y+5 = 24x+96$$

$$0 = 24x+96-10y-5$$

$$\boxed{24x-10y+91=0}$$

Slope of Normal: $\frac{-5}{12}$

Equation of Normal:

$$y+\frac{1}{2} = \frac{-5}{12}(x+4)$$

$$12y+6 = -5x-20$$

$$12y+6+5x+20=0$$

$$\boxed{5x+12y+26=0}$$

Required Equations

Question no 3: Check the position of the point (5,6) w.r.t circle

i) $x^2+y^2=81$

$x^2+y^2-81=0$ at $(5,6)$

$$(5)^2+(6)^2-81$$

$$25+36-81$$

$$-20<0$$

Inside the Circle

ii) $2x^2+2y^2+12x-8y+1=0$

$$2(5)^2+2(6)^2+12(5)-8(6)+1$$

$$2(25)+2(36)+60-48+1$$

$$50+72+60-48+1$$

$$135>0$$

Outside the Circle.

Question no 4: Find the length of the tangent drawn from the point (-5,4) to the circle $5x^2+5y^2-10x+15y-132=0$

$$5x^2+5y^2-10x+15y-131=0$$

Dividing by '5'

$$x^2+y^2-2x+3y-\frac{131}{5}=0.$$

Compare with

$$x^2+y^2+2gx+2fy+c=0$$

$$2g=-2 \; ; \; 2f=3 \; ; \; c=-\frac{131}{5}$$

$$x_1=-5; \; y_2=4.$$

$\therefore$ length of tangent $=\sqrt{x_1^2+y_1^2+2gx_1+2fy_1+c}$

$$=\sqrt{(-5)^2+(4)^2+(-2)(-5)+(3)(4)+\left(\frac{-131}{5}\right)}$$

$$=\sqrt{25+16+10+12-\frac{131}{5}}$$

$$=\sqrt{63-\frac{131}{5}}$$

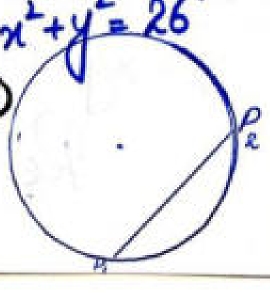
$$=\sqrt{\frac{315-131}{5}}$$

$$\boxed{=\sqrt{\frac{184}{5}} = \text{length of tangent}}$$

Question no 5: Find the length of the chord cut off from the line $2x+3y=13$ by the circle $x^2+y^2=26$

$x^2+y^2=26$ — ①

$|P_1P_2|=?$



Given line is $2x+3y=13$

$$\Rightarrow x=\frac{13-3y}{2} \text{ — Ⓐ}$$

Put Ⓐ in ①

$$\left(\frac{13-3y}{2}\right)^2+y^2=26$$

$$\frac{169+9y^2-78y}{4}+y^2=26$$

$$169+9y^2-78y+4y^2=104$$

$$169+9y^2-78y+4y^2-104=0$$

$$13y^2-78y+65=0$$

$$13y^2-13y-65y+65=0$$

$$13y(y-1)-65(y-1)=0$$

$$(y-1)(13y-65)=0$$

$$y-1=0 \quad ; \quad 13y-65=0$$

$$\boxed{y=1} \quad ; \quad y=\frac{65}{13}$$

$$\boxed{y=5}$$

Put y=1 in Ⓐ

$$x=\frac{13-3(1)}{2}$$

$$x=\frac{13-3}{2}$$

$$x=\frac{10}{2}$$

$$\boxed{x=5}$$

Put y=5 in Ⓐ

$$x=\frac{13-3(5)}{2}$$

$$x=\frac{13-15}{2}$$

$$x=-\frac{2}{2}$$

$$\boxed{x=-1}$$

So $P_1(5,1)$ and $P_2(-1,5)$ are points of contact.

Length of Chord $=|P_1P_2|$

$$|P_1P_2|=\sqrt{(-1-5)^2+(5-1)^2}$$

$$=\sqrt{(-6)^2+(4)^2}$$

$= \sqrt{36+16}$

$= \sqrt{52}$

$= \sqrt{13 \times 2^2}$

$\boxed{\text{length of chord} = 2\sqrt{13}}$

Question no 6: Find the coordinates of the points of intersection of the line $x+2y=6$ with the circle $x^2+y^2-2x-2y-39=0$.

$x^2+y^2-2x-2y-39=0$ — ①

Given line $x+2y=6$ is

$\Rightarrow x = 6-2y$ — Ⓐ

Put Ⓐ in ①

$(6-2y)^2+y^2-2(6-2y)-2y-39=0$

$36+4y^2-24y+y^2-12+4y-2y-39=0$

$5y^2-22y-15=0$

$5y^2-25y+3y-15=0$

$5y(y-5)+3(y-5)=0$

$(y-5)(5y+3)=0$

$y-5=0 \quad ; \quad 5y+3=0$

$\boxed{y=5} \; ; \; \boxed{y=-\frac{3}{5}}$

Put $y=5$ in Ⓐ ; Put $y=-\frac{3}{5}$ in Ⓐ

$x=6-2(5) \quad ; \quad x=6-2\left(-\frac{3}{5}\right)$

$x=6-10 \quad ; \quad x=6+\frac{6}{5}$

$\boxed{x=-4} \quad ;$

$x=\frac{30+6}{5}$

$\boxed{x=\frac{36}{5}}$

So points of contact are $P_1(-4,5)$ and $P_2\left(\frac{36}{5},-\frac{3}{5}\right)$

Question no 7: Find equation of the tangents to the circle $x^2+y^2=2$

i) parallel to the $x-2y+1=0$

$x^2+y^2=2$ is parallel to line $x-2y+1=0$

Given Circle:-

$x^2+y^2=(\sqrt{2})^2$

$r = a = \sqrt{2}$

Slope of Given line:-

$= -\frac{a}{b} = \frac{-1}{-2} = \frac{1}{2}$

So $m = \frac{1}{2}$ ($\because$ parallel)

Equation of tangent:-

$\because y = mx \pm a\sqrt{1+m^2}$

$y=\frac{1}{2}x \pm \sqrt{2}\sqrt{1+\left(\frac{1}{2}\right)^2}$

$y=\frac{1}{2}x \pm \sqrt{2}\sqrt{1+\frac{1}{4}}$

$y=\frac{1}{2}x \pm \sqrt{2}\sqrt{\frac{5}{4}}$

$y=\frac{1}{2}x \pm \sqrt{2}\frac{\sqrt{5}}{2}$

$2y = x \pm \sqrt{10}$

$x \pm \sqrt{10} - 2y = 0$

$x-2y \pm \sqrt{10} = 0$

$x-2y+\sqrt{10}=0 \; ; \; x-2y-\sqrt{10}=0$

Required tangents.

ii) perpendicular to the line $3x+2y=6$

$x^2+y^2=2$ is perpendicular to line $3x+2y=6$

Given circle:-

$$x^2+y^2=(\sqrt{2})^2$$

$$r=a=\sqrt{2}$$

Slope of Given line:-

$$=\frac{-a}{b}=-\frac{3}{2}$$

So $m=\frac{2}{3}$ ($\because$ perpendicular)

Equation of Tangents

$$\because y=mx\pm a\sqrt{1+m^2}$$

$$y=\frac{2}{3}x\pm\sqrt{2}\sqrt{1+\left(\frac{2}{3}\right)^2}$$

$$y=\frac{2}{3}x\pm\sqrt{2}\sqrt{1+\frac{4}{9}}$$

$$y=\frac{2}{3}x\pm\sqrt{2}\sqrt{\frac{13}{9}}$$

$$y=\frac{2}{3}x\pm\sqrt{2}\frac{\sqrt{13}}{3}$$

$$3y=2x\pm\sqrt{26}$$

$$2x-3y\pm\sqrt{26}=0$$

$$2x-3y+\sqrt{26}=0;\ 2x-3y-\sqrt{26}=0$$

Required Tangents.

Question no 8: Find equation of tangents drawn from

i) $(0,5)$ to $x^2+y^2=16$

Also find of Contact

$$x^2+y^2=(4)^2$$

$$r=a=4$$

Equation of tangents:

$$\because y=mx+a\sqrt{1+m^2} \quad (A$$

at point $(0,5)$

$$5=m(0)+4\sqrt{1+m^2}$$

$$5=4\sqrt{1+m^2}$$

Taking Square on B.S

$$25=16(1+m^2)$$

$$25=16+16m^2$$

$$25-16=16m^2$$

$$9=16m^2$$

$$m^2=\frac{9}{16}$$

$$m=\pm\frac{3}{4}$$

Now Ⓐ for $m=\pm\frac{3}{4}$ and $a=4$

$$y=\pm\frac{3}{4}x+4\sqrt{1+\left(\frac{3}{4}\right)^2}$$

$$y=\pm\frac{3}{4}x+4\sqrt{1+\frac{9}{16}}$$

$$y=\pm\frac{3}{4}x+4\sqrt{\frac{16+9}{16}}$$

$$y=\pm\frac{3}{4}x+4\sqrt{\frac{25}{16}}$$

$$y=\pm\frac{3}{4}x+4\left(\frac{5}{4}\right)$$

$$y=\pm\frac{3}{4}x+5$$

$$4y=\pm 3x+20$$

$$\pm 3x-4y+20=0$$

$$3x-4y+20=0;\ -3x-4y+20=0$$

$$\text{or } 3x+4y-20=0$$

Required Tangents.

Now We solve $3x-4y+20=0$ and $x^2+y^2=16$ for point of contact.

$$x^2+y^2=16 \quad \text{— (i)}$$

$$3x-4y+20=0$$

$$\Rightarrow x=\frac{4y-20}{3} \quad \text{— (ii)}$$

Put (ii) in (i)

$$\left(\frac{4y-20}{3}\right)^2+y^2=16$$

$$\frac{16y^2+400-160y}{9}+y^2=16$$

$$16y^2+400-160y+9y^2-114=0$$

$$25y^2-160y+256=0$$

$$25y^2-80y-80y+256=0$$

$$5y(5y-16)-16(5y-16)=0$$

$$(5y-16)(5y-16)=0$$

$$(5y-16)^2=0$$

$$5y-16=0$$

$$5y=16$$

$$\boxed{y=\frac{16}{5}}$$

Put in (ii)

$$x=\frac{4\left(\frac{16}{5}\right)-20}{3}$$

$$x=\frac{64/5-20}{3}=\frac{\frac{64-100}{5}}{3}$$

$$x=\frac{-36}{15} \Rightarrow \boxed{-\frac{12}{5}=x}$$

So point of Contact $\left(\frac{-12}{5},\frac{16}{5}\right)$

Now We solve $3x+4y-20=0$ and $x^2+y^2=16$ for point of contact

$$x^2+y^2=16 \quad \text{— (iii)}$$

$$3x+4y-20=0$$

$$\Rightarrow x=\frac{-4y+20}{3} \quad \text{— (iv)}$$

Put (iv) in (iii)

$$\left(\frac{-4y+20}{3}\right)^2+y^2=16$$

$$\frac{16y^2+400-160y}{9}+y^2=16$$

$$16y^2+400-160y+9y^2-114=0$$

$$25y^2-160y+256=0$$

$$25y^2-80y-80y+256=0$$

$$5y(5y-16)-16(5y-16)=0$$

$$(5y-16)(5y-16)=0$$

$$(5y-16)^2=0$$

$$5y-16=0$$

$$5y=16$$

$$\boxed{y=\frac{16}{5}}$$

Put in (iv)

$$x=\frac{-4\left(\frac{16}{5}\right)+20}{3}$$

$$x=\frac{-64/5+20}{3}=\frac{-64+100}{15}$$

$$x=\frac{36}{15} \Rightarrow \boxed{x=\frac{12}{5}}$$

So point of Contact

$\left(\frac{12}{5},\frac{16}{5}\right)$

ii) $(-1,2)$ to $x^2+y^2+4x+2y=0$

Given equation of circle

$$x^2+y^2+4x+2y=0$$

Compare with

$$x^2+y^2+2gx+2fy+c=0$$

$2g=4$; $2f=2$; $c=0$

$g=2$; $f=1$

Centre $(-g,-f) = (-2,-1)$

radius $= r = \sqrt{g^2+f^2-c}$

$= \sqrt{(2)^2+(1)^2-0}$

$= \sqrt{4+1} = \sqrt{5}$

Slope-point form

$y-2 = m(x+1)$ $\quad \because y-y_1=m(x-x_1)$

$y-2 = mx+m$

$mx+m-y+2=0$

$mx-y+m+2=0$ —(i)

$r = \frac{|mx-y+m+2|}{\sqrt{m^2+1}}$

Put $(-2,-1)$

$\sqrt{5} = \frac{|m(-2)-(-1)+2+m|}{\sqrt{m^2+1}}$

$\sqrt{5} = \frac{|-2m+1+2+m|}{\sqrt{m^2+1}}$

$\sqrt{5} = \frac{|-m+3|}{\sqrt{m^2+1}}$

Taking square on B.S

$5 = \frac{(-m+3)^2}{m^2+1}$

$5(m^2+1) = m^2+9-6m$

$5m^2+5-m^2-9+6m=0$

$4m^2+6m-4=0$

Dividing by '2'

$2m^2+3m-2=0$

$2m^2+4m-m-2=0$

$2m(m+2)-1(m+2)=0$

$(m+2)(2m-1)=0$

$m+2=0$; $2m-1=0$

$\boxed{m=-2}$; $\boxed{m=\frac{1}{2}}$

Put in (i) **Put in (i)**

$-2x-y+(-2)+2=0$; $\frac{1}{2}x-y+\frac{1}{2}+2=0$

$-2x-y-2+2=0$; $x-2y+1+4=0$

$-2x-y=0$; $x-2y+5=0$

$2x+y=0$;

Required Tangents

Now We solve $2x+y=0$ and $x^2+y^2+4x+2y=0$ for point of contact

$x^2+y^2+4x+2y=0$ —(ii)

$2x+y=0$

$\Rightarrow x=-\frac{y}{2}$ —(iii)

Put (iii) in (ii)

$\left(-\frac{y}{2}\right)^2+y^2+4\left(-\frac{y}{2}\right)+2y=0$

$\frac{y^2}{4}+y^2-\cancel{2y}+\cancel{2y}=0$

$y^2+4y^2=0$

$5y^2=0$

$y^2=0 \Rightarrow \boxed{y=0}$

Put in (iii)

$x = \frac{-0}{2} \Rightarrow \boxed{x = 0}$

So point of contact (0,0)

Now we solve $x-2y+5=0$ and $x^2+y^2+4x+2y=0$ for point of contact

$x^2+y^2+4x+2y=0$ — ⓘⓥ

$x-2y+5=0$

$\Rightarrow x = 2y-5$ — ⓥ

Put ⓥ in ⓘⓥ

$(2y-5)^2+y^2+4(2y-5)+2y=0$

$4y^2+25-20y+y^2+8y-20+2y=0$

$5y^2-10y+5=0$

Dividing by '5'

$y^2-2y+1=0$

$y^2-y-y+1=0$

$y(y-1)-1(y-1)=0$

$(y-1)(y-1)=0$

$(y-1)^2=0$

$y-1=0$

$\boxed{y=1}$

Put in ⓥ

$x=2(1)-5$

$x=2-5$

$\boxed{x=-3}$

So point of contact

(−3,1).

iii) $(-7,-2)$ to $(x+1)^2+(y-2)^2=26$

Given equation of Circle-

$(x+1)^2+(y-2)^2=26$

$(x-(-1))^2+(y-2)^2=(\sqrt{26})^2$

$(x-h)^2+(y-k)^2=r^2$

Centre $(-1,2)$; $r=\sqrt{26}$

Point-Slope form:-

$y+2=m(x+7)$ $\quad\because y-y_1=m(x-x_1$

$y+2=mx+7m$

$mx+7m-y-2=0$

$mx-y+7m-2=0$ —(i

$r=\frac{|mx-y+7m-2|}{\sqrt{m^2+1}}$

Put $(-1,2)$

$\sqrt{26}=\frac{|m(-1)-2+7m-2|}{\sqrt{m^2+1}}$

$\sqrt{26}=\frac{|-m-2+7m-2|}{\sqrt{m^2+1}}$

$\sqrt{26}=\frac{|6m-4|}{\sqrt{m^2+1}}$

Taking square on B.S

$26=\frac{(6m-4)^2}{m^2+1}$

$26(m^2+1) = 36m^2+16-48m$

$26m^2+26 = 36m^2+16-48m$

$36m^2+16-48m-26m^2-26=0$

$10m^2-48m-10=0$

Dividing by '2'

$5m^2-24m-5=0$

$5m^2+25m+m-5=0$

$$5m(m-5)+1(m-5)=0$$

$$(m-5)(5m+1)=0$$

$$m-5=0 \; ; \; 5m+1=0$$

$$\boxed{m=5} \; ; \; \boxed{m=-\frac{1}{5}}$$

Put $m=5$ in (i) Put $m=-\frac{1}{5}$ in (i)

$$5x-y+7(5)-2=0 \; ; \; -\frac{1}{5}x-y+7\left(-\frac{1}{5}\right)-2=0$$

$$5x-y+35-2=0 \; ; \; -x-5y-7-10=0$$

$$5x-y+33=0 \; ; \; -x-5y-17=0$$

$$x+5y+17=0$$

Required Tangents-

Now

We solve $5x-y+33=0$ and $(x+1)^2+(y-2)^2=26$.

$$(x+1)^2+(y-2)^2=26 \quad \text{— (ii)}$$

$$5x+33-y=0$$

$$\Rightarrow y=5x+33 \quad \text{— (iii)}$$

Put (iii) in (ii)

$$(x+1)^2+(5x+33-2)^2=26$$

$$(x+1)^2+(5x+31)^2=26$$

$$x^2+1+2x+25x^2+961+310x-26=0$$

$$26x^2+312x+936=0$$

Dividing by '26'

$$x^2+12x+36=0$$

$$(x+6)^2=0$$

$$x+6=0$$

$$\boxed{x=-6}$$

Put in (iii)

$$y=5(-6)+33$$

$$y=-30+33$$

$$\boxed{y=3}$$

So point of contact $(-6,3)$

Now

We solve $x+5y+17=0$ and $(x+1)^2+(y-2)^2=26$

$$x+5y+17=0$$

$$x=-5y-17 \quad \text{— (iv)}$$

Put in (ii)

$$(-5y-17+1)^2+(y-2)^2=26$$

$$(-5y-16)^2+(y-2)=26$$

$$25y^2+256+160y+y^2+4-4y-26=0$$

$$26y^2+156y+234=0$$

Dividing by '26'

$$y^2+6y+9=0$$

$$(y+3)^2=0$$

$$y+3=0$$

$$\boxed{y=-3}$$

Put $y=-3$ in (iv)

$$x=-5(-3)-17$$

$$x=15-17$$

$$\boxed{x=-2}$$

So point of contact $(-2,-3)$

Question no 9: Find an equation of chord of contact of the tangents drawn from (4,5) to the Circle

$$2x^2+2y^2-8x+12y+21=0$$

$$2x^2+2y^2-8x+12y+21=0$$

Dividing by '2'

$$x^2+y^2-4x+6y+\frac{21}{2}=0$$

Compare with

$$x^2+y^2+2gx+2fy+c=0$$

$2g=-4$; $2f=6$; $c=\frac{21}{2}$

$g=-2$; $f=3$;

Equation of Tangent:

$$\therefore xx_1+yy_1+g(x+x)+f(y+y_1)+c=0$$

$x_1=4$; $y_1=5$

By putting the values:

$$4x+5y-2(x+4)+3(y+5)+\frac{21}{2}=0$$

$$4x+5y-2x-8+3y+15+\frac{21}{2}=0$$

$$2x+8y+7+\frac{21}{2}=0$$

$$4x+16y+14+21=0$$

$$4x+16y+35=0$$

Required Equation

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Theory

Properties of a Circles

- Length of a diameter of the circle $x^2+y^2=a^2$ is 2a
- Perpendicular dropped from the centre of a circle on a chord bisects the chord.
- The perpendicular bisector of any chord of a circle passes through the centre of the circle.
- The line joining the centre of a circle to the midpoint of a chord is perpendicular to the chord.
- Congruent chords of a circle are equidistant from the centre.
- Measure of the central angle of a minor arc is double the measure of the angle subtended in the corresponding major arc.
- An angle in a semi-circle is a right angle
- The tangent to a circle at any point of the circle is perpendicular to the radical segment at that point.
- The perpendicular at the outer end of the radial segment is tangent to the circle
- Normal lines of a circle pass through the centre of a circle
- The Straight line drawn from the centre of a circle perpendicular to a tangent passes through the point of tangency.
- The midpoint of the hypotenuse of a right triangle is the circumcentre of the triangle.
- The perpendicular dropped from a point of a

circle on a diameter is a mean proportional between the segments into which it divides the diameter.

Conic Section:

Let L be a fixed line in a plane and F be a fixed point not on the line L. Suppose |PM| denotes the perpendicular distance of a point P(x, y) from the line L. The set of all points P in the plane such that

$$\frac{|PF|}{|PM|} = e \text{ (a positive constant)}$$

is called a conic section.

i) If $e = 1$, then the conic is parabola

ii) If $0 < e < 1$, then the conic is an ellipse

iii) If $e > 1$, then the conic is a hyperbola

The fixed line L is called a **directrix** and the fixed point F is called a **focus** of the conic. The number e is called the **eccentricity** of the conic.

• Parabola:

Let $e = 1$ and F be a fixed point and L is fixed line not containing F. Let P(x, y) be any point in the plane and |PM| be the perpendicular distance of a point P(x, y) from the line L. The set of all points P such that

$$\frac{|PF|}{|PM|} = 1 \text{ or } |PF| = |PM|$$

is called parabola.

OR The set of all points in a plane which is equidistant from a given fixed line in the plane is called parabola.

- The fixed point is called focus of the parabola
- The fixed line is called directrix of the parabola

• Standard Equation of Parabola:

$$y^2=(x+a)^2-(x-a)^2=4ax \text{ or } y^2=4ax$$

Definitions:

Axis: The line through the focus and perpendicular to the directrix is called axis of the parabola

Vertex: The point where the axis meets the parabola is called vertex of the parabola.

Focus: In parabola, the fixed point is called focus of the parabola.

Directrix: In parabola, the fixed line is called directrix of the parabola.

Tangent at vertex: A line passing through vertex and perpendicular to the axis of parabola is called tangent at vertex of parabola.

Chord: Line joining two distinct points on a parabola is called a chord of the parabola.

Focal chord: A chord passing through the focus of a parabola is called a focal chord of the parabola.

Latus rectum: The focal chord perpendicular to the axis of the parabola is called latusrectum

• Parametric Equation of Parabola:

The point $(at^2, 2at)$ lies on the parabola $y^2=4ax$ for any real t.

$x=at^2$, $y=2at$ are called parametric equation of the parabola $y^2 = 4ax$.

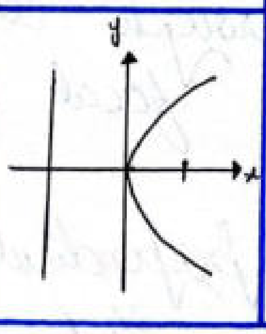
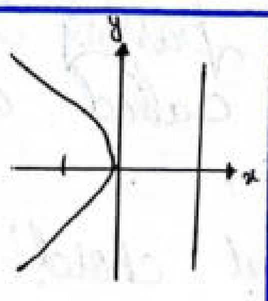
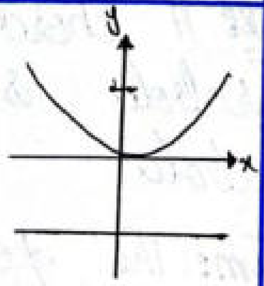
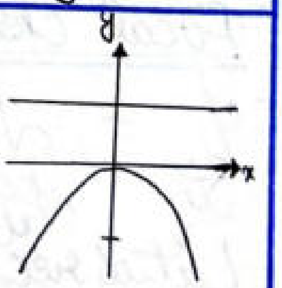
• General form of an equation of Parabola:

Let P(x,y) be any point on the parabola having F(h,k) as focus and M be the point on directrix $lx + my + n = 0$

By definition, equation of parabola is given by

$$|PM| = \frac{|lx+my+n|}{\sqrt{l^2+m^2}} \text{ or } (x-h)^2+(y-k)^2 = \left(\frac{lx+my+n}{l^2+m^2}\right)^2$$

Summary of Standard Parabolas

Serial. No.	1	2	3	4
Equation	$y^2 = 4ax$	$y^2 = -4ax$	$x^2 = 4ay$	$x^2 = -4ay$
Focus	$(a,0)$	$(-a,0)$	$(0,a)$	$(0,-a)$
Directrix	$x = -a$	$x = a$	$y = -a$	$y = a$
Vertex	$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$
Axis	$y = 0$	$y = 0$	$x = 0$	$x = 0$
Latusrectum	$x = a$	$x = -a$	$y = a$	$y = -a$
Graph				

Example #1: Analyze the parabola $x^2 = -16y$ and draw its graph.

$x^2 = -16y$ —— ①

Compare ① with

$x^2 = -4ay$

$-4a = -16$

$a = 4$

Focus:

$F(0,-a)$

$F(0,-4)$

Vertex:

$V(0,0)$

Directrix:

$y = a$

$y = 4$

Example # 2: Find an equation of the parabola whose focus is $F(-3,4)$ and directrix is $3x-4y+5=0$

Let $P(x,y)$ be any point on parabola.

Therefore $|PF|^2 = |PM|^2$

$$(x+3)^2+(y-4)^2 = \frac{(3x-4y+5)^2}{(3)^2+(-4)^2}$$

$$x^2+9+6x+y^2+16-8y = \frac{9x^2+16y^2+25-24xy \; 4+30x-40y}{9+16}$$

$$25(x^2+y^2+6x-8y+25) = 9x^2+16y^2+25-24xy+30x-40y$$

$$25x^2+25y^2+150x-200y+625-9x^2-16y^2-25+24xy-30x+40y=0.$$

$$16x^2+9y^2+120x-160y+24xy+600=0$$

Required Equation.

Example # 3: Analyze the parabola $x^2-4x-3y+13=0$.

$$x^2-4x-3y+13=0$$

$$x^2-4x-3y+9+4=0$$

$$x^2-4x+4 = 3y-9$$

$$(x-2)^2 = 3(y-3).$$

Compare with

$X^2 = 4aY.$

$X = x-2$; $Y = y-3$; $4a = 3$

$\Rightarrow a = \frac{3}{4}$

Focus: $F(0,a)$

Where $X=0$; $Y = a$

$x-2 = 0$; $y-3 = \frac{3}{4}$

$x = 2$; $y = \frac{3}{4}+3 = \frac{3+12}{4}$

$y = \frac{15}{4}$

So focus $(2, \frac{15}{4})$

Vertex:

$V(0,0)$

Where $X=0$; $Y=0$

$x-2=0$; $y-3=0$

$x=2$; $y=3$

So Vertex $(2,3)$.

Directrix:

$Y = -a$

$y-3 = -\frac{3}{4}$

$y = \frac{-3}{4}+3$

$y = \frac{-3+12}{4}$

$y = \frac{9}{4}$

Axis:

$X=0$

$x-2=0$

$x=2$

Exercise # 6.4

Question no 1: Find the focus, vertex and directrix of the parabola

i) $y^2 = 8x$

$y^2 = 8x$ — ①

Compare ① with

$y^2 = 4ax$

$4a = 8 \Rightarrow a = 2$

Focus:

$F(a,0)$

$F(2,0)$

Vertex:

$V(0,0)$

Directrix:

$x = -a$

$x = -2$

Axis: $y = 0$

ii) $x^2 = -16y$

$x^2 = -16y$ — ①

Compare ① with $x^2 = -4ay$

$-4a = -16$

$a = 4$

Focus: $F(0,-a)$

$F(0,-4)$

Vertex:

$V(0,0)$

Directrix:

$y = a$

$y = 4$

Axis:

$x = 0$

iii) $x^2 = 5y$

$x^2 = 5y$ — ①

Compare ① with $x^2 = 4ay$

$4a = 5$

$a = \frac{5}{4}$

Focus: $F(0,a)$

$F(0,\frac{5}{4})$

Vertex:

$V(0,0)$

Directrix: $x = -a$

$x = -\frac{5}{4}$

Axis:

$x = 0$

iv) $y^2 = -12x$

$y^2 = -12x$ — ①

Compare ① with $y^2 = -4ax$

$-4a = -12$

$a = 3$

Focus: $F(-a,0)$

$F(-3,0)$

Vertex:

$V(0,0)$

Directrix:

$x = a$

$x = 3$

Axis:

$y = 0$

v) $x^2 = 4(y-1)$

$x^2 = 4(y-1)$ — ①

Compare ① with

$X^2 = 4aY$

$Y = y-1$; $4a = 4$
$\Rightarrow a = 1$

Focus: $F(0,a) = F(X=0; Y=a)$
$X = 0$; $Y = a$
$x = 0$; $y-1 = 1$
; $y = 1+1 \Rightarrow y = 2$

So focus $(0,2)$

Vertex: $V(0,0) = V(X=0; Y=0)$
$X = 0$; $Y = 0$
$x = 0$; $y-1 = 0$
$y = 1$

So vertex$(0,1)$

Directrix: $Y = -a$ $\quad \therefore Y = y-1$
$y-1 = -1$ $\quad \therefore a = 1$
$y = -1+1 \Rightarrow y = 0$

Axis:
$X = 0$
$x = 0$

vi) $y^2 = -8(x-3)$
$y^2 = -8(x-3)$ —①

Compare ① with $Y^2 = -4aX$
$Y = y$; $X = x-3$; $-4a = -8$
$\Rightarrow a = 2$

Focus: $F(-a,0) = F(X=-a; Y=0)$
$X = -a$; $Y = 0$
$x-3 = -2$; $y = 0$
$x = -2+3$
$x = 1$

So focus $(1,0)$

Vertex: $V(0,0) = V(X=0; Y=0)$
$X = 0$; $Y = 0$
$x-3 = 0$; $y = 0$
$x = 3$

So Vertex $(3,0)$

Directrix: $X = a$ $\quad \therefore X = x-3$
$x-3 = 2$ $\quad \therefore a = 2$
$x = 2+3$
$x = 5$

Axis: $Y = 0$
$y = 0$

vii) $(x-1)^2 = 8(y+2)$
$(x-1)^2 = 8(y+2)$ —①

Compare ① with $X^2 = 4aY$
$X = x-1$; $Y = y+2$; $4a = 8$
$a = 2$

Focus: $F(0,a) = F(X=0; Y=a)$
$X = 0$; $Y = a$
$x-1 = 0$; $y+2 = 2$
$x = 1$; $y = 0$

So focus $(1,0)$

Vertex: $V(0,0) = V(X=0; Y=0)$
$X = 0$; $Y = 0$
$x-1 = 0$; $y+2 = 0$
$x = 1$; $y = -2$

So vertex $(1,-2)$

Directrix: $Y = -a$ $\quad \therefore Y = y+2$
$y+2 = -2$ $\quad \therefore a = 2$
$y = -2-2$
$y = -4$

Axis:
$X = 0$
$x-1 = 0$
$x = 1$.

viii) $y = 6x^2 - 1$
$6x^2 = y+1$
$x^2 = \frac{1}{6}(y+1)$ —①

Compare ① with $X^2 = 4aY$

$X = x$; $Y = y+1$; $4a = \frac{1}{6}$

$a = \frac{1}{24}$

Focus: $F(0,a) = F(X=0; Y=a)$

$X = 0$; $Y = a$

$x = 0$; $y+1 = \frac{1}{24}$

$y = \frac{1}{24} - 1$

$y = \frac{1-24}{24} = -\frac{23}{24}$

So focus $(0, -\frac{23}{24})$

Vertex: $V(0,0)$; $V(X=0; Y=0)$

$X = 0$; $Y = 0$

$x = 0$; $y+1 = 0$

$y = -1$

So Vertex $(0, -1)$

Directrix: $Y = -a$

$y+1 = -\frac{1}{24}$

$y = -\frac{1}{24} - 1 = \frac{-1-24}{24} = \frac{-25}{24}$

Axis: $X = 0$

$x = 0$

ix) $x + 8 - y^2 + 2y = 0$

$x + 8 = y^2 - 2y$

$x + 8 + 1 = y^2 - 2y + 1$

$x + 9 = (y-1)^2$

or $(y-1)^2 = 1(x+9)$ — ①

Compare ① with $Y^2 = 4aX$

$Y = y-1$; $X = x+9$; $4a = 1$

$\Rightarrow a = \frac{1}{4}$

Focus: $F(a,0) = F(X=a; Y=0)$

$X = a$; $Y = 0$

$x + 9 = \frac{1}{4}$; $y - 1 = 0$

$x = \frac{1}{4} - 9$; $y = 1$

$x = \frac{1-36}{4} = \frac{-35}{4}$;

So focus $(-\frac{35}{4}, 1)$

Vertex: $V(0,0) = V(X=0; Y=0)$

$X = 0$; $Y = 0$

$x + 9 = 0$; $y - 1 = 0$

$x = -9$; $y = 1$

So Vertex $(-9, 1)$

Directrix: $X = -a$

$x + 9 = -\frac{1}{4}$

$x = -\frac{1}{4} - 9$

$x = -\frac{1-36}{4} = -\frac{37}{4}$

Axis: $Y = 0$

$y - 1 = 0$

$y = 1$

x) $x^2 - 4x - 8y + 4 = 0$

$x^2 - 4x + 4 = 8y$

$(x-2)^2 = 8y$ — ①

Compare ① with $X^2 = 4aY$

$X = x-2$; $Y = y$; $4a = 8$

$\Rightarrow a = 2$

Focus: $F(0,a) = F(X=0; Y=a)$

$X = 0$; $Y = a$

$x - 2 = 0$; $y = 2$

$x = 2$;

So focus $(2,2)$

Vertex: $V(0,0) = V(X=0; Y=0)$

$X=0 \; ; \; Y=0$

$x-2=0 \; ; \; y=0$

$x=2$

So Vertex $(2,0)$

Directrix:

$Y=-a$

$y=-2$

Axis:

$X=0$

$x-2=0$

$x=2$

Question no 2: Write an equation of parabola with given elements.

i) Focus $(-3,1)$; directrix $x=3$

$F(-3,1)$

Directrix $= M = x-3$

Let $P(x,y)$ be any point on parabola

By the definition

$$|PF|^2 = |PM|^2$$

$$(x-(-3))^2+(y-1)^2 = \frac{(x-3)^2}{1^2+0^2}$$

$$(x+3)^2+(y-1)^2 = (x-3)^2$$

$$x^2+9+6x+y^2+1-2y = x^2+9-6x$$

$$x^2+6x+y^2-2y+10-x^2-9+6x=0$$

$$y^2-2y+12x+1=0$$

Required Equation

ii) Focus $(2,5)$; directrix $y=1$

$F(2,5)$

directrix $= M = y-1$

Let $P(x,y)$ be any point on parabola

By the definition

$$|PF|^2 = |PM|^2$$

$$(x-2)^2+(y-5)^2 = \frac{(y-1)^2}{0^2+1^2}$$

$$x^2+4-4x+y^2+25-10y = y^2+1-2y$$

$$x^2+y^2-4x-10y+29-y^2-1+2y=0$$

$$x^2-4x-8y+28=0$$

Required Equation.

iii) Focus $(-3,1)$; directrix $x-2y-3=0$

$F(-3,1)$

Directrix $= M = x-2y-3$

Let $P(x,y)$ be any point on parabola

By the definition

$$|PF|^2 = |PM|^2$$

$$(x-(-3))^2+(y-1)^2 = \frac{(x-2y-3)^2}{(1)^2+(-2)^2}$$

$$(x+3)^2+(y-1)^2 = \frac{(x-2y-3)^2}{1+4}$$

$$x^2+9+6x+y^2+1-2y = \frac{x^2+4y^2+9-4xy+12y-6x}{5}$$

$$\because (a+b+c)^2 = a^2+b^2+c^2+2ab+2bc+2ac$$

$$5(x^2+y^2+6x-2y+10) = x^2+4y^2+9-4xy+12y-6x$$

$$5x^2+5y^2+30x-10y+50 = x^2+4y^2+9-4xy+12y-6x$$

$$5x^2+5y^2+30x-10y+50-x^2-4y^2-9+4xy-12y+6x=0$$

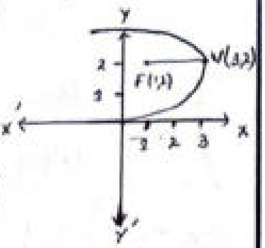
$$4x^2+y^2+36x-22y-4xy+41=0$$

Required Equation.

iv) Focus (1,2) ; Vertex (3,2)

$V(h,k) = V(3,2)$

From figure



$$Y^2 = -4aX$$

$$(y-k)^2 = -4a(x-h)$$

$$(y-2)^2 = -4a(x-3) - ⓘ$$

$\therefore$ $a = |FV|$ = Distance b/w Vertex & focus

$$a = \sqrt{(3-1)^2+(2-2)^2} \qquad \therefore d = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

$$a = \sqrt{(2)^2+(0)^2}$$

$$a = \sqrt{4}$$

$$a = 2$$

Put $a=2$ in ⓘ

$$(y-2)^2 = -4(2)(x-3)$$

$$y^2+4-4y = -8x+24$$

$$y^2+4-4y+8x-24=0$$

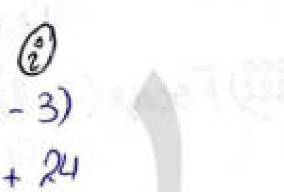
$$y^2-4y+8x-20=0$$

Required Equation.

v) Focus (-1,0) ; Vertex (-1,2)

$V(h,k) = V(-1,2)$

From figure



$$X^2 = -4aY$$

$$(x-h)^2 = -4a(y-k)$$

$$(x+1)^2 = -4a(y-2) - ⓘ$$

$\therefore$ $a = |FV|$ = Distance b/w force & vertex

$$a = \sqrt{(-1+1)^2+(2-0)^2}$$

$$a = \sqrt{(0)^2+(2)^2}$$

$$a = 2$$

Put $a=2$ in ⓘ

$$(x+1)^2 = -4(2)(y-2)$$

$$x^2+1+2x = -8y+16$$

$$x^2+1+2x+8y-16=0$$

$$x^2+2x+8y-15=0$$ Req. Equation

vi) Directrix $x=-2$; focus (2,2)

Let $P(x,y)$ be any point on parabola

$F(2,2)$

Directrix $x+2=0$

By the Definition

$$|PF|^2 = |PM|^2$$

$$(x-2)^2+(y-2)^2 = \frac{(x+2)^2}{1^2+0^2}$$

$$x^2+4-4x+y^2+4-4y = x^2+4+4x$$

$$\cancel{x^2}+y^2-4x-4y+8-\cancel{x^2}-4-4x=0$$

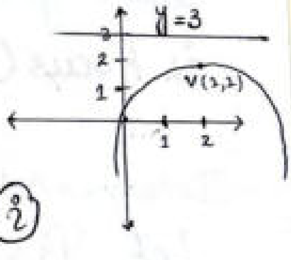
$$y^2-4y-8x+4=0$$

Required Equation

vii) Directrix $y=3$; vertex (2,2)

$V(h,k) = V(2,2)$

From figure:



$$X^2 = -4aY$$

$$(x-h)^2 = -4a(y-k)$$

$$(x-2)^2 = -4a(y-2) - ⓘ$$

directrix $y-3=0$

$\therefore$ $a = |VM|$

$$a = \frac{|0(2)+(1)(2)-3|}{\sqrt{0^2+1^2}} \qquad \therefore d = \frac{|ax_1+by_1+c|}{\sqrt{a^2+b^2}}$$

$$a = \frac{|0+2-3|}{\sqrt{1^2}}$$

$$a = \frac{|-1|}{1}$$

$$a = 1$$

Put $a=1$ in ⓘ

$$(x-2)^2 = -4(1)(y-2)$$

$$x^2+4-4x = -4y+8$$

$$x^2-4x+4+4y-8=0$$

$$x^2-4x+4y-4=0$$

Required Equation.

viii) Directrix $y=1$; length of latusrectum is 8. opens downward.

Directrix $y=1$

length of latusrectum $= 4a = 8$

$a = 2$

∵ Parbola opens downward

So equation

$$X^2 = -4aY$$

$$(x-h)^2 = -4a(y-k)$$

$$(x-h)^2 = -4(2)(y-k)$$

$$(x-h)^2 = -8(y-k) \text{ — (i)}$$

Compare (i) with $X^2 = -4aY$

$X = x-h$; $Y = y-k$

Directrix $Y = a$

$y - k = 2$ ∵ $a=2$

$1 - k = 2$ ∵ $y=1$

$1 - 2 = k$

$k = -1$

$V(h, -1)$

So equation (i)

$$(x-h)^2 = -8(y-(-1))$$

$$x^2+h^2-2hx = -8(y+1)$$

$$x^2+h^2-2hx = -8y-8$$

$$x^2+h^2-2hx+8y+8 = 0$$

Required Equation

ix) Axis $y=0$, through $(2,1)$ & $(11,-2)$

As equation of Parabola

$$(y-k)^2 = 4a(x-h)$$

∵ $y=0$ So the value of $k=0$

$$y^2 = 4a(x-h) \text{ — Ⓐ}$$

for $(2,1)$

$$(1)^2 = 4a(2-h)$$

$$1 = 8a - 4ah \text{ — (i)}$$

for $(11,-2)$

$$(-2)^2 = 4a(11-h)$$

$$4 = 44a - 4ah \text{ — (ii)}$$

(i) - (ii)

$$1 = 8a - \cancel{4ah}$$

$$\underset{-}{4} = \underset{-}{44a} \underset{+}{-} \cancel{4ah}$$

$$-3 = -36a$$

$$a = \frac{-3}{-36} = \frac{1}{12}$$

Put $a = \frac{1}{12}$ in (i)

$$1 = 8\left(\frac{1}{12}\right) - 4\left(\frac{1}{12}\right)h$$

$$1 = \frac{2}{3} - \frac{1}{3}h$$

$$1 = \frac{2-h}{3} \Rightarrow 3 = 2-h$$

$$h = 2-3 \Rightarrow h = -1$$

Eq Ⓐ become

$$y^2 = 4\left(\frac{1}{12}\right)(x+1)$$

$$y^2 = \frac{1}{3}(x+1)$$

$$3y^2 = x+1$$

Required Equation.

x) Axis parallel to y-axis, the point $(0,3)$; $(3,4)$ and $(4,11)$ lie on the graph.

As equation of Parabola

$(x-h)^2 = 4a(y-k)$ — Ⓐ

For (0,3)

$(0-h)^2 = 4a(3-k)$

$h^2 = 12a - 4ak$ — ⓘ

For (3,4)

$(3-h)^2 = 4a(4-k)$

$9+h^2-6h = 16a-4ak$ — (ii)

For (4,11)

$(4-h)^2 = 4a(11-k)$

$16+h^2-8h = 44a-4ak$ — (iii)

(iii) - (ii)

$16+\cancel{h^2}-8h = 44a-\cancel{4ak}$

$\underline{-9 \cancel{\pm h^2} \mp 6h = -16a \pm \cancel{4ak}}$

$7-2h = 28a$ — (iv)

(ii) - ⓘ

$9+\cancel{h^2}-6h = 16a-\cancel{4ak}$

$\underline{\cancel{h^2} \qquad = -12a \pm \cancel{4ak}}$

$9-6h = 4a$ — (v)

3(iv) - (v)

$21-\cancel{6h} = 84a$

$\underline{-9 \mp \cancel{6h} = -4a}$

$12 = 80a$

$a = \frac{12}{80} \Rightarrow a = \frac{3}{20}$

Put $a = \frac{3}{20}$ in (v)

$9-6h = 4\left(\frac{3}{20}\right)$

$9-6h = \frac{3}{5}$

$9-\frac{3}{5} = 6h \Rightarrow \frac{45-3}{5} = 6h$

$6h = \frac{42}{5} \Rightarrow h = \frac{42}{6\times5}$

$\boxed{h = \frac{7}{5}}$

Put $h = \frac{7}{5}$ & $a = \frac{3}{20}$ in ⓘ

$\left(\frac{7}{5}\right)^2 = 12\left(\frac{3}{20}\right) - 4\left(\frac{3}{20}\right)k$

$\frac{49}{25} = \frac{9}{5} - \frac{3}{5}k$

$\frac{3}{5}k = \frac{9}{5} - \frac{49}{25}$

$\frac{3}{5}k = \frac{45-49}{25}$

$\frac{3}{5}k = \frac{-4}{25}$

$k = \frac{-4}{25} \times \frac{5}{3}$

$\boxed{k = \frac{-4}{15}}$

Now equation Ⓐ become

$\left(x-\frac{7}{5}\right)^2 = 4\left(\frac{3}{20}\right)\left(y+\frac{4}{15}\right)$

$\left(\frac{5x-7}{5}\right)^2 = \frac{3}{5}\left(\frac{15y+4}{15}\right)$

$\frac{25x^2+49-70x}{25} = \frac{45y+12}{75}$

Multiply by '75' on Both sides

$3(25x^2+49-70x) = 45y+12$

$75x^2+147-210x = 45y+12$

$75x^2+147-210x-45y-12 = 0$

$75x^2-210x-45y+135 = 0$

Dividing by '15'

$5x^2-14x-3y+9 = 0$ Required Equation

Question no 3: Find an equation of parabola having its focus at the origin and directrix parallel to the i) the axis ii) the y-axis

i) the axis

Since directrix is parallel to x-axis M; $y=h$

$\Rightarrow y-h=0$

$F(0,0)$

Let $P(x,y)$ be any point on parabola

By the definition

Distance of P from F = Distance of P from Directrix

$$|PF| = |PM|$$

$$\sqrt{(x-0)^2+(y-0)^2} = \frac{|y-h|}{\sqrt{0^2+1^2}}$$

$$\sqrt{x^2+y^2} = \frac{|y-h|}{\sqrt{1^2}}$$

Taking square on B.S

$$x^2+y^2 = (y-h)^2$$

$$x^2+y^2 = y^2+h^2-2hy$$

$$x^2+\cancel{y^2}-\cancel{y^2}-h^2+2hy=0$$

$$x^2-h^2+2hy=0.$$

ii) the y-axis

Since directrix is parallel to y-axis M; $x=h$

$\Rightarrow x-h=0$

$F(0,0)$

Let $P(x,y)$ be any point on parabola

By the definition

$$|PF| = |PM|$$

$$\sqrt{(x-0)^2+(y-0)^2} = \frac{|x-h|}{\sqrt{1^2+0^2}}$$

$$\sqrt{x^2+y^2} = \frac{|x-h|}{\sqrt{1^2}}$$

Taking square on B.S

$$x^2+y^2 = (x-h)^2$$

$$x^2+y^2 = x^2+h^2-2hx$$

$$x^2+y^2-x^2-h^2+2hx=0$$

$$y^2-h^2+2hx=0$$

Required Equations-

Question no 4: Show that an equation of the parabola with focus at $(a\cos\alpha, a\sin\alpha)$ and directrix

$x\cos\alpha + y\sin\alpha + a = 0$ is

$(x\sin\alpha - y\cos\alpha)^2 = 4a(x\cos\alpha + y\sin\alpha).$

$F(a\cos\alpha,\ a\sin\alpha)$

Directrix; $x\cos\alpha + y\sin\alpha + a = 0$

Let $P(x,y)$ be any point on parabola

By the definition

$$|PF| = |PM|$$

$$\sqrt{(x-\cos\alpha)^2+(y-\sin\alpha)^2} = \frac{|x\cos\alpha + y\sin\alpha + a|}{\sqrt{\cos^2\alpha+\sin^2\alpha}}$$

$$\sqrt{(x-\cos\alpha)^2+(y-\sin\alpha)^2} = \frac{|x\cos\alpha + y\sin\alpha + a|}{\sqrt{1}}$$

$$\therefore \cos^2\alpha+\sin^2\alpha=1$$

Taking square on B.S

$$(x-a\cos\alpha)^2+(y-a\sin\alpha)^2=(x\cos\alpha+y\sin\alpha+a)^2$$

$$x^2+a^2\cos^2\alpha-2ax\cos\alpha+y^2+a^2\sin^2\alpha-2ay\sin\alpha$$
$$=x^2\cos^2\alpha+y^2\sin^2\alpha+a^2+2xy\cos\alpha\sin\alpha$$
$$+2ay\sin\alpha+2ax\cos\alpha$$

$$x^2+a^2\cos^2\alpha-2ax\cos\alpha+y^2+a^2\sin^2\alpha-2ay\sin\alpha$$
$$-x^2\cos^2\alpha-y^2\sin^2\alpha-a^2-2xy\cos\alpha\sin\alpha$$
$$-2ay\sin\alpha-2ax\cos\alpha=0$$

$$x^2+a^2(\cos^2\alpha+\sin^2\alpha)-2ax\cos\alpha+y^2-2ay\sin\alpha$$
$$-x^2\cos^2\alpha-y^2\sin^2\alpha-a^2-2xy\cos\alpha\sin\alpha$$
$$-2ay\sin\alpha-2ax\cos\alpha=0$$
$$\therefore \cos^2\alpha+\sin^2\alpha=1$$

$$x^2+\cancel{a^2}(1)-2ax\cos\alpha+y^2-2ay\sin\alpha$$
$$-x^2\cos^2\alpha-y^2\sin^2\alpha-\cancel{a^2}-2xy\cos\alpha\sin\alpha$$
$$-2ay\sin\alpha-2ax\cos\alpha=0$$

$$x^2(1-\cos^2\alpha)+y^2(1-\sin^2\alpha)-4ax\cos\alpha$$
$$-4ay\sin\alpha-2xy\cos\alpha\sin\alpha=0$$
$$\therefore 1-\cos^2\alpha=\sin^2\alpha\ ;\ 1-\sin^2\alpha=\cos^2\alpha$$

$$x^2\sin^2\alpha+y^2\cos^2\alpha-4a(x\cos\alpha+y\sin\alpha)$$
$$-2xy\cos\alpha\sin\alpha=0$$

$$x^2\sin^2\alpha+y^2\cos^2\alpha-2xy\cos\alpha\sin\alpha=$$
$$4a(x\cos\alpha+y\sin\alpha)$$

$$(x\sin\alpha-y\cos\alpha)^2=4a(x\cos\alpha+y\sin\alpha)$$

Hence proved -

Question no 5: Show that the ordinate at any point P of the parabola is an mean proportional between the lenght of the latesrectum and the absCissa of P.

Consider equation of parabola

$$y^2=4ax$$
$$yy=4ax$$
$$\frac{y}{x}=\frac{4a}{y}$$
$$\frac{4a}{y}=\frac{y}{x}$$

$$\frac{\text{latusrectum}}{\text{ordinate}}=\frac{\text{ordinate}}{\text{abscissa}}$$

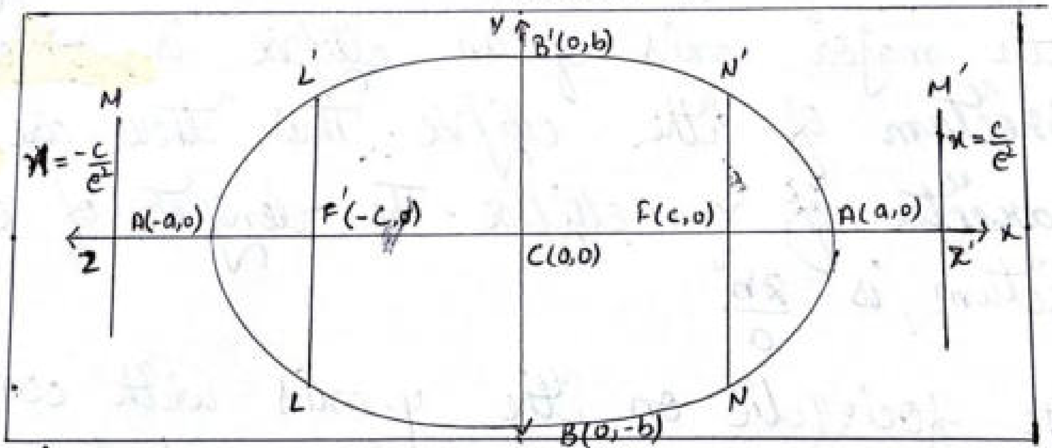
Ordinade is mean proportional ~~to~~ between latusrectum and abscissa

Theory

• Ellipse:

. Let $0<e<1$ and F be a fixed point and L is fixed line not containing F. Let $P(x,y)$ be the point in the plane and $|PM|$ be the perpendicular distance of a point from the line L. The set of all points P such that $\frac{|PF|}{|PM|} = e \ (0<e<1)$ is called ellipse

OR The set of all points P in a plane, such that distance of each point from a fixed point bears a constant ratio (less than one) to the distance from a fixed line is called an ellipse The number e is **eccentricity** of the ellipse, F a **focus** and L a **directrix**



• Definitions:

Let F' and F be two foci of the ellipse

- $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ ---- } ①$$

- The midpoint C of FF' is called the **Centre** of the ellipse. In case of ① centre is C(0,0)
- The intersection of ② with the line joining the

foci are obtained by setting $y=0$ into ①. These are the points $A'(a,0)$ and $A(a,0)$. The points A and A' are called **vertices** of the ellipse

- The line segment $AA' = 2a$ is called the **Major Axis** of the ellipse. The line through the centre of ① and perpendicular to the major axis has its equation as $x=0$. It meets ① at points $B'(0,b)$ and $B(0,-b)$. The line segment $BB' = 2b$ is called the **Minor Axis** of the ellipse and B', B are sometimes called the **coverticess** of the ellipse
- The length of the major axis is greater than the length of the minor axis
- Foci of an ellipse always lie on the major axis
- Each of the focal chords LFL' and NF'N' perpendicular to the major axis of an ellipse is called a **latusrectum** of the ellipse. Thus there are two **laterarecta** of an ellipse. The length of each latusrectum is $\frac{2b^2}{a}$
- If the foci lie on the y-axis with coordinates $(0,-ae)$ and $(0,ae)$, then equation of the ellipse is

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1 \; ; \; a > b$$

Note:

In each ellipse:
i) Length of major axis $= 2a$
ii) Length of minor axis $= 2b$
iii) Foci lie on the major axis
iv) Length of latusrectum $= \frac{2b^2}{a}$

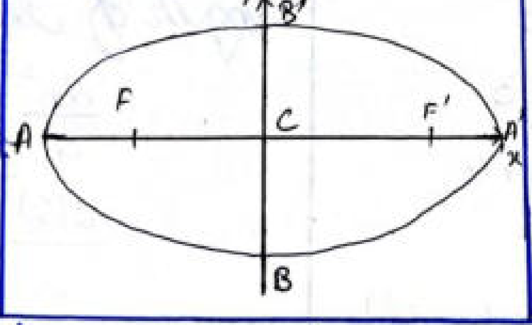
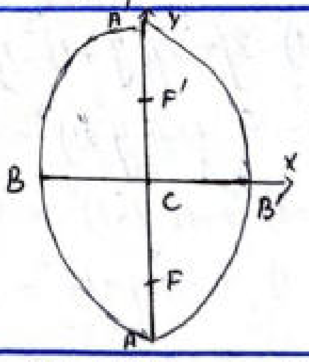
- **Standard Form of the Ellipse:**

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

- **Parametric equation of Ellipse:**

The point $(a\cos\theta, b\sin\theta)$ lies on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ for all real θ. $x=a\cos\theta$, $y=b\sin\theta$ are called parametric equation of the ellipse

Summary of Standard Ellipses

Equation	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, $a>b$ $c^2=a^2-b^2$	$\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$; $a>b$ $c^2=a^2-b^2$
Focus	$(\pm c,0)$	$(0,\pm c)$
Directrices	$x=\pm\frac{c}{e^2}$	$y=\pm\frac{c}{e^2}$
Major Axis	$y=0$	$x=0$
Vertices	$(\pm a,0)$	$(0,\pm a)$
Covertices	$(0,\pm b)$	$(\pm b,0)$
Centre	$(0,0)$	$(0,0)$
Eccentricity	$e=\frac{c}{a}<1$	$e=\frac{c}{a}<1$
Graph		

Example # 1: Find an equation of ellipse having centre (0,0), focus (0,-3); vertex (0,4).

C(0,0)

focus $(0,\pm c)=(0,\pm3)\Rightarrow c=3\Rightarrow c^2=9$

Vertices $(0,\pm a)=(0,\pm4)\Rightarrow a=4\Rightarrow a^2=16$

$b^2=a^2-c^2\Rightarrow b^2=16-9\Rightarrow b^2=7$

$\because$ foci have same abscissa so major axis is along y-axis $\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$

$\frac{x^2}{7}+\frac{y^2}{16}=1$

Example # 2: Analyze the equation $4x^2+9y^2=36$.

Dividing by 36

$$\frac{x^2}{9}+\frac{y^2}{4}=1$$

Compare with $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$a^2=9 \Rightarrow a=3 \; ; \; b^2=4 \Rightarrow b=2$

$\therefore c^2=a^2-b^2=9-4=5$

$\Rightarrow c^2=5 \Rightarrow c=\sqrt{5}$

Centre: $(0,0)$

Foci: $(\pm c,0)=(\pm\sqrt{5},0)$

Covertices: $(0,\pm b)=(0,\pm 2)$

Eccentricity: $e=\frac{c}{a}=\frac{\sqrt{5}}{3}$

Directrices: $x=\pm\frac{a}{e}=\pm\frac{3}{\sqrt{5}/3}=\pm\frac{9}{\sqrt{5}}$.

Vertices: $(\pm a,0)=(\pm 3,0)$.

Example # 3: Show that the equation $9x^2-18x+4y^2+8y-23=0$ represents an ellipse. Find its elements.

$$9x^2-18x+4y^2+8y-23=0$$

$$9(x^2-2x)+4(y^2+2y)=23$$

$$9(x^2-2x+1-1)+4(y^2+2y+1-1)=23$$

$$9[(x-1)^2-1]+4[(y+1)^2-1]=23$$

$$9(x-1)^2-9+4(y+1)^2-4=23$$

$$9(x-1)^2+4(y+1)^2=23+9+4$$

$$9(x-1)^2+4(y+1)^2=36$$

$$\frac{(x-1)^2}{4}+\frac{(y+1)^2}{9}=1 \quad (\div \text{ by } 36)$$

Compare with $\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$

$X=x-1 \; , \; Y=y-1$

$a^2=9 \Rightarrow a=3 \; ; \; b^2=4 \Rightarrow b=2$

$c^2=a^2-b^2=9-4=5 \Rightarrow c=\sqrt{5}$

Centre: $(X=0 \; ; \; Y=0)$

$x-1=0 \; ; \; y+1=0$

$x=1 \; ; \; y=-1$

So centre $(1,-1)$

Foci: $(X=0 \; ; \; Y=\pm c)$

$x-1=0 \; ; \; y+1=\pm\sqrt{5}$

$x=1 \; ; \; y=-1\pm\sqrt{5}$

So foci $(1,-1\pm\sqrt{5})$

Co-verties: $(X=\pm b \; ; \; Y=0)$

$x-1=\pm 2 \; ; \; y+1=0$

$x=1\pm 2 \; ; \; y=-1$

$x=3 ; x=-1 \; ;$

So covertices $(3,1)$ and $(-1,-1)$

Eccentricity: $e=\frac{c}{a}=\frac{\sqrt{5}}{3}$.

Directrices: $Y=\pm\frac{a}{e}$

$y+1=\pm\frac{3}{\sqrt{3}/3}=\pm\frac{9}{\sqrt{3}} \Rightarrow y=-1\pm\frac{9}{\sqrt{3}}$

Vertieces: $(X=0 \; ; \; Y=\pm a)$

$x-1=0 \; ; \; y+1=\pm 3$

$x=1 \; ; \; y=-1\pm 3$

$y=2 \; ; \; y=-4$

So vertices $(1,2)$, $(1,-4)$

Length of letusrectum:

$=\frac{2b^2}{a}$

$=\frac{2(2)^2}{3}$

$=\frac{8}{3}$

Exercise # 6.5

Question no 1: Find an equation of the ellipse with given data

i) Foci (±3,0) and minor axis of length 20.

Foci$(\pm c,0) = (\pm 3,0)$

$$\boxed{c=3}$$

Minor Axis $2b = 10$

$$\boxed{b=5}$$

$a^2 = b^2+c^2$ ∴ $c^2 = a^2-b^2$

$a^2 = (5)^2+(3)^2$

$a^2 = 25+9$

$$\boxed{a^2 = 34}$$

Equation of ellipse

$\frac{x^2}{34}+\frac{y^2}{25}=1$ ∴ $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

ii) Foci (0,-1) and (0,-5) and major axis of length 6.

Centre = Midpoint of foci

$(h,k) = \left(\frac{0+0}{2} ; \frac{-1-5}{2}\right)$

$= \left(\frac{0}{2} ; \frac{-6}{2}\right)$

$= (0,-3)$

∴ $2c$ = Distance between foci

$2c = \sqrt{(0-0)^2+(-5+1)^2}$

$2c = \sqrt{(0)^2+(-4)^2}$

$2c = \sqrt{16}$

$2c = 4$

$$\boxed{c=2}$$

Length of Major Axis $2a = 6$

$$\boxed{a=3}$$

$b^2 = a^2-c^2$ ∴ $c^2 = a^2-b^2$

$b^2 = (3)^2-(2)^2$

$b^2 = 9-4$

$b^2 = 5$

∴ foci have same abscissa so major axis is along y-axis

Now eq. of ellipse

∴ $\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1$

$\frac{(x-0)^2}{5}+\frac{(y+3)^2}{(3)^2}=1$

$\frac{x^2}{5}+\frac{(y+3)^2}{9}=1$

iii) Foci $(\pm 3\sqrt{3},0)$ and vertices $(\pm 6,0)$

Foci$(\pm c,0) = (\pm 3\sqrt{3},0)$

$c = 3\sqrt{3} \Rightarrow c^2 = 9(3)$

$c^2 = 27$

Centre = Midpoint of foci

$= \left(\frac{3\sqrt{3}-3\sqrt{3}}{2}, \frac{0-0}{2}\right)$

$= \left(\frac{0}{2}, \frac{0}{2}\right)$

$= (0,0)$

Vertices $(\pm a,0) = (\pm 6,0)$

$a = 6$

$a^2 = 36$

$b^2 = a^2 - c^2$ ∴ $c^2 = a^2 - b^2$

$b^2 = 36 - 27$

$b^2 = 9$

Equation of Ellipse

$$\frac{x^2}{36} + \frac{y^2}{9} = 1 \quad ∴ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

iv) Vertices (-1, 1), (5, 1); foci (4, 1) and (0, 1)

Centre = Midpoint of foci

$$(h,k) = \left(\frac{4+0}{2} ; \frac{1+1}{2}\right)$$

$$= \left(\frac{4}{2}, \frac{2}{2}\right)$$

$$= (2, 1)$$

∴ $2c$ = Distance between foci

$$2c = \sqrt{(4-0)^2 + (1-1)^2}$$

$$2c = \sqrt{(4)^2 + (0)^2}$$

$$2c = \sqrt{16}$$

$$2c = 4$$

$$c = 2$$

$$c^2 = 4$$

Also vertices $(-1,1), (5,1)$

$$2a = \sqrt{(-1-5)^2 + (1-1)^2}$$

$$2a = \sqrt{(-6)^2 + (0)^2}$$

$$2a = \sqrt{36}$$

$$2a = 6$$

$a = 3$

$a^2 = 9$

$b^2 = a^2 - c^2$ ∴ $c^2 = a^2 - b^2$

$b^2 = 9 - 4$

$b^2 = 5$

foci have same ordinate

So major axis is along x-axis

Equation of Ellipse

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

$$\frac{(x-2)^2}{9} + \frac{(y-1)^2}{5} = 1$$

v) Foci $(\pm\sqrt{5})$ and passing through the point $\left(\frac{3}{2}, \sqrt{3}\right)$

$F(+\sqrt{5}, 0)$; $F'(-\sqrt{5}, 0)$

$F(\pm c, 0) = (\pm\sqrt{5}, 0)$

$c = \sqrt{5}$

∴ $c^2 = a^2 - b^2$

$(\sqrt{5})^2 = a^2 - b^2$

$5 + b^2 = a^2$ —(i)

Centre = Midpoind of foci

$$(h,k) = \left(\frac{+\sqrt{5} - \sqrt{5}}{2}, \frac{0+0}{2}\right)$$

$$\left(\frac{0}{2}, \frac{0}{2}\right) = (0,0)$$

As ellipse is along x-axis

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{—(ii)}$$

Since $P\left(\frac{3}{2}, \sqrt{3}\right)$ lies on eq (ii)

$$\frac{\left(\frac{3}{2}\right)^2}{a^2} + \frac{(\sqrt{3})^2}{b^2} = 1$$

$$\frac{\frac{9}{4}}{a^2} + \frac{3}{b^2} = 1$$

$$\frac{9}{4a^2} + \frac{3}{b^2} = 1$$

Multiply by $4a^2b^2$

$9b^2 + 12a^2 = 4a^2b^2$ — (iii)

Put (i) in (iii)

$9b^2 + 12(5+b^2) = 4(5+b^2)b^2$

$9b^2 + 60 + 12b^2 = 20b^2 + 4b^4$

$21b^2 + 60 - 20b^2 - 4b^4 = 0$

$b^2 - 4b^4 + 60 = 0$

$-4b^4 + b^2 + 60 = 0$

$-(4b^4 - b^2 - 60) = 0$

$4b^4 - b^2 - 60 = 0$

$4b^4 - 16b^2 + 15b^2 - 60 = 0$

$4b^2(b^2 - 4) + 15(b^2 - 4) = 0$

$(b^2 - 4)(4b^2 + 15) = 0$

$b^2 - 4 = 0$; $4b^2 + 15 = 0$

$b^2 = 4$; $4b^2 = -15$

$b^2 = -\frac{15}{4}$

Not possible

Put $b^2 = 4$ in (i)

$5 + 4 = a^2$

$a^2 = 9$

So eq (ii) become

$$\frac{x^2}{9} + \frac{y^2}{4} = 1$$

vi) Vertices $(0, \pm 5)$, eccentricity $\frac{3}{5}$

Vertices$(0, \pm a) = (0, \pm 5)$

$a = 5 \Rightarrow a^2 = 25$

Centre = Midpoint of Vertices

$(h,k) = \left(\frac{0+0}{2}, \frac{5-5}{2}\right) = \left(\frac{0}{2}, \frac{0}{2}\right) = (0,0)$

eccentricity $= e = \frac{3}{5}$

$\therefore c = ea$

$c = \left(\frac{3}{5}\right)(5)$

$c = 3 \Rightarrow c^2 = 9$

$\therefore c^2 = a^2 - b^2$

$b^2 = a^2 - c^2$

$b^2 = 25 - 9$

$b^2 = 16$

$\therefore$ Vertices have same abscissa

So major axis is along y-axis

Eq of Ellipse $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$

$$\frac{x^2}{16} + \frac{y^2}{25} = 1$$

vii) $C(0,0)$, focus $(0,-3)$. Vertex$(0,4)$

$C(0,0)$

focus $= F'(0,-3)$

So other focus $= F(0,3)$

foci $(0, \pm c) = (0, \pm 3)$

$c = 3 \Rightarrow c^2 = 9$

Also vertex $= (0,4)$

Other vertex $= (0,-4)$

Vertices $(0, \pm a) = (0, \pm 4)$

$a = 4 \Rightarrow a^2 = 16$

∴ $c^2 = a^2 - b^2$

$b^2 = a^2 - c^2$

$b^2 = 16 - 9$

$b^2 = 7$

∴ foci have same abscissa
So major axis is along y-axis

Eq of Ellipse:

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$$

$$\frac{x^2}{7} + \frac{y^2}{16} = 1$$

viii) Centre (2,2), major axis parallel to y-axis and of length 8 units, minor axis parallel to x-axis and of length 6 units.

$C(h,k) = C(2,2)$

Length of major axis $2a = 8$

$a = 4$

$a^2 = 16$

Length of minor axis $2b = 6$

$b = 3$

$b^2 = 9$

Eq of Ellipse:

$$∴ \frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$$

$$\frac{(x-2)^2}{9} + \frac{(y-2)^2}{16} = 1$$

ix) Centre (0,0), symmetric with respect to both the axis and passing through the points (2,3), (6,1)

Eq of Ellipse is:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ —(i)}$$

∴ (i) passes through (2,3) so

$$\frac{(2)^2}{a^2} + \frac{(3)^2}{b^2} = 1$$

$$\frac{4}{a^2} + \frac{9}{b^2} = 1$$

$$4b^2 + 9a^2 = a^2b^2 \text{ —(ii)}$$

∴ Also (i) passes through (6,1) so

$$\frac{(6)^2}{a^2} + \frac{(1)^2}{b^2} = 1$$

$$\frac{36}{a^2} + \frac{1}{b^2} = 1$$

$$36b^2 + a^2 = a^2b^2 \text{ —(iii)}$$

By (ii) and (iii)

$36b^2 + a^2 = 4b^2 + 9a^2$

$36b^2 - 4b^2 = 9a^2 - a^2$

$32b^2 = 8a^2$

$4b^2 = a^2$ —(iv) ∴ Dividing by 8

Put (iv) in (ii)

$4b^2 + 9(4b^2) = (4b^2)b^2$

$4b^2 + 36b^2 = 4b^4$

$40b^2 = 4b^4$

$10 = b^2$ ∴ Dividing by $4b^2$

Put $b^2 = 10$ in (iv)

$a^2 = 4(10)$

$a^2 = 40$

Eq of Ellipse:

$$\frac{x^2}{40}+\frac{y^2}{10}=1$$

x) Centre (0,0), major axis horizontal, the points (3,1), (4,0) lie on the graph.

$C(0,0)$

Major axis horizontal (i-e., x-axis)

Eq. of req. ellipse is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \quad —(i)$$

∴ (i) passes through (3,1) so,

$$\frac{(3)^2}{a^2}+\frac{(1)^2}{b^2}=1$$

$$\frac{9}{a^2}+\frac{1}{b^2}=1$$

$$9b^2+a^2=a^2b^2 \quad —(ii)$$

∴ (i) passes through (4,0) so,

$$\frac{(4)^2}{a^2}+\frac{(0)^2}{b^2}=1$$

$$\frac{16}{a^2}=1$$

$$a^2=16 \Rightarrow a=4$$

Now (ii)

$$9b^2+16=16b^2$$

$$16=16b^2-9b^2$$

$$16=7b^2$$

$$b^2=\frac{16}{7}$$

So required ellipse:

$$\frac{x^2}{16}+\frac{y^2}{16/7}=1$$

Question no 2: Find the centre, foci, eccentricity, vertices and directrices of the ellipse whose equation is given:

i) $x^2+4y^2=16$

$$x^2+4y^2=16$$

$$\rightarrow \frac{x^2}{16}+\frac{y^2}{4}=1 \quad (\div \text{ by } 16)$$

Compare with $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$$a^2=16 \Rightarrow a=4$$

$$b^2=4 \Rightarrow b=2$$

$$\therefore c^2=a^2-b^2$$

$$c^2=16-4$$

$$c^2=12 \Rightarrow c=\sqrt{12}=2\sqrt{3}$$

Centre: (0,0)

Foci: $(\pm c,0)=(\pm 2\sqrt{3},0)$

Eccentricity: $e=\frac{c}{a}=\frac{2\sqrt{3}}{4}$

$$e=\frac{\sqrt{3}}{2}$$

Vertices: $(\pm a,0)=(\pm 4,0)$

Directrices: $x=\pm\frac{a}{e}$

$$x=\pm\frac{4}{\sqrt{3}/2}=\frac{\pm 8}{\sqrt{3}}$$

ii) $9x^2+y^2=18$

$$9x^2+y^2=18$$

$$\frac{x^2}{2}+\frac{y^2}{18}=1 \quad (\div \text{ by } 18)$$

Compare with $\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$

$b^2 = 2 \Rightarrow b = \sqrt{2}$

$a^2 = 18 \Rightarrow a = \sqrt{18} = 3\sqrt{2}$

$\therefore c^2 = a^2 - b^2$

$c^2 = 18 - 2$

$c^2 = 16 \Rightarrow c = 4$

Centre: $(0,0)$

Foci: $(0, \pm c) = (0, \pm 4)$

Eccentricity: $e = \frac{c}{a} = \frac{4}{3\sqrt{2}} = \frac{2\times 2}{3\sqrt{2}}$

$e = \frac{2\sqrt{2}}{3}$

Vertices: $(0, \pm a) = (0, \pm 3\sqrt{2})$

Directrices: $y = \pm \frac{a}{e}$

$y = \pm \frac{3\sqrt{2}}{2\sqrt{2}/3} = \pm \frac{9\cancel{\sqrt{2}}}{2\cancel{\sqrt{2}}} \Rightarrow y = \pm \frac{9}{2}$

iii) $25x^2 + 9y^2 = 225$

$25x^2 + 9y^2 = 225$

$\frac{x^2}{9} + \frac{y^2}{25} = 1$ (÷ by 225)

Compare with $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$

$b^2 = 9 \Rightarrow b = 3$

$a^2 = 25 \Rightarrow a = 5$

$\therefore c^2 = a^2 - b^2$

$c^2 = 25 - 9$

$c^2 = 16 \Rightarrow c = 4$

Centre: $(0,0)$

foci: $(0, \pm c) = (0, \pm 4)$

Eccentricity: $e = \frac{c}{a} = \frac{4}{5}$

Vertices: $(0, \pm a) = (0, \pm 5)$

Directrices: $y = \pm \frac{a}{e}$

$y = \pm \frac{5}{4/5} \Rightarrow y = \pm \frac{25}{4}$

iv) $\frac{(2x-1)^2}{4} + \frac{(y+2)^2}{16} = 1$

$\frac{(2x-1)^2}{4} + \frac{(y+2)^2}{16} = 1$

$\frac{\left[2(x-\frac{1}{2})\right]^2}{4} + \frac{(y+2)^2}{16} = 1$ —(i)

Compare with $\frac{X^2}{b^2} + \frac{Y^2}{a^2} = 1$

$\rightarrow X = x - \frac{1}{2}$; $Y = y + 2$

$b^2 = 4 \Rightarrow b = 2$

$a^2 = 16 \Rightarrow a = 4$

$\therefore c^2 = a^2 - b^2$

$c^2 = 16 - 4 = 12$

$c = \sqrt{12} = 2\sqrt{3}$

Centre: $(X = 0 ; Y = 0)$

$X = 0$; $Y = 0$

$x - \frac{1}{2} = 0$; $y + 2 = 0$

$x = \frac{1}{2}$; $y = -2$

So centre $(\frac{1}{2}, -2)$

Foci: Major axis is along y-axis

so Foci $(X = 0 ; Y = \pm c)$

$\therefore X = 0$; $Y = \pm 2\sqrt{3}$

$x - \frac{1}{2} = 0$; $y + 2 = \pm 2\sqrt{3}$

$x = \frac{1}{2}$; $y = -2 \pm 2\sqrt{3}$

So foci $\left(\frac{1}{2} ; -2 \pm 2\sqrt{3}\right)$

Eccentricity: $e = \frac{c}{a} = \frac{2\sqrt{3}}{4} = \frac{\sqrt{3}}{2}$

Vertices: $(X = 0 ; Y = \pm a)$

$X = 0 \quad ; \quad Y = \pm 4$

$x - \frac{1}{2} = 0 \quad ; \quad y + 2 = \pm 4$

$x = \frac{1}{2} \quad ; \quad y = -2 \pm 4$

$y = 2 \quad ; \quad y = -6$

So Vertices $\left(\frac{1}{2}, 2\right)$ and $\left(\frac{1}{2}, -6\right)$

Directrices: $Y = \pm \frac{a}{e}$

$y+2 = \pm \frac{4}{\sqrt{3}/2} \Rightarrow y + 2 = \pm \frac{8}{\sqrt{3}} \Rightarrow y = -2 \pm \frac{8}{\sqrt{3}}$

v) $x^2 + 16x + 4y^2 - 16y + 76 = 0.$

$x^2 + 16x + 4(y^2 - 4y) = -76$

$x^2 + 2(8)(x) + (8)^2 + 4(y^2 - 4y + 4 - 4) = -76 + (8)^2$

$(x + 8)^2 + 4[(y-2)^2 - 4] = -76 + 64$

$(x+8)^2 + 4(y-2)^2 - 16 = -12$

$(x+8)^2 + 4(y-2)^2 = 4$

$\therefore (\div \text{ by } 4)$

$\frac{(x+8)^2}{4} + \frac{(y-2)^2}{1} = 1$

Compare with $\frac{X^2}{a^2} + \frac{Y^2}{b^2} = 1$

$X = x + 8 \quad ; \quad Y = y - 2$

$a^2 = 4 \Rightarrow a = 2$

$b^2 = 1 \Rightarrow b = 1$

$c^2 = 4 - 1 \qquad \therefore c^2 = a^2 - b^2$

$c^2 = 3 \Rightarrow c = \sqrt{3}$

Centre: $(X = 0 ; Y = 0)$

$X = 0 \quad ; \quad Y = 0$

$x + 8 = 0 \quad ; \quad y - 2 = 0$

$x = -8 \quad ; \quad y = 2$

So centre $(-8, 2)$

Foci: As major axis is along x-axis. So Foci $(\pm c, 0)$

$X = \pm c \quad ; \quad Y = 0$

$x + 8 = \pm\sqrt{3} \quad ; \quad y - 2 = 0$

$x = -8 \pm \sqrt{3} \quad ; \quad y = 2$

So Foci $(-8 \pm \sqrt{3}, 2)$

Eccentricity: $e = \frac{c}{a} = \frac{\sqrt{3}}{2}$

Vertices: $(X = \pm a, Y = 0)$

$X = \pm a, \quad Y = 0$

$x + 8 = \pm 2 \quad ; \quad y - 2 = 0$

$x = -8 \pm 2 \quad ; \quad y = 2$

$x = -4, x = -10 \quad ;$

So vertices $(-4, 2), (-10, 2)$

Directrices: $X = \pm \frac{a}{e}$

$x + 8 = \pm \frac{2}{\sqrt{3}/2} \Rightarrow x + 8 = \frac{4}{\sqrt{3}} \Rightarrow x = -8 \pm \frac{4}{\sqrt{3}}$

vi) $25x^2 + 4y^2 - 250x - 16y + 541 = 0$

$25(x^2 - 10x) + 4(y^2 - 4y) = -541.$

$25(x^2 - 10x + 25 - 25) + 4(y^2 - 4y + 4 - 4) = -541$

$25[(x-5)^2 - 25] + 4[(y-2)^2 - 4] = -541$

$25(x-5)^2 - 625 + 4(y-2)^2 - 16 = -541$

$25(x-5)^2 + 4(y-2)^2 = -541 + 625 + 16$

$25(x-5)^2 + 4(y-2)^2 = 100 \qquad \therefore$ Dividing by 100

$\frac{(x-5)^2}{4} + \frac{(y-2)^2}{25} = 1$

Compare with $\frac{X^2}{b^2}+\frac{Y^2}{a^2}=1$

$X = x-5 \;;\; Y = y-2$

$b^2 = 4 \Rightarrow b = 2$

$a^2 = 25 \Rightarrow a = 5$

$\therefore c^2 = a^2 - b^2 \Rightarrow c^2 = 25-4$

$c^2 = 21 \Rightarrow c = \sqrt{21}$

Centre: $(X=0 \,;\, Y=0)$

$x-5=0 \;;\; y-2=0$

$x=5 \;;\; y=2$

So centre $(5,2)$

Foci: ∴ Major axis is along y-axis

Foci $(X=0 \,;\, Y=\pm c)$

$x-5=0 \;;\; y-2=\pm\sqrt{21}$

$x=5 \;;\; y=2\pm\sqrt{21}$

So foci $(5, 2\pm\sqrt{21})$

Eccentricity: $e=\frac{c}{a}=\frac{\sqrt{21}}{5}$

Vertices: $(X=0\,,\, Y=\pm a)$

$x-5=0 \;;\; y-2=\pm 5$

$x=5 \;;\; y=2\pm 5$

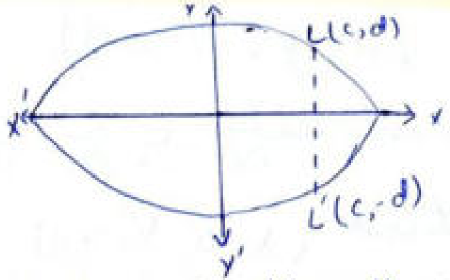
$y=7 \,;\, y=-3$

So Vertices $(5,7)\,;\,(5,-3)$

Directrices: $Y=\pm\frac{a}{e}$

$y-2=\frac{\pm 5}{\sqrt{21}/5} \Rightarrow y = 2\pm\frac{25}{\sqrt{21}}$

Question no 5: Prove that the latusrectum of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $\frac{2b^2}{a}$



Let $L(c,d)$ and $L'(c,-d)$ be end Points of latusrectum LL' of given ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ —(i)

∴ Length of letusrectum $=|LL'|=2d$

∴ $L(c,d)$ lies on (i) so

$$\frac{c^2}{a^2}+\frac{d^2}{b^2}=1$$

$$c^2b^2+a^2d^2=a^2b^2$$

$$a^2d^2=a^2b^2-c^2b^2$$

$$a^2d^2=b^2(a^2-c^2)$$

$$\because b^2=a^2-c^2$$

$$a^2d^2=b^2(b^2)$$

$$a^2d^2=b^4$$

$$d^2=\frac{b^4}{a^2}$$

$$\sqrt{d^2}=\sqrt{\frac{b^4}{a^2}}$$

$$d=\frac{b^2}{a}$$

Length of latusrectum $=2d$

$$2\left(\frac{b^2}{a}\right)$$

$$=\frac{2b^2}{a}$$

Hence proved.

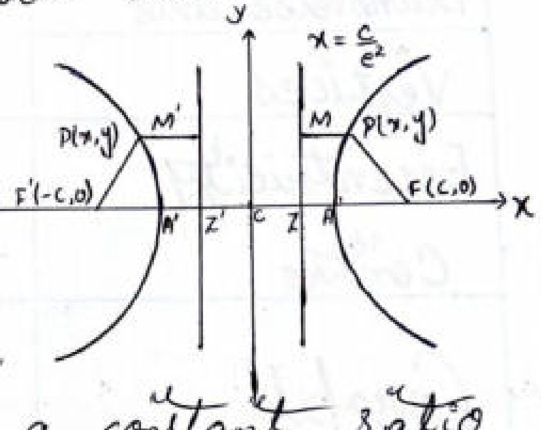
Theory

• Hyperbola:

Let $e > 1$ and F be a fixed point and L is fixed line not containing F. Let $P(x, y)$ be any point in the plane and $|PM|$ be the perpendicular distance of a point $P(x, y)$ from the line L. The set of all points P such that

$$\frac{|PF|}{|PM|} = e > 1$$

is called hyperbola



OR The set of all points P in a plane, such that distance of each point from a fixed point bears a constant ratio (greater than one) to the distance from a fixed line is called hyperbola.

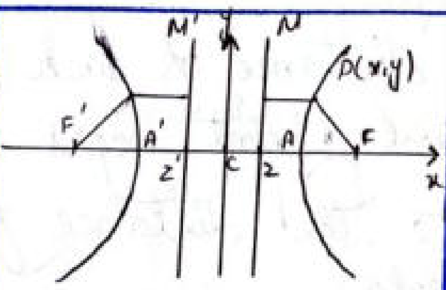
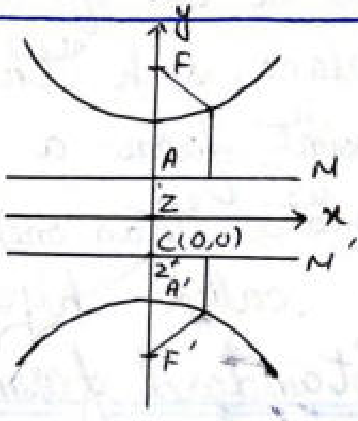
• Standard form of the hyperbola:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

• Parametric equation of the hyperbola:

The point $(a\sec\theta, b\tan\theta)$ lies on the ellipse $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ for all real θ $x = a\sec\theta$, $y = b\tan\theta$ are called parametric equation of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Summary of Standard Hyperbolas

Equation	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$
Foci	$(\pm c, 0)$	$(0, \pm c)$
Directrices	$x = \pm \frac{c}{e^2}$	$y = \pm \frac{c}{e^2}$
Transverse axis	$y = 0$	$x = 0$
Vertices	$(\pm a, 0)$	$(0, \pm a)$
Eccentricity	$e = \frac{c}{a} > 1$	$e = \frac{c}{a} > 1$
Centre	$(0,0)$	$(0,0)$
Graph		

Example # 1: Find an equation of the hyperbola whose foci are $(\pm 4, 0)$ and vertices $(\pm 2, 0)$

Foci $= (\pm c, 0) = (\pm 4, 0)$

$c = 4 \Rightarrow c^2 = 16$

Centre = midpoint of foci

$= \left(\frac{4-4}{2}, \frac{0+0}{2}\right) = \left(\frac{0}{2}, \frac{0}{2}\right) = (0,0)$

Vertices $= (\pm a, 0) = (\pm 2, 0)$

$a = 2 \Rightarrow a^2 = 4$

$\therefore c^2 = a^2 + b^2 \Rightarrow b^2 = c^2 - a^2$

$b^2 = 16 - 4 = 12$

Transverse axis is x-axis. So equation of required hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \Rightarrow \frac{x^2}{4} - \frac{y^2}{12} = 1$$

Example # 2: Discuss $25x^2 - 16y^2 = 400$

$25x^2 - 16y^2 = 400$

Dividing by 400

$$\frac{x^2}{16} - \frac{y^2}{25} = 1$$

Compare with $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$a^2 = 16 \Rightarrow a = 4$; $b^2 = 25 \Rightarrow b = 5$

$\therefore c^2 = a^2 + b^2$

$c^2 = 16 + 25 = 41$

$c = \sqrt{41}$

Centre: $(0,0)$

Eccentricity, $e = \frac{c}{a} = \frac{\sqrt{41}}{4}$

Foci: $(\pm c, 0) = (\pm\sqrt{41}, 0)$

Directrices: $x = \pm \frac{a}{e}$

$\Rightarrow x = \pm \frac{4}{\sqrt{41}/4} \Rightarrow x = \frac{\pm 16}{\sqrt{41}}$

Length of latusrectum: $\frac{2b^2}{a}$

$= \frac{2(25)}{4} = \frac{25}{2}$

Asymptotes: $y = \pm \frac{b}{a}x = \pm \frac{5}{4}x$

Example # 3: Find the eccentricity, the coordinates of the vertices and foci of hyperbola $\frac{y^2}{16} - \frac{x^2}{49} = 1$

$$\frac{y^2}{16} - \frac{x^2}{49} = 1$$

Compare with $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$

$a^2 = 16 \Rightarrow a = 4$; $b^2 = 49 \Rightarrow b = 7$

$\therefore c^2 = a^2 + b^2 = 16 + 49 = 65$

$\Rightarrow c = \sqrt{65}$

Vertices: $(0, \pm a) = (0, \pm 4)$

Foci: $(0, \pm c) = (0, \pm\sqrt{65})$

Eccentricity: $e = \frac{c}{a} = \frac{\sqrt{65}}{4}$

Example # 4: Discuss $4x^2 - 8x - y^2 - 2y - 1 = 0$

$4x^2 - 8x - y^2 - 2y - 1 = 0$

$4(x^2 - 2x) - (y^2 + 2y + 1) = 0$

$4(x^2 - 2x + 1 - 1) - (y+1)^2 = 0$

$4[(x-1)^2 - 1] - (y+1)^2 = 0$

$4(x-1)^2 - 4 - (y+1)^2 = 0$

$4(x-1)^2 - (y+1)^2 = 4$

$\frac{(x-1)^2}{1} - \frac{(y+1)^2}{4} = 1$ (÷ by 4)

Compare with $\frac{X^2}{a^2} - \frac{Y^2}{b^2} = 1$

$X = x - 1$; $Y = y + 1$; $a^2 = 1$; $b^2 = 4$

$a = 1$; $b = 2$

$\therefore c^2 = a^2 + b^2 = 1 + 4 = 5 \Rightarrow c = \sqrt{5}$

Centre: $(X = 0 ; Y = 0)$

$x - 1 = 0$; $y + 1 = 0$

$x = 1$; $y = -1$

So centre $(1, -1)$

Eccentricity: $e = \frac{c}{a} = \frac{\sqrt{5}}{1} = \sqrt{5}$

Foci: $(X = \pm c ; Y = 0)$

$x - 1 = \pm\sqrt{5}$; $y + 1 = 0$

$x = 1 \pm \sqrt{5}$; $y = -1$

So foci $(1 \pm \sqrt{5}, -1)$

Directrices: $X = \pm \frac{a}{e}$

$x - 1 = \pm \frac{1}{\sqrt{5}}$

$x = 1 \pm \frac{1}{\sqrt{5}}$

Vertices: $(X = \pm a, Y = 0)$

$x - 1 = \pm 1$; $y + 1 = 0$

$x = \pm 1 + 1$; $y = -1$

$x = +1 + 1, x = -1 + 1$

$x = 2 ; x = 0$

So vertices $(2, -1)$ and $(0, -1)$

Exercise # 6.6

Question no 1: Find an equation of hyperbola with the given data.

i) Centre (0,0). foci(6,0); vertex(4,0)

$C(0,0)$

foci$(6,0)$

other foci$(-6,0)$

$\text{Foci}(\pm c,0) = (\pm 6,0) \Rightarrow c=6$
$c^2=36$

Vertex$(4,0)$; other vertex$(-4,0)$

$\text{Vertex}(\pm a,0) = (\pm 4,0) \Rightarrow a=4$
$\Rightarrow a^2=16$

$\therefore c^2=a^2+b^2$

$b^2=c^2-a^2=36-16=20$

$\Rightarrow b=\sqrt{20}$

Equation of hyperbola:

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \Rightarrow \frac{x^2}{16}-\frac{y^2}{20}=1.$$

ii) Foci$(\pm 5,0)$; vertex$(3,0)$

$\text{Foci}(\pm c,0) = (\pm 5,0) \Rightarrow c=5$
$\Rightarrow c^2=25$

Centre = Midpoint of foci

$$=\left(\frac{5-5}{2},\frac{0+0}{2}\right)=\left(\frac{0}{2},\frac{0}{2}\right)=(0,0)$$

Vertex $(3,0)$; Other vertex$(-3,0)$

$\text{Vertex}(\pm a,0) = (\pm 3,0) \Rightarrow a=3$
$\Rightarrow a^2=9$

$\therefore c^2=a^2+b^2 \Rightarrow b^2=c^2-a^2$
$=25-9$
$=16$

Equation of hyperbola:

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \Rightarrow \frac{x^2}{9}-\frac{y^2}{16}=1.$$

iii) Foci$(2\pm 5\sqrt{2},-7)$. Length of transverse axis 20

Foci $(2\pm 5\sqrt{2},-7)$.

$F(2+5\sqrt{2},-7)$; $F'(2-5\sqrt{2},-7)$

Centre = Midpoint of foci

$$(h,k)=\left(\frac{2+5\sqrt{2}+2-5\sqrt{2}}{2};\frac{-7-7}{2}\right)$$

$$=\left(\frac{2+2}{2},\frac{-14}{2}\right) \Rightarrow (2,-7)$$

Length of transverse axis $=2a=10$

$\Rightarrow a=5 \rightarrow a^2=25$

Now distance b/w focus $F(2+5\sqrt{2},-7)$ and centre $(2,-7)$ is

$$c=\sqrt{(2+5\sqrt{2}-2)^2+(-7+7)^2}$$

$$=\sqrt{(5\sqrt{2})^2} = 5\sqrt{2}$$

$c^2=(5\sqrt{2})^2=25(2)=50$

$\therefore c^2=a^2+b^2 \Rightarrow b^2=c^2-a^2$

$b^2=50-25 \Rightarrow 25$

Equation of hyperbola:

$$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1 \Rightarrow \frac{(x-2)^2}{25}-\frac{(y+7)^2}{25}=1$$

iv) foci$(0,\pm 6)$, $e=2$

$\text{Foci}(0,\pm c)=(0,\pm 6)$

$c=6$; $c^2=36$

Centre = Midpoint of foci

$$=\left(\frac{0+0}{2},\frac{6-6}{2}\right)=\left(\frac{0}{2},\frac{0}{2}\right)=(0,0$$

$\therefore e=2 \Rightarrow e=\frac{c}{a}$ or $a=\frac{c}{e}$

$a=\frac{6}{2} \Rightarrow a=3$; $a^2=9$

∴ $b^2 = c^2 - a^2 = 36 - 9 = 27$

Equation of hyperbolas

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1 \Rightarrow \frac{y^2}{9} - \frac{x^2}{27} = 1$$

v) Foci(0, ±9) ; directrix y = ±4

Foci $(0, \pm c) = (0, \pm 9)$

$c = 9$; $c^2 = 81$

Centre = Midpoint of foci

$$= \left(\frac{0+0}{2} ; \frac{9-9}{2}\right) = \left(\frac{0}{2}, \frac{0}{2}\right) = (0,0)$$

Directrices: $y = \pm 4$

$$\pm\frac{a}{e} = \pm 4 \Rightarrow \frac{a}{e} = 4 \qquad \because y = \pm\frac{a}{e}$$

$a = 4e$ — (i

Also $c = ae$

$9 = ae$ ∵ $c = 9$

$e = \frac{9}{a}$

Put $e = \frac{9}{a}$ in (i)

$a = 4\left(\frac{9}{a}\right)$

$a^2 = 36$

∴ $b^2 = c^2 - a^2 = 81 - 36 = 45$

Equation of hyperbola:

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1 \Rightarrow \frac{y^2}{36} - \frac{x^2}{45} = 1$$

vi) Centre (2,2), horizontal transverse axis of length 6 and eccentricity e = 2

Centre $(h,k) = (2,2)$

Length of transverse axis $= 2a = 6$

$\Rightarrow a = 3 \Rightarrow a^2 = 9$

∴ $e = 2$; $a = 3$

∴ $c = ae$

$c = (3)(2)$

$c = 6$; $c^2 = 36$

∴ $b^2 = c^2 - a^2 = 36 - 9 = 27$

Equation of hyperbola:

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1 \Rightarrow \frac{(x-2)^2}{9} - \frac{(y-2)^2}{27} = 1$$

vii) Vertices (2, ±3) ; (0,5) lies on the curve.

Vertices = $(2, \pm 3)$

$A(2,3)$; $A'(2,-3)$

Centre = Midpoint of vertices

$$(h,k) = \left(\frac{2+2}{2} ; \frac{3-3}{2}\right) = \left(\frac{4}{2}, \frac{0}{2}\right) = (2,0)$$

∴ Distance b/w vertices = 2a

$$2a = \sqrt{(2-2)^2 + (-3-3)^2}$$

$$= \sqrt{(0)^2 + (-6)^2} = \sqrt{36}$$

$2a = 6 \Rightarrow a = 3$; $a^2 = 9$

∴ Vertices have same abscissa so transverse axis is along y-axis so equation of hyperbola is

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$$

$$\frac{y^2}{9} - \frac{(x-2)^2}{b^2} = 1 \text{ —(i}$$

∵ (i) passes through (0,5 so

$$\frac{(5)^2}{9} - \frac{(0-2)^2}{b^2} = 1$$

$$\frac{25}{9} - \frac{4}{b^2} = 1 \Rightarrow 25b^2 - 36 = 9b^2$$

$$25b^2 - 9b^2 = 36 \Rightarrow 16b^2 = 36 \Rightarrow b^2 = \frac{36}{16}$$

$$b^2 = \frac{9}{4}$$

Equation of hyperbola:

$$\frac{y^2}{9} - \frac{(x-2)^2}{9/4} = 1.$$

viii) Foci (5,-2), (5,4) and one Vertex (5,3)

Foci: $F'(5,-2)$; $F(5,4)$

Centre: Midpoint of foci

$$(h,k) = \left(\frac{5+5}{2}, \frac{-2+4}{2}\right) = \left(\frac{10}{2}, \frac{2}{2}\right) = (5,1)$$

∴ vertex $(5,3)$

∴ Distance b/w centre and vertex = a

$$a = \sqrt{(5-5)^2+(3-1)^2}$$

$$= \sqrt{(0)^2+(2)^2} = \sqrt{4} = 2$$

$$a^2 = 4$$

Also distance b/w foci = $2c$

$$2c = \sqrt{(5-5)^2+(4+2)^2}$$

$$= \sqrt{(0)^2+(6)^2} = 6$$

$$2c = 6 \Rightarrow c = 3 \; ; \; c^2 = 9$$

$$\therefore b^2 = c^2 - a^2 = 9 - 4 = 5$$

Equation of hyperbola:

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$$

$$\frac{(y-1)^2}{4} - \frac{(x-5)^2}{5} = 1.$$

Question no 2: Find the centre, foci, eccentricity, vertices and equation of directrices of each of the following.

i) $x^2 - y^2 = 9$

$$x^2 - y^2 = 9$$

Dividing by 9

$$\frac{x^2}{9} - \frac{y^2}{9} = 1$$

Here $a^2 = 9 \Rightarrow a = 3$; $b^2 = 9 \Rightarrow b = 3$

$$\therefore c^2 = a^2 + b^2 = 9 + 9 = 18$$

$$\Rightarrow c = \sqrt{18} = \sqrt{2 \times 3^2} = 3\sqrt{2}$$

Centre: $(0,0)$

Eccentricity: $e = \frac{c}{a} = \frac{3\sqrt{2}}{3} = \sqrt{2}$

Vertices: $(\pm a, 0) = (\pm 3, 0)$

foci: $(\pm c, 0) = (\pm 3\sqrt{2}, 0)$

Directrices: $x = \pm\frac{a}{e} = \pm\frac{3}{\sqrt{2}}$

ii) $\frac{x^2}{4} - \frac{y^2}{9} = 1$

Here $a^2 = 4 \Rightarrow a = 2$; $b^2 = 9 \Rightarrow b = 3$

$$\therefore c^2 = a^2 + b^2 = 4 + 9 = 13 \Rightarrow c = \sqrt{13}$$

Centre: $(0,0)$

foci: $(\pm c, 0) = (\pm\sqrt{13}, 0)$

Eccentricity: $e = \frac{c}{a} = \frac{\sqrt{13}}{2}$

Vertices: $(\pm a, 0) = (\pm 2, 0)$

Directrices: $x = \pm\frac{a}{e} = \pm\frac{2}{\sqrt{13}/2} = \pm\frac{4}{\sqrt{13}}$

iii) $\frac{y^2}{16} - \frac{x^2}{9} = 1$

Here $a^2 = 16 \Rightarrow a = 4$; $b^2 = 9 \Rightarrow b = 3$

$$\therefore c^2 = a^2 + b^2 = 16 + 9 = 25 \Rightarrow c = 5$$

Centre: (0,0)

Eccentricity: $e = \frac{c}{a} = \frac{5}{4}$

foci: $(0, \pm c) = (0, \pm 5)$

Directrices: $y = \pm \frac{a}{e} = \frac{\pm 4}{5/4} = \pm \frac{16}{5}$

Vertices: $(0, \pm a) = (0, \pm 4)$

iv) $\frac{y^2}{4} - x^2 = 1$

$$\frac{y^2}{4} - \frac{x^2}{1} = 1$$

$a^2 = 4 \Rightarrow a = 2 \;;\; b^2 = 1 \Rightarrow b = 1$

$\therefore c^2 = a^2 + b^2 = 4 + 1 = 5 \Rightarrow c = \sqrt{5}$

Centre: (0,0)

Eccentricity: $e = \frac{c}{a} = \frac{\sqrt{5}}{2}$

foci: $(0, \pm c) = (0, \pm\sqrt{5})$

Directrices: $y = \pm \frac{a}{e} = \pm \frac{2}{\sqrt{5}/2} = \pm \frac{4}{\sqrt{5}}$

Vertices: $(0, \pm a) = (0, \pm 2)$.

v) $\frac{(x-1)^2}{2} - \frac{(y-1)^2}{9} = 1$

$$\frac{(x-1)^2}{2} - \frac{(y-1)^2}{9} = 1$$

Compare with $\frac{X^2}{a^2} - \frac{Y^2}{b^2} = 1$

$X = x-1 \;;\; Y = y-1 \;;\; a^2 = 2 \;;\; b^2 = 9$

$a = \sqrt{2} \;;\; b = 3$

$\therefore c^2 = a^2 + b^2 = 2 + 9 = 11 \Rightarrow c = \sqrt{11}$

Centre: $(X = 0 \;;\; Y = 0)$

$x - 1 = 0 \;;\; y - 1 = 0$

$x = 1 \;;\; y = 1$

So centre (1,1)

Eccentricity: $e = \frac{c}{a} = \frac{\sqrt{11}}{\sqrt{2}} = \sqrt{\frac{11}{2}}$

Foci: $(X = \pm c \;;\; Y = 0)$

$x - 1 = \pm\sqrt{11} \;;\; y - 1 = 0$

$x = 1 \pm \sqrt{11} \;;\; y = 1$

So foci $(1 \pm \sqrt{11}, 1)$

Directrix: $X = \pm \frac{a}{e}$

$x - 1 = \pm \frac{\sqrt{2}}{\sqrt{11}/\sqrt{2}} \Rightarrow x - 1 = \pm\frac{2}{\sqrt{11}} \Rightarrow x = 1 \pm \frac{2}{\sqrt{11}}$

Vertices: $(X = \pm a, Y = 0)$

$x - 1 = \pm\sqrt{2} \;;\; y - 1 = 0$

$x = 1 \pm \sqrt{2} \;;\; y = 1$

So vertices $(1 \pm \sqrt{2}, 1)$

vi) $\frac{(y+2)^2}{9} - \frac{(x-2)^2}{16} = 1$.

$$\frac{(y+2)^2}{9} - \frac{(x-2)^2}{16} = 1$$

Compare with $\frac{Y^2}{a^2} - \frac{X^2}{b^2} = 1$

$Y = y+2 \;;\; X = x-2 \;;\; a^2 = 9 \;;\; b^2 = 16$

$a = 3 \;;\; b = 4$

$\therefore c^2 = a^2 + b^2 = 9 + 16 = 25 \Rightarrow c = 5$

Centre: $(X = 0 \;;\; Y = 0)$

$x - 2 = 0 \;;\; y + 2 = 0$

$x = 2 \;;\; y = -2$

So centre (2, -2)

Eccentricity: $e = \frac{c}{a} = \frac{5}{3}$

Foci: $(X = 0 \;;\; Y = \pm c)$

$x - 2 = 0 \;;\; y + 2 = \pm 5$

$x = 2 \;;\; y = -2 \pm 5$

$y = -7, 3$

So foci are $(2,3)$ and $(2,-7)$

Directrices: $Y = \pm \frac{a}{e}$

$y+2 = \pm \frac{3}{5/3} \Rightarrow y+2 = \pm \frac{9}{5} \Rightarrow y = -2 \pm \frac{9}{5}$

Vertices: $(X = 0 ; Y = \pm a)$

$x-2=0 \quad ; \quad y+2 = \pm 3$

$x = 2 \quad ; \quad y = -2 \pm 3$

$y = 1 ; -5$

So vertices are $(2,1)$ & $(2,-5)$

vii) $9x^2 - 12x - y^2 - 2y + 2 = 0.$

$$9x^2 - 12x - y^2 - 2y + 2 = 0$$

$$9\left(x^2 - \frac{12}{9}x\right) - (y^2 + 2y) = -2$$

$$9\left(x^2 - \frac{4}{3}x\right) - (y^2 + 2y) = -2$$

$$9\left\{x^2 - 2(x)\left(\frac{2}{3}\right) + \left(\frac{2}{3}\right)^2 - \left(\frac{2}{3}\right)^2\right\} - \left\{y^2 + 2y + 1 - 1\right\} = -2$$

$$9\left\{\left(x - \frac{2}{3}\right)^2 - \frac{4}{9}\right\} - \left\{(y+1)^2 - 1\right\} = -2$$

$$9\left(x - \frac{2}{3}\right)^2 - 4 - (y+1)^2 + 1 = -2$$

$$9\left(x - \frac{2}{3}\right)^2 - (y+1)^2 = -2 + 4 - 1$$

$$9\left(x - \frac{2}{3}\right)^2 - (y+1)^2 = 1$$

$$\frac{(x - 2/3)^2}{1/9} - \frac{(y+1)^2}{1} = 1$$

Compare with $\frac{X^2}{a^2} - \frac{Y^2}{b^2} = 1$

$X = x - \frac{2}{3} ; Y = y+1 ; a^2 = \frac{1}{9} ; b^2 = 1$

$a = \frac{1}{2} ; b = 1$

$\therefore c^2 = a^2 + b^2 = \frac{1}{9} + 1 = \frac{1+9}{9} = \frac{10}{9}$

$c = \sqrt{\frac{10}{9}} = \frac{\sqrt{10}}{3}$

Centre: $(X = 0 ; Y = 0)$

$x - \frac{2}{3} = 0 \quad ; \quad y + 1 = 0$

$x = \frac{2}{3} \quad ; \quad y = -1$

So centre $\left(\frac{2}{3}, -1\right)$

Eccentricity: $e = \frac{c}{a} = \frac{\sqrt{10}/3}{1/3} = \sqrt{10}$

Foci: $(X = \pm c ; Y = 0)$

$x - \frac{2}{3} = \pm \frac{\sqrt{10}}{3} \quad ; \quad y + 1 = 0$

$x = \frac{2}{3} \pm \frac{\sqrt{10}}{3} \quad ; \quad y = -1$

$x = \frac{2 \pm \sqrt{10}}{3}$

So foci $\left(\frac{2 \pm \sqrt{10}}{3} ; -1\right)$

Directrices: $X = \pm \frac{a}{e}$

$x - \frac{2}{3} = \pm \frac{1/3}{\sqrt{10}} \Rightarrow x - \frac{2}{3} = \pm \frac{1}{3\sqrt{10}} \Rightarrow x = \frac{2}{3} \pm \frac{1}{3\sqrt{10}}$

Vertices: $(X = \pm a, Y = 0)$

$x - \frac{2}{3} = \pm \frac{1}{3} \quad ; \quad y + 1 = 0$

$x = \frac{2}{3} \pm \frac{1}{3} \quad ; \quad y = -1$

$x = \frac{2 \pm 1}{3}$

$x = \frac{3}{3} ; x = \frac{1}{3}$

$x = 1 ; x = 1/3$

So vertices are $(1, -1)$ and $\left(\frac{1}{3}, -1\right)$

viii) $4y^2+12y-x^2+4x+1=0$

$4y^2+12y-x^2+4x+1=0$

$4(y^2+3y)-(x^2-4x)=-1$

$4\left\{y^2+2(y)(\frac{3}{2})+(\frac{3}{2})^2-(\frac{3}{2})^2\right\}-\left\{x^2-4x+4-4\right\}=-1$

$4\left\{(y+\frac{3}{2})^2-\frac{9}{4}\right\}-\left\{(x-2)^2-4\right\}=-1$

$4(y+\frac{3}{2})^2-9-(x-2)^2+4=-1$

$4(y+\frac{3}{2})^2-(x-2)^2=-1+9-4$

$4(y+\frac{3}{2})^2-(x-2)^2=4$

Dividing by 4

$$\frac{(y+\frac{3}{2})^2}{1}-\frac{(x-2)^2}{4}=1$$

compare with $\frac{Y^2}{a^2}-\frac{X^2}{b^2}=1$

$Y=y+\frac{3}{2}$; $X=x-2$; $a^2=1$; $b^2=4$

$a=1$; $b=2$

$\therefore c^2=a^2+b^2=1+4=5 \Rightarrow c=\sqrt{5}$

Centre: $(X=0\ ;\ Y=0)$

$x-2=0$; $y+\frac{3}{2}=0$

$x=2$; $y=-\frac{3}{2}$

So centre $(2,-\frac{3}{2})$

Eccentricity: $e=\frac{c}{a}=\frac{\sqrt{5}}{1}=\sqrt{5}$

Foci: $(X=0, Y=\pm c)$

$x-2=0$; $y+\frac{3}{2}=\pm\sqrt{5}$

$x=2$; $y=-\frac{3}{2}\pm\sqrt{5}$

So foci $(2, \frac{-3}{2}\pm\sqrt{5})$

Directrices: $Y=\pm\frac{a}{e}$

$y+\frac{3}{2}=\pm\frac{1}{\sqrt{5}} \Rightarrow y=-\frac{3}{2}\pm\frac{1}{\sqrt{5}}$

Vertices: $(X=0\ ;\ Y=\pm a)$

$x-2=0$; $y+\frac{3}{2}=\pm 1$

$x=2$; $y=-\frac{3}{2}\pm 1=\frac{-3\pm 2}{2}$

$y=-\frac{1}{2}\ ;\ \frac{-5}{2}$

So vertices are $(2,\frac{-1}{2})$ & $(2,-\frac{5}{2})$

ix) $x^2-y^2+8x-2y-10=0$.

$x^2+8x-y^2-2y-10=0$

$\left\{x^2+2(x)(4)+(4)^2-(4)^2\right\}-\left\{y^2+2(y)(1)+(1)^2-(1^2)\right\}=10$

$(x+4)^2-16-(y+1)^2+1=10$

$(x+4)^2-(y+1)^2=10+16-1$

$(x+4)^2-(y+1)^2=25$

Dividing by 25

$$\frac{(x+4)^2}{25}-\frac{(y+1)^2}{25}=1.$$

Compare with $\frac{X^2}{a^2}+\frac{Y^2}{b^2}=1$

$X=x+4$; $Y=y+1$, $a^2=25$; $b^2=25$

$a=5$; $b=5$

$\therefore c^2=a^2+b^2=25+25=50 \Rightarrow c=\sqrt{50}$

$c=5\sqrt{2}$

Centre: $(X=0\ ;\ Y=0)$

$x+4=0$; $y+1=0$

$x=-4$; $y=-1$

So centre $(-4,-1)$

Eccentricity: $e=\frac{c}{a}=\frac{5\sqrt{2}}{5}=\sqrt{2}$

Foci: $(X=\pm c, Y=0)$

$x+4=\pm 5\sqrt{2}$; $y+1=0$

$x=-4\pm 5\sqrt{2}$; $y=-1$

So foci $(-4\pm5\sqrt{2}, -1)$

Directrices: $X = \pm\frac{a}{e}$

$x+4 = \pm\frac{5}{\sqrt{2}} \Rightarrow x = -4\pm\frac{5}{\sqrt{2}}$

Vertices: $(X = \pm a, Y = 0)$

$x+4 = \pm5 \quad ; \quad y+1=0$

$x = -4\pm5 \quad ; \quad y=-1$

$x = 1 ; x=-9$

So vertices $(1,-1); (-9,-1)$

x) $9x^2-y^2-36x-6y+18=0$

$9x^2-y^2-36x-6y+18=0$

$9x^2-36x-y^2-6y+18=0$

$9(x^2-4x)-(y^2+6y)=-18$

$9\{x^2-2(x)(2)+(2)^2-(2)^2\}-\{y^2+6y+9-9\}=18$

$9\{(x-2)^2-4\}-(y+3)^2+9=-18$

$9(x-2)^2-36-(y+3)^2+9=-18$

$9(x-2)^2-(y+3)^2=-18+36-9$

$9(x-2)^2-(y+3)^2= 9$

Dividing by 9

$$\frac{(x-2)^2}{1}-\frac{(y+3)^2}{9}=1$$

Compare with $\frac{X^2}{a^2}-\frac{Y^2}{b^2}=1$

$X=x-2; \; Y=y+3; \; a^2=1 ; \; b^2=9$

$c^2=1+9=10; \; c=\sqrt{10} \quad a=1 ; \; b=3$

Centre: $(X=0 ; Y=0)$

$x-2=0 \quad ; \quad y+3=0$

$x=2 \quad ; \quad y=-3$

So centre $(2,-3)$.

Foci: $(X=\pm c ; Y=0)$

$x-2=\pm\sqrt{10} \quad ; \quad y+3=0$

$x=2\pm\sqrt{10} \quad ; \quad y=-3$

So foci $(2\pm\sqrt{10} ; -3)$

Eccentricity: $e=\frac{c}{a}=\frac{\sqrt{10}}{1}=\sqrt{10}$

Directrices: $X=\pm\frac{a}{e}$

$x-2=\pm\frac{1}{\sqrt{10}} \Rightarrow x=2\pm\frac{1}{\sqrt{10}}$

Vertices $(X=\pm a, Y=0)$

$x-2=\pm1 \quad ; \quad y+3=0$

$x=2\pm1 \quad ; \quad y=-3$

$x=2+1; x=2-1$

$x=3; x=1$

So vertices are $(3,-3)$ & $(1,-3)$

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Exercise # 6.7

Question no 1: Find equation of the tangents and normal to each of the following at the indicated point:

i) $y^2=4ax$ at $(at^2, 2at)$

$$y^2 = 4ax$$

Diff w.r.t 'x'

$$2y\frac{dy}{dx} = 4a$$

$$\frac{dy}{dx} = \frac{4a}{2y}$$

$$\frac{dy}{dx} = \frac{2a}{y}$$

Slope of tangent $= \frac{dy}{dx}\Big/_{(at^2,2at)}$

$$= \frac{\cancel{2a}}{\cancel{2a}t} = \frac{1}{t}$$

Equation of tangent:

$$\because y-y_1 = m(x-x_1)$$

$$y-2at = \frac{1}{t}(x-at^2)$$

$$ty-2at^2 = x-at^2$$

$$yt = x-at^2+2at^2$$

$$yt = x+at^2$$

Slope of Normal $= -t$

Equation of Normal:

$$\because y-y_1 = m(x-x_1)$$

$$y-2at = -t(x-at^2)$$

$$y-2at = -tx+at^3$$

$$y = -tx+at^3+2at$$

ii) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at $(a\cos\theta, b\sin\theta)$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

Diff w.r.t 'x'

$$\frac{2x}{a^2}+\frac{2y}{b^2}\frac{dy}{dx}=0$$

$$\frac{2y}{b^2}\frac{dy}{dx} = -\frac{2x}{a^2}$$

$$\frac{dy}{dx} = -\frac{2x}{a^2}\times\frac{b^2}{2y}$$

$$\frac{dy}{dx} = -\frac{b^2x}{a^2y}$$

Slope of tangent $= \frac{dy}{dx}\Big/_{(a\cos\theta, b\sin\theta)}$

$$= -\frac{b^2a\cos\theta}{a^2b\sin\theta} = -\frac{b\cos\theta}{a\sin\theta}$$

Equation of tangent:

$$\because y-y_1 = m(x-x_1)$$

$$y-b\sin\theta = -\frac{b\cos\theta}{a\sin\theta}(x-a\cos\theta)$$

$$a\sin\theta(y-b\sin\theta) = -b\cos\theta(x-a\cos\theta)$$

$$ay\sin\theta-ab\sin^2\theta = -bx\cos\theta+ab\cos^2\theta$$

$$ay\sin\theta+bx\cos\theta = ab\cos^2\theta+ab\sin^2\theta$$

$$ay\sin\theta+bx\cos\theta = ab(\cos^2\theta+\sin^2\theta)$$

$$ay\sin\theta+bx\cos\theta = ab \qquad \because \cos^2\theta+\sin^2\theta=1$$

Dividing by ab

$$\frac{y}{b}\sin\theta+\frac{x}{a}\cos\theta = 1$$

Slope of Normal $= \frac{a\sin\theta}{b\cos\theta}$

Equation of Normal:

$$y - b\sin\theta = \frac{a\sin\theta}{b\cos\theta}(x - a\cos\theta)$$

$$\frac{y - b\sin\theta}{a\sin\theta} = \frac{x - a\cos\theta}{b\cos\theta}$$

$$\frac{y}{a\sin\theta} - \frac{b\sin\theta}{a\sin\theta} = \frac{x}{b\cos\theta} - \frac{a\cos\theta}{b\cos\theta}$$

$$\frac{y}{a\sin\theta} - \frac{b}{a} = \frac{x}{b\cos\theta} - \frac{a}{b}$$

$$-\frac{b}{a} + \frac{a}{b} = \frac{x}{b\cos\theta} - \frac{y}{a\sin\theta}$$

$$\frac{-b^2+a^2}{ab} = \frac{x}{b}\sec\theta - \frac{y}{a}\operatorname{cosec}\theta$$

Multiply by ab

$$a^2-b^2 = ax\sec\theta - by\operatorname{cosec}\theta.$$

iii) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ **at** $(a\sec\theta, b\tan\theta)$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Diff w.r.t 'x'

$$\frac{2x}{a^2} - \frac{2y}{b^2}\frac{dy}{dx} = 0$$

$$-\frac{2y}{b^2}\frac{dy}{dx} = -\frac{2x}{a^2}$$

$$\frac{dy}{dx} = \frac{-2x}{a^2} \times \frac{-b^2}{2y}$$

$$\frac{dy}{dx} = \frac{b^2x}{a^2y}$$

Slope of tangent $= \left.\frac{dy}{dx}\right|_{(a\sec\theta,\, b\tan\theta)}$

$$= \frac{b^2 a\sec\theta}{a^2 b\tan\theta} = \frac{b\sec\theta}{a\tan\theta}$$

Equation of tangent:

$$\therefore y - y_2 = m(x - x_1)$$

$$y - b\tan\theta = \frac{b\sec\theta}{a\tan\theta}(x - a\sec\theta)$$

$$a\tan\theta(y - b\tan\theta) = b\sec\theta(x - a\sec\theta)$$

$$ay\tan\theta - ab\tan^2\theta = bx\sec\theta - ab\sec^2\theta$$

$$-ab\tan^2\theta + ab\sec^2\theta = bx\sec\theta - ay\tan\theta$$

$$ab(\sec^2\theta - \tan^2\theta) = bx\sec\theta - ay\tan\theta$$

$$ab(1) = bx\sec\theta - ay\tan\theta$$

Dividing by ab

$$1 = \frac{x}{a}\sec\theta - \frac{y}{b}\tan\theta$$

Slope of Normal: $\frac{-a\tan\theta}{b\sec\theta}$

Equation of Normal:

$$y - b\tan\theta = -\frac{a\tan\theta}{b\sec\theta}(x - a\sec\theta)$$

$$\frac{y - b\tan\theta}{a\tan\theta} = \frac{-(x - a\sec\theta)}{b\sec\theta}$$

$$\frac{y}{a\tan\theta} - \frac{b\tan\theta}{a\tan\theta} = \frac{-x}{b\sec\theta} + \frac{a\sec\theta}{b\sec\theta}$$

$$\frac{y}{a\tan\theta} - \frac{b}{a} = \frac{-x}{b\sec\theta} + \frac{a}{b}$$

$$\frac{y}{a\tan\theta} + \frac{x}{b\sec\theta} = \frac{a}{b} + \frac{b}{a}$$

$$\frac{y}{a}\cot\theta + \frac{x}{b}\cos\theta = \frac{a^2+b^2}{ab}$$

Multiply by ab

$$by\cot\theta + ax\cos\theta = a^2+b^2$$

Question no 8: Find the points of intersection of the given conics

i) $\frac{x^2}{18}+\frac{y^2}{8}=1$ and $\frac{x^2}{3}-\frac{y^2}{3}=1$

$\frac{x^2}{18}+\frac{y^2}{8}=1$; $\frac{x^2}{3}-\frac{y^2}{3}=1$

$\frac{4x^2+9y^2}{72}=1$; $\frac{x^2-y^2}{3}=1$

$4x^2+9y^2=72$ —(i ; $x^2-y^2=3$ —(ii

ⓘ + 9ⓘⓘ

$4x^2+\cancel{9y^2}=72$

$\underline{9x^2-\cancel{9y^2}=27}$

$13x^2=99$

$x^2=\frac{99}{13} \Rightarrow x=\pm\sqrt{\frac{99}{13}}$

Put in (ii)

$\frac{99}{13}-y^2=3$

$y^2=\frac{99}{13}-3$

$y^2=\frac{99-39}{13}$

$y^2=\frac{60}{13} \Rightarrow y=\pm\sqrt{\frac{60}{13}}$

$\left(\pm\sqrt{\frac{99}{13}}, \pm\sqrt{\frac{60}{13}}\right)$

ii) $x^2+y^2=8$ and $x^2-y^2=1$

$x^2+y^2=8$ —(i ; $x^2-y^2=1$ —(ii

ⓘ + ⓘⓘ

$x^2+\cancel{y^2}=8$

$\underline{x^2-\cancel{y^2}=1}$

$2x^2=9$

$x^2=\frac{9}{2} \Rightarrow x=\pm\frac{3}{\sqrt{2}}$

Put $x^2=\frac{9}{2}$ in ⓘ

$\frac{9}{2}+y^2=8$

$y^2=8-\frac{9}{2}$

$y^2=\frac{16-9}{2}$

$y^2=\frac{7}{2} \Rightarrow y=\pm\sqrt{\frac{7}{2}}$

$\left(\pm\frac{3}{\sqrt{2}}, \pm\sqrt{\frac{7}{2}}\right)$

iii) $3x^2+4y^2=12$ and $3y^2-2x^2=7$

$3x^2-4y^2=12$ —(i ; $-2x^2+3y^2=7$ —(ii

2ⓘ + 3ⓘⓘ

$\cancel{6x^2}-8y^2=24$

$\underline{-\cancel{6x^2}+9y^2=21}$

$y^2=45 \Rightarrow y=\pm\sqrt{45}$

Put $y^2=45$ in ⓘ

$3x^2-4(45)=12$

$3x^2-180=12$

$3x^2=192$

$x^2=64 \Rightarrow x=\pm 8$

$(\pm 8, \pm\sqrt{45})$

iv) $3x^2+5y^2=60$ and $9x^2+y^2=124$

$3x^2+5y^2=60$ —(i ; $9x^2+y^2=124$ (ii

ⓘ − 5ⓘⓘ

$3x^2+\cancel{5y^2}=60$

$\underline{\underset{-}{45x^2}\ \underset{-}{+}\cancel{5y^2}=\underset{-}{620}}$

$-42x^2=-560$

$x^2=\frac{40}{3} \Rightarrow x=\pm\sqrt{\frac{40}{3}}$

Put $x^2=\frac{40}{3}$ in ⓘ

$$3\left(\frac{40}{3}\right)+5y^2=60$$

$$40+5y^2=60$$

$$5y^2=60-40$$

$$5y^2=20$$

$$y^2=4 \Rightarrow y=\pm 2$$

$$\left(\pm\sqrt{\frac{40}{3}}\,;\ \pm 2\right)$$

v) $4x^2+y^2=16$ and $x^2+y^2+y+8=0$

$$4x^2+y^2=16 \text{ —(i)}$$

$$x^2+y^2+y+8=0 \text{ —(ii)}$$

(i) - 4 (ii)

$$\cancel{4x^2}+y^2=16$$

$$\underline{\cancel{\underset{-}{}4x^2}\underset{-}{+}4y^2\underset{-}{+}4y=\underset{+}{-}32}$$

$$-3y^2-4y=48$$

$$3y^2+4y+48=0$$

$$a=3\,;\ b=4\,;\ c=48$$

$$y=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$y=\frac{-4\pm\sqrt{(4)^2-4(3)(48)}}{2(3)}$$

$$y=\frac{-4\pm\sqrt{16-576}}{6}$$

$$y=\frac{-4\pm\sqrt{-560}}{6}$$

$$y=\frac{-4\pm\sqrt{560}\,i}{6}$$

At the value of y are complex (imaginary) so no real points of intersection of given conics exist.

Theory:

•Translation of Axes:

$$x=X+h$$
$$y=Y+k$$

These are called equations of transformation

$$X=x-h$$
$$Y=y-k$$

•Rotation of Axes:

$$x=X\cos\theta-Y\sin\theta$$
$$y=X\sin\theta+Y\cos\theta$$

•The General Equation of Second degree:

$$Ax^2+By^2+Gx+Fy+C=0$$

Note:

i) a circle if $A=B\neq 0$
ii) an ellipse if $A\neq B$ and both are of the same sign
iii) a hyperbola if $A\neq B$ and both are of opposite signs
iv) a parabola if either $A=0$ or $B=0$.

Note:

i) an ellipse or a circle if $h^2 - ab < 0$

ii) a parabola if $h^2 - ab = 0$

iii) a hyperbola if $h^2 - ab > 0$

Degenerate conic:

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \quad \text{— (i}$$

Under certain conditions ⓘ may not represent any conic. In such a case we say ⓘ represents a degenerate conic

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

Note:

An equation of the tangent to the general equation of the second degree at the point (x_1, y_1) may be obtained by replacing

x^2 by xx_1

y^2 by yy_1

$2xy$ by $xy_1 + yx_1$

$2x$ by $x + x_1$

$2y$ by $y + y_1$

in the equation of conic

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Example # 4: Show that $2x^2 - xy + 5x - 2y + 2 = 0$ represents a pair of lines. Also find and equation of each line.

$$2x^2 - xy + 5x - 2y + 2 = 0$$

Here $a = 2$

$b = 0$

$2h = -1 \Rightarrow h = -\frac{1}{2}$

$2g = 5 \Rightarrow g = \frac{5}{2}$

$2f = -2 \Rightarrow f = -1$

$c = 2$

$$= \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

$$= \begin{vmatrix} 2 & -1/2 & 5/2 \\ -1/2 & 0 & -1 \\ 5/2 & -1 & 2 \end{vmatrix}$$

Expand by R_2

$$= -\left(-\frac{1}{2}\right)\left[\left(-\frac{1}{2}\right)(2) - \left(\frac{5}{2}\right)(-1)\right] + 0 - (-1)\left[-2 + \frac{5}{4}\right]$$

$$= \frac{1}{2}\left[-1 + \frac{5}{2}\right] + 1\left[\frac{-8+5}{4}\right]$$

$$= \frac{1}{2}\left[\frac{-2+5}{2}\right] + 1\left[\frac{-3}{4}\right]$$

$$= \frac{1}{2}\left[\frac{3}{2}\right] + 1\left[\frac{-3}{4}\right]$$

$$= \frac{3}{4} - \frac{3}{4}$$

$$= 0$$

The given equation represent a degenerate conic

$$2x^2 + x(5-y) + (-2y+2) = 0$$

$a = 2$; $b = 5 - y$; $c = -2y + 2$

$$x = \frac{-(5-y) \pm \sqrt{(5-y)^2 - 4(2)(-2y+2)}}{2(2)}$$

$$x = \frac{y - 5 \pm \sqrt{25 + y^2 - 10y + 16y - 16}}{4}$$

$$x = \frac{y - 5 \pm \sqrt{y^2 + 6y + 9}}{4}$$

$$x = \frac{y - 5 \pm \sqrt{(y+3)^2}}{4}$$

$$x = \frac{y - 5 \pm (y+3)}{4}$$

$$x = \frac{y - 5 + y + 3}{4} \ ; \ x = \frac{y - 5 - y - 3}{4}$$

$$x = \frac{2y - 2}{4} \ ; \ x = \frac{-8}{4}$$

$$x = \frac{2(y-1)}{4} \ ; \ x = -2$$

$$2x = y - 1 \ ;$$

$$2x - y + 1 = 0 \ ; \ x + 2 = 0$$

Equations of the lines are $2x - y + 1 = 0$ and $x + 2 = 0$.

UNIT
7

Vectors

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Unit no 7

Vectors

Theory:

- **Scaler quantity:** A scalar quantity, or simply a scalar. is one that possesses only magnitude. It can be specified by a number alongwith unit. For example speed, work and volume
- **Vector quantity:** A vector quantity, or simply a vector is one that possesses both magnitude and direction. For example displacement, work, velocity and acceleration
- **Magnitude of a vector:** The magnitude or length or norm of a vector $\overrightarrow{AB}$ or $\underline{v}$, is its absalute value and is written as $|\overrightarrow{AB}|$ or simply AB or $|\underline{v}|$
- **Unit vector:** A unit vector is defined as a vector whose magnite is unity. Unit vector of vector $\underline{v}$ is written as $\hat{\underline{v}}$ (read as $\underline{v}$ hat) and is defined by $\hat{\underline{v}} = \frac{\underline{v}}{|\underline{v}|}$
- **Position vector:** The vector whose initial point is the origin O and whose terminal point is P, is called the position vector of the point P and is written as $\overrightarrow{OP}$
- **Null vector:** If terminal point B of a vector $\overrightarrow{AB}$ concides with its initial point A, then magnitude $AB=0$ and $\overrightarrow{AB}=\underline{0}$ which is called zero or null vector.
- **Equall vector:** Two vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are said to be equal, if they have the same magnitude and same direction i.e., $|\overrightarrow{AB}| = |\overrightarrow{CD}|$
- **Parallel vector:** Two vectors are parallel if and only if they are non-zero scaler multiple of each other. i.e., two vectors $\underline{u}$ and $\underline{v}$ are parallel to each other if $\underline{u}=c\underline{v}$ where c is any scaler quantity.

- **Addition and Subtraction of two vectors:**

Addition of vectors: For any two vectors $\underline{u} = [x, y]$ and $\underline{v} = [x', y']$ we have $\underline{u} + \underline{v} = [x, y] + [x', y'] = [x + x', y + y']$

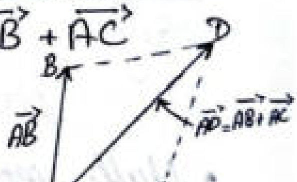
Subtraction of vectors: For any two vectors $\underline{u} = [x, y]$ and $\underline{v}[x', y']$ the subtraction of two vectors is defined as $\underline{u} - \underline{v} = \underline{u} + (-\underline{v})$.
$\underline{u} + (-\underline{v}) = [x, y] + [-x', -y'] = [x - x', y - y']$.

- **Scaler Multiplication:** For $\underline{u} = [x, y]$ and $\alpha \in R$, we have
$\alpha \underline{u} = \alpha[x, y] = [\alpha x, \alpha y]$.

- **Negative of a vector:** In scaler multiplication if $\alpha = -1$ and $\underline{u} = [x, y]$. the $\alpha \underline{u} = (-1)[x, y] = [-x, -y]$.
which is denoted by $-\underline{u}$ and is called additive inverse of $\underline{u}$

- **Triangle Law of addition:** If two vectors $\underline{u}$ and $\underline{v}$ are represented by the two sides AB and BC of a triangle such that the terminal point of $\underline{u}$ concides with the initial point of $\underline{v}$, then the third side AC of the triangle gives vector sum $\underline{u} + \underline{v}$, that is $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC} \Rightarrow \underline{u} + \underline{v} = \overrightarrow{AC}$.

- **Parallelogram Law of addition:** If two vectors $\underline{u}$ and $\underline{v}$ are represented by two adjacent sides AB and AC of a parallelogram, then diagonal AD gives the sum or resultant of $\overrightarrow{AB}$ and $\overrightarrow{AC}$. that is $\overrightarrow{AD} = \overrightarrow{AB} + \overrightarrow{AC}$



- **Set of vectors:** The set of all odered pairs $[x, y]$ of real numbers, together with the rules of addition and scaler multiplication, is called the set of vectors in R^2.

For a vector $\underline{u} = [x, y]$, x and y are called the components of $\underline{u}$

Note:

- The vector $[x, y]$ is an ordered pair of numbers, not a point (x, y) in the plane.
- $|\underline{v}| \geq 0$, and $|\underline{v}| = 0$ if and only if $\underline{v} = 0$
- $|c\underline{v}| = |c||\underline{v}|$
- Two vectors are said to be negative of each other if they have same magnitude but opposite direction If $\overrightarrow{AB} = \underline{v}$ then $\overrightarrow{BA} = -\overrightarrow{AB}$ and $|\overrightarrow{BA}| = |-\overrightarrow{AB}|$

- **The Ratio formula:** Let A and B be two points whose position vectors are $\underline{a}$ and $\underline{b}$ respectively. If a point P divides AB in the ratio p:q, then the position vector of P is given by $\underline{r} = \dfrac{q\underline{a}+p\underline{b}}{p+q}$

Example no 1: For $\underline{v}=[1,-3]$ and $\underline{w}=[2,5]$

i) $\underline{v}+\underline{w} = [1,-3]+[2,5] = \underline{i}-3\underline{j}+2\underline{i}+5\underline{j} = 3\underline{i}+2\underline{j} = [3,2]$

ii) $4\underline{v}+2\underline{w} = 4[1,-3]+2[2,5] = 4\underline{i}-12\underline{j}+4\underline{i}+10\underline{j} = 8\underline{i}-2\underline{j} = [8,-2]$

iii) $\underline{v}-\underline{w} = [1,-3]-[2,5] = \underline{i}-3\underline{j}-2\underline{i}-5\underline{j} = -\underline{i}-8\underline{j} = [-1,-8]$.

iv) $\underline{v}-\underline{v} = [1,-3]-[1,-3] = \underline{i}-3\underline{j}-\underline{i}+3\underline{j} = 0\underline{i}-0\underline{j} = [0,0]$

v) $|\underline{v}| = \underline{i}-3\underline{j} = \sqrt{(1)^2+(-3)^2} = \sqrt{1+9} = \sqrt{10}$

Example no 2: Find the unit vector in the same direction as the vector $\underline{v}=[3,-4]$

$\underline{v}=[3,-4]=3\underline{i}-4\underline{j}$

$|\underline{v}| = \sqrt{(3)^2+(-4)^2} = \sqrt{9+16} = \sqrt{25} = 5$

Now $\underline{u} = \dfrac{1}{|\underline{v}|}\underline{v} = \dfrac{1}{5}[3,-4]$ (∴ $\underline{u}$ is the unit vector in the direction of $\underline{v}$)

$\underline{u} = \left[\dfrac{3}{5}, \dfrac{-4}{5}\right]$

$|\underline{u}| = \sqrt{\left(\dfrac{3}{5}\right)^2+\left(\dfrac{-4}{5}\right)^2} = \sqrt{\dfrac{9}{25}+\dfrac{16}{25}} = \sqrt{\dfrac{9+16}{25}} = \sqrt{\dfrac{25}{25}} = \sqrt{1} = 1$

Example no 3: Find a unit vector in the direction of vector

i) $\underline{v}=2\underline{i}+6\underline{j}$

$|\underline{v}| = \sqrt{(2)^2+(6)^2} = \sqrt{4+36} = \sqrt{40}$

Unit vector $\hat{v} = \dfrac{\underline{v}}{|\underline{v}|} = \dfrac{2\underline{i}+6\underline{j}}{\sqrt{40}} = \dfrac{2\underline{i}}{\sqrt{40}}+\dfrac{6\underline{j}}{\sqrt{40}} = \dfrac{2\underline{i}}{2\sqrt{10}}+\dfrac{6\underline{j}}{2\sqrt{10}} = \dfrac{\underline{i}}{\sqrt{10}}+\dfrac{3\underline{j}}{\sqrt{10}}$

ii) $\underline{v}=[-2,4]$

$\underline{v} = -2\underline{i}+4\underline{j}$

$|\underline{v}| = \sqrt{(-2)^2+(4)^2} = \sqrt{4+16} = \sqrt{20}$

Unit vector $\hat{v} = \dfrac{\underline{v}}{|\underline{v}|} = \dfrac{-2\underline{i}+4\underline{j}}{\sqrt{20}} = \dfrac{-2}{\sqrt{20}}\underline{i}+\dfrac{4}{\sqrt{20}}\underline{j} = \dfrac{-2}{2\sqrt{5}}\underline{i}+\dfrac{4}{2\sqrt{5}}\underline{j} = \dfrac{-1}{\sqrt{5}}\underline{i}+\dfrac{2}{\sqrt{3}}\underline{j}$

Example no 4: If ABCD is a parallelogram such that the points A, B and C are (-2,-3); (1,4): (0,-5) respectively. Find the coordinates of D.

Suppose the coordinates of D are (x,y)

As ABCD is a parallelogram

so $\overrightarrow{AB} = \overrightarrow{DC}$ & $\overrightarrow{AB} \parallel \overrightarrow{DC}$

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = [1,4] - [-2,-3] = \underline{i} + 4\underline{j} - (-2\underline{i} - 3\underline{j}) = \underline{i} + 4\underline{j} + 2\underline{i} + 3\underline{j} = 3\underline{i} + 7\underline{j} = [3,7]$

$\overrightarrow{DC} = \overrightarrow{OC} - \overrightarrow{OD} = [0,-5] - [x,y] = 0\underline{i} - 5\underline{j} - (x\underline{i} + y\underline{j}) = 0\underline{i} - 5\underline{j} - x\underline{i} - y\underline{j} = -x\underline{i} + (-5-y)\underline{j}$

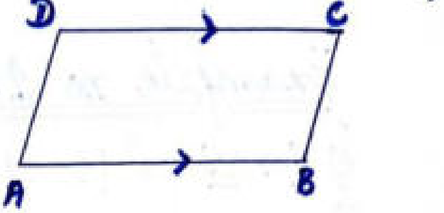
$\Rightarrow [-x, -5-y]$

$[3,7] = [-x, -y-5]$

$-x = 3 \quad ; \quad -y-5 = 7$

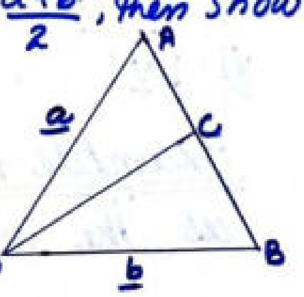
$\boxed{x = -3} \quad ; \quad -y = 12$

$\boxed{y = -12}$



Hence coordinates of D are **[-3, -12]**

Example no 5: If $\underline{a}$ and $\underline{b}$ be the p.vs of A & B respectively w.r.t origin O and C be a point on $\overline{AB}$ such that $\overline{OC} = \frac{a+b}{2}$, then show that C is the midpoint of AB.



Given that $\overrightarrow{OA} = \underline{a}$, $\overrightarrow{OB} = \underline{b}$ and $\overrightarrow{OC} = \frac{\underline{a}+\underline{b}}{2}$

In figure,

$\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{AC} \quad \Rightarrow \overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA}$ — ①

Similarly

$\overrightarrow{OB} = \overrightarrow{OC} + \overrightarrow{CB} \quad \Rightarrow \overrightarrow{CB} = \overrightarrow{OB} - \overrightarrow{OC}$ — ②

$\therefore 2\overrightarrow{OC} = \underline{a} + \underline{b}$ (given).

$\overrightarrow{OC} + \overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{OB}$

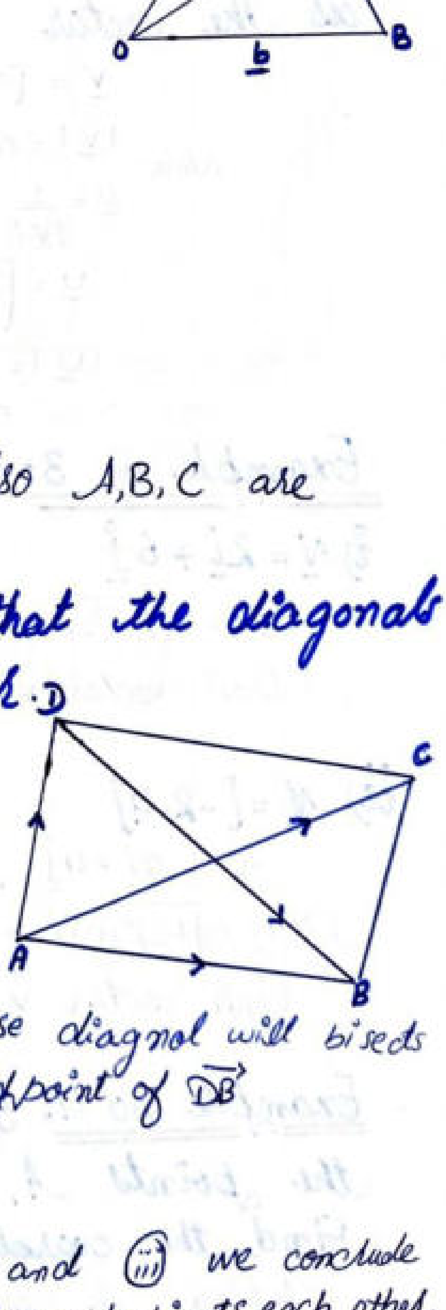
$\overrightarrow{OC} - \overrightarrow{OA} = \overrightarrow{OB} - \overrightarrow{OC}$

By ① & ②

$\overrightarrow{AC} = \overrightarrow{CB}$

C is equidistant from A and B. Also A, B, C are collinear. so C is midpoint of AB.

Example no 6: Use vectors, to prove that the diagonals of a parallelogram bisect each other.



Let ABCD be a parallelogram

Let $\underline{a}$, $\underline{b}$, $\underline{c}$ and $\underline{d}$ be position vectors of A, B, C and D respectively

so $\overrightarrow{AB} = \overrightarrow{DC} \Rightarrow \underline{b} - \underline{a} = \underline{c} - \underline{d} \Rightarrow \underline{b} + \underline{d} = \underline{c} + \underline{a}$ — (i)

from figure $\overrightarrow{AC}$ and $\overrightarrow{DB}$ are diagonals. Now these diagnol will bisects each other if P.V of midpoint of $\overrightarrow{AC}$ = P.V of midpoint of $\overrightarrow{DB}$

P.V of midpoint of $\overrightarrow{AC} = \frac{\underline{a}+\underline{c}}{2}$ — (ii)

P.V of midpoint of $\overrightarrow{BD} = \frac{\underline{b}+\underline{d}}{2}$

As $\underline{b} + \underline{d} = \underline{c} + \underline{a}$

P.V of midpoint of $\overrightarrow{BD} = \frac{\underline{c}+\underline{a}}{2}$ — (iii)

By (ii) and (iii) we conclude so that diagonals bisects each other

Exercise no 7.1

Question no 1: Write the vector $\overrightarrow{PQ}$ in the form $x\underline{i}+y\underline{j}$

i) $P=(2,3)\,;\,Q(6,-2)$

Position vector of $P = \overrightarrow{OP} = 2\underline{i}+3\underline{j}$

Position vector of $Q = \overrightarrow{OQ} = 6\underline{i}-2\underline{j}$

$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$

$\overrightarrow{PQ} = 6\underline{i}-2\underline{j}-(2\underline{i}+3\underline{j})$

$\overrightarrow{PQ} = 6\underline{i}-2\underline{j}-2\underline{i}-3\underline{j}$

$\boxed{\overrightarrow{PQ} = 4\underline{i}-5\underline{j}}$

ii) $P=(0,5)\,;\,Q(-1,-6)$

Position vector of $P = \overrightarrow{OP} = 0\underline{i}+5\underline{j}$

Position vector of $Q : \overrightarrow{OQ} = -\underline{i}-6\underline{j}$

$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$

$\overrightarrow{PQ} = -\underline{i}-6\underline{j}-(0\underline{i}+5\underline{j})$

$\overrightarrow{PQ} = -\underline{i}-6\underline{j}-5\underline{j}$

$\boxed{\overrightarrow{PQ} = -\underline{i}-11\underline{j}}$

Question no 2: Find the magnitude of vector $\underline{u}$:

i) $\underline{u} = 2\underline{i}-7\underline{j}$

$|\underline{u}| = \sqrt{(2)^2+(-7)^2}$

$\text{"} = \sqrt{4+49}$

$\boxed{|\underline{u}| = \sqrt{53}}$

ii) $\underline{u} = \underline{i}+\underline{j}$

$|\underline{u}| = \sqrt{(1)^2+(1)^2}$

$\text{"} = \sqrt{1+1}$

$\boxed{|\underline{u}| = \sqrt{2}}$

iii) $\underline{u} = [3,-4]$

$|\underline{u}| = \sqrt{(3)^2+(-4)^2}$

$\text{"} = \sqrt{9+16}$

$\text{"} = \sqrt{25}$

$\boxed{|\underline{u}| = 5}$

Question no 3: If $\underline{u} = 2\underline{i}-7\underline{j}$, $\underline{v} = \underline{i}-6\underline{j}$ & $\underline{w} = -\underline{i}+\underline{j}$. Find

i) $\underline{u}+\underline{v}-\underline{w}$

$= 2\underline{i}-7\underline{j}+\underline{i}-6\underline{j}-(-\underline{i}+\underline{j})$

$= 2\underline{i}-7\underline{j}+\underline{i}-6\underline{j}+\underline{i}-\underline{j}$

$\boxed{= 4\underline{i}-14\underline{j}}$

ii) $2\underline{u}-3\underline{v}+4\underline{w}$

$= 2(2\underline{i}-7\underline{j})-3(\underline{i}-6\underline{j})+4(-\underline{i}+\underline{j})$

$= 4\underline{i}-14\underline{j}-3\underline{i}+18\underline{j}-4\underline{i}+4\underline{j}$

$\boxed{= -3\underline{i}+8\underline{j}}$

iii) $\frac{1}{2}\underline{u}+\frac{1}{2}\underline{v}+\frac{1}{2}\underline{w}$

$= \frac{1}{2}[2\underline{i}-7\underline{j}+\underline{i}-6\underline{j}-\underline{i}+\underline{j}]$

$= \frac{1}{2}[2\underline{i}-12\underline{j}]$

$\boxed{= \underline{i}-6\underline{j}}$

Question no 4: Find the sum of the vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$, given the four points $A(1,-1)\,;B(2,0)\,;C(-1,3)\,;D(-2,2)$

$A(1,-1)\,;B(2,0)\,;C(-1,3)\,;D(-2,2)$

$\overrightarrow{AB}+\overrightarrow{CD} = ?$

First we find $\overrightarrow{AB}$ and $\overrightarrow{CD}$

$\overrightarrow{AB} = \overrightarrow{OB}-\overrightarrow{OA} = [2,0]-[1,-1] = 2\underline{i}+0\underline{j}-(\underline{i}-\underline{j}) = 2\underline{i}+0\underline{j}-\underline{i}+\underline{j} = \underline{i}+\underline{j}$

$\overrightarrow{CD} = \overrightarrow{OD}-\overrightarrow{OC} = [-2,2]-[-1,3] = -2\underline{i}+2\underline{j}-(-\underline{i}+3\underline{j}) = -2\underline{i}+2\underline{j}+\underline{i}-3\underline{j} = -\underline{i}-\underline{j}$

$\overrightarrow{AB}+\overrightarrow{CD} = \underline{i}+\underline{j}-\underline{i}-\underline{j}$

$= 0$ Null vector

Question no 5: Find the vector from the point A to the origin where $\overrightarrow{AB} = 4\underline{i}-2\underline{j}$ and B is the point $(-2;5)$

$\overrightarrow{AB} = 4\underline{i}-2\underline{j}$

$\overrightarrow{OB} = [-2;5] \Rightarrow -2\underline{i}+5\underline{j} \quad ; \quad \overrightarrow{AO} = ?$

$\vec{AB} = \vec{OB} - \vec{OA}$ = $\vec{OA} = \vec{OB} - \vec{AB}$ = $-2i + 5j - (4i - 2j) = -2i + 5j - 4i + 2j$

$\vec{OA} = -6i + 7j$

As we find $\vec{AO}$

so $\vec{AO} = -\vec{OA}$

$\vec{AO} = -(-6i + 7j)$

$$\boxed{\vec{AO} = 6i - 7j}$$

Question no 6: Find a unit vector in the direction of the vector given below:

i) $v = 2i - j$

$\therefore \hat{v} = \frac{v}{|v|}$

$|v| = \sqrt{(2)^2 + (-1)^2}$

$= \sqrt{4+1}$

$= \sqrt{5}$

$\hat{v} = \frac{2i - j}{\sqrt{5}}$

$$\boxed{\hat{v} = \frac{1}{\sqrt{5}}(2i - j)}$$

ii) $v = \frac{1}{2}i + \frac{\sqrt{3}}{2}j$

$\therefore \hat{v} = \frac{v}{|v|}$

$|v| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$

$= \sqrt{\frac{1}{4} + \frac{3}{4}}$

$= \sqrt{\frac{4}{4}} = \sqrt{1}$

$= 1$

$$\boxed{\hat{v} = \frac{1}{2}i + \frac{\sqrt{3}}{2}j}$$

iii) $-\frac{\sqrt{3}}{2}i - \frac{1}{2}j$

$\therefore \hat{v} = \frac{v}{|v|}$

$|v| = \sqrt{\left(-\frac{\sqrt{3}}{2}\right)^2 + \left(-\frac{1}{2}\right)^2}$

$= \sqrt{\frac{3}{4} + \frac{1}{4}}$

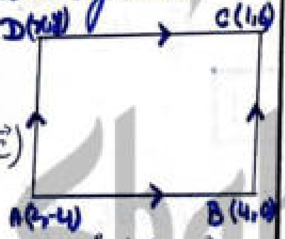
$= \sqrt{\frac{4}{4}} = \sqrt{1}$

$= 1$

$$\boxed{\hat{v} = -\frac{\sqrt{3}}{2}i - \frac{1}{2}j}$$

Question no 7: If A, B, C are respectively the points (2, -4); (4, 0); C(1, 6). Use vector method to find the coordinates of D

i) ABCD is a parallelogram



let D(x,y) be the required point

$\vec{AB} = \vec{DC}$ $(\because \vec{AB} \parallel \vec{DC})$

$\vec{AB} = \vec{OB} - \vec{OA}$

$= [4,0] - [2,-4] \Rightarrow 4i + 0j - (2i - 4j)$

$= 4i + 0j - 2i + 4j$

$= 2i + 4j$

$\vec{DC} = \vec{OC} - \vec{OD}$

$= [1,6] - [x,y] \Rightarrow i + 6j - (xi + yj)$

$= i + 6j - xi - yj$

$= (1-x)i + (6-y)j$

$2i + 4j = (1-x)i + (6-y)j$

$2 = 1 - x$; $4 = 6 - y$

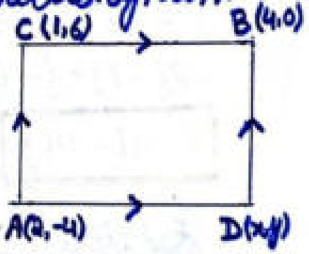
$2 - 1 = -x$; $4 - 6 = -y$

$1 = -x$; $-2 = -y$

$\boxed{x = -1}$; $\boxed{y = 2}$

$D(x,y) = D(-1, 2)$

ii) ADBC is a parallelogram



let D(x,y) be the required point

$\vec{AD} = \vec{CB}$ $(\because \vec{AD} \parallel \vec{CB})$

$\vec{AD} = \vec{OD} - \vec{OA}$

$= [x,y] - [2,-4] \Rightarrow xi + yj - (2i - 4j)$

$= xi + yj - 2i + 4j$

$= (x-2)i + (y+4)j$

$\vec{CB} = \vec{OB} - \vec{OC}$

$= [4,0] - [1,6] \Rightarrow 4i + 0j - (i + 6j)$

$= 4i + 0j - i - 6j$

$= 3i - 6j$

$(x-2)i + (y+4)j = 3i - 6j$

$x - 2 = 3$; $y + 4 = -6$

$x = 3 + 2$; $y = -6 - 4$

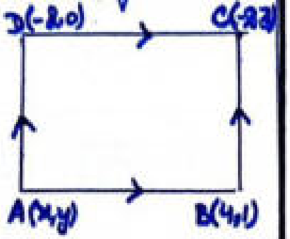
$\boxed{x = 5}$; $\boxed{y = -10}$

$D(x,y) = D(5, -10)$

Question no 8: If B; C; D are respectively (4,1); (-2,3); (-8,0).

Use vector method to find the coordinates of the point:

i) A if ABCD is a parallelogram



Let A(x,y) be the required point

$\vec{AB} = \vec{DC}$ $(\because \vec{AB} \parallel \vec{DC})$

$\vec{AB} = \vec{OB} - \vec{OA}$

$= [4,1] - [x,y]$

$= 4\underline{i} + \underline{j} - (x\underline{i} + y\underline{j})$

$= 4\underline{i} + \underline{j} - x\underline{i} - y\underline{j}$

$= (4-x)\underline{i} + (1-y)\underline{j}$

$\vec{DC} = \vec{OC} - \vec{OD}$

$= [-2,3] - [-8,0]$

$= -2\underline{i} + 3\underline{j} - (-8\underline{i} + 0\underline{j})$

$= -2\underline{i} + 3\underline{j} + 8\underline{i}$

$= 6\underline{i} + 3\underline{j}$

$(4-x)\underline{i} + (1-y)\underline{j} = 6\underline{i} + 3\underline{j}$

$4 - x = 6$; $1 - y = 3$

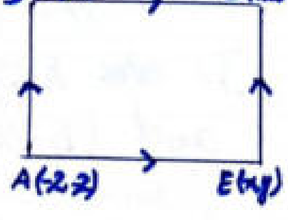
$-x = 6 - 4$; $-y = 3 - 1$

$-x = 2$; $-y = 2$

$\boxed{x = -2}$; $\boxed{y = -2}$

$A(x,y) = (-2,-2)$

ii) E if AEBD is a parallelogram



Let E(x,y) be the required point

$\vec{AE} = \vec{DB}$ $(\because \vec{AE} \parallel \vec{DB})$

$\vec{AE} = \vec{OE} - \vec{OA}$

$= [x,y] - [-2,-2]$

$= x\underline{i} + y\underline{j} - (-2\underline{i} - 2\underline{j})$

$= x\underline{i} + y\underline{j} + 2\underline{i} + 2\underline{j}$

$= (x+2)\underline{i} + (y+2)\underline{j}$

$\vec{DB} = \vec{OB} - \vec{OD}$

$= [4,1] - [-8,0]$

$= 4\underline{i} + \underline{j} - (-8\underline{i} + 0\underline{j})$

$= 4\underline{i} + \underline{j} + 8\underline{i} - 0\underline{j}$

$= 12\underline{i} + \underline{j}$

$(x+2)\underline{i} + (y+2)\underline{j} = 12\underline{i} + \underline{j}$

$x + 2 = 12$; $y + 2 = 1$

$x = 12 - 2$; $y = 1 - 2$

$\boxed{x = 10}$; $\boxed{y = -1}$

$E(x,y) = E(10,-1)$

Question no 9: If O is the origin and $\vec{OP} = \vec{AB}$, find the point P when A and B are (-3,7) and (1,0) respectively.

Given that $\vec{OP} = \vec{AB}$

A (-3,7)

B (1,0)

Let P(x,y) be the required point

Now $\vec{AB} = \vec{OB} - \vec{OA}$

$= [1,0] - [-3,7]$

$= \underline{i} + 0\underline{j} - (-3\underline{i} + 7\underline{j})$

$= \underline{i} + 3\underline{i} - 7\underline{j}$

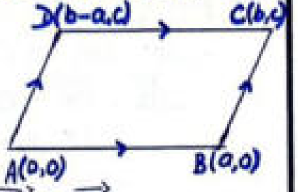
$= 4\underline{i} - 7\underline{j}$

As $\vec{OP} = \vec{AB}$

$[x-0, y-0] = 4\underline{i} - 7\underline{j}$

$[x, y] = [4,-7] \Rightarrow P(x,y) = P(4,-7)$

Question no 10: Use vectors, to show that ABCD is a llogram, when the points A, B, C and D are respectively (0,0); (a,0); (b,c) and (b-a,c).



If these points are the vertices of llogram then $\overrightarrow{AB} = \overrightarrow{DC}$ and $\overrightarrow{AD} = \overrightarrow{BC}$

Now $\overrightarrow{AB} = \overrightarrow{DC}$

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$= [a,0] - [0,0]$

$= a\underline{i} + 0\underline{j} - 0\underline{i} - 0\underline{j}$

$= a\underline{i}$

$\overrightarrow{DC} = \overrightarrow{OC} - \overrightarrow{OD}$

$= [b,c] - [b-a,c]$

$= b\underline{i} + c\underline{j} - [(b-a)\underline{i} + c\underline{j}]$

$= b\underline{i} + c\underline{j} - b\underline{i} + a\underline{i} - c\underline{j}$

$= a\underline{i}$

So $\overrightarrow{AB} = \overrightarrow{DC}$

Now $\overrightarrow{AD} = \overrightarrow{BC}$

$\overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA}$

$= [b-a,c] - [0,0]$

$= (b-a)\underline{i} + c\underline{j} - 0\underline{i} - 0\underline{j}$

$= b\underline{i} - a\underline{i} + c\underline{j}$

$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB}$

$= [b,c] - [a,0]$

$= b\underline{i} + c\underline{j} - a\underline{i} - 0\underline{j}$

$= b\underline{i} - a\underline{i} + c\underline{j}$

So $\overrightarrow{AD} = \overrightarrow{BC}$

Thus ABCD is a llogram

Question no 11: If $\overrightarrow{AB} = \overrightarrow{CD}$ find the coordinates of the point A when points B, C and D are (1,2); (-2,5); (4,11) respectively. Let A(x,y) be the required point

Given that $\overrightarrow{AB} = \overrightarrow{CD}$; B(1,2), C(-2,5), D(4,11)

Now we find $\overrightarrow{AB}$ and $\overrightarrow{CD}$

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$= [1,2] - [x,y]$

$= \underline{i} + 2\underline{j} - x\underline{i} - y\underline{j}$

$= (1-x)\underline{i} + (2-y)\underline{j}$

$\overrightarrow{CD} = \overrightarrow{OD} - \overrightarrow{OC}$

$= [4,11] - [-2,5]$

$= 4\underline{i} + 11\underline{j} - (-2\underline{i} + 5\underline{j})$

$= 4\underline{i} + 11\underline{j} + 2\underline{i} - 5\underline{j}$

$= 6\underline{i} + 6\underline{j}$

As $\overrightarrow{AB} = \overrightarrow{CD}$ (given)

$(1-x)\underline{i} + (2-y)\underline{j} = 6\underline{i} + 6\underline{j}$

$1 - x = 6$; $2 - y = 6$

$x = 1 - 6$; $2 - 6 = y$

$\boxed{x = -5}$; $\boxed{y = -4}$

$A[x,y] = A[-5,-4]$

Question no 12: Find the position vectors of the points of division of the line segments joining the following pair of points, in the given ratio:

i) $C = 2\underline{i} - 3\underline{j}$; $D = 3\underline{i} + 2\underline{j}$: $P:q = 4:3$

$\underline{a} = C = 2\underline{i} - 3\underline{j}$

$\underline{b} = D = 3\underline{i} + 2\underline{j}$

$P:q = 4:3$

$P = 4$, $q = 3$

$$\therefore \underline{r} = \frac{q\underline{a} + P\underline{b}}{P+q}$$

$$\underline{r} = \frac{3(2\underline{i} - 3\underline{j}) + 4(3\underline{i} + 2\underline{j})}{4+3}$$

$$\underline{r} = \frac{6\underline{i} - 9\underline{j} + 12\underline{i} + 8\underline{j}}{7}$$

$$\underline{r} = \frac{18\underline{i} - \underline{j}}{7}$$

$$\boxed{x = \frac{18}{7}\underline{i} - \frac{1}{7}\underline{j}}$$

ii) $E = 5\underline{j}$; $F = 4\underline{i} + \underline{j}$; $P : q = 2 : 5$

$\underline{a} = E = 5\underline{j}$

$\underline{b} = F = 4\underline{i} + \underline{j}$

$P : q = 2 : 5$

$P = 2$; $q = 5$.

$$\therefore \underline{x} = \frac{q\underline{a} + P\underline{b}}{P+q}$$

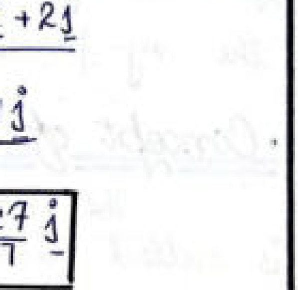
$$\underline{x} = \frac{5(5\underline{j}) + 2(4\underline{i} + \underline{j})}{2+5}$$

$$\underline{x} = \frac{25\underline{j} + 8\underline{i} + 2\underline{j}}{7}$$

$$\underline{x} = \frac{8\underline{i} + 27\underline{j}}{7}$$

$$\boxed{\underline{x} = \frac{8}{7}\underline{i} + \frac{27}{7}\underline{j}}$$

Question no 13: Prove that the line segment joining the midpoints of two sides of a triangle is parallel to third side and half as long.



Let $\underline{a}, \underline{b}, \underline{c}$ be the position vectors of A, B, C respectively then.

Position vector of D $= \frac{\underline{a} + \underline{b}}{2}$

Position vector of E $= \frac{\underline{a} + \underline{c}}{2}$

[∴ D and E are midpoints of $\overrightarrow{AB}$ and $\overrightarrow{AC}$ respectively]

Now we find $\overrightarrow{BC}$ and $\overrightarrow{DE}$

$$\therefore \overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB}$$

$$= \underline{c} - \underline{b} \text{ — ①}$$

and $\overrightarrow{DE} = \overrightarrow{OE} - \overrightarrow{OD}$

$$= \frac{\underline{a} + \underline{c}}{2} - \frac{\underline{a} + \underline{b}}{2}$$

$$= \frac{\underline{a} + \underline{c} - (\underline{a} + \underline{b})}{2}$$

$$= \frac{\underline{a} + \underline{c} - \underline{a} - \underline{b}}{2}$$

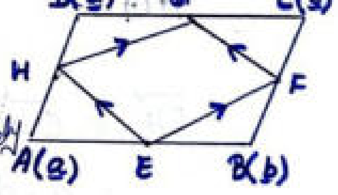
$$= \frac{\underline{c} - \underline{b}}{2}$$

$$\overrightarrow{DE} = \frac{1}{2}(\underline{c} - \underline{b})$$

$$\overrightarrow{DE} = \frac{1}{2}\overrightarrow{BC} \text{ — ②}$$

Thus from ① and ② $\overrightarrow{DE}$ and $\overrightarrow{BC}$ are $\overrightarrow{\parallel}$ and $\overrightarrow{DE}$ is half to $\overrightarrow{BC}$

Question no 24: Prove that the line segments joining the midpoints of the sides of a quadrilateral taken in order form a parallelogram



Suppose $\underline{a}, \underline{b}, \underline{c}, \underline{d}$ are position vectors of A, B, C, D respectively then so,

E, F, G, H are the midpoints of $\overrightarrow{AB}$; $\overrightarrow{BC}$, $\overrightarrow{CA}$ and $\overrightarrow{DA}$ respectively

Now

Position vector of E $= \frac{\underline{a} + \underline{b}}{2}$

Position vector of F $= \frac{\underline{b} + \underline{c}}{2}$

Position vector of G $= \frac{\underline{c} + \underline{d}}{2}$

Position vector of H $= \frac{\underline{a} + \underline{d}}{2}$

Now we proved that EFGH is a parallelogram.

$\vec{EF} = \vec{OF} - \vec{OE}$

$= \left(\frac{\underline{b}+\underline{c}}{2}\right) - \left(\frac{\underline{a}+\underline{b}}{2}\right)$

$= \frac{\underline{b}+\underline{c}-\underline{a}-\underline{b}}{2}$

$\vec{EF} = \frac{\underline{c}-\underline{a}}{2}$

$\vec{FG} = \vec{OG} - \vec{OF}$

$= \frac{\underline{c}+\underline{d}}{2} - \left(\frac{\underline{b}+\underline{c}}{2}\right)$

$= \frac{\underline{c}+\underline{d}-\underline{b}-\underline{c}}{2}$

$\vec{FG} = \frac{\underline{d}-\underline{b}}{2}$

$\vec{HG} = \vec{OG} - \vec{OH}$

$= \frac{\underline{c}+\underline{d}}{2} - \left(\frac{\underline{a}+\underline{d}}{2}\right)$

$= \frac{\underline{c}+\underline{d}-\underline{a}-\underline{d}}{2}$

$\vec{HG} = \frac{\underline{c}-\underline{a}}{2}$

$\vec{EH} = \vec{OH} - \vec{OE}$

$= \frac{\underline{a}+\underline{d}}{2} - \left(\frac{\underline{a}+\underline{b}}{2}\right)$

$= \frac{\underline{a}+\underline{d}-\underline{a}-\underline{b}}{2}$

$\vec{EH} = \frac{\underline{d}-\underline{b}}{2}$

$\because \vec{EF} = \vec{HG}$

$\Rightarrow \vec{EF}$ is $\parallel$ $\vec{HG}$

and $\because \vec{FG} = \vec{EH}$

$\vec{FG}$ is $\parallel$ to $\vec{EH}$

So EFGH is parallelogram.

Theory:

Intoduction of vectors in space

Right hand rule: If the fingers of the right hand, pointing in the direction of positive x-axis, are curled taward the positive y-axis, then the thumb will point in the direction of positive z-axis, perpendicular to the xy-plane.

Concept of a vector in space: The set $R^3 = \{(x,y,z) : x,y,z \in R\}$ is called 3-dimensional space. An element (x,y,z) of R^3 represent determined by its coordinates x, y and z.

We defined addition and scalar multiplication in R^3 by:

Addition: For any two vectors $\underline{u} = [x,y,z]$ and $\underline{v} = [x',y',z']$

$\underline{u}+\underline{v} = [x,y,z] + [x',y',z']$

$= [x+x', y+y' z+z']$

Scaler Multiplication: For $\underline{u} = [x,y,z]$ and $\alpha \in R$ we have

$\alpha\underline{u} = \alpha[x,y,z].$

$= [\alpha x; \alpha y : \alpha z].$

Set of vectors in R^3: The set of all ordered triples $[x,y,z]$ of real numbers, together with the rules of addition and scaler multiplication, is called the set of vectors in R^3.

• Negative of the vector:

The negative of the vector $\underline{u}=[x,y,z]$ as $-\underline{u}=(-1)\underline{u}=[-x,-y,-z]$.

• Subtraction of vectors:

The subtraction or difference of two vectors $\underline{v}=[x',y',z']$ and $\underline{w}=[x'',y'',z'']$ as

$$\underline{v}-\underline{w}=\underline{v}+(-w)=[x'-x'',y'-y'',z'-z'']$$

• Null vector:

The null or zero vector as $\underline{0}=[0,0,0]$.

• Equality of two vectors:

Equality of two vectors $\underline{v}=[x',y',z']$ and $\underline{w}=[x'',y'',z'']$ by $\underline{v}=\underline{w}$ if and only if $x'=x''$; $y'=y''$; $z'=z''$.

• Position vector:

For any Point $P(x,y,z)$ in R^3, a vector $\underline{u}=[x,y,z]$ is represented by a directed line segment $\overrightarrow{OP}$, whose initial point is at origin. Such vectors are called position vectors in R^3.

• Magnitude of a vector:

The magnitude or norm or lenght of a vector $\underline{u}$ in space by the distance of the point $P(x,y,z)$ from the origin O.

$$|\overrightarrow{OP}|=\sqrt{x^2+y^2+z^2}$$

• Properties of vectors:

Let $\underline{u}$, $\underline{v}$ and $\underline{w}$ be vectors in plane or in space and let $a,b\in R$ then they have the fallowing properties:

i) $\underline{u}+\underline{v}=\underline{v}+\underline{u}$
ii) $(\underline{u}+\underline{v})+\underline{w}=\underline{u}+(\underline{v}+\underline{w})$
iii) $\underline{u}+(-1)\underline{u}=\underline{u}-\underline{u}=\underline{0}$
iv) $a(\underline{v}+\underline{w})=a\underline{v}+a\underline{w}$
v) $a(b\underline{u})=(ab)\underline{u}$

• Distance b/w two points in space:

If $\overrightarrow{OP_1}$ and $\overrightarrow{OP_2}$ are the position vectors of the points $P_1(x_1,y_1,z_1)$; $P_2(x_2,y_2,z_2)$.

So the distance between P_1 and P_2

$$=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$$

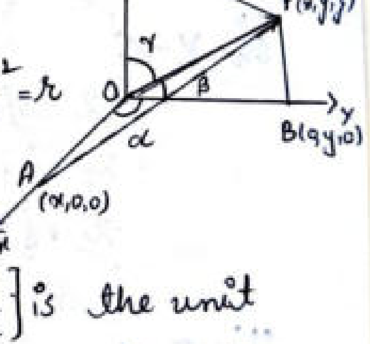
This is called the distance formula b/w two points P_1 and P_2 in R^3.

Prove: $\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1$

Let $\underline{r}=[x,y,z]$

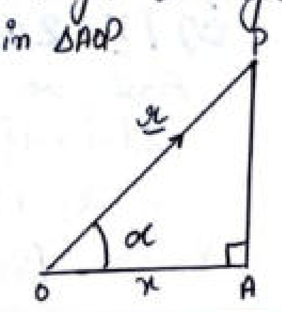
$$=x\underline{i}+y\underline{j}+z\underline{k}$$

$$|\underline{r}|=\sqrt{x^2+y^2+z^2}=r$$



then $\dfrac{\underline{r}}{|\underline{r}|}=\left[\dfrac{x}{r},\dfrac{y}{r},\dfrac{z}{r}\right]$ is the unit vector in the direction of the vector $\underline{r}=\overrightarrow{OP}$.

The triangle OAP is a right triangle with ∠A = 90°. Therefore in ΔAOP



$$\cos\alpha=\frac{\overline{OA}}{\overline{OP}}=\frac{x}{r}$$

$$\cos\beta=\frac{y}{r}$$

$$\cos\gamma=\frac{z}{r}$$

The numbers $\cos\alpha = \frac{x}{r}$; $\cos\beta = \frac{y}{r}$ and $\cos\gamma = \frac{z}{r}$ are called the direction of cosines of $\vec{OP}$.

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \frac{x^2}{r^2} + \frac{y^2}{r^2} + \frac{z^2}{r^2}$$

$$= \frac{x^2+y^2+z^2}{r^2}$$

$$= \frac{r^2}{r^2}$$

$$\therefore r = \sqrt{x^2+y^2+z^2}$$

$$\text{So } r^2 = x^2+y^2+z^2$$

$$= 1$$

Hence proved.

Example # 1: For the vectors, $\underline{v} = [2, 1, 3]$ and $\underline{w} = [-1, 4, 0]$, we have the following:

i) $\underline{v} + \underline{w}$

$$= [2, 1, 3] + [-1, 4, 0]$$

$$= 2\underline{i} + \underline{j} + 3\underline{k} - \underline{i} + 4\underline{j} + 0\underline{k}$$

$$= \underline{i} + 5\underline{j} + 3\underline{k}$$

ii) $\underline{v} - \underline{w}$

$$= [2,1,3] - [-1,4,0]$$

$$= 2\underline{i} + \underline{j} + 3\underline{k} - (-\underline{i} + 4\underline{j} + 0\underline{k})$$

$$= 2\underline{i} + \underline{j} + 3\underline{k} + \underline{i} - 4\underline{j}$$

$$= 3\underline{i} - 3\underline{j} + 3\underline{k}$$

iii) $2\underline{w}$

$$= 2(-\underline{i} + 4\underline{j} + 0\underline{k})$$

$$= -2\underline{i} - 8\underline{j} + 0\underline{k}$$

iv) $|\underline{v} - 2\underline{w}|$

First we find $2\underline{w}$

$$= 2(-\underline{i} + 4\underline{j} + 0\underline{k})$$

$$= -2\underline{i} + 8\underline{j} + 0\underline{k}$$

$$\underline{v} - 2\underline{w} = [2\underline{i} + \underline{j} + 3\underline{k}] - (-2\underline{i} + 8\underline{j} + 0\underline{k})$$

$$= 2\underline{i} + \underline{j} + 3\underline{k} + 2\underline{i} - 8\underline{j} - 0\underline{k}$$

$$= 4\underline{i} - 7\underline{j} + 3\underline{k}$$

$$|\underline{v} - 2\underline{w}| = \sqrt{(4)^2 + (-7)^2 + (3)^2}$$

$$= \sqrt{16+49+9}$$

$$= \sqrt{74}$$

Example # 2: If $\underline{u} = 2\underline{i} + 3\underline{j} + \underline{k}$; $\underline{v} = 4\underline{i} + 6\underline{j} + 2\underline{k}$ and $\underline{w} = -6\underline{i} - 9\underline{j} - 3\underline{k}$.

then (a) Find (i) $\underline{u} + 2\underline{v}$ (ii) $|\underline{u} - \underline{v} - \underline{w}|$

(b) Show that $\underline{u}, \underline{v}, \underline{w}$ are $\parallel$ to each other

(a)(i) $\underline{u} + 2\underline{v}$

$$= 2\underline{i} + 3\underline{j} + \underline{k} + 2(4\underline{i} + 6\underline{j} + 2\underline{k})$$

$$= 2\underline{i} + 3\underline{j} + \underline{k} + 8\underline{i} + 12\underline{j} + 4\underline{k}$$

$$= 10\underline{i} + 15\underline{j} + 5\underline{k}.$$

ii) $|\underline{u} - \underline{v} - \underline{w}|$

$$= (2\underline{i} + 3\underline{j} + \underline{k}) - (4\underline{i} + 6\underline{j} + 2\underline{k}) - (-6\underline{i} - 9\underline{j} - 3\underline{k})$$

$$= 2\underline{i} + 3\underline{j} + \underline{k} - 4\underline{i} - 6\underline{j} - 2\underline{k} + 6\underline{i} + 9\underline{j} + 3\underline{k}$$

$$= 4\underline{i} + 6\underline{j} + 2\underline{k}$$

$$|\underline{u} - \underline{v} - \underline{w}| = \sqrt{(4)^2 + (6)^2 + (2)}$$

$$= \sqrt{16+36+4}$$

$$= \sqrt{56}$$

(b)

$$\underline{v} = 4\underline{i} + 6\underline{j} + 2\underline{k}$$

$$= 2(2\underline{i} + 3\underline{j} + \underline{k})$$

$$\therefore \underline{v} = 2\underline{u}$$

$\Rightarrow$ $\underline{u}$ and $\underline{v}$ are parallel vectors and have same direction.

$$\underline{w} = -6\underline{i} - 9\underline{j} - 3\underline{k}$$

$$= -3(2\underline{i} + 3\underline{j} + \underline{k})$$

$$\therefore \underline{w} = -3\underline{u}$$

$\Rightarrow$ $\underline{u}$ and $\underline{w}$ are parallel vectars and have opposite direction.

Hence $\underline{u}$, $\underline{v}$ and $\underline{w}$ are parallel to each other.

Exercise no 7.2

Question no 1: Let

$A=(2,5)$; $B=(-1,1)$ and $C=(2,-6)$.

Find

i) $\overrightarrow{AB}$

$\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}$

$=[-1,1]-[2,5]$

$=-\underline{i}+\underline{j}-2\underline{i}-5\underline{j}$

$\boxed{\overrightarrow{AB}=-3\underline{i}-4\underline{j}}$

ii) $2\overrightarrow{AB}-\overrightarrow{CB}$

$\overrightarrow{AB}=[-1,1]-[2,5] \quad \because \overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}$

$=-\underline{i}+\underline{j}-2\underline{i}-5\underline{j}$

$=-3\underline{i}-4\underline{j}$

$2\overrightarrow{AB}=2(-3\underline{i}-4\underline{j})$

$=-6\underline{i}-8\underline{j}$

$\overrightarrow{CB}=[-1,1]-[2,-6] \quad \because \overrightarrow{CB}=\overrightarrow{OB}-\overrightarrow{OC}$

$=-\underline{i}+\underline{j}-2\underline{i}+6\underline{j}$

$=-3\underline{i}+7\underline{j}$

$2\overrightarrow{AB}-\overrightarrow{CB}=-6\underline{i}-8\underline{j}-(-3\underline{i}+7\underline{j})$

$\quad " \quad =-6\underline{i}-8\underline{j}+3\underline{i}-7\underline{j}$

$\boxed{2\overrightarrow{AB}-\overrightarrow{CB}=-3\underline{i}-15\underline{j}}$

iii) $2\overrightarrow{CB}-2\overrightarrow{CA}$

$\overrightarrow{CB}=[-1,1]-[2,-6] \quad \because \overrightarrow{CB}=\overrightarrow{OB}-\overrightarrow{OC}$

$=-\underline{i}+\underline{j}-2\underline{i}+6\underline{j}$

$=-3\underline{i}+7\underline{j}$

$2\overrightarrow{CB}=2(-3\underline{i}+7\underline{j})$

$=-6\underline{i}+14\underline{j}$

$\overrightarrow{CA}=[2,5]-[2,-6] \quad \because \overrightarrow{CA}=\overrightarrow{OA}-\overrightarrow{OC}$

$=2\underline{i}+5\underline{j}-2\underline{i}+6\underline{j}$

$=0\underline{i}+11\underline{j}$

$2\overrightarrow{CA}=2(0\underline{i}+11\underline{j})$

$=0\underline{i}+22\underline{j}$

$2\overrightarrow{CB}-2\overrightarrow{CA}=(-6\underline{i}+14\underline{j})-(0\underline{i}+22\underline{j})$

$=-6\underline{i}+14\underline{j}-0\underline{i}-22\underline{j}$

$\boxed{2\overrightarrow{CB}-2\overrightarrow{CA}=-6\underline{i}-8\underline{j}}$

Question no 2: Let $\underline{u}=\underline{i}+2\underline{j}-\underline{k}$; $\underline{v}=3\underline{i}-2\underline{j}+2\underline{k}$; $\underline{w}=5\underline{i}-\underline{j}+3\underline{k}$. Find

i) $\underline{u}+2\underline{v}+\underline{w}$

$=\underline{i}+2\underline{j}-\underline{k}+2(3\underline{i}-2\underline{j}+2\underline{k})+5\underline{i}-\underline{j}+3\underline{k}$

$=\underline{i}+2\underline{j}-\underline{k}+6\underline{i}-4\underline{j}+4\underline{k}+5\underline{i}-\underline{j}+3\underline{k}$

$=\boxed{12i-3\underline{j}+6\underline{k}}$

ii) $\underline{v}-3\underline{w}$

$=3\underline{i}-2\underline{j}+2k-3(5\underline{i}-\underline{j}+3\underline{k})$

$=3\underline{i}-2\underline{j}+2k-15\underline{i}+3\underline{j}-9\underline{k}$

$\boxed{=-12\underline{i}+\underline{j}-7\underline{k}}$

iii) $|3\underline{v}+\underline{w}|$

$=3(3\underline{i}-2\underline{j}+2\underline{k})+(5\underline{i}-\underline{j}+3\underline{k})$

$=9\underline{i}-6\underline{j}+6\underline{k}+5\underline{i}-\underline{j}+3\underline{k}$

$3\underline{v}+\underline{w}=14\underline{i}-7\underline{j}+9\underline{k}$

$|3\underline{v}+\underline{w}|=\sqrt{(14)^2+(-7)^2+(9)^2}$

$=\sqrt{196+49+81}$

$\boxed{=\sqrt{326}}$

Question no 3: Find the magnitude of the vector $\underline{v}$ and write the direction cosines of $\underline{v}$

i) $\underline{v}=2\underline{i}+3\underline{j}+4\underline{k}$

$|\underline{v}|=\sqrt{(2)^2+(3)^2+(4)^2}$

$=\sqrt{4+9+16}$

$=\sqrt{29}$

$\hat{v}=\dfrac{2\underline{i}+3\underline{j}+4\underline{k}}{\sqrt{29}} \quad \because \hat{v}=\dfrac{\underline{v}}{|\underline{v}|}$

$=\dfrac{2}{\sqrt{29}}\underline{i}+\dfrac{3}{\sqrt{29}}\underline{j}+\dfrac{4}{\sqrt{29}}\underline{k}$

$\cos\alpha=\dfrac{2}{\sqrt{29}}$; $\cos\beta=\dfrac{3}{\sqrt{29}}$; $\cos\gamma=\dfrac{4}{\sqrt{29}}$

ii) $\underline{v} = \underline{i} - \underline{j} - \underline{k}$

$$|\underline{v}| = \sqrt{(1)^2+(-1)^2+(-1)^2}$$
$$= \sqrt{1+1+1}$$
$$= \sqrt{3}$$
$$\hat{v} = \frac{i}{\sqrt{3}} - \frac{j}{\sqrt{3}} - \frac{k}{\sqrt{3}} \quad \therefore \hat{v} = \frac{v}{|v|}$$
$$\cos\alpha = \frac{1}{\sqrt{3}} ; \cos\beta = \frac{-1}{\sqrt{3}} ; \cos\gamma = \frac{-1}{\sqrt{3}}$$

iii) $\underline{v} = 4\underline{i} - 5\underline{j}$

$$|\underline{v}| = \sqrt{(4)^2+(-5)^2+(0)^2}$$
$$= \sqrt{16+25}$$
$$= \sqrt{41}$$
$$\hat{v} = \frac{4\underline{i} - 5\underline{j} + 0\underline{k}}{\sqrt{41}} \therefore \hat{v} = \frac{\underline{v}}{|\underline{v}|}$$
$$\hat{v} = \frac{4}{\sqrt{41}}\underline{i} - \frac{5}{\sqrt{41}}\underline{j} + \frac{0}{\sqrt{41}}\underline{k}$$
$$\cos\alpha = \frac{4}{\sqrt{41}} ; \cos\beta = \frac{-5}{\sqrt{41}} ; \cos\gamma = 0$$

Question no 4: Find α, so that $|\alpha\underline{i} + (\alpha+1)\underline{j} + 2\underline{k}| = 3$

$$|\alpha\underline{i} + (\alpha+1)\underline{j} + 2\underline{k}| = 3$$
$$\sqrt{(\alpha)^2+(\alpha+1)^2+(2)^2} = 3$$
$$\sqrt{\alpha^2+\alpha^2+2\alpha+1+4} = 3$$
$$\sqrt{2\alpha^2+2\alpha+5} = 3$$

By taking square on Both sides

$$(\sqrt{2\alpha^2+2\alpha+5})^2 = (3)^2$$
$$2\alpha^2+2\alpha+5 = 9$$
$$2\alpha^2+2\alpha+5-9 = 0$$
$$2\alpha^2+2\alpha-4 = 0$$

÷ing by 2

$$\alpha^2+\alpha-2 = 0$$
$$\alpha^2+2\alpha-\alpha-2 = 0$$
$$\alpha(\alpha+2) - 1(\alpha+2) = 0$$
$$(\alpha+2)(\alpha-1) = 0$$
$$\alpha+2 = 0 ; \alpha-1 = 0$$
$$\alpha = -2 ; \alpha = 1$$
$$\therefore \alpha = (1, -2)$$

Question no 5: Find a unit vector in the direction of $\underline{v} = \underline{i} + 2\underline{j} - \underline{k}$

$$|\underline{v}| = \sqrt{(1)^2+(2)^2+(-1)^2}$$
$$= \sqrt{1+4+1}$$
$$= \sqrt{6}$$
$$\hat{v} = \frac{\underline{i}+2\underline{j}-\underline{k}}{\sqrt{6}} \quad \therefore \hat{v} = \frac{v}{|v|}$$
$$\hat{v} = \frac{\underline{i}}{\sqrt{6}} + \frac{2\underline{j}}{\sqrt{6}} - \frac{\underline{k}}{\sqrt{6}}$$

Question no 6: If $\underline{a} = 3\underline{i} - \underline{j} - 4\underline{k}$, $\underline{b} = -2\underline{i} - 4\underline{j} - 3\underline{k}$ and $\underline{c} = \underline{i} + 2\underline{j} - \underline{k}$. Find a unit vector parallel to $3\underline{a} - 2\underline{b} + 4\underline{c}$.

$\underline{a} = 3\underline{i} - \underline{j} - 4\underline{k}$; $\underline{b} = -2\underline{i} - 4\underline{j} - 3\underline{k}$; $\underline{c} = \underline{i} + 2\underline{j} - \underline{k}$.

$$= 3\underline{a} - 2\underline{b} + 4\underline{c}$$
$$= 3(3\underline{i} - \underline{j} - 4\underline{k}) - 2(-2\underline{i} - 4\underline{j} - 3\underline{k}) + 4(\underline{i} + 2\underline{j} - \underline{k})$$
$$= 9\underline{i} - 3\underline{j} - 12\underline{k} + 4\underline{i} + 8\underline{j} + 6\underline{k} + 4\underline{i} + 8\underline{j} - 4\underline{k}$$
$$= 17\underline{i} - 13\underline{j} - 10\underline{k}$$

Let $\underline{v} = 17\underline{i} - 13\underline{j} - 10\underline{k}$

$$|\underline{v}| = \sqrt{(17)^2+(-13)^2+(-10)^2}$$
$$= \sqrt{289+169+100}$$
$$= \sqrt{558}$$
$$\therefore \hat{v} = \frac{\underline{v}}{|\underline{v}|}$$
$$\hat{v} = \frac{17\underline{i} - 13\underline{j} - 10\underline{k}}{\sqrt{558}}$$
$$\hat{v} = \frac{17}{\sqrt{558}}\underline{i} - \frac{13}{\sqrt{558}}\underline{j} - \frac{10}{\sqrt{558}}\underline{k}$$

Question no 7: Find a vector whose

i) magnitude is 4 and is Parallel to $2\underline{i}-3\underline{j}+6\underline{k}$

Let $\underline{a}=2\underline{i}-3\underline{j}+6\underline{k}$

$|\underline{a}|=\sqrt{(2)^2+(-3)^2+(6)^2}$

$|\underline{a}|=\sqrt{4+9+36}$

$|\underline{a}|=\sqrt{49}$

$|\underline{a}|=7$

$\hat{a}=\dfrac{2\underline{i}-3\underline{j}+6\underline{k}}{7}$ $\therefore\ \hat{a}=\dfrac{\underline{a}}{|\underline{a}|}$

$\hat{a}=\frac{2}{7}\underline{i}-\frac{3}{7}\underline{j}+\frac{6}{7}\underline{k}$

Let $\underline{b}$ is the required vector and $|\underline{b}|=4$

$\therefore$ $\underline{b}$ is parallel to $\underline{a}$

$\hat{b}=\hat{a}$

$\hat{b}=\frac{2}{7}\underline{i}-\frac{3}{7}\underline{j}+\frac{6}{7}\underline{k}$

$\therefore\ \underline{b}=|\underline{b}|\hat{b}$

$\underline{b}=4\left(\frac{2}{7}\underline{i}-\frac{3}{7}\underline{j}+\frac{6}{7}\underline{k}\right)$

$$\boxed{\underline{b}=\frac{8}{7}\underline{i}-\frac{12}{7}\underline{j}+\frac{24}{7}\underline{k}}$$

Required vector

ii) magnitude is 2 and is parallel to $-\underline{i}+\underline{j}+\underline{k}$

Let $\underline{a}=-\underline{i}+\underline{j}+\underline{k}$

$|\underline{a}|=\sqrt{(-1)^2+(1)^2+(1)^2}$

$|\underline{a}|=\sqrt{1+1+1}$

$|\underline{a}|=\sqrt{3}$

$\hat{a}=\dfrac{-\underline{i}+\underline{j}+\underline{k}}{\sqrt{3}}$ $\therefore\ \hat{a}=\dfrac{\underline{a}}{|\underline{a}|}$

$\hat{a}=\frac{-1}{\sqrt{3}}\underline{i}+\frac{1}{\sqrt{3}}\underline{j}+\frac{1}{\sqrt{3}}\underline{k}$

Let $\underline{b}$ is the required vector and $|\underline{b}|=2$

$\therefore$ $\underline{b}$ is parallel to $\underline{a}$

$\hat{b}=\hat{a}$

$\hat{b}=\frac{-1}{\sqrt{3}}\underline{i}+\frac{1}{\sqrt{3}}\underline{j}+\frac{1}{\sqrt{3}}\underline{k}$

$\therefore\ \underline{b}=|\underline{b}|\hat{b}$

$\underline{b}=2\left(\frac{-1}{\sqrt{3}}\underline{i}+\frac{1}{\sqrt{3}}\underline{j}+\frac{1}{\sqrt{3}}\underline{k}\right)$

$$\boxed{\underline{b}=\frac{-2}{\sqrt{3}}\underline{i}+\frac{2}{\sqrt{3}}\underline{j}+\frac{2}{\sqrt{3}}\underline{k}}$$

Required vector

Question no 8: If $\underline{u}=2\underline{i}+3\underline{j}+4\underline{k}$, $\underline{v}=-\underline{i}+3\underline{j}-\underline{k}$, $\underline{w}=\underline{i}+6\underline{j}+z\underline{k}$ represent the sides of a triangle Find the value of z.

$\underline{u}=2\underline{i}+3\underline{j}+4\underline{k}$

$\underline{v}=-\underline{i}+3\underline{j}-\underline{k}$

$\underline{w}=\underline{i}+6\underline{j}+z\underline{k}$

By head to tail rule

$\underline{w}=\underline{u}+\underline{v}$

$\underline{i}+6\underline{j}+z\underline{k}=2\underline{i}+3\underline{j}+4\underline{k}-\underline{i}+3\underline{j}-\underline{k}$

$\underline{i}+6\underline{j}+z\underline{k}=\underline{i}+6\underline{j}+3\underline{k}$

By Comparing

$$\boxed{z=3}$$

Question no 9: The position vectors of the points A, B, C and D are $2\underline{i}-\underline{j}+\underline{k}$, $3\underline{i}+\underline{j}$, $2\underline{i}+4\underline{j}-2\underline{k}$ and $-\underline{i}-2\underline{j}+\underline{k}$ respectively Show that $\overrightarrow{AB}$ is parallel to $\overrightarrow{CD}$.

Let

Position vector of $A = \vec{OA} = 2\underline{i} - \underline{j} + \underline{k}$

Position vector of $B = \vec{OB} = 3\underline{i} + \underline{j}$

Position vector of $C = \vec{OC} = 2\underline{i} + 4\underline{j} - 2\underline{k}$

Position vector of $D = \vec{OD} = -\underline{i} - 2\underline{j} + \underline{k}$

We have to prove that

$$\vec{AB} \parallel \vec{CD}$$

Now we find that $\vec{AB}$ and $\vec{CD}$.

$\vec{AB} = \vec{OB} - \vec{OA}$

$\quad " = 3\underline{i} + \underline{j} - (2\underline{i} - \underline{j} + \underline{k})$

$\quad " = 3\underline{i} + \underline{j} - 2\underline{i} + \underline{j} - \underline{k}$

$\vec{AB} = \underline{i} + 2\underline{j} - \underline{k}$

$\vec{CD} = \vec{OD} - \vec{OC}$

$\quad " = -\underline{i} - 2\underline{j} + \underline{k} - (2\underline{i} + 4\underline{j} - 2\underline{k})$

$\quad " = -\underline{i} - 2\underline{j} + \underline{k} - 2\underline{i} - 4\underline{j} + 2\underline{k}$

$\quad " = -3\underline{i} - 6\underline{j} + 3\underline{k}$

$\vec{CD} = -3($[illegible]$\underline{i} + 2\underline{j} + \underline{k})$

$\vec{CD} = -3(\vec{AB})$

So $\vec{AB} \parallel \vec{CD}$

* * *

Question no 10: We say that two - - - - - - - - if $c < 0$

a) Find two vectors of length 2 parallel to the vector $\underline{v} = 2\underline{i} - 4\underline{j} + 4\underline{k}$

$\underline{v} = 2\underline{i} - 4\underline{j} + 4\underline{k}$

$|\underline{v}| = \sqrt{(2)^2 + (-4)^2 + (4)^2}$

$|\underline{v}| = \sqrt{4 + 16 + 16}$

$|\underline{v}| = \sqrt{36}$

$|\underline{v}| = 6$

$\hat{v} = \dfrac{2\underline{i} - 4\underline{j} + 4\underline{k}}{6} \qquad \therefore \hat{v} = \dfrac{v}{|\underline{v}|}$

$\hat{v} = \frac{2}{6}\underline{i} - \frac{4}{6}\underline{j} + \frac{4}{6}\underline{k}$

$\hat{v} = \frac{1}{3}\underline{i} - \frac{2}{3}\underline{j} + \frac{2}{3}\underline{k}$

Let $2\hat{v}$ and $-2\hat{v}$ are required vectors

$2\hat{v} = 2\left(\frac{1}{3}\underline{i} - \frac{2}{3}\underline{j} + \frac{2}{3}\underline{k}\right)$

$= \frac{2}{3}\underline{i} - \frac{4}{3}\underline{j} + \frac{4}{3}\underline{k}$

$-2\hat{v} = -2\left(\frac{1}{3}\underline{i} - \frac{2}{3}\underline{j} + \frac{2}{3}\underline{k}\right)$

$= -\frac{2}{3}\underline{i} + \frac{4}{3}\underline{j} - \frac{4}{3}\underline{k}$

b) Find the constant a so that the vector's $\underline{v} = \underline{i} - 3\underline{j} + 4\underline{k}$ and $\underline{w} = a\underline{i} + 9\underline{j} - 12\underline{k}$ are parallel.

$\underline{v}$ and $\underline{w}$ are parallel (given)

$\underline{v} = c\underline{w}$

$\underline{i} - 3\underline{j} + 4\underline{k} = c(a\underline{i} + 9\underline{j} - 12\underline{k})$

$\underline{i} - 3\underline{j} + 4\underline{k} = ac\underline{i} + 9c\underline{j} - 12c\underline{k}$

By equalling

$ac = 1$ —— ①

$9c = -3$

$c = -\frac{3}{9}$

$c = -\frac{1}{3}$

Put $c = -\frac{1}{3}$ in ①

$a\left(-\frac{1}{3}\right) = 1$

$-a = 3$

$\boxed{a = -3}$

c) Find a vector of length 5 in the direction opposite that of $\underline{v} = \underline{i} - 2\underline{j} + 3\underline{k}$

$\underline{v} = \underline{i} - 2\underline{j} + 3\underline{k}$

$|\underline{v}| = \sqrt{(1)^2 + (-2)^2 + (3)^2}$

$= \sqrt{1 + 4 + 9}$

$|\underline{v}| = \sqrt{14}$

$\hat{v} = \dfrac{\underline{i} - 2\underline{j} + 3\underline{k}}{\sqrt{14}} \qquad \therefore \hat{v} = \dfrac{v}{|v|}$

$\hat{v} = \dfrac{1}{\sqrt{14}}\underline{i} - \dfrac{2}{\sqrt{14}}\underline{j} + \dfrac{3}{\sqrt{14}}\underline{k}$

Let $-5\hat{v}$ is the required vector

$-5\hat{v} = -5\left(\dfrac{1}{\sqrt{14}}\underline{i} - \dfrac{2}{\sqrt{14}}\underline{j} + \dfrac{3}{\sqrt{14}}\underline{k}\right)$

$= \dfrac{-5}{\sqrt{14}}\underline{i} + \dfrac{10}{\sqrt{14}}\underline{j} - \dfrac{15}{\sqrt{14}}\underline{k}$

d) Find a and b so that that the vectors $3\underline{i} - \underline{j} + 4\underline{k}$ and $a\underline{i} + b\underline{j} - 2\underline{k}$ are parallel.

let $\underline{u} = 3\underline{i} - \underline{j} + 4\underline{k}$

$\underline{v} = a\underline{i} + b\underline{j} - 2\underline{k}$

$\underline{u}$ and $\underline{v}$ are parallel (given)

$\underline{u} = c\underline{v}$

$3\underline{i} - \underline{j} + 4\underline{k} = c(a\underline{i} + b\underline{j} - 2\underline{k})$

$3\underline{i} - \underline{j} + 4\underline{k} = ac\underline{i} + bc\underline{j} - 2c\underline{k}$

By Comparing

$ac = 3$ — ①

$bc = -1$ — ②

$-2c = 4$

$c = -\dfrac{4}{2}$

$c = -2$

Put in ①

$a(-2) = 3$

$\boxed{a = -\dfrac{3}{2}}$

Put in ②

$b(-2) = -1$

$\boxed{b = \dfrac{1}{2}}$

Question no 11: Find the direction cosines for the given vector:

i) $\underline{v} = 3\underline{i} - \underline{j} + 2\underline{k}$

$|\underline{v}| = \sqrt{(3)^2 + (-1)^2 + (2)^2}$

$= \sqrt{9 + 1 + 4}$

$= \sqrt{14}$

$\hat{v} = \dfrac{3\underline{i} - \underline{j} + 2\underline{k}}{\sqrt{14}} \qquad \therefore \hat{v} = \dfrac{\underline{v}}{|\underline{v}|}$

$\hat{v} = \dfrac{3}{\sqrt{14}}\underline{i} - \dfrac{1}{\sqrt{14}}\underline{j} + \dfrac{2}{\sqrt{14}}\underline{k}$

$\cos\alpha = \dfrac{3}{\sqrt{14}}$; $\cos\beta = \dfrac{-1}{\sqrt{14}}$; $\cos\gamma = \dfrac{2}{\sqrt{14}}$

ii) $6\underline{i} - 2\underline{j} + \underline{k}$

let $\underline{v} = 6\underline{i} - 2\underline{j} + \underline{k}$

$|\underline{v}| = \sqrt{(6)^2 + (-2)^2 + (1)^2}$

$= \sqrt{36 + 4 + 1}$

$= \sqrt{41}$

$\hat{v} = \dfrac{6\underline{i} - 2\underline{j} + \underline{k}}{\sqrt{41}} \qquad \therefore \hat{v} = \dfrac{\underline{v}}{|\underline{v}|}$

$\hat{v} = \dfrac{6}{\sqrt{41}}\underline{i} - \dfrac{2}{\sqrt{41}}\underline{j} + \dfrac{1}{\sqrt{41}}\underline{k}$

$\cos\alpha = \dfrac{6}{\sqrt{41}}$; $\cos\beta = \dfrac{-2}{\sqrt{41}}$; $\cos\gamma = \dfrac{1}{\sqrt{41}}$

iii) $\overrightarrow{PQ}$, where $P = (2,1,5)$ and $Q = (1,3,1$

$P = \overrightarrow{OP} = 2\underline{i} + \underline{j} + 5\underline{k}$

$Q = \overrightarrow{OQ} = \underline{i} + 3\underline{j} + \underline{k}$

$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$

$= \underline{i} + 3\underline{j} + \underline{k} - (2\underline{i} + \underline{j} + 5\underline{k})$

$= \underline{i} + 3\underline{j} + \underline{k} - 2\underline{i} - \underline{j} - 5\underline{k}$

$\overrightarrow{PQ} = -\underline{i} + 2\underline{j} - 4\underline{k}$

$|\overrightarrow{PQ}| = \sqrt{(-1)^2+(2)^2+(-4)^2}$

$= \sqrt{1+4+16}$

$= \sqrt{21}$

$\hat{PQ} = \dfrac{-\underline{i}+2\underline{j}-4\underline{k}}{\sqrt{21}}$

$= \dfrac{-1}{\sqrt{21}}\underline{i} + \dfrac{2}{\sqrt{21}}\underline{j} - \dfrac{4}{\sqrt{21}}\underline{k}$

$\cos\alpha = \dfrac{-1}{\sqrt{21}}$; $\cos\beta = \dfrac{2}{\sqrt{21}}$; $\cos\gamma = \dfrac{-4}{\sqrt{21}}$

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Question no 12: Which of the fallowing triples can be the direction angles of a single vector.

i) 45°, 45°, 60°

$\therefore \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

$= (\cos 45°)^2 + (\cos 45°)^2 + (\cos 60°)^2$

$= \left(\dfrac{1}{\sqrt{2}}\right)^2 + \left(\dfrac{1}{\sqrt{2}}\right)^2 + \left(\dfrac{1}{2}\right)^2$

$= \dfrac{1}{2} + \dfrac{1}{2} + \dfrac{1}{4}$

$= \dfrac{2+2+1}{4}$

$= \dfrac{5}{4} \neq 1$

So the triples can not be direction angles of a single vector

ii) 30°, 45°, 60°

$\therefore \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

$= (\cos 30°)^2 + (\cos 45°)^2 + (\cos 60°)^2$

$= \left(\dfrac{\sqrt{3}}{2}\right)^2 + \left(\dfrac{1}{\sqrt{2}}\right)^2 + \left(\dfrac{1}{2}\right)^2$

$= \dfrac{3}{4} + \dfrac{1}{2} + \dfrac{1}{4}$

$= \dfrac{3+2+1}{4}$

$= \dfrac{6}{4}$

$= \dfrac{3}{2} \neq 1$

So the triples can not be the direction angles of a single vector

iii) 45°, 60°, 60°

$\therefore \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

$= (\cos 45°)^2 + (\cos 60°)^2 + (\cos 60°)^2$

$= \left(\dfrac{1}{\sqrt{2}}\right)^2 + \left(\dfrac{1}{2}\right)^2 + \left(\dfrac{1}{2}\right)^2$

$= \dfrac{1}{2} + \dfrac{1}{4} + \dfrac{1}{4}$

$= \dfrac{2+1+1}{4}$

$= \dfrac{4}{4}$

$= 1$

So the triples are the direction angles of a single vector

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Theory:

The Scalar Product of two Vectors.

The concept of angle between two vectors is expressed in terms of a scalar product of two vectors

Definition 1: Let two non-zero vectors $\underline{u}$ and $\underline{v}$, in the plane or in space, have same initial point. The dot product of $\underline{u}$ and $\underline{v}$, written as $\underline{u}\cdot\underline{v}$, is defined by

$$\underline{u}\cdot\underline{v} = |\underline{u}||\underline{v}|\cos\theta$$

where θ is the angle between $\underline{u}$ and $\underline{v}$ and $0\le\theta\le\pi$

Definition 2:

a) If $\underline{u}=a_1\underline{i}+b_1\underline{j}$ and $\underline{v}=a_2\underline{i}+b_2\underline{j}$ are two non-zero vectors in the plane then $\underline{u}\cdot\underline{v}$ is defined as $\underline{u}\cdot\underline{v} = a_1a_2+b_1b_2$

b) If $\underline{u}=a_1\underline{i}+b_1\underline{j}+c_1\underline{k}$ and $\underline{v}=a_2\underline{i}+b_2\underline{j}+c_2\underline{k}$ are two non-zero vectors in the space then $\underline{u}\cdot\underline{v}$ is defined as $\underline{u}\cdot\underline{v} = a_1a_2+b_1b_2+c_1c_2$

Note:

> The dot product is also reffered to the scalar product or the inner product

Deduction of the important Results:

a) $\underline{i}\cdot\underline{i} = |\underline{i}||\underline{i}|\cos 0° = 1$

$\underline{j}\cdot\underline{j} = |\underline{j}||\underline{j}|\cos 0° = 1$

$\underline{k}\cdot\underline{k} = |\underline{k}||\underline{k}|\cos 0° = 1$

b) $\underline{i}\cdot\underline{j} = |\underline{i}||\underline{j}|\cos 90° = 0$

$\underline{j}\cdot\underline{k} = |\underline{j}||\underline{k}|\cos 90° = 0$

$\underline{k}\cdot\underline{i} = |\underline{k}||\underline{i}|\cos 90° = 0$

c) $\underline{u}\cdot\underline{v} = |\underline{u}||\underline{v}|\cos\theta$

$= |\underline{v}||\underline{u}|\cos(-\theta)$

$= |\underline{v}||\underline{u}|\cos\theta$

$\Rightarrow \underline{u}\cdot\underline{v} = \underline{v}\cdot\underline{u}$

Properties of Dot Product:

Let $\underline{u}$, $\underline{v}$ and $\underline{w}$ be vectors and let c be the real number, then

i) $\underline{u}\cdot\underline{v} = 0 \Rightarrow \underline{u} = 0\ ;\ \underline{v} = 0$

ii) $\underline{u}\cdot\underline{v} = \underline{v}\cdot\underline{u}$ (commutative property)

iii) $\underline{u}\cdot(\underline{v}+\underline{w}) = \underline{u}\cdot\underline{v}+\underline{u}\cdot\underline{w}$ (distributive property)

iv) $(c\underline{u})\cdot\underline{v} = c(\underline{u}\cdot\underline{v})$ (c is scaler)

v) $\underline{u}\cdot\underline{u} = |\underline{u}|^2$

• Perpendicular (Orthogonal) Vectors:

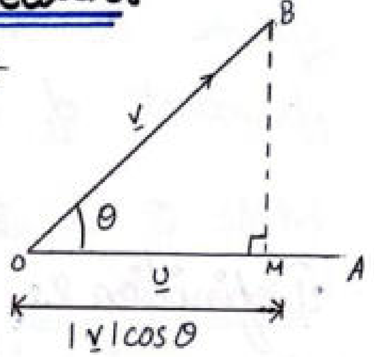
Definition: Two non-zero vectors $\underline{u}$ and $\underline{v}$ are perpendicular if and only if $\underline{u} \circ \underline{v} = 0$.

Since angle between $\underline{u}$ and $\underline{v}$ is $\frac{\pi}{2}$ and $\cos\frac{\pi}{2} = 0$.

So $\underline{u} \circ \underline{v} = |\underline{u}||\underline{v}|\cos\frac{\pi}{2}$

$\underline{u} \circ \underline{v} = 0$

• Projection of one vector upon another vector:



Let $\overline{OA} = \underline{u}$ and $\overline{OB} = \underline{v}$ be two vectors such that θ is the angle between them

$0 \le \theta \le \pi$.

Draw a perpendicular $\overline{BM}$ on $\overline{OA}$ ($\overline{BM} \perp \overline{OA}$)

Then $\overline{OM}$ is called projection of $\underline{v}$ along $\underline{u}$

- Projection of $\underline{v}$ along $\underline{u}$ = $\dfrac{\underline{u} \circ \underline{v}}{|\underline{u}|}$
- Projection of $\underline{u}$ along $\underline{v}$ = $\dfrac{\underline{u} \circ \underline{v}}{|\underline{v}|}$

• Angle between two vectors:

The angle between two vectors $\underline{u}$ and $\underline{v}$ is determined from the definition of dot product that is

$$\underline{u} \circ \underline{v} = |\underline{u}||\underline{v}|\cos\theta \Rightarrow \cos\theta = \frac{\underline{u} \circ \underline{v}}{|\underline{u}||\underline{v}|}$$

Corollaries:

i) If $\theta = 0$ or π, vectors $\underline{u}$ and $\underline{v}$ are collinear

ii) If $\theta = \frac{\pi}{2}$, then $\underline{u} \circ \underline{v} = 0$. Vectors $\underline{u}$ and $\underline{v}$ are perpendicular

Example no 1: i) If $\underline{v} = [x_1, y_1]$ and $\underline{w} = [x_2, y_2]$ are two vectors ~~d~~ in the plane, then find $\underline{v} \cdot \underline{w}$

$$\underline{v} \circ \underline{w} = (x_1\underline{i} + y_1\underline{j}) \circ (x_2\underline{i} + y_2\underline{j})$$

$$= x_1 x_2 + y_1 y_2$$

ii) If $\underline{v}$ and $\underline{w}$ are two non-zero vectors in the plane then $\underline{v} \circ \underline{w} = |\underline{v}||\underline{w}|\cos\theta$. where $0 \le \theta \le \pi$.

Example no 2:

If $\underline{u} = 3\underline{i} - \underline{j} - 2\underline{k}$; $\underline{v} = \underline{i} + 2\underline{j} - \underline{k}$ then find $\underline{v} \cdot \underline{u}$

$$\underline{v} \cdot \underline{u} = (\underline{i} + 2\underline{j} - \underline{k}) \cdot (3\underline{i} - \underline{j} - 2\underline{k})$$
$$= (1)(3) + (2)(-1) + (-1)(-2)$$
$$= 3 - 2 + 2$$
$$\boxed{\underline{v} \cdot \underline{u} = 3}$$

Example no 3:

If $\underline{u} = 2\underline{i} - 4\underline{j} + 5\underline{k}$; $\underline{v} = 4\underline{i} - 3\underline{j} - 4\underline{k}$ then find $\underline{u} \cdot \underline{v}$

$$\underline{u} \cdot \underline{v} = (2\underline{i} - 4\underline{j} + 5\underline{k}) \cdot (4\underline{i} - 3\underline{j} - 4\underline{k})$$
$$= (2)(4) + (-4)(-3) + (5)(-4)$$
$$= 8 + 12 - 20$$
$$\boxed{\underline{u} \cdot \underline{v} = 0}$$

Example no 4:

Find the angle between the vectors $\underline{u} = 2\underline{i} - \underline{j} + \underline{k}$; $\underline{v} = -\underline{i} + \underline{j}$

$$\therefore \cos\theta = \frac{\underline{u} \cdot \underline{v}}{|\underline{u}||\underline{v}|}$$

$$\underline{u} \cdot \underline{v} = (2\underline{i} - \underline{j} + \underline{k}) \cdot (-\underline{i} + \underline{j})$$
$$= (2)(-1) + (-1)(1) + (1)(0)$$
$$= -2 - 1 + 0$$
$$= -3$$

$$|\underline{u}| = \sqrt{(2)^2 + (-1)^2 + (1)^2}$$
$$= \sqrt{4 + 1 + 1}$$
$$= \sqrt{6}$$

$$|\underline{v}| = \sqrt{(-1)^2 + (1)^2}$$
$$= \sqrt{1 + 1}$$
$$= \sqrt{2}$$

Now $\cos\theta = \dfrac{-3}{\sqrt{6}\sqrt{2}}$

$$\cos\theta = -\frac{\sqrt{3}}{2}$$

$$\theta = \frac{5\pi}{6}$$

Example no 5:

Find a scalar α so that the vectors $2\underline{i} + \alpha\underline{j} + 5\underline{k}$ and $3\underline{i} + \underline{j} + \alpha\underline{k}$ are perpendicular

Let $\underline{u} = 2\underline{i} + \alpha\underline{j} + 5\underline{k}$

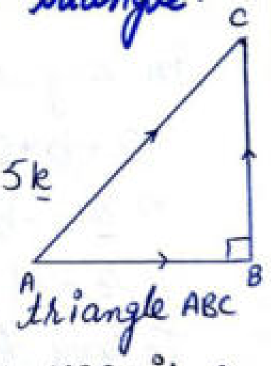
$\underline{v} = 3\underline{i} + \underline{j} + \alpha\underline{k}$

It is given that $\underline{u}$ and $\underline{v}$ are perpendicular

$$\therefore \underline{u} \cdot \underline{v} = 0$$
$$(2\underline{i} + \alpha\underline{j} + 5\underline{k}) \cdot (3\underline{i} + \underline{j} + \alpha\underline{k}) = 0$$
$$(2)(3) + (\alpha)(1) + (5)(\alpha) = 0.$$
$$6 + \alpha + 5\alpha = 0.$$
$$6 + 6\alpha = 0$$
$$6\alpha = -6$$
$$\boxed{\alpha = -1}$$

Example no 6:

Show that the vectors $2\underline{i} - \underline{j} + \underline{k}$, $\underline{i} - 3\underline{j} - 5\underline{k}$ and $3\underline{i} - 4\underline{j} - 4\underline{k}$ form the sides of a right triangle.



Let $\overrightarrow{AB} = 2\underline{i} - \underline{j} + \underline{k}$

$\overrightarrow{BC} = \underline{i} - 3\underline{j} - 5\underline{k}$

Now $\overrightarrow{AB} + \overrightarrow{BC} = 2\underline{i} - \underline{j} + \underline{k} + \underline{i} - 3\underline{j} - 5\underline{k}$

$= 3\underline{i} - 4\underline{j} - 4\underline{k} = \overrightarrow{AC}$

$\therefore \overrightarrow{AB}, \overrightarrow{BC}$ and $\overrightarrow{AC}$ form a triangle ABC

Further we prove that ΔABC is a right triangle.

$$\overrightarrow{AB} \cdot \overrightarrow{BC} = (2\underline{i} - \underline{j} + \underline{k}) \cdot (\underline{i} - 3\underline{j} - 5\underline{k}).$$
$$= (2)(1) + (-1)(-3) + (1)(-5).$$
$$= 2 + 3 - 5$$
$$= 0$$
$$\therefore \overrightarrow{AB} \perp \overrightarrow{BC}.$$

Hence ΔABC is the right triangle

Example no 7: Show that the components of a vector are the projections of that vector along $\underline{i}$. $\underline{j}$, $\underline{k}$ respectively.

Let $\underline{v} = a\underline{i} + b\underline{j} + c\underline{k}$

Projection of $\underline{v}$ along $\underline{i} = \frac{\underline{v}\cdot\underline{i}}{|\underline{i}|} = (a\underline{i}+b\underline{j}+c\underline{k})\cdot\underline{i} = a$

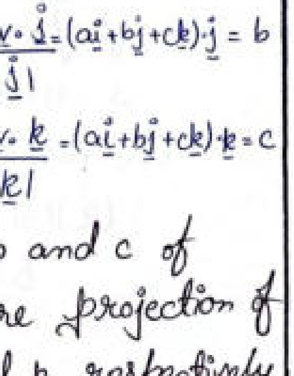
Projection of $\underline{v}$ along $\hat{\underline{j}} = \frac{\underline{v}\cdot\hat{\underline{j}}}{|\underline{j}|} = (a\underline{i}+b\underline{j}+c\underline{k})\cdot\underline{j} = b$

Projection of $\underline{v}$ along $\underline{k} = \frac{\underline{v}\cdot\underline{k}}{|\underline{k}|} = (a\underline{i}+b\underline{j}+c\underline{k})\cdot\underline{k} = c$

Hence components a, b and c of vector $\underline{v} = a\underline{i} + b\underline{j} + c\underline{k}$ are projection of vector along $\underline{i}$, $\hat{\underline{j}}$ and $\underline{k}$ respectively.

Example no 8: Prove that in any triangle ABC

i) $a^2 = b^2 + c^2 - 2bc\cos A$



Let $\overrightarrow{BC} = \underline{a}$, $\overrightarrow{CA} = \underline{b}$, $\overrightarrow{AB} = \underline{c}$ be sides of ΔABC.

In any triangle

$\underline{a} + \underline{b} + \underline{c} = 0$.

$\Rightarrow \underline{a} = -\underline{b} - \underline{c}$

$\underline{a} = -(\underline{b} + \underline{c})$.

Taking dot product with '$\underline{a}$'

$\underline{a}\cdot\underline{a} = -(\underline{b}+\underline{c})\cdot\underline{a}$

$a^2 = [-(\underline{b}+\underline{c})]\cdot[-(\underline{b}+\underline{c})]$

$a^2 = (\underline{b}+\underline{c})(\underline{b}+c)$.

$a^2 = \underline{b}\cdot\underline{b} + \underline{b}\cdot\underline{c} + \underline{c}\cdot\underline{b} + \underline{c}\cdot\underline{c}$

$a^2 = b^2 + \underline{b}\cdot\underline{c} + \underline{b}\cdot\underline{c} + c^2 \because \underline{c}\cdot\underline{b} = \underline{b}\cdot\underline{c}$

$a^2 = b^2 + c^2 + 2(\underline{b}\cdot\underline{c})$

According to the definition of

$a^2 = b^2 + c^2 + 2|\underline{b}||\underline{c}|\cos(\pi - A)$

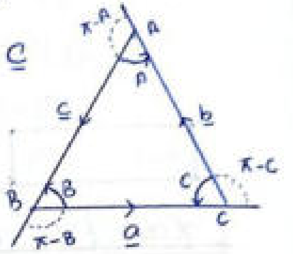
$a^2 = b^2 + c^2 + 2bc(-\cos A)$

$\because \cos(\pi - \theta) = -\cos\theta$

$a^2 \pm b^2 + c^2 - 2bc\cos A$

Hence proved

ii) $a = b\cos C + c\cos B$.



Let $\overrightarrow{BC} = \underline{a}$, $\overrightarrow{CA} = \underline{b}$, $\overrightarrow{AB} = \underline{c}$ be sides of ΔABC

In any triangle

$\underline{a} + \underline{b} + \underline{c} = 0$.

$\underline{a} = -\underline{b} - \underline{c}$

Taking dot product with '$\underline{a}$'

$\underline{a}\cdot\underline{a} = (-\underline{b} - \underline{c})\cdot\underline{a}$

$a^2 = -\underline{a}\cdot\underline{b} - \underline{a}\cdot\underline{c}$

According to the definition of dot product

$a^2 = -|\underline{a}||\underline{b}|\cos(\pi - C) - |\underline{a}||\underline{c}|\cos(\pi - B)$

$\because$ Angle from $\underline{a}$ to $\underline{b}$ is $\pi - C$

$\because$ Angle from $\underline{a}$ to $\underline{c}$ is $\pi - B$

$a^2 = -ab(\cos C) - ac(-\cos B)$

$a^2 = ab\cos C + ac\cos B$

Dividing by a on both sides

$a = b\cos C + c\cos B$.

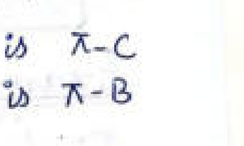
Hence proved.

Example no 9: Prove that:

$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

Suppose $\overrightarrow{OA}$ and $\overrightarrow{OB}$ are unit vector

So $\angle AOB = \alpha - \beta$



$|\overrightarrow{OA}| = |\overrightarrow{OB}| = 1$ As

$\overrightarrow{OA} = \cos\alpha\,\hat{\underline{i}} + \sin\alpha\,\hat{\underline{j}}$

$\overrightarrow{OB} = \cos\beta\,\hat{\underline{i}} + \sin\beta\,\hat{\underline{j}}$.

Now $\overrightarrow{OA}\cdot\overrightarrow{OB} = (\cos\alpha\,\hat{\underline{i}} + \sin\alpha\,\hat{\underline{j}})\cdot(\cos\beta\,\hat{\underline{i}} + \sin\beta\,\hat{\underline{j}})$

$\Rightarrow |\overrightarrow{OA}||\overrightarrow{OB}|\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$.

$(1)(1)\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$.

$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$.

Hence proved.

Exercise # 7.3

Question no 1: Find the cosine of the angle θ between $\underline{u}$ and $\underline{v}$:

i) $\underline{u} = 3\underline{i} + \underline{j} - \underline{k}$; $\underline{v} = 2\underline{i} - \underline{j} + \underline{k}$

$$\therefore \cos\theta = \frac{\underline{u}\cdot\underline{v}}{|\underline{u}||\underline{v}|}$$

$$|\underline{u}| = \sqrt{(3)^2 + (1)^2 + (-1)^2}$$

$$= \sqrt{9+1+1}$$

$$= \sqrt{11}$$

$$|\underline{v}| = \sqrt{(2)^2 + (-1)^2 + (1)^2}$$

$$= \sqrt{4+1+1}$$

$$= \sqrt{6}$$

$$\cos\theta = \frac{(3\underline{i} + \underline{j} - \underline{k})\cdot(2\underline{i} - \underline{j} + \underline{k})}{\sqrt{11}\sqrt{6}}$$

$$\cos\theta = \frac{6-1-1}{\sqrt{66}}$$

$$\boxed{\cos\theta = \frac{4}{\sqrt{66}}}$$

ii) $\underline{u} = \underline{i} - 3\underline{j} + 4\underline{k}$; $\underline{v} = 4\underline{i} - \underline{j} + 3\underline{k}$

$$\therefore \cos\theta = \frac{\underline{u}\cdot\underline{v}}{|\underline{u}||\underline{v}|}$$

$$|\underline{u}| = \sqrt{(1)^2 + (-3)^2 + (4)^2}$$

$$= \sqrt{1+9+16}$$

$$= \sqrt{26}$$

$$|\underline{v}| = \sqrt{(4)^2 + (-1)^2 + (3)^2}$$

$$= \sqrt{16+1+9}$$

$$= \sqrt{26}$$

$$\cos\theta = \frac{(\underline{i} - 3\underline{j} + 4\underline{k})\cdot(4\underline{i} - \underline{j} + 3\underline{k})}{\sqrt{26}\sqrt{26}}$$

$$\cos\theta = \frac{4+3+12}{(\sqrt{26})^2}$$

$$\boxed{\cos\theta = \frac{19}{26}}$$

iii) $\underline{u} = [-3, 5]$, $\underline{v} = [6, -2]$

$$\therefore \cos\theta = \frac{\underline{u}\cdot\underline{v}}{|\underline{u}||\underline{v}|}$$

$\underline{u} = -3\underline{i} + 5\underline{j}$; $\underline{v} = 6\underline{i} - 2\underline{j}$

$$|\underline{u}| = \sqrt{(-3)^2 + (5)^2}$$

$$= \sqrt{9+25}$$

$$= \sqrt{34}$$

$$|\underline{v}| = \sqrt{(6)^2 + (-2)^2}$$

$$= \sqrt{36+4}$$

$$= \sqrt{40}$$

$$\cos\theta = \frac{(-3\underline{i} + 5\underline{j})\cdot(6\underline{i} - 2\underline{j})}{\sqrt{34}\sqrt{40}}$$

$$\cos\theta = \frac{-18-10}{\sqrt{1360}}$$

$$\cos\theta = \frac{-28}{4\sqrt{85}}$$

$$\boxed{\cos\theta = \frac{-7}{\sqrt{85}}}$$

iv) $\underline{u} = [2, -3, 1]$; $\underline{v} = [2, 4, 1]$

$\underline{u} = 2\underline{i} - 3\underline{j} + \underline{k}$; $\underline{v} = 2\underline{i} + 4\underline{j} + \underline{k}$

$$|\underline{u}| = \sqrt{(2)^2 + (-3)^2 + (1)^2}$$

$$= \sqrt{4+9+1}$$

$$= \sqrt{14}$$

$$|\underline{v}| = \sqrt{(2)^2 + (4)^2 + (1)^2}$$

$$= \sqrt{4+16+1}$$

$$= \sqrt{21}$$

$$\cos\theta = \frac{(2\underline{i} - 3\underline{j} + \underline{k})\cdot(2\underline{i} + 4\underline{j} + \underline{k})}{\sqrt{14}\sqrt{21}}$$

$$\cos\theta = \frac{4-12+1}{\sqrt{294}} = \frac{-7}{7\sqrt{6}}$$

$$\boxed{\cos\theta = \frac{-1}{\sqrt{6}}}$$

Question no 2: Calculate the projection of $\underline{a}$ along $\underline{b}$ and projection of $\underline{b}$ along $\underline{a}$ when:

i) $\underline{a} = \underline{i} - \underline{k}$; $\underline{b} = \underline{j} + \underline{k}$

Projection of a along $\underline{b} = \dfrac{\underline{a} \circ \underline{b}}{|\underline{b}|}$

$$|\underline{b}| = \sqrt{(1)^2 + (1)^2} = \sqrt{1+1} = \sqrt{2}$$

$$= \frac{(\underline{i} - \underline{k}) \cdot (\underline{j} + \underline{k})}{\sqrt{2}} = \frac{0+0-1}{\sqrt{2}}$$

$$= \frac{-1}{\sqrt{2}}$$

Projection of $\underline{b}$ along $\underline{a} = \dfrac{\underline{a} \circ \underline{b}}{|\underline{a}|}$

$$|\underline{a}| = \sqrt{(1)^2 + (1)^2} = \sqrt{1+1} = \sqrt{2}$$

$$= \frac{(\underline{i} - \underline{k}) \cdot (\underline{j} + \underline{k})}{\sqrt{2}} = \frac{0+0-1}{\sqrt{2}}$$

$$= \frac{-1}{\sqrt{2}}.$$

ii) $\underline{a} = 3\underline{i} + \underline{j} - \underline{k}$; $\underline{b} = -2\underline{i} - \underline{j} + \underline{k}$

Projection of a along $\underline{b} = \dfrac{\underline{a} \circ \underline{b}}{|\underline{b}|}$

$$|\underline{b}| = \sqrt{(-2)^2 + (-1)^2 + (1)^2}$$

$$= \sqrt{4+1+1}$$

$$= \sqrt{6}$$

$$= \frac{(3\underline{i} + \underline{j} - \underline{k}) \circ (-2\underline{i} - \underline{j} + \underline{k})}{\sqrt{6}}$$

$$= \frac{-6-1-1}{\sqrt{6}} = \frac{-8}{\sqrt{6}}$$

Projection of b along $\underline{a} = \dfrac{\underline{a} \circ \underline{b}}{|\underline{a}|}$

$$|\underline{a}| = \sqrt{(3)^2 + (1)^2 + (-1)^2}$$

$$= \sqrt{9+1+1}$$

$$= \sqrt{11}$$

$$= \frac{(3\underline{i} + \underline{j} - \underline{k}) \cdot (-2\underline{i} - \underline{j} + \underline{k})}{\sqrt{11}}$$

$$= \frac{-6-1-1}{\sqrt{11}} = \frac{-8}{\sqrt{11}}.$$

Question no 3: Find a real number α so that the vectors $\underline{u}$ and $\underline{v}$ are perpendicular.

i) $\underline{u} = 2\alpha\underline{i} + \underline{j} - \underline{k}$; $\underline{v} = \underline{i} + \alpha\underline{j} + 4\underline{k}$

$\underline{u} \perp \underline{v}$ (given)

So $\underline{u} \circ \underline{v} = 0$

$$(2\alpha\underline{i} + \underline{j} - \underline{k}) \cdot (\underline{i} + \alpha\underline{j} + 4\underline{k}) = 0$$

$$2\alpha + \alpha - 4 = 0$$

$$3\alpha = 4$$

$$\boxed{\alpha = \frac{4}{3}}$$

ii) $\underline{u} = \alpha\underline{i} + 2\alpha\underline{j} - \underline{k}$; $\underline{v} = \underline{i} + \alpha\underline{j} + 3\underline{k}$

$\underline{u} \perp \underline{v}$ (given)

So $\underline{u} \circ \underline{v} = 0$

$$(\alpha\underline{i} + 2\alpha\underline{j} - \underline{k}) \cdot (\underline{i} + \alpha\underline{j} + 3\underline{k}) = 0$$

$$\alpha + 2\alpha^2 - 3 = 0$$

$$2\alpha^2 + \alpha - 3 = 0$$

$$2\alpha^2 - 2\alpha + 3\alpha - 3 = 0$$

$$2\alpha(\alpha - 1) + 3(\alpha - 1) = 0$$

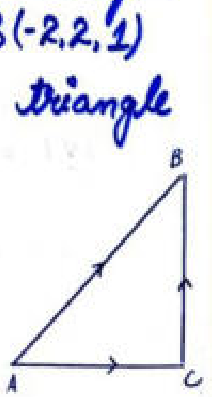
$$(\alpha - 1)(2\alpha + 3) = 0$$

$$\alpha - 1 = 0 \ ; \ 2\alpha + 3 = 0$$

$$\alpha = 1 \ ; \ \alpha = -\frac{3}{2}$$

$$\boxed{\alpha = 1 ; -\frac{3}{2}}$$

Question no 4: Find the number z so that the triangle with vertices A(1,-1,0); B(-2,2,1) and C(0,2,z) is a right triangle with right angle at C.



Given vectors

$A(1,-1,0)$; $\overrightarrow{OA} = \underline{i} - \underline{j} + 0\underline{k}$

$B(-2,2,1)$; $\overrightarrow{OB} = 2\underline{i} + 2\underline{j} + \underline{k}$

$C(0,2,z)$; $\overrightarrow{OC} = 0\underline{i} + 2\underline{j} + z\underline{k}$

$$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA}$$

$$= (0\underline{i} + 2\underline{j} + z\underline{k}) - (\underline{i} - \underline{j} + 0\underline{k}).$$

$= 0\underline{i} + 2\underline{j} + z\underline{k} - \underline{i} + \underline{j} - 0\underline{k}$

$\vec{AC} = -\underline{i} + 3\underline{j} + z\underline{k}$

$\vec{CB} = \vec{OB} - \vec{OC}$

$= -2\underline{i} + 2\underline{j} + \underline{k} - (0\underline{i} + 2\underline{j} + z\underline{k})$

$= -2\underline{i} + 2\underline{j} + \underline{k} - 0\underline{i} - 2\underline{j} - z\underline{k}$

$= -2\underline{i} + 0\underline{j} + (1 - z)\underline{k}.$

From triangle

$\vec{CB} \perp \vec{AC}$

than

$\vec{AC} \cdot \vec{CB} = 0.$

$(-\underline{i} + 3\underline{j} + z\underline{k}) \cdot (-2\underline{i} + 0\underline{j} + (1-z)\underline{k}) = 0$

$(-1)(-2) + (3)(0) + (z)(1 - z) = 0$

$2 + 0 + z - z^2 = 0.$

$-z^2 + z + 2 = 0.$

$-(z^2 - z - 2) = 0.$

$z^2 - z - 2 = 0$

$z^2 + z - 2z - 2 = 0$

$z(z + 1) - 2(z + 1) = 0.$

$z + 1 = 0 \ ; \ z - 2 = 0$

$z = -1 \ ; \ z = 2$

$\boxed{z = -1, 2}$

Question no 5: If $\underline{v}$ is a vector for which

$\underline{v} \cdot \underline{i} = 0 \ ; \ \underline{v} \cdot \underline{j} = 0 \ ; \ \underline{v} \cdot \underline{k} = 0.$ find $\underline{v}$.

Let $\underline{v} = a\underline{i} + b\underline{j} + c\underline{k}$ — ①

$\underline{v} \cdot \underline{i} = (a\underline{i} + b\underline{j} + c\underline{k}) \cdot \underline{i} = 0 \Rightarrow a = 0$

$\underline{v} \cdot \underline{j} = (a\underline{i} + b\underline{j} + c\underline{k}) \cdot \underline{j} = 0 \Rightarrow b = 0$

$\underline{v} \cdot \underline{k} = (a\underline{i} + b\underline{j} + c\underline{k}) \cdot \underline{k} = 0 \Rightarrow c = 0$

Put a, b, c in ①

$\underline{v} = 0\underline{i} + 0\underline{j} + 0\underline{k}$

$\boxed{\underline{v} = \vec{O}}$

Null vector.

Question no 6:

i) Show that the vectors $3\underline{i} - 2\underline{j} + \underline{k}$; $\underline{i} - 3\underline{j} + 5\underline{k}$ and $2\underline{i} + \underline{j} - 4\underline{k}$ form a right triangle.

Let $\underline{u} = 3\underline{i} - 2\underline{j} + \underline{k}$

$\underline{v} = \underline{i} - 3\underline{j} + 5\underline{k}$

$\underline{w} = 2\underline{i} + \underline{j} - 4\underline{k}$

First we show that $\underline{u}, \underline{v}, \underline{w}$ are the sides of triangle.

$\underline{v} + \underline{w} = \underline{i} - 3\underline{j} + 5\underline{k} + 2\underline{i} + \underline{j} - 4\underline{k}$

$= 3\underline{i} - 2\underline{j} + \underline{k}.$

$\underline{v} + \underline{w} = \underline{u}.$

Hence $\underline{u}, \underline{v}, \underline{w}$ are the sides of a triangle.

$\underline{u} \cdot \underline{w} = (3\underline{i} - 2\underline{j} + \underline{k}) \cdot (2\underline{i} + \underline{j} - 4\underline{k}).$

$= (3)(2) + (-2)(1) + (1)(-4)$

$= 6 - 2 - 4$

$= 6 - 6$

$\underline{u} \cdot \underline{w} = 0 \quad \Rightarrow \underline{u} \perp \underline{w}$

Hence $\underline{u}, \underline{v}$ and $\underline{w}$ are the sides of right triangle.

ii) Show that the set of Points $P = (1,3,2)$, $Q(4,1,4)$ and $R = (6,5,5)$ form a right triangle.

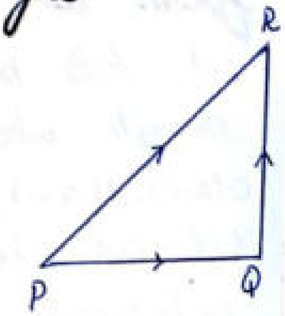
Given that

$P(1,3,2) \ ; \ \vec{OP} = \underline{i} + 3\underline{j} + 2\underline{k}$

$Q(4,1,4) \ ; \ \vec{OQ} = 4\underline{i} + \underline{j} + 4\underline{k}$

$R(6,5,5) \ ; \ \vec{OR} = 6\underline{i} + 5\underline{j} + 5\underline{k}$

Now we find $\vec{PQ}$, $\vec{QR}$ and $\vec{PR}$



$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$

$= 4\underline{i} + \underline{j} + 4\underline{k} - (\underline{i} + 3\underline{j} + 2\underline{k})$

$= 4\underline{i} + \underline{j} + 4\underline{k} - \underline{i} - 3\underline{j} - 2\underline{k}$

$= 3\underline{i} - 2\underline{j} + 2\underline{k}$

$\overrightarrow{QR} = \overrightarrow{OR} - \overrightarrow{OQ}$

$= 6\underline{i} + 5\underline{j} + 5\underline{k} - (4\underline{i} + \underline{j} + 4\underline{k})$

$= 6\underline{i} + 5\underline{j} + 5\underline{k} - 4\underline{i} - \underline{j} - 4\underline{k}$

$= 2\underline{i} + 4\underline{j} + \underline{k}$

$\overrightarrow{PR} = \overrightarrow{OR} - \overrightarrow{OP}$

$= 6\underline{i} + 5\underline{j} + 5\underline{k} - (\underline{i} + 3\underline{j} + 2\underline{k})$

$= 6\underline{i} + 5\underline{j} + 5\underline{k} - \underline{i} - 3\underline{j} - 2\underline{k}$

$= 5\underline{i} + 2\underline{j} + 3\underline{k}$

$\overrightarrow{PQ} + \overrightarrow{QR} = 3\underline{i} - 2\underline{j} + 2\underline{k} + 2\underline{i} + 4\underline{j} + \underline{k}$

$= 5\underline{i} + 2\underline{j} + 3\underline{k}$

$\overrightarrow{PQ} + \overrightarrow{QR} = \overrightarrow{PR}$

Hence $\overrightarrow{PQ}$, $\overrightarrow{QR}$ and $\overrightarrow{PR}$ are the sides of triangle.

$\overrightarrow{PQ} \cdot \overrightarrow{QR} = (3\underline{i} - 2\underline{j} + 2\underline{k}) \cdot (2\underline{i} + 4\underline{j} + \underline{k})$

$= (3)(2) + (-2)(4) + (2)(1)$

$= 6 - 8 + 2$

$= 0$

$\overrightarrow{PQ} \perp \overrightarrow{QR}$

Question no 7: Show that mid point of hypotenuse a right triangle is equidistant from its vertices.

Let AOB be a right triangle where

O(0,0); A(a,0); B(0, b)

Let M be the midpoint of hypotenuse $\overrightarrow{AB}$

So coordinates of M are

$= \left(\frac{a+0}{2}; \frac{b+0}{2}\right) = \left(\frac{a}{2}, \frac{b}{2}\right)$

Now

$\overline{OM} = \left[\frac{a}{2} - 0 ; \frac{b}{2} - 0\right]$

$= \left[\frac{a}{2} \cdot \frac{b}{2}\right] = \frac{a}{2}\underline{i} + \frac{b}{2}\underline{j}$

$|\overline{OM}| = \sqrt{\left(\frac{a}{2}\right)^2 + \left(\frac{b}{2}\right)^2}$

$= \sqrt{\frac{a^2}{4} + \frac{b^2}{4}}$ —— ①

$\overline{AM} = \left[\frac{a}{2} - a ; \frac{b}{2} - 0\right]$

$= \left[\frac{a-2a}{2} ; \frac{b-0}{2}\right]$

$= \left[-\frac{a}{2} ; \frac{b}{2}\right]$

$|\overline{AM}| = \sqrt{\left(-\frac{a}{2}\right)^2 + \left(\frac{b}{2}\right)^2}$

$= \sqrt{\frac{a^2}{4} + \frac{b^2}{4}}$ —— ②

$\overline{BM} = \left[\frac{a}{2} - 0 ; \frac{b}{2} - b\right]$

$= \left[\frac{a-0}{2} ; \frac{b-2b}{2}\right]$

$= \left[\frac{a}{2} ; -\frac{b}{2}\right]$

$|\overline{BM}| = \sqrt{\left(\frac{a}{2}\right)^2 + \left(-\frac{b}{2}\right)^2}$

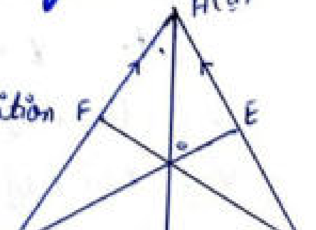
$= \sqrt{\frac{a^2}{4} + \frac{b^2}{4}}$ —— ③

from ①, ② and ③

$|\overline{OM}| = |\overline{AM}| = |\overline{BM}|$

Hence proved.

Question no 8: Prove that perpendicular bisectors of the sides of a triangle are concurrent.



Suppose $\underline{a}$, $\underline{b}$, $\underline{c}$ are position vectors of A, B, C then

Position vector of D $= \overrightarrow{OD} = \frac{\underline{b}+\underline{c}}{2}$

Position vector of E $= \overrightarrow{OE} = \frac{\underline{a}+\underline{c}}{2}$

Position vector of F : $\vec{OF} = \frac{\underline{a}+\underline{b}}{2}$

Also

$\vec{BA} = \underline{a} - \underline{b}$

$\vec{BC} = \underline{c} - \underline{b}$

$\vec{CA} = \underline{a} - \underline{c}$.

Now $\vec{OD} \perp \vec{BC}$ So $\vec{OD} \cdot \vec{BC} = 0$.

$\Rightarrow \left(\frac{\underline{b}+\underline{c}}{2}\right) \cdot (\underline{c} - \underline{b}) = 0$

$\Rightarrow (\underline{c}+\underline{b}) \cdot (\underline{c} - \underline{b}) = 0$

$c^2 - b^2 = 0$ —①

$\vec{OE} \perp \vec{CA}$ So $\vec{OE} \cdot \vec{CA} = 0$.

$\Rightarrow \left(\frac{\underline{a}+\underline{c}}{2}\right) \cdot (\underline{a} - \underline{c}) = 0$

$\Rightarrow (\underline{a}+\underline{c}) \cdot (\underline{a} - \underline{c}) = 0$

$\underline{a}^2 - \underline{c}^2 = 0$ —②

By adding ① and ②

$$\cancel{c^2} - \underline{b}^2 = 0$$
$$\underline{a}^2 - \cancel{\underline{c}^2}$$
$$\underline{a}^2 - \underline{b}^2 = 0.$$

$(\underline{a}+\underline{b})(\underline{a} - \underline{b}) = 0$.

$\left(\frac{\underline{a}+\underline{b}}{2}\right) \cdot (\underline{a} - \underline{b}) = 0$.

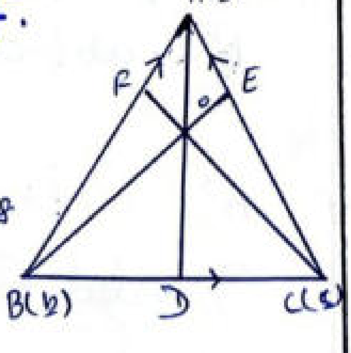
$\vec{OF} \cdot \vec{BA} = 0$.

$\vec{OF} \perp \vec{BA}$

Hence proved,

Question no 9: Prove that the altitudes of a triangle are concurrent.

Let $\underline{a}, \underline{b}, \underline{c}$ be the position vectors of A, B, C respectively



Let altitudes on $\vec{AB}$ and $\vec{BC}$ intersect at O(0,0).

$\vec{AB} = \vec{OB} - \vec{OA} = \underline{b} - \underline{a}$

$\vec{BC} = \vec{OC} - \vec{OB} = \underline{c} - \underline{b}$

$\vec{CA} = \vec{OA} - \vec{OC} = \underline{a} - \underline{c}$.

$\vec{OA} = \underline{a}$; $\vec{OB} = \underline{b}$; $\vec{OC} = \underline{c}$.

$\because \vec{AD} \perp \vec{BC}$ ($\because$ As we have no value of D)

$\vec{OA} \perp \vec{BC}$ then

$\vec{OA} \cdot \vec{BC} = 0$.

$\underline{a} \cdot (\underline{c} - \underline{b}) = 0$.

$\underline{a} \cdot \underline{c} - \underline{a} \cdot \underline{b} = 0$. —①

$\because \vec{BE} \perp \vec{CA}$

$\vec{OB} \perp \vec{CA}$ ($\because$ As we have no value of E)

then $\vec{OB} \cdot \vec{CA} = 0$.

$\underline{b} \cdot (\underline{a} - \underline{c}) = 0$

$\underline{b} \cdot \underline{a} - \underline{b} \cdot \underline{c} = 0$

$\because \underline{b} \cdot \underline{a} = \underline{a} \cdot \underline{b}$

$\underline{a} \cdot \underline{b} - \underline{b} \cdot \underline{c} = 0$. —②

By adding ① and ②

$\underline{a} \cdot \underline{c} - \cancel{\underline{a} \cdot \underline{b}} + \cancel{\underline{a} \cdot \underline{b}} - \underline{b} \cdot \underline{c} = 0$

$\underline{a} \cdot \underline{c} - \underline{b} \cdot \underline{c} = 0$.

$(\underline{a} - \underline{b}) \cdot \underline{c} = 0$

$-(\underline{b} - \underline{a}) \cdot \underline{c} = 0$

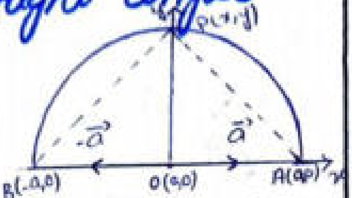
$(\underline{b} - \underline{a}) \cdot \underline{c} = 0$.

$\vec{AB} \cdot \vec{OC} = 0$.

$\vec{AB} \perp \vec{OC}$.

Hence altitudes of a triangle are concurrent.

Question no 10: Prove that the angle in a semi circle is a right angle.



Let O be the centre and $\underline{a}$ be radius of semi-circle

$\therefore x^2+y^2=a^2$ is the equation of circle. So,

$$\vec{AP} = [x-a,\ y-0]$$
$$= (x-a)\underline{i} + y\underline{j}.$$
$$\vec{BP} = [x+a,\ y-0]$$
$$= (x+a)\underline{i} + y\underline{j}$$

Now $\vec{AP}\cdot\vec{BP} = [(x-a)\underline{i}+y\underline{j}]\cdot[(x+a)\underline{i}+y\underline{j}]$

$$= (x-a)(x+a) + y^2.$$
$$= x^2 - a^2 + y^2$$
$$= x^2+y^2-a^2$$
$$= a^2-a^2 \qquad \because x^2+y^2=a^2$$
$$\vec{AP}\cdot\vec{BP} = 0$$
$$\vec{AP} \perp \vec{BP}$$

Hence angle in semicircle is a right angle.

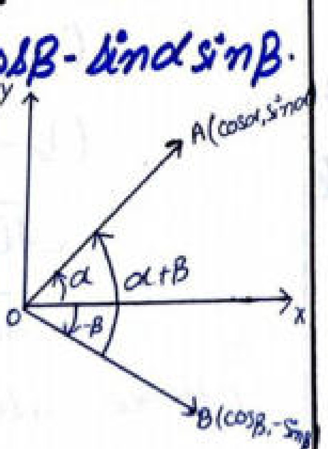
Question no 11: Prove that $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$.

Suppose $\vec{OA}$ and $\vec{OB}$ are unit vectors and.

$\angle XOA = \alpha$

$\angle XOB = -\beta$

$\angle BOA = \alpha+\beta$



Also

$|\vec{OA}| = |\vec{OB}| = 1$

As

$\vec{OA} = \cos\alpha\,\underline{i} + \sin\alpha\,\underline{j}$

$\vec{OB} = \cos(-\beta)\underline{i} + \sin(-\beta)\underline{j}$

$\vec{OB} = \cos\beta\,\underline{i} - \sin\beta\,\underline{j}$

Now.

$\vec{OA}\cdot\vec{OB} = (\cos\alpha\,\underline{i} + \sin\alpha\,\underline{j})\cdot(\cos\beta\,\underline{i} - \sin\beta\,\underline{j})$

$|\vec{OA}||\vec{OB}|\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta.$

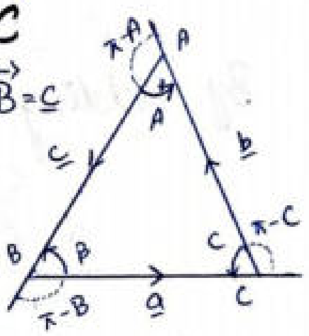
$(1)(1)\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta.$

Hence proved

Question no 12: Prove that in any triangle ΔABC.

i) $b = c\cos A + a\cos C$



Let $\vec{BC} = \underline{a}, \vec{CA} = \underline{b}, \vec{AB} = \underline{c}$ be sides of ΔABC

In any triangle

$\underline{a}+\underline{b}+\underline{c} = 0.$

$\underline{b} = -\underline{a} - \underline{c}.$

Taking dot product with '$\underline{b}$'

$\underline{b}\cdot\underline{b} = \underline{b}\cdot(-\underline{a}-\underline{c})$

$b^2 = -\underline{b}\cdot\underline{a} - \underline{b}\cdot\underline{c}$

$b^2 = -|\underline{b}||\underline{a}|\cos(\pi-C) - |\underline{b}||\underline{c}|\cos(\pi-A)$

$\because$ Angle from $\underline{b}$ to $\underline{a}$ is $\pi-C$

$\because$ Angle from b to $\underline{c}$ is $\pi-A$.

$b^2 = -ab(-\cos C) - bc(-\cos A).$

$\because \cos(\pi-\theta) = -\cos\theta$

$b^2 = ab\cos C + bc\cos A.$

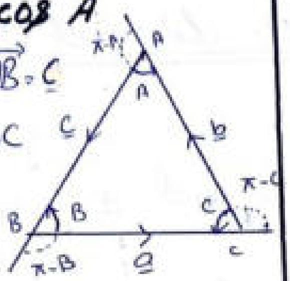
Divided by 'b' on both sides

$b = a\cos C + c\cos A.$

$b = c\cos A + a\cos C.$

Hence proved

ii) $c = a\cos B + b\cos A$



Let $\overrightarrow{BC}=\underline{a}, \overrightarrow{CA}=\underline{b}, \overrightarrow{AB}=\underline{c}$

be the sides of ΔABC

In any triangle

$\underline{a}+\underline{b}+\underline{c}=0$

$\underline{c}=-\underline{a}-\underline{b}$

Taking dot product with '$\underline{c}$'

$\underline{c}\cdot\underline{c} = \underline{c}\cdot(-\underline{a}-\underline{b})$.

$c^2 = -\underline{c}\cdot\underline{a} - \underline{c}\cdot\underline{b}$

$c^2 = -|\underline{c}||\underline{a}|\cos(\pi-B) - |\underline{c}||\underline{b}|\cos(\pi-A)$

∴ Angle from $\underline{a}$ to $\underline{c}$ is $\pi-B$.

∴ Angle from $\underline{b}$ to $\underline{c}$ is $\pi-A$.

$c^2 = -ac(-\cos B) - bc(-\cos A)$.

∴ $\cos(\pi-\theta) = -\cos\theta$

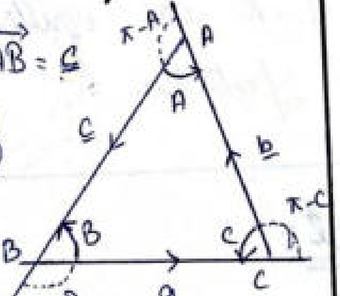
$c^2 = +ac\cos B + bc\cos A$.

Divided by 'c' on both sides

$c = a\cos B + b\cos A$.

Hence proved.

iii) $b^2 = c^2 + a^2 - 2ca\cos B$



Let $\overrightarrow{BC}=\underline{a}$; $\overrightarrow{CA}=\underline{b}$; $\overrightarrow{AB}=\underline{c}$

be the sides of

ΔABC.

In any triangle

$\underline{a}+\underline{b}+\underline{c}=0$.

$\underline{b} = -\underline{a}-\underline{c}$

$\underline{b} = -(\underline{a}+\underline{c})$.

Taking dot product with '$\underline{b}$'

$\underline{b}\cdot\underline{b} = -(\underline{a}+\underline{c})\cdot\underline{b}$

$b^2 = [-(\underline{a}+\underline{c})]\cdot[-(\underline{a}+\underline{c})]$.

$= (\underline{a}+\underline{c})\cdot(\underline{a}+\underline{c})$.

$= \underline{a}\cdot\underline{a} + \underline{a}\cdot\underline{c} + \underline{c}\cdot\underline{a} + \underline{c}\cdot\underline{c}$

$= a^2 + \underline{c}\cdot\underline{a} + \underline{c}\cdot\underline{a} + c^2$

$= a^2 + 2\underline{c}\cdot\underline{a} + c^2$

$= a^2+c^2+2|\underline{c}||\underline{a}|\cos(\pi-B)$

∴ Angle from $\underline{c}$ to $\underline{a}$ is $\pi-B$.

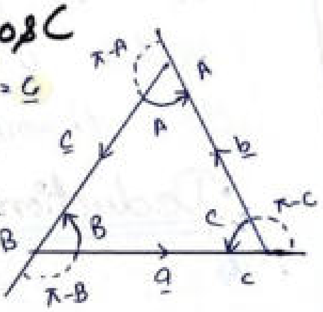
$b^2 = a^2+c^2+2ca(-\cos B)$.

∴ $\cos(\pi-\theta) = -\cos\theta$

$b^2 = a^2+c^2-2ca\cos B$.

Hence proved.

iv) $c^2 = a^2+b^2-2ab\cos C$



Let $\overrightarrow{BC}=\underline{a}$; $\overrightarrow{CA}=\underline{b}$; $\overrightarrow{AB}=\underline{c}$

be sides of triangle

ΔABC. In any triangle

$\underline{a}+\underline{b}+\underline{c}=0$.

$\underline{c} = -\underline{a}-\underline{b} = -(a+b)$

Taking dot product with 'c'

$\underline{c}\cdot\underline{c} = -(\underline{a}+\underline{b})\cdot\underline{c}$

$c^2 = [-(\underline{a}+\underline{b})]\cdot[-(\underline{a}+\underline{b})]$.

$= (\underline{a}+\underline{b})\cdot(\underline{a}+\underline{b})$.

$= \underline{a}\cdot\underline{a} + \underline{a}\cdot\underline{b} + \underline{b}\cdot\underline{a} + \underline{b}\cdot\underline{b}$

$= a^2 + \underline{a}\cdot\underline{b} + \underline{a}\cdot\underline{b} + b^2$

$= a^2 + 2a.\underline{b} + b^2$

$= a^2+b^2+2|\underline{a}||\underline{b}|\cos(\pi-C)$

∴ Angle from $\underline{b}$ to $\underline{a}$ is $\pi-C$

$c^2 = a^2+b^2+2ab(-\cos C)$

∴ $\cos(\pi-\theta) = -\cos$

$c^2 = a^2+b^2-2ab\cos C$.

Hence proved.

Theory:

The Cross Product or vector product of two vectors.

Definition: Let $\underline{u}$ and $\underline{v}$ be two non-zero vectors. The cross or vector product of $\underline{u}$ and $\underline{v}$, written as $\underline{u}\times\underline{v}$ and is defined as

$$\underline{u}\times\underline{v} = |\underline{u}||\underline{v}|\sin\theta\,\hat{\underline{n}}$$

where θ is the angle between the vectors, such that $0\le\theta\le\pi$ and $\hat{n}$ is a unit vector perpendicular to the plane of $\underline{u}$ and $\underline{v}$.

Right hand rule:

If the fingers of the right hand point along the vector $\underline{u}$ and then curl towards the vector $\underline{v}$, then the thumb will give the direction of $\hat{\underline{n}}$ which is $\underline{u}\times\underline{v}$.

Deduction of the important Results:

a) $\underline{i}\times\underline{i} = |\underline{i}||\underline{i}|\sin 0^\circ\hat{n} = 0$

$\underline{j}\times\underline{j} = |\underline{j}||\underline{j}|\sin 0^\circ\hat{n} = 0$

$\underline{k}\times\underline{k} = |\underline{k}||\underline{k}|\sin 0^\circ\hat{n} = 0$

b) $\underline{i}\times\underline{j} = |\underline{i}||\underline{j}|\sin 90^\circ\underline{k} = \underline{k}$

$\underline{j}\times\underline{k} = |\underline{j}||\underline{k}|\sin 90^\circ\underline{i} = \underline{i}$

$\underline{k}\times\underline{i} = |\underline{k}||\underline{i}|\sin 90^\circ\underline{j} = \underline{j}$

c) $\underline{u}\times\underline{v} = |\underline{u}||\underline{v}|\sin\theta\hat{n}$

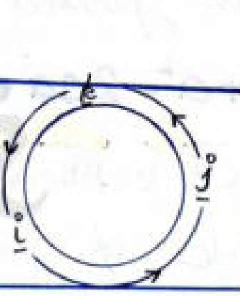
$\underline{v}\times\underline{u} = |\underline{v}||\underline{u}|\sin(-\theta)\hat{n}$

$= -|\underline{v}||\underline{u}|\sin\theta\hat{n}$

$\underline{u}\times\underline{v} = -|\underline{v}||\underline{u}|$

Note:

The cross product of $\underline{i}, \underline{j}, \underline{k}$ are written in The cyclic pattern.



Properties of Cross Product:

i) $\underline{u}\times\underline{v} = \underline{0}$ if $\underline{u} = \underline{0}$ or $\underline{v} = \underline{0}$

ii) $\underline{u}\times\underline{v} = -\underline{v}\times\underline{u}$

iii) $\underline{u}\times(\underline{v}+\underline{w}) = \underline{u}\times\underline{v} + \underline{u}\times\underline{w}$ (Distributive property)

iv) $\underline{u}\times(k\underline{v}) = (k\underline{u})\times\underline{u} = k(\underline{u}\times\underline{v})$ (k is scaler)

v) $\underline{u}\times\underline{u} = 0$

• Parallel Vectors:

If $\underline{u}$ and $\underline{v}$ are parallel vetors,

$$\theta = 0 \Rightarrow \sin 0^\circ = 0$$

then, $\underline{u} \times \underline{v} = |\underline{u}||\underline{v}| \sin\theta \hat{n}$.

$\underline{u} \times \underline{v} = \underline{0}$ or $|\underline{u} \times \underline{v}| = 0$.

if $\underline{u} \times \underline{v} = \underline{0}$, then

either $\sin\theta = 0$ or $|\underline{u}| = 0$ or $|\underline{v}| = 0$

i) If $\sin\theta = 0 \Rightarrow \theta = 0^\circ$ or 180°, which show that the vectors $\underline{u}$ and $\underline{v}$ are parallel.

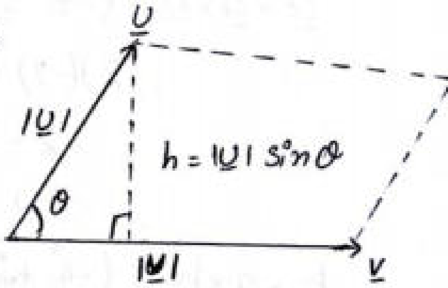
ii) If $\underline{u} = \underline{0}$ or $\underline{v} = \underline{0}$, then since the zero vector has no specific direction, we adopt the convention that the zero vector is parallel to every vector.

• Area of Parallelogram:

If $\underline{u}$ and $\underline{v}$ are two non-zero vectors and θ is the angle between $\underline{u}$ and $\underline{v}$, then $\underline{u}$ and $\underline{v}$ represent the lengths of the adjacent sides of a parallelogram. We know that

Area of parallelogram = base x height

$$= |\underline{u}||\underline{v}| \sin\theta = |\underline{u} \times \underline{v}|$$



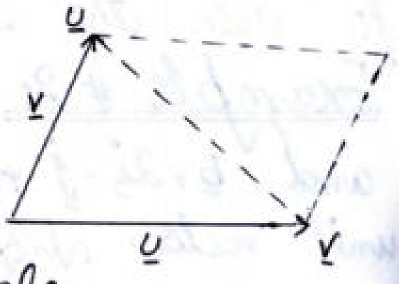
• Area of Triangle:

From figure it is clear that

Area of triangle $= \frac{1}{2}$(Area of Parallelogram)

Area of triangle $= \frac{1}{2} |\underline{u} \times \underline{v}|$.

Where $\underline{u}$ and $\underline{v}$ are vectors along two adjacent sides of the triangle.



Example # 1: Find a vector perpendicular to each of the vectors $\underline{a}=2\underline{i}-\underline{j}+\underline{k}$ and $\underline{b}=4\underline{i}+2\underline{j}-\underline{k}$

A vector perpendicular to both the vectors $\underline{a}$ and $\underline{b}$ is $\underline{a}\times\underline{b}$

$$\underline{a}\times\underline{b}=\begin{vmatrix}\underline{i} & \underline{j} & \underline{k}\\ 2 & -1 & 1\\ 4 & 2 & -1\end{vmatrix}$$

Expand R_1

$$=\underline{i}\begin{vmatrix}-1 & 1\\ 2 & -1\end{vmatrix}-\underline{j}\begin{vmatrix}2 & 1\\ 4 & -1\end{vmatrix}+\underline{k}\begin{vmatrix}2 & -1\\ 4 & 2\end{vmatrix}$$

$$=\underline{i}(1-2)-\underline{j}(-2-4)+\underline{k}(4+4)$$

$$\boxed{\underline{a}\times\underline{b}=-\underline{i}+6\underline{j}+8\underline{k}}$$

Verification:

$$\underline{a}\cdot\underline{a}\times\underline{b}=(2\underline{i}-\underline{j}+\underline{k})\cdot(-\underline{i}+6\underline{j}+8\underline{k})$$
$$=(2)(-1)+(-1)(6)+(1)(8)$$
$$=-2-6+8$$
$$=0$$

$$\underline{b}\cdot\underline{a}\times\underline{b}=(4\underline{i}+2\underline{j}-\underline{k})\cdot(-\underline{i}+6\underline{j}+8\underline{k})$$
$$=(4)(-1)+(2)(6)+(-1)(8)$$
$$=-4+12-8$$
$$=0$$

Hence $\underline{a}\times\underline{b}$ is perpendicular to both the vectors $\underline{a}$ and $\underline{b}$

Example # 2: If $\underline{a}=4\underline{i}+3\underline{j}+\underline{k}$ and $\underline{b}=2\underline{i}-\underline{j}+2\underline{k}$. Find a unit vector perpendicular to both $\underline{a}$ and $\underline{b}$. Also find the sine of the angle b/w the vectors $\underline{a}$ and $\underline{b}$.

$$\therefore\ \hat{\underline{n}}=\frac{\underline{a}\times\underline{b}}{|\underline{a}\times\underline{b}|}$$

$$\underline{a}\times\underline{b}=\begin{vmatrix}\underline{i} & \underline{j} & \underline{k}\\ 4 & 3 & 1\\ 2 & -1 & 2\end{vmatrix}$$

Expand R_1

$$=\underline{i}\begin{vmatrix}3 & 1\\ -1 & 2\end{vmatrix}-\underline{j}\begin{vmatrix}4 & 1\\ 2 & 2\end{vmatrix}+\underline{k}\begin{vmatrix}4 & 3\\ 2 & -1\end{vmatrix}$$

$$=\underline{i}(6+1)-\underline{j}(8-2)+\underline{k}(-4-6)$$

$$\underline{a}\times\underline{b}=7\underline{i}-6\underline{j}-10\underline{k}$$

$$|\underline{a}\times\underline{b}|=\sqrt{(7)^2+(-6)^2+(-10)^2}$$
$$=\sqrt{49+36+100}$$
$$=\sqrt{185}$$

$$\boxed{\hat{n}=\frac{7\underline{i}-6\underline{j}-10\underline{k}}{\sqrt{185}}}$$

Now

$$\therefore\ \sin\theta=\frac{|\underline{a}\times\underline{b}|}{|\underline{a}||\underline{b}|}$$

$$|\underline{a}|=\sqrt{(4)^2+(3)^2+(1)^2}$$
$$=\sqrt{16+9+1}$$
$$=\sqrt{26}$$

$$|\underline{b}|=\sqrt{(2)^2+(-1)^2+(2)^2}$$
$$=\sqrt{4+1+4}$$
$$=\sqrt{9}$$
$$=3$$

$$\boxed{\sin\theta=\frac{\sqrt{185}}{3\sqrt{26}}}$$

Example # 3: Prove that $\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$

Let $\overrightarrow{OA}$ and $\overrightarrow{OB}$ be unit vectors in the xy-plane making angles α and $-\beta$ with the positive x-axis respectively

$$\angle AOB=\alpha+\beta$$

Now

$\vec{OA} = \cos\alpha\underline{i} + \sin\alpha\underline{j}$

$\vec{OB} = \cos(-\beta)\underline{i} + \sin(-\beta)\underline{j}$

$= \cos\beta\underline{i} - \sin\beta\underline{j}$

$\therefore |\vec{OA}| = |\vec{OB}| = 1$

$\vec{OB} \times \vec{OA} = (\cos\beta\underline{i} - \sin\beta\underline{j}) \times (\cos\alpha\underline{i} + \sin\alpha\underline{j})$

$$|\vec{OB}||\vec{OA}| \sin(\alpha+\beta)\underline{k} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \cos\beta & -\sin\beta & 0 \\ \cos\alpha & \sin\alpha & 0 \end{vmatrix}$$

Expand C_3

$(1)(1)\sin(\alpha+\beta)\underline{k} = (\cos\beta\sin\alpha + \cos\alpha\sin\beta)\underline{k}$

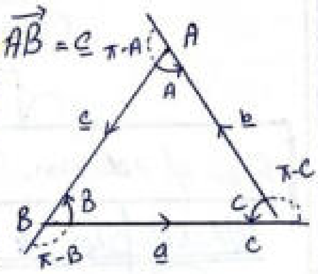
$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

Hence proved.

Example # 4: In any triangle ABC, prove that

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \text{ (Law of sines)}$$

Let $\vec{BC} = \underline{a}$, $\vec{CA} = \underline{b}$, $\vec{AB} = \underline{c}$ be the sides of ΔABC



In any triangle

$\underline{a} + \underline{b} + \underline{c} = 0$ — ①

Taking Cross product with 'c'

$\underline{a} \times \underline{c} + \underline{b} \times \underline{c} + \underline{c} \times \underline{c} = 0 \times \underline{c}$

$\underline{a} \times \underline{c} + \underline{b} \times \underline{c} = 0$ $\quad \therefore \underline{c} \times \underline{c} = 0$

$\underline{b} \times \underline{c} = -\underline{a} \times C$

$\underline{b} \times \underline{c} = \underline{c} \times \underline{a}$ $\quad \therefore -\underline{a} \times \underline{c} = \underline{c} \times \underline{a}$

$|\underline{b} \times \underline{c}| = |\underline{c} \times \underline{a}|$

$|\underline{b}||\underline{c}| \sin(\pi - A) = |\underline{c}||\underline{a}| \sin(\pi - B)$

$bc \sin A = ca \sin B$

Dividing by 'c'

$b \sin A = a \sin B$

$\frac{b}{\sin B} = \frac{a}{\sin A}$ — ②

Similarly by taking Cross product of ① with $\underline{b}$

$\underline{a} \times \underline{b} + \underline{b} \times \underline{b} + \underline{c} \times \underline{b} = 0 \times \underline{b}$

$\underline{a} \times \underline{b} + 0 + \underline{c} \times \underline{b} = 0$ $\quad \therefore \underline{b} \times \underline{b} = 0$

$\underline{a} \times \underline{b} = -\underline{c} \times \underline{b}$

$\underline{a} \times \underline{b} = \underline{b} \times \underline{c}$ $\quad \therefore -\underline{c} \times \underline{b} = \underline{b} \times \underline{c}$

$|\underline{a} \times \underline{b}| = |\underline{b} \times \underline{c}|$

$|\underline{a}||\underline{b}| \sin(\pi - C) = |\underline{b}||\underline{c}| \sin(\pi - A)$

$\therefore$ **Angle b/w $\underline{a}$ and $\underline{b}$ is $\pi - C$**

$\therefore$ **Angle b/w $\underline{b}$ and $\underline{c}$ is $\pi - A$**

$ab \sin C = bc \sin A$

Dividing by 'b'

$a \sin C = c \sin A$

$\frac{a}{\sin A} = \frac{c}{\sin C}$ — ③

From ② and ③

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Hence proved.

Example # 5: Find the area of the triangle with vertices A(1, -1, 1); B(2, 1, -1) and C(-1, 1; 2). Also find a unit vector perpendicular to the plane ABC.

$A(1,-1,1)$; $\vec{OA} = \underline{i} - \underline{j} + \underline{k}$

$B(2,1,-1)$; $\vec{OB} = 2\underline{i} + \underline{j} - \underline{k}$

$C(-1,1,2)$; $\vec{OC} = -\underline{i} + \underline{j} + 2\underline{k}$

$\vec{AB} = \vec{OB} - \vec{OA}$

$= (2\underline{i} + \underline{j} - \underline{k}) - (\underline{i} - \underline{j} + \underline{k})$

$= 2\underline{i} + \underline{j} - \underline{k} - \underline{i} + \underline{j} - \underline{k}$

$= \underline{i} + 2\underline{j} - 2\underline{k}$

$\vec{AC} = \vec{OC} - \vec{OA}$

$= (-\underline{i} + \underline{j} + 2\underline{k}) - (\underline{i} - \underline{j} + \underline{k})$

$= -\underline{i} + \underline{j} + 2\underline{k} - \underline{i} + \underline{j} - \underline{k}$

$= -2\underline{i} + 2\underline{j} + \underline{k}$

Now

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 2 & -2 \\ -2 & 2 & 1 \end{vmatrix}$$

Expand R_1

$\vec{AB} \times \vec{AC} = \underline{i}(2+4) - \underline{j}(1-4) + \underline{k}(2+4)$

$= 6\underline{i} + 3\underline{j} + 6\underline{k}$

$|\vec{AB} \times \vec{AC}| = \sqrt{(6)^2 + (3)^2 + (6)^2}$

$= \sqrt{36+9+36}$

$= \sqrt{81}$

$|\vec{AB} \times \vec{AC}| = 9$

∴ Area of triangle $= \frac{1}{2}|\vec{AB} \times \vec{AC}|$

$= \frac{1}{2}(9)$

$$\boxed{\text{Area of triangle} = \frac{9}{2}}$$

Now

∴ A unit vector ⊥ to the plane ABC $= \dfrac{\vec{AB} \times \vec{AC}}{|\vec{AB} \times \vec{AC}|}$

$= \frac{1}{9}(6\underline{i} + 3\underline{j} + 6\underline{k})$

$$\boxed{\text{Unit vector} \perp \text{ to the plane} = \frac{1}{3}(2\underline{i} + \underline{j} + 2\underline{k})}$$

Example # 6: Find area of parallelogram whose vertices are $P(0,0,0)$, $Q(-1,2,4)$, $R(2,-1,4)$ and $S(1,1,8)$.

$P(0,0,0)$; $Q(-1,2,4)$; $R(2,-1,4)$

$\vec{PQ} = (-1-0)\underline{i} + (2-0)\underline{j} + (4-0)\underline{k}$

$= -\underline{i} + 2\underline{j} + 4\underline{k}$

$\vec{PR} = (2-0)\underline{i} + (-1-0)\underline{j} + (4-0)\underline{k}$

$= 2\underline{i} - \underline{j} + 4\underline{k}$

Now

$$\vec{PQ} \times \vec{PR} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ -1 & 2 & 4 \\ 2 & -1 & 4 \end{vmatrix}$$

Expand R_1

$= \underline{i}(8+4) - \underline{j}(-4-8) + \underline{k}(1-4)$

$= 12\underline{i} + 12\underline{j} + 3\underline{k}$

∴ Area of parallelogram $= |\vec{PQ} \times \vec{PR}|$

$= \sqrt{(12)^2 + (12)^2 + (-3)^2}$

$= \sqrt{144+144+9}$

$$\boxed{\text{Area of ||elogram} = \sqrt{297}}$$

Example # 7: If $\underline{u} = 2\underline{i} - \underline{j} + \underline{k}$ and $\underline{v} = 4\underline{i} + 2\underline{j} - \underline{k}$. Find

i) $\underline{u} \times \underline{u}$

$$\underline{u} \times \underline{u} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & -1 & 1 \\ 2 & -1 & 1 \end{vmatrix}$$

$= 0$

∴ Two rows are same

ii) $\underline{u} \times \underline{v}$

$$\underline{u} \times \underline{v} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & -1 & 1 \\ 4 & 2 & -1 \end{vmatrix}$$

Expand R_1

$= \underline{i}(1-2) - \underline{j}(-2-4) + \underline{k}(4+4)$

$= -\underline{i} + 6\underline{j} + 8\underline{k}$

iii) $\underline{v} \times \underline{u}$

$$\underline{v} \times \underline{u} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 4 & 2 & -1 \\ 2 & -1 & 1 \end{vmatrix}$$

Expand R_1

$= \underline{i}(2-1) - \underline{j}(4+2) + \underline{k}(-4-4)$

$= \underline{i} - 6\underline{j} - 8\underline{k}$

Exercise # 7.4

Question no 1: Compute the cross product $\underline{a}\times\underline{b}$ and $\underline{b}\times\underline{a}$. Check your answer by showing that each $\underline{a}$ and $\underline{b}$ is perpendicular to $\underline{a}\times\underline{b}$ and $\underline{b}\times\underline{a}$.

i) $\underline{a} = 2\underline{i} + \underline{j} - \underline{k}$; $\underline{b} = \underline{i} - \underline{j} + \underline{k}$

$$\underline{a}\times\underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{vmatrix}$$

Expand by R_1

$$\underline{a}\times\underline{b} = \underline{i}(1-1) - \underline{j}(2+1) + \underline{k}(-2-1)$$

$$= 0 - 3\underline{j} - 3\underline{k}$$

$$\underline{a}\times\underline{b} = -3\underline{j} - 3\underline{k}$$

$$\therefore \underline{b}\times\underline{a} = -\underline{a}\times\underline{b}$$

$$\underline{b}\times\underline{a} = -(-3\underline{j} - 3\underline{k})$$

$$\underline{b}\times\underline{a} = 3\underline{j} + 3\underline{k}.$$

Cheaking:

$$\underline{a}\cdot\underline{a}\times\underline{b} = (2\underline{i} + \underline{j} - \underline{k})\cdot(-3\underline{j} - 3\underline{k})$$

$$= 0 - 3 + 3$$

$$= 0$$

$\underline{a}$ is $\perp$ to $\underline{a}\times\underline{b}$

$$\underline{a}\cdot\underline{b}\times\underline{a} = (2\underline{i} + \underline{j} - \underline{k})\cdot(3\underline{j} + 3\underline{k})$$

$$= 0 + 3 - 3$$

$$= 0$$

$\underline{a}$ is $\perp$ to $\underline{b}\times\underline{a}$

$$\underline{b}\cdot\underline{a}\times\underline{b} = (\underline{i} - \underline{j} + \underline{k})\cdot(-3\underline{j} - 3\underline{k})$$

$$= 0 + 3 - 3$$

$$= 0$$

$\underline{b}$ is $\perp$ to $\underline{a}\times\underline{b}$

$$\underline{b}\cdot\underline{b}\times\underline{a} = (\underline{i} - \underline{j} + \underline{k})\cdot(3\underline{j} + 3\underline{k})$$

$$= 0 - 3 + 3$$

$$= 0$$

$\underline{b}$ is $\perp$ to $\underline{b}\times\underline{a}$

ii) $\underline{a} = \underline{i} + \underline{j}$; $\underline{b} = \underline{i} - \underline{j}$.

$$\underline{a}\times\underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 1 & 0 \\ 1 & -1 & 0 \end{vmatrix}$$

Expand by R_1

$$= \underline{i}(0+0) - \underline{j}(0-0) + \underline{k}(-1-1)$$

$$= 0\underline{i} - 0\underline{j} - 2\underline{k}$$

$$\underline{a}\times\underline{b} = -2\underline{k}$$

$$\therefore \underline{b}\times\underline{a} = -\underline{a}\times\underline{b}$$

$$\underline{b}\times\underline{a} = -(-2\underline{k})$$

$$\underline{b}\times\underline{a} = 2\underline{k}$$

Cheaking:

$$\underline{a}\cdot\underline{a}\times\underline{b} = (\underline{i} + \underline{j})\cdot(-2\underline{k})$$

$$= 0$$

$\underline{a}$ is $\perp$ to $\underline{a}\times\underline{b}$

$$\underline{a}\cdot\underline{b}\times\underline{a} = (\underline{i} + \underline{j})\cdot(2\underline{k})$$

$$= 0$$

$\underline{a}$ is $\perp$ to $\underline{b}\times\underline{a}$

$$\underline{b}\cdot\underline{a}\times\underline{b} = (\underline{i} - \underline{j})\cdot(-2\underline{k})$$

$$= 0$$

$\underline{b}$ is $\perp$ to $\underline{a}\times\underline{b}$

$$\underline{b}\cdot\underline{b}\times\underline{a} = (\underline{i} - \underline{j})\cdot(2\underline{k})$$

$$= 0$$

$\underline{b}$ is $\perp$ to $\underline{b}\times\underline{a}$

iii) $\underline{a} = 3\underline{i} - 2\underline{j} + \underline{k}$; $\underline{b} = \underline{i} + \underline{j}$

$$\underline{a}\times\underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 3 & -2 & 1 \\ 1 & 1 & 0 \end{vmatrix}$$

Expand by R_1.

$$= \underline{i}(0-1) - \underline{j}(0-1) + \underline{k}(3+2)$$

$$\underline{a}\times\underline{b} = -\underline{i} + \underline{j} + 5\underline{k}$$

$\therefore \ \underline{b} \times \underline{a} = -\underline{a} \times \underline{b}$

$\underline{b} \times \underline{a} = -(-\underline{i} + \underline{j} + 5\underline{k})$

$\underline{b} \times \underline{a} = \underline{i} - \underline{j} - 5\underline{k}$

Cheaking:

$\underline{a} \cdot \underline{a} \times \underline{b} = (3\underline{i} - 2\underline{j} + \underline{k}) \cdot (-\underline{i} + \underline{j} + 5\underline{k})$

$= -3 - 2 + 5$

$= 0$

$\underline{a}$ is ⊥ to $\underline{a} \times \underline{b}$

$\underline{a} \cdot \underline{b} \times \underline{a} = (3\underline{i} - 2\underline{j} + \underline{k}) \cdot (\underline{i} - \underline{j} - 5\underline{k})$

$= 3 + 2 - 5$

$= 0$

$\underline{a}$ is ⊥ to $\underline{b} \times \underline{a}$

$\underline{b} \cdot \underline{a} \times \underline{b} = (\underline{i} + \underline{j}) \cdot (-\underline{i} + \underline{j} + 5\underline{k})$

$= -1 + 1 + 0$

$= 0$

$\underline{b}$ is ⊥ to $\underline{a} \times \underline{b}$

$\underline{b} \cdot \underline{b} \times \underline{a} = (\underline{i} + \underline{j}) \cdot (\underline{i} - \underline{j} - 5\underline{k})$.

$= 1 - 1 - 0$

$= 0$

$\underline{b}$ is ⊥ to $\underline{b} \times \underline{a}$.

iv) $\underline{a} = -4\underline{i} + \underline{j} - 2\underline{k}$; $\underline{b} = 2\underline{i} + \underline{j} + \underline{k}$

$$\underline{a} \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ -4 & 1 & -2 \\ 2 & 1 & 1 \end{vmatrix}$$

Expand by R_1

$= \underline{i}(1+2) - \underline{j}(-4+2) + \underline{k}(-4-2)$.

$\underline{a} \times \underline{b} = 3\underline{i} - 0\underline{j} - 6\underline{k}$

$-\underline{a} \times \underline{b} = \underline{b} \times \underline{a}$

$\underline{b} \times \underline{a} = -(3\underline{i} - 0\underline{j} - 6\underline{k})$

$= -3\underline{i} + 6\underline{k}$

Cheaking:

$\underline{a} \cdot \underline{a} \times \underline{b} = (-4\underline{i} + \underline{j} - 2\underline{k}) \cdot (3\underline{i} - 6\underline{k})$.

$= -12 + 0 + 12$

$= 0$

$\underline{a}$ is ⊥ to $\underline{a} \times \underline{b}$

$\underline{a} \cdot \underline{b} \times \underline{a} = (-4\underline{i} + \underline{j} - 2\underline{k}) \cdot (-3\underline{i} + 6\underline{k})$.

$= 12 + 0 - 12$

$= 0$

$\underline{a}$ is ⊥ to $\underline{b} \times \underline{a}$

$\underline{b} \cdot \underline{a} \times \underline{b} = (2\underline{i} + \underline{j} + \underline{k}) \cdot (3\underline{i} - 6\underline{k})$

$= 6 + 0 - 6$

$= 0$

$\underline{b}$ is ⊥ to $\underline{a} \times \underline{b}$

$\underline{b} \cdot \underline{b} \times \underline{a} = (2\underline{i} + \underline{j} + \underline{k}) \cdot (-3\underline{i} + 6\underline{k})$

$= -6 + 6$

$= 0$

$\underline{b}$ is ⊥ to $\underline{b} \times \underline{a}$.

Question # 2: Find a unit vector perpendicular to the plane containing $\underline{a}$ and $\underline{b}$ Also find sine of the angle between them.

i) $\underline{a} = 2\underline{i} - 6\underline{j} - 3\underline{k}$; $\underline{b} = 4\underline{i} + 3\underline{j} - \underline{k}$

$$\therefore \ \hat{n} = \frac{\underline{a} \times \underline{b}}{|\underline{a} \times \underline{b}|}$$

$$\underline{a} \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & -6 & -3 \\ 4 & 3 & -1 \end{vmatrix}$$

Expand by R_1

$= \underline{i}(6+9) - \underline{j}(-2+12) + \underline{k}(6+24)$

$\underline{a} \times \underline{b} = 15\underline{i} - 10\underline{j} + 30\underline{k}$

$|\underline{a} \times \underline{b}| = \sqrt{(15)^2 + (-10)^2 + (30)^2}$

$= \sqrt{225 + 100 + 900}$

$= \sqrt{1225}$

$= 35$

$$\hat{n} = \frac{15\underline{i} - 10\underline{j} + 30\underline{k}}{35}$$

$\hat{n} = \frac{15}{35}\underline{i} - \frac{10}{35}\underline{j} + \frac{30}{35}\underline{k}$

$$\boxed{\hat{n} = \frac{3}{7}\underline{i} - \frac{2}{7}\underline{j} + \frac{6}{7}\underline{k}}$$

$\therefore \sin\theta = \frac{|\underline{a}\times\underline{b}|}{|\underline{a}||\underline{b}|}$

$|\underline{a}| = \sqrt{(2)^2+(-6)^2+(-3)^2}$

$|\underline{a}| = \sqrt{4+36+9}$

$|\underline{a}| = \sqrt{49}$

$|\underline{a}| = 7$

$|\underline{b}| = \sqrt{(4)^2+(3)^2+(-1)^2}$

$= \sqrt{16+9+1}$

$|\underline{b}| = \sqrt{26}$

$\sin\theta = \frac{35}{7\sqrt{26}}$

$$\boxed{\sin\theta = \frac{5}{\sqrt{26}}}$$

ii) $\underline{a} = -\underline{i} - \underline{j} - \underline{k}$; $\underline{b} = 2\underline{i} - 3\underline{j} + 4\underline{k}$

$$\underline{a}\times\underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ -1 & -1 & -1 \\ 2 & -3 & 4 \end{vmatrix}$$

Expand by R_1

$= \underline{i}(-4-3) - \underline{j}(-4+2) + \underline{k}(3+2)$

$\underline{a}\times\underline{b} = -7\underline{i} + 2\underline{j} + 5\underline{k}$

$|\underline{a}\times\underline{b}| = \sqrt{(-7)^2+(2)^2+(5)^2}$

$= \sqrt{49+4+25}$

$= \sqrt{78}$

$\therefore \hat{n} = \frac{\underline{a}\times\underline{b}}{|\underline{a}\times\underline{b}|}$

$\hat{n} = \frac{-7\underline{i}+2\underline{j}+5\underline{k}}{\sqrt{78}}$

$$\boxed{\hat{n} = \frac{-7}{\sqrt{78}}\underline{i} + \frac{2}{\sqrt{78}}\underline{j} + \frac{5}{\sqrt{78}}\underline{k}}$$

$\therefore \sin\theta = \frac{|\underline{a}\times\underline{b}|}{|\underline{a}||\underline{b}|}$

$|\underline{a}| = \sqrt{(-1)^2+(-1)^2+(-1)^2}$

$= \sqrt{1+1+1}$

$= \sqrt{3}$

$|\underline{b}| = \sqrt{(2)^2+(-3)^2+(4)^2}$

$= \sqrt{4+9+16}$

$= \sqrt{29}$

$\sin\theta = \frac{\sqrt{78}}{\sqrt{3}\sqrt{29}} = \frac{\cancel{\sqrt{3}}\sqrt{26}}{\cancel{\sqrt{3}}\sqrt{29}}$

$$\boxed{\sin\theta = \sqrt{\frac{26}{29}}}$$

iii) $\underline{a} = 2\underline{i} - 2\underline{j} + 4\underline{k}$; $\underline{b} = -\underline{i} + \underline{j} - 2\underline{k}$

$\hat{n} = \frac{\underline{a}\times\underline{b}}{|\underline{a}\times\underline{b}|}$

$$\underline{a}\times\underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & -2 & 4 \\ -1 & 1 & -2 \end{vmatrix}$$

Expand by R_1

$= \underline{i}(4-4) - \underline{j}(-4+4) + k(2-2)$

$= 0\underline{i} - 0\underline{j} + 0\underline{k}$

Any unit vector

$\sin\theta = \frac{|\underline{a}\times\underline{b}|}{|\underline{a}||\underline{b}|}$

$|\underline{a}| = \sqrt{(2)^2+(-2)^2+(4)^2}$

$= \sqrt{4+4+16}$

$= \sqrt{24}$

$|\underline{b}| = \sqrt{(-1)^2+(1)^2+(-2)^2}$

$= \sqrt{1+1+4}$

$= \sqrt{6}$

$\sin\theta = \frac{0}{\sqrt{24}\sqrt{6}}$

$$\boxed{\sin\theta = 0}$$

iv) $\underline{a} = \underline{i} + \underline{j}$; $\underline{b} = \underline{i} - \underline{j}$

$\therefore \hat{n} = \frac{\underline{a} \times \underline{b}}{|\underline{a} \times \underline{b}|}$

$$\underline{a} \times \underline{b} = \begin{vmatrix} i & j & k \\ 1 & 1 & 0 \\ 1 & -1 & 0 \end{vmatrix}$$

Expand by R_1

$= \underline{i}(0+0) - \underline{j}(0-0) + \underline{k}(-1-1)$.

$= 0 + 0 - 2\underline{k}$

$\underline{a} \times \underline{b} = -2\underline{k}$

$|\underline{a} \times \underline{b}| = \sqrt{(-2)^2} = \sqrt{4}$

$= 2$

$\hat{n} = \frac{-2\underline{k}}{2} \Rightarrow \boxed{\hat{n} = -\underline{k}}$

$\therefore \sin\theta = \frac{|\underline{a} \times \underline{b}|}{|\underline{a}||\underline{b}|}$

$|\underline{a}| = \sqrt{(1)^2 + (1)^2} = \sqrt{1+1} = \sqrt{2}$

$|\underline{b}| = \sqrt{(1)^2 + (-1)^2} = \sqrt{1+1} = \sqrt{2}$

$\sin\theta = \frac{2}{\sqrt{2}\sqrt{2}} = \frac{2}{(\sqrt{2})^2}$

$\sin\theta = \frac{2}{2} \Rightarrow \boxed{\sin\theta = 1}$

Question no 3: Find the area of the triangle, determined by the point P, Q and R.

i) $P(0,0,0)$; $Q(2,3,2)$; $R(-1,1,4)$

$\overrightarrow{PQ} = [2,3,2] - [0,0,0]$

$= 2\underline{i} + 3\underline{j} + 2\underline{k} - 0\underline{i} - 0\underline{j} - 0\underline{k}$

$= 2\underline{i} + 3\underline{j} + 2\underline{k}$

$\overrightarrow{PR} = [-1,1,4] - [0,0,0]$.

$= -\underline{i} + \underline{j} + 4\underline{k} - 0\underline{i} - 0\underline{j} - 0\underline{k}$

$= -\underline{i} + \underline{j} + 4\underline{k}$

Now

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} i & j & k \\ 2 & 3 & 2 \\ -1 & 1 & 4 \end{vmatrix}$$

Expand by R_1

$= \underline{i}(12-2) - \underline{j}(8+2) + \underline{k}(2+3)$

$= 10\underline{i} - 10\underline{j} + 5\underline{k}$.

$|\overrightarrow{PQ} \times \overrightarrow{PR}| = \sqrt{(10)^2 + (-10)^2 + (5)^2}$

$= \sqrt{100 + 100 + 25}$

$= \sqrt{225}$

$= 15$

$\therefore$ Area of $\Delta = \frac{1}{2}|\overrightarrow{PQ} \times \overrightarrow{PR}|$

Area of $\Delta = \frac{1}{2}(15)$

$\boxed{\text{Area of } \Delta = \frac{15}{2} \text{ square units.}}$

ii) $P(1,-1,-1)$; $Q(2,0,-1)$; $R(0,2,1)$

$\overrightarrow{PQ} = [2,0,-1] - [1,-1,-1]$.

$= 2\underline{i} + 0\underline{j} - \underline{k} - \underline{i} + \underline{j} + \underline{k}$

$= \underline{i} + \underline{j} + 0\underline{k}$

$\overrightarrow{PR} = [0,2,1] - [1,-1,-1]$.

$= 0\underline{i} + 2\underline{j} + \underline{k} - \underline{i} + \underline{j} + \underline{k}$

$= -\underline{i} + 3\underline{j} + 2\underline{k}$

Now

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} i & j & k \\ 1 & 1 & 0 \\ -1 & 3 & 2 \end{vmatrix}$$

Expand by R_1

$= \underline{i}(2-0) - \underline{j}(2+0) + \underline{k}(3+1)$

$= 2\underline{i} - 2\underline{j} + 4\underline{k}$

$|\overrightarrow{PQ} \times \overrightarrow{PR}| = \sqrt{(2)^2 + (-2)^2 + (4)^2}$

$= \sqrt{4 + 4 + 16}$

$= \sqrt{24} \,.\, \sqrt{2 \times 2 \times 6}$

$= 2\sqrt{6}$

Area of $\Delta = \frac{1}{2}(2\sqrt{6})$

$$\boxed{\text{Area of } \Delta = \sqrt{6}}$$

Question # 4: Find the area of parallelogram, whose vertices are:

i) $A(0,0,0)$; $B(1,2,3)$; $C(2,-1,1)$; $D(3,1,4)$

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = [1,2,3]-[0,0,0]$

$= \underline{i}+2\underline{j}+3\underline{k}-0\underline{i}-0\underline{j}-0\underline{k}$.

$\overrightarrow{AB} = \underline{i}+2\underline{j}+3\underline{k}$.

$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = [2,-1,1]-[0,0,0]$

$= 2\underline{i}-\underline{j}+\underline{k}-0\underline{i}-0\underline{j}-0\underline{k}$

$\overrightarrow{AC} = 2\underline{i}-\underline{j}+\underline{k}$

Now

$$\overrightarrow{AB}\times\overrightarrow{AC} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 2 & 3 \\ 2 & -1 & 1 \end{vmatrix}$$

Expand by R_1

$= \underline{i}(2+3) - \underline{j}(1-6) + \underline{k}(-1-4)$

$= 5\underline{i} + 5\underline{j} - 5\underline{k}$.

$\therefore$ **Area of Parallelogram** $= |\overrightarrow{AB}\times\overrightarrow{AC}|$

$= \sqrt{(5)^2+(5)^2+(-5)^2}$

$= \sqrt{25+25+25}$

$= \sqrt{75}$

$$\boxed{\text{Area of ||elogram ABCD} = 5\sqrt{3} \text{ square units}}$$

ii) $A(1,2,-1)$; $B(4,2,-3)$; $C(6,-5,2)$ $D(9,-5,0)$.

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = [4,2,-3]-[1,2,-1]$.

$= 4\underline{i}+\cancel{2\underline{j}}-3\underline{k}-\underline{i}-\cancel{2\underline{j}}+\underline{k}$

$\overrightarrow{AB} = 3\underline{i}+0\underline{j}-2\underline{k}$

$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = [6,-5,2]-[1,2,-1]$.

$= 6\underline{i}-5\underline{j}+2\underline{k}-\underline{i}-2\underline{j}+\underline{k}$

$= 5\underline{i}-7\underline{j}+3\underline{k}$

Now

$$\overrightarrow{AB}\times\overrightarrow{AC} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 3 & 0 & -2 \\ 5 & -7 & 3 \end{vmatrix}$$

Expand by R_1

$= \underline{i}(0-14) - \underline{j}(9+10) + \underline{k}(-21-0)$

$= -14\underline{i} - 19\underline{j} - 21\underline{k}$.

$|\overrightarrow{AB}\times\overrightarrow{AC}| = \sqrt{(-14)^2+(-19)^2+(-21)^2}$

$= \sqrt{196+361+441}$

$= \sqrt{998}$

$$\boxed{\text{Area of Parallelogram} = \sqrt{998} \text{ sq units}}$$

iii) $A(-1,1,1)$; $B(-1,2,2)$; $C(-3,4,-5)$; $D(-3,5,-4)$

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = [-1,2,2]-[-1,1,1]$.

$= -\underline{i}+2\underline{j}+2\underline{k}+\underline{i}-\underline{j}-\underline{k}$.

$\overrightarrow{AB} = 0\underline{i}+\underline{j}+\underline{k}$

$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = [-3,4,-5]-[-1,1,1]$.

$= -3\underline{i}+4\underline{j}-5\underline{k}+\underline{i}-\underline{j}-\underline{k}$

$\overrightarrow{AC} = -2\underline{i}+3\underline{j}-6\underline{k}$.

Now

$$\overrightarrow{AB}\times\overrightarrow{AC} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 0 & 1 & 1 \\ -2 & 3 & -6 \end{vmatrix}$$

Expand by R_1.

$= \underline{i}(-6-3) - \underline{j}(0+2) + \underline{k}(0+2)$

$= -9\underline{i} - 2\underline{j} + 2\underline{k}$

$\therefore$ **Area of Parallelogram** $= |\overrightarrow{AB}\times\overrightarrow{AC}|$

$= \sqrt{(-9)^2+(-2)^2+(2)^2}$

$= \sqrt{81+4+4}$

$$\boxed{\text{Area of ||llagram ABCD} = \sqrt{89} \text{ square units}}$$

Question no 5: Which vectors if any, are perpendicular or parallel.

i) $\underline{u} = 5\underline{i} - \underline{j} + \underline{k}$; $\underline{v} = \underline{j} - 5\underline{k}$; $\underline{w} = -15\underline{i} + 3\underline{j} - 3\underline{k}$

$$\underline{u} \cdot \underline{v} = (5\underline{i} - \underline{j} + \underline{k}) \cdot (\underline{j} - 5\underline{k})$$
$$= (5)(0) + (-1)(1) + (1)(-5)$$
$$= 0 - 1 - 5$$
$$= -6 \neq 0$$

$\Rightarrow \underline{u}$ and $\underline{v}$ are not $\perp$ar

$$\underline{v} \cdot \underline{w} = (\underline{j} - 5\underline{k}) \cdot (-15\underline{i} + 3\underline{j} - 3\underline{k})$$
$$= (0)(-15) + (1)(3) + (-5)(-3)$$
$$= 0 + 3 + 15$$
$$= 18 \neq 0$$

$\Rightarrow \underline{v}$ and $\underline{w}$ are not $\perp$ar

$$\underline{u} \cdot \underline{w} = (5\underline{i} - \underline{j} + \underline{k}) \cdot (-15\underline{i} + 3\underline{j} - 3\underline{k})$$
$$= (5)(-15) + (-1)(3) + (1)(-3)$$
$$= -75 - 3 - 3$$
$$= -81 \neq 0$$

$\Rightarrow \underline{u}$ and $\underline{w}$ are not $\perp$ar

Now

$$\underline{w} = -15\underline{i} + 3\underline{j} - 3\underline{k}$$
$$\underline{w} = -3(5\underline{i} - \underline{j} + \underline{k})$$
$$\underline{w} = -3\underline{u}$$

$\Rightarrow \underline{u}$ and $\underline{w}$ are parallel

ii) $\underline{u} = \underline{i} + 2\underline{j} - \underline{k}$; $\underline{v} = -\underline{i} + \underline{j} + \underline{k}$; $\underline{w} = -\frac{\pi}{2}\underline{i} - \pi\underline{j} + \frac{\pi}{2}\underline{k}$

$$\underline{w} = -\frac{\pi}{2}\underline{i} - \pi\underline{j} + \frac{\pi}{2}\underline{k}$$
$$= \frac{-\pi\underline{i} - 2\pi\underline{j} + \pi\underline{k}}{2}$$
$$= \frac{-\pi}{2}(\underline{i} + 2\underline{j} - \underline{k})$$
$$\underline{w} = \frac{-\pi}{2}\underline{u}$$

$\Rightarrow \underline{u}$ and $\underline{w}$ are parallel

Now

$$\underline{u} \cdot \underline{v} = (\underline{i} + 2\underline{j} - \underline{k}) \cdot (-\underline{i} + \underline{j} + \underline{k})$$
$$= (1)(-1) + (2)(1) + (-1)(1)$$
$$= -1 + 2 - 1$$
$$= 0$$

$\Rightarrow \underline{u}$ and $\underline{v}$ are $\perp$ar

$$\underline{v} \cdot \underline{w} = (-\underline{i} + \underline{j} + \underline{k}) \cdot \left(-\frac{\pi}{2}\underline{i} - \pi\underline{j} + \frac{\pi}{2}\underline{k}\right)$$
$$= (-1)\left(-\frac{\pi}{2}\right) + (1)(-\pi) + (1)\left(\frac{\pi}{2}\right)$$
$$= \frac{\pi}{2} - \pi + \frac{\pi}{2}$$
$$= \frac{\pi - 2\pi + \pi}{2} = \frac{2\pi - 2\pi}{2}$$
$$\underline{v} \cdot \underline{w} = \frac{0}{2} = 0$$

$\Rightarrow \underline{v}$ and $\underline{w}$ are $\perp$ar.

For $\underline{u}$ and $\underline{w}$ no need to check because $\underline{u}$ and $\underline{w}$ are parallel.

Question no 6: Prove that:

$\underline{a} \times (\underline{b} + \underline{c}) + \underline{b} \times (\underline{c} + \underline{a}) + \underline{c} \times (\underline{a} + \underline{b}) = 0$

$$L.H.S = \underline{a} \times (\underline{b} + \underline{c}) + \underline{b} \times (\underline{c} + \underline{a}) + \underline{c} \times (\underline{a} + \underline{b})$$
$$= \underline{a} \times \underline{b} + \underline{a} \times \underline{c} + \underline{b} \times \underline{c} + \underline{b} \times \underline{a} + \underline{c} \times \underline{a} + \underline{c} \times \underline{b}$$
$$= \underline{a} \times \underline{b} + \underline{a} \times \underline{c} + \underline{b} \times \underline{c} + (-\underline{a} \times \underline{b}) + (-\underline{a} \times \underline{c}) + (-\underline{b} \times \underline{c})$$
$$= \underline{a} \times \underline{b} + \underline{a} \times \underline{c} + \underline{b} \times \underline{c} - \underline{a} \times \underline{b} - \underline{a} \times \underline{c} - \underline{b} \times \underline{c}$$
$$= 0 = R.H.S$$

Question no 7: If $\underline{a} + \underline{b} + \underline{c} = 0$ then prove that

$$\underline{a} \times \underline{b} = \underline{b} \times \underline{c} = \underline{c} \times \underline{a}$$

$$\underline{a} + \underline{b} + \underline{c} = 0 \quad \text{——} ①$$

Taking Cross product with 'a'

$$\underline{a} \times (\underline{a} + \underline{b} + \underline{c}) = \underline{a} \times 0$$
$$\underline{a} \times \underline{a} + \underline{a} \times \underline{b} + \underline{a} \times \underline{c} = 0.$$

$0 + \underline{a} \times \underline{b} + \underline{a} \times \underline{c} = 0$

$\because \underline{a} \times \underline{a} = 0$

$\underline{a} \times \underline{b} = -\underline{a} \times \underline{c}$

$\underline{a} \times \underline{b} = \underline{c} \times \underline{a}$ —— ②

$\because \underline{c} \times \underline{a} = -\underline{a} \times \underline{c}$

Now Taking Cross product with 'b'

$\underline{b} \times (\underline{a} + \underline{b} + \underline{c}) = \underline{b} \times 0.$

$\underline{b} \times \underline{a} + \underline{b} \times \underline{b} + \underline{b} \times \underline{c} = 0.$

$\underline{b} \times \underline{a} + 0 + \underline{b} \times \underline{c} = 0$

$\because \underline{b} \times \underline{b} = 0$

$\underline{b} \times \underline{a} = -\underline{b} \times \underline{c}$

$-\underline{a} \times \underline{b} = -\underline{b} \times \underline{c}$

$\underline{a} \times \underline{b} = \underline{b} \times \underline{c}$ —— ③

$\because \underline{b} \times \underline{a} = -\underline{a} \times \underline{b}$

From ② and ③

$\underline{a} \times \underline{b} = \underline{b} \times \underline{c} = \underline{c} \times \underline{a}$

Hence proved.

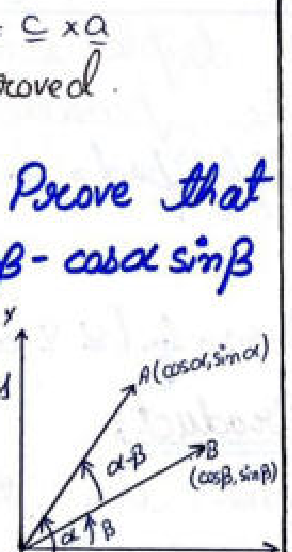
Question no 8: Prove that $\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

Suppose $\overrightarrow{OA}$ and $\overrightarrow{OB}$ are unit vectors

$\angle XOA = \alpha$

$\angle XOB = \beta$

$\angle BOA = \alpha - \beta.$



Also $|\overrightarrow{OA}| = |\overrightarrow{OB}| = 1$

As

$\overrightarrow{OA} = \cos\alpha\,\underline{i} + \sin\alpha\,\underline{j}$

$\overrightarrow{OB} = \cos\beta\,\underline{i} + \sin\beta\,\underline{j}.$

$$\overrightarrow{OB} \times \overrightarrow{OA} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \cos\beta & \sin\beta & 0 \\ \cos\alpha & \sin\alpha & 0 \end{vmatrix}$$

Expand by C_3

$|\overrightarrow{OB}||\overrightarrow{OA}| \sin(\alpha-\beta)\cancel{\underline{k}} = \cancel{\underline{k}}(\cos\beta\sin\alpha - \cos\alpha\sin\beta).$

$(1)(1)\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta.$

$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta.$

Hence proved.

Question no 9: If $\underline{a} \times \underline{b} = 0$ and $\underline{a} \cdot \underline{b} = 0$, what conclusion can be drawn about $\underline{a}$ and $\underline{b}$?

$\because \underline{a} \times \underline{b} = 0.$

$\Rightarrow |\underline{a}||\underline{b}| \sin\theta = 0.$

$\sin\theta = 0.$

$\theta = \sin^{-1}(0).$

$\theta = 0, \pi$

So $\underline{a}$ and $\underline{b}$ are parallel.

$\because \underline{a} \cdot \underline{b} = 0.$

$|\underline{a}||\underline{b}| \cos\theta = 0$

$\cos\theta = 0$

$\theta = \cos^{-1}(0).$

$\theta = 90°$

So $\underline{a}$ and $\underline{b}$ are perpendicular

At the same time parallel and perpendicular not possible, so one vector should be zero or null.

Theory:

Scalar Triple Product of vectors:

There are two types of triple product of vectors

i) Scalar triple Product: $(\underline{u}\times\underline{v})\cdot\underline{w}$ or $\underline{u}\cdot(\underline{v}\times\underline{w})$

ii) Vector triple Product: $\underline{u}\times(\underline{v}\times\underline{w})$

• **Definition:** Let $\underline{u}=a_1\underline{i}+b_1\underline{j}+c_1\underline{k}$, $\underline{v}=a_2\underline{i}+b_2\underline{j}+c_2\underline{k}$, $\underline{w}=a_3\underline{i}+b_3\underline{j}+c_3\underline{k}$ be three vectors. The scaler triple product of vectors $\underline{u}$, $\underline{v}$ and $\underline{w}$ is defined by $\underline{u}\cdot(\underline{v}\times\underline{w})$ or $\underline{v}\cdot(\underline{w}\times\underline{u})$ or $\underline{w}\cdot(\underline{u}\times\underline{v})$.

Note:

i) $(\underline{u}\times\underline{v})\cdot\underline{w}=\underline{u}\cdot(\underline{v}\times\underline{w})=[\underline{u}\ \underline{v}\ \underline{w}]$.

$(\underline{v}\times\underline{w})\cdot\underline{u}=\underline{v}\cdot(\underline{w}\times\underline{u})=[\underline{v}\ \underline{w}\ \underline{u}]$

$(\underline{w}\times\underline{u})\cdot\underline{v}=\underline{w}\cdot(\underline{u}\times\underline{v})=[\underline{w}\ \underline{u}\ \underline{v}]$.

ii) The value of the product changes if the order is not cyclic

iii) $\underline{u}\cdot\underline{v}\cdot\underline{w}$ and $\underline{u}\times(\underline{v}\cdot\underline{w})$ are meaningless

• **Volume of Parallelepiped:** The triple scalar product $(\underline{u}\times\underline{v})\cdot\underline{w}$ represents the volume of the parallelepiped.

$$\text{Volume of parallelepiped} = \underline{u}\cdot(\underline{v}\times\underline{w}).$$

• **Volume of Tetrahedron:**

$$\text{Volume of Tetrahedron} = \frac{1}{6}[\underline{u}\ \underline{v}\ \underline{w}]$$

• **Properties of triple scaler Product:**

• If $\underline{u}$, $\underline{v}$ and $\underline{w}$ are coplanar, then the volume of the parallelepiped so formed is zero, the vectors $\underline{u}$, $\underline{v}$, $\underline{w}$ are coplanar $\Leftrightarrow$ $(\underline{u}\times\underline{v})\cdot\underline{w}=0$ or $\underline{u}\cdot(\underline{v}\times\underline{w})=0$.

• If any two vectors of triple scalar product are equal, then its value is zero i.e; $[\underline{u}\ \underline{u}\ \underline{w}]=[\underline{u}\ \underline{v}\ \underline{v}]=0$.

• **Work done:**

Work done = (component of force along AB)(displacement

$=(F\cos\theta)(AB) = \underline{F}\cdot\overline{AB}$

• **Moment of Force:**

Moment of Force $= \underline{r}\times\underline{F}$

Example # 1: Find the volume of the parallelepiped determined by $\underline{u} = \underline{i}+2\underline{j}-\underline{k}$; $\underline{v} = \underline{i}-2\underline{j}+3\underline{k}$; $\underline{w} = \underline{i}-7\underline{j}-4\underline{k}$

Volume of the parallelepiped $= \underline{u}\cdot\underline{v}\times\underline{w} = \begin{vmatrix} 1 & 2 & -1 \\ 1 & -2 & 3 \\ 1 & -7 & -4 \end{vmatrix}$

Expand by R_1

$$\text{Volume} = 1(8+21)-2(-4-3)-1(-7+2)$$
$$= 1(29)-2(-7)-1(-5)$$
$$= 29+14+5$$
$$\boxed{\text{Volume} = 48}$$

Example # 2: Prove that four points $A(-3,5,-4)$; $B(-1,1,1)$; $C(-1,2,2)$; $D(-3,4,-5)$ are coplaner.

$$\overrightarrow{AB} = \overrightarrow{OB}-\overrightarrow{OA} = [-1,1,1]-[-3,5,-4]$$
$$= -\underline{i}+\underline{j}+\underline{k}+3\underline{i}-5\underline{j}+4\underline{k}$$
$$\overrightarrow{AB} = 2\underline{i}-4\underline{j}+5\underline{k}$$
$$\overrightarrow{AC} = \overrightarrow{OC}-\overrightarrow{OA} = [-1,2,2]-[-3,5,-4]$$
$$= -\underline{i}+2\underline{j}+2\underline{k}+3\underline{i}-5\underline{j}+4\underline{k}$$
$$\overrightarrow{AC} = 2\underline{i}-3\underline{j}+6\underline{k}$$
$$\overrightarrow{AD} = \overrightarrow{OD}-\overrightarrow{OA} = [-3,4,-5]-[-3,5,-4]$$
$$= -3\underline{i}+4\underline{j}-5\underline{k}+3\underline{i}-5\underline{j}+4\underline{k}$$
$$= 0\underline{i}-\underline{j}-\underline{k}$$

Volume of parallelepiped formed by $\overrightarrow{AB}$, $\overrightarrow{AC}$ and $\overrightarrow{AD}$ is

$$[\overrightarrow{AB}\ \overrightarrow{AC}\ \overrightarrow{AD}] = \begin{vmatrix} 2 & -4 & 5 \\ 2 & -3 & 6 \\ 0 & -1 & -1 \end{vmatrix}$$

Expand by R_1

$$= 2(3+6)+4(-2-0)+5(-2+0)$$
$$= 2(9)+4(-2)+5(-2)$$
$$= 18-8-10$$
$$= 0$$

As the volume is zero, so the points A, B, C and D are coplaner.

Example # 3: Find the volume of tetrahedron whose vertices are $A(2,1,8)$; $B(3,2,9)$ $C(2,1,4)$; $D(3,3,0)$.

$$\overrightarrow{AB} = \overrightarrow{OB}-\overrightarrow{OA} = [3,2,9]-[2,1,8]$$
$$= 3\underline{i}+2\underline{j}+9\underline{k}-2\underline{i}-\underline{j}-8\underline{k}$$
$$= \underline{i}+\underline{j}+\underline{k}$$
$$\overrightarrow{AC} = \overrightarrow{OC}-\overrightarrow{OA} = [2,1,4]-[2,1,8]$$
$$= 2\underline{i}+\underline{j}+4\underline{k}-2\underline{i}-\underline{j}-8\underline{k}$$
$$= 0\underline{i}+0\underline{j}-4\underline{k}$$
$$\overrightarrow{AD} = \overrightarrow{OD}-\overrightarrow{OA} = [3,3,0]-[2,1,8]$$
$$= 3\underline{i}+3\underline{j}+0\underline{k}-2\underline{i}-\underline{j}-8\underline{k}$$
$$= \underline{i}+2\underline{j}-8\underline{k}$$

$\therefore$ Volume of tetrahedron $= \frac{1}{6}[\overrightarrow{AB}\ \overrightarrow{AC}\ \overrightarrow{AD}]$

$$= \frac{1}{6}\begin{vmatrix} 1 & 1 & 1 \\ 0 & 0 & -4 \\ 1 & 2 & -8 \end{vmatrix}$$

Expand by R_2

$$= \frac{1}{6}[0+0+4(2-1)]$$
$$= \frac{1}{6}[4]$$
$$\boxed{\text{Volume of tetrahedron} = \frac{2}{3}}$$

Example # 4: Find the value of α, so that $\alpha\underline{i}+\underline{j}$; $\underline{i}+\underline{j}+3\underline{k}$ and $2\underline{i}+\underline{j}-2\underline{k}$ are coplaner

Let $\underline{u}=\alpha\underline{i}+\underline{j}$

$\underline{v}=\underline{i}+\underline{j}+3\underline{k}$

$\underline{w}=2\underline{i}+\underline{j}-2\underline{k}$

$$[\underline{u}\ \underline{v}\ \underline{w}]=\begin{vmatrix}\alpha & 1 & 0\\ 1 & 1 & 3\\ 2 & 1 & -2\end{vmatrix}$$

Expand by R_1.

$=\alpha(-2-3)-1(-2-6)+0$

$=-5\alpha+8$

The vectors will be coplaner if

$-5\alpha+8=0$

$-5\alpha=-8$

$$\boxed{\alpha=\frac{8}{5}}$$

Example # 5: Prove that the points whose position vectors are $A(-6\underline{i}+3\underline{j}+2\underline{k})$, $B(3\underline{i}-2\underline{j}+4\underline{k})$, $C(5\underline{i}+7\underline{j}+3\underline{k})$, $D(-13\underline{i}+17\underline{j}-\underline{k})$ are coplaner.

Let O be the origin.

$\overrightarrow{OA}=-6\underline{i}+3\underline{j}+2\underline{k}$

$\overrightarrow{OB}=3\underline{i}-2\underline{j}+4\underline{k}$

$\overrightarrow{OC}=5\underline{i}+7\underline{j}+3\underline{k}$

$\overrightarrow{OD}=-13\underline{i}+17\underline{j}-\underline{k}$

$\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=(3\underline{i}-2\underline{j}+4\underline{k})-(-6\underline{i}+3\underline{j}+2\underline{k})$

$=3\underline{i}-2\underline{j}+4\underline{k}+6\underline{i}-3\underline{j}-2\underline{k}$

$=9\underline{i}-5\underline{j}+2\underline{k}$

$\overrightarrow{AC}=\overrightarrow{OC}-\overrightarrow{OA}=(5\underline{i}+7\underline{j}+3\underline{k})-(-6\underline{i}+3\underline{j}+2\underline{k})$

$=5\underline{i}+7\underline{j}+3\underline{k}+6\underline{i}-3\underline{j}-2\underline{k}$

$=11\underline{i}+4\underline{j}+\underline{k}$

$\overrightarrow{AD}=\overrightarrow{OD}-\overrightarrow{OA}=(-13\underline{i}+17\underline{j}-\underline{k})-(-6\underline{i}+3\underline{j}+2\underline{k})$

$=-13\underline{i}+17\underline{j}-\underline{k}+6\underline{i}-3\underline{j}-2\underline{k}$

$=-7\underline{i}+14\underline{j}-3\underline{k}$

Now

$$\overrightarrow{AB}\cdot(\overrightarrow{AC}\times\overrightarrow{AD})=\begin{vmatrix}9 & -5 & 2\\ 11 & 4 & 1\\ -7 & 14 & -3\end{vmatrix}$$

Expand by R_1

$=9(-12-14)+5(-33+7)+2(154+28)$

$=9(-26)+5(-26)+2(182)$

$=-234-130+364$

$=0$

The points A, B, C and D are coplaner.

Example # 6: Find the work done by a constant force $\underline{F}=2\underline{i}+4\underline{j}$, if its point of application to a body moves it from A(1,1) to B(4,6)

$\underline{F}=2\underline{i}+4\underline{j}$

$\underline{d}=\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=[4,6]-[1,1]$

$=4\underline{i}+6\underline{j}-\underline{i}-\underline{j}=3\underline{i}+5\underline{j}$

$\therefore$ Work done $=\underline{F}\cdot\underline{d}$

$W=(2\underline{i}+4\underline{j})\cdot(3\underline{i}+5\underline{j})$

$W = (2)(3) + (4)(5)$

$W = 6+20$

$\boxed{W = 26\,J}$

Example # 7: The constant forces $2\underline{i}+5\underline{j}+6\underline{k}$ and $-\underline{i}+2\underline{j}+\underline{k}$ act on a body, which is displaced from position $P(4,-3,-2)$ to $Q(6,1,-3)$. Find the total work done.

let $F_1 = 2\underline{i}+5\underline{j}+6\underline{k}$

$F_2 = -\underline{i}+2\underline{j}+\underline{k}$

Total Force $= F_1 + F_2$

$= 2\underline{i}+5\underline{j}+6\underline{k}-\underline{i}+2\underline{j}+\underline{k}$

$= \underline{i}+7\underline{j}+7\underline{k}$

$\underline{d} = \overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$

$= (6\underline{i}+\underline{j}-3\underline{k}) - (4\underline{i}-3\underline{j}-2\underline{k})$

$= 6\underline{i}+\underline{j}-3\underline{k}-4\underline{i}+3\underline{j}+2\underline{k}$.

$\underline{d} = 2\underline{i}+4\underline{j}-\underline{k}$

∴ Work done $= \underline{F}\cdot\underline{d}$

$W = (\underline{i}+7\underline{j}+7\underline{k})\cdot(2\underline{i}+4\underline{j}-\underline{k})$

$= (1)(2)+(7)(4)+(7)(-1)$

$= 2+28-7$

$\boxed{W = 23\,J}$

Example # 8: Find the moment about the point $M(-2,4,-6)$ of the force represented by $\overrightarrow{AB}$, where coordinates of points A and B are $(1,2,-3)$ and $(3,-4,2)$ respectively.

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = [3,-4,2]-[1,2,-3]$

$= 3\underline{i}-4\underline{j}+2\underline{k}-\underline{i}-2\underline{j}+3\underline{k}$

$= 2\underline{i}-6\underline{j}+5\underline{k}$

$\overrightarrow{MA} = \overrightarrow{OA} - \overrightarrow{OM} = [1,2,-3]-[-2,4,-6]$

$= \underline{i}+2\underline{j}-3\underline{k}+2\underline{i}-4\underline{j}+6\underline{j}$

$= 3\underline{i}-2\underline{j}+3\underline{k}$

Moment of $\overrightarrow{AB}$ about $(-2,4,-6)$ $= \underline{r}\times\underline{F} = \overrightarrow{MA}\times\overrightarrow{AB}$

$$\overrightarrow{MA}\times\overrightarrow{AB} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 3 & -2 & 3 \\ 2 & -6 & 5 \end{vmatrix}$$

Expand by R_1

$= \underline{i}(-10+18) - \underline{j}(15-6) + \underline{k}(-18+4)$

$= 8\underline{i}-9\underline{j}-14\underline{k}$

Magnitude of the moment

$= \sqrt{(8)^2+(-9)^2+(-14)^2}$

$= \sqrt{64+81+196}$

$= \sqrt{341}$.

Exercise # 7.5

Question no 1: Find the volume of the parallelepiped for which the given vectors are three edges:

i) $\underline{u} = 3\underline{i} + 2\underline{k}$; $\underline{v} = \underline{i} + 2\underline{j} + \underline{k}$; $\underline{w} = -\underline{j} + 4\underline{k}$

$\therefore$ Volume of parallelepiped $= \underline{u}\cdot(\underline{v}\times\underline{w})$

$$\underline{u}\cdot(\underline{v}\times\underline{w}) = \begin{vmatrix} 3 & 0 & 2 \\ 1 & 2 & 1 \\ 0 & -1 & 4 \end{vmatrix}$$

Expand by R_1

$$= 3(8+1) - 0 + 2(-1-0)$$

$$= 3(9) + 2(-1)$$

$$= 27 - 2$$

$$\boxed{\text{Volume of } \parallel\text{elepiped} = 25}$$

ii) $\underline{u} = \underline{i} - 4\underline{j} - \underline{k}$; $\underline{v} = \underline{i} - \underline{j} - 2\underline{k}$; $\underline{w} = 2\underline{i} - 3\underline{j} + \underline{k}$

$\therefore$ Volume of parallelepiped $= \underline{u}\cdot(\underline{v}\times\underline{w})$

$$\underline{u}\cdot(\underline{v}\times\underline{w}) = \begin{vmatrix} 1 & -4 & -1 \\ 1 & -1 & -2 \\ 2 & -3 & 1 \end{vmatrix}$$

Expand by R_1

$$= 1(-1-6) + 4(1+4) - 1(-3+2)$$

$$= 1(-7) + 4(5) - 1(-1)$$

$$= -7 + 20 + 1$$

$$\boxed{\text{Volume of } \parallel\text{elepiped} = 14}$$

iii) $\underline{u} = \underline{i} - 2\underline{j} + 3\underline{k}$; $\underline{v} = 2\underline{i} - \underline{j} - \underline{k}$; $\underline{w} = \underline{j} + \underline{k}$

$\therefore$ Volume of parallelepiped $= \underline{u}\cdot(\underline{v}\times\underline{w})$

$$\underline{u}\cdot(\underline{v}\times\underline{w}) = \begin{vmatrix} 1 & -2 & 3 \\ 2 & -1 & -1 \\ 0 & 1 & 1 \end{vmatrix}$$

Expand by R_1

$$= 1(-1+1) + 2(2+0) + 3(2+0)$$

$$= 1(0) + 2(2) + 3(2)$$

$$= 0 + 4 + 6$$

$$\boxed{\text{Volume of } \parallel\text{elepiped} = 10}$$

Question no 2: Verify that

$\underline{a}\cdot\underline{b}\times\underline{c} = \underline{b}\cdot\underline{c}\times\underline{a} = \underline{c}\cdot\underline{a}\times\underline{b}$

if $\underline{a} = 3\underline{i} - \underline{j} + 5\underline{k}$, $\underline{b} = 4\underline{i} + 3\underline{j} - 2\underline{k}$ and $\underline{c} = 2\underline{i} + 5\underline{j} + \underline{k}$

We have to prove that

$$\underline{a}\cdot\underline{b}\times\underline{c} = \underline{b}\cdot\underline{c}\times\underline{a} = \underline{c}\cdot\underline{a}\times\underline{b}$$

Now

$$\underline{a}\cdot\underline{b}\times\underline{c} = \begin{vmatrix} 3 & -1 & 5 \\ 4 & 3 & -2 \\ 2 & 5 & 1 \end{vmatrix}$$

Expand by R_1

$$= 3(3+10) + 1(4+4) + 5(20-6)$$

$$= 3(13) + 1(8) + 5(14)$$

$$= 39 + 8 + 70$$

$$= 117 \text{ --- } ①$$

$$\underline{b}\cdot\underline{c}\times\underline{a} = \begin{vmatrix} 4 & 3 & -2 \\ 2 & 5 & 1 \\ 3 & -1 & 5 \end{vmatrix}$$

Expand by R_1

$$= 4(25+1) - 3(10-3) - 2(-2-15)$$

$$= 4(26) - 3(7) - 2(-17)$$

$$= 104 - 21 + 34$$

$$= 117 \text{ --- } ②$$

$$\underline{c}\cdot\underline{a}\times\underline{b} = \begin{vmatrix} 2 & 5 & 1 \\ 3 & -1 & 5 \\ 4 & 3 & -2 \end{vmatrix}$$

Expand by R_1

$$= 2(2-15) - 5(-6-20) + 1(9+4)$$

$$= 2(-13) - 5(-26) + 1(13)$$

$$= -26 + 130 + 13$$

$$= 117 \quad \text{——} ③$$

From ①, ② and ③

$$\underline{a}\cdot\underline{b}\times\underline{c} = \underline{b}\cdot\underline{c}\times\underline{a} = \underline{c}\cdot\underline{a}\times\underline{b}$$

Hence proved.

Question no 3: Prove that the vectors $\underline{i}-2\underline{j}+3\underline{k}$, $-2\underline{i}+3\underline{j}-4\underline{k}$ and $\underline{i}-3\underline{j}+5\underline{k}$ are coplaner.

Let

$\underline{u} = \underline{i}-2\underline{j}+3\underline{k}$

$\underline{v} = -2\underline{i}+3\underline{j}-4\underline{k}$

$\underline{w} = \underline{i}-3\underline{j}+5\underline{k}$

If the vectors are coplaner then $\underline{u}\cdot(\underline{v}\times\underline{w}) = 0$

$$\underline{u}\cdot(\underline{v}\times\underline{w}) = \begin{vmatrix} 1 & -2 & 3 \\ -2 & 3 & -4 \\ 1 & -3 & 5 \end{vmatrix}$$

Expand by R_1

$$= 1(15-12) + 2(-10+4) + 3(6-3)$$

$$= 1(3) + 2(-6) + 3(3)$$

$$= 3 - 12 + 9$$

$$= 12 - 12$$

$$= 0$$

So the vectors are coplaner.

Question no 4: Find the constant α such that the vectors are coplaner

i) $\underline{i}-\underline{j}+\underline{k}$, $\underline{i}-2\underline{j}-3\underline{k}$, $3\underline{i}-\alpha\underline{j}+5\underline{k}$

Let

$\underline{u} = \underline{i}-\underline{j}+\underline{k}$

$\underline{v} = \underline{i}-2\underline{j}-3\underline{k}$

$\underline{w} = 3\underline{i}-\alpha\underline{j}+5\underline{k}$

As these vectors are coplanar then $\underline{u}\cdot(\underline{v}\times\underline{w}) = 0$

$$\begin{vmatrix} 1 & -1 & 1 \\ 1 & -2 & -3 \\ 3 & -\alpha & 5 \end{vmatrix} = 0$$

Expand by R_1

$$1(-10-3\alpha)+1(5+9)+1(-\alpha+6)=0$$

$$-10-3\alpha+14-\alpha+6=0$$

$$-4\alpha+10=0$$

$$-4\alpha=-10$$

$$\alpha = -\frac{10}{-4}$$

$$\boxed{\alpha = \frac{5}{2}}$$

ii) $\underline{i}-2\alpha\underline{j}-\underline{k}$, $\underline{i}-\underline{j}+2\underline{k}$, $\alpha\underline{i}-2\underline{j}+\underline{k}$

Let

$\underline{u} = \underline{i}-2\alpha\underline{j}-\underline{k}$

$\underline{v} = \underline{i}-\underline{j}+2\underline{k}$

$\underline{w} = \alpha\underline{i}-2\underline{j}+\underline{k}$

As these vectors are coplanar then $\underline{u}\cdot(\underline{v}\times\underline{w}) = 0$

$$\begin{vmatrix} 1 & -2\alpha & -1 \\ 1 & -1 & 2 \\ \alpha & -2 & 1 \end{vmatrix} = 0$$

Expand by R_1

$1(-1+4)+2\alpha(1-2\alpha)-1(-2+\alpha)=0$

$3+2\alpha-4\alpha^2+2-\alpha=0$

$-4\alpha^2+\alpha+5=0$

$-(4\alpha^2-\alpha-5)=0$

$4\alpha^2-\alpha-5=0$

$4\alpha^2+4\alpha-5\alpha-5=0$

$4\alpha(\alpha+1)-5(\alpha+1)=0$

$(\alpha+1)(4\alpha-5)=0$

$\alpha+1=0\ ;\ 4\alpha-5=0$

$\alpha=-1\ ;\ \alpha=\frac{5}{4}$

$$\boxed{\alpha=-1,\frac{5}{4}}$$

Question no 5(a): Find the value of:

i) $2\underline{i}\times 2\underline{j}\cdot\underline{k}$

$=4\underline{k}\cdot\underline{k}$ ∴ $(\underline{i}\times\underline{j})=\underline{k}$

$=4(1)$ ∴ $(\underline{k}\cdot\underline{k})=1$

$=4$

ii) $3\underline{j}\cdot\underline{k}\times\underline{i}$

$=3\underline{j}\cdot\underline{j}$ ∴ $(\underline{k}\times\underline{i})=\underline{j}$

∴ $(\underline{j}\cdot\underline{j})=1$

$=3(1)$

$=3$

iii) $[\underline{k}\ \underline{i}\ \underline{j}]$

$=\underline{k}\cdot(\underline{i}\times\underline{j})$ ∴ $(\underline{i}\times\underline{j})=\underline{k}$

$=\underline{k}\cdot\underline{k}$ ∴ $\underline{k}\times\underline{k}=1$

$=1$

iv) $[\underline{i}\ \underline{i}\ \underline{k}]$

$=\underline{i}\cdot(\underline{i}\times\underline{k})$ ∴ $(\underline{i}\times\underline{k})=-\underline{j}$

$=\underline{i}\cdot-\underline{j}$ ∴ $\underline{i}\cdot-\underline{j}=0$

$=0$

(b) Prove that

$\underline{u}\cdot(\underline{v}\times\underline{w})+\underline{v}\cdot(\underline{w}\times\underline{u})+\underline{w}\cdot(\underline{u}\times\underline{v})=3\underline{u}\cdot(\underline{v}\times\underline{w})$

L.H.S

$=\underline{u}\cdot(\underline{v}\times\underline{w})+\underline{v}\cdot(\underline{w}\times\underline{u})+\underline{w}\cdot(\underline{u}\times\underline{v})$

$=\underline{u}\cdot(\underline{v}\times\underline{w})+(\underline{v}\times\underline{w})\cdot\underline{u}+(\underline{u}\times\underline{v})\cdot\underline{w}$

$=\underline{u}\cdot(\underline{v}\times\underline{w})+\underline{u}\cdot(\underline{v}\times\underline{w})+\underline{u}\cdot(\underline{v}\times\underline{w})$

$=3\underline{u}\cdot(\underline{v}\times\underline{w})=$ R.H.S

Hence proved

Question no 6: Find volume of the Tetrahedron with the Vertices

i) (0,1,2), (3,2,1), (1,2,1) and (5,5,6)

Let

$A(0,1,2)\ ;\ \overrightarrow{OA}=0\underline{i}+\underline{j}+2\underline{k}$

$B(3,2,1)\ ;\ \overrightarrow{OB}=3\underline{i}+2\underline{j}+\underline{k}$

$C(1,2,1)\ ;\ \overrightarrow{OC}=\underline{i}+2\underline{j}+\underline{k}$

$D(5,5,6)\ ;\ \overrightarrow{OD}=5\underline{i}+5\underline{j}+6\underline{k}$

∴ Volume of tetrahedron $=\frac{1}{6}[\overrightarrow{AB}\ \overrightarrow{AC}\ \overrightarrow{AD}]$

$\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=3\underline{i}+2\underline{j}+\underline{k}-(0\underline{i}+\underline{j}+2\underline{k})$

$=3\underline{i}+2\underline{j}+\underline{k}-0\underline{i}-\underline{j}-2\underline{k}=3\underline{i}+\underline{j}-\underline{k}$

$\overrightarrow{AC}=\overrightarrow{OC}-\overrightarrow{OA}=\underline{i}+2\underline{j}+\underline{k}-(0\underline{i}+\underline{j}+2\underline{k})$

$=\underline{i}+2\underline{j}+\underline{k}-0\underline{i}-\underline{j}-2\underline{k}=\underline{i}+\underline{j}-\underline{k}$

$\overrightarrow{AD}=\overrightarrow{OD}-\overrightarrow{OA}=5\underline{i}+5\underline{j}+6\underline{k}-(0\underline{i}+\underline{j}+2\underline{k})$

$=5\underline{i}+5\underline{j}+6\underline{k}-0\underline{i}-\underline{j}-2\underline{k}=5\underline{i}+4\underline{j}+4\underline{k}$

$$[\overrightarrow{AB}\ \overrightarrow{AC}\ \overrightarrow{AD}]=\begin{vmatrix}3 & 1 & -1\\ 1 & 1 & -1\\ 5 & 4 & 4\end{vmatrix}$$

Expand by R_1

$=3(4+4)-1(4+5)-1(4-5)$

$=3(8)-1(9)-1(-1)$

$=24-9+1=16$

Volume of tetrahedron $= \frac{1}{6}(16)$

$$\boxed{\text{Volume of tetrahedron} = \frac{8}{3}}$$

ii) $(2,1,8), (3,2,9), (2,1,4), (3,3,10)$

Let

$A(2,1,8)\,;\ \overrightarrow{OA} = 2\underline{i} + \underline{j} + 8\underline{k}$

$B(3,2,9)\,;\ \overrightarrow{OB} = 3\underline{i} + 2\underline{j} + 9\underline{k}$

$C(2,1,4)\,;\ \overrightarrow{OC} = 2\underline{i} + \underline{j} + 4\underline{k}$

$D(3,3,10)\,;\ \overrightarrow{OD} = 3\underline{i} + 3\underline{j} + 10\underline{k}$

∴ Volume of tetrahedron $= \frac{1}{6}[\overrightarrow{AB}\ \overrightarrow{AC}\ \overrightarrow{AD}]$

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = 3\underline{i} + 2\underline{j} + 9\underline{k} - (2\underline{i} + \underline{j} + 8\underline{k})$

$= 3\underline{i} + 2\underline{j} + 9\underline{k} - 2\underline{i} - \underline{j} - 8\underline{k} = \underline{i} + \underline{j} + \underline{k}$

$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = 2\underline{i} + \underline{j} + 4\underline{k} - (2\underline{i} + \underline{j} + 8\underline{k})$

$= 2\underline{i} + \underline{j} + 4\underline{k} - 2\underline{i} - \underline{j} - 8\underline{k} = 0\underline{i} + 0\underline{j} - 4\underline{k}$

$\overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA} = 3\underline{i} + 3\underline{j} + 10\underline{k} - (2\underline{i} + \underline{j} + 8\underline{k})$

$= 3\underline{i} + 3\underline{j} + 10\underline{k} - 2\underline{i} - \underline{j} - 8\underline{k} = \underline{i} + 2\underline{j} + 2\underline{k}$

$$[\overrightarrow{AB}\ \overrightarrow{AC}\ \overrightarrow{AD}] = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 0 & -4 \\ 1 & 2 & 2 \end{vmatrix}$$

Expand by R_2

$= 0 + 0 - (-4)(2-1)$

$= 4(1)$

$= 4$

$$\boxed{\text{Volume of tetrahedron} = \frac{1}{6}(4) = \frac{2}{3}}$$

Question no 7: Find the work done, if the point at which the constant force $\underline{F} = 4\underline{i} + 3\underline{j} + 5\underline{k}$ is applied to an object, moves from $P_1(3,1,-2)$ to $P_2(2,4,6)$.

Given data:

$\underline{F} = 4\underline{i} + 3\underline{j} + 5\underline{k}$

$P_1(3,1,-2)\,;\ \overrightarrow{OP_1} = 3\underline{i} + \underline{j} - 2\underline{k}$

$P_2(2,4,6)\,;\ \overrightarrow{OP_2} = 2\underline{i} + 4\underline{j} + 6\underline{k}$

$\underline{d} = \overrightarrow{P_1P_2} = \overrightarrow{OP_2} - \overrightarrow{OP_1}$

$\underline{d} = 2\underline{i} + 4\underline{j} + 6\underline{k} - (3\underline{i} + \underline{j} - 2\underline{k})$

$\underline{d} = 2\underline{i} + 4\underline{j} + 6\underline{k} - 3\underline{i} - \underline{j} + 2\underline{k}$

$\underline{d} = -\underline{i} + 3\underline{j} + 8\underline{k}$

∴ Work done $= \underline{F} \cdot \underline{d}$

$W = (4\underline{i} + 3\underline{j} + 5\underline{k}) \cdot (-\underline{i} + 3\underline{j} + 8\underline{k})$

$= (4)(-1) + (3)(3) + (5)(8)$

$= -4 + 9 + 40$

$$\boxed{\text{Work done} = 45}$$

Question no 8: A particle, acted by constant forces $4\underline{i} + \underline{j} - 3\underline{k}$ and $3\underline{i} - \underline{j} - \underline{k}$, is displacement from $A(1,2,3)$ to $B(5,4,1)$. Find the work done

Given Data:

Let $\underline{F}_1 = 4\underline{i} + \underline{j} - 3\underline{k}$

$\underline{F}_2 = 3\underline{i} - \underline{j} - \underline{k}$

Total Force $= \underline{F} = \underline{F}_1 + \underline{F}_2$

$= 4\underline{i} + \cancel{\underline{j}} - 3\underline{k} + 3\underline{i} - \cancel{\underline{j}} - \underline{k}$

$\underline{F} = 7\underline{i} - 4\underline{k}$

$A(1,2,3)\,;\ \overrightarrow{OA} = \underline{i} + 2\underline{j} + 3\underline{k}$

$B(5,4,1)\,;\ \overrightarrow{OB} = 5\underline{i} + 4\underline{j} + \underline{k}$

$\underline{d} = \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$\underline{d} = 5\underline{i} + 4\underline{j} + \underline{k} - (\underline{i} + 2\underline{j} + 3\underline{k})$

$\underline{d} = 5\underline{i} + 4\underline{j} + \underline{k} - \underline{i} - 2\underline{j} - 3\underline{k}$

$\underline{d} = 4\underline{i} + 2\underline{j} - 2\underline{k}$

∴ Work done $= \underline{F} \cdot \underline{d}$

$W = (7\underline{i} - 4\underline{k}) \cdot (4\underline{i} + 2\underline{j} - 2\underline{k})$

$w = (7)(4) + (0)(2) + (-4)(-2)$

$W = 28 + 0 + 8$

$\boxed{\text{Work done} = 36}$

Question no 9: Given Data

$A(5,-5,-7)$; $B(6,2,-2)$

$\underline{F}_1 = 10\underline{i} - \underline{j} + 11\underline{k}$; $\underline{F}_2 = 4\underline{i} + 5\underline{j} + 9\underline{k}$; $\underline{F}_3 = -2\underline{i} + \underline{j} - 9\underline{k}$

Show that the total work done by the forces is 102 units.

$\underline{F}_1 = 10\underline{i} - \underline{j} + 11\underline{k}$

$\underline{F}_2 = 4\underline{i} + 5\underline{j} + 9\underline{k}$

$F_3 = -2\underline{i} + \underline{j} - 9\underline{k}$

Total Force $= \underline{F} = \underline{F}_1 + \underline{F}_2 + \underline{F}_3$

$= 10\underline{i} - \underline{j} + 11\underline{k} + 4\underline{i} + 5\underline{j} + 9\underline{k} - 2\underline{i} + \underline{j} - 9\underline{k}$

$\underline{F} = 12\underline{i} + 5\underline{j} + 11\underline{k}$

$A(5,-5,7)$; $\overrightarrow{OA} = 5\underline{i} - 5\underline{j} - 7\underline{k}$

$B(6,2,-2)$; $\overrightarrow{OB} = 6\underline{i} + 2\underline{j} - 2\underline{k}$

$\underline{d} = \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$\underline{d} = 6\underline{i} + 2\underline{j} - 2\underline{k} - (5\underline{i} - 5\underline{j} - 7\underline{k})$

$\underline{d} = 6\underline{i} + 2\underline{j} - 2\underline{k} - 5\underline{i} + 5\underline{j} + 7\underline{k}$

$\underline{d} = \underline{i} + 7\underline{j} + 5\underline{k}$

$\therefore$ Workdone $= \underline{F} \cdot \underline{d}$

$W = (12\underline{i} + 5\underline{j} + 11\underline{k}) \cdot (\underline{i} + 7\underline{j} + 5\underline{k})$

$W = (12)(1) + (5)(7) + (11)(5)$

$W = 12 + 35 + 55$

$\boxed{\text{Work done} = 102}$

Question no 10: A force of magnitude 6 units acting parallel to $2\underline{i} - 2\underline{j} + \underline{k}$ displaces, the point of application from (1,2,3) to (5,3,7). Find the work done.

Required force $= \underline{F}$, $|\underline{F}| = 6$.

Let Given vector $= \underline{v} = 2\underline{i} - 2\underline{j} + \underline{k}$

As $\underline{F} \parallel \underline{v}$, so

$\underline{F} = |\underline{F}|\, \hat{v}$

$\underline{F} = |\underline{F}| \dfrac{\underline{v}}{|\underline{v}|}$

$\underline{F} = 6 \cdot \dfrac{2\underline{i} - 2\underline{j} + \underline{k}}{\sqrt{(2)^2 + (-2)^2 + (1)^2}}$

$\underline{F} = 6 \dfrac{2\underline{i} - 2\underline{j} + \underline{k}}{\sqrt{4+4+1}}$

$\underline{F} = 6 \dfrac{2\underline{i} - 2\underline{j} + \underline{k}}{\sqrt{9}}$

$\underline{F} = 6 \dfrac{2\underline{i} - 2\underline{j} + \underline{k}}{3}$

$\underline{F} = 2(2\underline{i} - 2\underline{j} + \underline{k})$

$\underline{F} = 4\underline{i} - 4\underline{j} + 2\underline{k}$

Let $A(1,2,3)$; $\overrightarrow{OA} = \underline{i} + 2\underline{j} + 3\underline{k}$

$B(5,3,7)$; $\overrightarrow{OB} = 5\underline{i} + 3\underline{j} + 7\underline{k}$

$\underline{d} = \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$\underline{d} = 5\underline{i} + 3\underline{j} + 7\underline{k} - (\underline{i} + 2\underline{j} + 3$

$\underline{d} = 5\underline{i} + 3\underline{j} + 7\underline{k} - \underline{i} - 2\underline{j} - 3\underline{k}$

$\underline{d} = 4\underline{i} + \underline{j} + 4\underline{k}$

Work done $= \underline{F} \cdot \underline{d}$

Work done $= (4\underline{i} - 4\underline{j} + 2\underline{k}) \cdot (4\underline{i} + \underline{j} + 4$

Work done $= (4)(4) + (-4)(1) + (2)(4)$

work done $= 16 - 4 + 8$

$\boxed{\text{Work done} = 20\text{units}}$

Question no 11: A force $\underline{F} = 3\underline{i} + 2\underline{j} - 4\underline{k}$ is applied at the point (1,-1,2). Find the moment of the force about the point (2,-1,3)

Given Data:

$$\underline{F} = 3\underline{i} + 2\underline{j} - 4\underline{k}$$

$$A(1,-1,2)\ ;\ \overrightarrow{OA} = \underline{i} - \underline{j} + 2\underline{k}$$

$$B(2,-1,3)\ ;\ \overrightarrow{OB} = 2\underline{i} - \underline{j} + 3\underline{k}$$

$$\underline{r} = \overrightarrow{BA} = \overrightarrow{OA} - \overrightarrow{OB}$$

$$\underline{r} = \underline{i} - \underline{j} + 2\underline{k} - (2\underline{i} - \underline{j} + 3\underline{k})$$

$$\underline{r} = \underline{i} - \underline{j} + 2\underline{k} - 2\underline{i} + \underline{j} - 3\underline{k}$$

$$\underline{r} = -\underline{i} + 0\underline{j} - \underline{k}$$

Moment $= \vec{r} \times \vec{F}$

$$= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ -1 & 0 & -1 \\ 3 & 2 & -4 \end{vmatrix}$$

Expand by R_1

$$= \underline{i}(0+2) - \underline{j}(4+3) + \underline{k}(-2-0)$$

$$\boxed{\text{Moment} = 2\underline{i} - 7\underline{j} - 2\underline{k}}$$

Question no 12: A force $\underline{F} = 4\underline{i} - 3\underline{k}$, passes through the point A(2,-2,5). Find the Moment of $\underline{F}$ about the point B(1,-3,1).

Given Data:

$$\underline{F} = 4\underline{i} - 3\underline{k}$$

$$A(2,-2,5)\ ;\ \overrightarrow{OA} = 2\underline{i} - 2\underline{j} + 5\underline{k}$$

$$B(1,-3,1)\ ;\ \overrightarrow{OB} = \underline{i} - 3\underline{j} + \underline{k}$$

$$\underline{r} = \overrightarrow{BA} = \overrightarrow{OA} - \overrightarrow{OB}$$

$$\underline{r} = 2\underline{i} - 2\underline{j} + 5\underline{k} - (\underline{i} - 3\underline{j} + \underline{k})$$

$$\underline{r} = 2\underline{i} - 2\underline{j} + 5\underline{k} - \underline{i} + 3\underline{j} - \underline{k}$$

$$\underline{r} = \underline{i} + \underline{j} + 4\underline{k}$$

Moment $= \underline{r} \times \underline{F}$

$$= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 1 & 4 \\ 4 & 0 & -3 \end{vmatrix}$$

Expand by R_1

$$= \underline{i}(-3-0) - \underline{j}(-3-16) + \underline{k}(0-4)$$

$$= \underline{i}(-3) - \underline{j}(-19) + \underline{k}(-4)$$

$$\boxed{\text{Moment} = -3\underline{i} + 19\underline{j} - 4\underline{k}}$$

Question no 13: Give a force $\underline{F} = 2\underline{i} + \underline{j} - 3\underline{k}$ acting at a point A(1,-2,1). Find the moment of $\underline{F}$ about the point B(2,0,-2)

Given Data:

$$\underline{F} = 2\underline{i} + \underline{j} - 3\underline{k}$$

$$A(1,-2,1);\ \overrightarrow{OA} = \underline{i} - 2\underline{j} + \underline{k}$$

$$B(2,0,-2);\ \overrightarrow{OB} = 2\underline{i} + 0\underline{j} - 2\underline{k}$$

$$\underline{r} = \overrightarrow{BA} = \overrightarrow{OA} - \overrightarrow{OB}$$

$$\underline{r} = \underline{i} - 2\underline{j} + \underline{k} - (2\underline{i} + 0\underline{j} - 2\underline{k})$$

$$\underline{r} = \underline{i} - 2\underline{j} + \underline{k} - 2\underline{i} - 0\underline{j} + 2\underline{k}$$

$$\underline{r} = -\underline{i} - 2\underline{j} + 3\underline{k}$$

Moment $= \underline{r} \times \underline{F}$

$$= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ -1 & -2 & 3 \\ 2 & 1 & -3 \end{vmatrix}$$

Expand by R_1

$$= \underline{i}(6-3) - \underline{j}(3-6) + \underline{k}(-1+4)$$

$= \underline{i}(3) - \underline{j}(-3) + \underline{k}(3)$

$\boxed{\text{Moment} = 3\underline{i} + 3\underline{j} + 3\underline{k}}$

Question no 14: Find the moment about $A(1,1,1)$ of each of the concurrent forces $\underline{i}-2\underline{j}$, $3\underline{i}+2\underline{j}-\underline{k}$, $5\underline{j}+2\underline{k}$, where $P(2,0,1)$ is their point of concurrency.

Given Data:

$A(1,1,1)$; $\vec{OA} = \underline{i}+\underline{j}+\underline{k}$

$P(2,0,1)$; $\vec{OP} = 2\underline{i}+0\underline{j}+\underline{k}$

$\underline{F}_1 = \underline{i} - 2\underline{j}$

$\underline{F}_2 = 3\underline{i}+2\underline{j}-\underline{k}$

$\underline{F}_3 = 5\underline{j}+2\underline{k}$

Total Force $= \underline{F} = \underline{F}_1+\underline{F}_2+\underline{F}_3$

$\underline{F} = \underline{i}-2\underline{j}+3\underline{i}+2\underline{j}-\underline{k}+5\underline{j}+2\underline{k}$

$\underline{F} = 4\underline{i}+5\underline{j}+\underline{k}$

Now we find $\underline{r}$

$\underline{r} = \vec{AP} = \vec{OP} - \vec{OA}$

$\underline{r} = 2\underline{i}+0\underline{j}+\underline{k}-(\underline{i}+\underline{j}-\underline{k})$

$\underline{r} = 2\underline{i}+0\underline{j}+\underline{k}-\underline{i}-\underline{j}-\underline{k}$

$\underline{r} = \underline{i}-\underline{j}+0\underline{k}$

Moment = $\underline{r} \times \underline{F}$

$$M = \underline{r}\times\underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & -1 & 0 \\ 4 & 5 & 1 \end{vmatrix}$$

Expand by R_1

$= \underline{i}(-1-0) - \underline{j}(1-0) + \underline{k}(5+4)$

$= \underline{i}(-1) - \underline{j}(1) + \underline{k}(9)$

$\boxed{\text{Moment} = -\underline{i}-\underline{j}+9\underline{k}}$

Question no 15: A force $\underline{F} = 7\underline{i}+4\underline{j}-3\underline{k}$ is applied at $P(1,-2,3)$. Find its moment about the point $Q(2,1,1)$.

Given Data:

$\underline{F} = 7\underline{i}+4\underline{j}-3\underline{k}$

$P(1,-2,3)$; $\vec{OP} = \underline{i}-2\underline{j}+3\underline{k}$

$Q(2,1,1)$; $\vec{OQ} = 2\underline{i}+\underline{j}+\underline{k}$

Now we find $\underline{r}$

$\underline{r} = \vec{QP} = \vec{OP} - \vec{OQ}$

$\underline{r} = \underline{i}-2\underline{j}+3\underline{k}-(2\underline{i}+\underline{j}+\underline{k})$

$\underline{r} = \underline{i}-2\underline{j}+3\underline{k}-2\underline{i}-\underline{j}-\underline{k}$

$\underline{r} = -\underline{i}-3\underline{j}+2\underline{k}$

Moment = $\underline{r} \times \underline{F}$

$$\text{Moment} = \underline{r}\times\underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ -1 & -3 & 2 \\ 7 & 4 & -3 \end{vmatrix}$$

Expand by R_1

$= \underline{i}(9-8) - \underline{j}(3-14) + \underline{k}(-4+21)$

$= \underline{i}(1) - \underline{j}(-11) + k(17)$

$\boxed{\text{Moment} = \underline{i}+11\underline{j}+17\underline{k}}$

The End

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