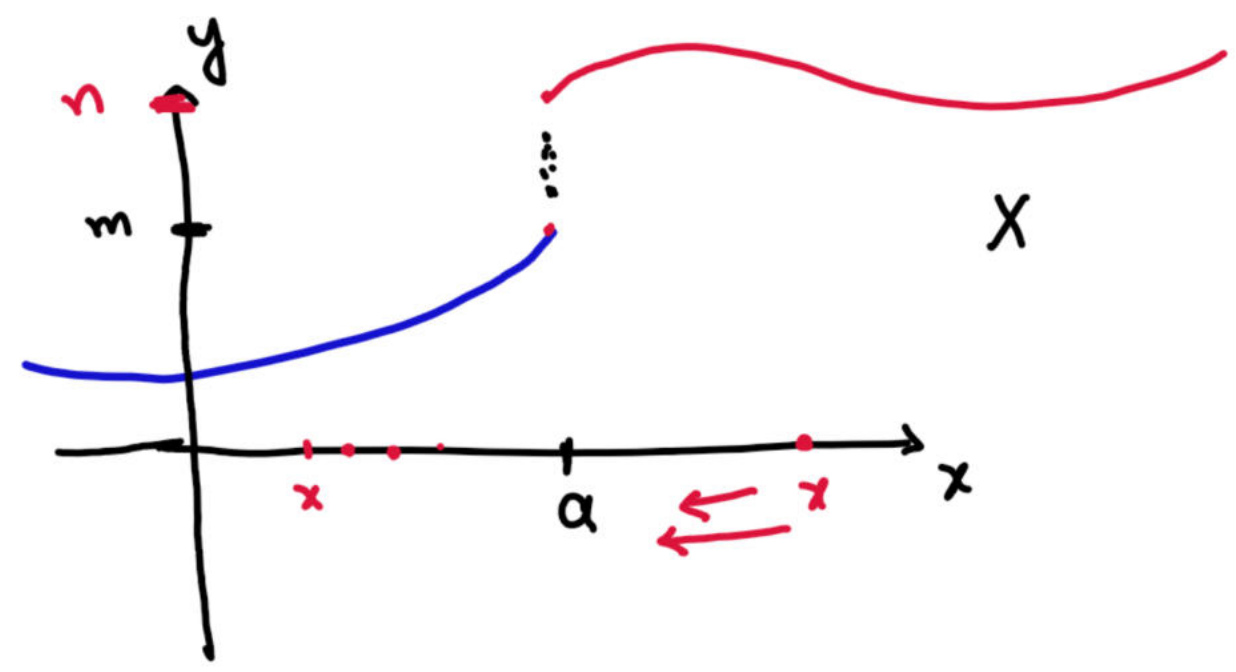
Important Points for Exercise 1.4

One Sided Limits



$$\lim_{x \to a^-} f(x) = m$$

$(x < a)$ (Left-hand limit)

$$\lim_{x \to a^+} f(x) = n$$

$(x > a)$ (Right-hand limit)

$\lim_{x \to a} f(x)$ exists if $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$.

Continuous Function

A function $f(x)$ is continuous at $x = a$ if

(i) $f(a)$ is defined. $(a \in Dom(f))$.

(ii) $\lim_{x \to a} f(x)$ exists. $\left(\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)\right)$.

(iii) $\lim_{x \to a} f(x) = f(a)$.

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Suppose Math.

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Exercise 2.4

1. Evaluate the following limits.

(i) $\lim_{x\to 2^+}\frac{x-2}{|x-2|}$ (ii) $\lim_{x\to 1^-}\frac{x^2+2x-3}{|x-1|}$ (iii) $\lim_{x\to 2}\frac{x^2+4x-12}{|x-2|}$

$$|x| = \begin{cases} x & \text{if } x\geq 0 \\ -x & \text{if } x<0 \end{cases}$$

$$|x-2| = \begin{cases} x-2 & \text{if } x-2\geq 0 \\ -(x-2) & \text{if } x-2<0 \end{cases}$$

$$= \begin{cases} x-2 & \text{if } x\geq 2 & \text{R.H.L} \\ -x+2 & \text{if } x<2 & \text{L.H.L} \end{cases}$$

$$|x-1| = \begin{cases} x-1 & \text{if } x-1\geq 0 \\ -(x-1) & \text{if } x-1<0 \end{cases}$$

$$= \begin{cases} x-1 & \text{if } x\geq 1 \\ -x+1 & \text{if } x<1 \end{cases}$$

(i) $\lim_{x\to 2^+}\frac{x-2}{|x-2|}$

$x>2$

$$= \lim_{x\to 2}\frac{\cancel{x-2}}{\cancel{x-2}} = \lim_{x\to 2}(1)$$

$$= 1 \quad \text{Ans}$$

(ii) $\lim_{x\to 1^-}\frac{x^2+2x-3}{|x-1|}$

$x<1$

$$= \lim_{x\to 1}\frac{x^2+2x-3}{-x+1}$$

$$= \lim_{x\to 1}\frac{x^2+3x-x-3}{-(x-1)} = \lim_{x\to 1}\frac{x(x+3)-1(x+3)}{-(x-1)}$$

$$= \lim_{x\to 1}\frac{\cancel{(x-1)}(x+3)}{-\cancel{(x-1)}} = -\lim_{x\to 1}(x+3)$$

$$= -(1+3) = -4 \quad \text{Ans}$$

(iii) $\lim_{x\to 2}\frac{x^2+4x-12}{|x-2|}$

L.H.L

$$\lim_{x\to 2^-}\frac{x^2+4x-12}{|x-2|}$$

$$= \lim_{x\to 2}\frac{x^2+4x-12}{-x+2} = \lim_{x\to 2}\frac{x^2+6x-2x-12}{-(x-2)}$$

$$= \lim_{x\to 2}\frac{x(x+6)-2(x+6)}{-(x-2)} = \lim_{x\to 2}\frac{\cancel{(x-2)}(x+6)}{-\cancel{(x-2)}}$$

$$= -(2+6) = -8$$

R.H.L

$$\lim_{x\to 2^+}\frac{x^2+4x-12}{|x-2|}$$

$$= \lim_{x\to 2}\frac{x^2+4x-12}{x-2}$$

$$= \lim_{x\to 2}\frac{\cancel{(x-2)}(x+6)}{\cancel{x-2}}$$

$$= 2+6 = 8$$

Since L.H.L $\neq$ R.H.L. so $\lim_{x\to 2}\frac{x^2+4x-12}{|x-2|}$ does not exist.

2. Determine whether $\lim_{x\to1} f(x)$, $\lim_{x\to2} f(x)$, $\lim_{x\to3} f(x)$ and $\lim_{x\to4} f(x)$ exit, when

$$f(x)=\begin{cases}2x+1 & if\ 0\le x\le 2\\ x-7 & if\ 2\le x\le 4\\ x & if\ 4\le x\le 6\end{cases}$$

At $x=1$

L.H.L $\lim_{x\to1^-} f(x) = \lim_{x\to1}(2x+1) = 2(1)+1 = 2+1 = 3$

R.H.L $\lim_{x\to1^+} f(x) = \lim_{x\to1}(2x+1) = 2(1)+1 = 2+1 = 3$

So $\lim_{x\to1} f(x)$ exists.

At $x=2$

L.H.L

$\lim_{x\to2^-} f(x) = \lim_{x\to2}(2x+1) = 2(2)+1 = 5$ ✓

R.H.L

$\lim_{x\to2^+} f(x) = \lim_{x\to2}(x-7) = 2-7 = -5$ ✓

Since L.H.L $\neq$ R.H.L, so

$\lim_{x\to2} f(x)$ does not exist.

At $x=3$

L.H.L

$\lim_{x\to3^-} f(x) = \lim_{x\to3}(x-7) = 3-7 = -4$

R.H.L

$\lim_{x\to3^+} f(x) = \lim_{x\to3}(x-7) = 3-7 = -4$

Since L.H.L = R.H.L, so

$\lim_{x\to3} f(x)$ exists.

At $x=4$

L.H.L

$\lim_{x\to4^-} f(x) = \lim_{x\to4}(x-7) = 4-7 = -3$

R.H.L

$\lim_{x\to4^+} f(x) = \lim_{x\to4}(x) = 4$

Since L.H.L $\neq$ R.H.L, so

$\lim_{x\to4} f(x)$ does not exist.

3. Test the continuity and discontinuity of the following functions.

(i) $f(x) = \sin(x^2 + \pi x) + 7x^2 + x$ at a point $x = 0$

(ii) $f(x) = \frac{2 - \cos 3x - \cos 4x}{x}$ at a point $x = 0$

(iii) $f(x) = \begin{cases} 7 + 3x, & \text{when } x < 1 \\ 1 - 5x, & \text{when } x \geq 1 \end{cases}$ at $x = 1$

(i) $f(x) = \sin(x^2 + \pi x) + 7x^2 + x$ at $x = 0$.

Value $f(0) = \sin(0^2 + 0) + 0 + 0 = 0$

L.H.L $\lim_{x \to 0^-} f(x) = \lim_{x \to 0} \sin(x^2 + \pi x) + 7x^2 + x = \sin(0+0) + 0 + 0 = 0$

R.H.L $\lim_{x \to 0^+} f(x) = \lim_{x \to 0} \sin(x^2 + \pi x) + 7x^2 + x = \sin(0+0) + 0 + 0 = 0$

So $\lim_{x \to 0} f(x) = 0$

Since $f(0) = \lim_{x \to 0} f(x)$, so function is continuous at $x = 0$.

(ii) $f(x) = \frac{2 - \cos 3x - \cos 4x}{x}$ at $x = 0$

Value $f(0) = \frac{2 - \cos 0 - \cos 0}{0} = \frac{2-1-1}{0} = \frac{0}{0}$ (indeterminate form)

$\Rightarrow f$ is not defined at $x = 0$.

So $f(x)$ is discontinuous at $x = 0$.

(iii) $f(x) = \begin{cases} 7 + 3x, & x < 1 \\ 1 - 5x, & x \geq 1 \end{cases}$ at $x = 1$.

Value $f(1) = 1 - 5(1) = 1 - 5 = -4$

L.H.L $\lim_{x \to 1^-} f(x) = \lim_{x \to 1} (7 + 3x) = 7 + 3(1) = 7 + 3 = 10$

R.H.L

$\lim_{x \to 1^+} f(x) = \lim_{x \to 1} (1 - 5x) = 1 - 5(1) = 1 - 5 = -4$

Since L.H.L $\neq$ R.H.L, so $\lim_{x \to 1} f(x)$ does not exist.

So $f(x)$ is discontinuous at $x = 1$.

4. Determine whether the following function are continuous at $x = 2$

(i) $f(x) = \frac{x^2-4}{x-2}$ (ii) $g(x) = \begin{cases} \frac{x^2-4}{x-2} & \text{when } x \neq 2 \\ 3 & \text{when } x = 2 \end{cases}$

(iii) $h(x) = \begin{cases} \frac{x^2-4}{x-2} & \text{when } x \neq 2 \\ 4 & \text{when } x = 2 \end{cases}$

(i) $f(x) = \frac{x^2-4}{x-2}$

Value $f(2) = \frac{2^2-4}{2-2} = \frac{4-4}{2-2} = \frac{0}{0}$

$\Rightarrow$ $f(2)$ is not defined.

So $f(x)$ is discontinuous at $x = 2$.

(ii) $f(x) = \begin{cases} \frac{x^2-4}{x-2} & \text{when } x \neq 2 \quad (x<2,\ x>2) \\ 3 & \text{when } x = 2 \end{cases}$

Value $f(2) = 3$

L.H.L $\lim_{x \to 2^-} f(x) = \lim_{x\to 2}\left(\frac{x^2-4}{x-2}\right) = \lim_{x \to 2}\left(\frac{(x+2)\cancel{(x-2)}}{\cancel{x-2}}\right) = 2+2 = 4.$

R.H.L $\lim_{x \to 2^+} f(x) = \lim_{x\to 2}\left(\frac{x^2-4}{x-2}\right) = 4$

Since L.H.L = R.H.L, so $\lim_{x\to 2} f(x) = 4.$

Since $f(2) \neq \lim_{x \to 2} f(x)$, so $f(x)$ is discontinuous at $x = 2$.

(iii) $f(x) = \begin{cases} \frac{x^2-4}{x-2}, & x \neq 2 \\ 4, & x = 2 \end{cases}$

Value $f(2) = 4$

$\lim_{x\to 2} f(x) = \lim_{x\to 2}\left(\frac{x^2-4}{x-2}\right) = \lim_{x\to 2}\left(\frac{(x+2)\cancel{(x-2)}}{\cancel{x-2}}\right) = 2+2 = 4$

Also $f(2) = \lim_{x \to 2} f(x)$

So $f(x)$ is continuous at $x = 2$.

5. Suppose that $f(x)=\begin{cases} -x^4+3 & \text{when } x\le 2 \\ x^2+9 & \text{when } x>2 \end{cases}$

Is continuous everywhere justify your conclusion?

$$f(x)=\begin{cases} -x^4+3 & \text{when } x\le 2 \\ x^2+9 & \text{when } x>2 \end{cases}$$

Since for $x<2$, $f(x)=-x^4+3$ (polynomial)

Since polynomials are continuous on their domain, so $f(x)$ is continuous on $x<2$.

Since for $x>2$, $f(x)=x^2+9$ (polynomial)

So $f(x)$ is continuous on $x>2$.

At $x=2$

L.H.L

$$\lim_{x\to 2^-} f(x)=\lim_{x\to 2}(-x^4+3) = -2^4+3 = -16+3 = -13$$

R.H.L

$$\lim_{x\to 2^+} f(x)=\lim_{x\to 2}(x^2+9) = 2^2+9 = 4+9 = 13$$

Since L.H.L $\neq$ R.H.L, so $\lim_{x\to 2} f(x)$ does not exist.

So $f(x)$ is discontinuous at $x=2$.

6. Find the value of k if $f(x) = \begin{cases} \frac{\sin kx}{x} & , x \neq 0 \\ 2 & , x = 0 \end{cases}$ is continuous at $x = 0$.

If $f(x)$ is continuous at $x = 0$

$$\Rightarrow \lim_{x \to 0} f(x) = f(0).$$

$$\boxed{\lim_{x \to 0} \frac{\sin x}{x} = 1}$$

$$\lim_{x \to 0} \frac{\sin kx}{x} = 2$$

$$\left(\lim_{x \to 0} \frac{\sin kx}{kx} \right) . k = 2$$

$$1 . k = 2$$

$$k = 2$$

7. Find the value of k if $f(x) = \begin{cases} kx - 9 & x < 5 \\ 9x - k & x > 5 \\ 36 & x = 5 \end{cases}$ is continuous at $x = 5$.

If $f(x)$ is continuous at $x = 5$.

$$\Rightarrow \lim_{x \to 5} f(x) = f(5)$$

$$\Rightarrow \lim_{x \to 5^-} f(x) = \lim_{x \to 5^+} f(x) = f(5)$$

$$\lim_{x \to 5} (kx - 9) = \lim_{x \to 5} (9x - k) = 36$$

$$k(5) - 9 = 9(5) - k = 36$$

$$5k - 9 = 45 - k = 36$$

$$5k - 9 = 36$$

$$5k = 36 + 9 = 45$$

$$k = \frac{45}{5}$$

$$k = 9$$

$$45 - k = 36$$

$$45 - 36 = k$$

$$9 = k$$

8. Find the values of m and n, so that given function f is continuous at $x = 3$.

(i) $f(x) = \begin{cases} mx & if\ x < 3 \\ n & if\ x = 3 \\ -2x+9 & if\ x > 3 \end{cases}$ (ii) $f(x) = \begin{cases} mx & if\ x < 3 \\ x^2 & if\ x \geq 3 \end{cases}$

(i) $f(x) = \begin{cases} mx & \text{if} \quad x<3 \\ n & \text{if} \quad x=3 \\ -2x+9 & \text{if} \quad x>3 \end{cases}$

Since $f(x)$ is continuous at $x = 3$, so

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^+} f(x) = f(3)$$

$$\lim_{x \to 3} (mx) = \lim_{x \to 3} (-2x+9) = n$$

$$3m = 3 = n$$

$3m = 3$

$\boxed{m = 1}$ $\boxed{3 = n}$

(ii) $f(x) = \begin{cases} mx & \text{if} \quad x<3 \\ x^2 & \text{if} \quad x \geq 3 \end{cases}$

Since $f(x)$ is continuous at $x = 3$,

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^+} f(x) = f(3)$$

$$\lim_{x \to 3} (mx) = \lim_{x \to 3} (x^2) = 3^2$$

$$3m = 9 = 9$$

$$3m = 9$$

$\boxed{m = 3}$

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9. If $f(x) = \begin{cases} \dfrac{\sqrt{2x+5}-\sqrt{x+7}}{2} & , \ x \neq 2 \\ k & , \ x = 2 \end{cases}$

Find the value of k so that f is continuous at $x = 2$.

Since $f(x)$ is continuous at $x = 2$, so

$$f(2) = \lim_{x \to 2} f(x)$$

$$k = \lim_{x \to 2} \frac{\sqrt{2x+5} - \sqrt{x+7}}{2}$$

$$k = \lim_{x \to 2} \frac{\sqrt{2x+5} - \sqrt{x+7}}{2} = \frac{\sqrt{2(2)+5} - \sqrt{2+7}}{2}$$

$$k = \frac{3-3}{2} = 0$$