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Chap# 8

Logic: Logic is a systematic method of reasoning that enables one to interpret the meanings of statements, examine their truth and deduce new information from existing facts. For example, we often draw general conclusions from limited number of observations or experiences.

Induction: If a person gets a Penicillin injection once or twice and experiences a reaction soon afterward. He generalises that he is allergic to Penicillin. This way of drawing conclusions is called induction.

Deduction: Sometimes we conclude from accepted or well-known facts. We often consult Lawyers or doctors based on their good reputation. This way of drawing conclusions from permises believed to be true, is called deduction.

Statement: A sentence or mathematical expression which may be true or false but not both is called statement. For example, the statement $a=b$ can be either true or false.

Logical Operators:

Inverse: Changing Sign only.

Converse: changing Place only.

Contra-Positive: changing both Sign and place.

Negation (NOT) $\sim$ F → T and T → F

Conjunction (and) $\wedge$ → Conjunction is True only when both are True, otherwise false.

Disjunction (or) $\vee$ → Disjunction is False only when both are False.

Implication (Condition): $P \rightarrow q$, If.....Then
where 'p' is antecedent or hypothesis and 'q' is Consequent or conclusion.

→ Implication is $\boxed{\text{False}}$ only when $\boxed{\text{antecedent}}$ is $\boxed{\text{True}}$ and $\boxed{\text{Consequent}}$ is $\boxed{\text{False}}$.

BiConditional: $P \longleftrightarrow q$
If and only if or is equivalent to.

Exercise # 8

Q#1 MCQ's (Correct answers)

(i) a (ii) d (iii) c (iv) a (v) b
(vi) a (vii) c (viii) b (ix) c (x) b

Q#2 Write Converse, inverse and Contrapositive of following:

Part	Statement	Converse	Inverse	Contrapositive
i	$\sim p \rightarrow q$	$q \rightarrow \sim p$	$p \rightarrow \sim q$	$\sim q \rightarrow p$
ii	$q \rightarrow p$	$p \rightarrow q$	$\sim q \rightarrow \sim p$	$\sim p \rightarrow \sim q$
iii	$\sim p \rightarrow \sim q$	$\sim q \rightarrow \sim p$	$p \rightarrow q$	$q \rightarrow p$
iv	$\sim q \rightarrow \sim p$	$\sim p \rightarrow \sim q$	$q \rightarrow p$	$p \rightarrow q$

Q#3 Write Truth table of following:

(i) $\sim(p \vee q) \vee (\sim q)$

p	q	$p \vee q$	$\sim(p \vee q)$	$\sim q$	$\sim(p \vee q) \vee \sim q$
T	T	T	F	F	F
T	F	T	F	T	T
F	T	T	F	F	F
F	F	F	T	T	T

(ii) $\sim(\sim q \vee \sim p)$

p	q	$\sim q$	$\sim p$	$\sim q \vee \sim p$	$\sim(\sim q \vee \sim p)$
T	T	F	F	F	T
T	F	T	F	T	F
F	T	F	T	T	F
F	F	T	T	T	F

(iii) $(p \vee q) \longleftrightarrow (p \wedge q)$

p	q	$p \vee q$	$p \wedge q$	$(p \vee q) \leftrightarrow (p \wedge q)$
T	T	T	T	T
T	F	T	F	F
F	T	T	F	F
F	F	F	F	T

Q#4 Differentiate between a mathematical statement and its Proofs. Given two examples.

ANS: Mathematical Statement:

"A mathematical statement is a declarative sentence that is either true or false. It expresses a fact or a relationship between mathematical objects."

example: The sum of angles in a triangle is always 180°.

Mathematical Proof: "A Proofs is a Logical sequence of steps that establishes the truth of a mathematical statement using definitions, axioms and theorems."

Example: Even number Property.

Explanation:

Statement: The Sum of two even numbers is always even.

Proof:

Suppose two even numbers

are $2a$ and $2b$.
Their Sum is
$$2a+2b = 2(a+b)$$
Since, $2(a+b)$ is also divisible by 2, So it is an even number.
Hence,
it is proved that, Sum of two even numbers is always even.

Q#5 What is difference between an axiom and a theorem? Give examples of each.

ANS: Theorem: "A theorem is a mathematical statement that has been proved true based on previously known facts".
Example: The sum of interior angles of a triangle is 180°.
Axiom: "An axiom is a mathematical statement that we believe to be true without any evidence or proof".
Example: Only one straight line can be drawn between two points.

Q#6. What is the importance of Logical reasoning in mathematical Proofs? Give an example.

ANS: Logical reasoning is important because:

i- It helps us to avoid mistakes in reasoning.

ii- it is applicable to all Kind of examples.

iii- it uses Known facts to Prove new results.

Example:

Statement: If two odd numbers are added, their Sum is always an even number.

Proof: (Using Logics)

Suppose two odd numbers are $2a+1$ and $2b+1$.

so,

$$2a+1+2b+1 = 2a+2b+2$$
$$= 2(a+b+1)$$

Since, $2(a+b+1)$ is divisible by 2

So, it is an even number.
Hence, Sum of two odd numbers is always an even number.

Q#7 Indicate whether it is an axiom, conjecture or theorem and explain your reasoning:

(i) There is exactly one straight line through any two points.

↳ It is an axiom.

Reasoning: This statement is a part of Euclidean geometry and known as axiom #1. It is the basic rule and foundation of geometry.

(ii) Every even number greater than 2 can be written as the sum of two prime numbers.

↳ It is a conjecture.

Reasoning: Given statement has not been proved in general. It has been widely believed to be true for a long time.

(iii) The Sum of the angles in a triangle is 180°.

↳ It is a theorem.

Reasoning: It can be proved by various methods such as extending one side of triangle and using the properties of Parallel Lines.

Q#8 Formulate Simple deductive Proofs for each of following algebraic expressions, Prove that L.H.S is equal to the R.H.S:

(i) Prove that: $(x-4)^2+9 = x^2-8x+25$

L.H.S

$(x-4)^2+9$

$\Rightarrow \; = (x-4)(x-4)+9$ $\qquad \because x^m \cdot x^n = x^{m+n}$

$= x(x-4)-4(x-4)+9$ $\qquad \because$ Right Distributive Law

$= x \cdot x - x \cdot 4 - 4 \cdot x + 4 \cdot 4 + 9$ $\qquad \because$ Right Distributive Law

$= x^2 - 4 \cdot x - 4 \cdot x + 16 + 9$ $\qquad \because$ Commutative Law & $x^m \cdot x^n = x^{m+n}$

$= x^2 - (4+4)x + 25$ ∵ Left Distributive Law

$= x^2 - 8x + 25 = R.H.S$

(ii) Prove that $(x+1)^2 - (x-1)^2 = 4x$

L.H.S

$(x+1)^2 - (x-1)^2$

$= (x+1)(x+1) - (x-1)(x-1)$ ∵ $x^m \cdot x^n = x^{m+n}$

$= x(x+1) + 1(x+1) - [x(x-1) - 1(x-1)]$ ∵ Right Distributive Law

$= x \cdot x + x \cdot 1 + 1 \cdot x + 1 \cdot 1 - [x \cdot x - x \cdot 1 - 1 \cdot x + 1 \cdot 1]$ ∵ Right Distributive Law

$= x^2 + 1 \cdot x + 1 \cdot x + 1 - [x^2 - 1 \cdot x - 1 \cdot x + 1]$ ∵ Commutative Law and $x^m \cdot x^n = x^{m+n}$

$= x^2 + (1+1)x + 1 - [x^2 - (1+1)x + 1]$ ∵ Left Distributive Law

$= x^2 + 2x + 1 - (x^2 - 2x + 1)$

$= x^2 + 2x + 1 - x^2 + 2x - 1$ ∵ Bracket off and sign change

$= \cancel{x^2} - \cancel{x^2} + 2x + 2x \cancel{+1} \cancel{-1}$ (Re-arrange)

$= 4x = R.H.S$

(iii) Prove that $(x+5)^2-(x-5)^2=20x$

L.H.S

$(x+5)^2-(x-5)^2$

$=(x+5)(x+5)-(x-5)(x-5)$ $\because x^m \cdot x^n = x^{m+n}$

$=x(x+5)+5(x+5)-[x(x-5)-5(x-5)]$ $\because$ Right Distributive Law

$=x\cdot x+x\cdot 5+5\cdot x+5\cdot 5-[x\cdot x-x\cdot 5-5\cdot x+5\cdot 5]$

$\because$ Right Distributive Law

$=x^2+5\cdot x+5\cdot x+25-[x^2-5\cdot x-5\cdot x+25]$

$\because$ Commutative Law and $x^m \cdot x^n = x^{m+n}$

$=x^2+(5+5)x+25-[x^2-(5+5)x+25]$

$\because$ Left Distributive Law

$=x^2+10x+25-(x^2-10x+25)$

$=x^2+10x+25-x^2+10x-25$ $\because$ Bracket off and Sign Change

$=\cancel{x^2}-\cancel{x^2}+10x+10x+\cancel{25}-\cancel{25}$ (Re-arrange)

$=20x=$ R.H.S

Q#9 Prove the following by justifying each step:

(i) $\frac{4+16x}{4} = 1+4x$

L.H.S $\frac{4+16x}{4}$

$= \frac{1}{4} \times (4+16x)$ $\quad \because \frac{a}{b} = \frac{1}{b} \times a$

$= \frac{1}{4} \cdot (4\times 1 + 4\times 4x)$ $\quad \because$ multiplicative identity

$= \frac{1}{4} \times 4 \cdot (1+4x)$ $\quad \because$ Distributive Law

$= (\frac{1}{4} \times 4) \cdot (1+4x)$ $\quad \because$ Associative Law

$= 1 \cdot (1+4x)$ $\quad \because$ multiplicative inverse

$= 1+4x$ $\quad \because$ multiplicative identity

$=$ R.H.S

(ii) $\frac{6x^2+18x}{3x^2-27} = \frac{2x}{x-3}$

L.H.S $\frac{6x^2+18x}{3x^2-27}$

$$= \frac{6x \cdot x + 6x \cdot 3}{3 \cdot x^2 - 3 \cdot 9} \quad \because (ab = a \cdot b)$$

$$= \frac{6x(x+3)}{3(x^2-9)} \quad \because \text{Common factors}$$

$$= \frac{2x(x+3)}{x^2-3^2} \quad \because \text{Cancellation Law}$$

$$= \frac{\cancel{2x(x+3)}}{(x+3)(x-3)} \quad \because a^2-b^2 = (a+b)(a-b)$$

$$= \frac{2x}{x-3} \quad \because \text{Cancellation law}$$

$$= R.H.S$$

(iii) $\dfrac{x^2+7x+10}{x^2-3x-10} = \dfrac{x+5}{x-5}$

L.H.S $\dfrac{x^2+7x+10}{x^2-3x-10}$

$$= \frac{x^2+5x+2x+10}{x^2-5x+2x-10} \quad \because \text{Factorizing}$$

$$= \frac{x\cdot x + 5\cdot x + 2\cdot x + 2\cdot 5}{x\cdot x - 5\cdot x + 2\cdot x - 2\cdot 5} \quad \because (ab = a\cdot b)$$

$$= \frac{x(x+5) + 2(x+5)}{x(x-5) + 2(x-5)} \quad \because \text{Common factors}$$

$$= \frac{(x+5)(x+2)}{(x-5)(x+2)} \quad \because \text{Common factors}$$

$$= \frac{x+5}{x-5} \quad \because \text{Cancellation Law}$$

$$= R.H.S$$

Q#10 Suppose x is an integer. If x is odd, then $9x+4$ is odd.

Let x be an integer.

Suppose x is odd, So,

$$x = 2K + 1 \quad \text{—— (1)}$$

given,

$$= 9x + 4$$

$$\Rightarrow \quad = 9(2K+1) + 4 \quad \because \text{using (1)}$$

$$= 18K + 9 + 4$$

$= 18K + 13$

Since, $18K = 2(9K)$ is divisible by 2, So it is even. Hence, $18K+13$ will be odd. (As Sum of an even and odd is always odd).

(Proved).

Q#11 Suppose x is an integer. If x is odd, then $7x+5$ is even.

Let x be an integer.

Suppose x is odd.

So, $x = 2K + 1$ —— ①

Given,

$= 7x + 5$

$= 7(2K+1) + 5$ ∵ using ①

$= 14K + 7 + 5$

$= 14K + 12$

Since, $14K = 2(7K)$ is divisible by 2, So it is even. Hence $14K+12$ is even. (As, Sum of two even no.s is always even).

$\Rightarrow$ $7x+5$ is an even.

Q#12. Prove following Statements:

(i) If x is an odd integer, then show that x^2-4x+6 is odd.

Proof:

Since, x is odd

i.e $x = 2k+1$ —(1), $x \in Z$

given,

$$= x^2 - 4x + 6$$

$$= (2k+1)^2 - 4(2k+1) + 6$$

$$= (2k)^2 + 2(2k)(1) + (1)^2 - 8k - 4 + 6$$

$$= 4k^2 + 4k + 1 - 8k + 2$$

$$= 4k^2 - 4k + 3$$

Since, $4k^2 = 2(2k^2)$ and $-4k = 2(-2k)$, both are even. So, Sum of even and odd is always odd. Hence x^2-4x+6 is odd.

(ii) If x is an even integer then show that x^2+2x+4 is even.

Proof:

Suppose x is even integer,

i.e $x = 2k$, $x \in Z$

given:

$$= x^2+2x+4$$

$$= (2k)^2+2(2k)+4$$

$$= 4k^2+4k+4$$

Since, $4k^2 = 2(2k^2)$ and $2(2k) = 4k$, both are even. So, Sum of two even number is always. Hence, x^2+2x+4 is even.

Q#13 Prove that for any two non-empty sets A and B.

$$(A \cap B)' = A' \cup B'$$

Proof:

Let $x \in (A \cap B)'$

$\Rightarrow \quad x \notin A \cap B$

$\Rightarrow \quad x \notin A$ and $x \notin B$

$\Rightarrow x \in A'$ and $x \in B'$

$\Rightarrow x \in A' \cup B'$

But x is any arbitrary element, so

$(A \cap B)' \subseteq A' \cup B'$ —— (1)

Now;

Let $y \in A' \cup B'$

$\Rightarrow y \in A'$ and $y \in B'$

$\Rightarrow y \notin A$ and $y \notin B$

$\Rightarrow y \notin A \cap B$

$\Rightarrow y \in (A \cap B)'$

But y is any arbitrary element. So,

$A' \cup B' \subseteq (A \cap B)'$ —— (2)

Hence, from (1) and (2),

$(A \cap B)' = A' \cup B'$

(Proved).

Q#14 If x and y are positive real numbers and $x^2 < y^2$ then $x < y$.

Sol:

Since, x and y are positive.

So, $x, y > 0$ (given)

and;

$$x^2 < y^2$$

taking square root on both sides,

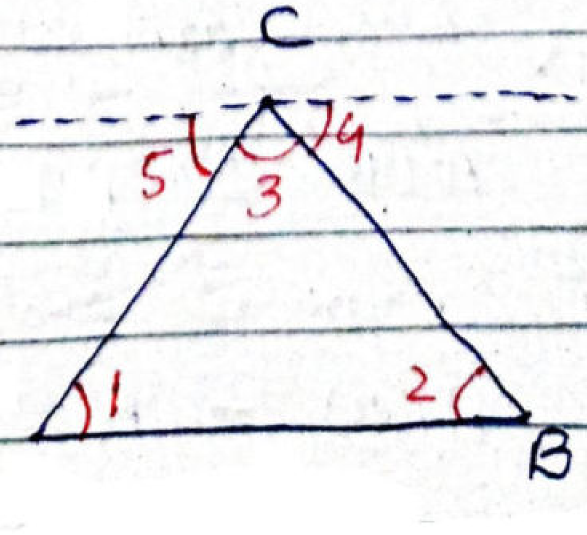
$$\sqrt{x^2} < \sqrt{y^2}$$

$$\Rightarrow \quad x < y$$

(Square root function is increasing for +ve integers).

Q#15 Prove that the sum of the interior angles of a triangle is $180°$.

Proof:



ABC is a triangle.
Draw a Line through Point C which is Parallel to AB.
Now; from figure,

$\angle 1 = \angle 5$ — ① (alternate angles)
$\angle 2 = \angle 4$ — ②

Also;
$\angle 5 + \angle 3 + \angle 4 = 180^\circ$ (Straight Line angles)
Using ① and ②,

$$\angle 1 + \angle 3 + \angle 2 = 180^\circ$$

$$\Rightarrow \angle 1 + \angle 2 + \angle 3 = 180^\circ$$

Thus,
Sum of interior angles of a triangle is 180°.

Q#16 If a, b and c are non-zero real numbers, Prove That:

(i) $\frac{a}{b} = \frac{c}{d} \Leftrightarrow ad = bc$

Left Side:

$$\frac{a}{b} = \frac{c}{d}$$

$$\frac{a}{b}\cdot(bd) = \frac{c}{d}\cdot(bd)$$ ∵ multiplication rule

$$a\cdot\frac{1}{b}(bd) = c\cdot\frac{1}{d}(bd)$$ ∵ $\frac{a}{b} = a\cdot\frac{1}{b}$

$$a\left(\frac{1}{b}\cdot b\right)d = c\cdot\frac{1}{d}(db)$$ ∵ Associative Law and Commutative Law

$$a\left(\frac{1}{b}\cdot b\right)d = c\cdot\left(\frac{1}{d}\cdot d\right)b$$ ∵ Associative Law

$$a(1)d = c(1)b$$ ∵ multiplicative inverse

$$ad = cb$$

$$ad = bc \quad (R.H.S)$$ ∵ Commutative Law

(ii) $\frac{a}{b}\cdot\frac{c}{d} = \frac{ac}{bd}$

L.H.S $\frac{a}{b}\cdot\frac{c}{d}$

$$= a\cdot\left(\frac{1}{b}\cdot c\right)\frac{1}{d} \qquad \because \frac{x}{y} = x\cdot\frac{1}{y}$$

$$= a\cdot c\cdot\frac{1}{b}\cdot\frac{1}{d} \qquad \because \text{Commutative Law}$$

$$= ac\cdot\frac{1}{bd} \qquad \because x\cdot\frac{1}{y} = \frac{x}{y}$$

$$= \frac{ac}{bd} \ (R.H.S)$$

(iii) $\frac{a}{b} + \frac{c}{b} = \frac{a+c}{b}$

L.H.S $\frac{a}{b} + \frac{c}{b}$

$$= a\cdot\frac{1}{b} + c\cdot\frac{1}{b} \qquad \because \frac{x}{y} = x\cdot\frac{1}{y}$$

$$= (a+c)\cdot\frac{1}{b} \qquad \because \text{Distributive Law}$$

$$= \frac{a+c}{b} \ (R.H.S)$$

{The End}