**ON STOLARSKY AND RELATED MEANS**

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ABSTRACT. We give a simple proof of the Stolarsky means inequality as well as some related inequalities for similar means of Stolarsky type.

1. **Introduction and Preliminaries**

Let us consider the following means

$$\begin{matrix}&E(x,y;r,s)=\left\{\frac{r\left(y^{s}-x^{s}\right)}{s\left(y^{r}-x^{r}\right)}\right\}^{\frac{1}{s-r}}\\&E(x,y;r,0)=E(0,r)=\left\{\frac{y^{r}-x^{r}}{r(ln⁡y-ln⁡x)}\right\}^{1/r}\\&E(x,y;r,r)=e^{-\frac{1}{r}}\left(\frac{x^{x^{r}}}{y^{y^{r}}}\right)^{1/\left(x^{r}-y^{r}\right)}\\&E(x,y;0,0)=\sqrt{xy}\end{matrix}$$

where $x$ and $y$ are positive real numbers $x\ne y,r$ and $s$ are any real numbers but 0 .

These means, known in literature, are called Stolarsky means. Namely Stolarsky[1] in 1975 (see also [2, p.120]) introduced these means. Stolarsky proved that the function $E(r,s)$ is increasing in both $r$ and $s$ i.e. for $r\leq u$ and $s\leq v$, we have

$$\begin{array}{c}E(x,y;r,s)\leq E(x,y;u,v).\#(1)\end{array}$$

In this paper, first we shall give a simple proof of inequality (1). Further we shall introduce two new classes of means of Stolarsky type.

2. **A Simple Proof of Stolarsky Means Inequality**

Note that $E(r,s)$ is continuous, this means it is enough to prove (1) in the case where $r,s,u,v\ne 0,r\ne s$ and $u\ne v$.

We consider the following function

$f(x)=p^{2}φ\_{r}(x)+2pqφ\_{t}(x)+q^{2}φ\_{s}(x) $ where $t=\frac{r+s}{2}$ and $p,q\in R$, and

$$φ\_{r}(x)=\left\{\begin{matrix}x^{r}/r,&r\ne 0\\ln⁡x,&r=0\end{matrix}\right.$$

Now

$$\begin{matrix}f^{'}(x)& =p^{2}x^{r-1}+2pqx^{t-1}+q^{2}x^{s-1}\\& =\left(px^{(r-1)/2}+qx^{(s-1)/2}\right)^{2}\geq 0\end{matrix}$$

This implies $f$ is monotonically increasing. So for $x\ne y$

$$\frac{f(x)-f(y)}{x-y}\geq 0$$

i.e.

$$p^{2}\frac{φ\_{r}(x)-φ\_{r}(y)}{x-y}+2pq\frac{φ\_{t}(x)-φ\_{t}(y)}{x-y}+q^{2}\frac{φ\_{s}(x)-φ\_{s}(y)}{x-y}\geq 0.$$

Let

$$ϕ(r)=\frac{φ\_{r}(x)-φ\_{r}(y)}{x-y}$$

then

$$p^{2}ϕ(r)+2pqϕ(t)+q^{2}ϕ(s)\geq 0$$

i.e.

$$ϕ^{2}(t)\leq ϕ(r)⋅ϕ(s)  where  t=\frac{r+s}{2}.$$

This implies $ϕ$ is log-convex in Jensen sense.

Also $lim\_{r\rightarrow 0} ϕ(r)=ϕ(0)$, which implies $ϕ$ is continuous for all $r\in R$. And therefore $log$-convex.

We need following lemma which proof can be found in [2].

Lemma 2.1. Let $f$ be log-convex function and if, $x\_{1}\leq y\_{1},x\_{2}\leq $ $y\_{2},x\_{1}\ne x\_{2},y\_{1}\ne y\_{2}$, then the following inequality is valid:

$$\begin{array}{c}\left(\frac{f\left(x\_{2}\right)}{f\left(x\_{1}\right)}\right)^{1/\left(x\_{2}-x\_{1}\right)}\leq \left(\frac{f\left(y\_{2}\right)}{f\left(y\_{1}\right)}\right)^{1/\left(y\_{2}-y\_{1}\right)}.\#(2)\end{array}$$

Applying Lemma 2.1 for $f=ϕ$, (let $r,s,u,v\ne 0)$ we get an inequality

$$\left\{\frac{r\left(y^{s}-x^{s}\right)}{s\left(y^{r}-x^{r}\right)}\right\}^{1/(s-r)}\leq \left\{\frac{u\left(y^{v}-x^{v}\right)}{v\left(y^{u}-x^{u}\right)}\right\}^{1/(v-u)}.$$

Since $E(r,s)$ is continuous, we have (1).

**CONCLUSION**

In the literature, many researchers have published so many results on different major generalizations of convex function. Many authors today focus on interval-valued functions, which is known as the $(h,m)$-convex interval-valued function. Additionally, we give the rigorous proof of the famous Hermite-Hadamard type inequality for $m$-convex in intervalvalued.

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