

ON STOLARSKY AND RELATED MEANS

K. GACŠETIĆ AND SAIMA NAZ KHAN

ABSTRACT. We give a simple proof of the Stolarsky means inequality as well as some related inequalities for similar means of Stolarsky type.

1. Introduction and Preliminaries

Let us consider the following means

$$\begin{aligned} E(x, y; r, s) &= \left\{ \frac{r(y^s - x^s)}{s(y^r - x^r)} \right\}^{\frac{1}{s-r}} \\ E(x, y; r, 0) &= E(0, r) = \left\{ \frac{y^r - x^r}{r(\ln y - \ln x)} \right\}^{1/r} \\ E(x, y; r, r) &= e^{-\frac{1}{r}} \left(\frac{x^{x^r}}{y^{y^r}} \right)^{1/(x^r - y^r)} \\ E(x, y; 0, 0) &= \sqrt{xy} \end{aligned}$$

where x and y are positive real numbers $x \neq y$, r and s are any real numbers but 0.

These means, known in literature, are called Stolarsky means. Namely Stolarsky[1] in 1975 (see also [2, p.120]) introduced these means. Stolarsky proved that the function $E(r, s)$ is increasing in both r and s i.e. for $r \leq u$ and $s \leq v$, we have

$$(1) \quad E(x, y; r, s) \leq E(x, y; u, v).$$

In this paper, first we shall give a simple proof of inequality (1). Further we shall introduce two new classes of means of Stolarsky type.

2. A Simple Proof of Stolarsky Means Inequality

Note that $E(r, s)$ is continuous, this means it is enough to prove (1) in the case where $r, s, u, v \neq 0, r \neq s$ and $u \neq v$. We consider the following function

$$f(x) = p^2 \varphi_r(x) + 2pq \varphi_t(x) + q^2 \varphi_s(x) \quad \text{where } t = \frac{r+s}{2} \text{ and } p, q \in \mathbb{R}, \text{ and}$$

Key words and phrases. convex function, log-convex function, Stolarsky means.

$$\varphi_r(x) = \begin{cases} x^r/r, & r \neq 0 \\ \ln x, & r = 0 \end{cases}$$

Now

$$\begin{aligned} f'(x) &= p^2 x^{r-1} + 2pqx^{t-1} + q^2 x^{s-1} \\ &= \left(px^{(r-1)/2} + qx^{(s-1)/2} \right)^2 \geq 0 \end{aligned}$$

This implies f is monotonically increasing. So for $x \neq y$

$$\frac{f(x) - f(y)}{x - y} \geq 0$$

i.e.

$$p^2 \frac{\varphi_r(x) - \varphi_r(y)}{x - y} + 2pq \frac{\varphi_t(x) - \varphi_t(y)}{x - y} + q^2 \frac{\varphi_s(x) - \varphi_s(y)}{x - y} \geq 0.$$

Let

$$\phi(r) = \frac{\varphi_r(x) - \varphi_r(y)}{x - y}$$

then

$$p^2 \phi(r) + 2pq \phi(t) + q^2 \phi(s) \geq 0$$

i.e.

$$\phi^2(t) \leq \phi(r) \cdot \phi(s) \quad \text{where} \quad t = \frac{r + s}{2}.$$

This implies ϕ is log-convex in Jensen sense.

Also $\lim_{r \rightarrow 0} \phi(r) = \phi(0)$, which implies ϕ is continuous for all $r \in \mathbb{R}$. And therefore log-convex.

We need following lemma which proof can be found in [2].

Lemma 2.1. Let f be log-convex function and if, $x_1 \leq y_1, x_2 \leq y_2, x_1 \neq x_2, y_1 \neq y_2$, then the following inequality is valid:

$$(2) \quad \left(\frac{f(x_2)}{f(x_1)} \right)^{1/(x_2 - x_1)} \leq \left(\frac{f(y_2)}{f(y_1)} \right)^{1/(y_2 - y_1)}.$$

Applying Lemma 2.1 for $f = \phi$, (let $r, s, u, v \neq 0$) we get an inequality

$$\left\{ \frac{r(y^s - x^s)}{s(y^r - x^r)} \right\}^{1/(s-r)} \leq \left\{ \frac{u(y^v - x^v)}{v(y^u - x^u)} \right\}^{1/(v-u)}.$$

Since $E(r, s)$ is continuous, we have (1).

CONCLUSION

In the literature, many researchers have published so many results on different major generalizations of convex function. Many authors today focus on interval-valued functions, which is known as the (h, m) -convex interval-valued function. Additionally, we give the rigorous proof of the famous Hermite-Hadamard type inequality for m -convex in intervalvalued.

REFERENCES

- [1] K. B. Stolarsky, Generalization of the logarithmic mean, *Math. Mag.* 48 (1975), 87-92.
- [2] J. E. Pečarić, F. Proschan and Y. C. Tong, *Convex functions, Partial Orderings and Statistical Applications*, Academic Press, New York, 1992.
- [3] J. E. Pečarić, I. Perić and H. M. Srivastava, A family of the Cauchy type meanvalue theorems, *J. Math. Anal. Appl.* 306 (2005) 730-739.

1 Department of Mathematics, University of Mars, Venus
Email address: Gac01@maths.com

2 School of Advanced Mathematical Sciences, Smith Town, WONDERLAND
Email address: wonderland008@gmail.com