According to the Curriculum of HEC

(Recommended for all Universities)

CLASSICAL MECHANICS

For BS & MSc Aspirants

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Objectives of the course:

To provide solid understanding of classical mechanics and enable the students to use this understanding while studying courses on quantum mechanics, statistical mechanics, electromagnetism, fluid dynamics, space-flight dynamics, astrodynamics and continuum mechanics.

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Books Recommended

- John R. Taylor. Classical Mechanics, University of Colorado.
- Goldstein H. Classical Mechanics, Addison-Wesley Publishing Co.
- Spiegel M. R. Theoretical Mechanics, Addison-Wesley Publishing Company.
- Mir K.L. Theoretical Mechanics: Ilmi Ketab Khana.
- Virtual University Lecture Handouts on Classical Mechanics.
- Mechanics by HRK.
- Mechanics, Waves and Oscillations by Kaleem Akhtar

Dedicated

To

Our beloved Teacher

Sir Tahir Nazir

University of Sargodha

CONTENTS

CHAPTER

1 GENERAL INTRODUCTION & NON INERTIAL REFERENCE SYSTEM

What is Mechanics?

Mechanics is the branch of science which studies the state of rest and motion of objects and laws governing rest, equilibrium and motion. Since material objects exist in the form of liquids gases and solids there are corresponding types of mechanics to deal with them.

i. Kinematics
ii. Dynamics
iii. Statics

Kinematics

Kinematics is the branch of mechanics which describe the motion of objects without consideration of their masses and force acting on them. It is the motion of objects without discussing the causes of motion.

Dynamics/Kinetics

Dynamics is the branch of mechanics concerned with the motion of objects under the action of force. It is the motion of objects with discussing the causes of motion.

Statics

Statics is the branch of mechanics concerned with objects at rest or in equilibrium under the action of forces.

Classical Mechanics

This is the branch of mechanics in which we study the mechanics of big bodies. It deals with the motion of physical objects at macroscopic level. It is based upon Newton's Law of Motion. It is also called Newtonian Mechanics because the bodies obey Newton's Law of Motion. The study of bodies on atomic scales falls in the category of Quantum Mechanics. The problems involving velocities which are not negligible when compared with the velocity of light or discussed on the basis of relativity. Galileo and Newtonian provide the base of classical mechanics in 17^{th} century.

Non – Relativistic Mechanics: Non – Relativistic Mechanics based on the laws of Newton's is concerned with bodies moving at speed and velocities negligibly small as compared to the speed of light. i.e. $c = 3 \times 10^8 ms^{-1}$

Relativistic Mechanics: Relativistic Mechanics is concerned with bodies moving at speed and velocities comparable to the speed of light. i.e. $c = 3 \times 10^8 ms^{-1}$

Division of Classical Mechanics

Three major divisions of classical mechanics are the following:

- **Mechanics of particles and rigid bodies:** It is based on newton's law. Basic concepts and terms are space, time and mass; particle and body; velocity, momentum and acceleration; force and energy.
- **Mechanics of fluid:** It is also based on newton's law and their extensions and deal with the behavior of the fluid (liquid and gases) in motion. Its two well-known branches are hydrodynamics (for fluid) and Aerodynamics (for gases).
- **Mechanics of elastic solids:** it deals with the behavior of solids when the undergo deformation under forces.

Macroscopic Objects: Visible objects through naked eyes are called Macroscopic Objects. e.g. Star, Table, Horse etc.

Microscopic Objects: Invisible objects through naked eyes are called Microscopic Objects. e.g. electron, proton, bacteria etc.

Classical Mechanics by Methodology and Approach

- **Newtonian Mechanics or Vector Mechanics:** In this type of mechanics vector quantities such as position vector, velocity, momentum etc. appear as basic physical entities. This is directly based on Newton Laws of motion.
- **Analytic Mechanics or Scalar Mechanics:** In this type of mechanics scalar quantities occupies the central position.

Rectangular Components

The process of splitting a vector into various parts or components is called "Resolution of vector" and these parts are called components of vector. If we split a vector in a rectangular plane OXY, such components are called rectangular components of a vector.

Component Along x-axis is called horizontal component of vector.

Component Along y-axis is called vertical component of vector.

Position vector: It is often convenient to describe the motion of a particle in terms of its x, y or rectangular components, relative to a fixed frame of reference. In a given reference system, the position of a particle can be specified by a single vector, namely, the displacement of the particle relative to the origin of the coordinate system. This vector is called the position vector of the particle. In rectangular coordinates, the position vector is simply $r = x\hat{\imath} + y\hat{\jmath}$
The components of the position vector of a moving particle are functions of the time, namely, $x = x(t)\ , y = y(t)$

CLASSIFICATION OF COORDINATES

Cartesian or Rectangular Coordinates: Let (x, y, z) be a point on surface S in R^3 then (x, y, z) are called Cartesian coordinates.

Polar Coordinates: Let $P(r, \theta)$ be a point on the curve in R^2 then (r, θ) are called Polar coordinates. Its parametric equations are $x = rCos\theta, y = rSin\theta$
Where $r = \sqrt{x^2 + y^2}$ and $\theta = tan^{-1}\frac{y}{x}$

Cylindrical Coordinate System $(\boldsymbol{r}, \boldsymbol{\theta}, \boldsymbol{z})$

Let $P(r, \theta, z)$ be a point on surface of cylinder S in R^3 then (r, θ, z) are called Cylindrical Polar coordinates.

Its parametric equations are $x = rCos\theta, y = rSin\theta, z = z$.

Where $r = \sqrt{x^2 + y^2}; r > 0$ and $\theta = tan^{-1}\frac{y}{x}$; $0 \leq \theta \leq 2\pi$ and $-\infty < z < \infty$

Fix	Vary	Locus
θ, z	r	Straight Line
r, θ	z	Line
r, z	θ	Circle
z	r, θ	Disk
θ	r, z	Plane
r	θ, z	Cylinder

Spherical Coordinate System $(\boldsymbol{r}, \boldsymbol{\theta}, \boldsymbol{\varphi})$

Let $P(r, \theta, \varphi)$ be a point on surface of sphere S in R^3 then (r, θ, φ) are called Cylindrical Polar coordinates.

Its parametric equations are $x = rSin\theta Cos\varphi, y = rSin\theta Sin\varphi, z = rCos\varphi.$

Where $r = \sqrt{x^2 + y^2 + z^2}; r > 0$ and $\theta = tan^{-1}\frac{\sqrt{x^2+y^2}}{z}$; $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$
and $\varphi = tan^{-1}\frac{y}{x}$; $0 \leq \varphi \leq 2\pi$

Fix	Vary	Locus
r, θ	φ	Circle
r, φ	θ	Semi-Circle
θ, φ	r	Line
r	θ, φ	Sphere
θ	r, φ	Cone
φ	r, θ	Plane

Framework

A framework that is used for the observation and mathematical description of physical phenomena and the formulation of physical laws, usually consisting of an observer, a coordinate system, and a clock or clocks assigning times at positions with respect to the coordinate system. A system of geometric axes in relation to which measurements of size, position, or motion can be made.

Frame of Reference

The system in which the clock and the meter scale used for the measurement are at rest. Such coordinate system is called a ***frame of reference***. There are two types of frame of references

- Inertial frames of reference (Newtonian Frames)
- Non – inertial frames of reference

Inertial Frames of Reference

Inertial frame of reference is that in which the law of inertia (Newton's first law of motion) holds, that is a frame in which a body that is acted upon by zero net external force moves with a constant velocity.

The law of inertia holds in any frame of reference, which happens to move with a constant velocity relative to a given inertial frame. Therefore, any frame of reference, which moves with a constant velocity relative to an inertial frame, is also an inertial frame. These frames are non – accelerated. i.e. $\vec{a} = 0$

Examples

- A frame of reference fixed with respect to the stars is an inertial frame.
- A spaceship drifting in outer space without spinning and with its engines shut off would be an ideal inertial frame.
- However for all practical purposes, any frame of reference fixed to the earth such as a railway station or a laboratory can be taken as an inertial frame. Thus a railway station is an inertial frame and a train travelling at constant velocity with respect to the railway station is also an inertial frame.

Non – Inertial Frames of Reference

Non – Inertial frame of reference is that in which the law of inertia (Newton's first law of motion) does not holds, that is a frame in which a body that is acted upon by zero net external force does not moves with a constant velocity. i.e. velocity remains change. E.g. person sitting in a moving train.

Newtonian or The Principle of Relativity (Galilean Invariance)

The Principle of Relativity (PR) applies to inertial frames of reference. This principle states that ***the laws of Physics take the same mathematical form in all inertial frames.***

Or the basic laws of Physics are identical in all frames of reference which are moving with uniform velocity (unaccelerated) relative to one another.

Or It is impossible by using any physical law to distinguish between inertial frames.

GALILEAN TRANSFORMATION (G.T) / NEWTONIAN TRANSFORMATION

This is the set of equations in classical physics that relate the space and time coordinates of two systems moving at a constant velocity relative to each other. The transformation equations which relate the time and space coordinates in frames S and S' and are called Galilean Transformations (G.T.) as follows;

$x' = x - vt, \qquad y' = y, \qquad z' = z\,, \qquad t' = t$

Nature of time and Space: According to G.T.

- the concept of time is absolute (invariant) ($t' = t$)
- the concept of space that is the concept of distance or length is also absolute (invariant) ($L' = L$).

Absolute (Invariant) Space

Space that is not affected by what occupies it or occurs within it and that provides a standard for distinguishing inertial system from other frames of references. For example, Bob on Earth, sitting at his telescope, catches sight of Alice in her rocket ship streaking at 9/10 the speed of light right towards the sun.

Application of G.T. to Mechanics

On the basis of G.T., it is possible to obtain relations between physical quantities measured by two inertial observers in relative motion. Some of these are merely listed below:

(a) If $\vec{u}$ and $\vec{u}'$ are the velocities of a particle as observed from frames S and S' respectively, then $\vec{u}' = \vec{u} - \vec{v}$
Where $\vec{v}$ is the velocity of S' relative S. This is the familiar ***'common sense'*** formula of relative velocity.

(b) Acceleration of a particle as measured in S and S' is the same. That is say $\vec{a}' = \vec{a}$

(c) The mass of a particle has the same value in different inertial frames. If m' and m are the masses of a particle as determined in frames S' and S respectively, then $m' = m$.

Hence equation of motion such as $\vec{F} = m\vec{a}$ in frame S is transformed into $\vec{F}' = m'\vec{a}'$ in frame S'. Not only this equation but in fact Newtonian Mechanics has the same form in different inertial frames according to pre-Einstein relativity.

Covariant

Laws which remain same in all inertial frame of references are called covariant laws. e.g. Newton law $\vec{F} = m\vec{a}$ is covariant in all inertial frame of references.

Invariant (Absolute)

Quantities which remain same in all inertial frame of references are called invariant quantities. e.g. mass, length, time etc.

Newton's 1^{st} law of motion (Galileo's Law)

An object continues its state of rest or of uniform motion in a straight line provided no net force or external force act on it. It is also called Law of inertia. This law measures the force of an object qualitatively.

The symbolic form of first law of motion is $\sum \vec{F} = 0$. i.e. $m\vec{a} = 0$

As the mass of the object is non-zero, therefore the acceleration of the concerned object must be zero. i.e. $\vec{a} = 0$. implies $\frac{d\vec{v}}{dt} = 0$. Then $\vec{v} =$ Constant.

Examples: A ball kicked in a ground, A car moving with constant velocity, A book lying in a book shelf.

Inertia: It is a property of a body due to which it resists any change in its state of rest or of its motion. It depends on mass of body. i.e. $I = mr^2$

Newton's 2^{nd} law of Motion

(Time rate of change of momentum equal to net force)

Newton's 2^{nd} law of motion describes the relationship among the force, mass and acceleration of the given object. We can states the 2^{nd} law of motion as;
Change of motion is proportional to the external applied force and takes place in the direction of the straight line in which the force acts. Or For any particle of mass m, the net force F on the particle of mass m times the particle's acceleration. i.e. $\vec{F} = m\vec{a}$.

The second law can be rephrased in terms of the particle's momentum, defined as $\vec{P} = m\vec{v}$ then $\vec{F} = m\vec{a} = m\frac{d\vec{v}}{dt} = \frac{d}{dt}(m\vec{v}) \Rightarrow \vec{F} = \frac{d\vec{P}}{dt}$
That is **the rate of change of linear momentum in the direction of applied force is equal to that force**.

Examples

When we apply same force to move a truck and a bicycle, the bicycle will have more acceleration than the truck, because the mass of bicycle is less than the truck. An empty shopping cart is much easier to move than a full one, because the empty one has less mass.

Newton's 3rd Law of motion

Newton's first two laws concern the response of a single object to applied forces. The third law addresses a quite different issue: Every force on an object inevitably involves a second object the object that exerts the force. The nail is hit by the hammer; the cart is pulled by the horse, and so on. Newton realized that if an object 1 exerts a force on another object 2, then object 2 always exerts a force (the "reaction" force) back on object 1.

Newton's third law can be stated very compactly:

To every action there is an equal and opposite reaction.

Examples

A fish's thrust through the water, A bird's fly in the air, A rocket's launch, The car moving on a road, The nail hit by hammer.

Gravitational Mass

Mass of the body define on the basis of gravitational properties is called Gravitational Mass.

Rigid Body

A rigid body is defined as a collection of particles such that distance between every pair of its constituent particles remains unchanged whatever the forces acting on it.

Constraint of Rigidity

The defining condition of a rigid body is called the constraint of rigidity. It can be expressed as $(r_i - r_j)^2 = (r_i - r_j).(r_i - r_j) = c_{ij}$ where r_i is the position vector of the i^{th} particle and c_{ij} is a constant. This definition implies that a rigid body will not undergo any deformation.

Question

A constant force $\vec{F}$ acting on a particle of mass m changes the velocity from $\vec{v}_1$ to $\vec{v}_2$ in time τ.

a) Prove that $\vec{F} = \frac{m(\vec{v}_2 - \vec{v}_1)}{\tau}$
b) Does above result hold if the force is variable? Explain.

Solution

By Newton's second law $\vec{F} = m\vec{a} = m\frac{d\vec{v}}{dt} \Rightarrow \frac{d\vec{v}}{dt} = \frac{\vec{F}}{m} \Rightarrow d\vec{v} = \frac{\vec{F}}{m}dt$

$$\Rightarrow \int_{\vec{v}_1}^{\vec{v}_2} d\vec{v} = \frac{\vec{F}}{m}\int_0^{\tau} dt \Rightarrow \vec{v}_2 - \vec{v}_1 = \frac{\vec{F}}{m}(\tau) \Rightarrow \vec{F} = \frac{m(\vec{v}_2 - \vec{v}_1)}{\tau}$$

Above result does not hold in general if the force is variable ($\vec{F}$ is not constant), since in such case we would not obtain the result of integration achieved above.

Question

What constant force is needed to bring a 900 *kg* mass moving at a speed of 100*km/h* to rest in 4*seconds*?

Solution

We shall assume that the motion takes place in a straight line which we choose as the positive direction of the x – axis. Then we have;

$$m = 900kg, \vec{v}_1 = 100\hat{\imath}kmh^{-1} = 27.78\hat{\imath}ms^{-1}, \vec{v}_2 = 0\hat{\imath}ms^{-1}, t = 4s$$

Using formula $\vec{F} = \frac{m(\vec{v}_2 - \vec{v}_1)}{t}$ we have

$$\vec{F} = m\vec{a} = \frac{m(\vec{v}_2 - \vec{v}_1)}{t} = -6.25 \times 10^3\hat{\imath} \text{ newtons}$$

Thus the force has magnitude 6.25×10^3 newtons in the negative x direction. i.e. in the direction opposite to the motion. This is of course to be expected.

Question

On the basis of G.T. show that the force acting on a particle is independent of the inertial frame in which it is measured. i.e. $\vec{F}' = \vec{F}$. **Or** Show that Newton's 2nd Law of motion is Covariant. **Or** Show that Newton's 2nd Law of motion is invariant under G.T.

Solution

If $\vec{u}$ and $\vec{u}'$ are the velocities of a particle as observed from frames S and S' respectively, then according to Galilean Transformations

$$x' = x - vt \Rightarrow \frac{dx'}{dt'} = \frac{d}{dt}(x - vt) \Rightarrow \frac{dx'}{dt'} = \frac{dx}{dt} - v\frac{dt}{dt} \qquad \text{in G.T. } t' = t$$

$$\Rightarrow u' = u - v \Rightarrow (u', 0,0) = (u, 0,0) - (v, 0,0) \Rightarrow \vec{\boldsymbol{u}}' = \vec{\boldsymbol{u}} - \vec{\boldsymbol{v}}$$

$$\Rightarrow \frac{d\vec{u}'}{dt'} = \frac{d}{dt}(\vec{u} - \vec{v}) \Rightarrow \frac{d\vec{u}'}{dt'} = \frac{d\vec{u}}{dt} - \frac{d\vec{v}}{dt} = \frac{d\vec{u}}{dt} \Rightarrow \vec{\boldsymbol{a}}' = \vec{\boldsymbol{a}}$$

Multiplying m' on both sides we get $\Rightarrow m'\vec{a}' = m'\vec{a}$

$\Rightarrow m'\vec{a}' = m\vec{a}$ In inertial frame $m' = m$

$\Rightarrow \vec{\boldsymbol{F}}' = \vec{\boldsymbol{F}}$

Equilibrium

A body is said to be in equilibrium if no net force acts on it. i.e. $\vec{F} = 0$

Types of Equilibrium

- **Stable Equilibrium/Stability of Equilibrium:** In Stable equilibrium the particle will return to its original position when slightly displaced to either side.
- **unStable Equilibrium:** In unStable equilibrium the particle will not return to its original position when slightly displaced to either side.
- **Neutral Equilibrium:** In neutral equilibrium the particle will return to its new position when slightly displaced to either side from its previous position.

Condition of Equilibrium State

There are two conditions for equilibrium state;

- Net force acting on a body is zero. i.e. $\sum \vec{F_i} = 0.$
- Net torque acting on a body is zero. i.e. $\sum \vec{\tau_i} = 0.$

Keep in Mind

- **Theorem:** If the force field is conservative with potential $\vec{V}$, then a Necessary and sufficient condition for a particle to be in equilibrium at a point is that $\nabla\vec{V} = 0.$ i.e. $\frac{\partial\vec{V}}{\partial x} = \frac{\partial\vec{V}}{\partial y} = \frac{\partial\vec{V}}{\partial z} = 0$
- **Theorem:** A Necessary and sufficient condition that and an equilibrium point be one of stability is that the potential V at the point be a minimum. i.e. $\frac{\partial\vec{V}}{\partial x} = \frac{\partial\vec{V}}{\partial y} = \frac{\partial\vec{V}}{\partial z} = 0$

Question

A particle is acted upon by the forces

$\vec{F}_1 = 5\hat{\imath} - 10\hat{\jmath} + 15\hat{k}$, $\vec{F}_2 = 10\hat{\imath} + 25\hat{\jmath} - 20\hat{k}$ and $\vec{F}_3 = 15\hat{\imath} - 20\hat{\jmath} + 10\hat{k}$

Find the force needed to keep the particle in equilibrium.

Solution

The resultant of forces is $\vec{R} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 = 30\hat{\imath} - 5\hat{\jmath} + 5\hat{k}$

Then the force needed to keep the particle in equilibrium is $-\vec{R} = -30\hat{\imath} + 5\hat{\jmath} - 5\hat{k}$

Stable Equilibrium/Stability of Equilibrium

In Stable equilibrium the particle will return to its original position when slightly displaced to either side.

Point of Stability

If a particle which is displaced slightly from an equilibrium point P tends to return to P, then we called P a point of stability or stable point and the equilibrium is said to be equilibrium. Otherwise we say that the point is one of instability and the equilibrium is **unstable**.

Theorem: A necessary and sufficient condition that an equilibrium point be one of stability is that the potential $\vec{V}$ at the point be a minimum.

Question

A particle moves along the x – axis in a force field having potential
$\vec{V} = \frac{1}{2}kx^2;\ k > 0$ then

a) Determine the point of equilibrium
b) Investigate the stability

Solution

a) Equilibrium point occur where $\nabla\vec{V} = 0$
$\Rightarrow \frac{d\vec{V}}{dx} = 0 \Rightarrow \frac{d}{dx}\left(\frac{1}{2}kx^2\right) = 0 \Rightarrow kx = 0 \Rightarrow x = 0$
Thus there is only one equilibrium point at $x = 0$
b) Since $\frac{d^2\vec{V}}{dx^2} = k > 0$, it follows that at $x = 0$, $\vec{V}$ is minimum then by using theorem "A necessary and sufficient condition that an equilibrium point be one of stability is that the potential $\vec{V}$ at the point be a minimum." $x = 0$ is a point of stability.

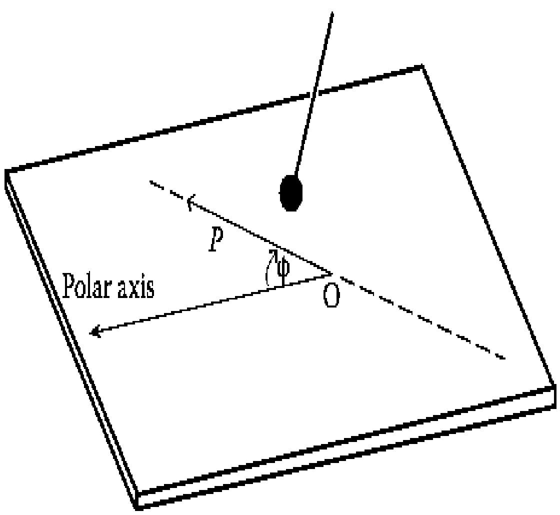
Bounded and Unbounded Motion of a Particle

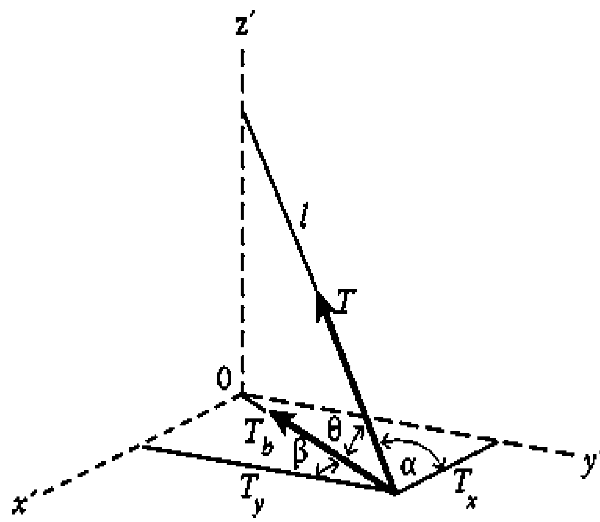
Dynamic system may be categorized as bounded or unbounded.

- If the sum of the kinetic and binding energies is less than zero, interacting entities are considered **bounded**. In this, system lies confined in a particular region of space. This generally happens when the energy of the particle is less than or equal to the total potential barrier at infinite separation. In other words, the particle has less energy than is required to escape the barrier. **In classical mechanics, a bounded system is one where the motion of all the objects in the system is restricted to some finite region of space.**
  For example consider an object moving in a Newtonian gravitational potential $V(r) = -\frac{GM}{r}$. The motion of this object is bounded if it has negative total energy. In this case, the object will move in a close orbit in the shape of a ellipse. We can draw an imaginary box of finite size that completely encloses the orbital ellipse of the object.
- If the sum of the kinetic and binding energies is greater than zero, interacting entities are considered **unbounded.** In this, system does not lies confined in a particular region of space. This generally happens when the energy of the particle is greater to the total potential barrier at infinite separation. In other words, the particle has greater energy than is required to escape the barrier. **In classical mechanics, an unbounded system is one where the motion of all the objects in the system is not restricted to some finite region of space.**
  For example consider an object moving in a Newtonian gravitational potential $V(r) = -\frac{GM}{r}$. The motion of this object is unbounded if it has positive total energy. In this case, the object will move along a hyperbolic escape trajectory. And there does not exist any finite sized trajectory. In this case, the motion is unbounded.

Foucault's Pendulum

The Coriolis effect resulting from the rotation of the Earth was dramatically demonstrated by Jean Foucault (1819 – 1868) in 1851, using <u>a long pendulum of 67 meter string with a very heavy bob (to reduce the effects of air currents) of 28km hung from a support designed to allow the pendulum to swing (rotated) freely in any direction (especially in a given vertical plane)</u>. His experiments showed that the plane in which the pendulum oscillates rotates slowly with time. The effect is very striking because, unlike previous examples, the motion takes place in a small region of space, and the velocity of the pendulum is not very great. The gravity force is, of course, much more important than the Coriolis force in determining the pendulum's motion. However, the direction of the small Coriolis force is out of the plane of oscillation; thus, despite its smallness, the Coriolis force has a significant effect on the motion of the pendulum.





As shown in Figure, ϕ is the angle between the line along which the pendulum oscillates and a reference polar axis. Foucault showed that the rate of rotation ϕ of the direction of swing of the bob is related to the latitude λ of the pendulum on the earth and the angular velocity ω of the Earth's rotation by the expression $= \omega sin\lambda$.

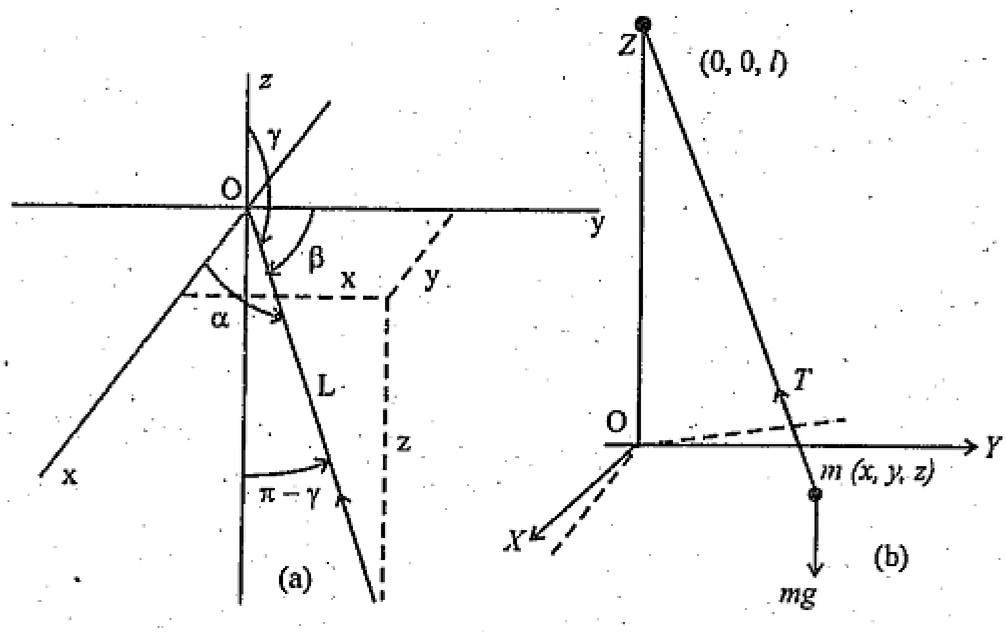
Statement: Foucault's Theorem (Foucault's Formula)

The plane of oscillation of the plane rotates with an angular frequency $\omega sin\lambda$

Proof

Consider a Foucault's Pendulum comprises a bob of mass m suspended by a light wire of length L from the point P on a high ceiling. The tension force on the bob is shown as $\vec{T}$ and its x and y components are T_x and T_y, for small oscillation the angle β is very small.

Consider a coordinate system OXYZ with origin O at the point of equilibrium and z – axis coincident with the local vertical; with point of suspension S on the z – axis. Then the xy – plane will be coincident with the local horizontal plane. We consider very small oscillation of the pendulum and therefore it is reasonable to assume that they take place in the horizontal plane.



Let $\vec{r}$ denote the position vector of the bob at any time t. If is the tension $\vec{T}$ in the string then the equation of motion of bob will be

$$m\ddot{\vec{r}} + 2m\vec{\omega} \times \dot{\vec{r}} = \vec{T} + m\vec{g} \qquad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Where the quantities on the L.H.S refer to the body (or rotating) coordinate system OXYZ, and those on the R.H.S to a fixed (or inertial) coordinate system with O as origin. Let α, β, γ be the angles which the line segment SP makes with coordinate axes, then the angles which the tension $\vec{T}$ makes with the same axes will be $\pi - \alpha, \pi - \beta, \pi - \gamma$. The component of $\vec{T}$ will therefore be

$$T_x = \vec{T}.\hat{\imath} = Tcos(\pi - \alpha) = -Tcos\alpha$$

$$T_y = \vec{T}.\hat{\jmath} = Tcos(\pi - \beta) = -Tcos\beta$$

$$T_z = \vec{T}.\hat{k} = Tcos(\pi - \gamma) = -Tcos\gamma$$

Let (x, y, z) be the coordinates of the point P. now we will use the following relations from the three – dimensional geometry

$$\frac{x_2 - x_1}{l} = cos\alpha \;\; ; \;\; \frac{y_2 - y_1}{l} = cos\beta \;\; ; \;\; \frac{z_2 - z_1}{l} = cos\gamma$$

Where l denotes the distance between the points (x_1, y_1, z_1) and (x_2, y_2, z_2). Noting that the end points of the string have coordinates $(00, l)$ and (x, y, z) we obtain

$$\frac{x - 0}{l} = cos\alpha \;\; ; \;\; \frac{y - 0}{l} = cos\beta \;\; ; \;\; \frac{z - 0}{l} = cos\gamma$$

$$\frac{x}{l} = cos\alpha \;\; ; \;\; \frac{y}{l} = cos\beta \;\; ; \;\; \frac{z}{l} = cos\gamma$$

Therefore on substitution $T_x = -T\frac{x}{l} \;\; ; \;\; T_y = -T\frac{y}{l} \;\; ; \;\; T_z = -T\frac{z}{l}$

We consider very small displacement in the YZ – plane. Then $x \ll l, y \ll l$ and $z \approx -l$. Under these circumstances $\vec{T} \approx m\vec{g}$, and for the components we can write

$$T_x = T\frac{x}{l} \;\; ; \;\; T_y = T\frac{y}{l} \;\; ; \;\; T_z = T \quad \text{(because } z \approx 0\text{)}$$

The angular velocity vector $\vec{\omega}$ can be written as $\vec{\omega} = (\omega cos\lambda)\hat{\imath} + (\omega sin\lambda)\hat{k}$ and $\vec{g} = -g\hat{k}$. Therefore

$$\vec{\omega} \times \dot{\vec{r}} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \omega cos\lambda & 0 & \omega sin\lambda \\ \dot{x} & \dot{y} & \dot{z} \end{vmatrix}$$

$$\vec{\omega} \times \dot{\vec{r}} = -(\omega\dot{y}sin\lambda)\hat{\imath} + \omega(\dot{x}sin\lambda - \dot{z}cos\lambda)\hat{\jmath} + (\omega\dot{y}cos\lambda)\hat{k}$$

Using in the equation (1) we have

$$m\ddot{\vec{r}} + 2m\vec{\omega} \times \dot{\vec{r}} = \vec{T} + m\vec{g}$$

$$m(\ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath} + \ddot{z}\hat{k}) + 2m[-(\omega\dot{y}sin\lambda)\hat{\imath} + \omega(\dot{x}sin\lambda - \dot{z}cos\lambda)\hat{\jmath} + (\omega\dot{y}cos\lambda)\hat{k}] = -\frac{T}{l}(x\hat{\imath} + y\hat{\jmath} + z\hat{k}) - mg\hat{k}$$

On equating the coefficients of $\hat{\imath}, \hat{\jmath}, \hat{k}$ we have

$$\ddot{x} + 2\omega(-sin\lambda)\dot{y} = -\frac{Tx}{ml}$$

$$\ddot{y} + 2\omega(\dot{x}sin\lambda - \dot{z}cos\lambda) = -\frac{Ty}{ml} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2) \quad \text{for all three equations}$$

$$\ddot{z} + 2\omega(cos\lambda)\dot{y} = -\frac{Tz}{ml} - g$$

Next we will make use of the assumption that the bob of the pendulum oscillates in the XY – plane. Because of this assumption $z \approx l$ and therefore $\dot{z} = \ddot{z} = 0$ also using $\vec{T} \approx m\vec{g}$ equation (2) reduces to the following form

$$\ddot{x} - (2\omega sin\lambda)\dot{y} = -\frac{gx}{l}\ ;\ \ddot{y} + (2\omega sin\lambda)\dot{x} = -\frac{gy}{l}\ ;\ (2\omega cos\lambda)\dot{y} = 0$$

Putting $\omega sin\lambda = \omega_z$, the motion of the pendulum in the XY – plane is given by

$$\ddot{x} + \frac{gx}{l} = 2\omega_z\dot{y} \text{ and } \ddot{y} + \frac{gy}{l} = -2\omega_z\dot{x} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

$$\left(\ddot{x} + \frac{gx}{l}\right) + i\left(\ddot{y} + \frac{gy}{l}\right) = (2\omega_z\dot{y}) + i(-2\omega_z\dot{x})$$

$$(\ddot{x} + i\ddot{y}) + \frac{g}{l}(x + iy) = 2\omega_z(\dot{y} - i\dot{x}) = 2\omega_z(-i^2\dot{y} - i\dot{x})$$

$(\ddot{x} + i\ddot{y}) + \frac{g}{l}(x + iy) = -2i\omega_z(\dot{x} + i\dot{y})$

$\ddot{\xi} + \frac{g}{l}\xi = -2i\omega_z\dot{\xi}$ using $\xi = x + iy$

$\ddot{\xi} + \frac{g}{l}\xi = 0$ is called equation of S.H.M. with period $2\pi\sqrt{\frac{g}{l}}$ and the terms $-2i\omega_z\dot{\xi}$ is called Damping Term

$\ddot{\xi} + 2i\omega_z\dot{\xi} + \frac{g}{l}\xi = 0$

$\left(D^2 + 2i\omega_z D + \frac{g}{l}\right)\xi = 0$

$D^2 + 2i\omega_z D + \frac{g}{l} = 0$

$D = \frac{1}{2}\left(-2i\omega_z \pm \sqrt{-4\omega_z^2 - 4\omega_0^2}\right) = -i\left(\omega_z \pm \sqrt{\omega_z^2 + \omega_0^2}\right)$

Where $\omega_0^2 = \frac{g}{l}$ is the angular frequency of the pendulum, in the absence of the damping term.

Since the roots are imaginary the general solution can be written as

$$\xi = e^{-i\omega_z}\left[Ae^{i\sqrt{\omega_z^2+\omega_0^2}\,t} + Be^{-i\sqrt{\omega_z^2+\omega_0^2}\,t}\right] \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

$\xi_0 = Ae^{\omega_0 t} + Be^{-\omega_0 t}$ in the absence of the damping term

The angular frequency (which is the same as angular velocity) ω_0 of the undamped oscillator is much greater than the angular velocity ω of earth's rotation. i.e. $\omega_0 \gg \omega_z$ therefore above solution (3) can be approximated as

$\xi \approx e^{-i\omega_z t}[Ae^{\omega_0 t} + Be^{-\omega_0 t}]$ or $\xi \approx e^{-i\omega_z t}\xi_0$

$\Rightarrow x + iy = (cos\omega_z t - isin\omega_z t)(x' + iy')$ using $\xi_0 = x' + iy'$

$\Rightarrow x = x'cos\omega_z t + y'sin\omega_z t$ and $y = -x'sin\omega_z t + y'cos\omega_z t$

Above equations describe a rotation through an angle $\omega' = -\omega_z = -\omega sin\lambda$. Thus the plane of oscillation of the plane rotates with an angular frequency $\omega sin\lambda$. This result clearly demonstrate the rotation of the earth.

- Division of Mechanics

- Most Fundamental Concepts
- i) Particles ii) Rigid Bodies

- Fundamental Laws and Principles of Mechanics
- The Newtonian and non - Newtonian Mechanics
- Frames of References

CHAPTER

2

KINETICS

Kinetics

The branch of Mechanics/Dynamics which deals with the geometry of motion of a body with reference to the force causing motion is called kinetics.

Conservation Laws of Mechanics

Certain quantities such as linear and angular momentum, under certain circumstances, remain constant during motion of mechanical system. They are called **constant of motion** or **conserved quantities**. The results expressing constancy of these quantities play fundamental role not only in mechanical but also in other areas of theoretical physics.

In mechanics these results follow as consequences of the fundamental laws of motion and therefore sometimes called **conservation axioms**. But in other branches of physical sciences, such as Elementary Particle Physics, where fundamental dynamics laws are not known, the results expressing conservation of quantities such as linear momentum, angular momentum and energy are regarded as fundamental postulates of the theory or fundamental laws of nature. Our belief in their universal validity is bases on the observation that in the areas of physics, such as mechanics, electromagnetic theory and thermodynamics, where fundamental laws are well – founded and well – understood, these conservation laws are found to be strictly valid.

Momentum (Linear Momentum)

If a body of mass m is moving with velocity, then the momentum of that body is equals to the product of mass and velocity of the specific body. Mathematically, we can write is as $\vec{P} = m\vec{v}$. It is a vector quantity. It S.I unit is kgms^{-1}.

Momentum of system of particles

The total momentum of a system of particles is the vector sum of the momenta of the individual particles:

$$\vec{P}_{sys} = \vec{p}_1 + \vec{p}_2 + \cdots + \vec{p}_n = \sum_{i=1}^{n} \vec{p}_i = \sum_{i=1}^{n} m_i \vec{v}_i$$

Where the system consists of n particles and m_i is the mass of ith particle and $\vec{v}_i$ is corresponding velocity.

Angular Momentum/ Moment of Momentum about Origin

Angular momentum of a particle of mass m, position vector $\vec{r}$ and linear momentum $\vec{P}$ is defined as $\vec{r} \times \vec{P}$. The angular momentum of a single particle is the cross product of linear momentum and position vector of concerned particle. It is also called moment of momentum. It is represented by L, H or Ω

Angular Momentum of System of Particles

The angular momentum L of a system of particles is defined accordingly, as the vector sum of the individual angular momentum, namely, $L = \sum_{i=1}^{n} r_i \times m_i \vec{v}_i$

Law of Conservation of Momentum (Linear Momentum)

Law of conservation of momentum can be stated as:

If the sum of the external forces on a system is zero, the total momentum of the system does not change. i.e. $\vec{P} =$ Constant.

Momentum is always conserved (even if forces are non-conservative).

Proof

We know that $\vec{F} = \frac{d\vec{P}}{dt}$

If $\vec{F} = 0$ then $\frac{d\vec{P}}{dt} = 0$ and hence $\vec{P} =$ Constant.

Explanation

In simple terms, momentum is considered to be a quantity of motion. This quantity is measurable because if an object is moving and has mass, then it has momentum. Something that has a large mass has a large momentum or something that is moving very fast has a large momentum. The momentum of individual component may change but the total momentum of system remains conserved.

Example

- A 3000 kg vehicle moving at 30 m/sec has a momentum of 90,000 kgm/sec as a result of product of the mass and the velocity.
- Two hockey players of equal mass are traveling towards each other, one is moving at 9 m/sec and the other at 5 m/sec. The one moving with the faster velocity has a greater momentum and will knock the other one backwards.
- A bullet fired from a gun, although small in mass, has a large momentum because of an extremely large velocity.

Law of Conservation of Angular Momentum

Total angular momentum of the system remains constant if external torque act on the system is zero. i.e. $L =$ Constant

Or The time rate change of angular momentum in the absence of some external forces is zero. Mathematically, we can write $\frac{dL}{dt} = 0$. i.e. $L =$ Constant

1st Proof

We know that $\tau = \frac{dL}{dt}$

If $\tau = 0$ then $\frac{dL}{dt} = 0$ and hence $L =$ Constant

2nd Proof

Let us calculate the time derivative of the angular momentum. Using the rule for differentiating the cross product, we find

$$\frac{dL}{dt} = \frac{d}{dt}\left(\sum_{i=1}^{n} r_i \times m_i \vec{v}_i\right) = \sum_{i=1}^{n}(v_i \times m_i \vec{v}_i) + \sum_{i=1}^{n}(r_i \times m_i \vec{a}_i)$$

Now the first term on the right vanishes, because, $v_i \times \vec{v}_i = 0$ and, because $m_i \vec{a}_i$ is equal to the total force acting on particle i, we can write

$$\frac{dL}{dt} = \sum_{i=1}^{n}(r_i \times m_i \vec{a}_i)$$

$$\frac{dL}{dt} = \sum_{i=1}^{n}\left(r_i \times \left(\sum_i F_i^{(ext)} + \sum_{i=1}^{n}\sum_{j=1}^{n} F_{ij}^{(int)}\right)\right) \qquad (2)$$

$$\frac{dL}{dt} = \sum_{i=1}^{n}\left(r_i \times F_i^{(ext)}\right) + \sum_{i=1}^{n}\sum_{j=1}^{n} F_{ij}^{(int)} \qquad (3)$$

Where F_i denotes the total external force on particle i, and F_{ij} denotes the (internal) force exerted on particle i by any other particle j. Now the double summation on the right consists of pairs of terms of the form $\left(r_i \times F_j\right) + \left(r_i \times F_{ji}\right)$ Denoting the vector displacement of particle j relative to particle i by r_{ij}, we have $r_{ij} = r_j - r_i$. Therefore, because $F_{ji} = -F_{ij}$, expression (3) reduces to $-r_{ij} \times F_{ij}$ Which clearly vanishes if the internal forces are central, that is, if they act along the lines connecting pairs of particles.

Hence, the double sum in Equation (3) vanishes. Now the cross product $(r_i \times F_i)$ is the moment of the external force F. The sum $\sum(r_i \times F_i)$ is, therefore, the total moment of all the external forces acting on the system. If we denote the total external torque, or moment of force, by N, Equation (3) takes the form $\frac{dL}{dt} = N.$

That is, the time rate of change of the angular momentum of a system is equal to the total moment of all the external forces acting on the system.

If a system is isolated, then $N = 0$, and the angular momentum remains constant in both magnitude and direction:

$$L = \sum_{i=1}^{n} r_i \times m_i \vec{v}_i = \text{Constant vector} \qquad (8)$$

This is a statement of the principle of conservation of angular momentum. It is a generalization for a single particle in a central field.

Applications (Examples) of angular momentum

- Planets move around the sun and satellites move around the earth are examples of angular momentum.
- If a car move with constant velocity then momentum of the car remains constant.

Examples

A particle moves in a force field given by $\vec{F} = r^2\vec{r}$ where $\vec{r}$ is the position vector of the particle. Prove that the angular momentum of the particle is conserved.

Solution

The torque acting on the particle is $\tau = \vec{r} \times \vec{F}$

$$\Rightarrow \tau = \vec{r} \times r^2\vec{r} = r^2(\vec{r} \times \vec{r}) = \mathbf{0}$$

Then by theorem "Total angular momentum of the system remains constant if external torque act on the system is zero. i.e. $L = \text{Constant}$"

The angular momentum is constant. i.e. The angular momentum is conserved.

Torque

Torque is defined as the turning effect of a body. It is trend of an acting force due to which the rotational motion of a body changes. It is a moment force acting on the particle about origin. It is also called **twist and rotational force on an object.** Mathematically, torque is defined as the cross product of the force vector to the distance vector, which causes rotational motion of the body. i.e. $\tau = \vec{r} \times \vec{F}$
The magnitude of torque depends upon the applied force, the length of the lever arm connecting the axis to the point where the force applied, and the angle between the force vector and the length of lever arm. Symbolically we can write it as:
$\tau = rFSin\theta$

Torque is a vector quantity implies that it has direction as well as magnitude. The SI unit for torque is the newton meter (Nm). The direction of torque can be approximate using Right Hand Rule.

Principal of angular momentum

Relationship between Torque and Angular Momentum

Or The moment of force or torque about the origin O of a coordinate system is equal to the time rate of change of angular momentum.

Proof: We know that

$\tau = \vec{r} \times \vec{F}$ ………(1)

$L = \vec{r} \times \vec{P}$ ………(2)

$(2) \Rightarrow \frac{dL}{dt} = \frac{d}{dt}\left(\vec{r} \times \vec{P}\right)$

$\Rightarrow \frac{dL}{dt} = \vec{r} \times \frac{d\vec{P}}{dt} + \frac{d\vec{r}}{dt} \times \vec{P}$

$\Rightarrow \frac{dL}{dt} = \vec{r} \times \vec{F} + \vec{v} \times m\vec{v}$

$\Rightarrow \frac{dL}{dt} = \tau + m(\vec{v} \times \vec{v})$

$\boldsymbol{\frac{dL}{dt} = \tau}$ $\because \vec{v} \times \vec{v} = 0$

Principal of angular momentum (Another Way)

If τ is the torque about the axis and $\vec{L}$ is angular momentum then $\tau = \frac{d\vec{L}}{dt}$

or If $\boldsymbol{L}$ is then angular momentum then show that the rate of change of angular momentum equal to the moment of torque or force. i.e. $\boldsymbol{G} = \frac{d\boldsymbol{L}}{dt}$.

Proof

$$G = r \times F = r \times \dot{P} = r \times m\dot{v} = mr \times \ddot{r} = mr \times \frac{d\dot{r}}{dt} = \frac{d}{dt}(mr \times \dot{r}) = \frac{d}{dt}(r \times P)$$

$$G = \frac{dL}{dt}$$

Work Energy relation in case of Plane Rotational Motion

The total work done in rotating a rigid body from an angle θ_1 where the angular speed is ω_1 to angle θ_2 where the angular speed is ω_2 is the difference in KE of rotation at ω_1 and ω_2.

Or Prove that $\int_{\theta_1}^{\theta_2} \mathrm{G}d\theta = \frac{1}{2}I\omega_2^2 - \frac{1}{2}I\omega_1^2$

Proof

From work done we have $W = \int_{\theta_1}^{\theta_2} \mathrm{G}d\theta$

$$\Rightarrow \int_{\theta_1}^{\theta_2} \mathrm{G}d\theta = \int_{t_1}^{t_2} \left(I\frac{d\omega}{dt}\right)(\omega dt)$$

$$\Rightarrow \int_{\theta_1}^{\theta_2} \mathrm{G}d\theta = \int_{\omega_1}^{\omega_2} \left(I\frac{dt}{dt}\right)(\omega d\omega)$$

$$\Rightarrow \int_{\theta_1}^{\theta_2} \mathrm{G}d\theta = \int_{\omega_1}^{\omega_2} I\omega d\omega$$

$$\Rightarrow \int_{\theta_1}^{\theta_2} \mathrm{G}d\theta = I\int_{\omega_1}^{\omega_2} \omega d\omega$$

$$\Rightarrow \int_{\theta_1}^{\theta_2} \mathrm{G}d\theta = \frac{1}{2}I\omega_2^2 - \frac{1}{2}I\omega_1^2$$

Work

When some external force is applied on an object, work is done by this force in the direction of force. Also when some work is done by the applied force, energy transferred from one place to another.

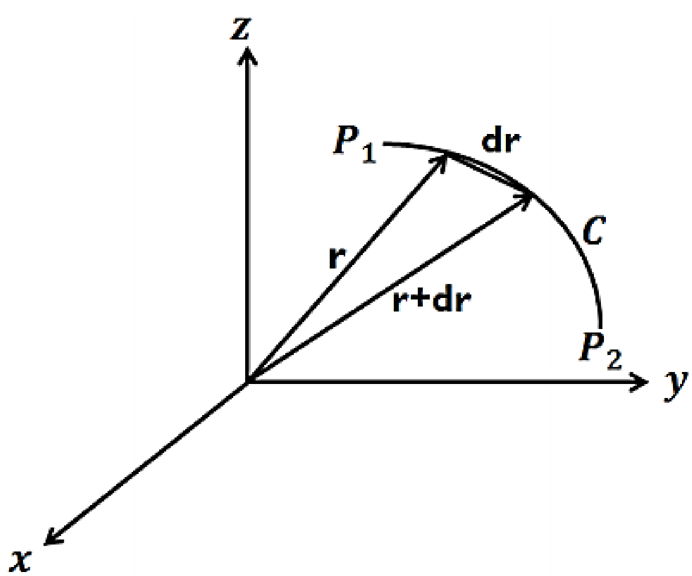
The work done can be defined as **a product of force and the displacement in the direction of applied force**. The amount of work done can be expressed as the following equation:

Work = Applied Force × Distance

The SI unit of work is the joule (J), which is defined as the work done by a force of one newton through a displacement of one meter.

If a force $\vec{F}$ acting on a particle gives it a displacement $d\vec{r}$, then the work done by the force on the particle is defined as $dW = \vec{F}.d\vec{r}$.

Since only the component of $\vec{F}$ in the direction of $d\vec{r}$ is effective in producing the motion.



The total work done by a force field (vector field) $\vec{F}$ in moving the particle from point P_1 to point P_2 along the curve C of Fig. is given by the line integral $W = \int_{P_1}^{P_2} \vec{F}.d\vec{r} = \int_{r_1}^{r_2} \vec{F}.d\vec{r}$, Where r_1 and r_2 are the position vectors of P_1 and P_2 respectively.

Energy

Energy is defined as the ability to do the work by the object. It is a measurable characteristic of a system which may be in the form of kinetic energy or potential. There exist many forms of energy. The energy neither can be created nor be destroyed but can be converted from one form to another. In mechanics, energy is the characteristic that transferred from one particle to another. The SI unit of energy is the joule; **1 joule can be defined as the energy transferred to an object by the work done of moving it a distance of 1 meter against a force of 1 newton**. The forms of energy include kinetic energy, potential energy, elastic energy, chemical energy, thermal energy and many others.

Potential Energy/ Potential/Scalar Potential

Energy possess by a body due to its position is called potential energy. It is a work done by a particle from its existing position to the standard position. It is denoted by V. Mathematically it is written as $V = \int_{p}^{p_0} \vec{F}.d\vec{r}$

Kinetic Energy (T)

Kinetic energy is the energy stored in a body due to its motion. It can be transferred from one objects to another and transformed into other kinds of energy. In classical mechanics, the kinetic energy is equal to 1/2 the product of the mass and the square of the speed. In formula form: $K.E = T = \frac{1}{2}mV^2$
The measuring unit of kinetic energy is the joule. It is denoted by T.

The kinetic energy increases with the square of the velocity. If a car is moving with double velocity then we can say that it has four times as much kinetic energy. As a consequence of this quadrupling, it takes four times the work to double the velocity.

If P denotes momentum of the object and m is the mass then we can symbolize the kinetic energy in the form of momentum as $T = \frac{P^2}{m}$

Gravitational Potential Energy

Gravitational Potential Energy is the energy possessed or acquired by an object due to a change in its position when it is present in a gravitational field. It is energy that is related to gravitational force or gravity.

Using Newton's Law of universal gravitation between two particles m_1 and m_2

$$\vec{F} = -G\frac{m_1 m_2}{r^3}\vec{r}$$

Where vector $\vec{r}$ is directed from m_1 to m_2.

If we replace m_1 by M and m_2 by m, thenpro

$$\vec{F} = -G\frac{Mm}{r^3}\vec{r} = GMm\nabla\left(\frac{1}{r}\right) = \nabla\left(\frac{GMm}{r}\right) = -\nabla\left(-\frac{GMm}{r}\right)$$

$$-\nabla \mathrm{V} = -\nabla\left(-\frac{GMm}{r}\right)$$

Which shows that gravitational potential energy between particles of masses M and m is given by

$$V(r) = -\frac{GMm}{r}$$

Electrostatic Potential Energy

Electrostatic Potential Energy is the electric potential energy per unit charge. It results from conservative coulomb forces and is associated with the configuration of a particular set of point charges within a defined system.

Using Coulomb's Law of the force between two charged particles q_1 and q_2

$$\vec{F} = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r^3}\vec{r} = -\frac{q_1 q_2}{4\pi\epsilon_0}\nabla\left(\frac{1}{r}\right) = -\nabla\left(\frac{q_1 q_2}{4\pi\epsilon_0}\frac{1}{r}\right)$$

$$-\nabla \mathrm{V} = -\nabla\left(\frac{q_1 q_2}{4\pi\epsilon_0}\frac{1}{r}\right)$$

Which shows that electrostatic potential function is given by

$$V(r) = \frac{q_1 q_2}{4\pi\epsilon_0}\frac{1}{r}$$

Conservative Force Field

A force field is said to conservative if the total work done by the particle moving along a curve is independent of the path taken by the particle and depend upon the end points of the curve only.

Necessary and sufficient conditions for a Conservative Force Field

Conservative force fields conserve the following properties:

i. A force field F is conservative if and only if there exists a continuously differentiable scalar field V such that $\vec{F} = -\nabla V$ or, equivalently, if and only if $curl\,\vec{F} = \nabla \times \vec{F} = 0$.
ii. A continuously differentiable force field F is conservative if and only if for any closed non-intersecting curve C (simple closed curve)
$W = \oint_C \vec{F}.d\vec{r} = 0$
i.e. the total work done in moving a particle around any closed path is zero.

Examples of Conservative Forces

- Gravitational force is an example of a conservative force.
- Elastic spring force is example of conservative force.
- The work done of a particle moving along a closed path is zero and the force which causes such motion is conservative.

Physical Significance of Conservative Force

For any conservative force $\vec{F}$ we have $\oint \vec{F}.d\vec{r} = 0$ for any closed path (in a simply connected region). This means that the force is not dissipative and any mechanical process taking place under its influence is reversible.

The property of reversibility can be described as follows;

If, at a certain moment, the velocities of all moving particles are reversed, then, following the same physical laws, a reversible mechanical process will retrace its former sequence of position and accelerations, in reverse order, as though time were running back.

Theorem

Show that a necessary and sufficient condition that $F_1dx + F_2dy + F_3dz$ be an exact differential is that $curl\,\vec{F} = \nabla \times \vec{F} = 0$ where $\vec{F} = F_1\hat{\imath} + F_2\hat{\jmath} + F_3\hat{k}$

Proof:

Suppose $F_1dx + F_2dy + F_3dz$ be an exact differential. Then x,y,z are independent variables. We know that

$$F_1dx + F_2dy + F_3dz = d\varphi = \frac{\partial\varphi}{\partial x}dx + \frac{\partial\varphi}{\partial y}dy + \frac{\partial\varphi}{\partial z}dz$$

$$F_1 = \frac{\partial\varphi}{\partial x} \; ; \; F_2 = \frac{\partial\varphi}{\partial y} \; ; \; F_3 = \frac{\partial\varphi}{\partial z}$$

$$\vec{F} = F_1\hat{\imath} + F_2\hat{\jmath} + F_3\hat{k} = \frac{\partial\varphi}{\partial x}\hat{\imath} + \frac{\partial\varphi}{\partial y}\hat{\jmath} + \frac{\partial\varphi}{\partial z}\hat{k} = \nabla\varphi$$

Thus $curl\,\vec{F} = \nabla \times \vec{F} = \nabla \times \nabla\varphi = 0$

Conversely suppose that $\nabla \times \vec{F} = 0$. Then $\vec{F} = \nabla\varphi$ and so $\vec{F}.d\vec{r} = \nabla\varphi.d\vec{r} = d\varphi$

That is $d\varphi = F_1dx + F_2dy + F_3dz$ be an exact differential.

Question

Show that $(y^2z^3cosx - 4x^3z)dx + 2z^3ysinxdy + (3y^2z^2sinx - x^4)dz$ be an exact differential of a function φ and find φ.

Solution

Given that $\vec{F} = (y^2z^3cosx - 4x^3z)\hat{\imath} + 2z^3ysinx\hat{\jmath} + (3y^2z^2sinx - x^4)\hat{k}$

Clearly $\nabla \times \vec{F} = 0$. Then according to result "$F_1dx + F_2dy + F_3dz$ be an exact differential iff $\nabla \times \vec{F} = 0$"
$(y^2z^3cosx - 4x^3z)dx + 2z^3ysinxdy + (3y^2z^2sinx - x^4)dz$ be an exact differential.

To find φ integrate these terms as needed and arrange to get required answer
$$F_1 = \frac{\partial\varphi}{\partial x} \; ; \; F_2 = \frac{\partial\varphi}{\partial y} \; ; \; F_3 = \frac{\partial\varphi}{\partial z}$$

Theorem: If $\vec{F} = -\nabla V$, where V is single valued and has continuous partial erivatives, show that the work done in moving the particle from one point $p_1 = (x_1, y_1, z_1)$ in this field to another point $p_2 = (x_2, y_2, z_2)$ is independent of the path joining the two points.

Proof: $\vec{W} = \int_{p_1}^{p_2} \vec{F}.d\vec{r} \Rightarrow \vec{W} = \int_{p_1}^{p_2} -\nabla V.d\vec{r}$

$$\vec{W} = -\int_{p_1}^{p_2}\left(\frac{\partial V}{\partial x}\hat{\imath} + \frac{\partial V}{\partial y}\hat{\jmath} + \frac{\partial V}{\partial z}\hat{k}\right).\left(dx\hat{\imath} + dy\hat{\jmath} + dz\hat{k}\right)$$

$$\vec{W} = -\int_{p_1}^{p_2}\frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz = -\int_{p_1}^{p_2} dV = -|V|_{p_1}^{p_2} = V(p_1) - V(p_2)$$

Integral depends only on points not on path joining the points.

Theorem: If $\int_C \vec{F}.d\vec{r}$ is independent of the path C joining any two points, show that there exists a function V such that $\vec{F} = -\nabla V$.

Proof: Let $\vec{F} = F_1\hat{\imath} + F_2\hat{\jmath} + F_3\hat{k}$ ……………(1)

If $\int_C \vec{F}.d\vec{r}$ is independent of the path C joining any two points which we take as (x_1, y_1, z_1) and (x, y, z) respectively then

$$V(x,y,z) = -\int_{(x_1,y_1,z_1)}^{(x,y,z)} \vec{F}.d\vec{r} = -\int_{(x_1,y_1,z_1)}^{(x,y,z)}(F_1dx + F_2dy + F_3dz)$$

Since $V = -\int_C \vec{F}.d\vec{r}$ is independent of the path C joining any two points, thus

$$V(x,y,z) = -\int_C[F_1(x,y,z)dx + F_2(x,y,z)dy + F_3(x,y,z)dz]$$

Let us choose a particular path the straight line segment from (x_1, y_1, z_1) to (x, y_1, z_1) to (x, y, z_1) to (x, y, z) and call $V(x, y, z)$ the work done along this path

$$V(x,y,z) = -\int_{x_1}^{x} F_1(x,y_1,z_1)dx - \int_{y_1}^{y} F_2(x,y,z_1)dy - \int_{z_1}^{z} F_3(x,y,z)dz$$

$$\frac{\partial V}{\partial z} = -F_3(x,y,z)$$

$$\frac{\partial V}{\partial y} = -F_2(x,y,z_1) - \int_{z_1}^{z}\frac{\partial F_3}{\partial y}(x,y,z)dz = -F_2(x,y,z)$$

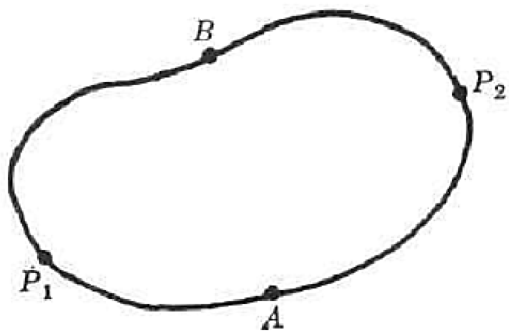
$$\frac{\partial V}{\partial x} = -F_1(x,y_1,z_1) - \int_{y_1}^{y} F_2(x,y,z_1)dy - \int_{z_1}^{z} F_3(x,y,z)dz = -F_1(x,y,z)$$

$$(1) \Rightarrow \vec{F} = -\frac{\partial V}{\partial x}\hat{\imath} - \frac{\partial V}{\partial y}\hat{\jmath} - \frac{\partial V}{\partial z}\hat{k} \Rightarrow \boldsymbol{\vec{F} = -\nabla V}$$

Theorem:

Prove that If $\int_{P_1}^{P_2} \vec{F}.d\vec{r}$ is independent of the path C joining any two points in a given region then $\oint \vec{F}.d\vec{r} = 0$ for all closed paths in the region and conversely.

Proof



Let $P_1AP_2BP_1$ be a closed curve then

$$\oint \vec{F}.d\vec{r} = \int_{P_1AP_2BP_1} \vec{F}.d\vec{r}$$

$$\oint \vec{F}.d\vec{r} = \int_{P_1AP_2} \vec{F}.d\vec{r} + \int_{P_2BP_1} \vec{F}.d\vec{r}$$

$$\oint \vec{F}.d\vec{r} = \int_{P_1AP_2} \vec{F}.d\vec{r} - \int_{P_1BP_2} \vec{F}.d\vec{r}$$

$$\oint \vec{F}.d\vec{r} = 0$$

Conversely

if $\oint \vec{F}.d\vec{r} = 0$

$$\int_{P_1AP_2} \vec{F}.d\vec{r} + \int_{P_2BP_1} \vec{F}.d\vec{r} = 0$$

$$\int_{P_1AP_2} \vec{F}.d\vec{r} - \int_{P_1BP_2} \vec{F}.d\vec{r} = 0$$

$$\int_{P_1AP_2} \vec{F}.d\vec{r} = \int_{P_1BP_2} \vec{F}.d\vec{r}$$

That is $\int_{P_1}^{P_2} \vec{F}.d\vec{r}$ is independent of the path C joining any two points in a given region.

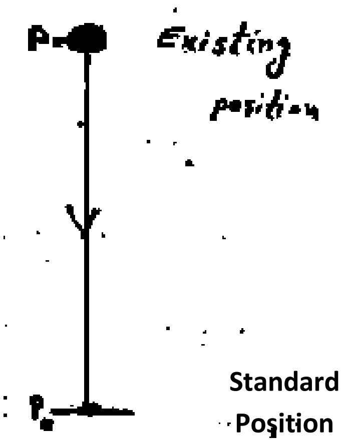
Theorem

If $\vec{F}$ is a conservative field force then there exist a scalar point function V such that $\vec{F} = -\nabla V.$

Proof

Consider a particle which is at existing position p and move towards the standard position p_0. At existing position potential energy is $V_p = \int_p^{p_0} \vec{F}.d\vec{r}$

At existing position potential energy is $V_{p_0} = 0$



$\Rightarrow V_p - V_{p_0} = \int_p^{p_0} \vec{F}.d\vec{r}$

$\Rightarrow \int_{p_0}^{p} dV = \int_p^{p_0} \vec{F}.d\vec{r} \Rightarrow \int_{p_0}^{p} dV = -\int_{p_0}^{p} \vec{F}.d\vec{r} \Rightarrow dV = -\vec{F}.d\vec{r}$

$\Rightarrow \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz = -\vec{F}.d\vec{r}$

$\Rightarrow \left(\frac{\partial V}{\partial x}\hat{\imath} + \frac{\partial V}{\partial y}\hat{\jmath} + \frac{\partial V}{\partial z}\hat{k}\right).\left(dx\hat{\imath} + dy\hat{\jmath} + dz\hat{k}\right) = -\vec{F}.d\vec{r}$

$\Rightarrow \nabla V.d\vec{r} = -\vec{F}.d\vec{r} \Rightarrow \vec{F} = -\nabla V$

Theorem

If the force acting on the particle is given by $\vec{F} = -\nabla\vec{V}$ then the total work done in moving the particle along the curve C from p_1 to p_2 is
$\overrightarrow{W} = \int_{p_1}^{p_2} \vec{F}.d\vec{r} = V(p_1) - V(p_2)$

Proof

$\overrightarrow{W} = \int_{p_1}^{p_2} \vec{F}.d\vec{r} \Rightarrow \overrightarrow{W} = \int_{p_1}^{p_2} -\nabla V.d\vec{r}$

$\overrightarrow{W} = -\int_{p_1}^{p_2} dV = -|V|_{p_1}^{p_2}$

$\overrightarrow{W} = V(p_1) - V(p_2)$

Or $\quad W_{12} = V_1 - V_2$

Work – Energy Theorem

A particle of constant mass m moves in space under the influence of a force field F. Assuming that at times t_1 and t_2 the velocity is $\vec{v}_1$ and $\vec{v}_2$ respectively, prove that the work done is the change in kinetic energy, i.e.,

$$W = \int_{t_1}^{t_2} \vec{F}.d\vec{r} = \frac{1}{2}m\vec{v}_2^2 - \frac{1}{2}m\vec{v}_1^2 = T_2 - T_1$$

Proof: Consider the work done by taking an external force $\vec{F}$, the force $\vec{F}$ moves the particle from position 1 to position 2 in the horizontal direction then

$$W = \int_1^2 \vec{F}.d\vec{r} \Rightarrow W = \int_1^2 m\vec{a}.\frac{d\vec{r}}{dt}dt \Rightarrow W = m\int_1^2 \frac{d\vec{v}}{dt}.\frac{d\vec{r}}{dt}dt = m\int_1^2 \vec{v}.\frac{d\vec{v}}{dt}dt$$

$$\Rightarrow W = m\int_1^2 \vec{v}d\vec{v} = m\left|\frac{\vec{v}^2}{2}\right|_1^2 = \frac{1}{2}m\vec{v}_2^2 - \frac{1}{2}m\vec{v}_1^2 \Rightarrow W = T_2 - T_1$$

Hence $W = \int_{t_1}^{t_2} \vec{F}.d\vec{r} = \frac{1}{2}m\vec{v}_2^2 - \frac{1}{2}m\vec{v}_1^2 = T_2 - T_1$

Question

Find the work done in moving a particle once around a circle C in the xy – plane, if the circle has center at the origin and radius 3 and if the force field is given by
$\vec{F} = (2x - y + z)\hat{\imath} + (x + y - z^2)\hat{\jmath} + (3x - 2y + 4z)\hat{k}$

Solution

In xy – plane we have $\vec{F} = (2x - y)\hat{\imath} + (x + y)\hat{\jmath} + (3x - 2y)\hat{k}$

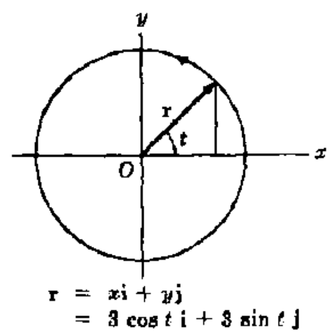
$$\Rightarrow W = \int_C \vec{F}.d\vec{r} = \int_C [(2x - y)\hat{\imath} + (x + y)\hat{\jmath} + (3x - 2y)\hat{k}].[dx\hat{\imath} + dy\hat{\jmath}]$$

$$\Rightarrow W = \int_C (2x - y)dx + (x + y)dy$$

$\Rightarrow W = 18\pi$ using $x = 3cost, y = 3sint\,; 0 \le t \le 2\pi$

If C were traversed in Counterclockwise (Clockwise) direction

The value of integral would be $18\pi(-18\pi)$



$\mathbf{r} = x\mathbf{i} + y\mathbf{j}$
$= 3\cos t\,\mathbf{i} + 3\sin t\,\mathbf{j}$

Conservation of Energy for a System of Particles in case of Conservative force

Principle of Conservation of Energy / Law of Conservation of Energy

The law of conservation of energy describes that the net energy of an isolated system remains conserved. Energy can neither be created nor destroyed; rather, it transforms from one form to another."

In case of conservative force field, the total energy is a constant. i.e. If T is for kinetic energy and V is for potential energy, then the total energy E is
$E = T + V =$ constant

Proof

Consider a particle move from position 1 to position 2. There will be two cases;

Case – I: Consider the work done by taking a conservative force $\vec{F}$ derived from a potential energy V, then

$$W_{12} = \int_1^2 \vec{F}.d\vec{r} \Rightarrow W_{12} = \int_1^2 -\nabla V.d\vec{r} \Rightarrow W_{12} = -\int_1^2 dV = -|V|_1^2$$

$$\Rightarrow W_{12} = V_1 - V_2 \qquad \ldots\ldots\ldots\ldots\ldots(1)$$

Case – II: Consider the work done by taking an external force $\vec{F}$, the force $\vec{F}$ moves the particle from position 1 to position 2 in the horizontal direction then

$$W_{12} = \int_1^2 \vec{F}.d\vec{r} \Rightarrow W_{12} = \int_1^2 m\vec{a}.\frac{d\vec{r}}{dt}dt \Rightarrow W_{12} = m\int_1^2 \frac{dV}{dt}.\frac{d\vec{r}}{dt}dt = m\int_1^2 V.\frac{dV}{dt}dt$$

$$\Rightarrow W_{12} = m\int_1^2 VdV = m\left|\frac{V^2}{2}\right|_1^2 = \frac{1}{2}mV_2^2 - \frac{1}{2}mV_1^2$$

$$\Rightarrow W_{12} = T_2 - T_1 \qquad \ldots\ldots\ldots\ldots\ldots(2)$$

From (1) and (2) we get

$$V_1 - V_2 = T_2 - T_1$$

$$\Rightarrow T_1 + V_1 = T_2 + V_2$$

$$\Rightarrow E = T + V = \text{constant}$$

Question: A particle of mass m moving along the x – axis under the influence of a conservative force field having potential V(x). If the particle is located at the position x_1 and x_2 at respective times t_1 and t_2, prove that if E is the total energy then $t_2 - t_1 = \sqrt{\frac{m}{2}}\int_{x_1}^{x_2}\frac{dx}{\sqrt{E-V(x)}}$

Solution: By the conservation of energy $T + V = E$

$$\Rightarrow \frac{1}{2}m\left(\frac{dx}{dt}\right)^2 + V(x) = E \Rightarrow \left(\frac{dx}{dt}\right)^2 = \frac{2}{m}(E - V(x)) \Rightarrow \frac{dx}{dt} = \sqrt{\frac{2}{m}}\sqrt{E - V(x)}$$

$$\Rightarrow \int_{t_1}^{t_2} dt = \sqrt{\frac{m}{2}}\int_{x_1}^{x_2}\frac{dx}{\sqrt{E-V(x)}} \Rightarrow t_2 - t_1 = \sqrt{\frac{m}{2}}\int_{x_1}^{x_2}\frac{dx}{\sqrt{E-V(x)}}$$

Conservative Systems and Orbits of Particles

A single particle moving in a conservative field of forces may perform an important type of motion. Suppose the total energy E of the system is a constant of motion i.e. $\frac{1}{2}m\dot{r}^2 + V(r) = E$

Where E is some constant denoting total energy of the system.

Suppose the particle's motion in such that it returns to the same position, represented by the position vector r_0, at a later time. Then it must have the same K.E. and therefore the same speed. It follows that in a conservative system it is possible for closed trajectories to occur. This fact is very relevant in the study of Earth's motion about the Sun.

Question: Is the force $\vec{F} = \vec{A} \times \vec{r}$ conservative?

Solution: Let $\vec{A} = A_1\hat{\imath} + A_2\hat{\jmath} + A_3\hat{k}$ and $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$ then

$$\vec{F} = \vec{A} \times \vec{r} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ A_1 & A_2 & A_3 \\ x & y & z \end{vmatrix} = (A_2 z - A_3 y)\hat{\imath} + (A_3 x - A_1 z)\hat{\jmath} + (A_1 y - A_2 x)\hat{k}$$

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial x} & \frac{\partial}{\partial x} \\ A_2 z - A_3 y & A_3 x - A_1 z & A_1 y - A_2 x \end{vmatrix} = 0$$

Thus the force $\vec{F} = \vec{A} \times \vec{r}$ is conservative.

Question

Find the potential energy function associated with the force

$\vec{F} = -yz\hat{\imath} - xz\hat{\jmath} - xy\hat{k}$

Solution

In this case $\vec{F} = -\nabla V$

$\Rightarrow -yz\hat{\imath} - xz\hat{\jmath} - xy\hat{k} = -\nabla V \Rightarrow \nabla V = yz\hat{\imath} + xz\hat{\jmath} + xy\hat{k}$

$\Rightarrow \frac{\partial V}{\partial x}\hat{\imath} + \frac{\partial V}{\partial y}\hat{\jmath} + \frac{\partial V}{\partial z}\hat{k} = yz\hat{\imath} + xz\hat{\jmath} + xy\hat{k}$

$\Rightarrow \frac{\partial V}{\partial x} = yz \;\ldots\ldots(1),\;\; \frac{\partial V}{\partial y} = xz \;\ldots\ldots(2),\;\; \frac{\partial V}{\partial z} = xy \;\ldots\ldots(3)$

$(1) \Rightarrow \int dV = yz\int dx \Rightarrow V = xyz + f(y,z)$ …………..(4)

$\Rightarrow \frac{\partial V}{\partial y} = xz + \frac{\partial f}{\partial y}(y,z)$ …………..(5) partially differentiating w.r.to y

$\Rightarrow xz = xz + \frac{\partial f}{\partial y}(y,z)$ from (2) and (5)

$\Rightarrow \frac{\partial f}{\partial y}(y,z) = 0 \Rightarrow f(y,z) = 0 + g(z) \Rightarrow f(y,z) = g(z)$ …………..(6)

$(4) \Rightarrow V = xyz + g(z)$ …………..(7) using (6) in (4)

$\Rightarrow \frac{\partial V}{\partial z} = xy + g'(z)$ partially differentiating w.r.to z

$\Rightarrow xy = xy + g'(z) \Rightarrow g'(z) = 0 \Rightarrow g(z) = K$ using (3)

Hence our required potential function is

$\Rightarrow V = xyz + K$

Question

Find the potential energy function associated with the force

$\vec{F} = ax\hat{\imath} + by\hat{\jmath} + cz\hat{k}$

Solution

In this case $\vec{F} = -\nabla V$

$\Rightarrow ax\hat{\imath} + by\hat{\jmath} + cz\hat{k} = -\nabla V \Rightarrow \nabla V = -\left(ax\hat{\imath} + by\hat{\jmath} + cz\hat{k}\right)$

$\Rightarrow \frac{\partial V}{\partial x}\hat{\imath} + \frac{\partial V}{\partial y}\hat{\jmath} + \frac{\partial V}{\partial z}\hat{k} = -ax\hat{\imath} - by\hat{\jmath} - cz\hat{k}$

$\Rightarrow \frac{\partial V}{\partial x} = -ax \;\ldots\ldots (1),\;\; \frac{\partial V}{\partial y} = -by \;\ldots\ldots (2),\;\; \frac{\partial V}{\partial z} = -cz \;\ldots\ldots (3)$

$(1) \Rightarrow \int dV = -a\int xdx \Rightarrow V = -\frac{1}{2}ax^2 + f(y,z)$ …………..(4)

$\Rightarrow \frac{\partial V}{\partial y} = \frac{\partial f}{\partial y}(y,z)$ …………..(5) partially differentiating w.r.to y

$\Rightarrow -by = \frac{\partial f}{\partial y}(y,z)$ from (2) and (5)

$\Rightarrow f = -\frac{1}{2}by^2 + h(z)$ …………..(6)

$(4) \Rightarrow V = -\frac{1}{2}ax^2 - \frac{1}{2}by^2 + h(z)$ …………..(7) using (6) in (4)

$\Rightarrow \frac{\partial V}{\partial z} = \frac{\partial h}{\partial z}$ partially differentiating w.r.to z

$\Rightarrow \frac{\partial h}{\partial z} = -cz \Rightarrow h = -\frac{1}{2}cz^2$

Hence our required potential function is

$\Rightarrow V = -\frac{1}{2}ax^2 - \frac{1}{2}by^2 - \frac{1}{2}cz^2$

Question: Discuss whether the following force is conservative, if so, find the potential energy function associated with the force

$\vec{F} = (ax + by^2)\hat{\imath} + (az + 2bxy)\hat{\jmath} + (ay + bz^2)\hat{k}$

Solution: For conservative force we will prove $\vec{\nabla} \times \vec{F} = 0$

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial x} & \frac{\partial}{\partial x} \\ ax + by^2 & az + 2bxy & ay + bz^2 \end{vmatrix} = 0$$

Thus the force $\vec{F}$ is conservative.

In this case $\vec{F} = -\nabla V$

$\Rightarrow (ax + by^2)\hat{\imath} + (az + 2bxy)\hat{\jmath} + (ay + bz^2)\hat{k} = -\nabla V$

$\Rightarrow -\left(\frac{\partial V}{\partial x}\hat{\imath} + \frac{\partial V}{\partial y}\hat{\jmath} + \frac{\partial V}{\partial z}\hat{k}\right) = (ax + by^2)\hat{\imath} + (az + 2bxy)\hat{\jmath} + (ay + bz^2)\hat{k}$

$\Rightarrow \frac{\partial V}{\partial x} = -ax - by^2 \;\ldots(1),\;\; \frac{\partial V}{\partial y} = -az - 2bxy \;\ldots(2),\;\; \frac{\partial V}{\partial z} = -ay - bz^2 \;\ldots(3)$

$(1) \Rightarrow V = -\frac{1}{2}ax^2 - bxy^2 + f(y,z)$ …………..(4)

$\Rightarrow \frac{\partial V}{\partial y} = -2bxy + \frac{\partial f}{\partial y}(y,z)$ …………..(5) partially differentiating w.r.to y

$\Rightarrow -az - 2bxy = -2bxy + \frac{\partial f}{\partial y}(y,z)$ from (2) and (5)

$\Rightarrow \frac{\partial f}{\partial y}(y,z) = -az \Rightarrow f(y,z) = -ayz + h(z)$ …………..(6)

$(4) \Rightarrow V = -\frac{1}{2}ax^2 - bxy^2 + -ayz + h(z)$ ………..(7) using (6) in (4)

$\Rightarrow \frac{\partial V}{\partial z} = -ay + h'(z)$ partially differentiating w.r.to z

$\Rightarrow -ay - bz^2 = -ay + h'(z) \Rightarrow h'(z) = -bz^2 \Rightarrow h(z) = -\frac{1}{3}bz^3$ using (3)

Hence our required potential function is $V = -\frac{1}{2}ax^2 - bxy^2 + -ayz - \frac{1}{3}bz^3$

Question: Find the work done by the force field

$\vec{F} = (y^2z^3 - 6xz^2)\hat{\imath} + 2xyz^3\hat{\jmath} + (3xy^2z^2 - 6x^2z)\hat{k}$

in moving a particle from the point $(-2,1,3)$ to $(1,-2,-1)$.

Solution: To find work done we will use the formula $W = \int_A^B \vec{F}.d\vec{r}$

$\Rightarrow W = \int_A^B \vec{F}.d\vec{r} = \int_A^B -\nabla V.d\vec{r} = -\int_{(-2,1,3)}^{(1,-2,-1)} dV = |-V|_{(-2,1,3)}^{(1,-2,-1)}$

$\Rightarrow W = |-3x^2z^2 + xy^2z^3 - c|_{(-2,1,3)}^{(1,-2,-1)} = 155$

Question:

Show that $\vec{F} = (2xy + z^3)\hat{\imath} + x^2\hat{\jmath} + 3xz^2\hat{k}$ is a conservative force field. Find the potential. Also find the work done in moving an object in this field from $(1,-2,1)$ to $(3,1,4)$.

Solution: For conservative force we will prove $\vec{\nabla} \times \vec{F} = 0$

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial x} & \frac{\partial}{\partial x} \\ 2xy + z^3 & x^2 & 3xz^2 \end{vmatrix} = 0.$$ Thus the force $\vec{F}$ is conservative.

To find potential we have $\vec{F} = -\nabla V$

$\Rightarrow (2xy + z^3)\hat{\imath} + x^2\hat{\jmath} + 3xz^2\hat{k} = -\nabla V$

$\Rightarrow -\left(\frac{\partial V}{\partial x}\hat{\imath} + \frac{\partial V}{\partial y}\hat{\jmath} + \frac{\partial V}{\partial z}\hat{k}\right) = (2xy + z^3)\hat{\imath} + x^2\hat{\jmath} + 3xz^2\hat{k}$

$\Rightarrow \frac{\partial V}{\partial x} = 2xy + z^3 \ldots\ldots(1), \quad \frac{\partial V}{\partial y} = x^2 \ldots\ \ldots(2), \quad \frac{\partial V}{\partial z} = 3xz^2 \ldots\ \ldots(3)$

Our required potential function is $V = -(x^2y^2 + xz^3)$

To find work done we will use the formula $W = \int_A^B \vec{F}.d\vec{r}$

$\Rightarrow W = \int_A^B \vec{F}.d\vec{r} = \int_A^B -\nabla V.d\vec{r} = -\int_{(1,-2,1)}^{(3,1,4)} dV = |-V|_{(1,-2,1)}^{(3,1,4)}$

$\Rightarrow W = |-(x^2y + xz^3)|_{(1,-2,1)}^{(3,1,4)} = -202$

Question

A particle of mass m moves under a force $\vec{F} = -cx^3$ where c is a positive constant. Then

i. Find potential energy function.
ii. If the particle starts from rest at $x = -a$ what is its velocity when it reaches at $x = 0$.
iii. Where in the subsequent motion does it come to the rest?

Solution

i. Given that $\vec{F} = -cx^3$

$$\Rightarrow \vec{F} = -\nabla V = -\frac{dV}{dx} = -cx^3 \Rightarrow \frac{dV}{dx} = cx^3 \Rightarrow dV = cx^3 dx \Rightarrow V = \frac{1}{4}cx^4 + A$$

ii. $W = \int_0^a \vec{F} dx$

$\frac{1}{2}mv^2 = -c\int_0^a x^3 dx$ since $W = K.E$

$v = \sqrt{\frac{c}{2m}}a^2$

iii. When body moves from $x = -a$ to $x = b$ it comes to rest then $W = 0$

$\Rightarrow \int_{-a}^{b} \vec{F} dx = 0 \Rightarrow -c\int_{-a}^{b} x^3 dx = 0 \Rightarrow b^4 - a^4 = 0$

$\Rightarrow b = a \ \ , \ b = -a(negelcted)$

When $b = a$ then it comes to the rest.

Motion of a Particle under a Constant Force

Let $\vec{F}$ be a constant force applied on a particle of mass 'm'. Then

$\vec{F} = m\vec{a} = \text{Constant} \Rightarrow \vec{a} = \text{Constant}$

Since $\frac{d\vec{v}}{dt} = \vec{a}$ Therefore $d\vec{v} = \vec{a}dt$

$\Rightarrow \vec{v} = \vec{a}t + \vec{A}$ on integrating

Initially using $t = 0, \vec{v} = \vec{v}_0$ we get $\vec{A} = \vec{v}_0$. Thus $\vec{v} = \vec{a}t + \vec{v}_0$

$\Rightarrow \frac{d\vec{r}}{dt} = \vec{v}_0 + \vec{a}t \Rightarrow d\vec{r} = \vec{v}_0 dt + \vec{a}t dt$

$\Rightarrow \vec{r} = \vec{v}_0 t + \vec{a}\frac{t^2}{2} + \vec{B}$ on integrating

Initially using $t = 0, \vec{r} = 0$ we get $\vec{B} = 0$. Thus $\boldsymbol{\vec{r} = \vec{v}_0 t + \frac{1}{2}\vec{a}t^2}$

Motion of a Particle under a Time Dependent Force

Let $\vec{F} = \vec{F}(t)$ be a time dependent force applied on a particle of mass 'm'. Then

$\vec{F} = m\vec{a} \Rightarrow \vec{F}(t) = m\vec{a} \Rightarrow \vec{a} = \frac{\vec{F}(t)}{m} \Rightarrow \frac{d\vec{v}}{dt} = \frac{\vec{F}(t)}{m} \Rightarrow d\vec{v} = \frac{\vec{F}(t)}{m} dt$

$\Rightarrow \int_{v_0}^{v} d\vec{v} = \frac{1}{m}\int_{t_0}^{t} \vec{F}(t)dt \Rightarrow |\vec{v}|_{v_0}^{v} = \frac{1}{m}\int_{t_0}^{t} \vec{F}(t)dt \Rightarrow \vec{v} - \vec{v}_0 = \frac{1}{m}\int_{t_0}^{t} \vec{F}(t)dt$

$\Rightarrow \vec{v} = \vec{v}_0 + \frac{1}{m}\int_{t_0}^{t} \vec{F}(t)dt$

$\Rightarrow \frac{d\vec{r}}{dt} = \vec{v}_0 + \frac{1}{m}\int_{t_0}^{t} \vec{F}(t)dt \Rightarrow d\vec{r} = \vec{v}_0 dt + \frac{1}{m}\left[\int_{t_0}^{t} \vec{F}(t)dt\right] dt$

$\Rightarrow \int_{r_0}^{r} d\vec{r} = \vec{v}_0 \int_{t_0}^{t} dt + \frac{1}{m}\int_{t_0}^{t}\left[\int_{t_0}^{t} \vec{F}(t)dt\right] dt$

$\Rightarrow |\vec{r}|_{r_0}^{r} = \vec{v}_0 |t|_{t_0}^{t} + \frac{1}{m}\int_{t_0}^{t}\left[\int_{t_0}^{t} \vec{F}(t)dt\right] dt$

$\Rightarrow \vec{r} - \vec{r}_0 = \vec{v}_0(t - t_0) + \frac{1}{m}\int_{t_0}^{t}\left[\int_{t_0}^{t} \vec{F}(t)dt\right] dt$

$\Rightarrow \boldsymbol{\vec{r} = \vec{r}_0 + \vec{v}_0(t - t_0) + \frac{1}{m}\int_{t_0}^{t}\left[\int_{t_0}^{t} \vec{F}(t)dt\right] dt}$

Motion of a Particle under a Velocity Dependent Force

Let $\vec{F} = \vec{F}(\vec{v})$ be a time dependent force applied on a particle of mass 'm'. Then

$$\vec{F} = m\vec{a} \Rightarrow \vec{F}(\vec{v}) = m\frac{d\vec{v}}{dt} \qquad \ldots\ldots\ldots\ldots(1)$$

$$\int_{t_0}^{t} dt = m\int_{v_0}^{v} \frac{1}{\vec{F}(\vec{v})} d\vec{v} \Rightarrow |t|_{t_0}^{t} = m\int_{v_0}^{v} \frac{1}{\vec{F}(\vec{v})} d\vec{v}$$

$$\Rightarrow t - t_0 = m\int_{v_0}^{v} \frac{1}{\vec{F}(\vec{v})} d\vec{v}$$

$$\Rightarrow t = t_0 + m\int_{v_0}^{v} \frac{1}{\vec{F}(\vec{v})} d\vec{v}$$

$$(1) \Rightarrow \vec{F}(\vec{v}) = m\frac{d\vec{v}}{dt} = m\frac{d\vec{v}}{dx}\frac{dx}{dt} = m\vec{v}\frac{d\vec{v}}{dx} \Rightarrow dx = \frac{m\vec{v}}{\vec{F}(\vec{v})} d\vec{v}$$

$$\Rightarrow \int_{x_0}^{x} dx = m\int_{v_0}^{v} \frac{\vec{v}}{\vec{F}(\vec{v})} d\vec{v}$$

$$\Rightarrow \int_{x_0}^{x} dx = m\int_{v_0}^{v} \frac{\vec{v}}{\vec{F}(\vec{v})} d\vec{v}$$

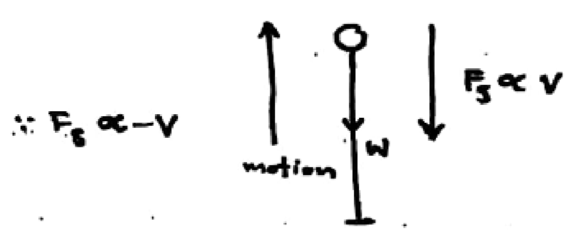
$$\Rightarrow x - x_0 = m\int_{v_0}^{v} \frac{\vec{v}}{\vec{F}(\vec{v})} d\vec{v}$$

$$\Rightarrow \boldsymbol{x = x_0 + m\int_{v_0}^{v} \frac{\vec{v}}{\vec{F}(\vec{v})} d\vec{v}}$$

Question

A particle of mass m is projected vertically up with an initial velocity v_0. If the force due to the friction of air is directly proportional to its instantaneous velocity, calculate velocity and position of the particle as a function of time.

Solution



Body move upward, so frictional force $= F_s = -kv$ and $W = -mg$

$F = F_s + W = -kv - mg \Rightarrow F(v) = -mg - kv$ where m, g, k are constants.

$\Rightarrow m\frac{dv}{dt} = -mg - kv \Rightarrow dv = \frac{-mg-kv}{m}dt \Rightarrow dv = \frac{-(mg+kv)}{m}dt$

$\Rightarrow \frac{1}{k}\int_{v_0}^{v}\frac{k}{mg+kv}dv = -\frac{1}{m}\int_0^t dt \Rightarrow \frac{1}{k}|ln(mg+kv)|_{v_0}^{v} = -\frac{1}{m}|t|_0^t$

$\Rightarrow \frac{1}{k}[ln(mg+kv) - ln(mg+kv_0)] = -\frac{t}{m} \Rightarrow -\frac{m}{k}\left[ln\left(\frac{mg+kv}{mg+kv_0}\right)\right] = t$

$\Rightarrow \boldsymbol{t = \frac{m}{k}\left[ln\left(\frac{mg+kv_0}{mg+kv}\right)\right]}$

Now $t = -\frac{m}{k}\left[ln\left(\frac{mg+kv}{mg+kv_0}\right)\right]$

$\Rightarrow ln\left(\frac{mg+kv}{mg+kv_0}\right) = -\frac{k}{m}t \Rightarrow \frac{mg+kv}{mg+kv_0} = e^{-\frac{k}{m}t} \Rightarrow mg + kv = (mg+kv_0)e^{-\frac{k}{m}t}$

$\Rightarrow kv = (mg+kv_0)e^{-\frac{k}{m}t} - mg \Rightarrow \boldsymbol{v = \frac{mg}{k}\left(e^{-\frac{k}{m}t} - 1\right) + v_0 e^{-\frac{k}{m}t}}$

Now $\frac{dx}{dt} = \frac{mg}{k}\left(e^{-\frac{k}{m}t} - 1\right) + v_0 e^{-\frac{k}{m}t}$

$\Rightarrow \int_{x_0}^{x}dx = \frac{mg}{k}\int_0^t\left(e^{-\frac{k}{m}t} - 1\right)dt + v_0\int_0^t e^{-\frac{k}{m}t}dt$

$\Rightarrow |x|_{x_0}^{x} = \frac{mg}{k}\left|\frac{e^{-\frac{k}{m}t}}{-\frac{k}{m}} - t\right|_0^t + v_0\left|\frac{e^{-\frac{k}{m}t}}{-\frac{k}{m}}\right|_0^t$

$\Rightarrow x - x_0 = \frac{mg}{k}\left(\frac{e^{-\frac{k}{m}t}}{-\frac{k}{m}} - t - \frac{1}{-\frac{k}{m}}\right) + v_0\left(\frac{e^{-\frac{k}{m}t}}{-\frac{k}{m}} - \frac{1}{-\frac{k}{m}}\right)$

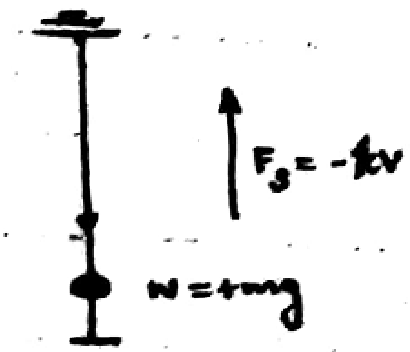
$\Rightarrow x = x_0 + \frac{mg}{k}\left(-\frac{me^{-\frac{k}{m}t}}{k} - t + \frac{m}{k}\right) + v_0\left(-\frac{me^{-\frac{k}{m}t}}{k} + \frac{m}{k}\right)$

$\Rightarrow \boldsymbol{x = x_0 - \frac{mgt}{k} + \left(\frac{m^2g}{k^2} + \frac{mv_0}{k}\right)\left(1 - e^{-\frac{k}{m}t}\right)}$

Question

A particle of mass m is falling under action of gravity near the surface of Earth. If the force due to the friction of air is directly proportional to its instantaneous velocity, calculate velocity and position of the particle as a function of time.

Solution



Body move downward, so frictional force $= F_s = -kv$ and $W = mg$

$F = W + F_s = mg - kv \Rightarrow F(v) = mg - kv$ where m, g, k are constants.

$$\Rightarrow m\frac{dv}{dt} = mg - kv \Rightarrow dv = \frac{mg-kv}{m}dt \Rightarrow dv = \frac{mg-kv}{m}dt$$

$$\Rightarrow -\frac{1}{k}\int_{v_0}^{v}\frac{-k}{mg-kv}dv = \frac{1}{m}\int_0^t dt \Rightarrow \frac{1}{k}|ln(mg-kv)|_{v_0}^{v} = -\frac{1}{m}|t|_0^t$$

$$\Rightarrow \frac{1}{k}[ln(mg-kv) - ln(mg-kv_0)] = -\frac{t}{m} \Rightarrow -\frac{m}{k}\left[ln\left(\frac{mg-kv}{mg-kv_0}\right)\right] = t$$

$$\Rightarrow \boldsymbol{t = \frac{m}{k}\left[ln\left(\frac{mg-kv_0}{mg-kv}\right)\right]}$$

Now $t = -\frac{m}{k}\left[ln\left(\frac{mg-kv}{mg-kv_0}\right)\right]$

$$\Rightarrow ln\left(\frac{mg-kv}{mg-kv_0}\right) = -\frac{k}{m}t \Rightarrow \frac{mg-kv}{mg-kv_0} = e^{-\frac{k}{m}t} \Rightarrow mg - kv = (mg-kv_0)e^{-\frac{k}{m}t}$$

$$\Rightarrow kv = mg - (mg-kv_0)e^{-\frac{k}{m}t} \Rightarrow \boldsymbol{v = \frac{mg}{k}\left(1 - e^{-\frac{k}{m}t}\right) - v_0 e^{-\frac{k}{m}t}}$$

Now $\frac{dx}{dt} = \frac{mg}{k}\left(1 - e^{-\frac{k}{m}t}\right) - v_0 e^{-\frac{k}{m}t}$

$$\Rightarrow \int_{x_0}^{x} dx = \frac{mg}{k}\int_0^t \left(1 - e^{-\frac{k}{m}t}\right) dt - v_0 \int_0^t e^{-\frac{k}{m}t} dt$$

$$\Rightarrow |x|_{x_0}^{x} = \frac{mg}{k}\left| t - \frac{e^{-\frac{k}{m}t}}{-\frac{k}{m}} \right|_0^t - v_0 \left| \frac{e^{-\frac{k}{m}t}}{-\frac{k}{m}} \right|_0^t$$

$$\Rightarrow x - x_0 = \frac{mg}{k}\left(t - \frac{e^{-\frac{k}{m}t}}{-\frac{k}{m}} + \frac{1}{-\frac{k}{m}} \right) - v_0 \left(\frac{e^{-\frac{k}{m}t}}{-\frac{k}{m}} - \frac{1}{-\frac{k}{m}} \right)$$

$$\Rightarrow x = x_0 + \frac{mg}{k}\left(t + \frac{me^{-\frac{k}{m}t}}{k} - \frac{m}{k} \right) - v_0 \left(-\frac{me^{-\frac{k}{m}t}}{k} + \frac{m}{k} \right)$$

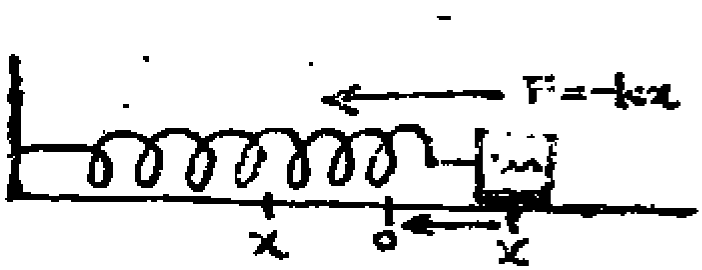
$$\Rightarrow x = x_0 + \frac{mg}{k}\left(t + \frac{me^{-\frac{k}{m}t}}{k} - \frac{m}{k} \right) - v_0 \left(-\frac{me^{-\frac{k}{m}t}}{k} + \frac{m}{k} \right)$$

$$\Rightarrow \boldsymbol{x = x_0 + \frac{mg}{k} t + \frac{m}{k}\left(\frac{mg}{k} - v_0\right)\left(e^{-\frac{k}{m}t} - 1\right)}$$

Question

A mass m tied to a spring having force constant k oscillate in one dimension. If the motion is subjected to the force $F = -kx$, find expression for displacement, velocity and period of oscillation.

Solution



Given $F = -kx$

In this case Law of Conservation of Energy holds as $K.E + P.E =$ Total energy

$T + V = E$ ……………..(1)

Since $T = \frac{1}{2}mv^2$ but $V = \int_x^0 F dx = -\int_0^x F dx = -\int_0^x (-kx) dx = \frac{1}{2}kx^2$ therefore

$(1) \Rightarrow \frac{1}{2}mv^2 + \frac{1}{2}kx^2 = E$

$\Rightarrow \frac{1}{2}mv^2 = E - \frac{1}{2}kx^2 \Rightarrow \frac{1}{2}mv^2 = E\left(1 - \frac{1}{2E}kx^2\right) \Rightarrow v^2 = \frac{2E}{m}\left(1 - \frac{1}{2E}kx^2\right)$

$\Rightarrow v^2 = \frac{2E}{m}\left(1 - \left(\sqrt{\frac{k}{2E}}x\right)^2\right)$

Put $\sqrt{\frac{k}{2E}}x = Sin\theta \Rightarrow dx = \sqrt{\frac{2E}{k}}Cos\theta d\theta$

$\Rightarrow v^2 = \frac{2E}{m}(1 - Sin^2\theta) \Rightarrow \left(\frac{dx}{dt}\right)^2 = \frac{2E}{m}Cos^2\theta \Rightarrow \frac{dx}{dt} = \sqrt{\frac{2E}{m}}Cos\theta$

$\Rightarrow \frac{dt}{dx} = \frac{1}{\sqrt{\frac{2E}{m}}Cos\theta} \Rightarrow dt = \frac{dx}{\sqrt{\frac{2E}{m}}Cos\theta} \Rightarrow dt = \frac{\sqrt{\frac{2E}{k}}Cos\theta d\theta}{\sqrt{\frac{2E}{m}}Cos\theta} \Rightarrow dt = \sqrt{\frac{m}{k}}d\theta$

$\Rightarrow dt = \frac{1}{\omega}d\theta$ with $\omega = \sqrt{\frac{k}{m}}$

$\Rightarrow \omega dt = d\theta \Rightarrow \int_{\theta_0}^{\theta} d\theta = \int_{t_0}^{t} \omega dt \Rightarrow \theta - \theta_0 = \omega t \Rightarrow \theta = \theta_0 + \omega t$

Then $\sqrt{\frac{k}{2E}}x = Sin\theta$ becomes $x = \sqrt{\frac{2E}{k}}Sin\theta$

$\Rightarrow x = \sqrt{\frac{2E}{k}}Sin(\theta_0 + \omega t)$

$\Rightarrow \frac{dx}{dt} = \sqrt{\frac{2E}{k}}\omega Cos(\theta_0 + \omega t) \Rightarrow v = \omega\sqrt{\frac{2E}{k}}Cos(\omega t + \theta_0)$

We know that $T = \frac{2\pi}{\omega}$

$\Rightarrow T = \frac{2\pi}{\sqrt{\frac{k}{m}}}$

$\Rightarrow T = 2\pi\sqrt{\frac{m}{k}}$

Question

A particle of mass m is at rest at the origin of the coordinate system. At $t = 0$ a force $F = F_0(1 - te^{-\lambda t})$ is applied to the particle. Find the velocity and position of the particle as a function of time.

Solution

Given $F = F_0(1 - te^{-\lambda t})$

In this case Newton's 2nd Law holds. i.e. $F = ma$

$$\Rightarrow m\vec{a} = F_0(1 - te^{-\lambda t}) \Rightarrow m\frac{d\vec{v}}{dt} = F_0(1 - te^{-\lambda t}) \Rightarrow d\vec{v} = \frac{F_0}{m}(1 - te^{-\lambda t})dt$$

$$\Rightarrow \int d\vec{v} = \frac{F_0}{m}\int(1 - te^{-\lambda t})dt \Rightarrow \vec{v} = \frac{F_0}{m}\left[t + \frac{t}{\lambda}e^{-\lambda t} + \frac{1}{\lambda^2}e^{-\lambda t} + A\right]$$

Initially $t = 0$ then $\vec{v} = 0$ we have $A = -\frac{1}{\lambda^2}$

$$\Rightarrow \vec{v} = \frac{F_0}{m}\left[t + \frac{t}{\lambda}e^{-\lambda t} + \frac{1}{\lambda^2}e^{-\lambda t} - \frac{1}{\lambda^2}\right] \Rightarrow \vec{v} = \frac{F_0}{m}\left[t + \frac{1}{\lambda}\left(te^{-\lambda t} + \frac{1}{\lambda}e^{-\lambda t} - \frac{1}{\lambda}\right)\right]$$

$$\Rightarrow \boldsymbol{\vec{v} = \frac{F_0 t}{m} + \frac{F_0}{\lambda m}\left(te^{-\lambda t} + \frac{1}{\lambda}e^{-\lambda t} - \frac{1}{\lambda}\right)}$$

$$\Rightarrow \frac{dx}{dt} = \frac{F_0 t}{m} + \frac{F_0}{\lambda m}\left(te^{-\lambda t} + \frac{1}{\lambda}e^{-\lambda t} - \frac{1}{\lambda}\right)$$

$$\Rightarrow \int dx = \int\left[\frac{F_0 t}{m} + \frac{F_0}{\lambda m}\left(te^{-\lambda t} + \frac{1}{\lambda}e^{-\lambda t} - \frac{1}{\lambda}\right)\right]dt$$

$$\Rightarrow x = \frac{F_0 t^2}{2m} - \frac{F_0 t}{\lambda^2 m}e^{-\lambda t} - \frac{F_0}{\lambda^3 m}e^{-\lambda t} - \frac{F_0}{\lambda^3 m}e^{-\lambda t} - \frac{F_0 t}{\lambda^2 m} + B$$

Initially $t = 0$ then $x = 0$ we have $B = \frac{2F_0}{\lambda^2 m}$

$$\Rightarrow x = \frac{F_0 t^2}{2m} - \frac{F_0 t}{\lambda^2 m}e^{-\lambda t} - \frac{F_0}{\lambda^3 m}e^{-\lambda t} - \frac{F_0}{\lambda^3 m}e^{-\lambda t} - \frac{F_0 t}{\lambda^2 m} + \frac{2F_0}{\lambda^2 m}$$

$$\Rightarrow \boldsymbol{x = \frac{2F_0 t}{\lambda^3 m}(1 - e^{-\lambda t}) - \frac{F_0 t}{\lambda^2 m}(1 + e^{-\lambda t}) + \frac{F_0}{2m}t^2}$$

Question

A particle having total energy E is moving in a potential field V(r). Show that the time taken by the particle to move from r_1 to r_2 is $t_2 - t_1 = \int_{t_1}^{t_2} \frac{dr}{\sqrt{\frac{2(E-V)}{m}}}$

Solution: T and V are position dependent energies and $T + V = E$

$$\Rightarrow \frac{1}{2}mv^2 + V = E \Rightarrow \frac{1}{2}m\left(\frac{dr}{dt}\right)^2 = E - V \Rightarrow \left(\frac{dr}{dt}\right)^2 = \frac{2(E-V)}{m} \Rightarrow \frac{dr}{dt} = \sqrt{\frac{2(E-V)}{m}}$$

$$\Rightarrow \int_{t_1}^{t_2} dt = \int_{t_1}^{t_2} \frac{dr}{\sqrt{\frac{2(E-V)}{m}}} \Rightarrow t_2 - t_1 = \int_{t_1}^{t_2} \frac{dr}{\sqrt{\frac{2(E-V)}{m}}}$$

Question

A block of mass m is at rest on a frictionless surface at the origin. At time $t = 0$ a force $\vec{F} = F_0 e^{-\lambda t}$ where λ is a small positive constant is applied. Calculate $x(t)$ and $v(t)$.

Solution: Given $\vec{F} = F_0 e^{-\lambda t}$

In this case Newton's 2nd Law holds. i.e. $\vec{F} = ma$

$$\Rightarrow m\vec{a} = F_0 e^{-\lambda t} \Rightarrow m\frac{d\vec{v}}{dt} = F_0 e^{-\lambda t} \Rightarrow d\vec{v} = \frac{F_0}{m} e^{-\lambda t} dt$$

$$\Rightarrow \int d\vec{v} = \frac{F_0}{m} \int e^{-\lambda t} dt \Rightarrow \vec{v} = \frac{F_0}{m}\left(\frac{e^{-\lambda t}}{-\lambda}\right) + A$$

Initially $t = 0$ then $\vec{v} = 0$ we have $A = \frac{1}{\lambda}$

$$\Rightarrow \vec{v} = \frac{F_0}{m}\left(\frac{e^{-\lambda t}}{-\lambda}\right) + \frac{1}{\lambda} \Rightarrow \boldsymbol{\vec{v}(t) = \frac{1}{\lambda}\left(1 - \frac{F_0}{m} e^{-\lambda t}\right)}$$

$$\Rightarrow \frac{dx}{dt} = \frac{1}{\lambda}\left(1 - \frac{F_0}{m} e^{-\lambda t}\right) \Rightarrow \int dx = \int \left(\frac{1}{\lambda} - \frac{F_0}{m\lambda} e^{-\lambda t}\right) dt \Rightarrow x = \frac{1}{\lambda} t + \frac{F_0}{m\lambda^2} e^{-\lambda t} + B$$

Initially $t = 0$ then $x = 0$ we have $B = -\frac{F_0}{m\lambda^2}$

$$\Rightarrow \boldsymbol{x(t) = \frac{1}{\lambda} t + \frac{F_0}{m\lambda^2}\left(e^{-\lambda t} - 1\right)}$$

Question

A particle of mass m having initial velocity $\vec{v}_0$ in horizontal direction is subjected to retarding force proportional to its instantaneous velocity. Calculate its velocity and position as a function of time.

Solution

In horizontal direction a retarding force is $= \vec{F} = -k\vec{v}$

$$\Rightarrow m\vec{a} = -k\vec{v} \Rightarrow m\frac{d\vec{v}}{dt} = -k\vec{v}$$

$$\Rightarrow \int_{v_0}^{v} \frac{1}{\vec{v}} d\vec{v} = -\frac{k}{m}\int_0^t dt \Rightarrow |ln\vec{v}|_{v_0}^{v} = -\frac{k}{m}|t|_0^t \Rightarrow ln\vec{v} - ln\vec{v}_0 = -\frac{k}{m}t$$

$$\Rightarrow ln\left(\frac{\vec{v}}{\vec{v}_0}\right) = -\frac{k}{m}t \Rightarrow \frac{\vec{v}}{\vec{v}_0} = e^{-\frac{k}{m}t} \Rightarrow \boldsymbol{\vec{v} = \vec{v}_0 e^{-\frac{k}{m}t}}$$

$$\Rightarrow \frac{dx}{dt} = \vec{v}_0 e^{-\frac{k}{m}t} \Rightarrow \int_{x_0}^{x} dx = \vec{v}_0 \int_0^t e^{-\frac{k}{m}t} dt$$

$$\Rightarrow |x|_{x_0}^{x} = \vec{v}_0 \left|\frac{e^{-\frac{k}{m}t}}{-\frac{k}{m}}\right|_0^t \Rightarrow x - x_0 = \vec{v}_0\left(-\frac{e^{-\frac{k}{m}t}}{\frac{k}{m}} + \frac{1}{\frac{k}{m}}\right)$$

$$\Rightarrow \boldsymbol{x = x_0 + \frac{m\vec{v}_0}{k}\left(1 - e^{-\frac{k}{m}t}\right)}$$

Question

A ball of mass m thrown with velocity on a horizontal surface, where the retarding force is proportional to the square root of instantaneous velocity. Calculate its velocity and position as a function of time.

Solution

Since $\vec{F} \propto \sqrt{\vec{v}} \Rightarrow \vec{F} = -k\sqrt{\vec{v}} \Rightarrow m\vec{a} = -k\sqrt{\vec{v}} \Rightarrow m\frac{d\vec{v}}{dt} = -k\sqrt{\vec{v}}$

$$\Rightarrow \int_{v_0}^{v} \frac{1}{\sqrt{\vec{v}}} d\vec{v} = -\frac{k}{m}\int_0^t dt \Rightarrow \left|\sqrt{\vec{v}}\right|_{v_0}^{v} = -\frac{k}{2m}|t|_0^t \Rightarrow \sqrt{\vec{v}} - \sqrt{\vec{v}_0} = -\frac{k}{2m}t$$

$$\Rightarrow \sqrt{\vec{v}} = \sqrt{\vec{v}_0} - \frac{k}{2m}t \Rightarrow \boldsymbol{\vec{v} = \left(\sqrt{\vec{v}_0} - \frac{k}{2m}t\right)^2}$$

$$\Rightarrow \frac{dx}{dt} = \left(\sqrt{\vec{v}_0} - \frac{k}{2m}t\right)^2 \Rightarrow \int_{x_0}^{x} dx = \int_0^t \left(\sqrt{\vec{v}_0} - \frac{k}{2m}t\right)^2 dt$$

$$\Rightarrow \int_{x_0}^{x} dx = \int_0^t \vec{v}_0 dt + \left(\frac{k}{2m}\right)^2 \int_0^t t^2 dt - \frac{k\sqrt{\vec{v}_0}}{m}\int_0^t t dt$$

$$\Rightarrow |x|_{x_0}^{x} = \vec{v}_0|t|_0^t + \left(\frac{k}{2m}\right)^2 \left|\frac{t^3}{3}\right|_0^t - \frac{k\sqrt{\vec{v}_0}}{m}\left|\frac{t^2}{2}\right|_0^t \Rightarrow x - x_0 = \vec{v}_0 t + \left(\frac{k}{2m}\right)^2 \frac{t^3}{3} - \frac{k\sqrt{\vec{v}_0}}{m}\frac{t^2}{2}$$

$$\Rightarrow \boldsymbol{x = x_0 + \vec{v}_0 t + \left(\frac{k}{2m}\right)^2 \frac{t^3}{3} - \frac{k\sqrt{\vec{v}_0}}{m}\frac{t^2}{2}}$$

Question

A particle of mass m is at rest at $t = 0$ when it is subjected to a force $\vec{F} = Asin\omega t$. Calculate values of $\vec{v}(t)$ and $x(t)$.

Solution

$$\vec{F} = Asin\omega t \Rightarrow m\vec{a} = Asin\omega t \Rightarrow m\frac{d\vec{v}}{dt} = Asin\omega t$$

$$\Rightarrow \int_0^v d\vec{v} = \frac{A}{m}\int_0^t sin\omega t dt \Rightarrow |\vec{v}|_0^v = \frac{A}{m}\left|-\frac{1}{\omega}cos\omega t\right|_0^t \Rightarrow \boldsymbol{\vec{v} = \frac{A}{m\omega}(1 - cos\omega t)}$$

$$\Rightarrow \frac{dx}{dt} = \frac{A}{m\omega}(1 - cos\omega t) \Rightarrow \int_0^x dx = \frac{A}{m\omega}\int_0^t (1 - cos\omega t)dt$$

$$\Rightarrow |x|_0^x = \frac{A}{m\omega}\left|t - \frac{1}{\omega}sin\omega t\right|_0^t \Rightarrow \boldsymbol{x = \frac{A}{m\omega}\left(t - \frac{1}{\omega}sin\omega t\right)}$$

Question

A particle of mass m is at rest at the origin of the coordinate system at $t = 0$, a force $\vec{F} = bt$ starts acting on the particle. Find velocity and position of the particle as a function of time.

Solution

$$\vec{F} = bt \Rightarrow m\vec{a} = bt \Rightarrow m\frac{d\vec{v}}{dt} = bt \Rightarrow \int_0^v d\vec{v} = \frac{b}{m}\int_0^t t dt \Rightarrow |\vec{v}|_0^v = \frac{b}{m}\left|\frac{t^2}{2}\right|_0^t$$

$$\Rightarrow \boldsymbol{\vec{v} = \frac{b}{2m}t^2} \Rightarrow \frac{dx}{dt} = \frac{b}{2m}t^2 \Rightarrow \int_0^x dx = \frac{b}{2m}\int_0^t t^2 dt \Rightarrow \boldsymbol{x = \frac{b}{6m}t^3}$$

Question

Find the displacement and velocity of a particle moving horizontally in a resistive medium in which the retarding force is proportional to the velocity.

Solution

In horizontal direction a retarding force is $\vec{F} = -k'\vec{v}$

$\Rightarrow m\vec{a} = -mk\vec{v} \Rightarrow \frac{d\vec{v}}{dt} = -k\vec{v}$

$\Rightarrow \int_{v_0}^{v} \frac{1}{\vec{v}} d\vec{v} = -k\int_0^t dt \Rightarrow |ln\vec{v}|_{v_0}^{v} = -k|t|_0^t \Rightarrow ln\vec{v} - ln\vec{v}_0 = -kt$

$\Rightarrow ln\left(\frac{\vec{v}}{\vec{v}_0}\right) = -kt \Rightarrow \frac{\vec{v}}{\vec{v}_0} = e^{-kt} \Rightarrow \boldsymbol{\vec{v} = \vec{v}_0 e^{-kt}}$

$\Rightarrow \frac{dx}{dt} = \vec{v}_0 e^{-kt} \Rightarrow \int_0^x dx = \vec{v}_0 \int_0^t e^{-kt}dt$

$\Rightarrow |x|_0^x = \vec{v}_0 \left|\frac{e^{-kt}}{-k}\right|_0^t \Rightarrow x = \vec{v}_0\left(-\frac{e^{-kt}}{k} + \frac{1}{k}\right) \Rightarrow \boldsymbol{x = \frac{\vec{v}_0}{k}\left(1 - e^{-kt}\right)}$

Question

A particle falling in a resistive medium in under a retarding force proportional to the velocity. Find its velocity and displacement.

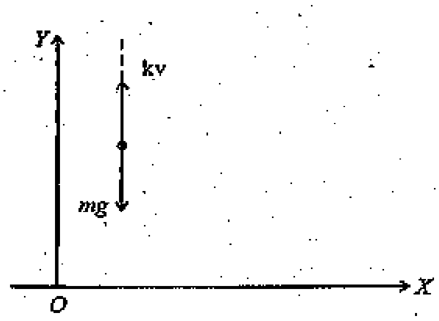
Solution

In particle falling downward a retarding force is $\vec{F} = -mg - k'\vec{v}$

$\Rightarrow m\vec{a} = -mg - mk\vec{v} \Rightarrow \frac{d\vec{v}}{dt} = -g - k\vec{v}$

$\Rightarrow \int \frac{d\vec{v}}{g+k\vec{v}} = -\int dt \Rightarrow \frac{1}{k}ln(g + k\vec{v}) = -t + C \Rightarrow g + k\vec{v} = e^{-kt+kC} = e^{-kt}e^{kC}$

$\Rightarrow g + k\vec{v} = Ae^{-kt}$



Suppose that initially the particle has velocity $\vec{v}_0$ and

Position y_0.i.e. $\vec{v} = \vec{v}_0$ when $y = y_0$ at $t = 0$

We have $A = e^{kC} = g + k\vec{v}_0$

$\Rightarrow g + k\vec{v} = (g + k\vec{v}_0)e^{-kt} \Rightarrow \vec{v} = -\frac{g}{k} + \left(\frac{g+k\vec{v}_0}{k}\right)e^{-kt}$

$\Rightarrow \frac{dy}{dt} = -\frac{g}{k} + \left(\frac{g+k\vec{v}_0}{k}\right)e^{-kt} \Rightarrow y = -\frac{g}{k}t - \left(\frac{g+k\vec{v}_0}{k^2}\right)e^{-kt} + B$

Using initial conditions. i.e. $y = y_0$ when $t = 0$ We have $B = y_0 + \frac{g+k\vec{v}_0}{k^2}$

$\Rightarrow y = -\frac{g}{k}t - \left(\frac{g+k\vec{v}_0}{k^2}\right)e^{-kt} + y_0 + \frac{g+k\vec{v}_0}{k^2}$

$\Rightarrow \boldsymbol{y = y_0 - \frac{g}{k}t + \frac{g+k\vec{v}_0}{k^2}(1 - e^{-kt})}$

Equation shows that in the limit $t \to \infty$ (i.e. after the passage of long enough time) $v \to -\frac{g}{k}$. This velocity is called terminal velocity.

Question

Discuss equilibrium for the particle subject to the force $\vec{F} = -a^2x^2$.

Solution

Since $\vec{F} = -\frac{dV}{dx}$

$\Rightarrow V(x) = -\int \vec{F}dx + C \Rightarrow V(x) = -\int(-a^2x^2)dx + C = a^2\int x^2dx + C$

$\Rightarrow V(x) = \frac{1}{3}a^2x^3 + C$

Now since $\vec{F} = 0$ at $x = 0$, the particle is in a state of equilibrium at $x = 0$.

To see further if the equilibrium is stable or unstable, we calculate $\frac{d^2V}{dx^2}$

If $\frac{dV}{dx} = a^2x^2$ then $\frac{d^2V}{dx^2} = 2a^2x$

We note that $\frac{d^2V}{dx^2} = 0$ at $x = 0$, $\frac{d^2V}{dx^2} > 0$ at $x > 0$, $\frac{d^2V}{dx^2} < 0$ at $x < 0$, then these results show that the equilibrium is stable for positive displacement and unstable for negative displacement.

Non-Conservative / Dissipative Forces

Forces that cannot be expressed in the term of a potential energy function are called non-conservative forces. We can also state that forces that do not store energy are called non-conservative or dissipative forces. If there is no scalar function V such that $F = -\Delta V$ [or, equivalently, if $V \times F = 0$], then F is called a non-conservative force field. Friction is a non-conservative force, and there are others. It is always opposed to the direction of motion and is not a single valued function of position alone. Similarly the impulse (time dependent force) is also non-conservative and cannot be derived from a scalar point function. An example of non-conservative force, we have $F = kv$, where v is the velocity of the particle, then $\oint \vec{F}.d\vec{r} = \int \vec{F}.\frac{d\vec{r}}{dt}dt = \int_{t_1}^{t_2} \vec{F}.\vec{v}\,dt = k\int_{t_1}^{t_2} \vec{v}.\vec{v}\,dtt = k\int_{t_1}^{t_2} v^2 dt > 0$
Which shows the integral is not equals to zero. Hence the force is non-conservative.

Work-Energy relation and Non-conservative Forces

We have already shown that for any general force F: $\int_{P_1}^{P_2} \vec{F}.d\vec{r} = T_2 - T_1$

When the force F can be broken into conservative and non-conservative parts

$\vec{F} = \vec{F}^{(c)} + \vec{F}^{(nc)}$

Then we have $\int_{P_1}^{P_2} \vec{F}^{(c)}.d\vec{r} + \int_{P_1}^{P_2} \vec{F}^{(nc)}.d\vec{r} = T_2 - T_1$

$\Rightarrow V_1 - V_2 + \int_{P_1}^{P_2} \vec{F}^{(nc)}.d\vec{r} = T_2 - T_1 \qquad \because \vec{F}^{(c)} = -\Delta V$

$\Rightarrow T_1 + V_1 + \int_{P_1}^{P_2} \vec{F}^{(nc)}.d\vec{r} = T_2 + V_2$

The work done in overcoming friction is always negative, because $\vec{F}^{(nc)}$ is opposite to the displacement relation above proves that the influence of friction is dissipative and therefore decrease the total mechanical energy of the system. Alternatively above can be expressed as

$\Rightarrow (V_2 - V_1) + (T_2 - T_1) = \int_{P_1}^{P_2} \vec{F}^{(nc)}.d\vec{r} \Rightarrow \Delta(\boldsymbol{V} + \boldsymbol{T}) = \int_{\boldsymbol{P_1}}^{\boldsymbol{P_2}} \vec{\boldsymbol{F}}^{(\boldsymbol{nc})}.\boldsymbol{d\vec{r}}$
It is interesting to remember that the process in which work is converted into internal energy (due to friction) are irreversible.

Impulse

Impulse is a special type of force defined by applying the integral of a force $\vec{F}$, over the time interval, t, for which it acts on the body. Impulse is a directional (vector) quantity in the same direction of force as force is also a directional quantity. When Impulse is applied to a rigid body, it results a corresponding vector change in its linear momentum along the same direction. The SI unit of impulse is the newton second (Ns), and the dimensionally equivalent unit of momentum is the kilogram meter per second ($kgms^{-1}$). The particle is located at P_1 and P_2 at times t_1 and t_2 where it has velocities v_1 and v_2 respectively. The time integral of the force F given by $\int_{t_1}^{t_2} \vec{F} dt$ is called the impulse of the force $\vec{F}$.

Angular Impulse

The time integral of the torque $I = \int_{t_1}^{t_2} \Lambda dt$ is called the angular impulse.

Theorem

Impulse is equal to the change in momentum $\int_{t_1}^{t_2} \vec{F} dt = m\vec{v}_2 - m\vec{v}_1 = \vec{P}_2 - \vec{P}_1$

Proof

We have to prove that the impulse of a force is equal to the change in momentum.

By definition of impulse and Newton's second law, we have

$$\int_{t_1}^{t_2} \vec{F} dt = \int_{t_1}^{t_2} m\frac{d\vec{v}}{dt} dt = \int_{t_1}^{t_2} m d\vec{v} = m|\vec{v}|_{t_1}^{t_2} = m\vec{v}_2 - m\vec{v}_1 = \vec{P}_2 - \vec{P}_1$$

Where we use the conditions $\vec{v}(t_1) = \vec{v}_1$ and $\vec{v}(t_2) = \vec{v}_2$

The theorem is true even when the mass is variable and the force is non-conservative.

Theorem

Prove that $\int_{t_1}^{t_2} \Lambda dt = \Omega_2 - \Omega_1$

Proof

$$\int_{t_1}^{t_2} \Lambda dt = \int_{t_1}^{t_2} \frac{d\Omega}{dt} dt = \int_{t_1}^{t_2} d\Omega = \Omega_2 - \Omega_1$$

Question

A mass of 5000kg moves on a straight line from a speed of 540km/h to 720km/h in 2 minutes. What is the impulse developed in this time?

Solution

Assume that the mass travel in the direction of positive x direction. In SI system

$$\vec{v}_1 = 540\hat{\imath}kmh^{-1} = \frac{540\hat{\imath}\times 1000m}{3600s} = 1.5 \times 10^2\hat{\imath}ms^{-1}$$

$$\vec{v}_2 = 720\hat{\imath}kmh^{-1} = \frac{720\hat{\imath}\times 1000m}{3600s} = 2.0 \times 10^2\hat{\imath}ms^{-1}$$

$$I = m(\vec{v}_2 - \vec{v}_1) = (5000kg)(0.5 \times 10^2\hat{\imath}ms^{-1}) = 2.5 \times 10^5\hat{\imath}kgms^{-1}$$

$$I = 2.5 \times 10^5\hat{\imath}Ns$$

Thus the impulse has magnitude in the positive x direction.

Power

The rate of doing work is called power. If an agent does work ΔW in time Δt, then the average power is defined as the ration to total work done to the total time. It is described mathematically as:

$$P_{av} = \langle P \rangle = \frac{\Delta W}{\Delta t}$$

If the power is variable, then the instantaneous power is given by the expression:

$$P_{ins} = \lim_{\Delta t \to 0}\frac{\Delta W}{\Delta t} = \frac{dW}{dt}$$

Watt

The SI unit of power is watt which can be defined as:

"If an agent does work of one joule of work per second, the power of that agent will be 1 watt"

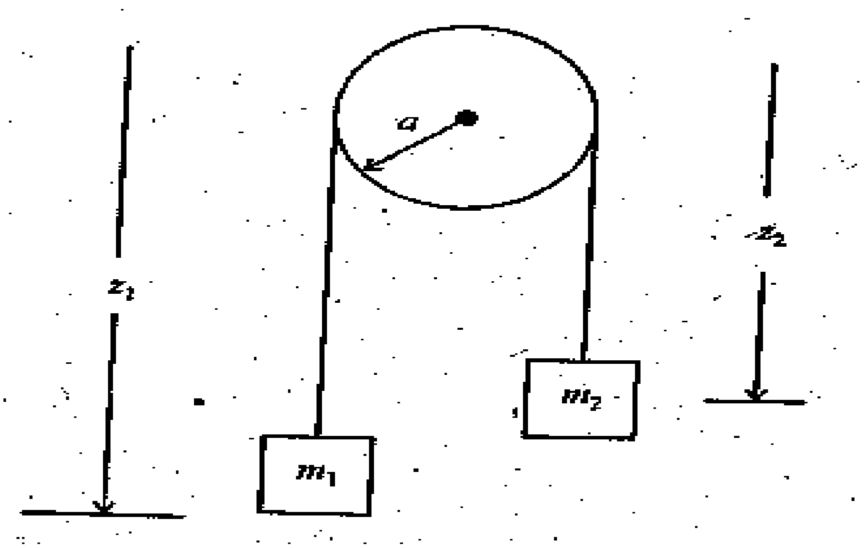
Question: Prove that $\vec{P} = \vec{F}.\vec{v}$

Proof: As $\vec{P} = \frac{d\overrightarrow{W}}{dt}$

$\Rightarrow \vec{P} = \frac{\vec{F}.d\vec{r}}{dt} = \vec{F}.\frac{d\vec{r}}{dt} = \vec{F}.\vec{v}$ using $d\overrightarrow{W} = \vec{F}.d\vec{r}$

Atwood Machine

As an example of a two – particle system we discuss the motion of the Atwood machine. **It is a mechanical system consisting of two particles connected by a string passing over a pulley. It is an idealized mechanical system used to gain insight about the behaviour of a two – particles system.**



Here is assumed that

a) The string is massless and inextensible.
b) The pulley has no inertia and rotates on frictionless bearings.

We first determine the acceleration of each particle by a simple application of Newton's 2nd Law;

$\tau - m_1 g = m_1 \ddot{z}_1$ ……………(i)

$\tau - m_2 g = m_2 \ddot{z}_2$ ……………(ii)

Where τ is the tension in the string, supposed to be constant at all points of the string and z_1, z_2 are the instantaneous distances from the centre of the pulley to the respective particles. Since the string is inextensible, we must have

$l = z_1 + z_2 + 2\pi R$ ……………(iii) where l is the length of the string.

$(iii) \Rightarrow \dot{z}_2 = -\dot{z}_1$; $\ddot{z}_2 = -\ddot{z}_2$ ……………(iv)

Eliminating $\ddot{z}_2$ and from (i) and (ii) with the help of (iv) we have

$\ddot{z}_1 = g\frac{m_2 - m_1}{m_2 + m_1}$ ……………(v)

For $m_2 > m_1$; $\ddot{z}_1 > 0$. i.e. the particle m_1 moves upwards.

For $m_2 < m_1$; $\ddot{z}_1 < 0$. i.e. the particle m_2 moves upwards.

For $m_2 = m_1$; $\ddot{z}_1 = \ddot{z}_2 = 0$. i.e. no motion.

In each case the acceleration remains constant.

To calculate the tension τ in the string we have from (i)

$$\tau = m_1 g + m_1 \ddot{z}_1 = m_1 g + m_1 \left(g \frac{m_2 - m_1}{m_2 + m_1} \right)$$

$$\tau = \frac{2m_1 m_2}{m_1 + m_2} g \quad \ldots\ldots\ldots\ldots\ldots(\text{vi})$$

From this expression it follows that in case of $m_1 = m_2$; $\tau = m_1 g = m_2 g$ and the system will be in the state of equilibrium.

On the other hand, if $m_1 \gg m_2$ then $\ddot{z}_1 \sim g$ and the acceleration is nearly the same as in the state of free fall.

In spite of the fact that one particle is accelerated upwards and the other downwards, the net system acceleration will be downwards, as long as the two masses are unequal.

To see this we consider the acceleration of the c.m. Regarding the c.m. as a particle of mass $m_1 + m_2$, with acceleration $\ddot{z}_c$ we write its equation of motion as

$$2\tau - m_1 g - m_2 g = (m_1 + m_2)\ddot{z}_c$$

$$2\left(\frac{2m_1 m_2}{m_1 + m_2} g \right) - m_1 g - m_2 g = (m_1 + m_2)\ddot{z}_c$$

$$\ddot{z}_c = -g \left(\frac{m_1 - m_2}{m_1 + m_2} \right)^2$$

This shows that the acceleration of the system is always $(m_1 \neq m_2)$ downwards.

Virial Theorem

This theorem has to do with time averaged behavior of an isolated system of N particles. According to this theorem

$\langle T\rangle = -\frac{1}{2}\langle \sum_i \vec{F}_i.\vec{r}_i\rangle$

Here the quantity $-\frac{1}{2}\langle \sum_i \vec{F}_i.\vec{r}_i\rangle$ is called the Virial of the system. Where angle brackets represent the average over time of the enclosed quantity.

Proof

Let us consider a scalar function $S = \sum_{i=1}^{N} \vec{P}_i.\vec{r}_i$ ………….(1)

where $\vec{P}_i$ and $\vec{r}_i$ denote the linear momentum and position vector of the i^{th} particle of the system. Assume that system is bounded for all time. i.e. the system remains confined with fixed boundaries.

$(1) \Rightarrow \frac{dS}{dt} = \sum_{i=1}^{N}\left(\dot{\vec{P}}_i.\vec{r}_i + \vec{P}_i.\dot{\vec{r}}_i\right)$ …………(2)

Define the time average of a function $\psi(t)$ over an interval $[0,\tau]$ as follows

$\bar{\psi}(t) = \langle\psi(t)\rangle = \frac{1}{\tau}\int_0^{\tau}\psi(t)dt$ then (2) in view of this definition becomes

$\Rightarrow \langle\frac{dS}{dt}\rangle = \frac{1}{\tau}\int_0^{\tau} S'(t)dt = \frac{S(\tau)-S(0)}{\tau}$ …………(3)

If the system is periodic and τ issome multiple of the period p (i.e. $\tau = np \ ; n \in Z$) then $\langle\frac{dS}{dt}\rangle = 0$. If the system is not periodic, then by the assumption of boundedness, (3) becomes $\lim_{\tau\to\infty}\langle\frac{dS}{dt}\rangle = 0$

Therefore whether the system is periodic or not, we have

$\langle\sum_{i=1}^{N}\left(\dot{\vec{P}}_i.\vec{r}_i + \vec{P}_i.\dot{\vec{r}}_i\right)\rangle = 0$

$\Rightarrow \langle\sum_{i=1}^{N}\vec{P}_i.\dot{\vec{r}}_i\rangle = -\langle\sum_{i=1}^{N}\dot{\vec{P}}_i.\vec{r}_i\rangle$ …………(4)

Now $\sum_{i=1}^{N}\vec{P}_i.\dot{\vec{r}}_i = \sum_{i=1}^{N} m_i\vec{v}_i.\vec{v}_i = 2\sum_{i=1}^{N}\left(\frac{1}{2}m_i\vec{v}_i^2\right) = 2T$

Where T is the total K.E. of the system, then

$(4) \Rightarrow 2\langle T\rangle = -\langle\sum_{i=1}^{N}\dot{\vec{P}}_i.\vec{r}_i\rangle$ $(4) \Rightarrow \langle \boldsymbol{T}\rangle = -\frac{\mathbf{1}}{\mathbf{2}}\langle\sum_{\boldsymbol{i}}\vec{\boldsymbol{F}}_{\boldsymbol{i}}.\vec{\boldsymbol{r}}_{\boldsymbol{i}}\rangle$ where $\dot{\vec{P}}_i = \vec{F}$

The word "virial" derives from *vis* or *viris*, the Latin word for "force" or "energy", and was given its technical definition by Clausius in 1870.

The significance of the virial theorem is that it allows the average total kinetic energy to be calculated even for very complicated systems that defy an exact solution, such as those considered in statistical mechanics; this average total kinetic energy is related to the temperature of the system by the equipartition theorem.

However, the virial theorem does not depend on the notion of temperature and holds even for systems that are not in thermal equilibrium. The virial theorem has been generalized in various ways, most notably to a tensor form. Definitions of the virial and its time derivative

Virial of the System

For a collection of N point particles, the scalar moment of inertia I about the origin is defined by the equation $I = \sum_{i=1}^{N} m_i |\vec{r}_i|^2 = \sum_{i=1}^{N} m_i r_i^2$ where m_i and r_i represent the mass and position of the kth particle. $r_i = |\vec{r}_i|$ is the position vector magnitude.

The scalar **virial G** is defined by the equation $G = \sum_{i=1}^{N} \vec{P}_i . \vec{r}_i$ where $\vec{P}_i$ is the momentum vector of the kth particle.

Assuming that the masses are constant, the virial G is one-half the time derivative of this moment of inertia

$$\frac{1}{2}\frac{dI}{dt} = \frac{1}{2}\frac{d}{dt}\sum_{i=1}^{N} m_i \vec{r}_i . \vec{r}_i = \frac{1}{2}\sum_{i=1}^{N} m_i \frac{d\vec{r}_i}{dt} . \vec{r}_i = \frac{1}{2}\sum_{i=1}^{N} m_i \vec{v}_i . \vec{r}_i = \sum_{i=1}^{N} \vec{P}_i . \vec{r}_i = G$$

Virial Radius

In astronomy, the term virial radius is used to refer to the radius of a sphere, centered on a galaxy or a galaxy cluster, within which virial equilibrium holds.

CHAPTER

3

KINEMATICS

Kinematics is the branch of mechanics deals with the moving objects without reference to the forces which cause the motion. In other words we can say those kinematics are the features or properties of motion of concerned with system of particles (rigid bodies).

Here some features of rigid body motion are

- Displacement
- Position
- Velocity
- Linear Velocity & Angular Velocity
- Linear Acceleration & Angular Acceleration
- Motion of a Rigid Body (Translation & Rotation)

From everyday experience, we all have some idea as to the meaning of each of the following terms or concepts. However, we would certainly find it difficult to formulate completely satisfactory definitions. We take them as undefined concepts.

- **Space.** This is closely related to the concepts of point, position,' direction and displacement. Measurement in space involves the concepts of length or distance, with which we assume familiarity. Units of length are feet, meters, miles, etc.
- **Time.** This concept is derived from our experience of having one event taking place after, before or simultaneous with another event. Measurement of time is achieved, for example, by use of clocks. Units of time are seconds, hours, years, etc.
- **Matter.** Physical objects are composed of "small bits of matter" such as atoms and molecules. From this we arrive at the concept of a material object called a particle which can be considered as occupying a point in space and perhaps moving as time goes by. A measure of the "quantity of matter" associated with a particle is called

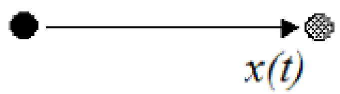
its mass. Units of mass are grams, kilograms, etc. Unless otherwise stated we shall assume that the mass of a particle does not change with time.

Rectilinear Motion

When a moving particle remains on a single straight line, the motion is said to be rectilinear. In this case, without loss of generality we can choose the x-axis as the line of motion. The general equation of motion is then

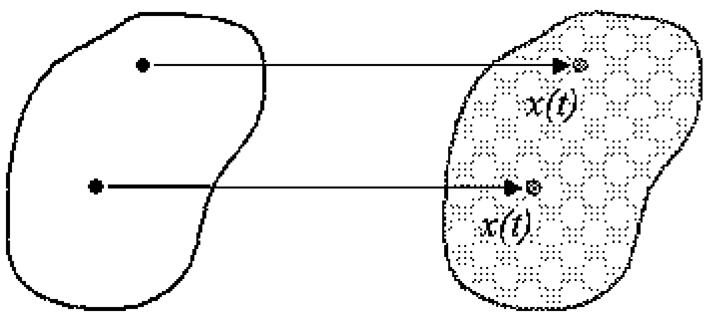
$\vec{F} = m\vec{a} \Longrightarrow \vec{F}(x, \dot{x}, \ddot{x}) = m\ddot{x}$

Rectilinear Motion of Particles



Rectilinear motion of a body is defined by considering the two point of a body covered the same distance in the parallel direction. The figures below illustrate rectilinear motion for a particle and body.

Rectilinear motion for a body



In the above figures, $x(t)$ represents the position of the particles along the direction of motion, as a function of time t. An example of linear motion is an athlete running g along a straight track.

The rectilinear motion can be of two types:

i. Uniform rectilinear motion
ii. Non uniform rectilinear motion

Uniform Rectilinear Motion

Uniform rectilinear motion is a type of motion in which the body moves with uniform velocity or zero acceleration.

In contrast, Non uniform rectilinear motion is such type of motion with variable velocity or non-zero acceleration.

Uniformly Accelerated Rectilinear Motion

Uniformly accelerated rectilinear motion is a special case of non-uniform rectilinear motion along a line is that which arises when an object is subjected to constant acceleration. This kind of motion is called uniformly accelerated motion.

Uniformly accelerated motion is a type of motion in which the velocity of an object changes by an equal amount in every equal intervals of time. An example of uniformly accelerated body is freely falling object in which the amount of gravitational acceleration remains same. $\vec{F} = m\vec{g}$

Curvilinear Motion of Particle

The motion of a particle moving in a curved path is called curvilinear motion. Example: A stone thrown into the air at an angle.

Importance/Purpose: Curvilinear motion describes the motion of a moving particle that conforms to a known or fixed curve. The study of such motion involves the use of two co-ordinate systems, the first being planar motion and the latter being cylindrical motion.

Velocity of Curvilinear motion

If the tangential and normal unit vectors are $\vec{e}_t$ and $\vec{e}_n$ respectively, then the velocity will be $\vec{v} = \frac{d\vec{r}}{dt}\vec{e}_t$.

You have already learnt that $\vec{v} = |\vec{v}|\mathrm{T} = v\mathrm{T}$

Acceleration of Curvilinear Motion

If the tangential and normal unit vectors are $\vec{e}_t$ and $\vec{e}_n$ respectively, then the acceleration will be $\vec{a} = \frac{d^2\vec{r}}{dt^2}\vec{e}_t + \frac{\left(\frac{d\vec{r}}{dt}\right)^2}{\rho}\vec{e}_n$

You have already learnt that $\vec{a} = v'\mathrm{T} + \kappa v^2 N$

Example

- A stone thrown into the air at an angle.
- A car driving along a curved road.
- Throwing paper airplanes or paper darts is an example of curvilinear motion.

Example

For the rectilinear motion of a particle moving with a velocity $u\sqrt{\frac{a^2-x^2}{x^2}}$ at a distance x from a fixed point. Show that particle attracted towards the fixed point with a force $F(x) \propto \frac{1}{x^3}$

Solution

$$\vec{v} = u\sqrt{\frac{a^2-x^2}{x^2}} \Rightarrow v^2 = u^2\left(\frac{a^2-x^2}{x^2}\right) \Rightarrow v^2 = u^2\left(\frac{a^2}{x^2}-1\right)$$

$$\Rightarrow 2v\frac{dv}{dx} = u^2\left(\frac{-2a^2}{x^3}\right) \Rightarrow v\frac{dv}{dx} = \frac{-u^2a^2}{x^3}$$

$$\Rightarrow mv\frac{dv}{dx} = \frac{-mu^2a^2}{x^3} \qquad \ldots\ldots\ldots\ldots(1)$$

Using the fact $F = ma$

$$F = m\frac{dv}{dt} = m\frac{dv}{dx}\frac{dx}{dt}$$

$$F = mv\frac{dv}{dx}$$

$(1) \Rightarrow F(x) \propto \frac{1}{x^3}$ and $-mu^2$ is constant.

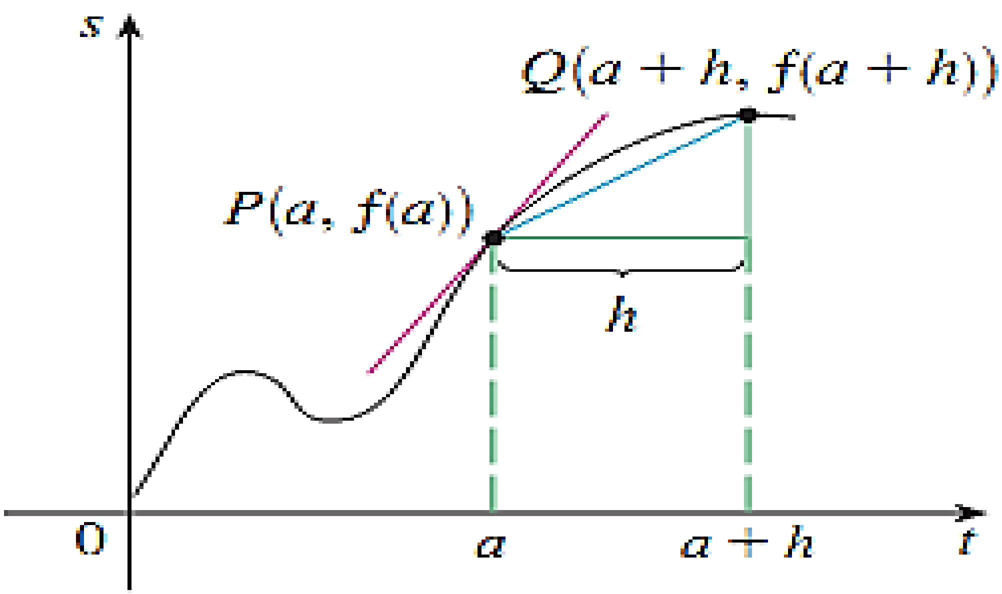
Velocity

Suppose an object moves along a straight line according to an equation of motion $s = f(t)$, where s is the displacement (directed distance) of the object from the origin at time t. The function f that describes the motion is called the **position function** of the object. In the time interval from $t = a$ to $t = a + h$ the change in position is $f(a+h) - f(a)$.

The average velocity over this time interval is

average velocity $= \frac{\text{displacement}}{\text{time}} = \frac{f(a+h)-f(a)}{h}$

which is the same as the slope of the secant line PQ in Figure.



$$m_{PQ} = \frac{f(a+h) - f(a)}{h}$$
$$= \text{average velocity}$$

Now suppose we compute the average velocities over shorter and shorter time intervals$[a, a+h]$. In other words, we let h approach 0.

We define the **velocity** (or **instantaneous velocity**) $\vec{v}(a)$ at time $t = a$ to be the limit of these average velocities:

$\vec{v}(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$

This means that the velocity at time $t = a$ is equal to the slope of the tangent line at P.

Relative Velocity

If two particles P_1 and P_2 are moving with respective velocities $\vec{v}_1$ and $\vec{v}_2$, then the vector $\vec{v}\ |_{P_2/P_1} = \vec{v}_2 - \vec{v}_1$ is called the relative velocity of P_2 with respect to P_1.

Acceleration

If $s = s(t)$ is the position function of an object that moves in a straight line, we know that its first derivative represents the velocity $\vec{v}(t)$ of the object as a function of time. Then'

The instantaneous rate of change of velocity with respect to time is called the **acceleration** $\vec{a}(t)$ of the object. Thus the acceleration function is the derivative of the velocity function and is therefore the second derivative of the position function:

$\vec{a}(t) = \vec{v}'(t) = \vec{r}''(t) = \frac{d^2\vec{r}}{dt^2}$

Relative Acceleration

If two particles P_1 and P_2 are moving with respective accelerations $\vec{a}_1$ and $\vec{a}_2$, then the vector $\vec{a}\ |_{P_2/P_1} = \vec{a}_2 - \vec{a}_1$ is called the relative acceleration of P_2 with respect to P_1.

Cartesian Components of Velocity and Acceleration

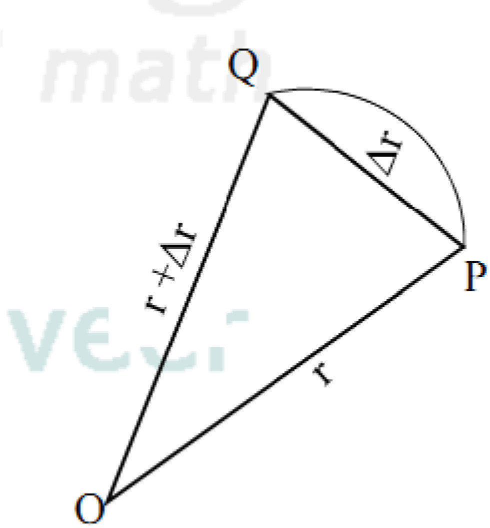
Let $\vec{r} = x\hat{\imath} + y\hat{\jmath}$ be a position vector of a particle then

$\vec{v}(t) = \frac{d\vec{r}}{dt} = \frac{dx}{dt}\hat{\imath} + \frac{dy}{dt}\hat{\jmath}$ and $\vec{a}(t) = \vec{v}'(t) = \frac{d^2\vec{r}}{dt^2} = \frac{d^2x}{dt^2}\hat{\imath} + \frac{d^2y}{dt^2}\hat{\jmath}$, Then

$v_x = x -$ component of velocity $= \frac{dx}{dt}$, $v_y = y -$ component of velocity $= \frac{dy}{dt}$

$a_x = x -$ component of velocity $= \frac{d^2x}{dt^2}$, $a_y = y -$ component of velocity $= \frac{d^2y}{dt^2}$

Cartesian Components of Velocity and Acceleration



Let $\vec{r} = x\hat{\imath} + y\hat{\jmath}$ be a position vector of a particle then $PQ = \Delta r$

And therefore, velocity of a fluid particle denoted as '$\vec{V}$'.

$\vec{V} = \lim_{\Delta t \to 0} \frac{\Delta r}{\Delta t} = \frac{d\vec{r}}{dt}$ and $\vec{a} = \lim_{\Delta t \to 0} \frac{\Delta V}{\Delta t} = \frac{d^2\vec{r}}{dt^2}$

$\Rightarrow \vec{V} = \frac{d}{dt}(x\hat{\imath} + y\hat{\jmath}) = \frac{dx}{dt}\hat{\imath} + \frac{dx}{dt}\hat{\jmath}$

$\Rightarrow \vec{a} = \frac{d^2\vec{r}}{dt^2} = \frac{d^2}{dt^2}(x\hat{\imath} + y\hat{\jmath}) = \frac{d^2x}{dt^2}\hat{\imath} + \frac{d^2y}{dt^2}\hat{\jmath}$

$v_x = x -$ component of velocity $= \frac{dx}{dt}$

$v_y = y -$ component of velocity $= \frac{dy}{dt}$

$a_x = x -$ component of velocity $= \frac{d^2x}{dt^2}$

$a_y = y -$ component of velocity $= \frac{d^2y}{dt^2}$

Tangential and Normal/Centripetal Components of Velocity

When we study the motion of a particle, it is often useful to resolve the acceleration in two components, one in the direction of the tangent and the other in the direction of the normal. If we write $v = |\vec{v}|$ for the speed of the particle, then

$$\mathrm{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|} = \frac{\vec{v}(t)}{|\vec{v}(t)|} = \frac{\vec{v}(t)}{v}$$

And so $\vec{v} = v\mathrm{T} + 0\mathrm{N}$ with $\vec{v}_T = v$ and $\vec{v}_N = 0$

Question (Tangential and Normal/Centripetal Components of Acceleration)

Show that acceleration of a particle travels along a space curve with velocity $\vec{v}$ is given by $\vec{a} = v'\mathrm{T} + \kappa v^2 N = \frac{dv}{dt}\mathrm{T} + \frac{v^2}{R}N$. We may use ρ instead R.

Solution

Since $\vec{a} = \frac{d\vec{v}}{dt}$

$$\Rightarrow \vec{a} = \frac{d}{dt}(|v|T) = \frac{d}{dt}(vT) = \frac{dv}{dt}\mathrm{T} + \mathrm{v}\frac{dT}{dt} \quad \ldots\ldots\ldots\ldots\ldots(1)$$

$$\frac{dT}{dt} = \frac{dT}{ds}\frac{ds}{dt} = N\kappa v \qquad \because N = \frac{1}{\kappa}\frac{dT}{ds} \Rightarrow N\kappa = \frac{dT}{ds} \Rightarrow N\kappa = \frac{dT}{ds} \text{ also } \frac{ds}{dt} = v$$

$$\frac{dT}{dt} = N\frac{1}{R}v$$

$$(1) \Rightarrow \vec{a} = \frac{dv}{dt}\mathrm{T} + \mathrm{v}.N\frac{1}{R} \Rightarrow \boldsymbol{\vec{a}} = \boldsymbol{\frac{dv}{dt}}\mathbf{T} + \boldsymbol{\frac{v^2}{R}N} = \boldsymbol{\frac{dv}{dt}}\mathbf{T} + \boldsymbol{\kappa v^2 N}$$

Writing $\vec{a}_T$ and $\vec{a}_N$ for the tangential and normal components of acceleration, we have $\boldsymbol{\vec{a}} = \boldsymbol{\vec{a}_T}\mathbf{T} + \boldsymbol{\vec{a}_N N}$ where $\vec{a}_T = v'$ and $\vec{a}_N = \kappa v^2 = \frac{v^2}{\rho} = \omega^2\rho$

Note

Although we have expressions for the tangential and normal components of acceleration above, it's desirable to have expressions that depend only on $\vec{r}, \vec{r}'$, and $\vec{r}''$. To this end we take the dot product of $\vec{v} = v\mathrm{T}$ with $\vec{a}$:

$$\vec{v}.\vec{a} = v\mathrm{T}.(v'\mathrm{T} + \kappa v^2 N) = vv'\mathrm{T}.\mathrm{T} + \kappa v^3\mathrm{T}.N = vv'$$

Therefore $\vec{a}_T = v' = \frac{\vec{v}.\vec{a}}{v} = \frac{\vec{r}'(t).\vec{r}''(t)}{|\vec{r}'(t)|}$

Using the formula for curvature, we have

$$\vec{a}_N = \kappa v^2 = \frac{|\vec{r}'(t)\times\vec{r}''(t)|}{|\vec{r}'(t)|^3}.|\vec{r}'(t)|^2 = \frac{|\vec{r}'(t)\times\vec{r}''(t)|}{|\vec{r}'(t)|}$$

radial and transverse components of velocity and acceleration.

Circular Motion

Consider an object is revolving along a circular path with constant angular velocity ω. The position of the body revolving in a circle is given by:

$\vec{r} = r\hat{r}$

Suppose that the center of the circle is at origin O. Now the magnitude of $\vec{r}$ remains constant and the unit vector $\hat{r}$ rotates at a constant rate. A circular motion is an example of a motion in two dimension. i.e. in a plane. So $\hat{r}$ can be written as:

$\hat{r} = \frac{\vec{r}}{r} = Cos\theta\hat{\imath} + Sin\theta\hat{\jmath}$

$\hat{r} = Cos\omega t\hat{\imath} + Sin\omega t\hat{\jmath}$

Where ω is the angular velocity (speed) which is constant.

Radial and Transversal Components of Velocity and Acceleration

In polar coordinates, the position of a particle is specified by a radius vector r and the polar angle θ which are related to x and y through the relations

$x = rCos\theta$ and $y = rSin\theta$

Provided the two coordinate frames have the same origin and the x – axis and the initial line coincide. The direction of radius vector is known as **radial direction** and that perpendicular to it in the direction of the increasing θ is called **transverse direction**.

If $\hat{r} = \frac{\vec{r}}{r} = Cos\theta\hat{\imath} + Sin\theta\hat{\jmath}$ then

$\omega =$ angular speed (velocity) $= \frac{d\theta}{dt} = \dot{\theta}$

$\propto =$ angular acceleration $= \frac{d^2\theta}{dt^2} = \ddot{\theta}$

$v_r =$ radial component of velocity $= \frac{dr}{dt} = \dot{r}$

$v_\theta =$ transversal component of velocity $= r\frac{d\theta}{dt} = r\dot{\theta}$

$a_r =$ radial component of acceleration $= \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = \ddot{r} - r\dot{\theta}^2$

$a_\theta =$ transversal component of acceleration $= 2\frac{dr}{dt}\left(\frac{d\theta}{dt}\right) + r\frac{d^2\theta}{dt^2} = 2\dot{r}\dot{\theta} + r\ddot{\theta}$

Radial and Transversal Components of Velocity and Acceleration

In polar coordinates, the position of a particle is specified by a radius vector $\vec{r}$ and the polar angle θ which are related to x and y through the relations

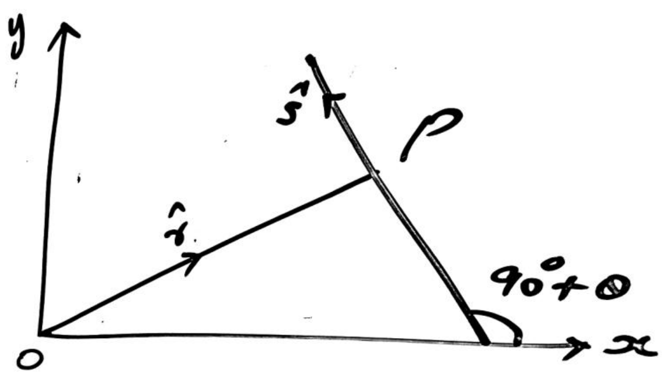
$x = rCos\theta$ and $y = rSin\theta$

Let $\hat{r}$ and $\hat{s}$ be unit vectors in the radial and transverse directions respectively as shown in figure. Then

$\hat{r} = Cos\theta\hat{\imath} + Sin\theta\hat{\jmath}$ …………..(i)

$\hat{s} = Cos(90° + \theta)\hat{\imath} + Sin(90° + \theta)\hat{\jmath}$

$\hat{s} = -Sin\theta\hat{\imath} + Cos\theta\hat{\jmath}$ …………..(ii)



$\Rightarrow \frac{d\hat{r}}{dt} = \frac{d}{d\theta}(Cos\theta\hat{\imath} + Sin\theta\hat{\jmath})\frac{d\theta}{dt}$

$\Rightarrow \frac{d\hat{r}}{dt} = \frac{d\theta}{dt}\hat{s}$ …………..(iii)

$\Rightarrow \frac{d\hat{s}}{dt} = \frac{d}{d\theta}(-Sin\theta\hat{\imath} + Cos\theta\hat{\jmath})\frac{d\theta}{dt}$

$\Rightarrow \frac{d\hat{s}}{dt} = -\frac{d\theta}{dt}\hat{r}$ …………..(iv)

We know that $\hat{r} = \frac{\vec{r}}{r}$ implies $\vec{r} = r\hat{r}$

$\Rightarrow \vec{v} = \frac{d\vec{r}}{dt} = \frac{d}{dt}(r\hat{r}) = \frac{dr}{dt}\hat{r} + r\frac{d\hat{r}}{dt} = \frac{dr}{dt}\hat{r} + r\frac{d\theta}{dt}\hat{s}$

v_r = radial component of velocity $= \frac{dr}{dt} = \dot{r}$

v_θ = transversal component of velocity $= r\frac{d\theta}{dt} = r\dot{\theta}$

Let $\vec{a}$ be the acceleration then

$$\Rightarrow \vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left(\frac{dr}{dt}\hat{r} + r\frac{d\theta}{dt}\hat{s}\right) = \frac{d}{dt}\left(\frac{dr}{dt}\hat{r}\right) + \frac{d}{dt}\left(r\frac{d\theta}{dt}\hat{s}\right)$$

$$\Rightarrow \vec{a} = \frac{d^2r}{dt^2}\hat{r} + \frac{dr}{dt}\frac{d\hat{r}}{dt} + \frac{dr}{dt}\left(\frac{d\theta}{dt}\right)\hat{s} + \frac{d^2\theta}{dt^2}r\hat{s} + \frac{d\hat{s}}{dt}\left(\frac{d\theta}{dt}\right)r$$

$$\Rightarrow \vec{a} = \frac{d^2r}{dt^2}\hat{r} + \frac{dr}{dt}\left(\frac{d\theta}{dt}\hat{s}\right) + \frac{dr}{dt}\left(\frac{d\theta}{dt}\right)\hat{s} + \frac{d^2\theta}{dt^2}r\hat{s} + \left(-\frac{d\theta}{dt}\hat{r}\right)\left(\frac{d\theta}{dt}\right)r$$

$$\Rightarrow \vec{a} = \frac{d^2r}{dt^2}\hat{r} - r\left(\frac{d\theta}{dt}\right)^2\hat{r} + 2\frac{dr}{dt}\left(\frac{d\theta}{dt}\right)\hat{s} + \frac{d^2\theta}{dt^2}r\hat{s}$$

$$\Rightarrow \vec{a} = \left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right]\hat{r} + \left[2\frac{dr}{dt}\left(\frac{d\theta}{dt}\right) + \frac{d^2\theta}{dt^2}r\right]\hat{s}$$

$a_r = \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = \ddot{r} - r\dot{\theta}^2$ and $a_\theta = 2\frac{dr}{dt}\left(\frac{d\theta}{dt}\right) + r\frac{d^2\theta}{dt^2} = 2\dot{r}\dot{\theta} + r\ddot{\theta}$

Question

A particle moves so that its position vector is given by $\vec{r} = Cos\omega t\hat{\imath} + Sin\omega t\hat{\jmath}$ where ω is constant. Then show that

i. The velocity $\vec{v}$ of the particle is perpendicular to $\vec{r}$.
ii. The acceleration $\vec{a}$ is directed toward the origin and has magnitude proportional to the distance from the origin.
iii. $\vec{r} \times \vec{v}$ is constant vector.

Solution

$\vec{r} = Cos\omega t\hat{\imath} + Sin\omega t\hat{\jmath}$

$\vec{v} = \frac{d\vec{r}}{dt} = -\omega Sin\omega t\hat{\imath} + \omega Cos\omega t\hat{\jmath} \Rightarrow \vec{a} = \frac{d\vec{v}}{dt} = -\omega^2 Cos\omega t\hat{\imath} - \omega^2 Sin\omega t\hat{\jmath}$

i. $\vec{r}.\vec{v} = (Cos\omega t\hat{\imath} + Sin\omega t\hat{\jmath}).(-\omega Sin\omega t\hat{\imath} + \omega Cos\omega t\hat{\jmath}) = 0$
The velocity $\vec{v}$ of the particle is perpendicular to $\vec{r}$.

ii. $\vec{a} = -\omega^2 Cos\omega t\hat{\imath} - \omega^2 Sin\omega t\hat{\jmath} = -\omega^2(Cos\omega t\hat{\imath} + Sin\omega t\hat{\jmath}) = -\omega^2\vec{r}$
The acceleration $\vec{a}$ is directed toward the origin and has magnitude proportional to the distance from the origin.

iii. $\vec{r} \times \vec{v} = (Cos\omega t\hat{\imath} + Sin\omega t\hat{\jmath}) \times (-\omega Sin\omega t\hat{\imath} + \omega Cos\omega t\hat{\jmath})$

$$\vec{r} \times \vec{v} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ Cos\omega t & Sin\omega t & 0 \\ -\omega Sin\omega t & \omega Cos\omega t & 0 \end{vmatrix} = \omega\hat{k}$$

$\vec{r} \times \vec{v}$ is constant vector.

Question

Given a space with position vector $\vec{r} = 3Cos2t\hat{\imath} + 3Sin2t\hat{\jmath} + (8t-4)\hat{k}$. Find unit tangent vector to the curve. Also verify that $\vec{v} = vT$.

Solution

$\vec{r} = 3Cos2t\hat{\imath} + 3Sin2t\hat{\jmath} + (8t-4)\hat{k}$

$\mathrm{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|} = -\frac{3}{5}Sin2t\hat{\imath} + \frac{3}{5}Cos2t\hat{\jmath} + \frac{4}{5}\hat{k}$

$\vec{v} = -6Sin2t\hat{\imath} + 6Cos2t\hat{\jmath} + 8\hat{k} \Rightarrow v = |\vec{v}| = 10$

Cearly $\vec{v} = vT$.   Prove it by $L.H.S = R.H.S$

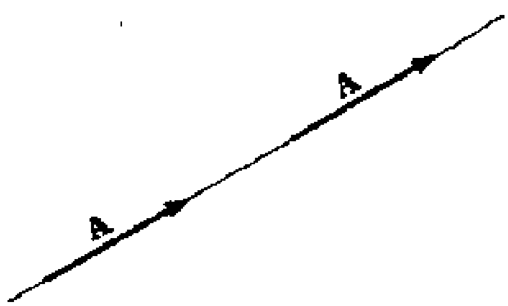
Free Vectors

Vectors which are specified by magnitude and direction only are called **free vectors**. Few types of such vectors given as follows;

- **Equal free Vectors:** Any two free vectors are equal if they have the same magnitude and direction.

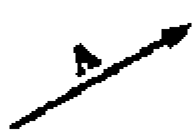
(a) Equal free vectors

- **Equal Sliding Vectors:** Any two free vectors are equal sliding iff they have the same magnitude, direction and line of action.



(b) Equal sliding vectors

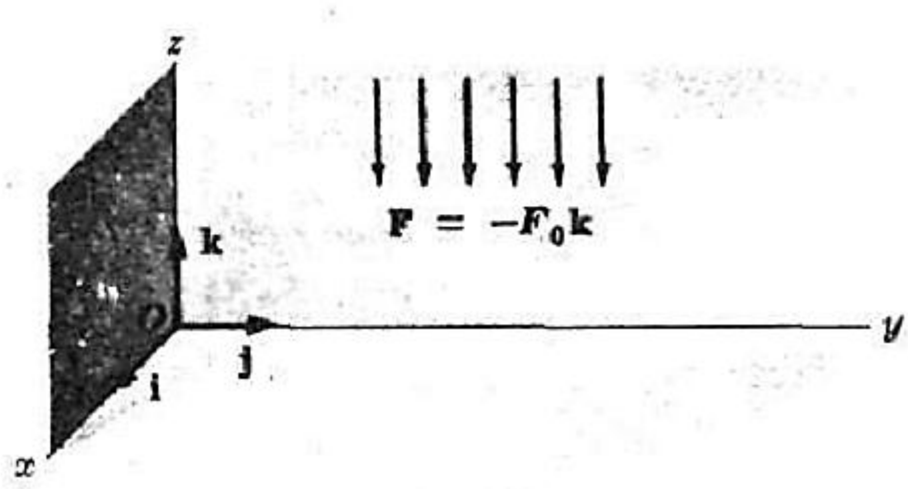
- **Equal Bound Vectors:** Any two free vectors are equal bound vectors iff they have the same magnitude, direction and point of action. i.e identical.



(c) Bound vector

Uniform Force Field

A force field which has constant magnitude and direction is called a uniform or constant force field. If the direction of this field is taken as the negative z direction and the magnitude is the constant F_0 then the force field will be $\vec{F} = F_0\hat{k}$



Uniformly Accelerated Motion

If a particle of constant mass m moves in a uniform field, then its acceleration is uniform or constant. The motion is then described as uniformly accelerated motion. Its formula is given by $\vec{a} = \frac{F_0}{m}\hat{k}$.

Accelerated due to Gravity

Near the earth's surface an object fall with a vertical acceleration which is constant provided that air resistance is negligible. This acceleration is denoted by $\vec{g}$ and called the acceleration due to gravity or the gravitational acceleration. Its value is given by $\vec{g} = 9.8ms^{-2}$.

Freely Falling Bodies

If an object moves so that the only force acting upon it is its weight, or force due to gravity, then the object is often called a freely falling body. If $\vec{r}$ is the position vector and m is the mass of the body, then using Newton's 2nd Law and $\vec{W} = mg$ assuming the motion in xy – plane we have

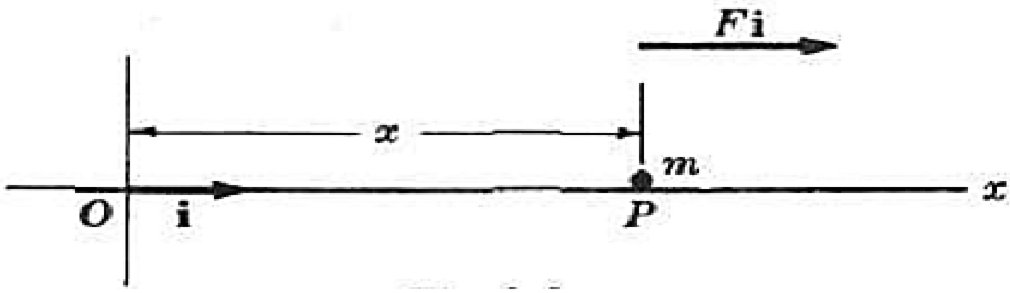
$$\vec{F} = m\vec{a} \Rightarrow \vec{F} = \frac{\vec{W}}{g}\vec{a} \Rightarrow m\frac{d^2\vec{r}}{dt^2} = \frac{-mg\hat{k}}{g} \Rightarrow \boldsymbol{\frac{d^2\vec{r}}{dt^2} = -g\hat{k}}$$

Equations shows that motion of freely falling body is independent of mass.

Question

A particle of mass m moves along a straight line under the influence of a constant force of magnitude F. If its initial speed is $\vec{v}_0$, find the speed, the velocity and the distance travelled after time t.

Solution



Assume that the straight line along which the particle P moves is the x – axis as shown in figure. Suppose that at time t the particle is at a distance x from origin O. If $\hat{\imath}$ is a unit vector in the direction OP and v is the speed at time t, then the velocity is $\vec{v}\hat{\imath}$. Then we have

$$\vec{F} = m\vec{a} \Rightarrow m\frac{d}{dt}(v\hat{\imath}) = F\hat{\imath} \Rightarrow m\frac{dv}{dt} = F \Rightarrow \frac{dv}{dt} = \frac{F}{m} \Rightarrow dv = \frac{F}{m}dt \Rightarrow v = \frac{F}{m}t + A$$

Initially using $t = 0, v = v_0$ we get $A = v_0$. Thus $v = \frac{F}{m}t + v_0$

$$\Rightarrow v = v_0 + \frac{F}{m}t$$

To find velocity

Since we have $v = v_0 + \frac{F}{m}t$

$$\Rightarrow v\hat{\imath} = v_0\hat{\imath} + \frac{F\hat{\imath}}{m}t \Rightarrow \vec{v} = \vec{v}_0 + \frac{\vec{F}}{m}t$$

To find distance

Since we have $v = v_0 + \frac{F}{m}t$

$$\Rightarrow \frac{dx}{dt} = v_0 + \frac{F}{m}t \Rightarrow dx = \left(v_0 + \frac{F}{m}t\right)dt \Rightarrow x = v_0t + \frac{F}{2m}t^2 + B$$

Initially using $x = 0, t = 0$ we get $B = 0$.

Thus $x = v_0t + \frac{F}{2m}t^2$

Trajectory

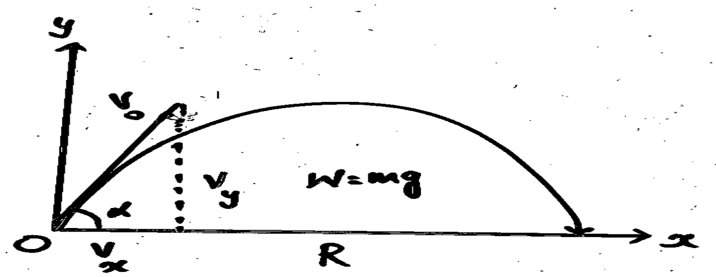
The curve traced by a moving particle is called the trajectory or path of the particle.

Projectile Motion of a Particle

An object fired from a gun or dropped from a moving airplane is often called a **projectile**. If a ball is thrown from one person to another or an object is dropped from a moving plane, then their path of traveling/motion is often called a **projectile**.

Position vector of Projectile at any time t

Consider a body of mass m projected with velocity $\vec{v}_0$ at angle $\propto$ with the horizontal. Derive the expression for the P.V. of the projectile.



Solution

Since $\vec{F} = F_x\hat{\imath} + F_y\hat{\jmath} = 0\hat{\imath} + (-mg)\hat{\jmath}$

$\Rightarrow F_x = 0 \Rightarrow ma_x = 0 \Rightarrow m\frac{d^2x}{dt^2} = 0 \Rightarrow m \neq 0, \frac{d^2x}{dt^2} = 0 \Rightarrow v_x = \frac{dx}{dt} = c_1$

$\Rightarrow v_x = \frac{dx}{dt} = v_0 Cos \propto$ initially using $t = 0, v_x = v_0 Cos \propto, c_1 = v_0 Cos \propto$

$\Rightarrow dx = v_0 Cos \propto dt \Rightarrow x = (v_0 Cos \propto)t + c_2$

$\Rightarrow x = (v_0 Cos \propto)t$ initially using $t = 0, x = 0, c_2 = 0$

$\Rightarrow F_y = -mg \Rightarrow ma_y = -mg \Rightarrow \frac{d^2y}{dt^2} = -g \Rightarrow v_y = \frac{dy}{dt} = -gt + c_3$

$\Rightarrow v_y = \frac{dy}{dt} = -gt + v_0 Sin \propto$ initially using $t = 0, v_y = v_0 Sin \propto, c_3 = v_0 Sin \propto$

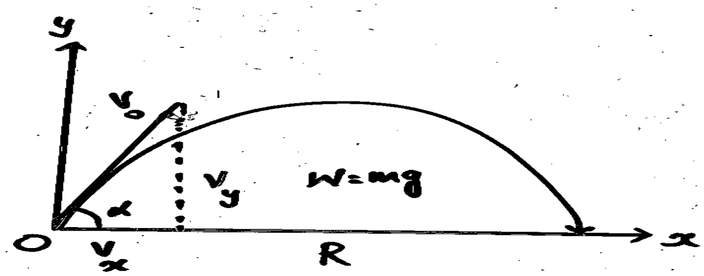
$\Rightarrow dy = -gtdt + v_0 Sin \propto dt \Rightarrow y = -\frac{1}{2}gt^2 + (v_0 Sin \propto)t + c_4$

$\Rightarrow y = -\frac{1}{2}gt^2 + (v_0 Sin \propto)t$ initially using $t = 0, y = 0, c_4 = 0$

$\Rightarrow \boldsymbol{\vec{r} = x\hat{\imath} + y\hat{\jmath} = (v_0 Cos \propto)t\hat{\imath} + \left[(v_0 Sin \propto)t - \frac{1}{2}gt^2\right]\hat{\jmath}}$

Range of Flight/ Range of Projectile/ Horizontal Range of Projectile

Consider a body of mass m projected with velocity $\vec{v}_0$ at angle $\propto$ with the horizontal. Derive the expression for the range of flight.



Solution

Since $\vec{F} = F_x\hat{\imath} + F_y\hat{\jmath} = 0\hat{\imath} + (-mg)\hat{\jmath}$

$\Rightarrow F_x = 0 \Rightarrow ma_x = 0 \Rightarrow m\frac{d^2x}{dt^2} = 0 \Rightarrow m \neq 0, \frac{d^2x}{dt^2} = 0 \Rightarrow v_x = \frac{dx}{dt} = c_1$

$\Rightarrow v_x = \frac{dx}{dt} = v_0 Cos \propto$ initially using $t = 0, v_x = v_0 Cos \propto, c_1 = v_0 Cos \propto$

$\Rightarrow dx = v_0 Cos \propto dt \Rightarrow x = (v_0 Cos \propto)t + c_2$

$\Rightarrow \boldsymbol{x = (v_0 Cos \propto)t}$ initially using $t = 0, x = 0, c_2 = 0$

$\Rightarrow F_y = -mg \Rightarrow ma_y = -mg \Rightarrow \frac{d^2y}{dt^2} = -g \Rightarrow v_y = \frac{dy}{dt} = -gt + c_3$

$\Rightarrow v_y = \frac{dy}{dt} = -gt + v_0 Sin \propto$ initially using $t = 0, v_y = v_0 Sin \propto, c_3 = v_0 Sin \propto$

$\Rightarrow dy = -gtdt + v_0 Sin \propto dt \Rightarrow y = -\frac{1}{2}gt^2 + (v_0 Sin \propto)t + c_4$

$\Rightarrow \boldsymbol{y = -\frac{1}{2}gt^2 + (v_0 Sin \propto)t}$ initially using $t = 0, y = 0, c_4 = 0$

$\Rightarrow y = -\frac{1}{2}g\left(\frac{x}{v_0 Cos\propto}\right)^2 + (v_0 Sin \propto)\left(\frac{x}{v_0 Cos\propto}\right)$ Using $t = \frac{x}{v_0 Cos\propto}$

$\Rightarrow y = -\frac{gx^2}{2v_0^2}\left(\frac{1}{Cos^2\propto}\right) + xtan \propto \Rightarrow y = xtan \propto - \frac{gx^2}{2v_0^2} Sec^2 \propto$

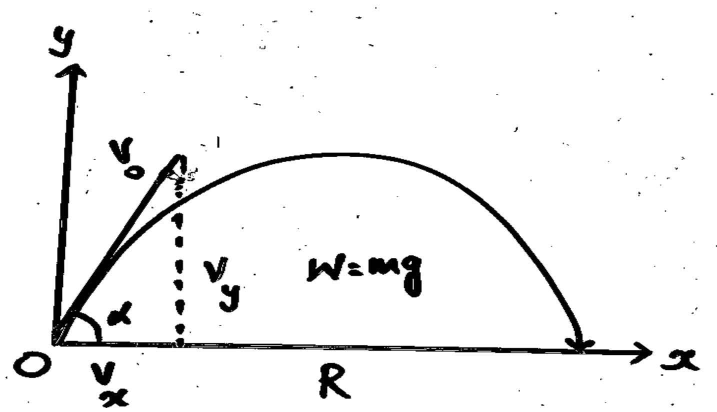
$\Rightarrow 0 = Rtan \propto - \frac{gR^2}{2v_0^2} Sec^2 \propto$ Using $x = R, y = 0$

$\Rightarrow Rtan \propto = \frac{gR^2}{2v_0^2} Sec^2 \propto = \frac{2v_0^2}{g}\frac{tan\propto}{Sec^2\propto} = \frac{2v_0^2}{g} Sin \propto Cos \propto$

$\Rightarrow \boldsymbol{R = \frac{v_0^2}{g} Sin2 \propto}$

Time of Flight/ Time of Projectile/ The Time of flight back to Earth

Consider a body of mass m projected with velocity $\vec{v}_0$ at angle $\propto$ with the horizontal. Derive the expression for the time of flight.



Solution

Since $\vec{F} = F_x\hat{\imath} + F_y\hat{\jmath} = 0\hat{\imath} + (-mg)\hat{\jmath}$

$\Rightarrow F_x = 0 \Rightarrow ma_x = 0 \Rightarrow m\frac{d^2x}{dt^2} = 0 \Rightarrow m \neq 0, \frac{d^2x}{dt^2} = 0 \Rightarrow v_x = \frac{dx}{dt} = c_1$

$\Rightarrow v_x = \frac{dx}{dt} = v_0 Cos \propto$ initially using $t = 0, v_x = v_0 Cos \propto, c_1 = v_0 Cos \propto$

$\Rightarrow dx = v_0 Cos \propto dt \Rightarrow x = (v_0 Cos \propto)t + c_2$

$\Rightarrow \boldsymbol{x = (v_0 Cos \propto)t}$ initially using $t = 0, x = 0, c_2 = 0$

$\Rightarrow R = (v_0 Cos \propto)T$ Using $x = R, t = T$

$\Rightarrow T = \frac{R}{v_0 Cos\propto}$

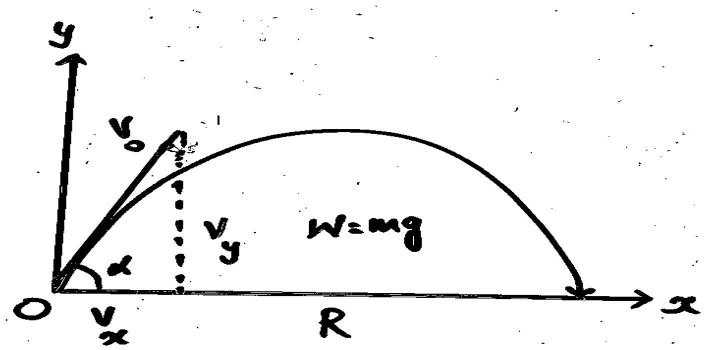
$\Rightarrow T = \frac{\frac{v_0^2}{g}Sin2\propto}{v_0 Cos\propto} = \frac{v_0^2.2Sin\propto Cos\propto}{v_0 g Cos\propto}$ $\because R = \frac{v_0^2}{g}Sin2 \propto$

$\Rightarrow \boldsymbol{T = \frac{2v_0}{g}Sin} \propto$

Remember that time of flight depends on $v_0 Sin \propto$ which is the vertical component of the velocity of the projection.

Path of Projectile is a Parabola

Consider a body of mass m projected with velocity $\vec{v}_0$ at angle $\propto$ with the horizontal.



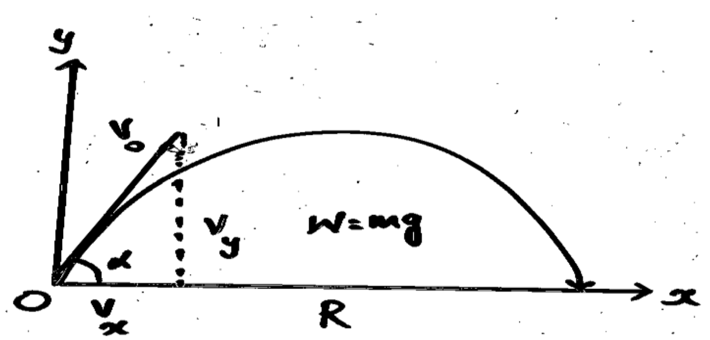
We know that $t = \frac{R}{v_0 Cos\propto}$ and $y = -\frac{1}{2}gt^2 + (v_0 Sin \propto)t$ then using both equations we have

$$\Rightarrow y = -\frac{1}{2}g\left(\frac{R}{v_0 Cos\propto}\right)^2 + (v_0 Sin \propto)\left(\frac{R}{v_0 Cos\propto}\right) \Rightarrow y = Rtan \propto - \frac{g}{2v_0^2}R^2 sec^2 \propto$$

Which is a Parabola.

Maximum Range of Projectile / Maximum Horizontal Range of Projectile

Consider a body of mass m projected with velocity $\vec{v}_0$ at angle $\propto$ with the horizontal.



We know that $R = \frac{v_0^2}{g}Sin2 \propto$

The range of the projectile will be maximum, when $Sin2 \propto = 1$

$\Rightarrow 2 \propto = Sin^{-1}(1) \Rightarrow \propto = 45°$

Thus the projectile will have the maximum range when it will be projected at an angle of $45°$, therefore

$$R = \frac{v_0^2}{g}$$

Question

An object of mass m is thrown vertically upward from the earth's surface with speed v_0, find the position at any time, the time taken to reach the highest point and the maximum height reached.

Solution

Let the position vector of m at any time t be $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$. Assume that the object starts at $\vec{r} = 0$ when $t = 0$. Since the force acting on the object is $-mg\hat{k}$, we have by Newton's Law;

$$\Rightarrow \vec{F} = -mg\hat{k} \Rightarrow m\frac{d^2\vec{r}}{dt^2} = -mg\hat{k} \Rightarrow \frac{d\vec{v}}{dt} = -g\hat{k} \Rightarrow \vec{v} = -gt\hat{k} + A$$

Using $\vec{v} = v_0\hat{k}$ at time $t = 0$ we have $A = v_0\hat{k}$

$$\Rightarrow \vec{v} = -gt\hat{k} + v_0\hat{k} \Rightarrow \vec{v} = (v_0 - gt)\hat{k}$$

$$\Rightarrow \frac{d\vec{r}}{dt} = (v_0 - gt)\hat{k} \Rightarrow \vec{r} = \left(v_0 t - \frac{1}{2}gt^2\right)\hat{k} + B$$

Using $\vec{r} = 0$ at time $t = 0$ we have $B = 0$

$$\Rightarrow \boldsymbol{\vec{r} = \left(v_0 t - \frac{1}{2}gt^2\right)\hat{k}}$$

Or equivalently $\boldsymbol{x = 0}$; $\boldsymbol{y = 0}$; $\boldsymbol{z = v_0 t - \frac{1}{2}gt^2}$

The highest point is reached when $\vec{v} = (v_0 - gt)\hat{k} = 0$ that is at time $t = \frac{v_0}{g}$

At time $t = \frac{v_0}{g}$ the maximum height reached is from $z = v_0 t - \frac{1}{2}gt^2$ as follows

$$z = \frac{v_0^2}{2g}$$

Maximum Height of Projectile Reached

A the highest point of the path the component of the velocity is zero thus using $(v_0 Sin \propto) - gt = 0$ and we get $t = \frac{v_0}{g}Sin \propto$

$$\Rightarrow y = (v_0 Sin \propto)t - \frac{1}{2}gt^2$$

$$\Rightarrow \text{Maximum Height Reached} = y = (v_0 Sin \propto)\left(\frac{v_0}{g}Sin \propto\right) - \frac{1}{2}g\left(\frac{v_0}{g}Sin \propto\right)^2$$

$$\Rightarrow \text{Maximum Height Reached} = \frac{v_0^2 Sin^2 \propto}{2g}$$

Question

A projectile is launched with initial speed v_0 at an angle $\propto$ with the horizontal acting upon a force due to air resistance equal to $-\beta v$ where β is constant. Find the position and velocity vector at any time.

Solution

Body move downward, so frictional force $= \vec{F_s} = -k'\vec{v} = -\beta\vec{v}$ and $\overrightarrow{W} = -m\vec{g}$

$\vec{F} = \overrightarrow{W} + \vec{F_s} = -m\vec{g} - \beta\vec{v} \Rightarrow m\vec{a} = -m\vec{g} - \beta\vec{v}$

$\Rightarrow \frac{d\vec{v}}{dt} = -\vec{g} - \left(\frac{\beta}{m}\right)\vec{v} \Rightarrow \frac{d\vec{v}}{dt} + \left(\frac{\beta}{m}\right)\vec{v} = -\vec{g}(x)$

$\Rightarrow \frac{d\vec{v}}{dt}e^{\left(\frac{\beta}{m}\right)t} + \left(\frac{\beta}{m}\right)\vec{v}e^{\left(\frac{\beta}{m}\right)t} = -\vec{g}(x)e^{\left(\frac{\beta}{m}\right)t} \Rightarrow \frac{d}{dt}\left(\vec{v}e^{\left(\frac{\beta}{m}\right)t}\right) = -ge^{\left(\frac{\beta}{m}\right)t}$

$\Rightarrow \vec{v}e^{\left(\frac{\beta}{m}\right)t} = -g\int e^{\left(\frac{\beta}{m}\right)t}\,dt + A \Rightarrow \vec{v}e^{\left(\frac{\beta}{m}\right)t} = -\left(\frac{mg}{\beta}\right)e^{\left(\frac{\beta}{m}\right)t} + A$

using initially $t = 0, v = v_0 cos \propto \hat{j} + v_0 sin \propto \hat{k}$

we get $A = v_0 cos \propto \hat{j} + v_0 sin \propto \hat{k} + \frac{mg}{\beta}$

$\Rightarrow \vec{v}e^{\left(\frac{\beta}{m}\right)t} = -\left(\frac{mg}{\beta}\right)e^{\left(\frac{\beta}{m}\right)t} + v_0 cos \propto \hat{j} + v_0 sin \propto \hat{k} + \frac{mg}{\beta}$

$\Rightarrow \boldsymbol{\vec{v} = (v_0 cos \propto \hat{j} + v_0 sin \propto \hat{k})e^{-\left(\frac{\beta}{m}\right)t} + \frac{mg}{\beta}\left(1 - e^{-\left(\frac{\beta}{m}\right)t}\right)}$ **required velocity**

$\Rightarrow \frac{d\vec{r}}{dt} = (v_0 cos \propto \hat{j} + v_0 sin \propto \hat{k})e^{-\left(\frac{\beta}{m}\right)t} + \frac{mg}{\beta}\left(1 - e^{-\left(\frac{\beta}{m}\right)t}\right)$

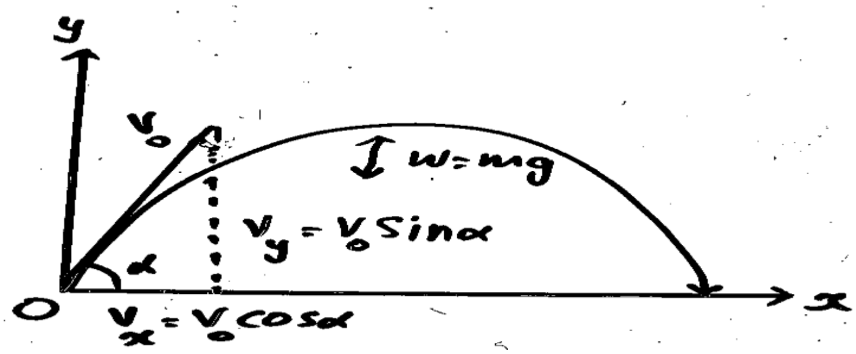
$\Rightarrow \vec{r} = -\left(\frac{m}{\beta}\right)(v_0 cos \propto \hat{j} + v_0 sin \propto \hat{k})e^{-\left(\frac{\beta}{m}\right)t} + \frac{mg}{\beta}\left(t + \left(\frac{m}{\beta}\right)e^{-\left(\frac{\beta}{m}\right)t}\right) + B$

using initially $t = 0, r = 0$ we get $B = \frac{m}{\beta}\left(v_0 cos \propto \hat{j} + v_0 sin \propto \hat{k} + \frac{m^2 g}{\beta^2}\right)$

$\Rightarrow \boldsymbol{\vec{r} = \left(\frac{mv_0}{\beta}\right)(cos \propto \hat{j} + sin \propto \hat{k})\left(1 - e^{-\left(\frac{\beta}{m}\right)t}\right) - \frac{mg}{\beta}\left(t + \left(\frac{m}{\beta}\right)e^{-\left(\frac{\beta}{m}\right)t} - \frac{m}{\beta}\right)}$

Projectile Motion (of a particle) with air Resistance

Consider a body of mass m projected with velocity $\vec{v}_0$ at angle $\propto$ with the horizontal.



Body move downward, so frictional force $= \vec{F}_s = -k'\vec{v}$ and $\vec{W} = -m\vec{g}$

$\vec{F} = \vec{W} + \vec{F}_s = -m\vec{g} - k'\vec{v} \Rightarrow m\vec{a} = -m\vec{g} - mk\vec{v}$ where m, g, k are constants.

$\Rightarrow \vec{a} = -\vec{g} - k\vec{v} \Rightarrow (x''\hat{\imath} + y''\hat{\jmath}) = -g\hat{\jmath} - k(x'\hat{\imath} + y'\hat{\jmath})$

$\Rightarrow x''\hat{\imath} + y''\hat{\jmath} = -kx'\hat{\imath} + (-g - ky')\hat{\jmath}$

$\Rightarrow x'' = -kx' \ldots\ldots\ldots.(i),\ y'' = -g - ky' \ldots\ldots\ldots.(ii)$

$(i) \Rightarrow x'' = -kx' \Rightarrow z' = -kz$ using $x' = z,\ x'' = z'$

$\Rightarrow \frac{dz}{dt} = -kz \Rightarrow \int \frac{1}{z} dz = -k \int dt \Rightarrow lnz = -kt + A \Rightarrow z = e^{-kt+A} = e^{-kt}e^{A}$

$\Rightarrow z = c_1 e^{-kt} \Rightarrow z = u_1 e^{-kt}$ using initially $t = 0, v = u_1$ we get $c_1 = u_1$

$\Rightarrow \frac{dx}{dt} = u_1 e^{-kt} \Rightarrow \int dx = u_1 \int e^{-kt}\, dt \Rightarrow x = u_1 \frac{e^{-kt}}{-k} + B$

$\Rightarrow x = u_1 \frac{e^{-kt}}{-k} + \frac{u_1}{k}$ using initially $t = 0, x = 0$

$\Rightarrow \boldsymbol{x = \frac{u_1}{k}\left(1 - e^{-kt}\right)}$

$(ii) \Rightarrow y'' = -g - ky' \Rightarrow s' = -ks - g$ using $y' = s,\ y'' = s'$

$\Rightarrow \frac{ds}{dt} + ks = -g \Rightarrow \frac{ds}{dt}e^{kt} + kse^{kt} = -ge^{kt}$ using integrating factor

$\Rightarrow \frac{d}{dt}(se^{kt}) = -ge^{kt} \Rightarrow \int d(se^{kt}) = -g \int e^{kt}dt \Rightarrow se^{kt} = -g\frac{e^{kt}}{k} + D$

$\Rightarrow v_1 = -\frac{g}{k} + D$ using initially $t = 0, s = y' = v = v_1$

$\Rightarrow D = v_1 + \frac{g}{k}$ then $se^{kt} = -g\frac{e^{kt}}{k} + v_1 + \frac{g}{k} \Rightarrow s = -\frac{g}{k} + e^{-kt}\left(v_1 + \frac{g}{k}\right)$

$$\Rightarrow \frac{dy}{dt} = -\frac{g}{k} + e^{-kt}\left(v_1 + \frac{g}{k}\right) \Rightarrow \int dy = \int\left[-\frac{g}{k} + e^{-kt}\left(v_1 + \frac{g}{k}\right)\right]dt$$

$$\Rightarrow y = -\frac{g}{k}\int dt + \left(v_1 + \frac{g}{k}\right)\int e^{-kt}\,dt \Rightarrow y = -\frac{g}{k}t + \left(v_1 + \frac{g}{k}\right)\frac{e^{-kt}}{-k} + E$$

using initially $t = 0, t = 0$ we get $E = \frac{1}{k}\left(v_1 + \frac{g}{k}\right)$

$$\Rightarrow y = -\frac{g}{k}t - \left(v_1 + \frac{g}{k}\right)\frac{e^{-kt}}{k} + \frac{1}{k}\left(v_1 + \frac{g}{k}\right)$$

$$\Rightarrow \boldsymbol{y = -\frac{g}{k}t + \frac{1}{k}\left(v_1 + \frac{g}{k}\right)\left(1 - e^{-kt}\right)}$$

Time of Flight/ Time of Projectile with Air Resistance

Since we know that $y = -\frac{g}{k}t + \frac{1}{k}\left(v_1 + \frac{g}{k}\right)\left(1 - e^{-kt}\right)$

Using $y = 0, t = T$

$$\Rightarrow 0 = -\frac{g}{k}T + \frac{1}{k}\left(v_1 + \frac{g}{k}\right)\left(1 - e^{-kT}\right) \Rightarrow \frac{g}{k}T = \frac{1}{k}\left(v_1 + \frac{g}{k}\right)\left(1 - e^{-kT}\right)$$

$$\Rightarrow gT = \left(\frac{kv_1 + g}{k}\right)\left(1 - e^{-kT}\right) \Rightarrow T = \left(\frac{kv_1 + g}{gk}\right)\left[1 - \left(1 - kT + \frac{k^2T^2}{2!} - \frac{k^3T^3}{3!} + \cdots\right)\right]$$

$$\Rightarrow T = \left(\frac{kv_1 + g}{gk}\right)\left(kT - \frac{k^2T^2}{2!} + \frac{k^3T^3}{3!} + \cdots\right)$$

$$\Rightarrow T = \left(\frac{kv_1 + g}{gk}\right)kT\left(1 - \frac{kT}{2!} + \frac{k^2T^2}{3!} + neglecting\right) \quad ; k \ll 1$$

$$\Rightarrow 1 = \left(\frac{kv_1 + g}{g}\right)\left(1 - \frac{kT}{2!} + \frac{k^2T^2}{3!}\right) \Rightarrow \frac{g}{kv_1 + g} = 1 - \frac{kT}{2!} + \frac{k^2T^2}{3!}$$

$$\Rightarrow \frac{kT}{2} = 1 + \frac{k^2T^2}{6} - \frac{g}{kv_1 + g} \Rightarrow \frac{kT}{2} = \left(\frac{kv_1 + g - g}{kv_1 + g}\right) + \frac{k^2T^2}{6}$$

$$\Rightarrow \frac{kT}{2} = \left(\frac{kv_1}{kv_1 + g}\right) + \frac{k^2T^2}{6} \Rightarrow T = \frac{2}{k}\left(\frac{kv_1}{kv_1 + g} + \frac{k^2T^2}{6}\right) \Rightarrow T = \frac{2v_1}{kv_1 + g} + \frac{kT^2}{3} \ldots\ldots\ldots(iii)$$

For ideal condition friction is zero so $k = 0$ then $T = \frac{2v_1}{g}$

$\Rightarrow T = \frac{2v_0 sin\propto}{g}$ using $v_1 = v_0 sin \propto$

$\Rightarrow T_0 = \frac{2v_0 sin\propto}{g}$ for small value of k using $T = T_0$

$(iii) \Rightarrow T = \frac{2v_1}{kv_1 + g} + \frac{kT_2^2}{3}$

$\Rightarrow T = \frac{2v_1}{kv_1 + g} + \frac{k}{3}\left(\frac{2v_1}{g}\right)^2 \Rightarrow T = \frac{2v_1}{g\left(1+\frac{kv_1}{g}\right)} + \frac{k}{3}\left(\frac{2^2 v_1^2}{g^2}\right)$

$\Rightarrow T = \frac{2v_1}{g}\left[\left(1 + \frac{kv_1}{g}\right)^{-1} + \frac{2kv_1}{3g}\right] \Rightarrow T = \frac{2v_1}{g}\left[\left(1 - \frac{kv_1}{g} + \frac{k^2 v_1^2}{g} + \cdots\right) + \frac{2kv_1}{3g}\right]$

$\Rightarrow T = \frac{2v_1}{g}\left[1 - \frac{kv_1}{g} + neglecting + \frac{2kv_1}{3g}\right] \Rightarrow T = \frac{2v_1}{g}\left[1 - \left(\frac{kv_1}{g} - \frac{2kv_1}{3g}\right)\right]$

$\Rightarrow T = \frac{2v_1}{g}\left[1 - \left(\frac{3kv_1 - 2kv_1}{3g}\right)\right] \Rightarrow \boldsymbol{T = \frac{2v_1}{g}\left(1 - \frac{kv_1}{3g}\right)}$

Range of Projectile with Air Resistance

Since we know that $x = \frac{u_1}{k}(1 - e^{-kt})$

$\Rightarrow R' = \frac{u_1}{k}(1 - e^{-kT})$ using $x = R',\ t = T$

$\Rightarrow R' = \frac{u_1}{k}\left[1 - \left(1 - kT + \frac{k^2T^2}{2!} - \frac{k^3T^3}{3!} + \cdots\right)\right]$

$\Rightarrow R' = \frac{u_1}{k}\left[1 - 1 + kT - \frac{k^2T^2}{2!} + \frac{k^3T^3}{3!} - \cdots\right]$

$\Rightarrow R' = \frac{u_1}{k}\left[kT - \frac{k^2T^2}{2!} + \frac{k^3T^3}{3!} - \cdots\right]$

using $T = \frac{2v_1}{g}\left(1 - \frac{kv_1}{3g}\right)$

$\Rightarrow R' = \frac{u_1}{k}\left[k\left(\frac{2v_1}{g}\left(1 - \frac{kv_1}{3g}\right)\right) - \frac{k^2}{2}\left(\frac{2v_1}{g}\left(1 - \frac{kv_1}{3g}\right)\right)^2 + neglecting\right]$

$$\Rightarrow R' = \frac{u_1}{k}\left[\frac{2kv_1}{g}\left(1-\frac{kv_1}{3g}\right)-\frac{2^2k^2v_1^2}{2g^2}\left(1-\frac{kv_1}{3g}\right)^2\right]$$

$$\Rightarrow R' = \frac{u_1}{k}\left[\frac{2kv_1}{g}\left(1-\frac{kv_1}{3g}\right)-\frac{2k^2v_1^2}{g^2}\left(1+\frac{2k^2v_1^2}{9g^2}+\cdots\right)\right]$$

$$\Rightarrow R' = \frac{u_1}{k}.\frac{2kv_1}{g}\left[\left(1-\frac{kv_1}{3g}\right)-\frac{kv_1}{g}(1+neglecting)\right]$$

$$\Rightarrow R' = \frac{2u_1v_1}{g}\left[1-\frac{kv_1}{3g}-\frac{kv_1}{g}\right] \Rightarrow R' = \frac{2u_1v_1}{g}\left[1-\frac{kv_1+3kv_1}{3g}\right]$$

$$\Rightarrow R' = \frac{2u_1v_1}{g}\left[1-\frac{4kv_1}{3g}\right] \Rightarrow \boldsymbol{R' = R\left[1-\frac{4kv_1}{3g}\right]}$$

$$\Rightarrow R' = R-\frac{4kv_1}{3g}R \qquad R' < R \text{ (Due to friction)}$$

$$\Rightarrow \frac{4kv_1}{3g}R = R-R' \Rightarrow \text{decrease in Range} = \Delta R = R-R' = \frac{4kv_1}{3g}R$$

$$\Rightarrow \Delta R = \frac{4k(v_0 sin\propto)}{3g}\frac{v_0 sin2\propto}{g} \Rightarrow \boldsymbol{\Delta R = \frac{4kv_2^2 sin\propto sin2\propto}{3g^2}}$$

Ideal condition

$k = 0$

$v_1 = v_0 sin \propto$

$u_1 = v_0 cos \propto$

$R = \dfrac{2u_1v_1}{g}$

$R = \dfrac{v_0 sin2\propto}{g}$

Question

Show that $x = u_1 t$ if force of friction is zero.

Solution

Let $x = \frac{u_1}{k}(1-e^{-kt})$

$$\Rightarrow x = \frac{u_1}{k}\left[1-\left(1-kt+\frac{k^2t^2}{2!}-\frac{k^3t^3}{3!}+\cdots\right)\right]$$

$$\Rightarrow x = \frac{u_1}{k}\left[1-1+kt-\frac{k^2t^2}{2!}+\frac{k^3t^3}{3!}-\cdots\right] = \frac{u_1}{k}\left[kt-\frac{k^2t^2}{2!}+\frac{k^3t^3}{3!}-\cdots\right]$$

$$\Rightarrow x = \frac{u_1}{k}.k\left[t-\frac{kt^2}{2!}+\frac{k^2t^3}{3!}-\cdots\right] \Rightarrow x = u_1\left[t-\frac{kt^2}{2!}+\frac{k^2t^3}{3!}-\cdots\right]$$

When force of friction is zero it means k = 0

$$\Rightarrow x = u_1[t-0+0-\cdots]$$

$$\Rightarrow x = u_1 t$$

CHAPTER

4

RESISTED MOTION AND DAMPED FORCE OSCILLATOR

Motion in a Resisting Medium

In practice an object is acted upon not only by is weight but by other forces as well. An important class of forces are those which tend to oppose the motion of an object and reduce the magnitude of successive oscillations about the equilibrium position. Such forces, which generally arises because of motion in some medium such as air or water, are often called **resisting, damping** or **dissipative** force and the corresponding medium is said to be a **resisting, damping** or **dissipative** medium. A useful approximated damping force is given as follows;

$$\vec{F}_D = -\beta\vec{v} = -\beta v\hat{\imath} = -\beta\frac{dx}{dt}\hat{\imath}$$

Where the descript D stands for the damping force and β is the positive constant called the damping coefficient. Note the $\vec{F}_D$ and $\vec{v}$ are in opposite direction.

Friction Force

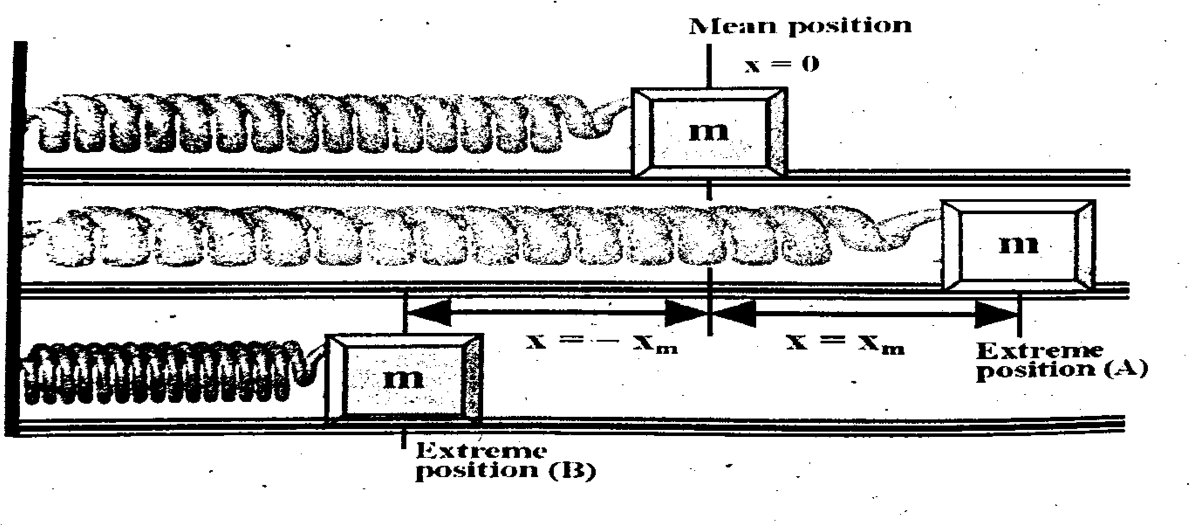
Friction forces play an important role in damping or retarding motion initiated by other forces friction force between two bodies results from the interaction between the surface molecules of the two bodies and involves a very large number of such iteration. The phenomenon is therefore complex and depends on factor such as the condition and nature of the surfaces and their relative velocity.

Some Useful Definitions

- **Simple Harmonic Motion and Simple Harmonic Oscillator:** SHM occur when the net force is directly proportional to the displacement from the mean position and is always directed towards the mean position. The body executing SHM is called *Simple Harmonic Oscillator*. The motion of simple pendulum and the motion of mass spring system is SHM.
  Simple Harmonic Motion is an oscillatory motion that occurs whenever a force acts on a body in the opposite direction to its displacement from its equilibrium position , with the magnitude of the force , proportional to the magnitude of the displacement. i.e. $\vec{F} \propto -x$ or $\vec{F} = -kx$
  Where k is the constant of proportionality often called the spring constant, elastic constant, stiffness factor or modulus of elasticity
- **Restoring Force:** A force that always pushes of pulls the object performing oscillatory motion towards the mean position.
- **Vibration:** One complete round trip of a vibrating body about its mean position is called one vibration.
- **Time Period:** The time taken by a vibrating body to complete one vibration is called time period.
- **Frequency:** The number of vibrations or cycles of a vibrating body in one second is called its frequency. It is reciprocal of time period.
- **Amplitude:** The maximum displacement of a vibrating body on either side from its mean position is called its amplitude.
- **Oscillations/Vibrations:** A body is said to be vibrating (oscillating) if it moves back and forth or to and fro about a point.
- **Damped forced oscillations/ Damped oscillations:** The oscillations of a system in the presence of some resistive force.
- **Linear frequency:** The amount of vibrations completed in unit time is called linear frequency. Its SI unit is called hertz (Hz).
- **Angular frequency:** The amount of rotations completed in unit time is called linear frequency. The linear frequency f and the angular frequency ω are related as $f = \frac{\omega}{2\pi}$

Equation of Motion of Simple Harmonic Oscillator

Consider a block of mass m is attached with one end of a string. The other end of spring is fixed to a support. The block is free to move to and fro over a frictionless horizontal surface as shown in figure.



The point x = 0 when block is at rest is called **mean position** because spring is not exerting any force on the block. The block attached with spring having constant k takes to and fro motion under restoring force F given as

$\vec{F} = -kx$ ……………….(1)

$\vec{F} = m\vec{a} = m\frac{d^2x}{dt^2}$ ……………….(2) by Newton's 2nd Law

Comparing (1) and (2) we have

$$m\frac{d^2x}{dt^2} = -kx$$

$m\ddot{x} + kx = 0$ Or $\ddot{x} + \frac{k}{m}x = 0$

This is called equation of motion of simple harmonic oscillator or linear Harmonic Oscillator. This type of motion is often called Simple Harmonic Motion.

Damped Harmonic Oscillator

The oscillator which moves in a resistive medium under a restoring force is called the Damped Harmonic Oscillator and equation of motion of the harmonic oscillator is given as

$$m\frac{d^2x}{dt^2} = -kx - \beta\frac{dx}{dt} \qquad \text{or} \qquad m\frac{d^2x}{dt^2} + \beta\frac{dx}{dt} + kx = 0$$

We may write it as follows;

$$\frac{d^2x}{dt^2} + \frac{\beta}{m}\frac{dx}{dt} + \frac{k}{m}x = 0$$

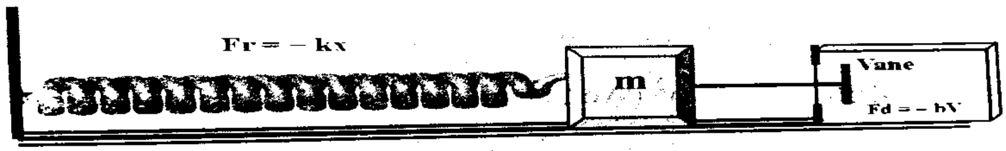
$$\ddot{x} + 2\gamma\dot{x} + \omega^2 x = 0 \qquad \text{using } \frac{\beta}{m} = 2\gamma \; ; \; \frac{k}{m} = \omega^2$$

Remark

- Damped Harmonic Oscillation $\ddot{x} + 2y\dot{x} + \omega^2 x = 0$ represent **over damped motion** if $\gamma^2 > \omega^2$. i.e. $\beta^2 > 4km$ and in this case equation $\ddot{x} + 2y\dot{x} + \omega^2 x = 0$ has the general solution of the following form
  $x = e^{-\gamma t}(Ae^{\propto t} + Be^{-\propto t})$ where $\propto = \sqrt{\gamma^2 - \omega^2}$
  And A,B are arbitrary constants can be found from the initial conditions.
- Damped Harmonic Oscillation $\ddot{x} + 2y\dot{x} + \omega^2 x = 0$ represent **critically damped motion** if $\gamma^2 = \omega^2$. i.e. $\beta^2 = 4km$ and in this case equation $\ddot{x} + 2y\dot{x} + \omega^2 x = 0$ has the general solution of the following form
  $x = e^{-\gamma t}(A + Bt)$
  And A,B are arbitrary constants can be found from the initial conditions.
- Damped Harmonic Oscillation $\ddot{x} + 2y\dot{x} + \omega^2 x = 0$ represent **under damped** or **damped oscillatory motion** if $\gamma^2 < \omega^2$. i.e. $\beta^2 < 4km$ and in this case equation $\ddot{x} + 2y\dot{x} + \omega^2 x = 0$ has the general solution of the following form
  $x = e^{-\gamma t}(Asin\lambda t + Bcos\lambda t) = Ce^{-\gamma t}cos(\lambda t - \varphi)$ where $\propto = \sqrt{\omega^2 - \gamma^2}$
  And where $C = \sqrt{A^2 + B^2}$ called the **amplitude**, and φ called the **phase angle** or **epoch**, can be determined from the initial conditions.

Equation of Motion of Damped Harmonic Oscillator

Consider a block of mass m is attached with one end of a string. The other end is connected with a mass less vane. The block is free to move to and fro over a frictionless horizontal surface as shown in figure.



Now displace the block towards right through some displacement and release. The block attached with spring having constant k takes to and fro motion under restoring force F given as

$$\vec{F}_r = -kx$$

The damping force experienced by vane when it moves in resistive medium is

$$\vec{F}_d = -\beta\vec{v}$$

Net Force $= \vec{F} = \vec{F}_r + \vec{F}_d = -kx - \beta\vec{v}$ ……………….(1)

$\vec{F} = m\vec{a} = m\frac{d^2x}{dt^2}$ ……………….(2) by Newton's 2nd Law

Comparing (1) and (2) we have

$m\frac{d^2x}{dt^2} = -kx - \beta\frac{dx}{dt}$ or $m\frac{d^2x}{dt^2} + \beta\frac{dx}{dt} + kx = 0$

We may write it as follows;

$$\frac{d^2x}{dt^2} + \frac{\beta}{m}\frac{dx}{dt} + \frac{k}{m}x = 0$$

$\ddot{x} + 2\gamma\dot{x} + \omega^2 x = 0$ using $\frac{\beta}{m} = 2\gamma$; $\frac{k}{m} = \omega^2$

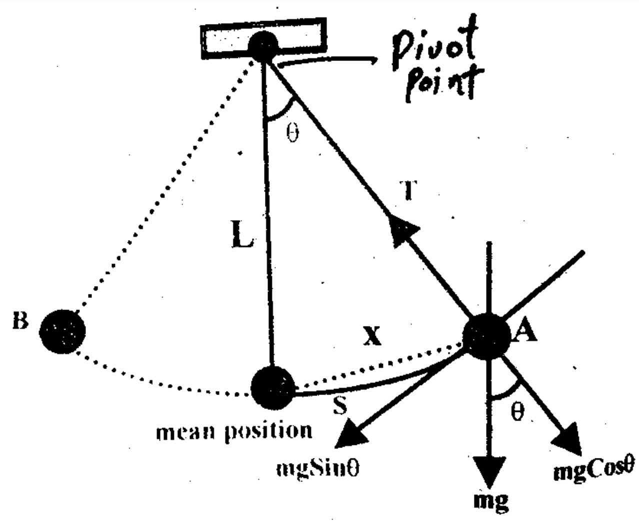
This is called equation of motion of damped harmonic oscillator . This type of motion is often called damped Harmonic Motion.

Simple Pendulum

The metallic bob suspended by a weightless inextensible string is called **simple pendulum**. The distance between point of suspension and center of bob is called **length of simple pendulum**. The bob at rest when no resultant force acts on it is called **mean position** or **equilibrium position**.

Equation of motion of a Simple Pendulum

Consider a bob of mass m attached with a string. The string is hanged vertically from a support as shown in figure;



Pull the pendulum from mean position to position A such that string makes a small angle θ with vertical. The bob starts moving toward mean position under restoring force when released. It gets maximum velocity at mean position and does not stop due to inertia but continues to move towards extreme position B. The velocity of bob becomes zero at position B due to restoring force.

The path followed by bob when it moves from mean position to position A is called an arc of circle having radius l. The arc length S and chord length x are approximately equal for small angle.

The forces acting on bob when it is at position A are

- Weight of bob acting vertically downward
- Tension acting along the string

Resolving weight force into components we get $\vec{F} = -mgsin\theta$

The negative sign means direction of $\vec{F}$ is opposite to direction of increasing θ and for small amplitude we have $sin\theta \approx \theta$

$\vec{F} = -mg\theta$ ……………….(1)

$\vec{F} = m\vec{a}$ ……………….(2) by Newton's 2nd Law

Comparing (1) and (2) we have

$m\vec{a} = -mg\theta \Rightarrow \vec{a} = -g\theta$

The relation $s = r\theta$ for circular path gives $x = l\theta$ then

$$\vec{a} = -g\left(\frac{x}{l}\right) \Rightarrow \vec{a} = -\left(\frac{g}{l}\right)x \Rightarrow \frac{d^2x}{dt^2} = -\left(\frac{g}{l}\right)x$$

$\ddot{x} + g\left(\frac{x}{l}\right) = 0$ (Equation of motion of a Simple Pendulum)

Resonance / Resonance Frequency

Resonant frequency is the oscillation of a system at its natural or unforced resonance. Resonance occurs when a system is able to store and easily transfer energy between different storage modes, such as Kinetic energy or Potential energy as you would find with a simple pendulum. A familiar example is a playground swing, which acts as a pendulum.

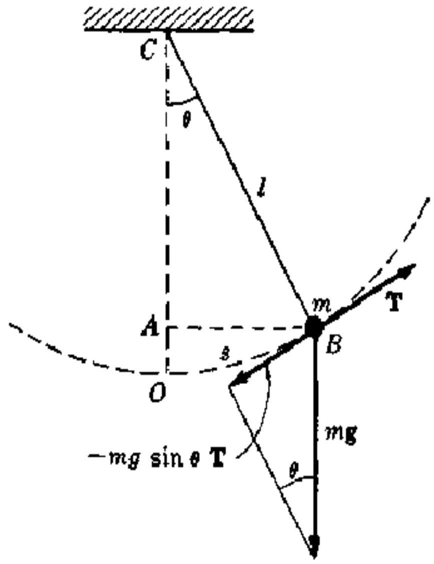
Forced Vibrations

Forced vibration occurs when motion is sustained or driven by an applied periodic force in either damped or undamped systems. Vibration of vehicles during the running on uneven roads, vibration of air compressors and musical instruments etc. are some of the examples for forced vibrations.

Question

Determine the motion of simple pendulum of length l and mass m assuming small vibrations and no resisting force.

Solution



Let the position of m at any time be determined by s,

the arc length measured from the equilibrium position O.

Let θ be the angle made by the pendulum string with the

vertical. If $\vec{T}$ is a unit tangent vector to the circular path of

the pendulum bob m, then by Newton's second law

$\vec{F} = m\vec{a} = m\frac{d^2s}{dt^2}\vec{T}$ ……………….(1)

Resolving force into components we get $\vec{F} = -mgsin\theta$

The negative sign means direction of $\vec{F}$ is opposite to direction of increasing θ and for small amplitude we have $sin\theta \approx \theta$

$\vec{F} = -mg\theta$ ……………….(2)

Comparing (1) and (2) we have

$$m\frac{d^2s}{dt^2}\vec{T} = -mg\theta \Rightarrow \frac{d^2s}{dt^2} = -g\theta \Rightarrow \frac{d^2}{dt^2}(l\theta) = -g\theta \qquad \because s = l\theta$$

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\theta \Rightarrow \frac{d^2\theta}{dt^2} + \frac{g}{l}\theta = 0$$

Which has solution $\theta = Acos\sqrt{\frac{g}{l}}t + Bsin\sqrt{\frac{g}{l}}t$

Using initial conditions $\theta = \theta_0, \frac{d\theta}{dt} = 0$ at $t = 0$ we get $A = \theta_0, B = 0$

$\Rightarrow \theta = \theta_0 cos\sqrt{\frac{g}{l}}t$. Here is time period $2\pi\sqrt{l/g}$

Energy of a Simple Harmonic Oscillator

If T is the kinetic energy, V the potential energy and E = T + V the total energy of a simple harmonic oscillator then we have

$$E = T + V = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$$

Question

Prove that the force $\vec{F} = -kx\hat{\imath}$ acting on a simple harmonic oscillator is conservative.

Solution: Given that $\vec{F} = -kx\hat{\imath}$ then

$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial x} & \frac{\partial}{\partial x} \\ -kx & 0 & 0 \end{vmatrix} = 0$. Thus the force $\vec{F} = \vec{A} \times \vec{r}$ is conservative.

Question

Find the potential energy of a simple harmonic oscillator.

Solution

In this case the potential or potential energy is given as $\vec{F} = -\nabla V$

$\Rightarrow -kx\hat{\imath} = -\nabla V \Rightarrow \nabla V = kx\hat{\imath} \Rightarrow \frac{\partial V}{\partial x}\hat{\imath} + \frac{\partial V}{\partial y}\hat{\jmath} + \frac{\partial V}{\partial z}\hat{k} = kx\hat{\imath}$

$\Rightarrow \frac{\partial V}{\partial x} = kx \ \dots\dots(1),\ \ \frac{\partial V}{\partial y} = 0 \ \dots\dots(2),\ \ \frac{\partial V}{\partial z} = 0 \ \dots\dots(3)$

$(1) \Rightarrow V = \frac{1}{2}kx^2 + c$

$\Rightarrow V = \frac{1}{2}kx^2$ using $V = 0$ for $x = 0$ we get $c = 0$

Question

Express in symbol the principal of conservation of energy for a simple harmonic oscillator.

Solution

We know that $m\frac{d^2x}{dt^2} = -kx$

$\Rightarrow m\frac{dv}{dt} = -kx \Rightarrow m\frac{dv}{dx}\frac{dx}{dt} = -kx \Rightarrow mv\frac{dv}{dx} = -kx$

$\Rightarrow E = T + V = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$ after integration

CHAPTER

5 CENTRAL FORCES AND PLANETARY MOTION

Central Force Fields

Suppose that a force acting on a particle of mass m is such that

i. It is always directed from m toward or away from fixed point.
ii. Its magnitude depends only on the distance from fixed point.

Then we call the force a central force or central force field. Mathematically it can be written as $\vec{F} = f(r)\frac{\vec{r}}{r}$. The central force is one of the attraction towards origin if $f(r) < 0$ or repulsion from origin if $f(r) > 0$.

Or If a particle is moving in an orbit under the influence of a force whose line of action passes through some fixed point, then such a force is called a central force or central force field and the fixed point is called its centre. The central force may be attractive or repulsive.

Properties of a Central Force Fields

If a particle moves in a central force field, then the following properties are valid;

i. The path or orbit of the particle must be a plane curve. i.e. particle moves in a plane.
ii. The angular momentum of the particle is conserved. i.e. constant.
iii. The particle moves in such a way that the position vector or radius vector drawn from Origin to the particle sweeps out equal areas in equal times. In other words, the time rate of change in area is constant. This is sometime called the law of areas.

Property

The path or orbit of the particle must be a plane curve. i.e. particle moves in a plane.

Proof

Let $\vec{F} = f(r)\frac{\vec{r}}{r}$ be the central force field then

$$\vec{r} \times \vec{F} = \vec{r} \times f(r)\frac{\vec{r}}{r} = 0 \Rightarrow \vec{r} \times m\frac{d\vec{v}}{dt} = 0$$

$$\Rightarrow \vec{r} \times \frac{d\vec{v}}{dt} = 0 \qquad \ldots\ldots\ldots\ldots\ldots(i)$$

Now $\vec{v} \times \vec{v} = 0 \Rightarrow \frac{d\vec{r}}{dt} \times \vec{v} = 0$

$$\Rightarrow \frac{d\vec{r}}{dt} \times \vec{v} = 0 \qquad \ldots\ldots\ldots\ldots\ldots(ii)$$

Adding (i) And (ii) we get

$$\vec{r} \times \frac{d\vec{v}}{dt} + \frac{d\vec{r}}{dt} \times \vec{v} = 0$$

$$\Rightarrow \frac{d}{dt}(\vec{r} \times \vec{v}) = 0$$

$\Rightarrow \vec{r} \times \vec{v} = \vec{h}$ where $\vec{h}$ is a constant vector.

$$\Rightarrow \vec{r}.(\vec{r} \times \vec{v}) = \vec{r}.\vec{h} \Rightarrow \vec{r}.\vec{h} = 0 \qquad \because \vec{a}.\left(\vec{a} \times \vec{b}\right) = 0$$

$$\Rightarrow \vec{r} \perp \vec{h}$$

This shows that the position vector of the particle at any time is perpendicular to the fixed constant vector $\vec{h}$ and Thus the path or orbit of the particle must be a plane curve. i.e. particle moves in a plane.

Property

The angular momentum of the particle is conserved. i.e. constant.

Or Prove that for a particle moving in a central force field the angular momentum is conserved.

Proof

Let $\vec{F} = f(r)\frac{\vec{r}}{r}$ be the central force field then

$$\vec{r} \times \vec{F} = \vec{r} \times f(r)\frac{\vec{r}}{r} = 0 \Rightarrow \vec{r} \times m\frac{d\vec{v}}{dt} = 0$$

$$\Rightarrow \vec{r} \times \frac{d\vec{v}}{dt} = 0 \qquad \ldots\ldots\ldots\ldots\ldots(i)$$

Now $\vec{v} \times \vec{v} = 0 \Rightarrow \frac{d\vec{r}}{dt} \times \vec{v} = 0$

$$\Rightarrow \frac{d\vec{r}}{dt} \times \vec{v} = 0 \qquad \ldots\ldots\ldots\ldots\ldots(ii)$$

Adding (i) And (ii) we get

$$\vec{r} \times \frac{d\vec{v}}{dt} + \frac{d\vec{r}}{dt} \times \vec{v} = 0$$

$$\Rightarrow \frac{d}{dt}(\vec{r} \times \vec{v}) = 0$$

$\Rightarrow \vec{r} \times \vec{v} = \vec{h}$ where $\vec{h}$ is a constant vector.

$$\Rightarrow m(\vec{r} \times \vec{v}) = m\vec{h} \Rightarrow \vec{r} \times m\vec{v} = m\vec{h} \Rightarrow \vec{r} \times \vec{P} = m\vec{h}$$

$$\Rightarrow \vec{L} = m\vec{h}$$

This shows that the angular momentum of the particle is conserved. i.e. constant. That is always constant in magnitude and direction.

Equation of motion for a particle in a Central Force Fields

Since we know that the path or orbit of the particle must be a plane curve. i.e. particle moves in a plane. Choose this plane to be the xy and the coordinates describing the position of the particle at any time t to be polar coordinates (r,θ). We have $\vec{a} = (\ddot{r} - r\dot{\theta}^2)\vec{r}_1 + (r\ddot{\theta} - 2\dot{r}\dot{\theta})\theta_1$ then

$$\vec{F} = m\vec{a} \Rightarrow \vec{F} = m\left[(\ddot{r} - r\dot{\theta}^2)\vec{r}_1 + (r\ddot{\theta} - 2\dot{r}\dot{\theta})\theta_1\right]$$

$$\Rightarrow m\left[(\ddot{r} - r\dot{\theta}^2)\vec{r}_1 + (r\ddot{\theta} - 2\dot{r}\dot{\theta})\theta_1\right] = f(r)\frac{\vec{r}}{r} = f(r)\vec{r}_1$$

Thus the required equations of motion are

$$m(\ddot{r} - r\dot{\theta}^2) = f(r) \qquad \text{and} \qquad m(r\ddot{\theta} - 2\dot{r}\dot{\theta}) = 0$$

Property (many questions covered)

The particle moves in such a way that the position vector or radius vector drawn from Origin to the particle sweeps out equal areas in equal times. In other words, the time rate of change in area is constant. This is sometime called the law of areas.

Or Prove that for a particle in central force field the areal velocity is constant.

Or Show that $r^2\dot{\theta} = h$, a constant. **Or** Show that $r^2\dot{\theta} = 2\dot{A}$.

Proof

From equations of motion awe have $m(r\ddot{\theta} - 2\dot{r}\dot{\theta}) = 0$

$$\Rightarrow \frac{m}{r}(r^2\ddot{\theta} - 2r\dot{r}\dot{\theta}) = 0 \Rightarrow \frac{m}{r}\frac{d}{dt}(r^2\dot{\theta}) = 0 \Rightarrow \boldsymbol{r^2\dot{\theta} = h}\text{, a constant.}$$

Also we know that $\Delta A = \frac{1}{2}|\vec{r}\times\Delta\vec{r}|$ for a parallelogram

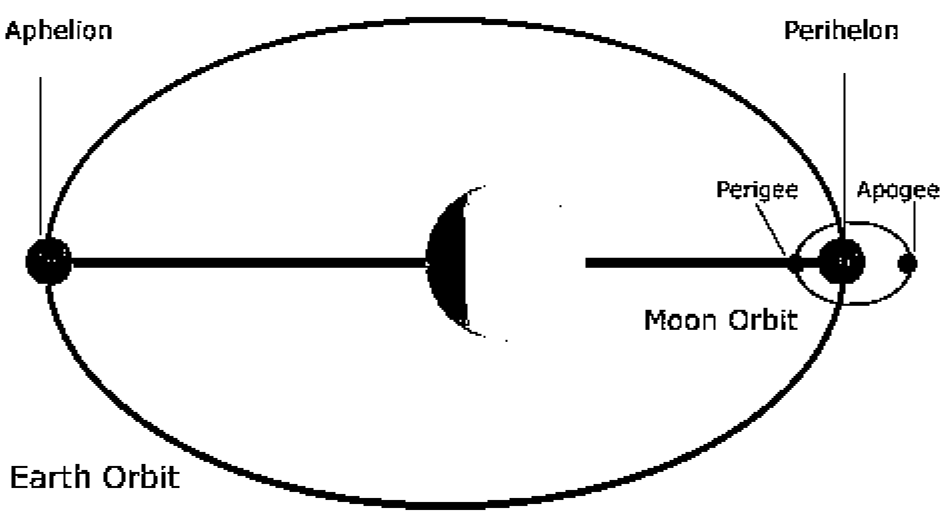
$$\Rightarrow \lim_{\Delta t\to 0}\frac{\Delta A}{\Delta t} = \frac{1}{2}\lim_{\Delta t\to 0}\left|\vec{r}\times\frac{\Delta\vec{r}}{\Delta t}\right| \Rightarrow \dot{A} = \frac{1}{2}|\vec{r}\times\vec{v}| = \frac{1}{2}r^2\dot{\theta} \Rightarrow \boldsymbol{r^2\dot{\theta} = 2\dot{A}}$$

$\dot{A} = \frac{1}{2}h$ combining above both equations.

This proves that for a particle in central force field the areal velocity is constant. Here $\dot{A} = \dot{A}\hat{k}$ is called areal velocity.

Useful Definitions

- **Orbits:** The path of planet or satellite is called its orbit. An orbit is a regular, repeating path that one object in space takes around another one. An object in an orbit is called a satellite**.** Orbit comes from the Latin orbita, "course," or "track."
- **Solar System:** A Solar System is composed of a star and objects called planets which revolve around it.
- **Satellites:** The **star** is an object which emits its own light, while the **planets** are the objects that do not emit light but can reflect it. And the objects revolving about the planets are called **satellites**.
- **Aphelion and Perihelion:** The largest and smallest distances of a planet from the sun about which it revolves are called the Aphelion and Perihelion respectively.
- **Apogee and Perigee:** The largest and smallest distances of a satellite around a planet about which it revolves are called the Apogee and Perigee respectively.
- **Period/Sidereal Period:** The time for one complete revolution of a body in an orbit is called its period. Sometime it is called sidereal period to distinguish it from other periods such as the period of earth's motion about its axis, etc.



Determination of the Orbit from the Central Force

If the central force is prescribed. i.e. if $f(r)$ is given, it is possible to determine the orbit or path of the particle. This orbit can be obtained in the form $r = r(\theta)$ or $r = r(t)$; $\theta = \theta(t)$ which are parametric equations in terms of time parameter.

Determination of Central Force from the Orbit

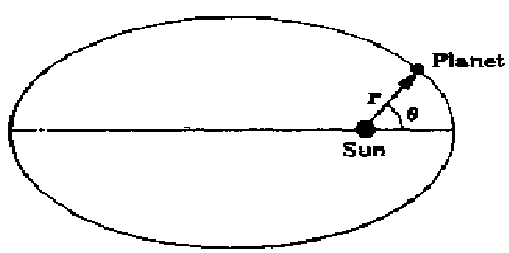
If we know the orbit or path of the particle, it is possible to determine the central force of the orbit. If the orbit is given by $r = r(\theta)$ or $u = u(\theta)$ where $u = \frac{1}{r}$, then the central force can be found by using the following equations;

$$f(r) = \frac{mh^2}{r^4}\left\{\frac{d^2r}{d\theta^2} - \frac{2}{r}\left(\frac{dr}{d\theta}\right)^2\right\} - r \qquad \text{Or} \qquad f\left(\frac{1}{u}\right) = -mh^2u^2\left\{\frac{d^2u}{d\theta^2} + u\right\}$$

Kepler's Law of Planetary Motion

Kepler's Three Laws of Planetary Motion are as follows;

1. Every planet moves in an orbit which is an ellipse with the sun at one focus.
2. The radius vector drawn from the sun to any planet sweeps out equal areas in equal time. (the law of areas)
3. The square of the periods of revolution of the planets are proportional to the cubes of the semi major axes of their orbits.



Remember

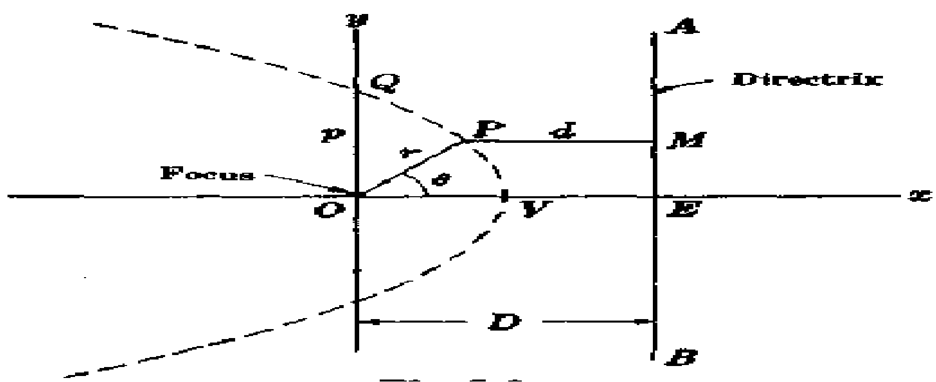
- **Equation of Conics** is $\frac{l}{r} = \epsilon(1 + cos\theta)$ or $\frac{p}{r} = \epsilon(1 + cos\theta)$
- If $\epsilon = 0$ we have $x^2 + y^2 = l^2$ a circle. If $\epsilon = 1$ we have $x^2 + y^2 = (l - x)^2$ or $y^2 = l^2 - 2lx$ a parabola. If $\epsilon < 1$ or $\epsilon > 1$ we have $x^2 + y^2 = (l - \epsilon x)^2$ or $(1 - \epsilon^2)x^2 + y^2 = l^2 + 2lx$ which is an ellipse if $\epsilon < 1$ and is a hyperbola if $\epsilon > 1$.

Question (Inverse Square Law of Attraction)

Prove that if a planet is to revolve around the sun in an elliptical path with the sun at a focus, then the central force necessary varies inversely as the square of the distance of the planet from the sun.

Solution

Consider a fixed point O and a fixed line AB distance D from O. Suppose that a point P in the plane of O and AB moves so that the ratio of its distance from point O to its distance from line AB is always equal to the positive constant ϵ, then the curve described by P is given by $r = \frac{p}{1+\epsilon cos\theta}$.



Similarly if the path is an ellipse with the sun at a focus, then calling r the distance from the sun, we have

$r = \frac{p}{1+\epsilon cos\theta}$ or $u = \frac{1}{r} = \frac{1}{p} + \frac{\epsilon}{p} cos\theta$

Where $\epsilon < 1$. Then the central force is given by

$$f\left(\frac{1}{u}\right) = -mh^2u^2\left\{\frac{d^2u}{d\theta^2} + u\right\} = -\frac{mh^2u^2}{p}$$

$f(r) = -\frac{mh^2}{pr^2} = -\frac{K}{r^2}$ replacing u by $\frac{1}{r}$

Proved that if a planet is to revolve around the sun in an elliptical path with the sun at a focus, then the central force necessary varies inversely as the square of the distance of the planet from the sun.

Kepler's First Law of Planetary Motion/Law of Orbit

Every planet moves in an elliptical orbit with the sun at one focus.

Proof

To derive the first law we will assume that the force of attraction between the planet and the sun is not only central but also obeys Newton's law of gravitation. For an inverse square law $f(r) = -k/r^2$ and $V(r) = -\int f(r)dr = -k/r$ where $k = GmM$ for the gravitational force; and with a similar expression for electrical force. Moreover k is positive for an attractive force and negative for a repulsive force.

The equation of the orbit can be obtained from the conservation laws for energy E and angular momentum L. The total energy E of the planet is given by

$$E = \frac{1}{2}m\dot{r}^2 + \frac{L^2}{(2mr^2)} + V(r) \quad \text{or} \quad \dot{r} = \sqrt{\frac{2}{m}\left(E - V - L^2/(2mr^2)\right)} \qquad (3.3.2)$$

and for angular momentum we have

$$\frac{d\theta}{dt} = \frac{L}{mr^2} \quad \text{or} \quad \theta = \int \frac{Ldt}{mr^2} + \text{constant} \qquad (3.3.3)$$

From (3.3.2)

$$dt = \frac{dr}{\sqrt{(2/m)(E - V(r)) - L^2/(m^2r^2)}}$$

Also $d\theta = (L/mr^2)dt$. On substituting the value of dt in this relation, writing $V = -k/r$, and integrating we obtain

$$\begin{aligned}\theta - \theta_0 &= \int_{r_0}^{r_1} \frac{Ldr}{mr^2\sqrt{(2/m)(E + k/r) - L^2/(m^2r^2)}} \\ &= \int_{r_0}^{r_1} \frac{Ldr}{r^2\sqrt{2mE + 2mk/r) - L^2/r^2}}\end{aligned}$$

With $u = 1/r$ we can express the last equation in the form

$$\theta = \theta_0 - \int_{u_0}^{u_1} \frac{du}{\sqrt{2mE/L^2 + 2mku/L^2 - u^2}} \qquad (3.3.4)$$

To perform the integration on the right side of (3.3.4), we simplify the integrand as follows.

$$
\begin{aligned}
\frac{2mE}{L^2} + 2mk\frac{u}{L^2} - u^2 &= -\left(u^2 - \frac{2mku}{L^2} - \frac{2mE}{L^2}\right) \\
&= -\left[u^2 - \frac{2mku}{L^2} + \left(\frac{mk}{L^2}\right)^2\right] + \left(\frac{mk}{L^2}\right)^2 + \frac{2mE}{L^2} \\
&= -\left(u - \frac{mk}{L^2}\right)^2 + \frac{2mE}{L^2} + \frac{m^2k^2}{L^4} \\
&= -\left[\left(u - \frac{mk}{L^2}\right)^2 - A^2\right]
\end{aligned}
$$

where $A^2 = m^2k^2/L^4 + 2mE/L^2$ Therefore (3.3.2) now becomes

$$
\begin{aligned}
-(\theta - \theta_0) &= \int \frac{d(u - mk/L^2)}{\sqrt{u - mk/L^2 - A^2}} \\
&= \int \frac{d(u - mk/L^2)}{\sqrt{u - mk/L^2 - A^2}} \\
&= \int \frac{d\xi}{\sqrt{\xi - A^2}}, \quad \xi = u - mk/L^2 \\
&= \cos^{-1}\frac{\xi}{A}
\end{aligned}
$$

Hence we can write

$$
\begin{aligned}
\cos(\theta - \theta_0) &= \frac{u - mk/L^2}{A}, \quad A^2 = \frac{2mE}{L^2} + \frac{m^2k^2}{L^4} \\
&= \frac{(L/mkr) - 1/L}{\sqrt{1 + 2L^4/mk^2}}
\end{aligned}
$$

wherefrom we obtain

$$
\frac{L}{mk}\frac{1}{r} = \frac{1}{L} + \sqrt{1 + 2L^4/mk^2}\cos(\theta - \theta_0)
$$

Comparing this equation with the equation of the ellipse

$$
\frac{\ell}{r} = 1 + e\cos(\theta - \theta_0) \tag{3.3.5}
$$

we find that $\ell = L^2/(mk)$, $e = \sqrt{1 + 2L^4/mk^2}$ are respectively the latus-rectum and eccentricity of the elliptical orbit.

Kepler's Second Law of Planetary Motion/ Law of Areas

The radius vector drawn from the sun to any planet sweeps out equal areas in equal time. In other words areal velocity of radius vector is a constant of motion.

Proof

From equations of motion awe have $m\left(r\ddot{\theta} - 2\dot{r}\dot{\theta}\right) = 0$

$\Rightarrow \frac{m}{r}\left(r^2\ddot{\theta} - 2r\dot{r}\dot{\theta}\right) = 0 \Rightarrow \frac{m}{r}\frac{d}{dt}\left(r^2\dot{\theta}\right) = 0 \Rightarrow \boldsymbol{r^2\dot{\theta} = h}$, a constant.

Also we know that $\Delta A = \frac{1}{2}|\vec{r} \times \Delta\vec{r}|$ for a parallelogram

$\Rightarrow \lim_{\Delta t\to 0}\frac{\Delta A}{\Delta t} = \frac{1}{2}\lim_{\Delta t\to 0}\left|\vec{r} \times \frac{\Delta\vec{r}}{\Delta t}\right| \Rightarrow \dot{A} = \frac{1}{2}|\vec{r} \times \vec{v}| = \frac{1}{2}r^2\dot{\theta} \Rightarrow \boldsymbol{r^2\dot{\theta} = 2\dot{A}}$

$\dot{A} = \frac{1}{2}h$ combining above both equations.

This proves that for a particle in central force field the areal velocity is constant. Here $\dot{A} = \dot{A}\hat{k}$ is called areal velocity.

The particle moves in such a way that the position vector or radius vector drawn from sun to the particle sweeps out equal areas in equal times. In other words, the time rate of change in area is constant. This is sometime called the law of areas.

Kepler's Third Law of Planetary Motion/ Law of Periods

The square of the periods of revolution of the planets are proportional to the cubes of the semi major axes of their orbits.

Proof

If a and b are the lengths of the semi – major and semi – minor axes, then the area of the ellipse is πab. Since the areal velocity has the magnitude $\frac{h}{2}$, the time taken to sweep over area πab, the period, is

$P = \frac{\pi ab}{h/2} = \frac{2\pi ab}{h} \Rightarrow P = \frac{2\pi m^{1/2}a^{3/2}}{K^{1/2}}$ using $b = a\sqrt{1-\epsilon^2}\,, p = a(1-\epsilon^2) = \frac{mh^2}{K}$

$\Rightarrow P^2 = \frac{4\pi^2 m a^3}{K}$

Hence the square of the periods of revolution of the planets are proportional to the cubes of the semi major axes of their orbits.

Apsides and Apsidal Angles for Nearly Circular Orbits

Apsides, Also called: apse. either of two points lying at the extremities of an eccentric orbit of a planet, satellite, etc, such as the aphelion and perihelion of a planet or the apogee and perigee of the moon. An apsis is the farthest or nearest point in the orbit of a planetary body about its primary body. The line of apsides is the line connecting the two extreme values.In physics Angle through which the radius vector rotates in going between two consecutive apsides is called the apsidal angle.

Motion in an Inverse Square Field

As we have seen, the planets revolve in elliptical orbits about the sun which is at one focus of the ellipse. In a similar manner, satellite (natural or man made) may revolve around planets in elliptical orbits. However, the motion of an object in an inverse square field of attraction need not always be elliptical but may be parabolic or hyperbolic. In such cases the object, such as a comet or meteorite, would enter the solar system and then leave but never return again.

Question

Prove that the speed v of the particle moving in an elliptical path in an inverse square field is given by $v^2 = \frac{K}{m}\left(\frac{2}{r} - \frac{1}{a}\right)$ where a is the semi major axis.

Solution

From theory (Spiegel book) we have $p = \frac{mh^2}{K} = a(1-\epsilon^2) = a\left(-\frac{2Emh^2}{K^2}\right)$ where $E = -\frac{K}{2a}$. And by conservation of energy using $V = -\frac{K}{r}$ we have

$$\frac{1}{2}mv^2 = E - V = -\frac{K}{2a} - \frac{K}{r}$$

$$v^2 = \frac{K}{m}\left(\frac{2}{r} - \frac{1}{a}\right)$$

Similarly we can show for a hyperbola $\qquad v^2 = \frac{K}{m}\left(\frac{2}{r} + \frac{1}{a}\right)$

While for a parabola $\qquad v^2 = \frac{2K}{mr}$

Example

Suppose a particle is subject to an inverse cube attractive force. Calculate the time taken by the particle to move from a distance d to the centre of force.

Solution

We choose the X-axis along the line of force, and the origin at the centre of force. Then the equation of motion can be written as

$$m\frac{d^2x}{dt^2} = -\frac{mk^2}{x^3} \tag{1}$$

To solve (1) we multiply both sides with $2dx/dt$ and integrate w.r.t. t

$$\begin{aligned}
\int \left(2\frac{dx}{dt}\frac{d^2x}{dt^2}\right) dt &= 2k^2 \int \frac{dx}{dt}\frac{1}{x^3}\, dt \\
\int \frac{d}{dt}\left(\frac{dx}{dt}\right)^2 dt &= -2k^2 \int \frac{dx}{dt}\frac{1}{x^3}\, dt + \text{constant} \\
\left(\frac{dx}{dt}\right)^2 &= -2k^2 \int \frac{dx}{x^3} + c \\
&= \frac{k^2}{x^2} + c
\end{aligned}$$

Now $dx/dt = 0$ when $x = d$, therefore

Escape velocity / Gravitational Escape

This is an application of energy conservation method, as an illustration of energy conservation methods, we consider the problem of the gravitational escape of a particle from the earth. The gravitational potential energy due to earth's attraction on a particle of mass m at a distance $r > r_e$ (earth's radius) from the earth's center with mass of earth M is

$$V(r) = -\int_r^\infty \frac{GMm}{r^2}dr$$

$$V(r) = -\frac{GMm}{r}$$ after simplification

According to the law of conservation of energy $\frac{1}{2}mv^2 - \frac{GMm}{r} = c$ (Constant)

Using initial $r = r_e, v = v_0$ we have $\frac{1}{2}mv_0^2 - \frac{GMm}{r_e} = c$ then

$$\frac{1}{2}mv^2 - \frac{GMm}{r} = \frac{1}{2}mv_0^2 - \frac{GMm}{r_e} \Rightarrow \frac{1}{2}v^2 - \frac{GM}{r} = \frac{1}{2}v_0^2 - \frac{GM}{r_e}$$

$$\Rightarrow v^2 = v_0^2 + 2\frac{GM}{r} - 2\frac{GM}{r_e} \Rightarrow v = \sqrt{v_0^2 + 2\frac{GM}{r} - 2}$$

$$\Rightarrow v = \sqrt{v_0^2 + 2GM\left(\frac{1}{r} - \frac{1}{r_e}\right)} \Rightarrow 0 = \sqrt{v_0^2 - \frac{2GM}{r_e}}$$ when $r \to \infty, v \to 0$

$$\Rightarrow v_0^2 = \frac{2GM}{r_e} \Rightarrow \vec{v}_0 = \sqrt{\frac{2GM}{r_e}} \quad \ldots\ldots..(1)$$

Now weight of a particle is equal to the gravitational force exerted on it by the earth. Therefore $\frac{GMm}{r_e^2} = mg$ which gives $GM = gr_e^2$ then

$$(1) \Rightarrow \vec{v}_0 = \sqrt{\frac{2gr_e^2}{r_e}} \Rightarrow \vec{v}_0 = \sqrt{2gr_e}$$

The particle will escape to infinity and $\vec{v}_0 = \sqrt{2gr_e}$ is called escape velocity of the particle.

Remember the magnitude of the escape velocity of an object from the earth's surface using $g = 9.80ms^{-2}, r_e = 6.38 \times 10^6 m$ is about $11kms^{-1}$.

CHAPTER

6

PLANER MOTION OF RIGID BODIES

Rigid Body

A rigid body is defined as a collection of particles such that distance between every pair of its constituent particles remains unchanged whatever the forces acting on it. This is a body which cannot be deformed by the external force acting on it.

- When a force is applied to an object/ system of particles, and if the object maintains its overall shape, then the object is called a rigid body.
- Gap between two fixed points on the rigid body remains same regardless of external forces exerted on it.
- We can neglect the deformation of such bodies.
- A rigid body usually has continuous distribution of mass.

Rigid Body – I: Those bodies in which angular momentum and angular velocities have different directions are called rigid bodies of type I.

Rigid Body – II: Those bodies in which angular momentum and angular velocities have same directions are called rigid bodies of type II.

Elastic Bodies

A body that regains its original dimension and shape when the externally applied force is removed is an Elastic body.

When a force is applied to a system of particles, it changes the distance be individual particles. Such systems are often called deformable or elastic bodies.

Examples

- A spring and rubber band are some common examples of elastic bodies.
- A wheel is a common example of rigid body.

Properties of Rigid Bodies

Following are some of the properties of the rigid bodies.

Degree of freedom

The number of coordinates required to specify the position of a system of one or more particles is called the number of degrees of freedom of the system. For example a particle moving freely in space requires 3 coordinates, e.g. (x, y, z), to specify its position. Thus the number of degrees of freedom is 3.
Similarly, a system consisting of N particles moving freely in space requires 3N coordinates to specify its position. Thus the number of degrees of freedom is 3N.

Translations/ Translational Motion of Rigid Body

Motion of a rigid body in a straight or curved line on the smooth or rough surface. A displacement of a rigid body is a direct change of position of its particles. Translational motion is the displacement of all particles of the body by the same amount and the line segment joining the initial and the final position of the particles represented by parallel vectors. Examples of translational motion are particles freely falling down to earth and the motion of a bullet fired from a gun.

Rotations/ Rotational Motion of Rigid Body

Motion of a rigid body about a fixed line or fixed point (centre of mass) in the space. Circular motion of a body about a fixed point or axis is called rotation. If during a displacement the points of the rigid body on some line remains fixed and all other are displaced through the same angle, then this displacement is called rotation. A rigid performs rotations around an imaginary line called a rotation axis. If the axis of rotation passes through the center of mass of the rigid body then body is said to spin or rotate upon itself. If a body rotates about some external fixed point is called revolution orbital motion of the rigid body. The example of revolution is the rotation of earth around sun and motion of moon around sun. Rotational motion concerns only with rigid bodies. The reverse rotation of a body (inverse rotation) is also a rotation. A wheel is common examples of rotation.

Introduction to General Plane Motion

The general plane motion of a rigid body can be considered as:

- Translational motion along the given fixed plane and rotational motion about a suitable axis perpendicular to the plane.
- This fixed axis is specifically chosen to pass through the center of mass of the rigid body.

Instantaneous Axis of Rotation

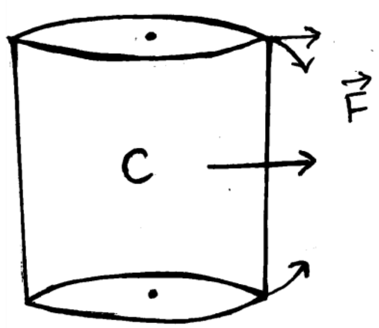
The axis about which the rigid body rotates is called instantaneous axis of rotation, where this axis is perpendicular to the plane. The line referred to is called the instantaneous axis of rotation. Rotations can be considered as finite or infinitesimal. Finite rotations cannot be represented by vectors since the commutative law fails. However, infinitesimal rotations can be represented by vectors.

Instantaneous Centre of Rotation

The point where instantaneous axis meets the fixed plane along which the body performs translation motion is described as the instantaneous centre of rotation.

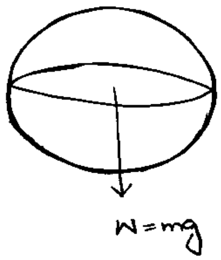
The Centre of Mass (c.m.) / Centroid of System

The centre of mass (c.m.) or centroid of system of particles is a hypothetical particle such that if the entire mass of the system were concentrated there, the mechanical properties would remain the same. In particular expression of linear momentum, angular momentum and kinetic energy assume simpler or more convenient forms when referred to the coordinated of this hypothetical particle and the equation of motion can be reduced to simpler equation of a single particle. **Centre of Mass** is a point where an applied force causes the system to move without any rotation. Its formula is $\vec{r}_{cm} = \frac{\sum_1^n r_i m_i}{\sum_1^n m_i}$



The Centre of Gravity

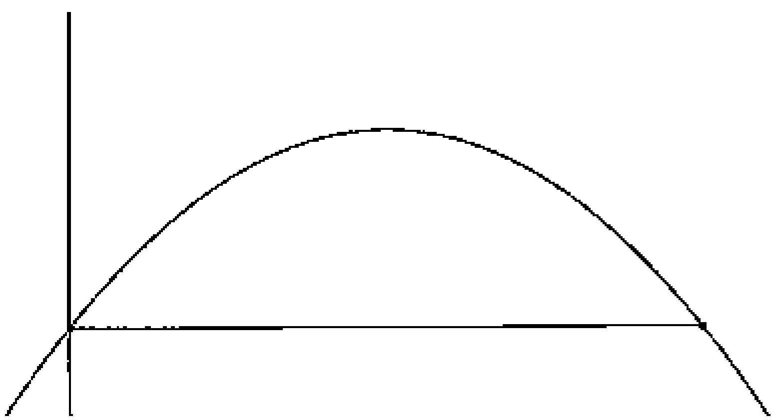
Centre of Gravity is a point where the whole weight of the system acts in the downward direction.



Motion of Centre of Mass

Motion of centre of mass can be examined by considering the following points:

1. If a system experiences no external force, the center-of-mass of the system will remain at rest, or will move at constant velocity if it is already moving.
2. If there is an external force, the center of mass accelerates according to $\vec{F} = m\vec{a}$.
3. Basically, the centre-of-mass of a system can be treated as a point mass, following Newton's Laws.
4. If an object is thrown into the air, different parts of the object can follow quite complicated paths, but the centre-of-mass will follow a parabola.



5. If an object explodes, the different pieces of the object will follow seemingly independent paths after the explosion. The centre of mass, however, will keep doing what it was doing before the explosion. This is because an explosion involves only internal forces.

Example

Find the center of mass of 3 particles having masses 2,4 and 3 grams are placed at points with position vectors $\hat{\imath}, 2\hat{\imath}-\hat{\jmath}, 3\hat{\imath}+4\hat{\jmath}-\hat{k}$ respectively.

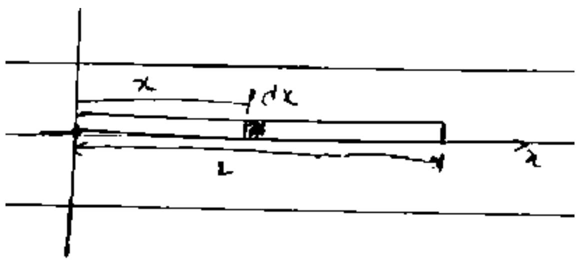
Solution

$$\vec{r}_{cm}=\frac{\sum_1^3 r_i m_i}{\sum_1^3 m_i}=\frac{r_1m_1+r_2m_2+r_3m_3}{m_1+m_2+m_3}=\frac{19}{9}\hat{\imath}-\frac{1}{9}\hat{\jmath}-\frac{12}{9}\hat{k}$$

Center of Mass when body is uniformly (continuous) distributed

When a body of mass M is uniformly distributed then it c.m is

- Along x – axis $=\bar{X}=\frac{\int xdm}{M}=\frac{\int xdm}{\sum_1^n m_i}$
- Along y – axis $=\bar{Y}=\frac{\int ydm}{M}=\frac{\int ydm}{\sum_1^n m_i}$
- Along z – axis $=\bar{Z}=\frac{\int zdm}{M}=\frac{\int zdm}{\sum_1^n m_i}$

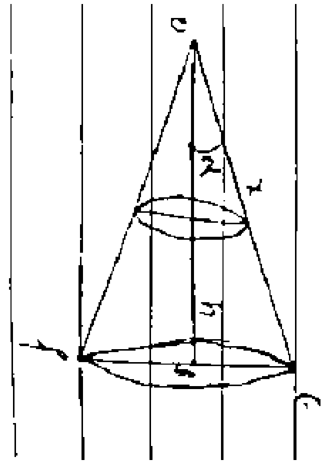


Example

Find the center of mass of rod of length l.

Solution

$$X_{cm}=\frac{\int_0^L xdm}{\int_0^L dm}=\frac{\int_0^L x\rho dx}{\int_0^L \rho dx}=\frac{\int_0^L x\frac{M}{L}dx}{\int_0^L\frac{M}{L}dx}=\frac{\int_0^L xdx}{\int_0^L dx}=\frac{L}{2}$$



Example

Find the center of mass of hollow right circular cone.

Solution

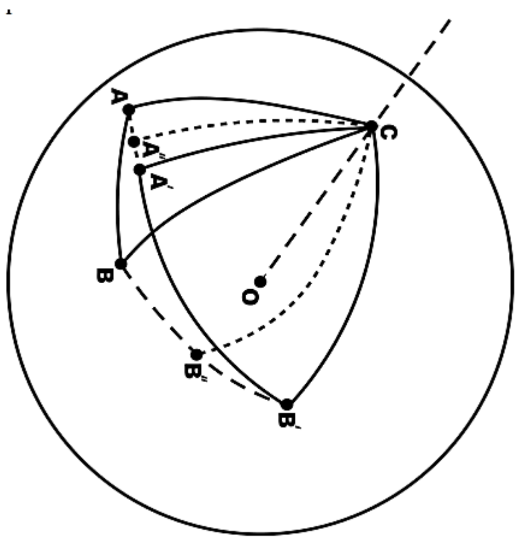
$$X_{cm}=\frac{\int_0^h xdm}{\int_0^h dm}=\frac{\int_0^h x\rho 2\pi x Cosx dx}{\int_0^h \rho 2\pi x Cosx dx}=\frac{\int_0^h x^2dx}{\int_0^h xdx}=\frac{2}{3}h$$

Euler's Theorem

A rotation of a rigid body about a fixed point of the body is equivalent to a rotation about a line which passes through the (fixed) point.

Proof

Let O be the fixed point in the body, which we take as a sphere S. Further, we take O at the center of the sphere. Let A, B be two distinct points on the sphere. As the body moves, the point O (on the axis) remains foxed and A and B suffer displacement.



Let A' and B' be the new locations of the points A and B after an infinitesimal time interval δt respectively. We join (A, B) and (A', B') by great circular areas. Also we join (A, B') and (B, B') by mean of great circular arcs. Let A'' and B'' draw axes at right angles, which meat at the point C on the sphere. We join C with A, B, A', B' by means of great circular arcs.

Consider the spherical triangles $\Delta CA'A''$ and $\Delta CAA''$. Obviously

i.	$m < CA''A \cong CA''A'$	each angle is right angle
ii.	$AA'' = A''A'$	A'' is the midpoint of AA'
iii.	CA'' is common to triangle.	
iv.	$\Delta CA''A \cong \Delta CA''A'$	(S.A.S) Postulate
v.	$CA = CA'$	corresponding sides of congruent triangles

…………………… (1)

Similarly Consider the spherical triangles $\Delta CB''B$ and $\Delta CB''B'$. Obviously

i.	$m < CB''B \cong CB''B'$	each angle is right angle
ii.	$BB'' = B''B'$	B'' is the midpoint of BB'
iii.	CB'' is common to triangle.	
iv.	$\Delta CB''B \cong \Delta CB''B'$	(S.A.S) Postulate
v.	$CB = CB'$	corresponding sides of congruent triangles

…………………… (2)

And $AB = A'B'$ distance between the fixed point on the sphere remain fixed

…………………… (3)

From (1), (2) and (3) we have

$AB = A'B'$

$CA = CA'$

$CB = CB'$

Then $\Delta CAB \cong \Delta CA'B'$

The portion of rigid body lying in ΔCAB has moved to $\Delta CA'B'$.

In this process the point O and C have remained fixed, although the later was at rest only instantaneously. Therefore the body has under gone a rotation about the axis OC.

Hence A rotation of a rigid body about a fixed point of the body is equivalent to a rotation about a line which passes through the (fixed) point.

Chasle's Theorem/ Mozzi - Chasle's Theorem

The most general rigid body displacement can be produced by a translation along a line (called its screw axis/ mozzi axis) followed (or preceded) by a rotation about that line.

Or The most general motion of a rigid body is that of translation and rotation.

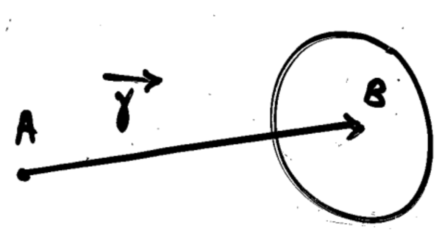
Or The most general motion of a rigid body is composed of pure translation followed by a rotation about some base point (fixed point).

Or Let $\vec{r}$ be a position vector of a base point A and $\vec{\omega}$ is angular velocity of any rigid body then motion $\vec{v}$ of rigid body is composed of pure translation followed by a rotation about some base point. i.e. $\vec{v} = \vec{v}_A + \vec{\omega} \times \vec{r}$

Explanation:

- A rigid body has six degrees of freedom.
- By Euler's theorem, three of these are associated with pure rotation.
- The remaining three must be associated with translation.
- To describe the general motion of a rigid body, think of the general motion as translation of a fixed point O in the body to a point O′ followed by the rotation about an axis through O′.

.**Proof**



Let $\vec{r}$ be a position vector of B from a base point A and $\vec{\omega}$ is angular velocity of rigid body then

$$\vec{r} = \vec{r}_{translation} + \vec{r}_{rotation}$$

$$\frac{d\vec{r}}{dt} = \frac{d\vec{r}_{tra}}{dt} + \frac{d\vec{r}_{rot}}{dt}$$

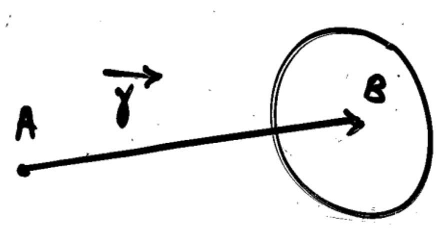
$$\boldsymbol{\vec{v} = \vec{v}_A + \vec{\omega} \times \vec{r}}$$

Question (Equation of Axis of Rotation in Screw Motion such that $\vec{v} \parallel \vec{\omega}$)

Explain the term Screw Motion, also show that the general motion of a rigid body is screw motion.

Solution

The motion which consists of translation and rotation about a line along the translation is called **Screw Motion**. Or the motion of an object in which linear and angular velocities are in the same direction (or Parallel) is called **Screw Motion**. In this motion linear velocity of each particle on the axis of rotation is parallel (or antiparallel) to the angular velocity. In case of screw motion we have $\vec{r} = \vec{a} + \lambda\vec{\omega}$. To prove this consider a rigid body in general motion.



Let $\vec{r}$ be a position vector of B from a base point A and $\vec{\omega}$ is angular velocity of rigid body then linear velocity of B is as follows

$\vec{v}_B = \vec{v}_A + \vec{\omega} \times \vec{r}$ ……………..(1)

In general $\vec{v}$ and $\vec{\omega}$ are not parallel, that we can choose B such that the linear velocity $\vec{v}_B$ of B is parallel to the angular velocity $\vec{\omega}$ of the rigid body.

$\Rightarrow \vec{\omega} \times \vec{v}_B = \vec{\omega} \times \vec{v}_A + \vec{\omega} \times (\vec{\omega} \times \vec{r})$ taking cross product of $\vec{\omega}$ with (1)

$\Rightarrow 0 = \vec{\omega} \times \vec{v}_A + \vec{\omega} \times (\vec{\omega} \times \vec{r})$ since $\vec{\omega} \parallel \vec{v}_B$

$\Rightarrow \vec{\omega} \times \vec{v}_A + (\vec{\omega}.\vec{r})\vec{\omega} - (\vec{\omega}.\vec{\omega})\vec{r} = 0 \Rightarrow \vec{\omega} \times \vec{v}_A + (\vec{\omega}.\vec{r})\vec{\omega} - \omega^2\vec{r} = 0$

$\Rightarrow \omega^2\vec{r} = \vec{\omega} \times \vec{v}_A + (\vec{\omega}.\vec{r})\vec{\omega} \Rightarrow \vec{r} = \frac{\vec{\omega}\times\vec{v}_A}{\omega^2} + \frac{(\vec{\omega}.\vec{r})\vec{\omega}}{\omega^2}$

$\Rightarrow \vec{r} = \vec{a} + \lambda\vec{\omega}$ putting $\vec{a} = \frac{\vec{\omega}\times\vec{v}_A}{\omega^2}$, $\lambda = \frac{(\vec{\omega}.\vec{r})\vec{\omega}}{\omega^2}$

This is called the Equation of Axis of Rotation in Screw Motion such that $\vec{v} \parallel \vec{\omega}$

Question

A particle moves in a plane with constant angular speed (velocity). Show that its acceleration is perpendicular to its velocity.

Solution

$$\vec{v} = \vec{\omega} \times \vec{r} \Rightarrow \frac{d\vec{v}}{dt} = \frac{d}{dt}(\vec{\omega} \times \vec{r}) \Rightarrow \frac{d\vec{v}}{dt} = \vec{\omega} \times \frac{d\vec{r}}{dt} \Rightarrow \vec{a} = \vec{\omega} \times \vec{v}$$

This shows that in rotational motion acceleration is perpendicular to the velocity.

Question

A particle moves in a plane elliptical orbit by the position vector

$\vec{r} = 2bsin\omega t\hat{e}_1 + bcos\omega t\hat{e}_2$, then Find velocity and acceleration.

Solution

$$\vec{r} = 2bsin\omega t\hat{e}_1 + bcos\omega t\hat{e}_2 \Rightarrow \frac{d\vec{r}}{dt} = \vec{v} = 2b\omega cos\omega t\hat{e}_1 - b\omega sin\omega t\hat{e}_2$$

$$\Rightarrow \frac{d\vec{v}}{dt} = \vec{a} = -2b\omega^2 sin\omega t\hat{e}_1 - b\omega^2 cos\omega t\hat{e}_2$$

Question Calculate angular speed of the Earth.

Solution

Time of Earth relation $= 24$ hour $= 24 \times 60 \times 60$ sec $= 84600$ sec

Rotating angle $= \theta = 2\pi$

Angular speed $= \omega = \frac{\theta}{t} = \frac{2\pi}{84600} = 0.000073$ rad/sec

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Question Calculate angular speed of the second hand of a watch.

Solution

Time $= 60$ sec ; Rotating angle $= \theta = 2\pi$

Angular speed $= \omega = \frac{\theta}{t} = \frac{2\pi}{60} = 0.105$ rad/sec

Varignon's Theorem

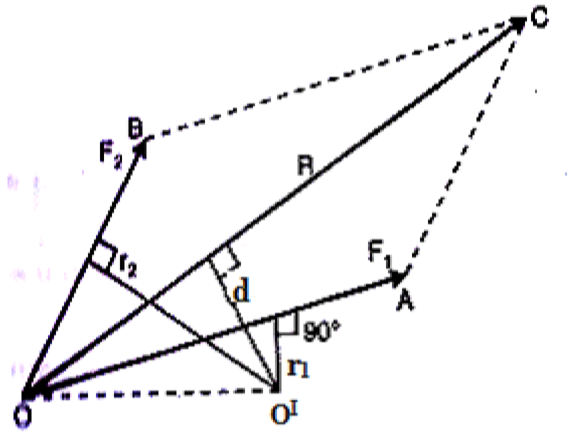
The moment of a force about any point is equal to the algebraic sum of the moments of its components about that point.

Or The moment of the resultant of a number of forces about any point is equal to the algebraic sum of the moments of all the forces of the system about the same point.

Or Torque acting on the system of particle is equal to the sum of all torque acting on each particle. i.e. $\tau = \sum_1^n \tau_i = \tau_1 + \tau_2 + \cdots + \tau_n$

This property was originally established by the French mathematician Varignon (1654–1722) long before the introduction of vector algebra, is known as Varignon's theorem.

Proof



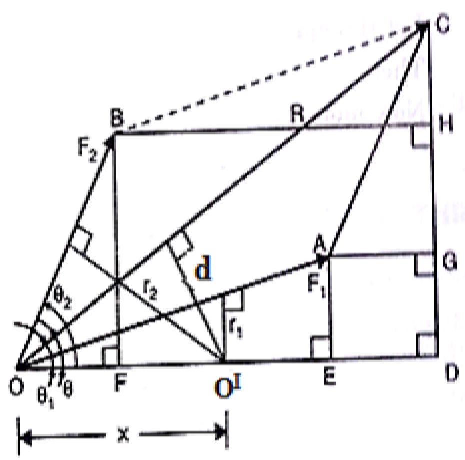


Fig. shows two forces F_1 and F_2 acting at point O. These forces are represented in magnitude and direction by OA and OB. Their resultant R is represented in magnitude and direction by OC which is the diagonal of parallelogram OACB. Let O' is the point in the plane about which moments of F_1, F_2 and R are to be determined. From point O', draw perpendiculars on OA,OC and OB.

Let r_1= Perpendicular distance between F_1 and O'.

d = Perpendicular distance between R and O'.

r_2 = Perpendicular distance between F_2 and O'.

Then according to Varignon's principle;

Moment of R about O' must be equal to algebraic sum of moments of F_1 and F_2 about O'.

$R \times d = F_1 \times r_1 + F_2 \times r_2$

Now refer to Fig. (b). Join OO' and produce it to D. From points C, A and B draw perpendiculars on OD meeting at D,E and F respectively. From A and B also draw perpendiculars on CD meeting the line CD at G and H respectively.

Let θ_1 = Angle made by F; with OD, θ = Angle made by R with OD, and θ_2 = Angle made by F_2 with OD.

In Fig.(b), OA = BC and also OA parallel to BC, hence the projection of OA and BC on the same vertical line CD will be equal i.e., GD = CH as GD is the projection of OA on CD and CH is the projection of BC on CD.

Then from Fig. (b), we have

$P_1 \sin\theta_1 = AE = GD = CH$

$F_1 \cos\theta_1 = OE$

$F_2 \sin\theta_1 = BF = HD$

$F_2 \cos\theta_2 = OF = ED$

(OB = AC and also OB || AC. Hence projections of OB and AC on the same horizontal line OD will be equal i.e., OF = ED)

$R\sin\theta = CD$

$R\cos\theta = OD$

Let the length $OO' = x$.

Then

$x\sin\theta_1 = r_1$, $x\sin\theta = d$ and $x\sin\theta_2 = r_2$

Now

Moment of R about O' = R × (distance between O' and R)

= R × d = R × xsinθ (d = xsinθ) = (R sin θ) × x

= CD × x

= (CH +HD)× x (R sin θ = CD) = (CH +HD)× x

= (F_1 sinθ_1 + F_2 sinθ_2) × x (CH = F_1 sin θ_1 and HD = F_2 sin θ_2)

= F_1 × x sinθ_1 + F_2 × x sinθ_2

= F_1 × r_1 + F_2 × r_2 (x sinθ_1 = r_1 and x sinθ_2 = r_2)

= Moment of F_1 about O' + Moment of F_2 about O'.

Hence moment of R about any point in the algebraic sum of moments of its components F_1 and F_2 about the same point.

Hence Varignon’s principle is proved.

The principle of moments (or Varignon’s principle) is not restricted to only two concurrent forces but is also applicable to any coplanar force system, i.e., concurrent or non-concurrent or parallel force system.

The Moment of Inertia

The moment of inertia of a rigid body is a property which depends upon its mass and shape, (i.e. the mass distribution of the body) and determines its behavior in rotational motion. In rotational motion, the moment of inertia plays the same role as the mass in linear motion.

Formally the moment of inertia I of the particle of mass m about a line is defined by $I = md^2$ where d is the perpendicular distance between the particle and the line (called the axis).

Moment of Inertia of System of particles

The moment of inertia of a system of particles, with masses $m_1, m_2, m_3, \dots, m_n$ about the axis AB is defined as $I = \sum_{i=1}^{n} m_i d_i^2$ and for continuous mass distribution (sum of partition of a function) we may use it as $I = \int r^2 \, dm$ where r is the perpendicular distance between the particle and the line (called the axis).

In dimensions, the moment of inertia can be expressed as $[I] = [M][L^2]$

Examples of the Moment of Inertia

- The moments of inertia of a ring of radius a about an axis through center is Ma^2
- The moment of inertia of a hoop of mass M and radius a about an axis passing through its center is Ma^2
- The moment of inertia of the sphere is $\frac{2}{5}Ma^2$
- Calculate the moment of inertia of a right circular cone about its axis of symmetry is $\frac{3}{10}Ma^2$
- The moment of inertia of a uniform rod of length l about an axis perpendicular to the rod and passing through an end point is $\frac{1}{3}Ma^2$
- The moment of inertia of a uniform triangular lamina of mass M about one of its sides is $\frac{1}{6}Mh^2$

Moment of Inertia in Coordinate System

The moment of inertia of a particle of mass m with coordinates (x, y, z) relative to the orthogonal Cartesian coordinate system $OXYZ$ about X, Y, Z axes will be

$$I_{xx} = \int (y^2 + z^2)dm = m(y^2 + z^2)$$

$$I_{yy} = \int (x^2 + z^2)dm = m(x^2 + z^2)$$

$$I_{zz} = \int (x^2 + y^2)dm = m(x^2 + y^2)$$

Product of Inertia

The product of inertia for the same particle w.r.to the pair of coordinate axes are defined as

$$I_{xy} = \int xydm = mxy\ ;\ I_{yz} = \int yzdm = myz\ ;\ I_{zx} = \int zxdm = mzx$$

It may be **positive**, may be **negative** or may be **zero**, depending on coordinate axes. These definitions can be easily generalized to a system of particle and a rigid body.

Parallel Axis Theorem

The rotational inertia about an axis is equal to the inertia about parallel axis through centre of mass plus mass time the square of the distance between two parallel axis.

i.e. $I = I' + Md^2$

Perpendicular Axis Theorem

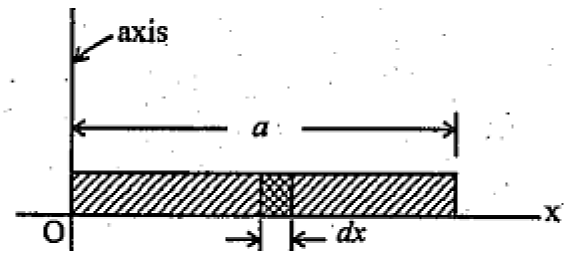
The moment of inertia of a plane rigid body about an axis perpendicular to the body is equal to the sum of the moment of inertia about two mutually perpendicular axes lying in the plane of the body and meeting at the common point with the given axis.

i.e. $I_{zz} = I_{xx} + I_{yy}$

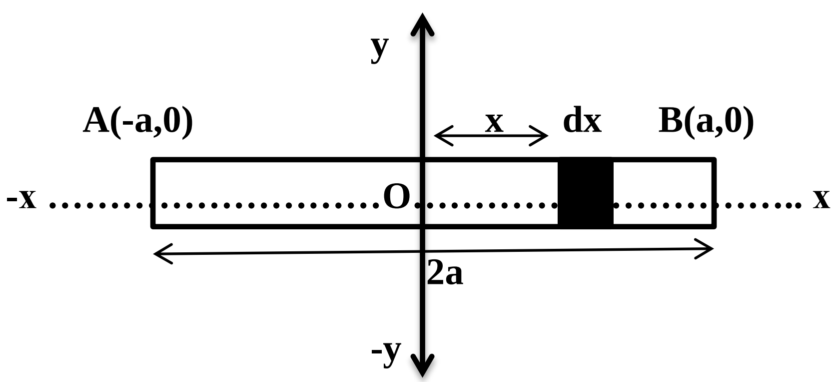
Question

Find moment of inertia of a thin rod of mass M of length 2a about a line through its centre and perpendicular to its length.

Or Calculate the moment of inertia of a uniform (rigid) rod of length l about an axis perpendicular to the rod and passing through an end point.



Solution



Consider a rod of length 2a along x – axis. Centre of their rod is origin as shown in figure.

Moment of inertia about y – axis for total length $= I = \int_{-a}^{a} x^2 \, dm$ ……………..(i)

Consider a small portion of the rod whose mass is dm and length dx, then linear mass density is $\rho = \frac{dm}{dx}$. i.e. $dm = \rho dx$

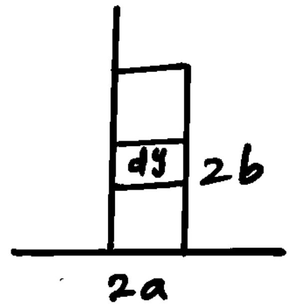
$(i) \Rightarrow I = \rho \int_{-a}^{a} x^2 \, dx \Rightarrow I = \frac{2}{3}\rho a^3$ ……………..(ii)

For whole mass of the rod $\rho = \frac{M}{2a}$. Then

$(ii) \Rightarrow I = \frac{2}{3} . \frac{M}{2a} . a^3 \Rightarrow \boldsymbol{I = \frac{1}{3} M a^2}$

Question

Show that moment of inertia of a uniform rectangular plate of sides 2a,2b about a corner are $\frac{4}{3}Ma^2, \frac{4}{3}Mb^2$. Also find same quantities at the centre.



Solution

We know that for a thin rod or strip, $dI = \frac{1}{3}l^2 dm$

$\Rightarrow dI = \frac{1}{3}(2a)^2 dm \Rightarrow dI = \frac{4}{3}a^2 dm$

Moment of inertia of plate about y – axis $= I_y = \frac{4}{3}\int_0^{2b} a^2\, dm$ ……………..(i)

Now by using area mass density $\rho = \frac{dm}{dA}$. i.e. $dm = \rho dA \Rightarrow dm = \rho 2ady$

$(i) \Rightarrow I_y = \frac{4}{3}\int_0^{2b} a^2\,(\rho 2ady) \Rightarrow I_y = \frac{16}{3}\rho b a^3$ ……………..(ii)

For whole mass of the plate $\rho = \frac{M}{A} = \frac{M}{2a.2b} = \frac{M}{4ab}$. Then

$(ii) \Rightarrow I_y = \frac{16}{3}.\frac{M}{4ab}.b.a^3 \Rightarrow \boldsymbol{I_y = \frac{4}{3}Ma^2}$

Moment of inertia of plate about x – axis $= I_x = \frac{4}{3}\int_0^{2a} b^2\, dm$ ……………..(iii)

Now by using area mass density $\rho = \frac{dm}{dA}$. i.e. $dm = \rho dA \Rightarrow dm = \rho 2bdx$

$(iii) \Rightarrow I_x = \frac{4}{3}\int_0^{2a} b^2\,(\rho 2bdx) \Rightarrow I_x = \frac{16}{3}\rho a b^3$ ……………..(iv)

For whole mass of the plate $\rho = \frac{M}{A} = \frac{M}{2a.2b} = \frac{M}{4ab}$. Then

$(iv) \Rightarrow I_x = \frac{16}{3}.\frac{M}{4ab}.a.b^3 \Rightarrow \boldsymbol{I_x = \frac{4}{3}Mb^2}$

Now by using perpendicular axis theorem

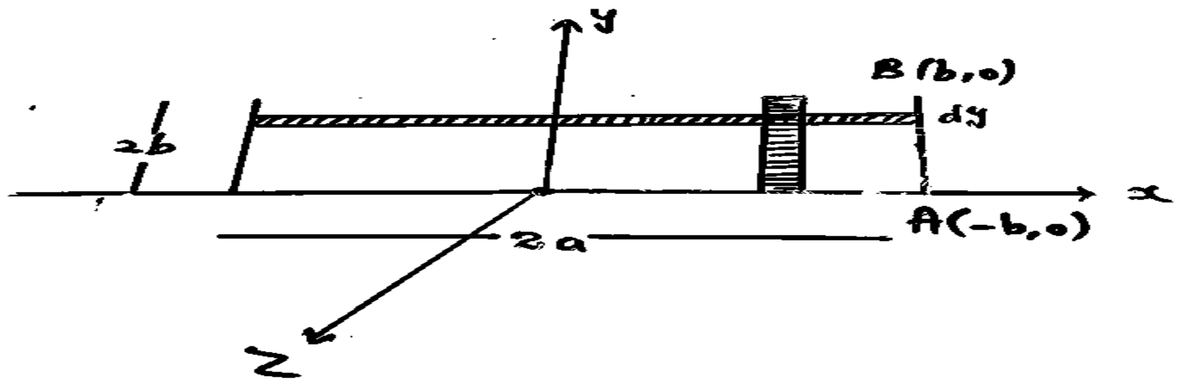
$I = I_{cm} + Md^2 \Rightarrow I_{cm} = I - Md^2$

$\Rightarrow I = \frac{4}{3}Ma^2 - Ma^2 \Rightarrow \boldsymbol{I = \frac{1}{3}Ma^2}$

Question

Find moment of inertia of a uniform rectangular plate of mass M and edges of lengths 2a,2b about a line passing through its centre, parallel to sides 2a,2b and perpendicular to its plane.

Solution



We know that for a thin rod or strip, $I = \frac{1}{3}Ma^2$

Moment of inertia of a strip of thickness dy at a distance y to the origin is given by $I_{oy} = \frac{1}{3}\int_{-b}^{b} a^2\, dm$ ……………..(i)

Now by using area mass density $\rho = \frac{dm}{dA}$. i.e. $dm = \rho dA \Rightarrow dm = \rho 2ady$

$(i) \Rightarrow I_{oy} = \frac{1}{3}\int_{-b}^{b} a^2\,(\rho 2ady) \Rightarrow I = \frac{4}{3}\rho ba^3$ ……………..(ii)

For whole mass of the plate $\rho = \frac{M}{A} = \frac{M}{2a.2b} = \frac{M}{4ab}$. Then

$(ii) \Rightarrow I_{oy} = \frac{4}{3}.\frac{M}{4ab}.b.a^3 \Rightarrow \boldsymbol{I_{oy} = \frac{1}{3}Ma^2}$

Moment of inertia of a strip of thickness dx at a distance x to the origin is given by $I_{ox} = \frac{1}{3}\int_{-a}^{a} b^2\, dm$ ……………..(iii)

Now by using area mass density $\rho = \frac{dm}{dA}$. i.e. $dm = \rho dA \Rightarrow dm = \rho 2bdx$

$(iii) \Rightarrow I_{ox} = \frac{1}{3}\int_{-a}^{a} b^2\,(\rho 2bdx) \Rightarrow I = \frac{4}{3}\rho ab^3$ ……………..(iv)

For whole mass of the plate $\rho = \frac{M}{A} = \frac{M}{2a.2b} = \frac{M}{4ab}$. Then

$(iv) \Rightarrow I_{ox} = \frac{4}{3}.\frac{M}{4ab}.a.b^3 \Rightarrow \boldsymbol{I_{ox} = \frac{1}{3}Mb^2}$

Now by using perpendicular axis theorem

Moment of inertia perpendicular to the plane $= I_{oz} = I_{ox} + I_{oy}$

$\Rightarrow I_{oz} = \frac{1}{3}Mb^2 + \frac{1}{3}Ma^2 \Rightarrow \boldsymbol{I_{oz} = \frac{1}{3}M(a^2 + b^2)}$

Question

Find moment of inertia of a square plate of mass M and length of each edge is 2a perpendicular to its plane.

Solution

Since we know that for a rectangular plate we have moment of inertia along x,y,z axes as follows;

$I_{ox} = \frac{1}{3}Mb^2$; $I_{oy} = \frac{1}{3}Ma^2$; $I_{oz} = \frac{1}{3}M(a^2 + b^2)$

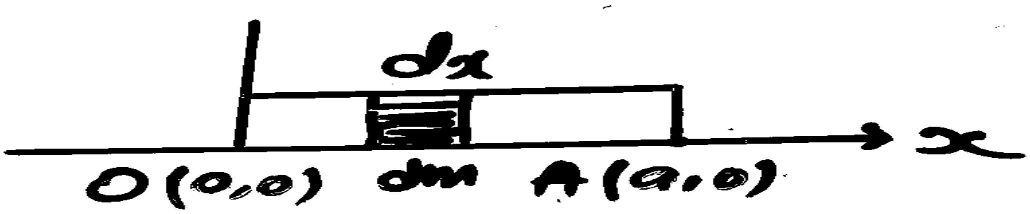
Then using $a = b$ in I_{oz} we have $I_{oz} = \frac{1}{3}M(a^2 + a^2) \Rightarrow I_{oz} = \frac{2}{3}Ma^2$

Question

Find the M.I. of a uniform rod AB of length a at the end of its extreme points.

Solution

Consider a uniform rod of length a along x – axis as shown in figure



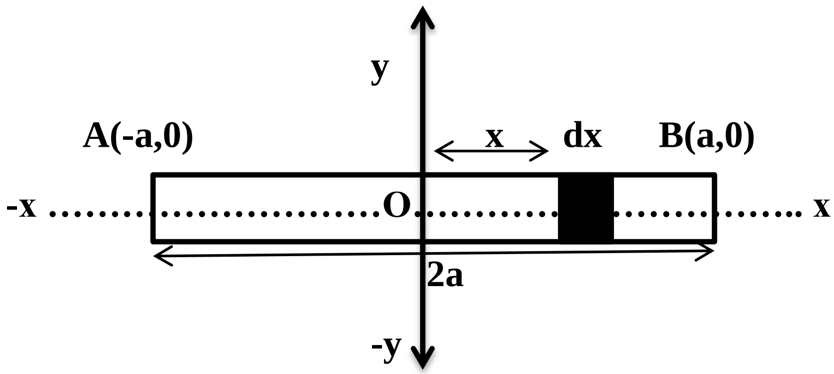
M.I. about y – axis $= I_{yy} = \int x^2 + z^2\, dm = \int x^2\, dm \qquad \because z = 0$ in xy - plane

$\Rightarrow I_{yy} = \int_0^a x^2 \rho dx \Rightarrow I_{yy} = \frac{1}{3}Ma^2$

Question

Calculate the moment of inertia of a uniform (rigid) rod of length l about an axis perpendicular to the rod and passing through a mid-point.

Solution



Consider a rod of length l along x – axis. Centre of their rod is origin as shown in figure.

Moment of inertia about y – axis for total length $= I = \int_{-a}^{a} x^2\, dm$ ……………..(i)

$\Rightarrow I = \rho \int_{-a}^{a} x^2\, dx \Rightarrow I = \frac{2}{3}\rho a^3 = \frac{1}{3}\boldsymbol{Ma^2}$ using $\rho = \frac{M}{2a}$

M.I. passing through mid – point

Using parallel axis theorem $I = I_0 + Md^2$

$\Rightarrow I_0 = I - Md^2 \Rightarrow I_0 = \frac{1}{3}Ma^2 - \frac{1}{4}Ma^2 \Rightarrow \boldsymbol{I_0 = \frac{1}{12}Ma^2}$

Question

Calculate the moment of inertia of a uniform (rigid) rod of length l about an axis passing through center without using parallel axis theorem.

Solution

Moment of inertia about y – axis for total length $= I = \int_{-\frac{l}{2}}^{\frac{l}{2}} x^2\, dm$ ……………..(i)

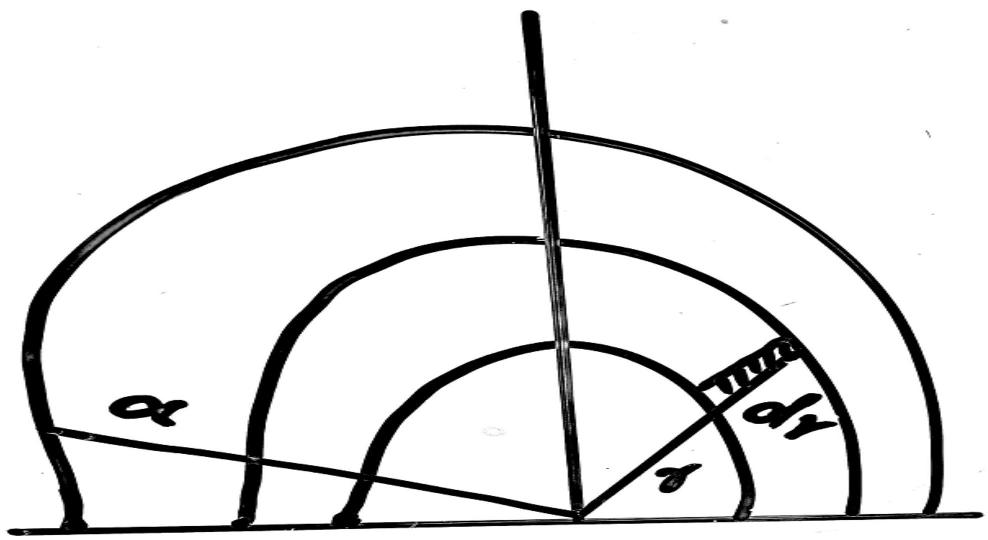
Consider a small portion of the rod whose mass is dm and length dx, then linear mass density is $\rho = \frac{dm}{dx}$. i.e. $dm = \rho dx$

$(i) \Rightarrow I = \rho \int_{-\frac{l}{2}}^{\frac{l}{2}} x^2\, dx \Rightarrow I = \frac{1}{12}Ml^2$ ……………..(ii) using $\rho = \frac{M}{l}$

Question

Find the moment of inertia of diameters through centre and perpendicular to the centre for semicircular lamina of mass m and radius a.

Solution



Consider a ring of radius r and thickness dr

M.I. of ring about its diameter $= dI = \frac{1}{2}r^2 dm$

M.I. of semi disk about x – axis $= I = \frac{1}{2}\int_0^a r^2\, dm$ ……………..(i)

Now by using area mass density $\rho = \frac{dm}{dA}$. i.e. $dm = \rho dA \Rightarrow dm = \rho\pi r dr$

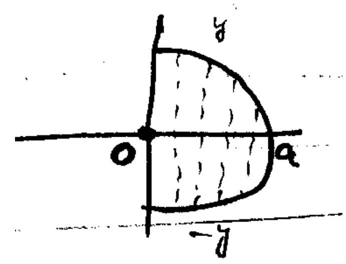
$(i) \Rightarrow I = \frac{1}{2}\int_0^a r^2 . \rho\pi r dr \Rightarrow I = \frac{\rho\pi}{2}\int_0^a r^3\, dr$

$\Rightarrow \boldsymbol{I = \frac{1}{4}Ma^2}$ using $\rho = \frac{M}{A} = \frac{M}{\frac{1}{2}\pi a^2}$

Question

Find the moment of inertia of diameters through centre and perpendicular to the centre for semi elliptical lamina of mass M and semi axes a, b

Solution



About x – axis

$I_{xx} = \int_{-y}^{y}\int_{0}^{a}(y^2 + z^2)\,dm$

$I_{xx} = \rho\int_{-y}^{y}\int_{0}^{a} y^2\,dxdy \Rightarrow I_{xx} = 2\rho\int_{0}^{a}\int_{0}^{y} y^2\,dydx$

$\Rightarrow I_{xx} = 2.\frac{M}{\frac{1}{2}\pi ab}.\int_{0}^{a}\int_{0}^{y} y^2\,dydx \Rightarrow I_{xx} = \frac{2}{3}.\frac{M}{\frac{1}{2}\pi ab}.\int_{0}^{a} y^3\,dx$

$\Rightarrow I_{xx} = \frac{2}{3}.\frac{M}{\frac{1}{2}\pi ab}.\int_{0}^{a}\left[\frac{b}{a}\sqrt{a^2 - x^2}\right]^3 dx$

For ellipse

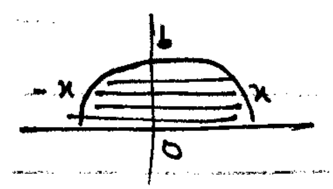
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$y = \frac{b}{a}\sqrt{a^2 - x^2}$$

Using $x = aSin\theta, dx = aCos\theta d\theta$

If $x \to 0, a$ then $\theta \to 0, \frac{\pi}{2}$ using all these assumptions we have $\boldsymbol{I_{xx} = \frac{1}{4}Mb^2}$

About y – axis



$I_{yy} = \int_{0}^{b}\int_{-x}^{x}(x^2 + z^2)\,dm$

$\Rightarrow I_{yy} = 2\rho\int_{0}^{b}\int_{0}^{x} x^2\,dxdy$

$\Rightarrow I_{yy} = 2.\frac{M}{\frac{1}{2}\pi ab}.\int_{0}^{b}\int_{0}^{x} x^2\,dxdy \Rightarrow I_{yy} = \frac{2}{3}.\frac{M}{\frac{1}{2}\pi ab}.\int_{0}^{b} x^3\,dy$

$\Rightarrow I_{yy} = \frac{2}{3}.\frac{M}{\frac{1}{2}\pi ab}.\int_{0}^{b}\left[\frac{a}{b}\sqrt{b^2 - x^2}\right]^3 dy$

For ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$x = \frac{a}{b}\sqrt{b^2 - x^2}$$

Using $x = bSin\theta, dx = bCos\theta d\theta$

If $x \to 0, b$ then $\theta \to 0, \frac{\pi}{2}$ using all these assumptions we have

$\Rightarrow \boldsymbol{I_{yy} = \frac{1}{4}Ma^2}$

Question

Find the moment of inertia about an axis through centre and perpendicular to the plane of the lamina.

Solution

We know that for semi elliptical lamina we have

$I_{xx} = \frac{1}{4}Mb^2$ and $I_{yy} = \frac{1}{4}Ma^2$ **(if long Q then find separately)**

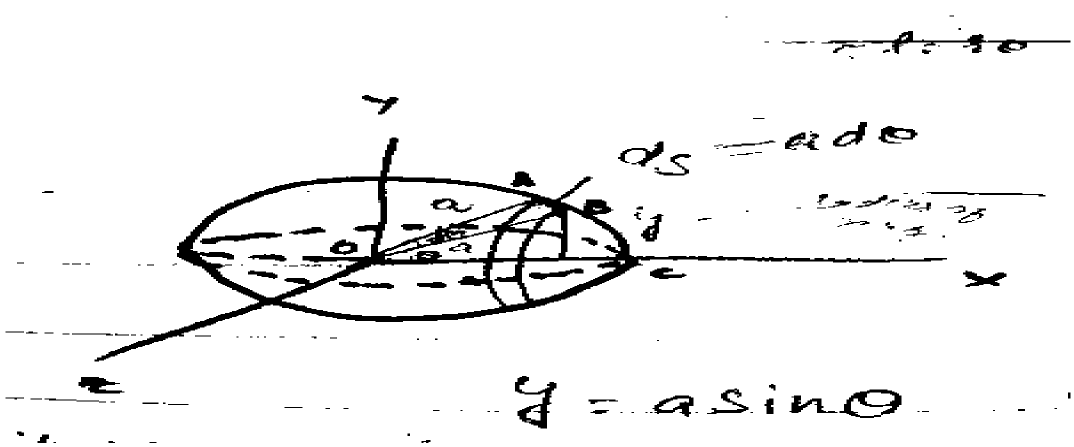
Using perpendicular axis theorem

$I_{zz} = I_{xx} + I_{yy} \Rightarrow I_{zz} = \frac{1}{4}Mb^2 + \frac{1}{4}Ma^2 \Rightarrow I_{zz} = \frac{1}{4}(a^2 + b^2)$

Question

Find the moment of inertia of a uniform spherical shell of mass M and radius a about any diameter.

Solution



Spherical shells consist of circular rings of different radii but same thickness.

Moment of inertia of one ring about x – axis diameter $= dI_x = y^2 dm$

M.I of spherical shell about its x – axis diameter $= I_x = \int_0^{\pi} y^2 \, dm$

Using $\rho = \frac{dm}{dA}$. i.e. $dm = \rho dA = \rho(2\pi y)ds = 2\pi\rho(aSin\theta)(ad\theta)$

$I_x = 2\pi\rho \int_0^{\pi}(aSin\theta)^2 \, (aSin\theta)(ad\theta)$

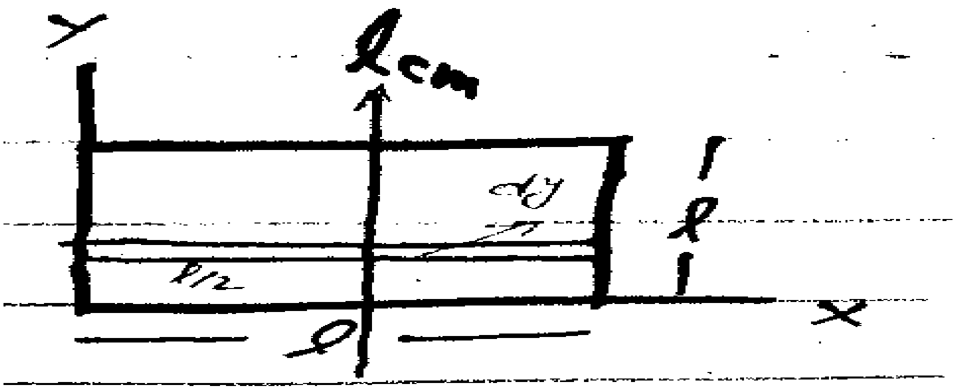
$\Rightarrow I_x = \frac{2}{3}Ma^2$ using $\rho = \frac{M}{4\pi a^2}$

Question

Find the moment of inertia of a uniform square plate about any axis through its centre and lying in the plane of the plate.

Or Prove that the moment of inertia about all lines through the centre of mass of a uniform square lamina and lying in its plane are equal.

Solution



Square plate consists of parallel plates (strips) with thickness dy with length l.

M.I. of one strip about y axis $= dI = \frac{1}{3}l^2 dm$

M.I. of square plate about an axis $= I = \frac{1}{3}\int_0^l l^2\, dm$ ……………..(i)

Now by using area mass density $\rho = \frac{dm}{dA}$. i.e. $dm = \rho dA \Rightarrow dm = \rho l dy$

$(i) \Rightarrow I = \frac{1}{3}\int_0^l l^2 . \rho l dy \Rightarrow I = \frac{\rho}{3}\int_0^l l^3\, dy$

$\Rightarrow \boldsymbol{I = \frac{1}{3}Ml^2}$ using $\rho = \frac{M}{A} = \frac{M}{l^2}$

Using parallel axis theorem

$I = I_{cm} + Md^2$

$I_{cm} = I - Md^2$

$I_{cm} = \frac{1}{3}Ml^2 - M\left(\frac{l}{2}\right)^2$

$\Rightarrow \boldsymbol{I_{cm} = \frac{1}{12}Ml^2}$

Question

Find the M.I. of a uniform elliptical plate with semi-major and semi-minor axes a, b (respectively) about (i) a major axis (ii) a minor axis (iii) about an axis through the centre and perpendicular to the plate.

Solution

We take the plate as $x^2/a^2 + y^2/b^2 = 1$. Then

$$\begin{aligned}
I_{xx} &= \int (\Delta m) y^2 = \int_{\text{plate}} \rho dS y^2 \\
&= \rho \int_{-a}^{a} \left(\int_{-y_1}^{y_1} y^2 dy \right) dx, \qquad y_1 = \frac{b}{a}\sqrt{a^2 - x^2} \\
&= \frac{2\rho}{a} \int_{-a}^{a} (y_1^2/3) dx \\
&= \frac{2\rho}{3} \int_{-a}^{a} \frac{b^3}{a^3} (a^2 - x^2)^{3/2} dx \\
&= \frac{4\rho b^3}{3a^3} \int_0^a (a^2 - x^2)^{3/2} dx
\end{aligned}$$

with $x = a\sin\theta$, this integral becomes

$$\begin{aligned}
I_{xx} &= \frac{4\rho b^3}{3a^3} \int_0^{\pi/2} a^3 \cdot \cos^3\theta (a\cos\theta d\theta) \\
&= \frac{4}{3}\rho b^3 a \int_0^{\pi/2} \cos\theta d\theta (a\cos\theta d\theta) \\
&= \frac{4}{3}\rho b^3 a \frac{1\times 3}{2\times 4} \times \frac{\pi}{2} \\
&= \frac{1}{4}\rho a b^3 \pi \\
&= \frac{1}{4}\pi a b^3 \times \frac{m}{\pi a b} = \frac{1}{4} M b^2
\end{aligned}$$

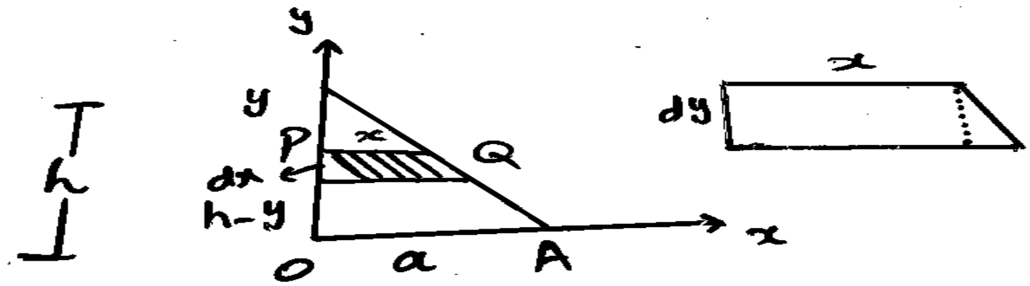
Similarly

$$I_{yy} = \frac{1}{4} M a^2, \qquad I_{zz} = \frac{1}{4} M (a^2 + b^2)$$

Question

Calculate the moment of inertia of a uniform triangular lamina about one of its edges (sides).

Solution



Consider a uniform triangular lamina AOB. Let $OB = h, OA = a$. Consider a strip of length x and thickness dy at a distance of $r = h - y$.

M.I. of strip about x – axis $= I_{xx} = \int r^2\, dm = \int (h-y)^2\, dm$ ……………..(i)

For uniform triangular lamina using $\rho = \frac{dm}{dA} = \frac{dm}{xdy} \Rightarrow dm = \rho x dy$

$(i) \Rightarrow I_{xx} = \int_0^h (h-y)^2 dm = \int_0^h (h-y)^2 \rho x dy$

$\Rightarrow I_{xx} = \rho \int_0^h (h-y)^2 x dy$ ……………..(ii)

Since ΔAOB and ΔQPB are similar, so $\frac{|PQ|}{|PB|} = \frac{|OA|}{|OB|} \Rightarrow \frac{x}{y} = \frac{a}{h} \Rightarrow x = \frac{ay}{h}$

$(ii) \Rightarrow I_{xx} = \rho \int_0^h (h-y)^2 . \frac{ay}{h} . dy$

$\Rightarrow I_{xx} = \frac{\rho a}{h} \int_0^h y(h^2 + y^2 - 2hy). dy = \frac{\rho a}{h} \left| h^2 \frac{y^2}{2} + \frac{y^4}{4} - 2h \frac{y^3}{3} \right|_0^h$

$\Rightarrow I_{xx} = \frac{\rho a}{h} \left(\frac{h^4}{12} \right) \Rightarrow \boldsymbol{I_{xx} = \frac{1}{12} a h^3 \rho}$

For whole triangle using $\rho = \frac{M}{A} = \frac{M}{\frac{1}{2} a h}$

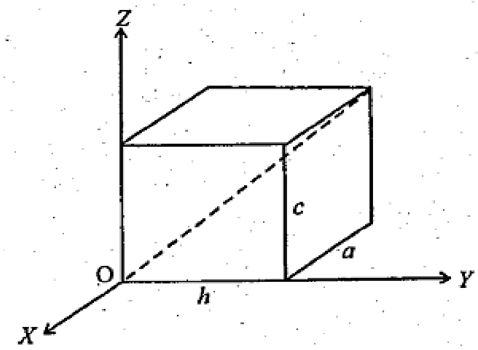
$\Rightarrow I_{xx} = \frac{1}{12} a h^3 . \frac{M}{\frac{1}{2} a h} \Rightarrow \boldsymbol{I_{xx} = \frac{1}{6} M h^2}$

Question

Calculate the inertia matrix of a uniform solid rectangular box (parallelepiped or cuboid) at one of its corners.

Solution

Let the lengths of the edges be a, b, c and let the axes be chosen along the edges as shown in the figure. By definition



$$I_{11} \equiv I_{xx} = \int \rho(\mathbf{r})(y^2 + z^2)dV$$

Since the box is made of uniform material, the density ρ must be constant. Therefore

$$\begin{aligned}
I_{xx} &= \rho \int_0^c \int_0^b \int_0^a (y^2 + z^2)dx\, dy\, dz \\
&= \rho \int_0^b \int_0^c (y^2 + z^2)dy\, dz \int_0^a dx \\
&= \rho a \left[\int_0^c \int_0^b y^2\, dy\, dz + \int_0^c \int_0^b z^2 dz\, dy\right] \\
&= \rho a \left(c\frac{b^3}{3} + b\frac{c^3}{3}\right) = \rho \frac{abc}{3}(b^2 + c^2) = \frac{M}{3}(b^2 + c^2)
\end{aligned}$$

Similarly

$$I_{22} = \frac{M}{3}(a^2 + c^2), \quad I_{33} = \frac{M}{3}(a^2 + b^2)$$

For the products of inertia we have

$$\begin{aligned} I_{12} &= -\int \rho\, x\, y\, dV = -\rho \int_0^c \int_0^b \int_0^a xy\, dx\, dy\, dz \\ &= -\rho \int_0^c dz \int_0^a x dx \int_0^b y dy \\ &= -\rho\, c \frac{a^2}{2}\frac{b^2}{2} = -\frac{a^2\, b^2\, c}{4}\frac{M}{abc} = -\frac{Mab}{4} \end{aligned}$$

Similarly $I_{23} = -Mbc/4$, $\;I_{31} = -Mca/4$.

Corollary

For a cube of side a, (taking $b=a$, $c=a$)

$$I_{11} = \frac{M}{3}(a^2 + a^2) = \frac{2}{3}Ma^2$$

Because of symmetry

$$I_{22} = I_{33} = \frac{2}{3}Ma^2$$

Similarly

$$I_{12} = I_{23} = I_{31} = -\frac{1}{4}Ma^2$$

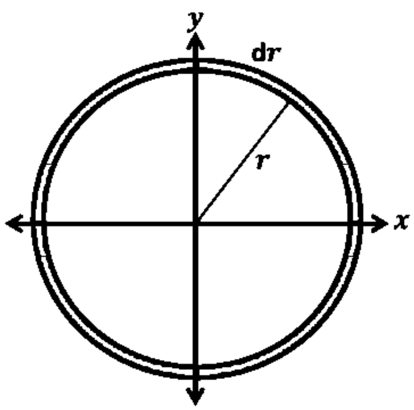
The inertia matrix for the cube can be displayed as

$$(I_{ij}) = \begin{bmatrix} 2Ma^2/3 & -Ma^2/4 & -Ma^2/4 \\ -Ma^2/4 & 2Ma^2/3 & -Ma^2/4 \\ -Ma^2/4 & -Ma^2/4 & 2Ma^2/3 \end{bmatrix} = \frac{1}{12}Ma^2 \begin{bmatrix} 8 & -3 & -3 \\ -3 & 8 & -3 \\ -3 & -3 & 8 \end{bmatrix}$$

Question

Calculate the moment of inertia of a hoop (circular disk, ring) of mass M and radius r about an axis passing through its center.

Solution



Consider a hoop of radius a and mass M.

Moment of inertia of the small portion of the hoop of mass dm about an axis through center and perpendicular to the plane of the ring equals

$I = \int r^2 \, dm$ ……………..(i)

We consider this hoop to be composed of small masses (dm) each of length ds.

$\rho = \frac{M}{A} = \frac{M}{2\pi r dr} = \frac{dm}{drds} \Rightarrow dm = \frac{M}{2\pi r} ds$

$(i) \Rightarrow I = \int r^2 \left(\frac{M}{2\pi r} ds\right)$

$\Rightarrow I = \frac{M}{2\pi r} r^2 \int ds = \frac{Mr}{2\pi} \int ds$

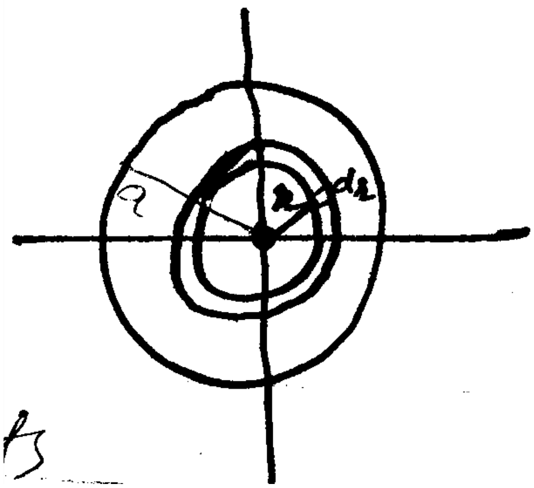
$\Rightarrow I = \frac{Mr}{2\pi} . 2\pi r$

$\Rightarrow \boldsymbol{I = Mr^2}$

Question

Find moment of inertia of a uniform circular plate or disk about its any diameter.

Solution



Disk consists of circular rings. Consider one ring of radius r and thickness dr.
Moment of inertia of a rings about its any diameter is $dI = \frac{1}{2}r^2 dm$

M.I. of a disk about its x – axis diameter $= I_x = \frac{1}{2}\int_0^a r^2\, dm$

$\rho = \frac{dm}{dA} = \frac{dm}{2\pi r dr} \Rightarrow dm = \rho 2\pi r dr$

$\Rightarrow I_x = \frac{\rho}{2}\int_0^a r^2\,(2\pi r dr)$

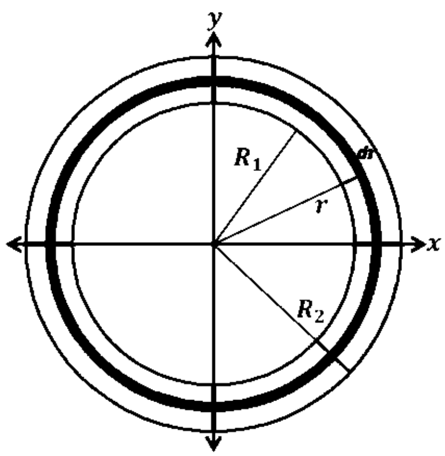
$\Rightarrow I_x = \frac{M/\pi a^2}{2}\int_0^a r^2\,(2\pi r dr)$

$\Rightarrow \boldsymbol{I_x = \frac{1}{4}Ma^2}$ similarly $\boldsymbol{I_y = \frac{1}{4}Ma^2}$

Question

Calculate the moment of inertia of annular disk of mass M. The inner radius of the annulus is R_1 and the outer radius is R_2 about an axis passing through its center.

Solution



Subdivide the annular disk into concentric rings one of which is shown in the fig. Let the mass of the ring is dm, and the radius be r, then the moment of inertia of the ring will be:

$I = \int r^2\, dm$ ……………..(i)

The Surface area of the ring is; Area $= (2\pi r)dr = 2\pi r dr$

Since the surface area of the annulus is $\pi(R_2^2 - R_1^2)$

Therefore, we can have $\frac{dm}{M} = \frac{2\pi r dr}{\pi(R_2^2 - R_1^2)} \Rightarrow dm = \frac{2 r dr}{(R_2^2 - R_1^2)} M$

$(i) \Rightarrow I = \int r^2 \left(\frac{2rdr}{(R_2^2 - R_1^2)} M\right) \Rightarrow I = \frac{2M}{R_2^2 - R_1^2} \int r^3 dr$

Thus the total M.I of the annulur disk will be

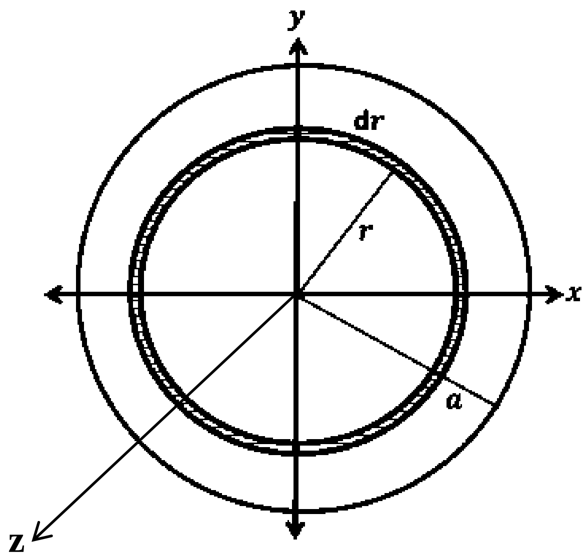
$\Rightarrow I = \frac{2M}{R_2^2 - R_1^2} \int_{r=R_1}^{R_2} r^3 dr$

$\Rightarrow \boldsymbol{I = \frac{1}{2} M\left(R_1^2 + R_2^2\right)}$

Question

Find the moment of inertia of a uniform circular disk of radius a, and mass M about the (axis of the disk) line through its centre and perpendicular to its plane.

Solution



Consider a uniform circular disk of radius a and mass M. Consider a ring on circular disk. Thickness of ring is dr and the distance from the origin is r.

Moment of inertia about z – axis $= I_z = \int_0^a r^2\, dm$ ……………..(i)

Now by using area mass density $\rho = \frac{dm}{dA}$. i.e. $dm = \rho dA \Rightarrow dm = \rho 2\pi r dr$

$(i) \Rightarrow I_z = \int_0^a r^2\, (\rho 2\pi r dr) \Rightarrow I_z = \frac{1}{2}\rho\pi a^4$ ……………..(ii) $\quad \because A = \pi r^2$

For whole mass $\rho = \frac{M}{A} = \frac{M}{\pi r^2}$. Then

$(ii) \Rightarrow I_z = \frac{1}{2}.\frac{M}{\pi r^2}.\pi a^4 \Rightarrow \boldsymbol{I_z = \frac{1}{2}Ma^2}$

In case of circular disk $I_x = I_y$ then by using perpendicular axis theorem

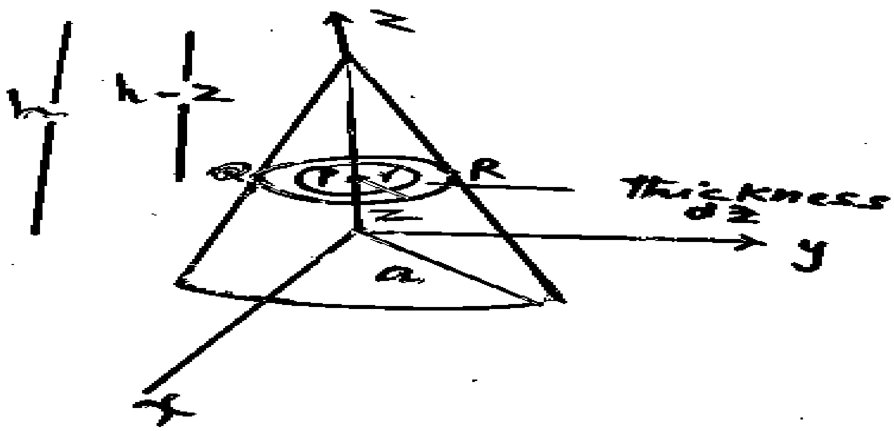
$I_z = I_x + I_y \Rightarrow I_z = I_x + I_x = 2I_x$

$\Rightarrow I_x = \frac{1}{2}I_z \Rightarrow \boldsymbol{I_x = \frac{1}{4}Ma^2}$ also $\boldsymbol{I_y = \frac{1}{4}Ma^2}$

Question

Calculate the moment of inertia of a right circular cone of height h and radius about its axis.

Solution



Let M be the mass, a the radius and h the height of right circular cone. We regard the cone as composed of elementary circular cylindrical discs of small thickness each parallel to the base of the cone. We choose the z-axis along the axis of symmetry, and consider a typical disc of radius r and width dz at a distance z from the base.

Moment of inertia of disc $= I = \frac{1}{2}\pi\rho h a^4$

Moment of inertia of disc $= I = \frac{1}{2}\pi\rho dz r^4$ ……………..(i) for our disc

From figure $\frac{h-z}{h} = \frac{r}{a} \Rightarrow r = a\left(\frac{h-z}{h}\right)$

$(i) \Rightarrow I = \frac{1}{2}\pi\rho dz\left(a\left(\frac{h-z}{h}\right)\right)^4$

Moment of inertia for whole cone about z axis $= I = \frac{1}{2}\frac{\pi\rho a^4}{h^4}\int_{z=0}^{h}(h-z)^4\,dz$

$\Rightarrow I = \frac{1}{10}\pi\rho a^4 h$ using $h - z = u$ with $u \to h, 0$ as $z \to 0, h$

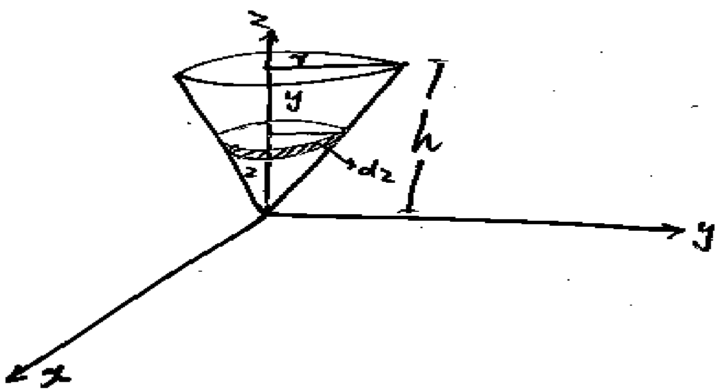
For whole mass of the cone $\rho = \frac{M}{V} = \frac{M}{\frac{1}{3}\pi a^2 h}$. Then

$\Rightarrow I = \frac{1}{10}\pi a^4 . \frac{M}{\frac{1}{3}\pi a^2 h} . h \Rightarrow I = \frac{3}{10}Ma^2$

Question

Calculate the moment of inertia of a right circular cone about its axis of symmetry.

Solution



Let M be the mass, y the radius and h the height of right circular cone. We regard the cone as composed of elementary circular discs of small thickness each parallel to the base of the cone. We choose the z-axis along the axis of symmetry, and consider a typical disc of radius r and width dz at a distance z from the base.

Moment of inertia about z axis $= I_{zz} = \frac{1}{2}\int_0^h y^2\, dm$ ……………..(i)

Now by using volume mass density $\rho = \frac{dm}{dV}$. i.e. $dm = \rho dV \Rightarrow dm = \pi y^2 \rho dz$

From figure $\frac{y}{z} = \frac{r}{h} \Rightarrow y = \frac{rz}{h}$

$(i) \Rightarrow I_{zz} = \frac{1}{2}\int_0^h y^4\, \pi\rho dz \Rightarrow I_{zz} = \frac{1}{2}\int_0^h \left(\frac{rz}{h}\right)^4 \pi\rho dz \Rightarrow I_{zz} = \frac{1}{10}\pi r^2 \rho h$ ……….(ii)

For whole mass of the cone $\rho = \frac{M}{V} = \frac{M}{\frac{1}{3}\pi r^2 h}$. Then

$(ii) \Rightarrow I_{zz} = \frac{1}{10}\pi r^2 . \frac{M}{\frac{1}{3}\pi r^2 h} . h \Rightarrow I_{zz} = \frac{3}{10} M r^2$ پچھلاسوال اس حصہ تک کیاجاسکتاہے

If θ be the semi vertical angle of the cone then $tan\theta = \frac{r}{h} \Rightarrow r = htan\theta$ then

$\Rightarrow \boldsymbol{I_{zz} = \frac{3}{10} M h^2 tan^2\theta}$

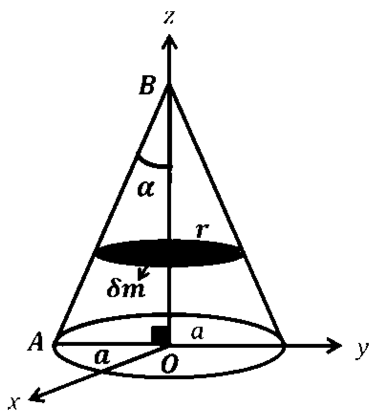
In this $I_{xx} = I_{yy}$ then by using perpendicular axis theorem
$I_{zz} = I_{xx} + I_{yy} \Rightarrow I_{zz} = I_{xx} + I_{xx} = 2I_{xx}$

$\Rightarrow I_{xx} = \frac{1}{2} I_{zz} \Rightarrow \boldsymbol{I_{xx} = \frac{3}{20} M h^2 tan^2\theta}$ also $\boldsymbol{I_{yy} = \frac{3}{20} M h^2 tan^2\theta}$

Question

Prove that the moment of inertia of a uniform right circular cone using parallel axis theorem of mass m, height h and semi vertical angle $\propto$ about a diameter of its base is $\frac{1}{20}Mh^2(3tan^2 \propto +2) = \frac{3M}{20}(3a^2 + 2h^2)$

Solution

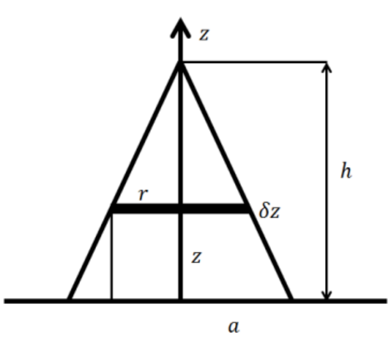


In the case of M.I about its diameter, we consider the elementary disc of mass δm whose moment of inertia about a diameter will be $\delta I_0 = \frac{1}{4}r^2\delta m.$

We note that the diameter passes through the center (which is also the centroid) of the elementary disc. Hence by parallel axis theorem, the M.I. δI of the elementary disc about a parallel axis (parallel diameter) at the base is given by

$$\delta I = \delta I_0 + (\delta m)z^2 = \frac{1}{4}r^2\delta m + \delta m z^2 = \delta m\left(\frac{1}{4}r^2 + z^2\right) = \pi r^2 \rho \delta z\left(\frac{1}{4}r^2 + z^2\right)$$

$$\delta I = \rho\pi\left(\frac{1}{4}r^4 + r^2z^2\right)\delta z$$



From the similar triangles, we have

$$\frac{r}{a} = \frac{h-z}{h} \quad \text{or} \quad r = a\left(\frac{h-z}{h}\right)$$

Therefore

$$\delta I = \rho\pi\left(\frac{1}{4}\left[a\left(\frac{h-z}{h}\right)\right]^4 + \left[a\left(\frac{h-z}{h}\right)\right]^2 z^2\right)\delta z$$

$$\delta I=\rho\pi\left[\frac{a^4}{4h^4}(h-z)^4+\frac{a^2}{h^4}(h-z)^2z^2\right]\delta z$$

$$\delta I=\rho\pi\left[\frac{a^4}{4h^4}(h-z)^4+\frac{a^2}{h^2}(h^2z^2-2hz^3+z^4)\right]\delta z$$

Therefore M.I of complete right circular cone about a diameter is given by

$$I=\rho\pi\int_0^h\left[\frac{a^4}{4h^4}(h-z)^4+\frac{a^2}{h^2}(h^2z^2-2hz^3+z^4)\right]\delta z$$

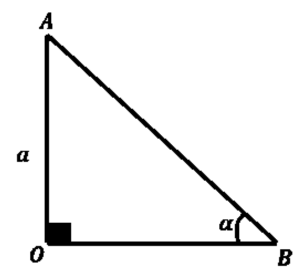
$$I=\rho\pi\left[\frac{a^4}{4h^4}\frac{h^5}{5}+\frac{a^2}{h^2}\frac{h^5}{30}\right]$$

$$I=\rho\pi\left[\frac{a^4h}{20}+\frac{a^2h^3}{30}\right]$$

Since we know that $\rho=\frac{M}{\frac{1}{3}\pi a^2h}$ therefore

$$I=\frac{M}{20}(3a^2+2h^2)$$

Since the semi vertical angle of the right circular cone is $\propto$, So by right triangle AOB, we have $tan\propto=\frac{a}{h}\Rightarrow a=htan\propto$ then



$$I=\frac{M}{20}(3(htan\propto)^2+2h^2)$$

$$I=\frac{1}{20}Mh^2(3tan^2\propto+2)$$

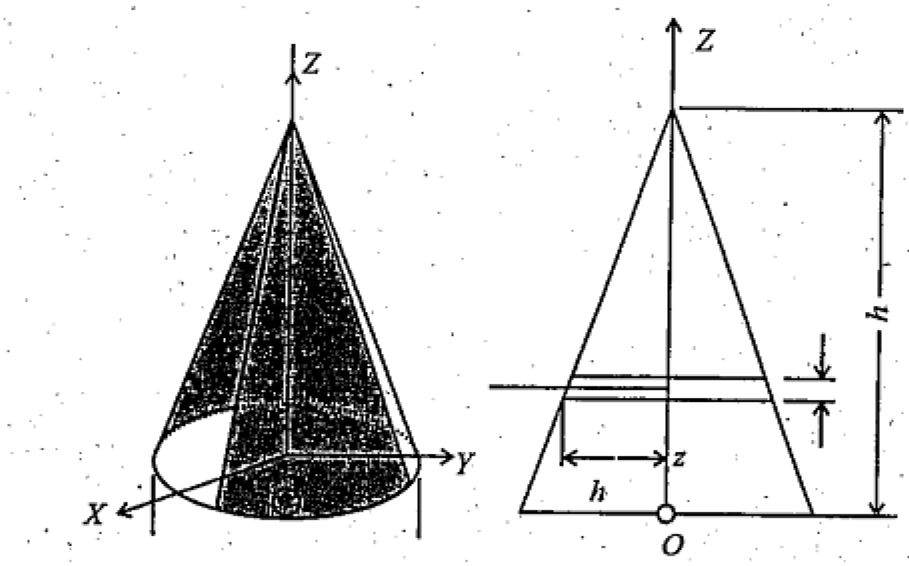
Question

Calculate the moment of inertia of a right circular cone about its axis of symmetry and about any diameter of the base.

Solution

Let M be the mass, a the radius and h the height of the right circular cone. We regard the cone as composed of elementary circular discs of small thickness each parallel to the base of the cone. We choose the Z-axis along the axis of symmetry, and consider a typical disc of radius r and width δz at a distance z from the base, (see figure).

Mass of the disc is given by $\delta m=\rho\pi r^2\delta z.$



From the similar triangles we have

$$\frac{r}{a} = \frac{h-z}{h} \quad \text{or} \quad r = a\,\frac{h-z}{h}$$

Therefore

$$\delta m = \rho\pi\left(a\,\frac{h-z}{h}\right)^2 \delta z$$

If δI denotes the M.I. of the elementary disc about the axis of symmetry, then (by example 3)

$$\delta I = \frac{1}{2}\,\delta m\, r^2$$

On substituting for δm and r, the M.I. of the cone about its axis of symmetry will be

$$\begin{aligned}
I_{sa} &= \frac{1}{2}\int_0^h \rho\pi\left(a\,\frac{h-z}{h}\right)^2 \delta z\left(a\,\frac{h-z}{h}\right)^2 \\
&= \frac{1}{2}\,\frac{\rho\pi a^4 h}{h^4}\int_0^h (h-z)^4\, dz \\
&= \frac{1}{2}\,\frac{\rho\pi a^4 h}{h^4}\left.\frac{(z-h)^5}{5}\right]_0^h \\
&= \frac{1}{2}\,\frac{\rho\pi a^4 h}{h^4}\,\frac{h^5}{5} = \frac{3}{10}\,M a^2
\end{aligned}$$

where we have used the result

$$\rho = \text{density of the cone} = \frac{M}{(1/3)\pi a^2 h}$$

M.I. of the cone about a diameter

In this case the M.I. of the elementary disc of mass δm about a diameter will be $\delta I_0 = (1/4)r^2\delta m$. We note that the diameter passes through the centre (which is also the centroid) of the elementary disc. Hence by parallel axis theorem, the M.I. δI of the elementary disc about a parallel axis (parallel diameter) at the base is given by

$$\begin{aligned}
\delta I &= \delta I_0 + (\delta m)z^2. \\
&= (1/4)r^2\delta m + (\delta m)z^2 = \delta m(r^2/4 + z^2) \\
&= \rho\pi(r^4/4 + r^2 z^2)\delta z \\
&= \rho\pi\left[\frac{a^4}{4h^4}(z-h)^4 + \frac{a^2}{h^2}(z^4 - 2hz^3 + 2h^2z^2)\right]\delta z
\end{aligned}$$

Therefore moment of inertia about a diameter is given by

$$\begin{aligned}
I_d &= \rho\pi\int_0^h\left\{\frac{a^4}{4h^4}(z-h)^4\right\}dz \\
&+ \rho\pi\int_0^h\left\{\frac{a^2}{h^2}(z^4 - 2hz^3 + 2h^2z^2)\right\}dz \\
&= \rho\pi\left(\frac{a^4}{4h^4}\frac{h^5}{5} + \frac{a^2}{h^2}\frac{h^5}{60}\right) \\
&= \frac{3M}{20}(3a^2 + 2h^2)
\end{aligned}$$

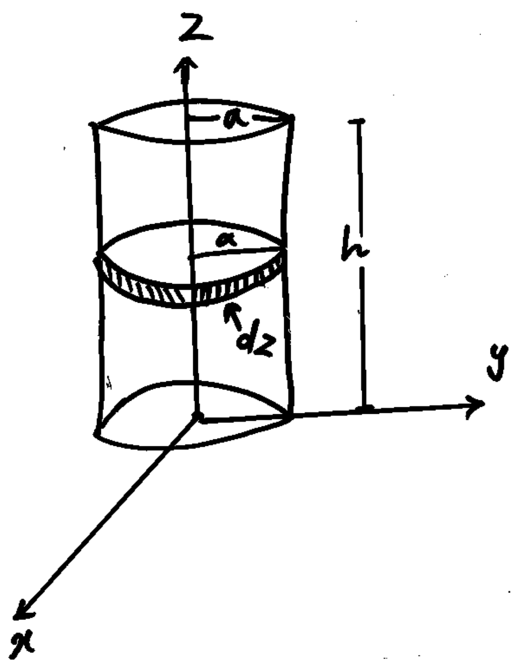
where we have substituted the value of ρ.

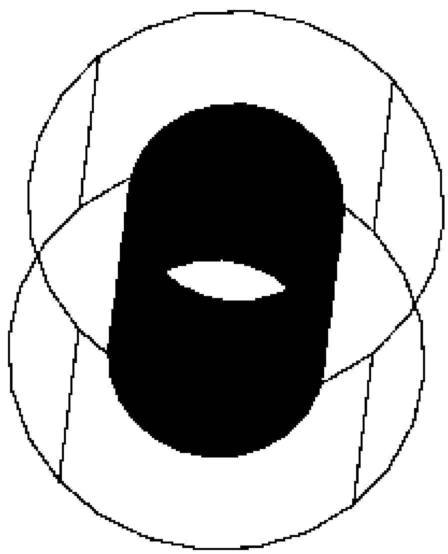
Question

To find the moment of inertia of a solid circular cylinder of radius a, mass M and the height of the cylinder h about the axis of the cylinder.

Or Calculate the moment of inertia of a uniform circular cylinder of height h and radius a with respect to its longitudinal axis.

Solution





Consider a cylinder of radius a, mass M and the height of the cylinder is h.
Consider a small disk of cylinder of thickness dz and z length from the origin.

Moment of inertia about z axis $= I_{zz} = \frac{1}{2}\int_0^h a^2\, dm$ ……………..(i)

Now by using volume mass density $\rho = \frac{dm}{dV}$. i.e. $dm = \rho dV \Rightarrow dm = \pi a^2 \rho dz$

$(i) \Rightarrow I_{zz} = \frac{1}{2}\int_0^h a^4\, \pi\rho dz \Rightarrow I_{zz} = \frac{\pi\rho a^4}{2}\int_0^h dz \Rightarrow I_{zz} = \frac{1}{2}\pi a^4 \rho h$ ……….(ii)

For whole mass of the cylinder $\rho = \frac{M}{V} = \frac{M}{\pi a^2 h}$. Then

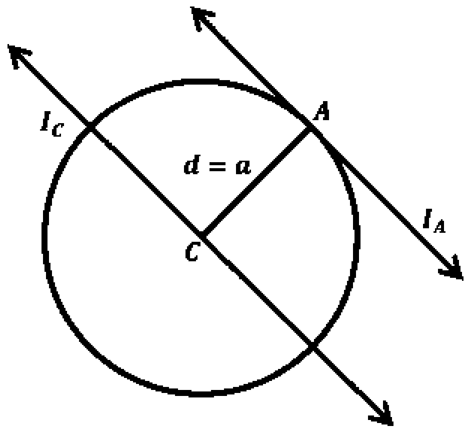
$(ii) \Rightarrow I_{zz} = \frac{1}{2}\pi a^4 . \frac{M}{\pi a^2 h} . h$

$\Rightarrow \boldsymbol{I_{zz} = \frac{1}{2}Ma^2}$

Question

Use the parallel axis theorem to find the moment of inertia of a solid circular cylinder about a line on the surface of the cylinder and parallel to axis of cylinder.

Solution



Suppose the cross section of cylinder as in figure. Then the axis of the cylinder is passing through the point C, while the line on the surface of cylinder is passing through A. So, we have to find out M.I of circular cylinder about a line passing through the point A whose radius is a (radius of circular cylinder) and mass is M.

By parallel axis theorem $I_A = I_C + Ma^2$ ………….(1)

Since I_C which is the moment of inertia of a solid circular cylinder about an axis passing from the center of mass is defined by $I_C = \frac{1}{2}Ma^2$ where a is the radius of a solid circular cylinder. Then

$(1) \Rightarrow I_A = \frac{1}{2}Ma^2 + Ma^2$

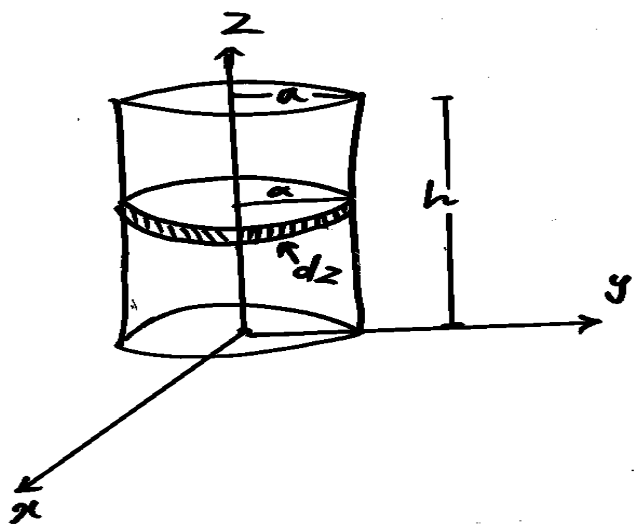
$\Rightarrow I_A = \frac{3}{2}Ma^2$

Question

Find the moment of inertia of a uniform circular cylinder of length h and radius a about an axis through the center and perpendicular to the central axis, namely I_{xx} or I_{yy}.

Or Calculate the moment of inertia of a uniform circular cylinder of height h and radius a about an axis through its centre of mass and perpendicular to its axis.

Solution



Consider a cylinder of radius a, mass M and the height of the cylinder is h. Consider a small disk of cylinder of thickness dz and z length from the origin.

Moment of inertia about z axis = $I_{zz} = \frac{1}{2}\int_0^h a^2\, dm$ ……………..(i)

Now by using volume mass density $\rho = \frac{dm}{dV}$. i.e. $dm = \rho dV \Rightarrow dm = \pi a^2 \rho dz$

$(i) \Rightarrow I_{zz} = \frac{1}{2}\int_0^h a^4\, \pi\rho dz \Rightarrow I_{zz} = \frac{\pi\rho a^4}{2}\int_0^h dz \Rightarrow I_{zz} = \frac{1}{2}\pi a^4 \rho h$ ………..(ii)

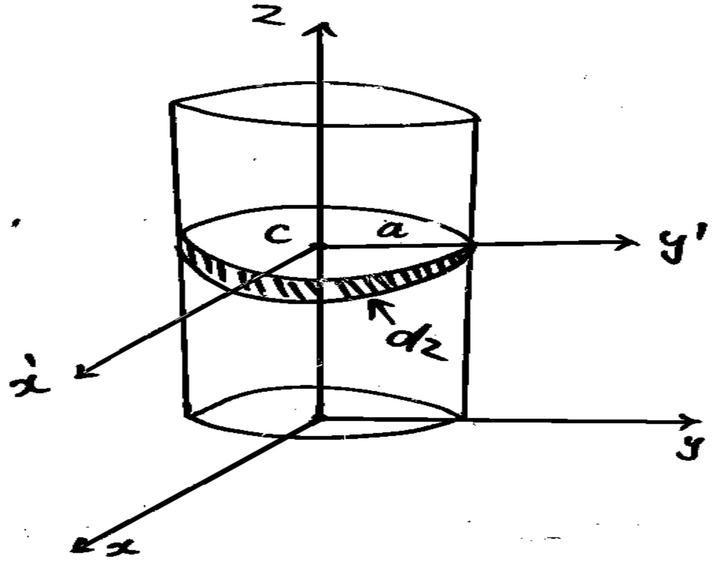
For whole mass of the cylinder $\rho = \frac{M}{V} = \frac{M}{\pi a^2 h}$. Then

$(ii) \Rightarrow I_{zz} = \frac{1}{2}\pi a^4 . \frac{M}{\pi a^2 h} . h \Rightarrow \boldsymbol{I_{zz} = \frac{1}{2}Ma^2}$

In this $I_{xx} = I_{yy}$ then by using perpendicular axis theorem
$I_{zz} = I_{xx} + I_{yy} \Rightarrow I_{zz} = I_{xx} + I_{xx} = 2I_{xx}$

$\Rightarrow I_{xx} = \frac{1}{2}I_{zz} \Rightarrow \boldsymbol{I_{xx} = \frac{1}{4}Ma^2}$ also $\boldsymbol{I_{yy} = \frac{1}{4}Ma^2}$

Let c be the centre of mass of the cylinder if the disc considered in the distance z from c.



Then moment of inertia about cy' is (by parallel axis theorem)

$\Rightarrow I_{cy'} = I_{cy} + Md^2$

$\Rightarrow I_{cy'} = \frac{1}{4}Ma^2 + Mz^2 \Rightarrow I_{cy'} = \frac{1}{4}\int dm\, a^2 + \int dm\, z^2$

$\Rightarrow I_{cy'} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \left(\frac{1}{4}a^2 + z^2\right) dm$ ……….(iii)

By using volume mass density $\rho = \frac{dm}{dV}$. i.e. $dm = \rho dV \Rightarrow dm = \pi a^2 \rho dz$

$(iii) \Rightarrow I_{cy'} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \left(\frac{1}{4}a^2 + z^2\right) \pi a^2 \rho dz$

$\Rightarrow I_{cy'} = \rho\pi a^2 \int_{-\frac{h}{2}}^{\frac{h}{2}} \left(\frac{1}{4}a^2 + z^2\right) dz = \rho\pi a^2 \left|\frac{1}{4}a^2 z + \frac{1}{3}z^3\right|_{-\frac{h}{2}}^{\frac{h}{2}}$

$\Rightarrow I_{cy'} = \frac{1}{12}\rho\pi a^2 h(3a^2 + h^2)$

For whole mass of the cylinder $\rho = \frac{M}{V} = \frac{M}{\pi a^2 h}$. Then

$\Rightarrow I_{cy'} = \frac{1}{12}.\frac{M}{\pi a^2 h}.\pi a^2 h(3a^2 + h^2)$

$\Rightarrow I_{cy'} = \frac{M}{12}(3a^2 + h^2)$

Question

Find the moment of inertia of a uniform hemisphere of mass M and radius a about

(i) Its axis of symmetry.

(ii) An axis perpendicular to the axis of symmetry and passing through the centre of the base.

Solution

We will use spherical polar coordinates (r, θ, ϕ). Their use makes computational work simpler. Their range of variation for the hemisphere will be

$$0 \le r < a, \qquad 0 \le \theta \le \pi/2, \qquad 0 \le \phi \le 2\pi$$

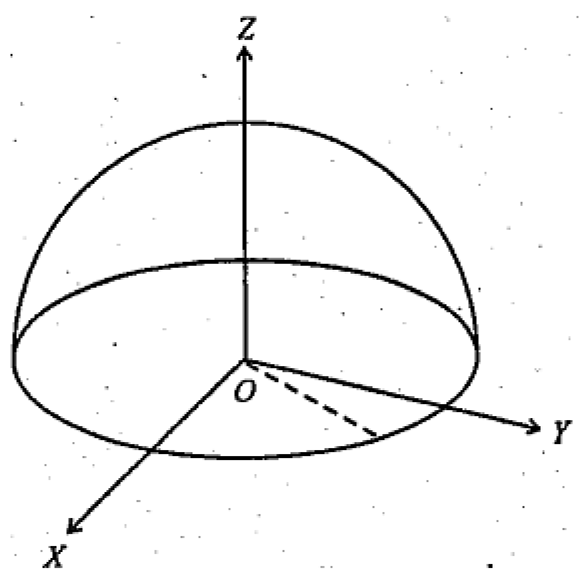
$$x = r\sin\theta \;\cos\phi, \quad y = r\sin\theta \;\sin\phi, \quad z = r\cos\phi$$

We choose the coordinate axes as shown in the figure.

M.I. about the axis of symmetry

Obviously the Z-axis is the axis of symmetry. Hence

$$I_{zz} = \int \rho(\mathbf{r})\,(x^2 + y^2)dV = \rho \int (x^2 + y^2)dV$$



Now in terms of spherical polar coordinates

$$\begin{aligned} x^2+y^2 &= r^2(\sin^2\theta\ \cos^2\phi + \sin^2\theta\ \sin^2\phi) \\ &= r^2\sin^2\theta(\cos^2\phi+\sin^2\phi) = r^2\sin^2\theta \end{aligned}$$

and the element of volume in spherical polar coordinates is given by

$$dV = dr(rd\theta)(r\sin\theta d\phi) = r^2\sin\theta\, dr\, d\theta\, d\phi$$

Therefore

$$\begin{aligned} I_{zz} &= \rho\int_0^a\int_0^{\pi/2}\int_0^{2\pi} r^4\sin^3\theta\ dr\ d\theta\ d\phi \\ &= \rho\int_0^a r^4 dr\int_0^{\pi/2}\sin^3\theta\ d\theta\int_0^{2\pi} d\phi \\ &= 2\pi\ \rho\ \frac{a^5}{5}\int_0^{\pi/2}\sin^3\theta\ d\theta \end{aligned}$$

$$\begin{aligned} \int_0^{\pi/2}\sin^3\theta\ d\theta &= \int_0^{\pi/2}\frac{3\sin\theta-\sin 3\theta}{4}\ d\theta \\ &= -\frac{1}{4}\left(\frac{3\sin\theta-\sin 3\theta}{4}\right) d\theta = \frac{2}{3} \qquad (1) \end{aligned}$$

On substitution we obtain

$$I_{zz} = \frac{16\pi a^5\rho}{40} = \frac{2}{5}M\,a^2$$

where we have used the relation $M = (2/3)\pi a^3\rho$ between the mass and volume of a hemisphere.

M.I. about a diameter through the base

Our X-axis is an axis through the base. By definition

$$I_{xx} = \int\rho(\mathbf{r})(y^2+z^2)dV = \rho\int(y^2+z^2)dV$$

In spherical polar coordinates $(r,\ \theta,\ \phi)$,

$$\begin{aligned} y^2+z^2 &= r^2\sin^2\theta\sin^2\phi + r^2\cos^2\theta \\ &= r^2(\sin^2\theta\sin^2\phi + \cos^2\theta) \end{aligned}$$

and $dV = r^2 \sin\theta \, dr \, d\theta \, d\phi$. Therefore.

$$\begin{aligned}
I_{xx} &= \rho \int_0^{\pi} \int_{\theta}^{\pi/2} \int_0^{a} r^2(\sin^2\theta \sin^2\phi + \cos^2\theta) r^2 \sin\theta \, dr \, d\theta \, d\phi \\
&= \rho \int_0^{\pi} \int_0^{\pi/2} (\sin^3\theta \sin^2\phi + \sin\theta \cos^2\theta) \frac{a^5}{5} \, d\theta \, d\phi \\
&= \frac{a^5\rho}{5} \int_0^{\pi} \int_0^{\pi/2} \left[\sin^3\theta \sin^2\phi \, d\theta \, d\phi + \sin^3\theta \, d\theta \, d\phi\right] \\
&= \frac{a^5\rho}{5}(I_1 + I_2)
\end{aligned}$$

where

$$I_2 = \int_0^{\pi} \int_0^{\pi/2} \sin^3\theta \, d\theta \, d\phi$$

and

$$I_1 = \int_0^{2\pi} \int_0^{\pi/2} \sin^3\theta \sin^2\phi \, d\theta \, d\phi$$

To evaluate these integrals, we use the results

$$\int_0^{2\pi} d\phi = 2\pi, \quad \int_0^{\pi/2} \sin^3\theta \, d\theta = \frac{2}{3}, \quad \int_0^{2\pi} \sin^2\phi \, d\phi = \pi$$

and after substitution obtain

$$I_1 = \pi, \quad I_2 = 2\pi/3$$

Therefore

$$I_{xx} = \frac{a^5\rho}{5}\left(\frac{2\pi}{3} + \frac{2\pi}{3}\right) = \frac{4\pi a^5}{15} \frac{M}{(2\pi a^3/3)} = \frac{2}{5} M a^2$$

Because of symmetry, the M.I. of the hemisphere about any other diameter or axis through the centre of the base will be the same. Hence

$$I_{yy} = \frac{2}{5} M a^2$$

Question

Find the moments and products of inertia for a uniform sphere of radius a w.r.t. axes through its centre.

Solution

We use the same notation as in example 7. The difference here is that now θ will vary over 0 to π.

Because of symmetry we expect that $I_{xx} = I_{yy} = I_{zz}$. Similarly we expect that $I_{xy} = I_{yz} = I_{zx}$. It is more convenient to calculate I_{zz} rather than I_{xx} or I_{yy}. Therefore

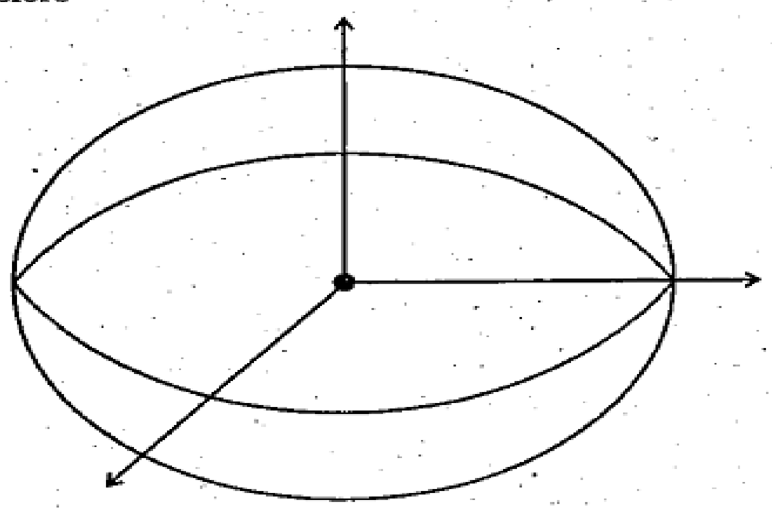


Figure 6.9: A sphere with origin chosen at the centre.

$$I_{zz} \equiv I_{33} = \rho \int\int\int_{\text{sphere}} (x^2 + y^2)dV$$

where

$$x^2 + y^2 = r^2(\sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi) = r^2\sin^2\theta$$

We have

$$I_{11} = \rho\int_0^{2\pi}\int_0^{\pi}\int_0^{a} r^2\sin^2\theta\, r^2\sin\theta\, dr\, d\theta\, d\phi$$
$$= 2\pi\rho\int_0^{a} r^4 dr \int_0^{\pi}\sin^3\theta\, d\theta$$

On making the substitutions

$$\int_0^a r^4 dr = \frac{a^5}{5}, \quad \int_0^{\pi}\sin^3\theta\, d\theta = \frac{4}{3}$$

we obtain

$$I_{xx} \equiv I_{11} = \frac{4\pi a^5\rho}{15} = \frac{2}{5}Ma^2$$

Corollary

$$\int_{\text{sphere}} x^2 dV = \int_{\text{sphere}} y^2 dV = \int_{\text{sphere}} z^2 dV$$
$$= \frac{1}{2}\int_{\text{sphere}} (x^2 + y^2) dV = \frac{4\pi}{15} a^5$$

Question

Find moments and products of inertia for the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

w.r.t. its axes of symmetry.

Solution

We refer to the figure for example

$$I_{11} = \rho \int_{\text{elliposid}} (y^2 + z^2) dV$$

We put $x/a = x'$, $y/b = y'$, $z/c = z'$. Then

$$dx = a\,dx', \quad dy = b\,dy', \quad dz = c\,dz'$$

and $dV = dx\,dy\,dz = a\,b\,c\,dx'\,dy'\,dz'$.

Now under the above transformation, the given ellipsoid is transformed into the unit sphere S:$x'^2 + y'^2 z'^2 = 1$. The integration is now over the region enclosed by a unit sphere.

$$I_{11} = \rho \int_S (b^2 y'^2 + c^2 z'^2) a\,b\,c\,dx'\,dy'\,dz'$$

Now because of symmetry

$$\int_S y'^2\,dV' = \int_S z'^2\,dV'$$

and each integral equals

$$\frac{1}{2}\int_S (y'^2 + z'^2)\,dV' = \frac{4\pi}{15}, \qquad \text{(as shown in the corollary above)}$$

Therefore on substitution

$$I_{11} = \rho\, a\, b\, c\left(b^2 \times \frac{4\pi}{15} + c^2 \times \frac{4\pi}{15}\right)$$

$$= \frac{M}{(4/3)a\,b\,c\,\pi} \times a\,b\,c \times \frac{4\pi}{15}(b^2 + c^2) = \frac{M}{5}(b^2 + c^2)$$

Similarly

$$I_{22} = \frac{M}{10}(c^2 + a^2), \quad I_{33} = \frac{M}{10}(a^2 + b^2)$$

(ii) Products of Inertia

$$I_{12} = -\rho \int_{\text{ellipsoid}} xy\,dV = -\rho \int a\,x'\,by'\,(abc\,dx'\,dy'\,dz')$$

$$= -\rho a^2 b^2 c \int\int\int_S x'\,y'\,dx'\,dy'\,dz'$$

Using the polar coordinate (r, θ, ϕ), we have

$$I_{12} = -\rho\, a^2 b^2 c \int_0^{2\pi}\int_0^{\pi}\int_0^{1} r\sin\theta\cos\phi(r\sin\theta\sin\phi)r^2\sin\theta\,dr\,d\theta\,d\phi = 0$$

Similarly

$$I_{23} = 0 = I_{31}$$

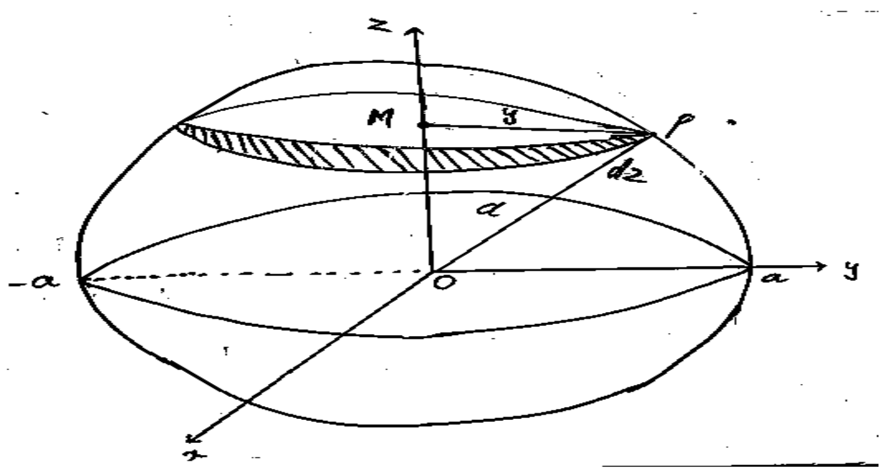
These results should not be surprising because the symmetry axes of an ellipsoid are also its principal axes, (see the section on principal axes).

Question

Find the moment of inertia of a solid homogeneous sphere with respect to any geometrical axis.

Or Find the moment of inertia of a uniform solid sphere of radius a and mass M about an axis (thez-axis) passing through the center.

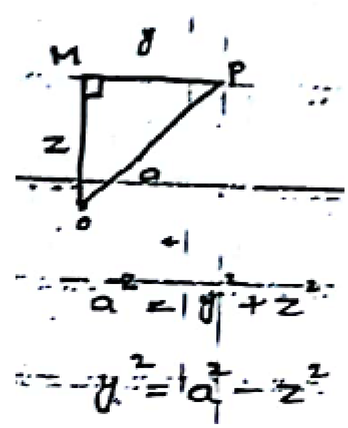
Solution



Consider a sphere of radius a, mass M. Consider a circular disk of thickness dz and z length from the origin. Radius of circular disk is y as shown in figure.

Moment of inertia about z axis $= I_{zz} = \frac{1}{2}\int_{-a}^{a} y^2\, dm$ ……………..(i)

Now by using volume mass density $\rho = \frac{dm}{dV}$. i.e. $dm = \rho dV \Rightarrow dm = \pi y^2 \rho dz$



$(i) \Rightarrow I_{zz} = \frac{1}{2}\int_{-a}^{a} y^2\, \pi y^2 \rho dz \Rightarrow I_{zz} = \frac{\pi\rho}{2}\int_{-a}^{a} y^4 dz$

$\Rightarrow I_{zz} = \frac{\pi\rho}{2}\int_{-a}^{a}(a^2 - z^2)^2 dz = \frac{\pi\rho}{2}\int_{-a}^{a}(a^4 + z^4 - 2a^2z^2)dz$

$\Rightarrow I_{zz} = \frac{8}{15}\pi a^5 \rho$ ………..(ii)

For whole mass of the sphere $\rho = \frac{M}{V} = \frac{M}{\frac{4}{3}\pi a^3}$. Then

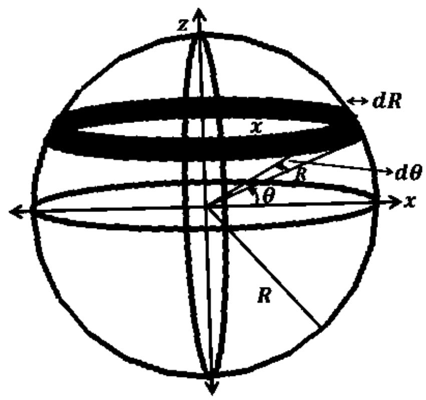
$(ii) \Rightarrow I_{zz} = \frac{8}{15}\pi a^5 . \frac{M}{\frac{4}{3}\pi a^3} \Rightarrow \boldsymbol{I_{zz} = \frac{2}{5} M a^2}$

For a uniform solid sphere, due to symmetry, we have $\boldsymbol{I_{zz} = I_{xx} = I_{yy} = \frac{2}{5} M a^2}$

Question

A thin uniform hollow sphere has a radius R and mass M. Calculate its moment of inertia about any axis through its center.

Solution



In order to calculate the moment of inertia of the hollow sphere, we split the hollow sphere into thin hoops (rings), as shown in Figure. We have already derived the expression for the moment of inertia of a representative hoop of radius x, which is $I = dmx^2$ of an elementary ring of mass dm and the radius x.
The volume of the elementary ring is $dV = 2\pi xRd\theta dR$
and $dm = \rho dV \Rightarrow dm = \rho 2\pi xRd\theta dR$

Moment of inertia of the small ring of radius x $= I_{ring} = dmx^2 = \rho 2\pi x^3 Rd\theta dR$

Moment of inertia for the whole hollow sphere $= I = \int I_{ring} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \rho 2\pi x^3 Rd\theta dR$

$\Rightarrow I = 4\rho\pi RdR \int_0^{\frac{\pi}{2}} x^3 d\theta$

To solve the integral, we need to write x in terms of θ.
From fig we have $x = Rcos\theta$ then the integral becomes,

$\Rightarrow I = 4\rho\pi RdR \int_0^{\frac{\pi}{2}} (Rcos\theta)^3 d\theta = 4\rho\pi R^4 dR \int_0^{\frac{\pi}{2}} cos^3\theta d\theta$

$\Rightarrow I = 4\rho\pi R^4 dR \int_0^{\frac{\pi}{2}} cos\theta . cos^2\theta d\theta = 4\rho\pi R^4 dR \int_0^{\frac{\pi}{2}} cos\theta(1 - sin^2\theta) d\theta$

$\Rightarrow I = \frac{8}{3}\pi\rho R^4 dR$

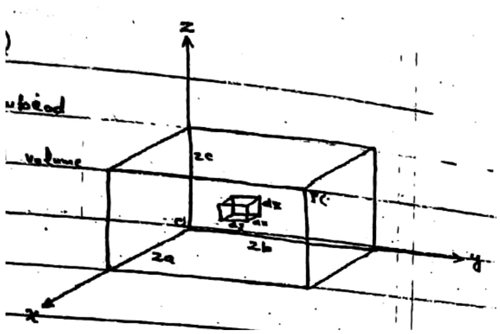
For whole mass of the sphere $\rho = \frac{M}{V} = \frac{M}{4\pi R^2 dR}$. Then $I = \frac{8}{3}\pi . \frac{M}{4\pi R^2 dR} . R^4 dR$

$\Rightarrow I = \frac{2}{3} MR^2$

Question

Find the moment and product of inertia about the concurrent edges OX,OY,OZ of a uniform regular block with dimensions $0 \leq x \leq 2a, 0 \leq y \leq 2b, 0 \leq z \leq 2c.$

Solution



Consider a uniform rectangular block of length 2a, width 2b and height 2c as shown in figure. Consider a small portion in this cuboid of mass dm and volume $dV = dxdydz$ then

Moment of inertia about x axis = $I_{xx} = \int_V (y^2 + z^2)dm$

Moment of inertia about x axis = $I_{xx} = \int_0^{2c}\int_0^{2b}\int_0^{2a}(y^2 + z^2)\, dm$ ……………..(i)

Now by using volume mass density $\rho = \frac{dm}{dV}$. i.e. $dm = \rho dV \Rightarrow dm = \rho dxdydz$

$(i) \Rightarrow I_{xx} = \int_0^{2c}\int_0^{2b}\int_0^{2a}(y^2 + z^2)\, \rho dxdydz$

$\Rightarrow I_{xx} = \frac{32abc}{3}(b^2 + c^2)\rho$ ……….(ii)

For whole mass of the sphere $\rho = \frac{M}{V} = \frac{M}{2a2b2c} = \frac{M}{8abc}$. Then

$(ii) \Rightarrow I_{xx} = \frac{32abc}{3}(b^2 + c^2).\frac{M}{8abc} \Rightarrow \boldsymbol{I_{xx} = \frac{4}{3}M(b^2 + c^2)}$

Similarly, we have $\boldsymbol{I_{yy} = \frac{4}{3}M(a^2 + c^2)}$, $\boldsymbol{I_{zz} = \frac{4}{3}M(a^2 + b^2)}$

Now for product of inertia consider $I_{xy} = \int_V xydm$

$I_{xy} = \int_0^{2c}\int_0^{2b}\int_0^{2a} xy\, dm$ ……………..(iii)

Now by using volume mass density $\rho = \frac{dm}{dV}$. i.e. $dm = \rho dV \Rightarrow dm = \rho dxdydz$

$(iii) \Rightarrow I_{xy} = \int_0^{2c}\int_0^{2b}\int_0^{2a} xy\,\rho dxdydz$

$\Rightarrow I_{xy} = 8a^2b^2c\rho$ ....(iv)

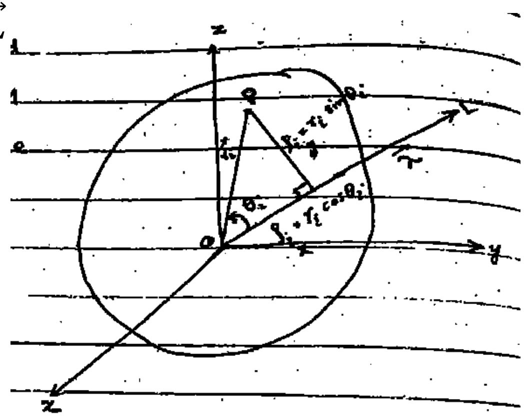
For whole mass of the sphere $\rho = \frac{M}{V} = \frac{M}{2a2b2c} = \frac{M}{8abc}$. Then

$(iv) \Rightarrow I_{xy} = 8a^2b^2c.\frac{M}{8abc} \Rightarrow \boldsymbol{I_{xy} = Mab}$

Similarly, we have $\boldsymbol{I_{yz} = Mbc}$, $\boldsymbol{I_{zx} = Mca}$

Moment of Inertia of Rigid Body about any Line through the Origin/ in Space

Consider a rigid body of mass M rotates along line $\overrightarrow{OL}$ as shown in figure. Let $P_i(x_i, y_i, z_i)$ be any point on the rigid body then a position vector of $\overrightarrow{OP}$ is



$\vec{r}_i = x_i\hat{\imath} + y_i\hat{\jmath} + z_i\hat{k}$

Let $\vec{\lambda}$ represents the direction of line $\overrightarrow{OL}$ as follows

$\vec{\lambda} = l\hat{\imath} + m\hat{\jmath} + n\hat{k}$

Where l, m, n are direction cosines with $l^2 + m^2 + n^2 = 1$; $|\vec{\lambda}| = 1$ then

Moment of inertia about line $\overrightarrow{OL} = I = \int P_i^2\,dm$ ....(i)

where P_i is perpendicular distance of line.

From figure $P_i = r_i sin\theta_i = r_i\lambda sin\theta = |\vec{r}_i \times \vec{\lambda}|$

$$P_i = |\vec{r}_i \times \vec{\lambda}| = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ x_i & y_i & z_i \\ l & m & n \end{vmatrix} = (ny_i - mz_i)\hat{\imath} + (lz_i - nx_i)\hat{\jmath} + (mx_i - ly_i)\hat{k}$$

$P_i^2 = (ny_i - mz_i)^2 + (lz_i - nx_i)^2 + (mx_i - ly_i)^2$

$\Rightarrow I = \int[(ny_i - mz_i)^2 + (lz_i - nx_i)^2 + (mx_i - ly_i)^2]\,dm$

$\Rightarrow I = \int[n^2y_i^2 + m^2z_i^2 - 2nmy_iz_i + l^2z_i^2 + n^2x_i^2 - 2lnx_iz_i + m^2x_i^2 + l^2y_i^2 - 2lmx_iy_i]\,dm$

$\Rightarrow I = l^2 \int (y_i^2 + z_i^2)\, dm + m^2 \int (x_i^2 + z_i^2)\, dm + n^2 \int (x_i^2 + y_i^2)\, dm - 2lm \int x_i y_i \, dm - 2mn \int y_i z_i \, dm - 2ln \int x_i z_i \, dm$

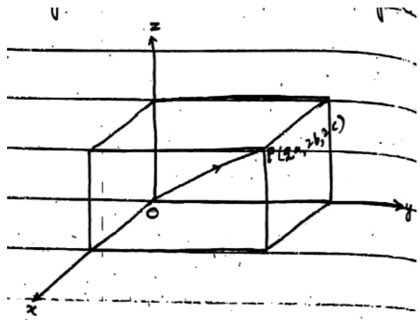
$\Rightarrow I = l^2 I_{xx} + m^2 I_{yy} + n^2 I_{zz} - 2lmI_{xy} - 2mnI_{yz} - 2lnI_{zx}$

This is the required expression for the Moment of Inertia of a Rigid Body about any Line through the Origin (in space)

Question

Find moment of inertia of a rectangular block about a diagonal. Dimensions of rectangular block are 2a,2b,2c respectively.

Solution



Consider a rectangular block of length 2a,width 2b and height 2c as shown in figure. Now by using expression of M.I about any line $\overrightarrow{OP}$. i.e.

$\Rightarrow I = l^2 I_{xx} + m^2 I_{yy} + n^2 I_{zz} - 2lmI_{xy} - 2mnI_{yz} - 2lnI_{zx}$

Since we know that M.I and P.I of a rectangular bloc are

$I_{xx} = \frac{4}{3}M(b^2 + c^2), I_{yy} = \frac{4}{3}M(a^2 + c^2), I_{zz} = \frac{4}{3}M(a^2 + b^2)$

$I_{xy} = Mab, I_{yz} = Mbc, I_{zx} = Mca$

For direction cosines l, m, n we have a position vector of line $O(0,0,0)$ to $P(2a, 2b, 2c)$ as $\vec{r} = 2a\hat{i} + 2b\hat{j} + 2c\hat{k}$ then

$|\vec{r}| = \sqrt{(2a)^2 + (2b)^2 + (2c)^2} = 2\sqrt{a^2 + b^2 + c^2}$

Now direction cosines will become

$l = \frac{x}{r} = \frac{2a}{2\sqrt{a^2+b^2+c^2}} = \frac{a}{\sqrt{a^2+b^2+c^2}} \Rightarrow l^2 = \frac{a^2}{a^2+b^2+c^2}$

$m = \frac{y}{r} = \frac{2b}{2\sqrt{a^2+b^2+c^2}} = \frac{b}{\sqrt{a^2+b^2+c^2}} \Rightarrow m^2 = \frac{b^2}{a^2+b^2+c^2}$

$n = \frac{z}{r} = \frac{2c}{2\sqrt{a^2+b^2+c^2}} = \frac{c}{\sqrt{a^2+b^2+c^2}} \Rightarrow n^2 = \frac{c^2}{a^2+b^2+c^2}$

Using all above values we have

$$\Rightarrow I = l^2.\frac{4}{3}M(b^2+c^2) + m^2.\frac{4}{3}M(a^2+c^2) + n^2.\frac{4}{3}M(a^2+b^2) - 2lm.Mab - 2mn.Mbc - 2ln.Mca$$

$$\Rightarrow I = \frac{4}{3}M[l^2(b^2+c^2) + m^2(a^2+c^2) + n^2(a^2+b^2)] - 2M[lmab + mnbc + lnca]$$

$$\Rightarrow I = \frac{4}{3}M\left[\frac{a^2(b^2+c^2)}{a^2+b^2+c^2} + \frac{b^2(a^2+c^2)}{a^2+b^2+c^2} + \frac{c^2(a^2+b^2)}{a^2+b^2+c^2}\right] - 2M\left[\frac{a}{\sqrt{a^2+b^2+c^2}}\frac{b}{\sqrt{a^2+b^2+c^2}}.ab + \frac{b}{\sqrt{a^2+b^2+c^2}}\frac{c}{\sqrt{a^2+b^2+c^2}}.bc + \frac{c}{\sqrt{a^2+b^2+c^2}}\frac{a}{\sqrt{a^2+b^2+c^2}}.ca\right]$$

$$\Rightarrow I = \frac{4}{3}M\left[\frac{a^2(b^2+c^2)}{a^2+b^2+c^2} + \frac{b^2(a^2+c^2)}{a^2+b^2+c^2} + \frac{c^2(a^2+b^2)}{a^2+b^2+c^2}\right] - 2M\left[\frac{(ab)(ab)}{a^2+b^2+c^2} + \frac{(bc)(bc)}{a^2+b^2+c^2} + \frac{(ca)(ca)}{a^2+b^2+c^2}\right]$$

$$\Rightarrow I = \frac{4}{3}M\left[\frac{a^2b^2+a^2c^2+b^2a^2+b^2c^2+c^2a^2+c^2b^2}{a^2+b^2+c^2}\right] - 2M\left[\frac{a^2b^2+b^2c^2+c^2a^2}{a^2+b^2+c^2}\right]$$

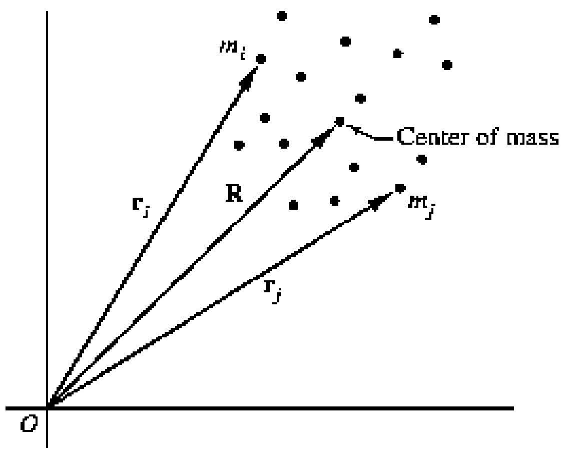
$$\Rightarrow I = \frac{4}{3}M\left[\frac{2a^2b^2+2b^2c^2+2c^2a^2}{a^2+b^2+c^2}\right] - 2M\left[\frac{a^2b^2+b^2c^2+c^2a^2}{a^2+b^2+c^2}\right]$$

$$\Rightarrow I = 2M\left[\frac{4a^2b^2+4b^2c^2+4c^2a^2-3a^2b^2-3b^2c^2-3c^2a^2}{3(a^2+b^2+c^2)}\right]$$

$$\Rightarrow \boldsymbol{I = \frac{2}{3}M\left[\frac{a^2b^2+b^2c^2+c^2a^2}{a^2+b^2+c^2}\right]}$$

Position Vector of Center of Mass of the System of Particles

Let $\vec{r}_1, \vec{r}_2, \vec{r}_3, \dots, \vec{r}_n = \sum_{i=1}^{n} \vec{r}_i$ be the position vectors of a system of n particles of masses $M = m_1, m_2, m_3, \dots, m_n = \sum_{i=1}^{n} m_i$ respectively [see Fig.].



The center of mass or centroid of the system of particles is defined as that point C having position vector $\vec{R}$. And $\vec{r}_i'$ is a position vector of each particle about centre of mass C. then by varignon's theorem

Torque acting on the system of particle is equal to the sum of all torque acting on each particle. i.e. $\tau = \sum_1^n \tau_i = \tau_1 + \tau_2 + \cdots + \tau_n$

$$\Rightarrow \vec{R} \times \vec{F} = \vec{r}_1 \times \vec{F}_1 + \vec{r}_2 \times \vec{F}_2 + \cdots + \vec{r}_n \times \vec{F}_n$$

$$\Rightarrow \vec{R} \times M\vec{a} = \vec{r}_1 \times m_1\vec{a} + \vec{r}_2 \times m_2\vec{a} + \cdots + \vec{r}_n \times m_n\vec{a}$$

$$\Rightarrow M\vec{R} \times \vec{a} = m_1\vec{r}_1 \times \vec{a} + m_2\vec{r}_2 \times \vec{a} + \cdots + m_n\vec{r}_n \times \vec{a}$$

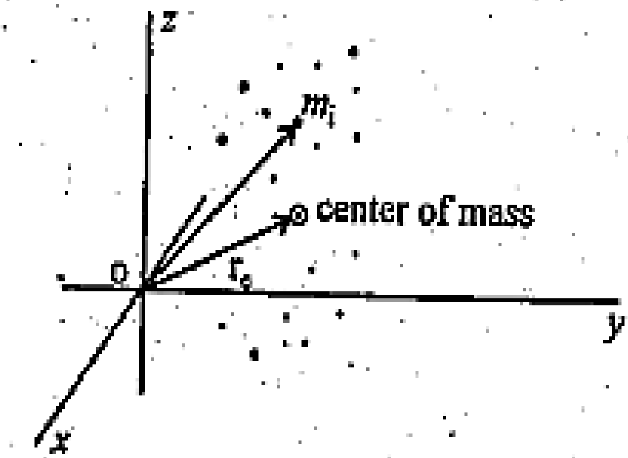
$$\Rightarrow M\vec{R} \times \vec{a} = (m_1\vec{r}_1 + m_2\vec{r}_2 + \cdots + m_n\vec{r}_n) \times \vec{a}$$

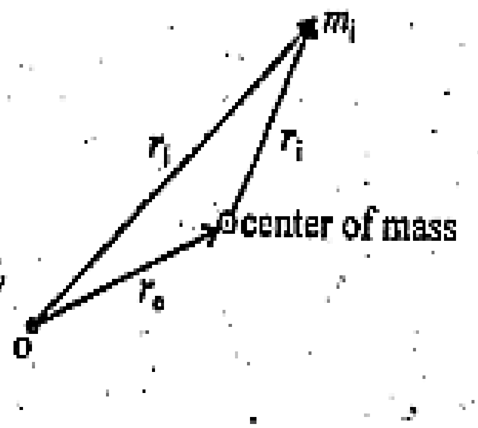
$$\Rightarrow M\vec{R} = m_1\vec{r}_1 + m_2\vec{r}_2 + \cdots + m_n\vec{r}_n = \sum_i m_i\vec{r}_i$$

$$\Rightarrow \vec{R} = \vec{r}_c = \frac{\sum_i m_i\vec{r}_i}{M = \sum_i m_i}$$

Question: Show that $\sum \boldsymbol{m_i \vec{r}_i' = 0}$

Solution





Solution

For this we will use the position vector of the system of particles

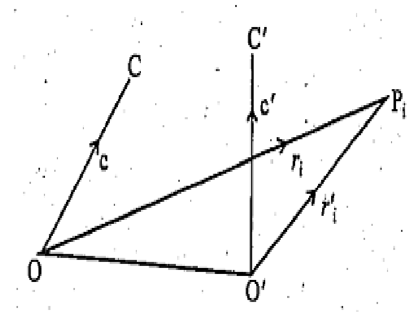
$R = \vec{r}_c = \frac{\sum_i m_i \vec{r}_i}{\sum_i m_i}$ ……………..(1)

By using Head to Tail rule $\vec{r}_i = \vec{r}_c + \vec{r}_i'$

$\Rightarrow \vec{r}_c = \frac{\sum_i m_i(\vec{r}_c + \vec{r}_i')}{M}$

$\Rightarrow \vec{r}_c = \frac{(\sum_i m_i)\vec{r}_c + \sum_i m_i \vec{r}_i{}'}{M} \Rightarrow \vec{r}_c = \frac{M\vec{r}_c}{M} + \frac{\sum_i m_i \vec{r}_i{}'}{M} \Rightarrow \vec{r}_c = \vec{r}_c + \frac{\sum_i m_i \vec{r}_i{}'}{M}$

$\Rightarrow \frac{\sum_i m_i \vec{r}_i{}'}{M} = 0 \Rightarrow \sum_i m_i \vec{r}_i{}' = 0$ or $\sum_{i=1}^{n} m_i \vec{r}_i' = 0$



Note that $\sum_{i=1}^{n} m_i \dot{\vec{r}}_i' = 0$ or $\sum_{i=1}^{n} m_i \vec{v}_i{}' = 0$

Then $\sum_{i=0}^{n} m_i \ddot{\vec{r}}_i' = 0$ or $\sum_{i=1}^{n} m_i \vec{a}_i{}' = 0$

Uniqueness of the c.m.

Let $\vec{r}_c = \frac{\sum_i m_i \vec{r}_i}{M}$ and $\vec{r}_c' = \frac{\sum_i m_i \vec{s}_i}{M}$ be p.v. of C and C' respectively, then from figure

$\overline{CC'} = \overline{CO} + \overline{OO'} + \overline{O'C'} \Rightarrow \overline{CC'} = -\vec{r}_c + \vec{a} + \vec{r}_c \Rightarrow \overline{CC'} = -\frac{\sum_i m_i \vec{r}_i}{M} + \vec{a} + \frac{\sum_i m_i \vec{s}_i}{M}$

$\Rightarrow \overline{CC'} = \frac{\sum_i m_i(\vec{s}_i - \vec{r}_i)}{M} + \vec{a} \Rightarrow \overline{CC'} = -\vec{a} + \vec{a} \Rightarrow \overline{CC'} = 0 \Rightarrow C$ is the same point as C'

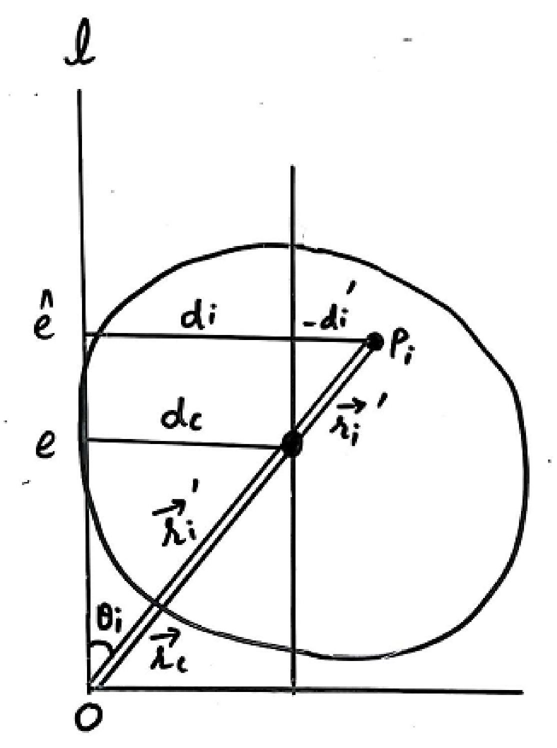
Parallel Axis Theorem / Huygens Steiner Theorem / Steiner Theorem

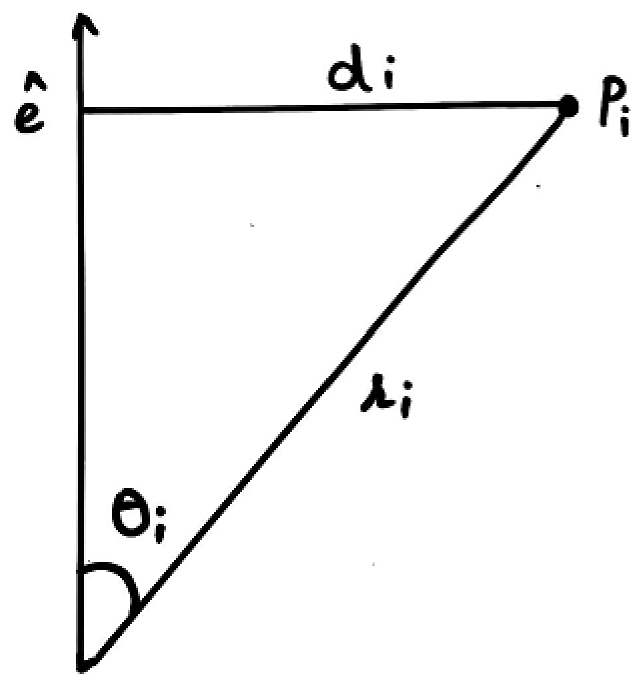
The rotational inertia about an axis is equal to the inertia about parallel axis through centre of mass plus mass time the square of the distance between two parallel axis. i.e. $I = I' + Md^2$

This theorem also known as Huygens Steiner Theorem or just Steiner Theorem, named after Christian Huygens and Jakob Steiner.

Importance

This theorem helps us in calculating moment of inertia matrix of a rigid body at any point in terms of information about the same body at some other point. This theorem is used to find rotation of Earth about its own axis and sun axis.





Proof

Consider a rigid body of mass M. Let I denotes the moment of inertia of body about l. Let us take its i^{th} particle of mass m_i at a distance of d_i from the central axis then

$$I = \sum m_i d_i^2 \qquad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Now using $d_i = r_i Sin\theta_i = |\hat{e}_i||\vec{r}_i|Sin\theta_i = |\hat{e}_i \times \vec{r}_i|$ and $\vec{r}_i = \vec{r}_c + \vec{r}_i'$

$$\Rightarrow d_i = |\hat{e}_i \times (\vec{r}_c + \vec{r}_i')| = |\hat{e}_i \times \vec{r}_c + \hat{e}_i \times \vec{r}_i'|$$

$$(1) \Rightarrow I = \sum m_i |\hat{e}_i \times \vec{r}_c + \hat{e}_i \times \vec{r}_i'|^2$$

$$\Rightarrow I = \sum m_i |(\hat{e}_i \times \vec{r}_c)^2 + (\hat{e}_i \times \vec{r}_i')^2 + 2(\hat{e}_i \times \vec{r}_c).(\hat{e}_i \times \vec{r}_i')|$$

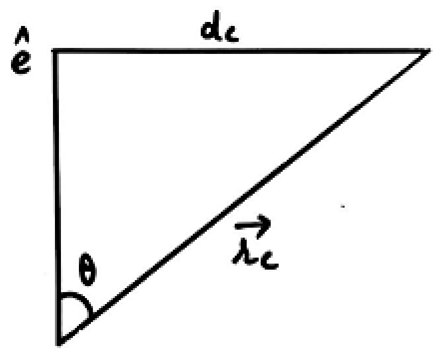
$$\Rightarrow I = \sum m_i |\hat{e}_i \times \vec{r}_c|^2 + \sum m_i |\hat{e}_i \times \vec{r}_i'|^2 + 2|\hat{e}_i \times \vec{r}_c|.\hat{e}_i \times \sum m_i |\vec{r}_i'|$$

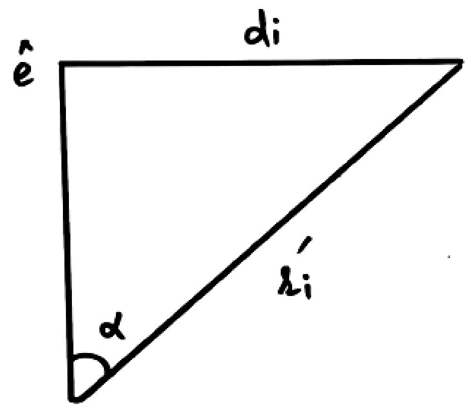
$$\Rightarrow I = \sum m_i (\hat{e}_i \times \vec{r}_c)^2 + \sum m_i (\hat{e}_i \times \vec{r}_i')^2 + 2(\hat{e}_i \times \vec{r}_c).\hat{e}_i \times \sum m_i \vec{r}_i'$$

$$\Rightarrow I = \sum m_i d_c^2 + \sum m_i {d_i'}^2 + 2(\hat{e}_i \times \vec{r}_c).\hat{e}_i \times (0)$$

$$\Rightarrow I = Md^2 + I'$$

$$I = I' + Md^2 \qquad \text{proved}$$





$d_c = \vec{\lambda}_c \, Sin\theta$
$= |\hat{e} \times \vec{\lambda}_i|$

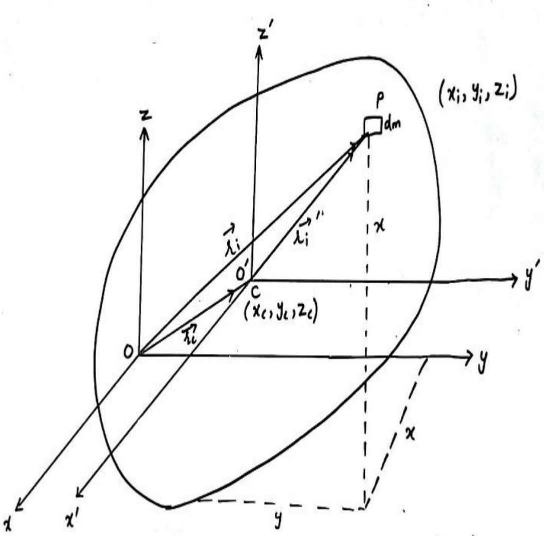
$d_i' = \lambda_i' \, Sin\alpha$
$= |\hat{e} \times \lambda_i'|$

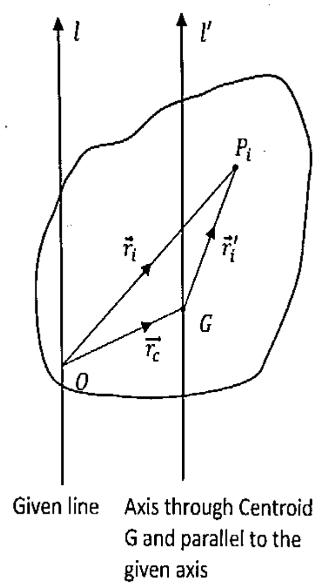
Parallel Axis Theorem (another Proof)

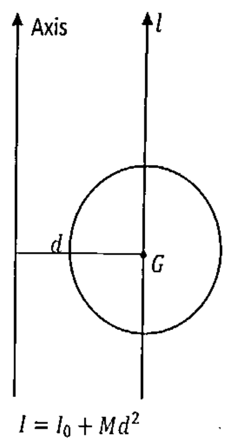
The rotational inertia about an axis is equal to the inertia about parallel axis through centre of mass plus mass time the square of the distance between two parallel axis. i.e. $I = I' + Md^2$

Importance

This theorem helps us in calculating moment of inertia matrix of a rigid body at any point in terms of information about the same body at some other point. This theorem is used to find rotation of Earth about its own axis and sun axis.







Proof

Consider a rigid body of mass M. Let I' denotes the inertia of body about its central axis. Let us take its i^{th} particle of mass m_i at a distance of x_i from the central axis then $I' = \sum m_i x_i^2$

Now consider a parallel axis at a distance d from the central axis. The rotational inertia about this parallel axis is given by

$$I = \sum m_i (d + x_i)^2 = \sum m_i \left(d^2 + x_i^2 + 2dx_i\right) = \left(\sum m_i\right) d^2 + \sum m_i x_i^2 + 2d \sum m_i x_i$$

$$I = Md^2 + I' + 2d(0)$$

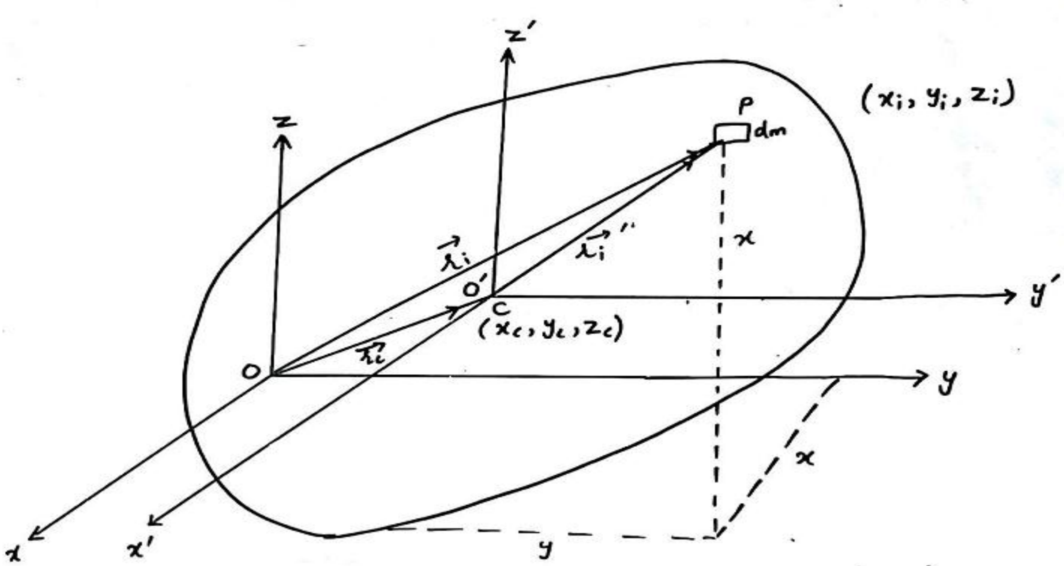
$$I = I' + Md^2 \qquad \text{proved}$$

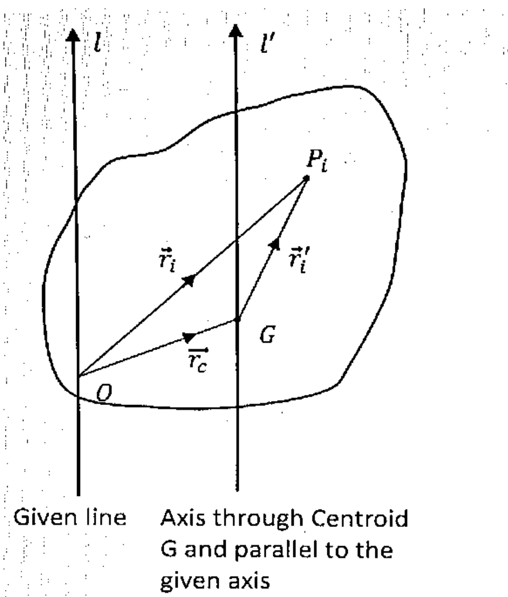
Parallel Axis Theorem (another Proof)

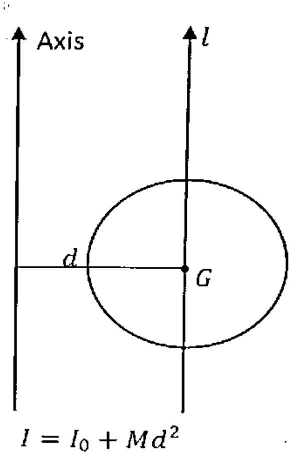
The rotational inertia about an axis is equal to the inertia about parallel axis through centre of mass plus mass time the square of the distance between two parallel axis. i.e. $I = I' + Md^2$

Importance

This theorem helps us in calculating moment of inertia matrix of a rigid body at any point in terms of information about the same body at some other point. This theorem is used to find rotation of Earth about its own axis and sun axis.







Proof

Consider a body whose centre of mass is located at the origin O' of the prime coordinate system that is at point (x_0, y_0, z_0) relative to the unprimed system. Consider an infinitesimal particle of mass dm which is located at $P_i(x_i, y_i, z_i)$ relative to the unprime system and $P_i'(x_i', y_i', z_i')$ relative to the prime system as shown in figure. Then

Moment of inertia about x – axis $= I_{xx} = \int\left(y_i^2 + z_i^2\right)dm$ …………..(1)

By using head to tail rule $\vec{r}_i = \vec{r}_c + \vec{r}_i'$

$\Rightarrow (x_i, y_i, z_i) = (x_c, y_c, z_c) + (x_i', y_i', z_i') = (x_c + x_i', y_c + y_i', z_c + z_i')$

$\Rightarrow x_i = x_c + x_i',\ y_i = y_c + y_i',\ z_i = z_c + z_i'$

$(1) \Rightarrow I_{xx} = \int[(y_c + y_i')^2 + (z_c + z_i')^2]dm$

$\Rightarrow I_{xx} = \int\left[y_c^2 + {y_i'}^2 + 2y_c y_i' + z_c^2 + {z_i'}^2 + 2z_c z_i'\right]dm$

$\Rightarrow I_{xx} = \int\left[(y_c^2 + +z_c^2) + \left({y_i'}^2 + {z_i'}^2\right) + 2y_c y_i' + 2z_c z_i'\right]dm$

$\Rightarrow I_{xx} = \int(y_c^2 + +z_c^2)dm + \int\left({y_i'}^2 + {z_i'}^2\right)dm + 2y_c\int y_i' dm + 2z_c\int z_i' dm$

$\Rightarrow I_{xx} = \int(y_c^2 + +z_c^2)dm + \int\left({y_i'}^2 + {z_i'}^2\right)dm + 2y_c(0) + 2z_c(0)$

$\Rightarrow I_{xx} = (y_c^2 + +z_c^2)\int dm + \int\left({y_i'}^2 + {z_i'}^2\right)dm$

$\Rightarrow I_{xx} = (y_c^2 + +z_c^2)M + I_{x'x'}$

$\Rightarrow I_{xx} = I_{x'x'} + (y_c^2 + +z_c^2)M$ result of parallel axis theorem about x – axis

Similarly

$\Rightarrow I_{yy} = I_{y'y'} + (x_c^2 + +z_c^2)M$ result of parallel axis theorem about y – axis

$\Rightarrow I_{zz} = I_{z'z'} + (x_c^2 + +y_c^2)M$ result of parallel axis theorem about z – axis

Respectively.

Now consider

Product of inertia about x,y – axis = $I_{xy} = \int (x_i y_i) dm$ …………..(2)

By using head to tail rule $\vec{r}_i = \vec{r}_c + \vec{r}_i'$

$\Rightarrow (x_i, y_i, z_i) = (x_c, y_c, z_c) + (x_i', y_i', z_i') = (x_c + x_i', y_c + y_i', z_c + z_i')$

$\Rightarrow x_i = x_c + x_i',\ y_i = y_c + y_i',\ z_i = z_c + z_i'$

$(2) \Rightarrow I_{xy} = \int (x_c + x_i')(y_c + y_i') dm \Rightarrow I_{xy} = \int (x_c y_c + x_c y_i' + x_i' y_c + x_i' y_i') dm$

$\Rightarrow I_{xy} = \int (x_c y_c) dm + \int (x_c y_i') dm + \int (x_i' y_c) dm + \int (x_i' y_i') dm$

$\Rightarrow I_{xy} = x_c y_c \int dm + x_c \int (y_i') dm + y_c \int (x_i') dm + \int (x_i' y_i') dm$

$\Rightarrow I_{xy} = x_c y_c \int dm + x_c (0) + y_c (0) + \int (x_i' y_i') dm \quad \because \sum m_i r_i = 0, \int r_i' dm = 0$

$\Rightarrow I_{xy} = \int (x_i' y_i') dm + x_c y_c \int dm$

$\Rightarrow I_{xy} = I_{x'y'} + x_c y_c M$

Similarly $I_{yz} = I_{y'z'} + y_c z_c M$; $I_{zx} = I_{z'x'} + z_c z_c M$

In vector form we know that $I = \int r_i^2 dm$

$\Rightarrow I = \int (\vec{r}_i . \vec{r}_i) dm = \int (\vec{r}_c + \vec{r}_i').(\vec{r}_c + \vec{r}_i') dm$

$\Rightarrow I = \int (\vec{r}_c . \vec{r}_c + \vec{r}_c . \vec{r}_i' + \vec{r}_i' . \vec{r}_c + \vec{r}_i' . \vec{r}_i') dm = \int \left(\vec{r}_c^{\,2} + {\vec{r}_i'}^2 + 2\vec{r}_i' . \vec{r}_c \right) dm$

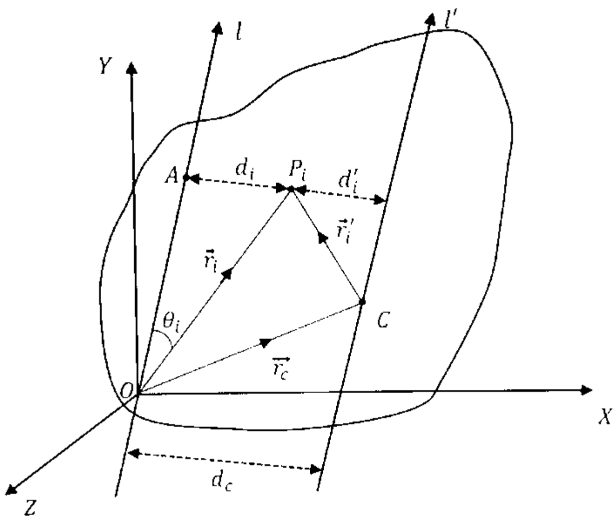
$\Rightarrow I = \vec{r}_c^{\,2} \int dm + \int {\vec{r}_i'}^2 dm + 2\vec{r}_c . \int \vec{r}_i' dm$

$\Rightarrow I = \vec{r}_c^{\,2} M + I' + 2\vec{r}_c . (0)$

$\Rightarrow I = I' + M d_c^2$

Parallel Axis Theorem (For discrete mass distribution)

The moment of inertia of a rigid body in the form of discrete mass distribution (set of particles) about a given axis is equal to the sum of moment of inertia of the same body about a parallel axis (to the given axis) through the centre of mass of the body and moment of inertia due to total mass of the body placed at is centre of mass, about the given axis. i.e. $I_l = I_{l'} + Md_c^2$



Proof: Consider a rigid body of mass M. Let I denotes the moment of inertia of body about l. Let us take its i^{th} particle of mass m_i at a distance of d_i from the central axis then $I = \sum m_i d_i^2$ ………………(1)

Now using $d_i = r_i Sin\theta_i = |\hat{e}_i||\vec{r}_i|Sin\theta_i = |\hat{e}_i \times \vec{r}_i|$ and $\vec{r}_i = \vec{r}_c + \vec{r}_i'$

$\Rightarrow d_i = |\hat{e}_i \times (\vec{r}_c + \vec{r}_i')| = |\hat{e}_i \times \vec{r}_c + \hat{e}_i \times \vec{r}_i'|$

$(1) \Rightarrow I = \sum m_i |\hat{e}_i \times \vec{r}_c + \hat{e}_i \times \vec{r}_i'|^2$

$\Rightarrow I = \sum m_i |(\hat{e}_i \times \vec{r}_c)^2 + (\hat{e}_i \times \vec{r}_i')^2 + 2(\hat{e}_i \times \vec{r}_c).(\hat{e}_i \times \vec{r}_i')|$

$\Rightarrow I = \sum m_i |\hat{e}_i \times \vec{r}_c|^2 + \sum m_i |\hat{e}_i \times \vec{r}_i'|^2 + 2|\hat{e}_i \times \vec{r}_c|.\hat{e}_i \times \sum m_i |\vec{r}_i'|$

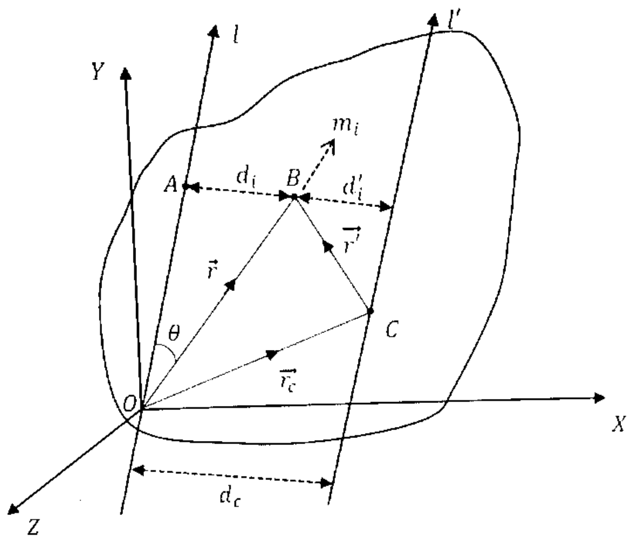
$\Rightarrow I = \sum m_i (\hat{e}_i \times \vec{r}_c)^2 + \sum m_i (\hat{e}_i \times \vec{r}_i')^2 + 2(\hat{e}_i \times \vec{r}_c).\hat{e}_i \times \sum m_i \vec{r}_i'$

$\Rightarrow I = \sum m_i d_c^2 + \sum m_i {d_i'}^2 + 2(\hat{e}_i \times \vec{r}_c).\hat{e}_i \times (0) \Rightarrow I = Md^2 + I'$

$\Rightarrow I = I' + Md^2$ proved

Parallel Axis Theorem (For Continuous mass distribution)

The moment of inertia of a rigid body in the form of continuous mass distribution about a given axis is equal to the sum of moment of inertia of the same body about a parallel axis (to the given axis) through the centre of mass of the body and moment of inertia due to total mass of the body placed at is centre of mass, about the given axis. i.e. $I_l = I_{l'} + Md_c^2$



Proof: Consider a rigid body of mass $M = \int dm$. Let I denotes the moment of inertia of body about l. Let us take its i^{th} particle of mass m_i at a distance of d_i from the central axis then $I = \int d_i^2 dm$ ………………(1)

Now using $d_i = \vec{r}Sin\theta_i = |\hat{e}_i||\vec{r}|Sin\theta_i = |\hat{e}_i \times \vec{r}|$ and $\vec{r} = \vec{r}_c + \vec{r}_i'$

$\Rightarrow d_i = |\hat{e}_i \times (\vec{r}_c + \vec{r}_i')| = |\hat{e}_i \times \vec{r}_c + \hat{e}_i \times \vec{r}_i'|$

$(1) \Rightarrow I = \int |\hat{e}_i \times \vec{r}_c + \hat{e}_i \times \vec{r}_i'|^2 dm$

$\Rightarrow I = \int |(\hat{e}_i \times \vec{r}_c)^2 + (\hat{e}_i \times \vec{r}_i')^2 + 2(\hat{e}_i \times \vec{r}_c).(\hat{e}_i \times \vec{r}_i')| dm$

$\Rightarrow I = \int |\hat{e}_i \times \vec{r}_c|^2 dm + \int |\hat{e}_i \times \vec{r}_i'|^2 dm + 2|\hat{e}_i \times \vec{r}_c|.\hat{e}_i \times \int |\vec{r}_i'| dm$

$\Rightarrow I = \int (\hat{e}_i \times \vec{r}_c)^2 dm + \int (\hat{e}_i \times \vec{r}_i')^2 dm + 2(\hat{e}_i \times \vec{r}_c).\hat{e}_i \times \int (\vec{r}_i') dm$

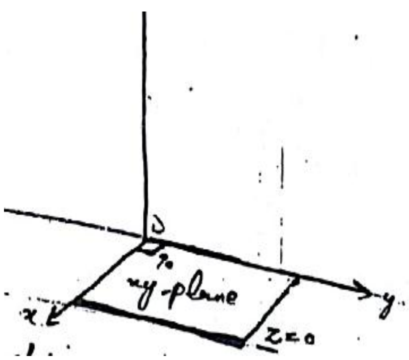
$\Rightarrow I = \int dm\, d_c^2 + \int dm\, {d_i'}^2 + 2(\hat{e}_i \times \vec{r}_c).\hat{e}_i \times (0) \Rightarrow I = Md^2 + I'$

Perpendicular Axis Theorem/ Perpendicular Axis Theorem (for a particle) /Plane Figure Theorem

The moment of inertia of a plane rigid body about an axis perpendicular to the body is equal to the sum of the moment of inertia about two mutually perpendicular axes lying in the plane of the body and meeting at the common point with the given axis. i.e. $I_{zz} = I_{xx} + I_{yy}$

Or The moment of inertia of a plane rigid body about a perpendicular axis is equal to the sum of the moment of inertias about the orthogonal axes of the plane. i.e. $I_{zz} = I_{xx} + I_{yy}$

Importance: This theorem helps us in calculating moment of inertia matrix of a rigid body at any point in terms of information about the same body at some other point.



Proof

Let us consider a rectangular frame of reference OXYZ. If there is a distribution of matter in xy – plane. i.e. z = 0, then

Moment of inertia about x – axis = $I_{xx} = \int (y^2 + z^2)dm = \int y^2 dm$ ……..(1)

Moment of inertia about y – axis = $I_{yy} = \int (x^2 + z^2)dm = \int x^2 dm$ ……..(2)

Moment of inertia about z – axis = $I_{zz} = \int (x^2 + y^2)dm$ ……..(3)

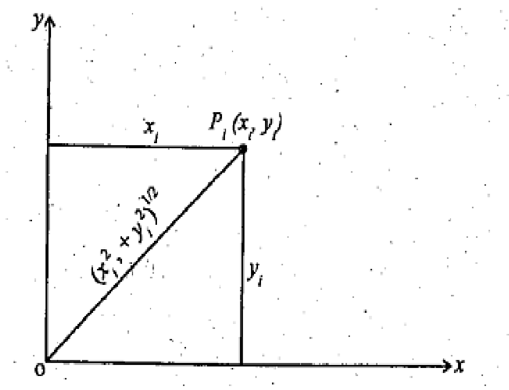
Adding (1) and (2)

$I_{xx} + I_{yy} = \int y^2 dm + \int x^2 dm = \int (x^2 + y^2)dm$

$\boldsymbol{I_{zz} = I_{xx} + I_{yy}}$ similarly we may write $\boldsymbol{I_{xx} = I_{yy} + I_{zz}}$, $\boldsymbol{I_{yy} = I_{xx} + I_{zz}}$

Perpendicular Axis Theorem (for discrete mass distribution)

The moment of inertia of a plane rigid body in the form of discrete mass distribution (set of particles) about an axis perpendicular to the body is equal to the sum of the moment of inertia about two mutually perpendicular axes lying in the plane of the body and meeting at the common point with the given axis. i.e.
$I_{zz} = I_{xx} + I_{yy}$



Proof

Let us consider a rectangular frame of reference OXYZ. If there is a distribution of matter in xy – plane. i.e. z = 0, then

Moment of inertia about x – axis = $I_{xx} = \sum m_i\left(y_i^2 + z_i^2\right) = \sum m_i y_i^2$ ……..(1)

Moment of inertia about y – axis = $I_{yy} = \sum m_i\left(x_i^2 + z_i^2\right) = \sum m_i x_i^2$ ……..(2)

Moment of inertia about z – axis = $I_{zz} = \sum m_i\left(x_i^2 + y_i^2\right)$ ……..(3)

Adding (1) and (2)

$I_{xx} + I_{yy} = \sum m_i y_i^2 + \sum m_i x_i^2 = \sum m_i\left(x_i^2 + y_i^2\right)$

$\boldsymbol{I_{zz} = I_{xx} + I_{yy}}$

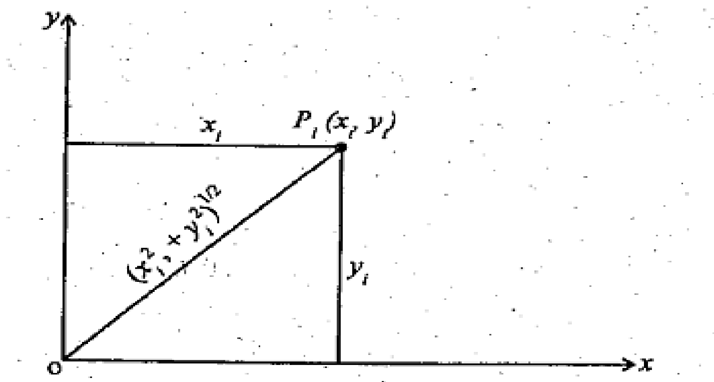
Similarly we may write

$\boldsymbol{I_{xx} = I_{yy} + I_{zz}}$

$\boldsymbol{I_{yy} = I_{xx} + I_{zz}}$

Perpendicular Axis Theorem (for continuous mass distribution)

The moment of inertia of a plane rigid body in the form of continuous mass distribution about an axis perpendicular to the body is equal to the sum of the moment of inertia about two mutually perpendicular axes lying in the plane of the body and meeting at the common point with the given axis. i.e. $I_{zz} = I_{xx} + I_{yy}$



Proof

Let us consider a rectangular frame of reference OXYZ. If there is a distribution of matter in xy – plane. i.e. z = 0, then

Moment of inertia about x – axis = $I_{xx} = \int (y_i^2 + z_i^2)dm_i = \int y_i^2 dm_i$ ……..(1)

Moment of inertia about y – axis = $I_{yy} = \int (x_i^2 + z_i^2)dm_i = \int x_i^2 dm_i$ ……..(2)

Moment of inertia about z – axis = $I_{zz} = \int (x_i^2 + y_i^2)dm_i$ ……..(3)

Adding (1) and (2)

$I_{xx} + I_{yy} = \int y_i^2 dm_i + \int x_i^2 dm_i = \int (x_i^2 + y_i^2)dm_i$

$\boldsymbol{I_{zz} = I_{xx} + I_{yy}}$

Similarly we may write

$\boldsymbol{I_{xx} = I_{yy} + I_{zz}}$

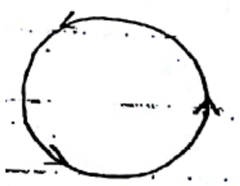
$\boldsymbol{I_{yy} = I_{xx} + I_{zz}}$

Linear and Angular Variables in Scalar Form

When a body moves along a straight line, then we use linear variables. i.e.

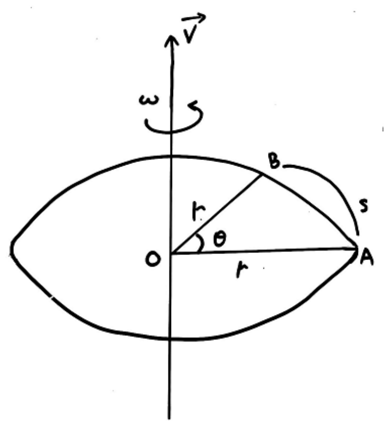
Linear displacement (S), Linear Velocity (v) and Linear Acceleration (a)

When a body moves along a circular path, then we use angular variables. i.e.



Angular displacement (θ), Angular Velocity (ω) and Angular Acceleration (α)

Linear and Angular Velocity of a Rigid Body about a Fixed Axis/ Linear and Angular Velocity (Speed) in Scalar Form.



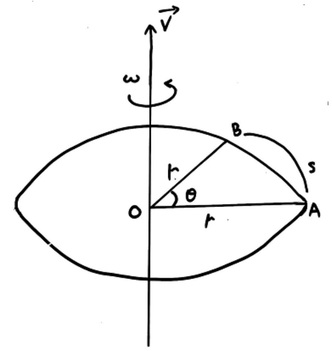
Let a body moves along a circular path, moving in a circle with constant radius $OA = r$ from point A to B length of arc will be s. i.e. $\overset{\frown}{AB} = s$ and angle between two radii is θ. i.e. $\angle AOB = \theta$ then we know that

$$s = r\theta$$

$$\Rightarrow \frac{ds}{dt} = r\frac{d\theta}{dt}$$

$$\Rightarrow v = r\omega$$

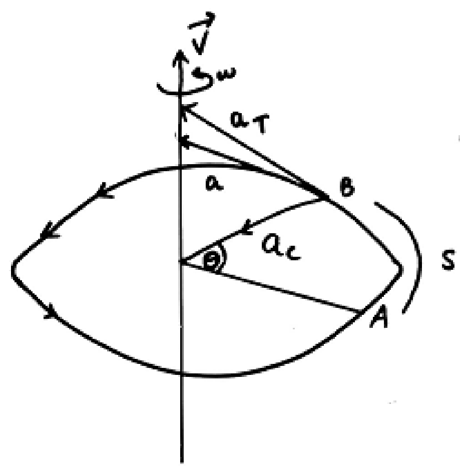
Linear and Angular Acceleration of a Rigid Body about a Fixed Axis/ Linear and Angular Acceleration in Scalar Form.



Let a body moves along a circular path, moving in a circle with constant radius $OA = r$ from point A to B length of arc will be s. i.e. $AB = s$ and angle between two radii is θ. i.e. $\angle AOB = \theta$ then we know that

$$s = r\theta \Rightarrow \frac{ds}{dt} = r\frac{d\theta}{dt} \Rightarrow v = r\omega \Rightarrow \frac{dv}{dt} = r\frac{d\omega}{dt} \Rightarrow a = r\alpha$$

Total Acceleration produced by a body with moves with Angular Speed.



In case of circular motion body moves with a centripetal acceleration a_c which is towards the centre. We know that $F_c = ma_c$ also $F_c = m\frac{v^2}{r}$ therefore $ma_c = m\frac{v^2}{r}$

$$\Rightarrow a_c = \frac{v^2}{r}$$

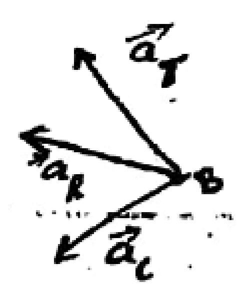
since body moves with angular speed therefore $v = r\omega$

$$\Rightarrow a_c = \frac{(r\omega)^2}{r} \Rightarrow a_c = \frac{r^2\omega^2}{r} \Rightarrow a_c = r\omega^2$$

If a_T is a tangential acceleration and since $a_T \perp a_c$ then by Pythagoras Theorem

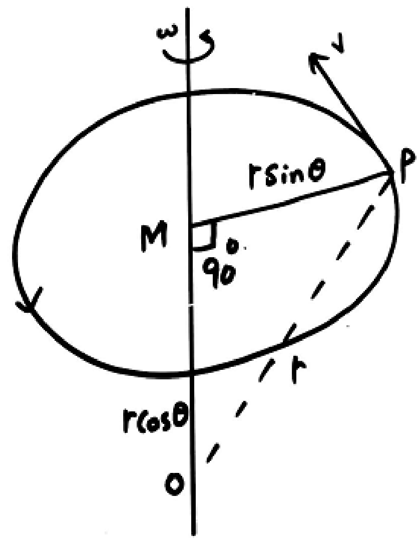
$$a_R^2 = a_T^2 + a_c^2$$

$$\Rightarrow a_R = \sqrt{a_T^2 + a_c^2}$$



This is expression of resultant acceleration in case of circular motion.

Linear and Angular Variables in Vector Form



Since $v = r\omega = (radius)\omega$ therefore

from figure $radius \; = \overrightarrow{MP} = rsin\theta$

$v = r\omega = (rsin\theta)\omega$

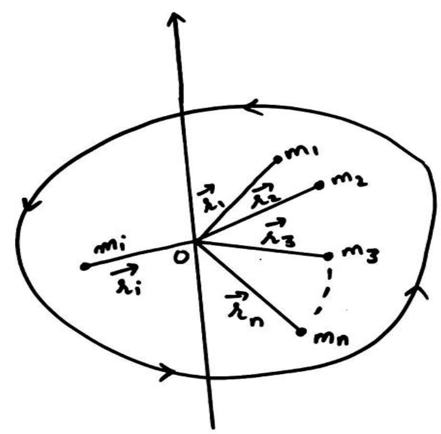
$\Rightarrow v = \omega rsin\theta$

$\Rightarrow v = |\vec{\omega} \times \vec{r}|$ by Right Hand Rule

$\Rightarrow v\hat{n} = |\vec{\omega} \times \vec{r}|\hat{n}$

$\Rightarrow \vec{v} = \vec{\omega} \times \vec{r}$

Kinetic Energy of Rotation



A rigid body consists of n – particles, each of mass $m_i; i = 1,2,3, \dots, n$ with position vector of each is $\vec{r}_i$. Then total kinetic energy of body is

$K.E = \frac{1}{2}\sum_{i=1}^{n} m_i v_i^2$

Since rotation is angular so $v_i = r_i\omega$

We use angular velocity as ω because it remains same for all particles of a rigid body. Then

$K.E = \frac{1}{2}\sum_{i=1}^{n} m_i r_i^2 \,\omega^2 = \frac{1}{2}\omega^2 \sum_{i=1}^{n} m_i r_i^2$

By using $I = \sum_{i=1}^{n} m_i r_i^2$ which is called rotational inertia of a body or moment of inertia of a body w.r.to axes of rotation. It plays same role in angular motion as mass in linear motion. So,

$\boldsymbol{K.E = \frac{1}{2}I\omega^2}$

Result **Prove that $\boldsymbol{I = mr^2}$**

Proof

In linear motion momentum of force is $\Rightarrow \tau = rF = r(ma)$

In rotational motion momentum of force is $\Rightarrow \tau = I\alpha$

$\Rightarrow I\alpha = r(ma) \Rightarrow I(\frac{a}{r}) = r(ma)$ using $v = r\omega \Rightarrow \frac{dv}{dt} = r\frac{d\omega}{dt} \Rightarrow a = r\alpha$

$\Rightarrow \boldsymbol{I = mr^2}$

Angular Momentum of System of Particles

Since we know that moment of linear momentum of a system of particles is called angular momentum, therefore

$\vec{L} = \sum_{i=1}^{n} \vec{r}_i \times \vec{P}_i$

$\Rightarrow \vec{L} = \sum_{i=1}^{n} \vec{r}_i \times m_i\vec{v}_i \Rightarrow \vec{L} = \sum_{i=1}^{n} \vec{r}_i \times m_i\dot{\vec{r}}_i \Rightarrow \vec{L} = \sum_{i=1}^{n} m_i\left(\vec{r}_i \times \dot{\vec{r}}_i\right)$

$\Rightarrow \vec{L} = \sum_{i=1}^{n} m_i \left[\left(\vec{R} + \vec{r}_i'\right) \times \left(\dot{\vec{R}} + \dot{\vec{r}}_i'\right)\right]$ using $\vec{r}_i = \vec{R} + \vec{r}_i', \dot{\vec{r}}_i = \dot{\vec{R}} + \dot{\vec{r}}_i'$

$\Rightarrow \vec{L} = \sum_{i=1}^{n} m_i \left[\vec{R} \times \dot{\vec{R}} + \vec{R} \times \dot{\vec{r}}_i' + \vec{r}_i' \times \dot{\vec{R}} + \vec{r}_i' \times \dot{\vec{r}}_i'\right]$

$\Rightarrow \vec{L} = \vec{R} \times (\sum_{i=1}^{n} m_i)\dot{\vec{R}} + \vec{R} \times \sum_{i=1}^{n} m_i\, \dot{\vec{r}}_i' + (\sum_{i=1}^{n} m_i\, \vec{r}_i') \times \dot{\vec{R}} + \sum_{i=1}^{n} \vec{r}_i' \times m_i\dot{\vec{r}}_i'$

Using $\sum_{i=1}^{n} m_i = M, \dot{\vec{R}} = \vec{v}_{cm}, \sum_{i=1}^{n} m_i\, \dot{\vec{r}}_i' = 0, \sum_{i=1}^{n} m_i\, \vec{r}_i' = 0$

$\Rightarrow \vec{L} = \vec{R} \times M\vec{v}_{cm} + 0 + 0 + \sum_{i=1}^{n} \vec{r}_i' \times m_i\dot{\vec{r}}_i'$

$\Rightarrow \vec{L} = \vec{R} \times \vec{P}_{cm} + \sum_{i=1}^{n} \vec{r}_i' \times \vec{P}_i' \Rightarrow \vec{L} = \vec{L}_0 + \vec{L}_i'$

Hence **total angular momentum $\vec{L}$ of a system of particles is equal to the sum of angular momentum $\vec{L}_0$ about origin and angular momentum $\vec{L}_i'$ about centre of mass of a system of particle.**

Question

A rigid body is pivoted at a point O and is rotating at the rate of 90 radian per second about a fixed line in the direction of the vector $(-1, 2, 2)$ to an observer looking in the direction of this vector. The sense of rotation of the body is clockwise. Find the velocity of P with position vector $(1/3, 2/3, -1/3)$.

Solution

The direction of the axis of rotation along the vector $-\mathbf{i}+2\mathbf{j}+2\mathbf{k}$ is given by the unit vector $\mathbf{e}=(-\mathbf{i}+2\mathbf{j}+2\mathbf{k})/3$, which also gives the direction of the angular velocity $\vec{\omega}$.

$$\vec{\omega}=\omega\mathbf{e}=\frac{90}{3}(-1, 2, 2) \text{ radian/sec}$$

This will be the same for all points of the rotating body.

Position vector of $P=(1/3, 2/3, -1/3)\equiv\mathbf{r}$.

If $\mathbf{v}_P$ denotes the linear velocity of P, then

$$\mathbf{v}_P=\vec{\omega}\times\mathbf{r}=30(-1, 2, 2)\times(1/3, 2/3, -1/3)$$

or

$$\mathbf{v}_P=10\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 2 & 2 \\ 1 & 2 & -1 \end{vmatrix}=10(-6, 1, -4)$$

Question

A rigid body is rotating about a fixed origin O. The points A and B have position vectors $A(0, -1, 2)$ and $B(2, 0, 0)$ respectively. Find the angular velocity of the body. (The units are in metres and seconds).

Solution

Position vector of $A=\mathbf{r}_A=(0, -1, 2)$

Linear vector of A viz. $\mathbf{v}_A=(7, -2, -1)$ is related to the angular velocity of the body by $\mathbf{v}_A=\vec{\omega}\times\mathbf{r}_A$. Hence

$$\mathbf{v}_A=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \omega_1 & \omega_2 & \omega_3 \\ 0 & -1 & 2 \end{vmatrix}$$

which can also be written as

$$(7,\ -2,\ -1) = (2\omega_2 + \omega_3,\ -2\omega_1,\ -\omega_1)$$

Comparing coefficients of i, j, and k, we obtain

$$2\omega_2 + \omega_3 = 7 \text{ and } 2\omega_1 = 2,\ -\omega_1 = -1 \qquad (1)$$

Both of these give the same solution, *viz.*, $\omega_1 = 1$.

Position vector of $B = \mathbf{r}_B = (2,\ 0,\ 0)$

Linear velocity of $B = \mathbf{v}_B = (0,\ 6,\ -4)$ is related to angular velocity by

$$\mathbf{v}_B = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \omega_1 & \omega_2 & \omega_3 \\ 2 & 0 & 0 \end{vmatrix}$$

which gives

$$(0,\ 6,\ -4) = \mathbf{i}(0) - \mathbf{j}(0 - 2\omega_3) + \mathbf{k}(-2\omega_2)$$

Comparing coefficients of i, j, and k on both sides, we obtain $0 = 0$, $2\omega_3 = 6$ *i.e.* $\omega_3 = 3$.

and $2\omega_2 = 4$ *i.e.* $\omega_2 = 2$.

Substituting for ω_2 and ω_3 in equation (1), we obtain $2\omega_2 + \omega_3 = 7$ *i.e.* $7 = 7$, which shows that the system is consistent. Hence

$$\vec{\omega} = \omega_1\mathbf{i} + \omega_2\mathbf{j} + \omega_3\mathbf{k} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$$

Question

The instantaneous velocities of particle at points

$$(a,0,0),\quad (0,\ a/\sqrt{3},\ 0),\quad (0,\ 0,\ 2a)$$

of a rigid body are $(u,\ 0,\ 0)$, $(u,\ 0,\ v)$, $(u+v,\ -\sqrt{3}v,\ v/2)$ respectively w.r.t. a rectangular coordinate system. Find the magnitude and direction of spin of the body and the point at which the central axis cuts the XZ-plane.

Solution

Let $A,\ B,\ C$ be the given points. Then

$$\mathbf{r}_A = (a,\ 0,\ 0),\quad \mathbf{r}_B = (0,\ a/\sqrt{3},\ 0),\quad \mathbf{r}_C = (0,\ 0,\ 2a)$$

The corresponding velocities are given by

$$\mathbf{v}_A = (u,\ 0,\ 0),\quad \mathbf{v}_B = (u,\ 0,\ v),\quad \mathbf{v}_C = (u+v,\ -\sqrt{3}v,\ v/2)$$

We take A as a reference point. Then

$$\mathbf{r}_1 = \text{position vector of } B \text{ w.r.t. } A = \mathbf{r}_B - \mathbf{r}_A = (-a,\ a/\sqrt{3},\ 0)$$

$$\mathbf{r}_2 = \text{position vector of } C \text{ w.r.t. } A = \mathbf{r}_C - \mathbf{r}_A = (-a,\ 0,\ 2a)$$

Let

$$\mathbf{v}_1 = \text{velocity of } B \text{ w.r.t. } A = \mathbf{v}_B - \mathbf{v}_A = (0,\ 0,\ v)$$

and

$$\mathbf{v}_2 = \text{velocity of } C \text{ w.r.t. } A = \mathbf{v}_C - \mathbf{v}_A = (v,\ -\sqrt{3}v,\ v/2)$$

Let $\vec{\omega}$ be the angular velocity of the rigid body w.r.t. any reference point. Then

$$\mathbf{v}_1 = \vec{\omega}\times\mathbf{r}_1,\quad \mathbf{v}_2 = \vec{\omega}\times\mathbf{r}_2$$

Therefore using the formula $\mathbf{v} = \vec{\omega}\times\mathbf{r}$, we obtain the following equations.

$$(0,0,\ v) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \omega_1 & \omega_2 & \omega_3 \\ -a & a/\sqrt{3} & 0 \end{vmatrix}$$

and

$$(v,\ -\sqrt{3}v,\ v/2) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \omega_1 & \omega_2 & \omega_3 \\ -a & 0 & 2a \end{vmatrix}$$

which are equivalent to

$$(0,0,\ v) = \left(\frac{-a\omega_3}{\sqrt{3}},\ -a\omega_3,\ \frac{a\omega_1}{\sqrt{3}} + a\omega_2\right)$$

and

$$\left(v, \quad -\sqrt{3}v, \quad v/2\right) \quad = \quad (2a\omega_2, \quad -a\,\omega_3 + 2a\omega_1, \quad a\omega_2)$$

From these equations we obtain

$$-\frac{a\omega_3}{\sqrt{3}} = 0, \quad \frac{a\,\omega_1}{\sqrt{3}} + a\,\omega_2 = v, \quad v = 2a\omega_2$$

On solving for ω_2, ω_3 we obtain $\omega_3 = 0$, and $\omega_2 = v/(2a)$.

To find ω_1, we have

$$\frac{a}{\sqrt{3}}\omega_1 + \frac{v}{2} = v$$

which gives $\omega_1 = \sqrt{3}v/(2a)$. Therefore

$$\vec{\omega} = \left(\frac{\sqrt{3}v}{2a}, \ \frac{v}{2a}, \ 0\right)$$

and

$$|\vec{\omega}| = \frac{v}{2a}(\sqrt{3+1}) = \frac{v}{a} \quad (v > 0, \quad a > 0)$$

Question

Show that equal and opposite rotations of a rigid body about distinct axes are equivalent to a translation of the body.

Solution

Let a rigid body be subjected to rotation with angular velocities $\vec{\omega}$ and $-\vec{\omega}$ about parallel axes passing through points O_1 and O_2. Let P be any particle of the body with such that its position vectors w.r.t. O_1 and O_2 are $\mathbf{r}_1$ and $\mathbf{r}_2$ and the directed line segment O_1O_2 is represented by the vectors. If $\mathbf{v}_P$ denotes the velocity of the particle P, then

$$\mathbf{v}_P = (\vec{\omega}\times\mathbf{r}_1) + (-\vec{\omega}\times\mathbf{r}_2) = \vec{\omega}\times(\mathbf{r}_1 - \mathbf{r}_2) \quad \text{or} \quad \mathbf{v}_P = \vec{\omega}\times\mathbf{s}$$

which is independent of P, i.e. this result is true for all particles of the rigid body. Hence the rigid body undergoes a translation.

Question

The points $(a,2a,-a)$, $(-a, -a, a)$, (a, a, a) of a rigid body have instantaneous velocities

$$(\sqrt{3}/2v,\ 0,\ \sqrt{3}/2),\ (-v/\sqrt{3},\ 0,\ -v/\sqrt{3}),\ (0,\ -v/\sqrt{3},\ v/\sqrt{3})$$

respectively w.r.t. a rectangular coordinate system. Show that the body has line through the origin having direction cosines $(1, -1, -1)/\sqrt{3}$ as instantaneous axis of rotation and that the magnitude of angular velocity is$v/2a$.

Solution

Let$A$, $B$and$C$ be the three points on the rigid body then

$$\mathbf{r}_A = (a,\ 2a,\ -a), \quad \mathbf{r}_B = (-a,\ -a,\ a), \quad \mathbf{r}_C = (a,\ a,\ a)$$

with velocities

$$\mathbf{v}_A = \left(\frac{\sqrt{3}v}{2},\ 0,\ \frac{\sqrt{3}v}{2}\right), \quad \mathbf{v}_B = \left(\frac{-v}{\sqrt{3}},\ 0,\ \frac{-v}{\sqrt{3}}\right), \quad \mathbf{v}_C = \left(0,\ \frac{-v}{\sqrt{3}},\ \frac{v}{\sqrt{3}}\right)$$

Let$A$be taken as the reference point. Then

$$\mathbf{r}_1 = \text{position vector of } B \text{w.r.t. } A = \mathbf{r}_B - \mathbf{r}_A = (-2a,\ -3a,\ 2a)$$

and

$$\mathbf{r}_2 = \text{position vector of } C \text{ w.r.t. } A = \mathbf{r}_C - \mathbf{r}_A = (0,\ -a,\ -2a)$$

Similarly the relative velocities$\mathbf{v}_1$ and$\mathbf{v}_2$ are given by

$$\mathbf{v}_1 = \mathbf{v}_B - \mathbf{v}_A = \left(-\frac{5v}{2\sqrt{3}},\ 0,\ -\frac{5v}{2\sqrt{3}}\right)$$

and

$$\mathbf{v}_2 = \mathbf{v}_C - \mathbf{v}_A = \left(-\frac{\sqrt{3}v}{2},\ -\frac{v}{\sqrt{3}},\ -\frac{v}{2\sqrt{3}}\right)$$

Let $\vec{\omega}$ be the angular velocity of the rigid body w.r.t. any point. Then

$$\mathbf{v}_1 = \vec{\omega} \times \mathbf{r}_1, \qquad \mathbf{v}_2 = \vec{\omega} \times \mathbf{r}_2$$

These relations can also be written as

$$\mathbf{v}_1 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \omega_1 & \omega_2 & \omega_3 \\ -2a & -3a & 2a \end{vmatrix} \tag{1}$$

and

$$\mathbf{v}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \omega_1 & \omega_2 & \omega_3 \\ 0 & -a & -2a \end{vmatrix} \tag{2}$$

Equating the coefficients of $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$ on both sides of (1), we obtain the equations

$$-\frac{5v}{2\sqrt{3}} = 2a\omega_2 + 3a\omega_3 \tag{3}$$

$$0 = -2a(\omega_3 + \omega_1) \tag{4}$$

and

$$-\frac{5v}{2\sqrt{3}} = 2a\omega_2 - 3a\omega_1 \tag{5}$$

From (4) $\omega_3 = -\omega_1$, and on substituting in (3) and (5) they become identical. Therefore all the components of angular velocity cannot be calculated from relation (1). Therefore we now use relation (2) and equate the coefficients of $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$ on both sides of (2). This gives

$$-\frac{\sqrt{3}v}{2} = 2a\omega_2 + a\omega_3 \tag{6}$$

$$-\frac{v}{\sqrt{3}} = -2a\omega_1 \tag{7}$$

and

$$-\frac{v}{2\sqrt{3}} = -a\omega_1 \tag{8}$$

Here equations (7) and (8) are the same and give $\omega_1 = v/(2\sqrt{3}a)$. On substituting this in (6) or (3), we obtain $\omega_2 = -v/(2\sqrt{3}a)$. Hence the angular velocity vector is given by

$$\vec{\omega} = \left(\frac{v}{2\sqrt{3}a},\ -\frac{v}{2\sqrt{3}a},\ -\frac{v}{2\sqrt{3}a}\right)$$

The angular speed is given by

$$|\vec{\omega}| = \sqrt{\omega_1^2 + \omega_2^2 + \omega_3^2} = \sqrt{3 \times \frac{v^2}{12a^2}} = \frac{v}{2a}$$

The direction cosines of the axis of rotation are the components of a unit vector along the angular velocity and are given by

$$\frac{\vec{\omega}}{\omega} \text{ or } \left(\frac{1}{\sqrt{3}}, \ -\frac{1}{\sqrt{3}}, \ -\frac{1}{\sqrt{3}}\right)$$

Question

Masses of 1,2and 3kg are located at positions $\hat{\imath}+\hat{\jmath}+\hat{k}, 4\hat{\jmath}+\hat{k}, 2\hat{\imath}+2\hat{k}$ respectively. If their velocities are $7\hat{\imath}, -6\hat{\jmath}, -3\hat{\imath}$. Find the position and velocity of centre of mass. Also find the angular momentum of the system with respect to the origin.

Solution

Given that $m_1 = 1kg, m_2 = 2kg, m_3 = 3kg$

$\vec{r}_1 = \hat{\imath}+\hat{\jmath}+\hat{k}, \vec{r}_2 = 4\hat{\jmath}+\hat{k}, \vec{r}_3 = 2\hat{\imath}+2\hat{k}$ and $\vec{v}_1 = 7\hat{\imath}, \vec{v}_2 = -6\hat{\jmath}, \vec{v}_3 = -3\hat{\imath}$

Radius vector of centre of mass is given by $\vec{R} = \frac{\sum_{i=1}^{3} m_i\vec{r}_i}{M=\sum_{i=1}^{3} m_i}$

$\Rightarrow \vec{R} = \frac{m_1\vec{r}_1+m_2\vec{r}_2+m_3\vec{r}_3}{m_1+m_2+m_3} = \frac{7\hat{\imath}+9\hat{\jmath}+13\hat{k}}{6}$

Velocity of centre of mass is given by $\vec{v}_{cm} = \frac{\sum_{i=1}^{3} m_i\vec{v}_i}{M=\sum_{i=1}^{3} m_i}$

$\Rightarrow \vec{v}_{cm} = \frac{m_1\vec{v}_1+m_2\vec{v}_2+m_3\vec{v}_3}{m_1+m_2+m_3} = \frac{-\hat{\imath}-6\hat{\jmath}}{3}$

Angular momentum of system of particles is given by $\vec{L} = \sum_{i=1}^{3} \vec{r}_i \times \vec{P}_i$

$\Rightarrow \vec{L} = \sum_{i=1}^{3} \vec{r}_i \times m_i\vec{v}_i = \sum_{i=1}^{3} m_i(\vec{r}_i \times \vec{v}_i)$

$\Rightarrow \vec{L} = m_1(\vec{r}_1 \times \vec{v}_1) + m_2(\vec{r}_2 \times \vec{v}_2) + m_3(\vec{r}_3 \times \vec{v}_3)$

$\Rightarrow \vec{L} = (1)\big(\hat{\imath}+\hat{\jmath}+\hat{k} \times 7\hat{\imath}\big) + (2)\big(4\hat{\jmath}+\hat{k} \times -6\hat{\jmath}\big) + (3)\big(2\hat{\imath}+2\hat{k} \times -3\hat{\imath}\big)$

$\Rightarrow \vec{L} = 36\hat{\imath} - 11\hat{\jmath} - 7\hat{k}$

Question

Masses of 4,3and 1kg moves under a force such that their position vectors at time t are $\vec{r}_1 = 3\hat{j} + 2t^2\hat{k}, \vec{r}_2 = 3t\hat{i} - \hat{k}, \vec{r}_3 = 4t\hat{i} + t^2\hat{j}$ respectively. Find the position vector and velocity of the centre of mass and angular momentum of the system with respect to the origin at $t = 2s$.

Solution

Given that $m_1 = 4kg, m_2 = 3kg, m_3 = 1kg$

$\vec{r}_1 = 3\hat{j} + 2t^2\hat{k}, \vec{r}_2 = 3t\hat{i} - \hat{k}, \vec{r}_3 = 4t\hat{i} + t^2\hat{j}$

$\Rightarrow \vec{v}_1 = 4t\hat{k}, \vec{v}_2 = 3\hat{i}, \vec{v}_3 = 4\hat{i} + 2t\hat{j}$

Radius vector of centre of mass is given by $\vec{R} = \frac{\sum_{i=1}^{3} m_i\vec{r}_i}{M=\sum_{i=1}^{3} m_i}$

$\Rightarrow \vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3}{m_1+m_2+m_3} = \frac{26\hat{i}+16-29\hat{k}}{8}$ at $t = 2s$

Velocity of centre of mass is given by $\vec{v}_{cm} = \frac{\sum_{i=1}^{3} m_i\vec{v}_i}{M=\sum_{i=1}^{3} m_i}$

$\Rightarrow \vec{v}_{cm} = \frac{m_1\vec{v}_1 + m_2\vec{v}_2 + m_3\vec{v}_3}{m_1+m_2+m_3} = \frac{13\hat{i}+4\hat{j}+32\hat{k}}{8}$ at $t = 2s$

Angular momentum of system of particles is given by $\vec{L} = \sum_{i=1}^{3} \vec{r}_i \times \vec{P}_i$

$\Rightarrow \vec{L} = \sum_{i=1}^{3} \vec{r}_i \times m_i\vec{v}_i = \sum_{i=1}^{3} m_i(\vec{r}_i \times \vec{v}_i)$

$\Rightarrow \vec{L} = m_1(\vec{r}_1 \times \vec{v}_1) + m_2(\vec{r}_2 \times \vec{v}_2) + m_3(\vec{r}_3 \times \vec{v}_3)$

$\Rightarrow \vec{L} = 96\hat{i} + 9\hat{j} + 16\hat{k}$ at $t = 2s$

Question

Particle of Masses 1,2and 4kg moves under a force such that their position vectors at time t are $\vec{r}_1 = 2\hat{\imath} + 4t^2\hat{k}, \vec{r}_2 = 4t\hat{\imath} - \hat{k}, \vec{r}_3 = cos\pi t\hat{\imath} + sin\pi t\hat{\jmath}$ respectively. Find the angular momentum of the system with respect to the origin at $t = 1s$.

Solution Given that $m_1 = 1kg, m_2 = 2kg, m_3 = 4kg$

$\vec{r}_1 = 2\hat{\imath} + 4t^2\hat{k}, \vec{r}_2 = 4t\hat{\imath} - \hat{k}, \vec{r}_3 = cos\pi t\hat{\imath} + sin\pi t\hat{\jmath}$

$\Rightarrow \vec{v}_1 = 8t\hat{k}, \vec{v}_2 = 4\hat{\imath}, \vec{v}_3 = -\pi sin\pi t\hat{\imath} + \pi cos\pi t\hat{\jmath}$

Angular momentum of system of particles is given by $\vec{L} = \sum_{i=1}^{3} \vec{r}_i \times \vec{P}_i$

$\Rightarrow \vec{L} = \sum_{i=1}^{3} \vec{r}_i \times m_i\vec{v}_i = \sum_{i=1}^{3} m_i(\vec{r}_i \times \vec{v}_i)$

$\Rightarrow \vec{L} = m_1(\vec{r}_1 \times \vec{v}_1) + m_2(\vec{r}_2 \times \vec{v}_2) + m_3(\vec{r}_3 \times \vec{v}_3) = -24\hat{\jmath} + 4\pi\hat{k}$ at $t = 1s$

Question

The position vectors and velocities of Masses 2,3and 4kg are respectively $2\hat{\imath} - 3\hat{\jmath}$, $\hat{\imath} + \hat{\jmath} + \hat{k}, 4\hat{\jmath} + 3\hat{k}$. If their velocities are $-3\hat{\imath}, -6\hat{\jmath}, 2\hat{\imath} + 3\hat{k}$. Find the position and velocity of centre of mass. Also find the total angular momentum of the system with respect to the origin.

Solution Given that $m_1 = 2kg, m_2 = 3kg, m_3 = 4kg$

$\vec{r}_1 = 2\hat{\imath} - 3\hat{\jmath}, \vec{r}_2 = \hat{\imath} + \hat{\jmath} + \hat{k}, \vec{r}_3 = 4\hat{\jmath} + 3\hat{k} \Rightarrow \vec{v}_1 = -3\hat{\imath}, \vec{v}_2 = -6\hat{\jmath}, \vec{v}_3 = 2\hat{\imath} + 3\hat{k}$

Radius vector of centre of mass is given by $\vec{R} = \frac{\sum_{i=1}^{3} m_i\vec{r}_i}{M=\sum_{i=1}^{3} m_i}$

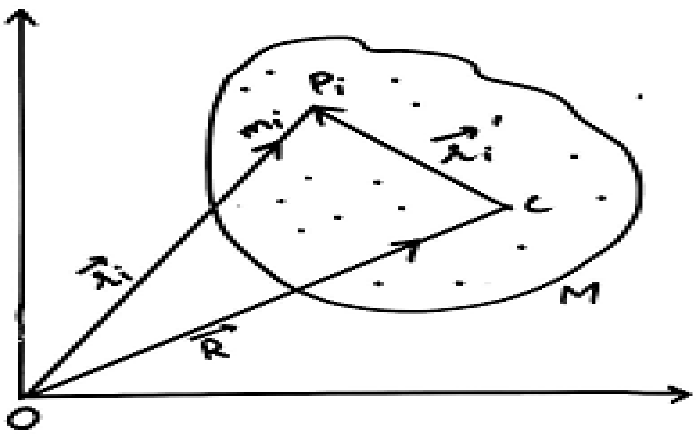
$\Rightarrow \vec{R} = \frac{m_1\vec{r}_1+m_2\vec{r}_2+m_3\vec{r}_3}{m_1+m_2+m_3} = \frac{7\hat{\imath}+13\hat{\jmath}+15\hat{k}}{9}$

Velocity of centre of mass is given by $\vec{v}_{cm} = \frac{\sum_{i=1}^{3} m_i\vec{v}_i}{M=\sum_{i=1}^{3} m_i} = \frac{-18\hat{\jmath}+12\hat{k}}{9}$

Angular momentum of system of particles is given by $\vec{L} = \sum_{i=1}^{3} \vec{r}_i \times \vec{P}_i$

$\Rightarrow \vec{L} = 48\hat{\imath} + 24\hat{\jmath} - 18\hat{k}$

Kinetic Energy of System of Particles



For a system of particles $K.E = \frac{1}{2}\sum_{i=1}^{n} m_i \times v_i^2$

$\Rightarrow K.E = \frac{1}{2}\sum_{i=1}^{n} m_i \times (\vec{v}_i.\vec{v}_i)$ ………………(1)

From figure by Head to Tail rule

$\vec{r}_i = \vec{R} + \vec{r}_i' \Rightarrow \frac{d\vec{r}_i}{dt} = \frac{d\vec{R}}{dt} + \frac{d\vec{r}_i'}{dt} \Rightarrow \vec{v}_i = \vec{v}_{cm} + \vec{v}_i'$

$(1) \Rightarrow K.E = \frac{1}{2}\sum_{i=1}^{n} m_i \times (\vec{v}_{cm} + \vec{v}_i').(\vec{v}_{cm} + \vec{v}_i')$

$\Rightarrow K.E = \frac{1}{2}\sum_{i=1}^{n} m_i \times (\vec{v}_{cm}.\vec{v}_{cm} + \vec{v}_{cm}.\vec{v}_i' + \vec{v}_i'.\vec{v}_{cm} + \vec{v}_i'.\vec{v}_i')$

$\Rightarrow K.E = \frac{1}{2}\sum_{i=1}^{n} m_i \times \left(v_{cm}^2 + 2\vec{v}_{cm}.\vec{v}_i' + {v_i'}^2\right)$

$\Rightarrow K.E = \frac{1}{2}(\sum_{i=1}^{n} m_i)v_{cm}^2 + \frac{1}{2}2\vec{v}_{cm}.\sum_{i=1}^{n} m_i\,\vec{v}_i' + \frac{1}{2}\sum_{i=1}^{n} m_i\,{v_i'}^2$

$\Rightarrow K.E = \frac{1}{2}Mv_{cm}^2 + \frac{1}{2}2\vec{v}_{cm}.(0) + \frac{1}{2}\sum_{i=1}^{n} m_i\,{v_i'}^2 \Rightarrow K.E = \frac{1}{2}Mv_{cm}^2 + \frac{1}{2}\sum_{i=1}^{n} m_i\,{v_i'}^2$

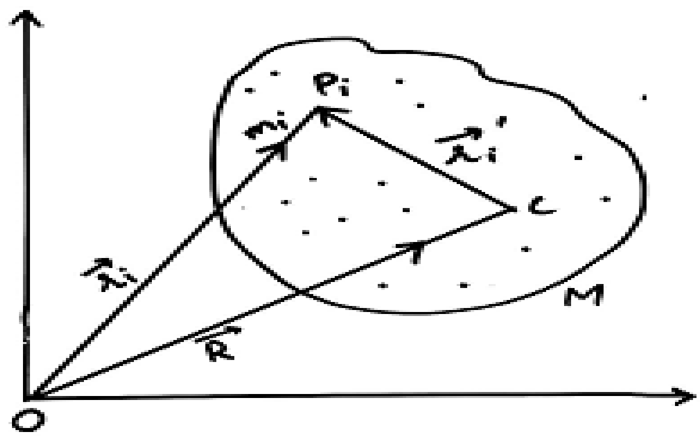
$\Rightarrow \vec{T} = \vec{T}_0 + \vec{T}_i'$

Hence **total K.E. $\vec{T}$ of a system of particles is equal to the sum of K.E. $\vec{T}_0$ of centre of mass w.r.to origin and K.E. $\vec{T}_i'$ of i[th] particle w.r.to centre of mass of a system of particle.**

Remark

- **Translational Motion:** Motion of a body in a straight line on the plane or rough surface.
- **Rotational Motion:** Motion of a body about a fixed axis in the space.

Kinetic Energy of a Rigid Body in General (Konig Theorem)



For a system of particles $K.E = T = \frac{1}{2}\int v_i^2\, dm$ ………………(1)

From figure by Head to Tail rule

$\vec{r}_i = \vec{R} + \vec{r}_i' \Rightarrow \frac{d\vec{r}_i}{dt} = \frac{d\vec{R}}{dt} + \frac{d\vec{r}_i'}{dt} \Rightarrow \vec{v}_i = \dot{\vec{R}} + \dot{\vec{r}}_i'$

Now $v_i^2 = \vec{v}_i.\vec{v}_i = \left(\dot{\vec{R}} + \dot{\vec{r}}_i'\right).\left(\dot{\vec{R}} + \dot{\vec{r}}_i'\right) = \dot{R}^2 + \dot{r}_i'^2 + 2\dot{\vec{R}}.\dot{\vec{r}}_i'$

$(1) \Rightarrow T = \frac{1}{2}\int v_i^2\, dm = \frac{1}{2}\int \left[\dot{R}^2 + \dot{r}_i'^2 + 2\dot{\vec{R}}.\dot{\vec{r}}_i'\right] dm$

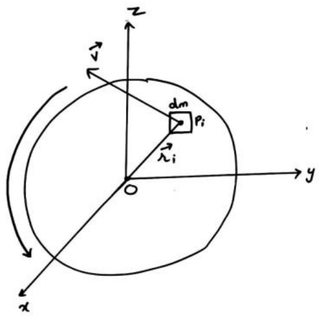
$\Rightarrow T = \frac{1}{2}\dot{R}^2 \int dm + \frac{1}{2}\int \dot{r}_i'^2\, dm + \dot{\vec{R}}.\int \dot{\vec{r}}_i'\, dm$

$\Rightarrow T = \frac{1}{2}\dot{R}^2 M + \frac{1}{2}\int \dot{r}_i'^2\, dm + \dot{\vec{R}}.(0) \Rightarrow T = \frac{1}{2}M\dot{R}^2 + \frac{1}{2}\int \dot{r}_i'^2\, dm$

$\Rightarrow \vec{T} = \vec{T}_0 + \vec{T}'$

$\Rightarrow \vec{T} = \vec{T}_{translational} + \vec{T}'_{rotational}$

Kinetic Energy of a Rigid Body Rotating about a Fixed Point



Consider a rigid body rotate about a fixed point O. Consider a point $P_i(x_i, y_i, z_i)$ which rotate with the motion of rigid body then $K.E = T = \frac{1}{2}Mv^2$

Kinetic Energy for single particle at P_i of mass dm is given by $dT = \frac{1}{2}v^2dm$

For whole body we get $\int dT = \frac{1}{2}\int v^2dm$

$\Rightarrow T_{rot} = \frac{1}{2}\int v^2dm$ …………………..(i)

In case of rotation $\vec{v} = \vec{\omega}\times\vec{r}$ where $\vec{\omega} = constant$ is angular velocity

$$\Rightarrow \vec{v} = \left(\omega_x\hat{\imath} + \omega_y\hat{\jmath} + \omega_z\hat{k}\right)\times\left(x\hat{\imath} + y\hat{\jmath} + z\hat{k}\right)$$

$$\Rightarrow \vec{v} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \omega_x & \omega_y & \omega_z \\ x & y & z \end{vmatrix}$$

$$\Rightarrow \vec{v} = \left(z\omega_y - y\omega_z\right)\hat{\imath} + (x\omega_z - z\omega_x)\hat{\jmath} + \left(y\omega_x - x\omega_y\right)\hat{k}$$

$$\Rightarrow v^2 = \left(z\omega_y - y\omega_z\right)^2 + (x\omega_z - z\omega_x)^2 + \left(y\omega_x - x\omega_y\right)^2$$

$$(i) \Rightarrow T_{rot} = \frac{1}{2}\int\left[\left(z\omega_y - y\omega_z\right)^2 + (x\omega_z - z\omega_x)^2 + \left(y\omega_x - x\omega_y\right)^2\right]dm$$

$$\Rightarrow T_{rot} =$$

$$\frac{1}{2}\int\begin{bmatrix} z^2\omega_y^2 + y^2\omega_z^2 - 2yz\omega_y\omega_z + x^2\omega_z^2 + z^2\omega_x^2 - 2xz\omega_x\omega_z + y^2\omega_x^2 \\ +x^2\omega_y^2 - 2xy\omega_x\omega_y \end{bmatrix}dm$$

$$\Rightarrow T_{rot} = \frac{1}{2}\begin{bmatrix} \omega_x^2\int(y^2 + z^2)dm + \omega_y^2\int(x^2 + z^2)dm + \omega_z^2\int(x^2 + y^2)dm \\ -2\omega_x\omega_y\int xydm - 2\omega_y\omega_z\int yzdm - 2\omega_z\omega_x\int zxdm \end{bmatrix}$$

$$\Rightarrow \boldsymbol{T_{rot} = \frac{1}{2}\left[\omega_x^2 I_{xx} + \omega_y^2 I_{yy} + \omega_z^2 I_{zz} - 2\omega_x\omega_y I_{xy} - 2\omega_y\omega_z I_{yz} - 2\omega_z\omega_x I_{zx}\right]}$$

In terms of matrix we have $\Rightarrow \boldsymbol{T_{rot}} = \frac{\mathbf{1}}{\mathbf{2}}\begin{bmatrix}\boldsymbol{\omega_x}\\ \boldsymbol{\omega_y}\\ \boldsymbol{\omega_z}\end{bmatrix}^t \begin{bmatrix}\boldsymbol{I_{xx}} & \boldsymbol{-I_{xy}} & \boldsymbol{-I_{xz}}\\ \boldsymbol{-I_{yx}} & \boldsymbol{I_{yy}} & \boldsymbol{-I_{yz}}\\ \boldsymbol{-I_{zx}} & \boldsymbol{-I_{zy}} & \boldsymbol{I_{zz}}\end{bmatrix}\begin{bmatrix}\boldsymbol{\omega_x}\\ \boldsymbol{\omega_y}\\ \boldsymbol{\omega_z}\end{bmatrix}$

Special Case: When the body rotates about the principle axis then

Product of inertias $= I_{xy} = I_{yz} = I_{zx} = 0$ then

$\Rightarrow \boldsymbol{T_{rot}} = \frac{\mathbf{1}}{\mathbf{2}}\left[\boldsymbol{\omega_x^2 I_{xx} + \omega_y^2 I_{yy} + \omega_z^2 I_{zz}}\right]$ required expression

Question

Find the K.E of homogeneous circular cylinder of mass m and radius a rolling on a plane with linear velocity.

Solution

Since we know that $\vec{T} = \vec{T}_{translational} + \vec{T}'_{rotational}$

$\Rightarrow \vec{T} = \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2$ ……………….(i)

In case of cylinder moment of inertia $= I = \frac{1}{2}Ma^2$

$(i) \Rightarrow \vec{T} = \frac{1}{2}Mv^2 + \frac{1}{2}.\frac{1}{2}Ma^2.\frac{v^2}{a^2}$ since $v = r\omega$

$\Rightarrow \vec{T} = \frac{1}{2}Mv^2 + \frac{1}{4}Mv^2 \Rightarrow \vec{T} = \frac{3}{4}Mv^2$

Kinetic Energy in terms of Rotational and Angular Momentum

Consider a rigid body rotating about an axis passing through a fixed point in it with an angular velocity $\vec{\omega}$ consisting of n – particles of mass m_i where position vector is $\vec{r}_i$ moving with velocity $\vec{v}_i$. Then expressing of kinetic energy is given by

$K.E = T = \frac{1}{2}\sum_{i=1}^{n} m_i v_i^2$

$\Rightarrow K.E = \frac{1}{2}\sum_{i=1}^{n} m_i(\vec{v}_i.\vec{v}_i) = \frac{1}{2}\sum_{i=1}^{n} m_i(\vec{\omega}\times\vec{r}_i.\vec{v}_i) = \frac{1}{2}\vec{\omega}.\sum_{i=1}^{n}\vec{r}_i \times m_i\vec{v}_i$

$\Rightarrow K.E = \frac{1}{2}\vec{\omega}.\sum_{i=1}^{n}\vec{r}_i \times \vec{P}_i \Rightarrow \boldsymbol{K.E = T_{rot} = \frac{1}{2}\vec{\omega}.L}$

Radius of Gyration of Various Bodies

Radius of gyration of a body is defined as the distance from the reference axis at which the given area is assumed to be compressed and kept as a thin strip, such that there is no change in its moment of inertia. It specifies the distribution of the elements of body around the axis in terms of the mass moment of inertia, As it is the perpendicular distance from the axis of rotation to a point mass m that gives an equivalent inertia to the original object m The nature of the object does not affect the concept, which applies equally to a surface bulk mass.

Mathematically the radius of gyration is the root mean square distance of the object's parts from either its center of mass or the given axis, depending on the relevant application.

Let $I = \sum m_i d_i^2$ be the moment of inertia of a system of particles about AB, and $M = \sum m_i$ be the total mass of the system. Then the quantity K such that

$$K^2 = \frac{I}{M} = \frac{\sum m_i d_i^2}{\sum m_i} \text{ or } K = \sqrt{\frac{I}{M}} = \sqrt{\frac{\sum m_i d_i^2}{\sum m_i}}$$

is called the radius of gyration of the system AB.

Example

Find the radius of gyration, K, of the triangular lamina of mass M and moment of inertia $I = \frac{1}{6}Mh^2$.

Solution

Since formula for radius of gyration is given by

$$K^2 = \frac{I}{M} = \frac{\frac{1}{6}Mh^2}{M} \text{ or } K = \sqrt{\frac{1}{6}h^2}$$

$$K = \frac{h}{\sqrt{6}}$$

The Compound Pendulum

The compound pendulum provides a simple example of motion about a fixed axis. By fixed axis we mean that the direction of the axis of rotation is always along the same line; the axis itself may move along this line. For example a car wheel attached to an axle undergoes fixed axis rotation as long as the car drives straight ahead. If the car turns, the wheel must rotate about a vertical axis while simultaneously spinning on the axle. In this case the motion is not a rotation about a fixed axis.

For simplification, the axis of rotation may be chosen along any of the coordinate axes. If we choose Z-axis as the axis of rotation, then

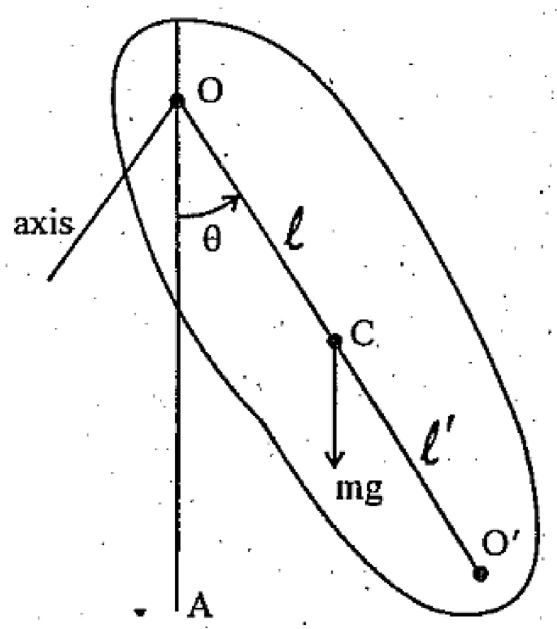
$$\mathbf{L} = L_z \mathbf{k} = I_{zz}\omega_z \mathbf{k} \equiv I_{33}\omega_3 \mathbf{k}$$

i.e.

$$L_z = I_{zz}\omega_z$$

For a simple pendulum of mass m and length l, (suspended by a massless string), the period of oscillation is given by $\tau = 2\pi\sqrt{l/g}$.

As an example of a rigid body motion about a fixed axis, we consider a compound pendulum. The pendulum is free to rotate about a horizontal axis through a point O. The symbols are defined in the diagram. Taking



moments about the point of suspension O, and denoting the torque of external forces about O by G_0, we have

$$G_0 = \dot{L}_0 = I_0\dot{\omega} = I_0\dot{\omega} = I_0\ddot{\theta} \tag{8.2.1}$$

Since the only force acting in this case is mg acting downwards, we have

$$G_0 = -mgl_1 \sin\theta \tag{8.2.2}$$

From (8.2.1) and (8.2.2)

$$\ddot{\theta} = -\frac{mgl}{I_0}\sin\theta$$

For motion with small angular displacement, the last equation reduces to

$$\ddot{\theta} + \frac{mgl}{I_0}\theta = 0 \tag{8.2.3}$$

General solution of (8.2.3) can be written as $\theta = \theta_0 \cos(\omega_0 t + \alpha_0)$ where θ_0 is the amplitude of the motion and ω_0 denotes the natural angular (or circular) frequency given by

$$\omega_0 = \sqrt{\frac{mgl}{I_0}} \tag{8.2.4}$$

The natural cyclic frequency ν_0 and the time interval τ_0 are therefore given by

$$\nu_0 = \frac{1}{2\pi}\sqrt{\frac{mgl}{I_0}} \tag{8.2.5a}$$

and

$$\tau_0 = \frac{1}{\nu_0} = 2\pi\sqrt{\frac{I_0}{mgl}} \tag{9.2.5b}$$

Now if we put $I_0/m = K^2$ where K is the radius of gyration, then

$$\tau_0 = 2\pi\sqrt{\frac{K^2}{gl}}$$

By comparing this result with the corresponding result for the simple harmonic motion, *viz.* $\tau_0 = 2\pi\sqrt{l/g}$, we find that the period of a physical pendulum is equal to that of a simple pendulum of length $l' = K^2/l$.

Next we want to investigate whether there exists other axes of suspension of the physical pendulum corresponding to which the period is the same as given by (9.2.5 b). If l' denotes the distance of the other point of suspension from the centroid C, then

$$2\pi\sqrt{\frac{I_0}{mgl}} = 2\pi\sqrt{\frac{I_0'}{mgl'}} \tag{8.2.6}$$

Now applying the parallel-axis theorem, we obtain

$$I_0 = I_{cm} + ml^2 = mK_{cm}^2 + ml^2 \tag{8.2.7a}$$

$$I_0' = I_{cm} + ml'^2 = mK_{cm}^2 + ml'^2 \tag{8.2.7b}$$

where K_{cm} is the radius of gyration about the centroid and is given by $I_{cm} = mK_{cm}^2$, where I_{cm} is the moment of inertia about the centroid. On substitution from (8 a, b) into (8.2. 6), we have

$$\frac{K_{c.m.}^2 + l^2}{l} = \frac{K_{c.m.}^2 + l'^2}{l'}$$

or

$$l l' = K_{c.m.}^2 \tag{8.2.8}$$

The result (8.2.8) shows that the alternative axis of rotation is located at a distance $l' = K_{c.m.}^2/l$ from the mass centre. This axis passes through the point O' in the figure. The point O' is called *centre of oscillation* or *centre of suspension* w.r.t. O. Similarly O is the centre of suspension w.r.t. O'.

Case of uniform rod

For a uniform rod of length l suspended at one end, $I_c = ml^2/12$. Also $l' = l/2$, $l' = K_{c.m.}^2 = l/6$.

The location of the centre of oscillation O' is therefore $l/2 + l/6 = 2l/3$ from O, the point of support. This is identical with the location of the centre of percussion.

Question

Difference between simple and compound pendulums.

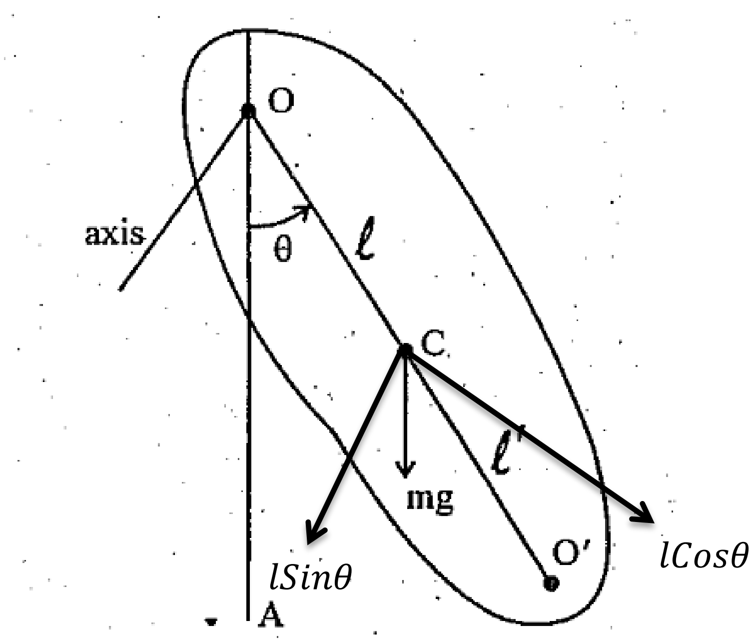
Answer

- The metallic bob suspended by a weightless inextensible string is called **simple pendulum**. The distance between point of suspension and center of bob is called **length of simple pendulum**. The bob at rest when no resultant force acts on it is called **mean position** or **equilibrium position**. But a **physical or compound pendulum** is a rigid body that oscillates due to its own weight about a horizontal axis that does not pass through the center of mass of the body.
- In simple pendulum we have point mass/single mass particle but in compound pendulum we have not a point mass, we have distribution of mass. In compound pendulum we first define center of gravity, we define all particles distribution by centre of mass.

Question

Obtain the equation of motion of compound pendulums.

Answer



K.E. of rotation $= T = \frac{1}{2}I_0\omega^2 = \frac{1}{2}I_0\dot{\theta}^2$

P.E. of rotation $= V = \frac{\partial V}{\partial x}\hat{\imath} + \frac{\partial V}{\partial y}\hat{\jmath} + \frac{\partial V}{\partial z}\hat{k} = -Mg\hat{\jmath}$

P.E. of rotation in component form $= V = -MglCos\theta$

By the principal of conservation of energy $\quad T + V =$ Constant

$\frac{1}{2}I_0\dot{\theta}^2 - MglCos\theta = C \Rightarrow \frac{1}{2}I_0\left(2\theta\ddot{\theta}\right) + MglSin\theta\dot{\theta} = 0$

$\Rightarrow \ddot{\theta} + \frac{Mgl}{I_0}Sin\theta = 0$ after simplification

Question

Show that length of simple pendulum is equivalent to compound pendulum.

Answer

By equation of motion of simple pendulum we have $\ddot{\theta} + \frac{g}{l}Sin\theta = 0$

For small vibration $Sin\theta = \theta$ then $\quad \ddot{\theta} + \frac{g}{l}\theta = 0$

$\ddot{\theta} = -\frac{g}{l}\theta$ ……………..(1)

By equation of motion of compound pendulum we have $\ddot{\theta} + \frac{Mgl}{I_0}Sin\theta = 0$

For small vibration $Sin\theta = \theta$ then $\quad \ddot{\theta} + \frac{Mgl}{I_0}\theta = 0$

$\ddot{\theta} = -\frac{Mgl}{I_0}\theta$ ……………..(2)

Comparing (1) and (2) we have $\quad -\frac{g}{l}\theta = -\frac{Mgl}{I_0}\theta$

$\frac{g}{l} = \frac{Mgl}{I_0}$

$l = \frac{I_0}{Ml}$ which is equal to compound pendulum.

CHAPTER

7

MOTION OF RIGID BODIES IN THREE DIMENSIONS AND MOMEMNT OF INERTIA OF RIGID BODIES

Relation b/w Angular Momentum and Moment of Inertia
Or Angular Momentum in Terms of Moment of Inertia

Consider a rigid body consisting of n – particles of $m_i; i = 1,2,3,\dots,n$ which rotate and translate then angular momentum about origin is

$\vec{L} = \sum_{i=1}^{n} \vec{r}_i \times \vec{P}_i$

$\Rightarrow \vec{L} = \sum_{i=1}^{n} \vec{r}_i \times m_i\vec{v}_i \Rightarrow \vec{L} = \sum_{i=1}^{n} m_i(\vec{r}_i \times \vec{v}_i)$

In case of rotation $\vec{v}_i = \vec{\omega}_i \times \vec{r}_i$

Since angular velocity remains same for each particle of a rigid body

$\Rightarrow \vec{L} = \sum_{i=1}^{n} m_i\big(\vec{r}_i \times (\vec{\omega}_i \times \vec{r}_i)\big)$

$\Rightarrow \vec{L} = \sum_{i=1}^{n} m_i[(\vec{r}_i.\vec{r}_i)\vec{\omega}_i - (\vec{r}_i.\vec{\omega}_i)\vec{r}_i]$

$\Rightarrow \vec{L} = \sum_{i=1}^{n} m_i[r_i^2\vec{\omega}_i - (\vec{r}_i.\vec{\omega}_i)\vec{r}_i]$ ……………..(i)

Consider position vector for each particle is

$\vec{r}_i = x_i\hat{\imath} + y_i\hat{\jmath} + z_i\hat{k} \Rightarrow r_i^2 = x_i^2 + y_i^2 + z_i^2$ and $\vec{\omega}_i = \omega_x\hat{\imath} + \omega_y\hat{\jmath} + \omega_z\hat{k}$

$\vec{r}_i.\vec{\omega}_i = x_i\omega_x + y_i\omega_y + z_i\omega_z$ and $\vec{L} = L_x\hat{\imath} + L_y\hat{\jmath} + L_z\hat{k}$

$(i) \Rightarrow L_x\hat{\imath} + L_y\hat{\jmath} + L_z\hat{k} = \sum_{i=1}^{n} m_i\big[\big(x_i^2 + y_i^2 + z_i^2\big)\big(\omega_x\hat{\imath} + \omega_y\hat{\jmath} + \omega_z\hat{k}\big) - \big(x_i\omega_x + y_i\omega_y + z_i\omega_z\big)\big(x_i\hat{\imath} + y_i\hat{\jmath} + z_i\hat{k}\big)\big]$

$$\Rightarrow L_x\hat{\imath} + L_y\hat{\jmath} + L_z\hat{k} =$$

$$\sum_{i=1}^{n} m_i \begin{bmatrix} \{(x_i^2 + y_i^2 + z_i^2)\omega_x - (x_i\omega_x + y_i\omega_y + z_i\omega_z)x_i\}\hat{\imath} \\ +\{(x_i^2 + y_i^2 + z_i^2)\omega_y - (x_i\omega_x + y_i\omega_y + z_i\omega_z)y_i\}\hat{\jmath} \\ +\{(x_i^2 + y_i^2 + z_i^2)\omega_z - (x_i\omega_x + y_i\omega_y + z_i\omega_z)z_i\}\hat{k} \end{bmatrix}$$

$$\Rightarrow L_x\hat{\imath} + L_y\hat{\jmath} + L_z\hat{k} =$$

$$\sum_{i=1}^{n} m_i \begin{bmatrix} \{(x_i^2 + y_i^2 + z_i^2)\omega_x - x_i^2\omega_x - x_iy_i\omega_y - x_iz_i\omega_z\}\hat{\imath} \\ +\{(x_i^2 + y_i^2 + z_i^2)\omega_y - x_iy_i\omega_x - y_i^2\omega_y - y_iz_i\omega_z\}\hat{\jmath} \\ +\{(x_i^2 + y_i^2 + z_i^2)\omega_z - x_iz_i\omega_x + y_iz_i\omega_y + z_i^2\omega_z\}\hat{k} \end{bmatrix}$$

$$\Rightarrow L_x\hat{\imath} + L_y\hat{\jmath} + L_z\hat{k} =$$

$$\sum_{i=1}^{n} m_i \begin{bmatrix} \{x_i^2\omega_x + y_i^2\omega_x + z_i^2\omega_x - x_i^2\omega_x - x_iy_i\omega_y - x_iz_i\omega_z\}\hat{\imath} \\ +\{x_i^2\omega_y + y_i^2\omega_y + z_i^2\omega_y - x_iy_i\omega_x - y_i^2\omega_y - y_iz_i\omega_z\}\hat{\jmath} \\ +\{x_i^2\omega_z + y_i^2\omega_z + z_i^2\omega_z - x_iz_i\omega_x + y_iz_i\omega_y + z_i^2\omega_z\}\hat{k} \end{bmatrix}$$

$$\Rightarrow L_x\hat{\imath} + L_y\hat{\jmath} + L_z\hat{k} = \sum_{i=1}^{n} m_i \begin{bmatrix} \{y_i^2\omega_x + z_i^2\omega_x - x_iy_i\omega_y - x_iz_i\omega_z\}\hat{\imath} \\ +\{x_i^2\omega_y + z_i^2\omega_y - x_iy_i\omega_x - y_iz_i\omega_z\}\hat{\jmath} \\ +\{x_i^2\omega_z + y_i^2\omega_z - x_iz_i\omega_x + y_iz_i\omega_y\}\hat{k} \end{bmatrix}$$

$$\Rightarrow L_x\hat{\imath} + L_y\hat{\jmath} + L_z\hat{k} =$$

$$\begin{bmatrix} \{\sum_{i=1}^{n} m_i(y_i^2 + z_i^2)\omega_x + (-\sum_{i=1}^{n} m_ix_iy_i)\omega_y + (-\sum_{i=1}^{n} m_ix_iz_i)\omega_z\}\hat{\imath} \\ \{\sum_{i=1}^{n} m_i(x_i^2 + z_i^2)\omega_y + (-\sum_{i=1}^{n} m_ix_iy_i)\omega_x + (-\sum_{i=1}^{n} m_iy_iz_i)\omega_z\}\hat{\jmath} \\ \{\sum_{i=1}^{n} m_i(x_i^2 + y_i^2)\omega_z + (-\sum_{i=1}^{n} m_ix_iz_i)\omega_x + (\sum_{i=1}^{n} m_iy_iz_i)\omega_y\}\hat{k} \end{bmatrix}$$

$$\Rightarrow L_x\hat{\imath} + L_y\hat{\jmath} + L_z\hat{k} = \begin{bmatrix} \{I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z\}\hat{\imath} \\ \{I_{yy}\omega_y + I_{xy}\omega_x + I_{yz}\omega_z\}\hat{\jmath} \\ \{I_{zz}\omega_z + I_{xz}\omega_x + I_{yz}\omega_y\}\hat{k} \end{bmatrix}$$

Product of inertia may be positive, may be negative or zero.

$$\Rightarrow L_x\hat{\imath} + L_y\hat{\jmath} + L_z\hat{k} = \begin{bmatrix} \{I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z\}\hat{\imath} \\ \{I_{xy}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z\}\hat{\jmath} \\ \{I_{xz}\omega_x + I_{yz}\omega_y + I_{zz}\omega_z\}\hat{k} \end{bmatrix}$$

On comparing we have

$L_x = I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z$

$L_y = I_{xy}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z$

$L_z = I_{xz}\omega_x + I_{yz}\omega_y + I_{zz}\omega_z$

Inertia Matrix

In matrix form we have $\begin{bmatrix} L_x \\ L_y \\ L_z \end{bmatrix} = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \Rightarrow \boldsymbol{\vec{L} = I\vec{\omega}}$

Here $I = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix}$ is called inertia matrix.

Results

- **M.I about x – axis** then $\vec{\omega} = (\omega_x, 0,0)$ and
  $L_x = \omega_x I_{xx}$; $L_y = \omega_x I_{xy}$; $L_z = \omega_x I_{xz}$
- **M.I about y – axis** then $\vec{\omega} = (0, \omega_y, 0)$ and
  $L_x = \omega_y I_{xy}$; $L_y = \omega_y I_{yy}$; $L_z = \omega_y I_{yz}$
- **M.I about z – axis** then $\vec{\omega} = (0,0, \omega_z)$ and
  $L_x = \omega_z I_{xz}$; $L_y = \omega_z I_{yz}$; $L_z = \omega_z I_{zz}$
  L is not parallel to ω

Rotational Kinetic Energy in terms of Inertia Matrix

Since we know that $T_{rot} = \frac{1}{2}\vec{\omega}.\vec{L}$ but $\vec{L} = I\vec{\omega}$ then

$T_{rot} = \frac{1}{2}\vec{\omega}.I\vec{\omega} = \frac{1}{2}I(\vec{\omega}.\vec{\omega}) \Rightarrow \boldsymbol{T_{rot} = \frac{1}{2}I\omega^2}$

Principal Axes: The axes along which angular momentum and angular velocities are parallel (coincident) vectors are called principal axes. **Or** axis relative to which products of inertia are equal to zero known as principal axes.

Principal Axes and Principal Moments of Inertia

In Inertia Matrix form we have

$$\begin{bmatrix} L_x \\ L_y \\ L_z \end{bmatrix} = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \Rightarrow \vec{\boldsymbol{L}} = \boldsymbol{I}\vec{\boldsymbol{\omega}}$$

Here $I = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix}$ is called inertia matrix.

In $I = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix}$ the off diagonal elements are zero. i.e.

$$I_{xy} = I_{xz} = I_{yz} = I_{yz} = I_{zx} = I_{zy} = 0$$

Then we get principle axes.

1st Principle axes (x – axes) $\quad I_{xx} = I_1$

2nd Principle axes (y – axes) $\quad I_{yy} = I_2$

3rd Principle axes (z – axes) $\quad I_{zz} = I_3$

Then the matrix $I = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix}$ is called **Principal Moments of Inertia Matrix.**

Keep in mind: When a rigid body is rotating about a fixed point O, the angular velocity vector $\vec{\omega}$ and the angular momentum vector $\vec{L}$ (about O) are not in general in the same direction. However it can be proved that at each point in the body there exists distinct directions, which are fixed relative to the body, along which the two vectors are aligned i.e. coincident. Such directions are called **principal directions** and the axes along them are referred to as principal axes of inertia. The corresponding moments of inertia are called **principal moments of inertia**. **Or** inertia relative to the principal axis is called **principal moments of inertia**.

Remarks

- Inertia matrix is symmetric.
- Axes of this coordinate system are called **Principal Axes**.
- The origin of the Principle Axis is called **Principal Point**.
- The three coordinate planes each passes through the two principal axes is called **Principal Plane**.
- **Why we use $\boldsymbol{I_1, I_2, I_3}$ instead of $\boldsymbol{I_{xx}, I_{yy}, I_{zz}}$?** Single subscript use in I_1, I_2, I_3 is used to distinguish the moment of inertia about arbitrary axis.
- **Orthogonality of Principal Axes**
  If the principal axes at each point of the body exist, then their orthogonality can be proved by stating that axes relative to which product of inertia are zero are the principal axes.
- **Why $\boldsymbol{I_1, I_2, I_3}$ do not change with time?** If Principal axes are attached to the rigid body then I_1, I_2, I_3 do not change with time. So they are treated as a constant.

Angular Momentum in Terms of Inertia using Principal Axes

In case of Principal Axes system we have

$\vec{L} = L_1 e_1 + L_2 e_2 + L_3 e_3$ …………………(1)

For M.I about x – axis we have $\vec{\omega} = (\omega_1, 0,0)$ then $L_1 = I_1 \omega_1$

For M.I about y – axis we have $\vec{\omega} = (0, \omega_2, 0)$ then $L_2 = I_2 \omega_2$

For M.I about z – axis we have $\vec{\omega} = (0,0, \omega_3)$ then $L_3 = I_3 \omega_3$

$(1) \Rightarrow \vec{L} = I_1 \omega_1 e_1 + I_2 \omega_2 e_2 + I_3 \omega_3 e_3$ …………………(2)

Equating (1) and (2) we have

$$L_1 e_1 + L_2 e_2 + L_3 e_3 = I_1 \omega_1 e_1 + I_2 \omega_2 e_2 + I_3 \omega_3 e_3$$

In Inertia Matrix form we have

$$\begin{bmatrix} L_1 \\ L_2 \\ L_3 \end{bmatrix} = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} \Rightarrow \vec{\boldsymbol{L}} = \boldsymbol{I} \vec{\boldsymbol{\omega}}$$

Theorem

Show that Products of Inertia for Principal Axis are equal to zero.

Proof

We know that $\vec{L} = I\vec{\omega}$

Also $\vec{L} = \sum_{i=1}^{n} \vec{r}_i \times \vec{P}_i$

$\Rightarrow I\vec{\omega} = \sum_{i=1}^{n} \vec{r}_i \times m_i\vec{v}_i \Rightarrow I\vec{\omega} = \sum_{i=1}^{n} m_i(\vec{r}_i \times \vec{v}_i)$

$\Rightarrow I\vec{\omega} = \sum_{i=1}^{n} m_i\big(\vec{r}_i \times (\vec{\omega} \times \vec{r}_i)\big)$

$\Rightarrow I\vec{\omega} = \sum_{i=1}^{n} m_i[(\vec{r}_i.\vec{r}_i)\vec{\omega} - (\vec{r}_i.\vec{\omega})\vec{r}_i]$

$\Rightarrow I\vec{\omega} = \sum_{i=1}^{n} m_i[r_i^2\vec{\omega} - (\vec{r}_i.\vec{\omega})\vec{r}_i] \Rightarrow I\vec{\omega} = \sum_{i=1}^{n} m_i r_i^2\vec{\omega} - \sum_{i=1}^{n} m_i(\vec{r}_i.\vec{\omega})\vec{r}_i$

$\Rightarrow \sum_{i=1}^{n} m_i(\vec{r}_i.\vec{\omega})\vec{r}_i = \sum_{i=1}^{n} m_i r_i^2\vec{\omega} - I\vec{\omega}$

$\Rightarrow \sum_{i=1}^{n} m_i(\vec{r}_i.\vec{\omega})\vec{r}_i = [\sum_{i=1}^{n} m_i r_i^2 - I]\vec{\omega}$ ……………….(1)

Consider $\vec{r}_i = x_i\hat{\imath} + y_i\hat{\jmath} + z_i\hat{k} \Rightarrow r_i^2 = x_i^2 + y_i^2 + z_i^2$

and $\vec{\omega} = \omega_x\hat{\imath} + \omega_y\hat{\jmath} + \omega_z\hat{k}$ then $\vec{r}_i.\vec{\omega} = x_i\omega_x + y_i\omega_y + z_i\omega_z$

$(1) \Rightarrow \sum_{i=1}^{n} m_i\big(x_i\omega_x + y_i\omega_y + z_i\omega_z\big)\big(x_i\hat{\imath} + y_i\hat{\jmath} + z_i\hat{k}\big)$

$= \big[\sum_{i=1}^{n} m_i\big(x_i^2 + y_i^2 + z_i^2\big) - I\big]\big(\omega_x\hat{\imath} + \omega_y\hat{\jmath} + \omega_z\hat{k}\big)$

$\Rightarrow \big[\sum_{i=1}^{n} m_i\big(x_i^2\omega_x + x_iy_i^2\omega_y + x_iz_i^2\omega_z\big)\big]\hat{\imath} + \big[\sum_{i=1}^{n} m_i\big(x_iy_i\omega_x + y_i^2\omega_y + y_iz_i\omega_z\big)\big]\hat{\jmath} + \big[\sum_{i=1}^{n} m_i\big(x_iz_i\omega_x + y_iz_i\omega_y + z_i^2\omega_z\big)\big]\hat{k}$

$= \big[\sum_{i=1}^{n} m_i\big(x_i^2 + y_i^2 + z_i^2\big) - I\big]\omega_x\hat{\imath} + \big[\sum_{i=1}^{n} m_i\big(x_i^2 + y_i^2 + z_i^2\big) - I\big]\omega_y\hat{\jmath} + \big[\sum_{i=1}^{n} m_i\big(x_i^2 + y_i^2 + z_i^2\big) - I\big]\omega_z\hat{k}$ ……………….(2)

Comparing coefficients of $\hat{\imath}$ in (2)

$\sum_{i=1}^{n} m_i\big(x_i^2\omega_x + x_iy_i^2\omega_y + x_iz_i^2\omega_z\big) = \big[\sum_{i=1}^{n} m_i\big(x_i^2 + y_i^2 + z_i^2\big) - I\big]\omega_x$

Comparing coefficients of ω_x; ω_y ; ω_z

$\sum_{i=1}^{n} m_i x_i^2 = \sum_{i=1}^{n} m_i x_i^2 + \sum_{i=1}^{n} m_i\left(y_i^2 + z_i^2\right) - I \Rightarrow I_{xx} = I = \sum_{i=1}^{n} m_i\left(y_i^2 + z_i^2\right)$

And $I_{xy} = \sum_{i=1}^{n} m_i x_i y_i = 0$ Also $I_{xz} = \sum_{i=1}^{n} m_i x_i z_i = 0$

Comparing coefficients of $\hat{j}$ in (2)

$\sum_{i=1}^{n} m_i\left(x_i y_i \omega_x + y_i^2 \omega_y + y_i z_i \omega_z\right) = \left[\sum_{i=1}^{n} m_i\left(x_i^2 + y_i^2 + z_i^2\right) - I\right]\omega_y$

Comparing coefficients of ω_x; ω_y ; ω_z

$\sum_{i=1}^{n} m_i y_i^2 = \sum_{i=1}^{n} m_i y_i^2 + \sum_{i=1}^{n} m_i\left(x_i^2 + z_i^2\right) - I \Rightarrow I_{yy} = I = \sum_{i=1}^{n} m_i\left(x_i^2 + z_i^2\right)$

And $I_{yz} = \sum_{i=1}^{n} m_i y_i z_i = 0$

Hence prove $\boldsymbol{I_{xy} = I_{yz} = I_{zx} = 0}$

Theorem

Show that in matrix notation $\left[\dot{\vec{L}}\right] = \left[\vec{\omega} \times \vec{L}\right] + [I]\left[\dot{\vec{\omega}}\right]$ where I is the inertia matrix.

Proof

We know that $\left[\vec{L}\right] = [I][\vec{\omega}]$

Also $\vec{L} = \sum_{i=1}^{n} \vec{r}_i \times \vec{P}_i$

$\Rightarrow \vec{L} = \sum_{i=1}^{n} \vec{r}_i \times m_i \vec{v}_i \Rightarrow \vec{L} = \sum_{i=1}^{n} m_i(\vec{r}_i \times \vec{v}_i) \Rightarrow \vec{L} = \sum_{i=1}^{n} m_i\left(\vec{r}_i \times (\vec{\omega} \times \vec{r}_i)\right)$

$\Rightarrow [I][\vec{\omega}] = \sum_{i=1}^{n} m_i\left(\vec{r}_i \times (\vec{\omega} \times \vec{r}_i)\right)$

$\Rightarrow [I]\left[\dot{\vec{\omega}}\right] = \sum_{i=1}^{n} m_i \vec{r}_i \times \left(\dot{\vec{\omega}} \times \vec{r}_i\right)$ ……………..(1)

Now $\vec{L} = \sum_{i=1}^{n} \vec{r}_i \times \vec{P}_i$

$\Rightarrow \vec{L} = \sum_{i=1}^{n} \vec{r}_i \times m_i \vec{v}_i \Rightarrow \vec{L} = \sum_{i=1}^{n} m_i(\vec{r}_i \times \vec{v}_i) \Rightarrow \frac{d\vec{L}}{dt} = \frac{d}{dt}\left[\sum_{i=1}^{n} m_i(\vec{r}_i \times \vec{v}_i)\right]$

$\Rightarrow \dot{\vec{L}} = \sum_{i=1}^{n} m_i\left(\dot{\vec{r}}_i \times \vec{v}_i + \vec{r}_i \times \frac{d\vec{v}_i}{dt}\right) \Rightarrow \dot{\vec{L}} = \sum_{i=1}^{n} m_i\left(\vec{v}_i \times \vec{v}_i + \vec{r}_i \times \frac{d}{dt}(\vec{\omega} \times \vec{r}_i)\right)$

$\Rightarrow \dot{\vec{L}} = \sum_{i=1}^{n} m_i\left(0 + \vec{r}_i \times \left(\vec{\omega} \times \vec{v}_i + \dot{\vec{\omega}} \times \vec{r}_i\right)\right)$

$\Rightarrow \dot{\vec{L}} = \sum_{i=1}^{n} m_i \vec{r}_i \times (\vec{\omega} \times \vec{v}_i) + \sum_{i=1}^{n} m_i \vec{r}_i \times \left(\dot{\vec{\omega}} \times \vec{r}_i\right)$

$\Rightarrow \dot{\vec{L}} = \sum_{i=1}^{n} m_i \vec{r}_i \times (\vec{\omega} \times \vec{v}_i) + [I]\left[\dot{\vec{\omega}}\right]$ ……………..(2) using (1)

Now

$\Rightarrow \sum_{i=1}^{n} m_i \vec{r}_i \times (\vec{\omega} \times \vec{v}_i) = \sum_{i=1}^{n} m_i \vec{r}_i \times \left(\vec{\omega} \times (\vec{\omega} \times \vec{r}_i)\right)$

$\Rightarrow \sum_{i=1}^{n} m_i \vec{r}_i \times (\vec{\omega} \times \vec{v}_i) = \sum_{i=1}^{n} m_i \vec{r}_i \times [(\vec{\omega}.\vec{r}_i)\vec{\omega} - (\vec{\omega}.\vec{\omega})\vec{r}_i]$

$\Rightarrow \sum_{i=1}^{n} m_i \vec{r}_i \times (\vec{\omega} \times \vec{v}_i) = \sum_{i=1}^{n} m_i \vec{r}_i \times [(\vec{\omega}.\vec{r}_i)\vec{\omega} - \vec{\omega}^2 \vec{r}_i]$

$\Rightarrow \sum_{i=1}^{n} m_i \vec{r}_i \times (\vec{\omega} \times \vec{v}_i) = \sum_{i=1}^{n} m_i (\vec{\omega}.\vec{r}_i)(\vec{r}_i \times \vec{\omega}) - \sum_{i=1}^{n} m_i \vec{\omega}^2 (\vec{r}_i \times \vec{r}_i)$

$\Rightarrow \sum_{i=1}^{n} m_i \vec{r}_i \times (\vec{\omega} \times \vec{v}_i) = \sum_{i=1}^{n} m_i (\vec{\omega}.\vec{r}_i)(\vec{r}_i \times \vec{\omega})$ ……………..(3)

Similarly $\sum_{i=1}^{n} m_i \vec{\omega} \times (\vec{r}_i \times \vec{v}_i) = \sum_{i=1}^{n} m_i \vec{\omega} \times \left(\vec{r}_i \times (\vec{\omega} \times \vec{r}_i)\right)$

$\Rightarrow \sum_{i=1}^{n} m_i \vec{\omega} \times (\vec{r}_i \times \vec{v}_i) = \sum_{i=1}^{n} m_i \vec{\omega} \times [(\vec{r}_i.\vec{r}_i)\vec{\omega} - (\vec{r}_i.\vec{\omega})\vec{r}_i]$

$\Rightarrow \sum_{i=1}^{n} m_i \vec{\omega} \times (\vec{r}_i \times \vec{v}_i) = \sum_{i=1}^{n} m_i \vec{\omega} \times [\vec{r}_i^{\,2} \vec{\omega} - (\vec{r}_i.\vec{\omega})\vec{r}_i]$

$\Rightarrow \sum_{i=1}^{n} m_i \vec{\omega} \times (\vec{r}_i \times \vec{v}_i) = \sum_{i=1}^{n} m_i \vec{r}_i^{\,2} (\vec{\omega} \times \vec{\omega}) - \sum_{i=1}^{n} m_i (\vec{r}_i.\vec{\omega})(\vec{\omega} \times \vec{r}_i)$

$\Rightarrow \sum_{i=1}^{n} m_i \vec{\omega} \times (\vec{r}_i \times \vec{v}_i) = -\sum_{i=1}^{n} m_i (\vec{r}_i.\vec{\omega})(\vec{\omega} \times \vec{r}_i)$

$\Rightarrow \sum_{i=1}^{n} m_i \vec{\omega} \times (\vec{r}_i \times \vec{v}_i) = \sum_{i=1}^{n} m_i (\vec{r}_i.\vec{\omega})(\vec{r}_i \times \vec{\omega})$

$(3) \Rightarrow \sum_{i=1}^{n} m_i \vec{r}_i \times (\vec{\omega} \times \vec{v}_i) = \sum_{i=1}^{n} m_i \vec{\omega} \times (\vec{r}_i \times \vec{v}_i)$

$\Rightarrow \sum_{i=1}^{n} m_i \vec{r}_i \times (\vec{\omega} \times \vec{v}_i) = \vec{\omega} \times \sum_{i=1}^{n} (\vec{r}_i \times m_i \vec{v}_i) = \vec{\omega} \times \sum_{i=1}^{n} \left(\vec{r}_i \times \vec{P}_i\right) = \vec{\omega} \times \vec{L}$

$(2) \Rightarrow \dot{\vec{L}} = \vec{\omega} \times \vec{L} + [I]\left[\dot{\vec{\omega}}\right]$

$\Rightarrow \left[\dot{\vec{\boldsymbol{L}}}\right] = \left[\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{L}}\right] + [\boldsymbol{I}]\left[\dot{\vec{\boldsymbol{\omega}}}\right]$

Theorem: For a rigid body, there exist a set of three mutually orthogonal axes called principal axes relative to which the product of inertia are zero and angular velocities and angular momentum are oriented along the same direction.

Or Prove that there are three principal moments of inertia (eigenvalues) relative to the principal axis.

Proof: Since we know that $\vec{L} = I\vec{\omega}$

$$\Rightarrow \begin{bmatrix} L_x \\ L_y \\ L_z \end{bmatrix} = \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

$$L_x = I_{xx}\omega_x = I\omega_x \; ; \; L_y = I_{yy}\omega_y = I\omega_y \; ; \; L_z = I_{zz}\omega_z = I\omega_z \quad \ldots\ldots\ldots(1)$$

Also from general theory of angular momentum $L_i = \sum_j I_{ij}\omega_j$

$$L_x = I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z$$

$$L_y = I_{xy}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z \quad \ldots\ldots\ldots(2)$$

$$L_z = I_{xz}\omega_x + I_{yz}\omega_y + I_{zz}\omega_z$$

Comparing (1) and (2)

$$L_x = I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z = I\omega_x$$

$$I_{xy}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z = I\omega_y$$

$$I_{xz}\omega_x + I_{yz}\omega_y + I_{zz}\omega_z = I\omega_z$$

After rearranging we have

$$(I_{xx} - I)\omega_x + I_{xy}\omega_y + I_{xz}\omega_z = 0$$

$$I_{xy}\omega_x + \left(I_{yy} - I\right)\omega_y + I_{yz}\omega_z = 0$$

$$I_{xz}\omega_x + I_{yz}\omega_y + (I_{zz} - I)\omega_z = 0$$

This is the homogeneous system of equations which have the non – trivial solution

So $\begin{vmatrix} I_{xx} - I & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} - I & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} - I \end{vmatrix} = 0$ which is cubic in I gives three principal M.I.

Determination of Principal Axes by Diagonalizing the Inertia Matrix

How to find the Principal Axes

Since we know that

$L_x = I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z$

$L_y = I_{xy}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z$

$L_z = I_{xz}\omega_x + I_{yz}\omega_y + I_{zz}\omega_z$

And for Principal Axes we have $L_x = I_1\omega_x$; $L_y = I_2\omega_y$; $L_z = I_3\omega_z$

Then

$I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z = I_1\omega_x$

$I_{xy}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z = I_2\omega_y$

$I_{xz}\omega_x + I_{yz}\omega_y + I_{zz}\omega_z = I_3\omega_z$

After rearranging we have

$(I_{xx} - I_1)\omega_x + I_{xy}\omega_y + I_{xz}\omega_z = 0$

$I_{xy}\omega_x + (I_{yy} - I_2)\omega_y + I_{yz}\omega_z = 0$

$I_{xz}\omega_x + I_{yz}\omega_y + (I_{zz} - I_3)\omega_z = 0$

This is the homogeneous system of equations which have the non – trivial solution

So $\begin{vmatrix} I_{xx} - I_1 & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} - I_2 & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} - I_3 \end{vmatrix} = 0$ this is the required result to find the Principal Axes and the matrix $\begin{bmatrix} I_{xx} - I_1 & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} - I_2 & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} - I_3 \end{bmatrix}$ is called diagonalizable inertia matrix.

Determination of Principal Axes by Diagonalizing the Inertia Matrix (another way)

Suppose a rigid body has no axis of symmetry. Even so, the tensor that represents the moment of inertia of such a body is characterized by a real, symmetric 3 × 3 matrix that can be diagonalized. The resulting diagonal elements are the values of the principal moments of inertia of the rigid body.

The axes of the coordinate system, in which this matrix is diagonal, are the principal axes of the body, because all products of inertia have vanished. Thus, finding the principal axes and corresponding moments of inertia of any rigid body, symmetric or not, is virtually the same as to diagonalizing its moment of inertia matrix.

Explanation

There are a number of ways to diagonalize a real, symmetric matrix. We present here a way that is quite standard.

First, suppose that we have found the coordinate system (principal axes) in which all products of inertia vanish and the resulting moment of inertia tensor is now represented by a diagonal matrix whose diagonal elements are the principal moments of inertia.

Let e_i be the unit vectors that represent this coordinate system, that is, they point along the direction along the three principal axes of the rigid body. If the moment of inertia tensor is "dotted" with one of these unit vectors, the result is equivalent to a simple multiplication of the unit vector by a scalar quantity, i.e.

$$Ie_i = \lambda_i e_i \qquad (1)$$

The quantities λ_i are just the principal M.I about their respective principal axes. The problem of finding the principal axes is one of finding those vectors e_i that satisfy the condition

$$(I - \lambda_i)e_i = 0 \qquad (2)$$

In general this condition is not satisfied for any arbitrary set of orthonormal unit vectors e_i. It is satisfied only by a set of unit vectors aligned with the principal axes of the rigid body.

Any arbitrary *xyz* coordinate system can always be rotated such that the coordinate axes line up with the principal axes. The unit vectors specifying these coordinate axes then satisfy the condition in equation (2). This condition is equivalent to vanishing of the following determinant

$$|I - \lambda I| = 0 \qquad (3)$$

Explicitly, this equation reads

$$\begin{vmatrix} I_{11} - I & I_{12} & I_{13} \\ I_{21} & I_{22} - I & I_{23} \\ I_{31} & I_{32} & I_{33} - I \end{vmatrix} = 0$$

It is a cubic in λ, namely, $-\lambda^3 + A\lambda^2 + B\lambda^3 + C = 0$ (4)

In which A,B, and C are functions of the I's. The three roots λ_1, λ_2 and λ_3 are the three principal moments of inertia.

We now have the principal moments of inertia, but the task of specifying the components of the unit vectors representing the principal axes in terms of our initial coordinate system remains to be solved.

Here we can make use of the fact that when the rigid body rotates about one of its principal axes; the angular momentum vector is in the same direction as the angular velocity vector.

Let the angles of one of the principal axes relative to the initial *xyz* coordinate system be α, β and γ and let the body rotate about this axis. Therefore, a unit vector pointing in the direction of this principal axis has components $(cos\alpha, cos\beta, cos\gamma)$.

Using equation (1), $Ie_1 = \lambda_1 e_1$

where λ_1, the first principal moment of the three $(\lambda_1, \lambda_2, \lambda_3)$, is obtained by solving eq (4).

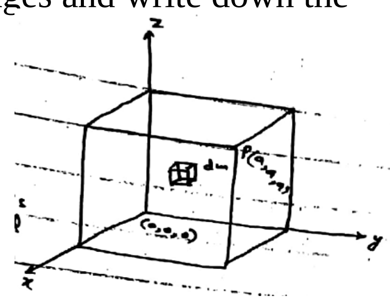
In matrix form

$$\begin{bmatrix} I_{11} - \lambda_1 & I_{12} & I_{13} \\ I_{21} & I_{22} - \lambda_1 & I_{23} \\ I_{31} & I_{32} & I_{33} - \lambda_1 \end{bmatrix} \begin{bmatrix} \cos\alpha \\ \cos\beta \\ \cos\gamma \end{bmatrix} = 0$$

- The direction cosines may be found by solving the above equations.
- The solutions are not independent. They are subject to the constraint
  $cos^2\alpha + cos^2\beta + cos^2\gamma = 1$
- In other words the resultant vector e_1 specified by these components is a unit vector.

Question

Find the moment of inertia and product of inertia of a homogeneous cube of side a and for an origin at corner with axes directly along the edges and write down the inertia matrix.



Solution

Since inertias of cube of side a are

$I_{xx}, I_{yy}, I_{zz}, I_{xy}, I_{yz}, I_{zx}$

M.I. about x axis $= I_{xx} = \int_V (y^2 + z^2) dm = \int_0^a \int_0^a \int_0^a (y^2 + z^2)\, dm$ ……..(i)

Now by using volume mass density $\rho = \frac{dm}{dV}$. i.e. $dm = \rho dV \Rightarrow dm = \rho dxdydz$

$(i) \Rightarrow I_{xx} = \int_0^a \int_0^a \int_0^a (y^2 + z^2)\, \rho dxdydz \Rightarrow I_{xx} = \rho a \left(\frac{2a^4}{3}\right)$

For whole mass of the cube $\rho = \frac{M}{V} = \frac{M}{a.a.a} = \frac{M}{a^3}$. Then

$\Rightarrow I_{xx} = \frac{M}{a^3} . a \left(\frac{2a^4}{3}\right) \Rightarrow \boldsymbol{I_{xx} = \frac{2}{3} Ma^2}$

When mass is not given then use integration in solution

For cubical shape (with equal length and edges), $\boldsymbol{I_{xx} = I_{yy} = I_{zz} = \frac{2}{3} Ma^2}$

Product of inertia $= I_{xy} = \int_V xydm = \int_0^a \int_0^a xy\, dm$ ……..(ii)

Now by using volume mass density $\rho = \frac{dm}{dV}$. i.e. $dm = \rho dV \Rightarrow dm = \rho dxdydz$

$(ii) \Rightarrow I_{xy} = \int_0^a \int_0^a xy\, \rho dxdydz \Rightarrow I_{xy} = \rho\left(\frac{a^5}{4}\right)$

For whole mass of the cube $\rho = \frac{M}{V} = \frac{M}{a.a.a} = \frac{M}{a^3}$. Then

$\Rightarrow I_{xy} = \frac{M}{a^3}.\left(\frac{a^5}{4}\right) \Rightarrow \boldsymbol{I_{xy} = \frac{1}{4}Ma^2}$

For cubical shape (with equal length and edges), $\boldsymbol{I_{xy} = I_{yz} = I_{zx} = \frac{1}{4}Ma^2}$

Now inertia matrix will be written as $I = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix}$

$$\Rightarrow I = \begin{bmatrix} \frac{2}{3}Ma^2 & \frac{1}{4}Ma^2 & -\frac{1}{4}Ma^2 \\ \frac{1}{4}Ma^2 & \frac{2}{3}Ma^2 & -\frac{1}{4}Ma^2 \\ \frac{1}{4}Ma^2 & \frac{1}{4}Ma^2 & \frac{2}{3}Ma^2 \end{bmatrix}$$

Question

Four particles of masses m,2m,3m,4m are located at $(a,a,a),(a,-a,-a),(-a,a,-a)$ and $(-a,-a,a)$ respectively. Calculate its principal moment of inertia.

Solution

Given masses are $m_1 = m, m_2 = 2m, m_3 = 3m, m_4 = 4m$. Given points for each masses $A(a,a,a), B(a,-a,-a), C(-a,a,-a)$ and $D(-a,-a,a)$ and Required Principal moment of inertia are I_1, I_2, I_3. First of all we find all moment of inertia.

M.I. about x axis $= I_{xx} = \sum_{i=1}^{4} m_i\left(y_i^2 + z_i^2\right)$

$\Rightarrow I_{xx} = m_1(y_1^2 + z_1^2) + m_2(y_2^2 + z_2^2) + m_3(y_3^2 + z_3^2) + m_4(y_4^2 + z_4^2)$

$\Rightarrow I_{xx} = m(a^2 + a^2) + 2m(a^2 + a^2) + 3m(a^2 + a^2) + 4m(a^2 + a^2)$

$\Rightarrow \boldsymbol{I_{xx} = 20ma^2}$

And in this case $\boldsymbol{I_{xx} = I_{yy} = I_{zz} = 20ma^2}$

Product of Inertia $= I_{xy} = \sum_{i=1}^{4} m_i(x_i y_i)$

$\Rightarrow I_{xy} = m_1(x_1y_1) + m_2(x_2y_2) + m_3(x_3y_3) + m_4(x_4y_4)$

$\Rightarrow I_{xy} = m(a.a) + 2m(a.-a) + 3m(-a.a) + 4m(-a.-a)$

$\Rightarrow I_{xy} = m(a^2) + 2m(-a^2) + 3m(-a^2) + 4m(a^2)$

$\Rightarrow \boldsymbol{I_{xy} = 0}$

Also $I_{yz} = \sum_{i=1}^{4} m_i(y_i z_i)$

$\Rightarrow I_{yz} = m_1(y_1z_1) + m_2(y_2z_2) + m_3(y_3z_3) + m_4(y_4z_4)$

$\Rightarrow I_{yz} = m(a.a) + 2m(-a.-a) + 3m(a.-a) + 4m(-a.a)$

$\Rightarrow I_{yz} = m(a^2) + 2m(a^2) + 3m(-a^2) + 4m(-a^2)$

$\Rightarrow \boldsymbol{I_{yz} = -4ma^2}$

And $I_{zx} = \sum_{i=1}^{4} m_i(z_i x_i)$

$\Rightarrow I_{zx} = m_1(z_1x_1) + m_2(z_2x_2) + m_3(z_3x_3) + m_4(z_4x_4)$

$\Rightarrow I_{zx} = m(a.a) + 2m(-a.-a) + 3m(-a.-a) + 4m(a.-a)$

$\Rightarrow I_{zx} = m(a^2) + 2m(a^2) + 3m(a^2) + 4m(-a^2)$

$\Rightarrow \boldsymbol{I_{zx} = 2ma^2}$

Now inertia matrix will be written as $I_A = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix}$

$\Rightarrow I_A = \begin{bmatrix} 20ma^2 & 0 & 2ma^2 \\ 0 & 20ma^2 & -4ma^2 \\ 2ma^2 & -4ma^2 & 20ma^2 \end{bmatrix} \Rightarrow I_A = 2ma^2 \begin{bmatrix} 0 & 0 & 1 \\ 0 & 10 & -2 \\ 1 & -2 & 10 \end{bmatrix}$

$\Rightarrow I_A = \beta \begin{bmatrix} 0 & 0 & 1 \\ 0 & 10 & -2 \\ 1 & -2 & 10 \end{bmatrix} \Rightarrow I_A = \begin{bmatrix} 0 & 0 & \beta \\ 0 & 10\beta & -2\beta \\ \beta & -2\beta & 10\beta \end{bmatrix}$ using $\beta = 2ma^2$

Now for Principal Moment of Inertia we have $\begin{vmatrix} I_{xx} - I & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} - I & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} - I \end{vmatrix} = 0$

$$\Rightarrow \begin{vmatrix} 10\beta - I & 0 & \beta \\ 0 & 10\beta - I & -2\beta \\ \beta & -2\beta & 10\beta - I \end{vmatrix} = 0 \Rightarrow (10\beta - I)[(10\beta - I)^2 - 5\beta^2] = 0$$

$$\Rightarrow (10\beta - I) = 0, (10\beta - I)^2 - 5\beta^2 = 0$$

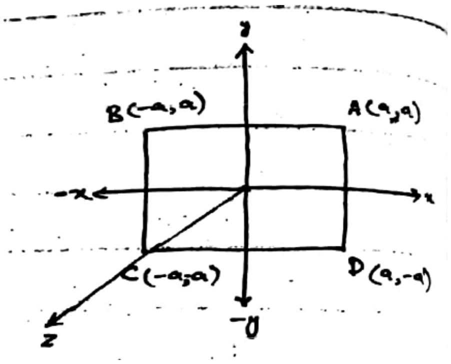
$$\Rightarrow I = 10\beta,\ \ I = 10\beta \mp \sqrt{5}\beta \Rightarrow I_1 = 10\beta,\ I_2 = 10\beta - \sqrt{5}\beta, I_3 = 10\beta + \sqrt{5}\beta$$

$$\Rightarrow I_1 = 20ma^2,\ I_2 = 2\left(10 - \sqrt{5}\right)ma^2, I_3 = 2\left(10 + \sqrt{5}\right)ma^2 \quad \text{using } \beta = 2ma^2$$

Question

A square of side 2a has particles of masses m,2m,3m,4m at its vertices. Calculate its principal moment of inertia at the centre of square.

Solution



Given masses are $m_1 = m, m_2 = 2m, m_3 = 3m, m_4 = 4m$. Given points for each masses $A(a,a), B(-a,a), C(-a,-a)$ and $D(a,-a)$ and Required Principal moment of inertia are I_1, I_2, I_3. First of all we find all moment of inertia.

In case of square $z_i = 0$

M.I. about x axis $= I_{xx} = \sum_{i=1}^{4} m_i\left(y_i^2 + z_i^2\right) = \sum_{i=1}^{4} m_i y_i^2$

$$\Rightarrow I_{xx} = m_1 y_1^2 + m_2 y_2^2 + m_3 y_3^2 + m_4 y_4^2$$

$$\Rightarrow I_{xx} = ma^2 + 2ma^2 + 3ma^2 + 4ma^2$$

$$\Rightarrow \boldsymbol{I_{xx} = 10ma^2}$$

And in this case $\boldsymbol{I_{xx} = I_{yy} = 10ma^2}$

M.I. about z axis $= I_{zz} = \sum_{i=1}^{4} m_i\left(x_i^2 + y_i^2\right)$ or using perpendicular axis theorem

$$I_{zz} = I_{xx} + I_{yy} = 20ma^2$$

Product of Inertia $= I_{xy} = \sum_{i=1}^{4} m_i(x_i y_i)$

$\Rightarrow I_{xy} = m_1(x_1y_1) + m_2(x_2y_2) + m_3(x_3y_3) + m_4(x_4y_4)$

$\Rightarrow I_{xy} = m(a^2) + 2m(-a^2) + 3m(a^2) + 4m(-a^2)$

$\Rightarrow \boldsymbol{I_{xy} = -2ma^2}$

Also $\boldsymbol{I_{yz} = I_{zx} = 0}$ In case of square $z_i = 0$

Now inertia matrix will be written as $I_A = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix}$

$\Rightarrow I_A = \begin{bmatrix} 10ma^2 & -2ma^2 & 0 \\ -2ma^2 & 10ma^2 & 0 \\ 0 & 0 & 20ma^2 \end{bmatrix} \Rightarrow I_A = 2ma^2 \begin{bmatrix} 5 & -1 & 0 \\ -1 & 5 & 0 \\ 0 & 0 & 10 \end{bmatrix}$

$\Rightarrow I_A = \beta \begin{bmatrix} 5 & -1 & 0 \\ -1 & 5 & 0 \\ 0 & 0 & 10 \end{bmatrix} \Rightarrow I_A = \begin{bmatrix} 5\beta & -\beta & 0 \\ -\beta & \beta & 0 \\ 0 & 0 & 10\beta \end{bmatrix}$ using $\beta = 2ma^2$

Now for Principal Moment of Inertia we have $\begin{vmatrix} I_{xx} - I & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} - I & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} - I \end{vmatrix} = 0$

$\Rightarrow \begin{vmatrix} 5\beta - I & -\beta & 0 \\ -\beta & \beta - I & 0 \\ 0 & 0 & 10\beta - I \end{vmatrix} = 0 \Rightarrow (10\beta - I)[(5\beta - I)^2 - \beta^2] = 0$

$\Rightarrow (10\beta - I) = 0, (5\beta - I)^2 - \beta^2 = 0$

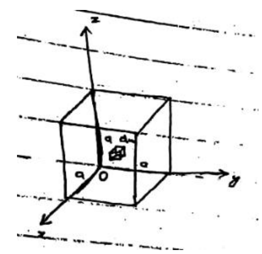
$\Rightarrow I = 10\beta,\ I = 5\beta \mp \beta \Rightarrow I_1 = 10\beta,\ I_2 = 5\beta - \beta, I_3 = 5\beta + \beta$

$\Rightarrow I_1 = 20ma^2,\ I_2 = 2(5\beta - \beta)ma^2, I_2 = 2(5\beta + \beta)ma^2$ using $\beta = 2ma^2$

$\Rightarrow I_1 = 20ma^2,\ I_2 = 8ma^2, I_3 = 12ma^2$

Question

Find the moment of inertia for a cube of mass M and side a and for an origin at one corner.



Solution

Since inertias of cube of side a are

$I_{xx}, I_{yy}, I_{zz}, I_{xy}, I_{yz}, I_{zx}$

M.I. about x axis $= I_{xx} = \int_V (y^2 + z^2) dm = \int_0^a \int_0^a \int_0^a (y^2 + z^2)\, dm$ ……..(i)

Now by using volume mass density $\rho = \frac{dm}{dV}$. i.e. $dm = \rho dV \Rightarrow dm = \rho dxdydz$

$(i) \Rightarrow I_{xx} = \int_0^a \int_0^a \int_0^a (y^2 + z^2)\, \rho dxdydz \Rightarrow I_{xx} = \rho a \left(\frac{2a^4}{3}\right)$

For whole mass of the cube $\rho = \frac{M}{V} = \frac{M}{a.a.a} = \frac{M}{a^3}$. Then

$\Rightarrow I_{xx} = \frac{M}{a^3} . a \left(\frac{2a^4}{3}\right) \Rightarrow \boldsymbol{I_{xx} = \frac{2}{3} Ma^2}$

For cubical shape (with equal length and edges), $\boldsymbol{I_{xx} = I_{yy} = I_{zz} = \frac{2}{3} Ma^2}$

Product of inertia $= I_{xy} = \int_V xy dm = \int_0^a \int_0^a xy\, dm$ ……..(ii)

Now by using volume mass density $\rho = \frac{dm}{dV}$. i.e. $dm = \rho dV \Rightarrow dm = \rho dxdydz$

$(ii) \Rightarrow I_{xy} = \int_0^a \int_0^a xy\, \rho dxdydz \Rightarrow I_{xy} = \rho \left(\frac{a^5}{4}\right)$

For whole mass of the cube $\rho = \frac{M}{V} = \frac{M}{a.a.a} = \frac{M}{a^3}$. Then

$\Rightarrow I_{xy} = \frac{M}{a^3} . \left(\frac{a^5}{4}\right) \Rightarrow \boldsymbol{I_{xy} = \frac{1}{4} Ma^2}$

For cubical shape (with equal length and edges), $\boldsymbol{I_{xy} = I_{yz} = I_{zx} = \frac{1}{4} Ma^2}$

Now inertia matrix will be written as $I = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix}$

$$\Rightarrow I = \begin{bmatrix} \frac{2}{3}Ma^2 & \frac{1}{4}Ma^2 & -\frac{1}{4}Ma^2 \\ \frac{1}{4}Ma^2 & \frac{2}{3}Ma^2 & -\frac{1}{4}Ma^2 \\ \frac{1}{4}Ma^2 & \frac{1}{4}Ma^2 & \frac{2}{3}Ma^2 \end{bmatrix} \Rightarrow I_A = \frac{1}{12}Ma^2 \begin{bmatrix} 8 & 3 & 3 \\ 3 & 8 & 3 \\ 3 & 3 & 8 \end{bmatrix}$$

$$\Rightarrow I_A = \beta \begin{bmatrix} 8 & 3 & 3 \\ 3 & 8 & 3 \\ 3 & 3 & 8 \end{bmatrix} \Rightarrow I_A = \begin{bmatrix} 8\beta & 3\beta & 3\beta \\ 3\beta & 8\beta & 3\beta \\ 3\beta & 3\beta & 8\beta \end{bmatrix} \qquad \text{using } \beta = \frac{1}{12}ma^2$$

Now for Principal Moment of Inertia we have $\begin{vmatrix} I_{xx} - I & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} - I & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} - I \end{vmatrix} = 0$

$$\Rightarrow \begin{vmatrix} 8\beta - I & 3\beta & 3\beta \\ 3\beta & 8\beta - I & 3\beta \\ 3\beta & 3\beta & 8\beta - I \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 5\beta - I & -(5\beta - I) & 0 \\ 3\beta & 8\beta - I & 3\beta \\ 3\beta & 3\beta & 8\beta - I \end{vmatrix} = 0 \quad \sim R_1 - R_2$$

$$\Rightarrow (5\beta - I) \begin{vmatrix} 1 & -1 & 0 \\ 3\beta & 8\beta - I & 3\beta \\ 3\beta & 3\beta & 8\beta - I \end{vmatrix} = 0$$

$$\Rightarrow (5\beta - I) = 0 \;;\; \begin{vmatrix} 1 & -1 & 0 \\ 3\beta & 8\beta - I & 3\beta \\ 3\beta & 3\beta & 8\beta - I \end{vmatrix} = 0$$

$$\Rightarrow (5\beta - I) = 0 \;;\; \begin{vmatrix} 1 & 0 & 0 \\ 3\beta & 11\beta - I & 3\beta \\ 3\beta & 6\beta & 8\beta - I \end{vmatrix} = 0 \qquad \sim C_2 + C_1$$

$$\Rightarrow (5\beta - I)[(11\beta - I)(8\beta - I) - 18\beta] = 0$$

$$\Rightarrow (5\beta - I) = 0, (11\beta - I)(8\beta - I) - 18\beta = 0$$

$$\Rightarrow (5\beta - I) = 0, (5\beta - I)(14\beta - I) = 0$$

$$\Rightarrow I = 5\beta,\; I = 5\beta, I = 14\beta$$

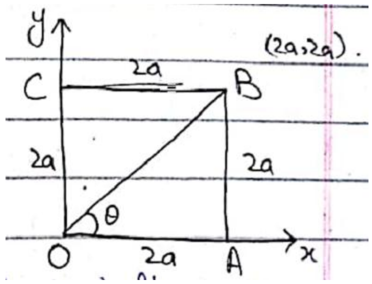
$$\Rightarrow I_1 = \frac{5}{12}ma^2,\; I_2 = \frac{5}{12}ma^2, I_3 = \frac{7}{6}ma^2 \qquad \text{using } \beta = \frac{1}{12}ma^2$$

$$\Rightarrow I_1 = 20ma^2,\; I_2 = 8ma^2, I_2 = 12ma^2$$

Question

A uniform square plate OABC which has sides of length $2a$ is cut in half along the diagonal OB. Calculate Principal M.I. of triangular plate OAB relative to the corner.

Solution



Consider A uniform square plate OABC

which has sides of length $2a$ is cut in half along

the diagonal OB as shown in figure.

Since square plate is in xy – plane, so z = 0 and $I_{xz} = I_{yz} = 0$

M.I. about x axis $= I_{xx} = \int_R (y^2 + z^2) dm = \int_0^{2a} \int_0^{2a} (y^2 + z^2)\, dm$

M.I. about x axis in xy – plane $= I_{xx} = \int_R y^2 dm = \int_0^{2a} \int_0^{2a} y^2\, dm$ ……..(i)

Now by using area mass density $\rho = \frac{dm}{dA} = \frac{dm}{\frac{1}{2}dxdy}$. i.e. $dm = \frac{\rho}{2} dxdy$

$(i) \Rightarrow I_{xx} = \frac{\rho}{2} \int_0^{2a} \int_0^{2a} y^2\, dxdy \Rightarrow I_{xx} = \frac{8}{3}\rho a^4$

For whole mass $\rho = \frac{M}{A} = \frac{M}{\frac{1}{2}(2a\times 2a)} = \frac{M}{2a^2}$. Then

$\Rightarrow I_{xx} = \frac{8}{3} a^4 \left(\frac{M}{2a^2}\right) \Rightarrow \boldsymbol{I_{xx} = \frac{4}{3}Ma^2}$ and In case of square $\boldsymbol{I_{yy} = I_{xx} = \frac{4}{3}Ma^2}$

By using Perpendicular axis theorem $\boldsymbol{I_{zz} = I_{xx} + I_{yy} = \frac{8}{3}Ma^2}$

Product of inertia $= I_{xy} = \int_R xy dm = \int_0^{2a} \int_0^{2a} xy\, dm$ ……..(ii)

Now by using area mass density $\rho = \frac{dm}{dA} = \frac{dm}{\frac{1}{2}dxdy}$. i.e. $dm = \frac{\rho}{2} dxdy$

$(ii) \Rightarrow I_{xy} = \frac{\rho}{2} \int_0^{2a} \int_0^{2a} xy\, dxdy \Rightarrow I_{xy} = 2a^4\rho$

For whole mass $\rho = \frac{M}{A} = \frac{M}{\frac{1}{2}(2a\times 2a)} = \frac{M}{2a^2}$. Then

$\Rightarrow I_{xy} = 2a^4.\left(\frac{M}{2a^2}\right) \Rightarrow \boldsymbol{I_{xy} = Ma^2}$. Here $\boldsymbol{I_{yz} = I_{zx} = 0}$

Now inertia matrix will be written as $I = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix}$

$$\Rightarrow I = \begin{bmatrix} \frac{4}{3}Ma^2 & Ma^2 & 0 \\ Ma^2 & \frac{4}{3}Ma^2 & 0 \\ 0 & 0 & \frac{8}{3}Ma^2 \end{bmatrix} \Rightarrow I_A = \frac{1}{3}Ma^2 \begin{bmatrix} 4 & 3 & 0 \\ 3 & 4 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$

$$\Rightarrow I_A = \beta \begin{bmatrix} 4 & 3 & 0 \\ 3 & 4 & 0 \\ 0 & 0 & 8 \end{bmatrix} \Rightarrow I_A = \begin{bmatrix} 4\beta & 3\beta & 0 \\ 3\beta & 4\beta & 0 \\ 0 & 0 & 8\beta \end{bmatrix} \qquad \text{using } \beta = \frac{1}{3}Ma^2$$

Now for Principal Moment of Inertia we have $\begin{vmatrix} I_{xx} - I & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} - I & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} - I \end{vmatrix} = 0$

$$\Rightarrow \begin{vmatrix} 4\beta - I & 3\beta & 0 \\ 3\beta & 4\beta - I & 0 \\ 0 & 0 & 8\beta - I \end{vmatrix} = 0 \Rightarrow (8\beta - I)\begin{vmatrix} 4\beta - I & 3\beta \\ 3\beta & 4\beta - I \end{vmatrix} = 0$$

$$\Rightarrow (8\beta - I) = 0 \, ; \begin{vmatrix} 4\beta - I & 3\beta \\ 3\beta & 4\beta - I \end{vmatrix} = 0 \Rightarrow (8\beta - I) = 0, (4\beta - I)^2 - (3\beta)^2 = 0$$

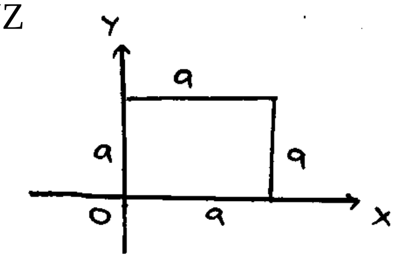
$$\Rightarrow (8\beta - I) = 0, (4\beta - I - 3\beta)(14\beta - I + 3\beta) = 0$$

$$\Rightarrow (8\beta - I) = 0, (\beta - I)(7\beta - I) = 0 \Rightarrow I = 8\beta,\ I = \beta, I = 7\beta$$

$$\Rightarrow I_1 = \frac{8}{3}Ma^2,\ I_2 = \frac{1}{3}Ma^2, I_3 = \frac{7}{3}Ma^2 \qquad \text{using } \beta = \frac{1}{3}Ma^2$$

Question

Find the inertia matrix for a uniform square plate of length a about a pair of adjacent edges taken as OX,OY axes and calculate the principal moments and principal axes at the origin of the coordinate system OXYZ



Solution

M.I. about x axis $= I_{xx} = \int_R (y^2 + z^2) dm$

M.I. about x axis $= I_{xx} = \int_0^a \int_0^a (y^2 + z^2)\, dm$

M.I. about x axis in xy – plane $= I_{xx} = \int_R y^2 dm = \int_0^a \int_0^a y^2\, dm$ ……..(i)

Now by using area mass density $\rho = \frac{dm}{dA} = \frac{dm}{dxdy}$. i.e. $dm = \rho dxdy$

$(i) \Rightarrow I_{xx} = \rho \int_0^a \int_0^a y^2\, dxdy \Rightarrow \boldsymbol{I_{xx} = \frac{1}{3}Ma^2}$ using $\rho = \frac{M}{A} = \frac{M}{(a\times a)} = \frac{M}{a^2}$

In case of square $\boldsymbol{I_{yy} = I_{xx} = \frac{1}{3}Ma^2}$

By using Perpendicular axis theorem $\boldsymbol{I_{zz} = I_{xx} + I_{yy} = \frac{2}{3}Ma^2}$

Product of inertia $= I_{xy} = -\int_R xydm = -\int_0^a \int_0^a xy\, dm$ ……..(ii)

Now by using area mass density $\rho = \frac{dm}{dA} = \frac{dm}{dxdy}$. i.e. $dm = \rho dxdy$

$(ii) \Rightarrow I_{xy} = -\rho \int_0^a \int_0^a xy\, dxdy \Rightarrow \boldsymbol{I_{xy} = -\frac{1}{4}Ma^2}$ using $\rho = \frac{M}{A} = \frac{M}{(a\times a)} = \frac{M}{a^2}$

Here $\boldsymbol{I_{yz} = I_{zx} = 0}$ for xy – plane.

Now inertia matrix will be written as $I = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix}$

$$\Rightarrow I = \begin{bmatrix} \frac{1}{3}Ma^2 & -\frac{1}{4}Ma^2 & 0 \\ -\frac{1}{4}Ma^2 & \frac{1}{3}Ma^2 & 0 \\ 0 & 0 & \frac{2}{3}Ma^2 \end{bmatrix} \Rightarrow I_A = \begin{bmatrix} \frac{1}{3}\beta & -\frac{1}{4}\beta & 0 \\ -\frac{1}{4}\beta & \frac{1}{3}\beta & 0 \\ 0 & 0 & \frac{2}{3}\beta \end{bmatrix} \text{ using } \beta = Ma^2$$

Now for Principal Moment of Inertia we have $\begin{vmatrix} I_{xx}-I & I_{xy} & I_{xz} \\ I_{yx} & I_{yy}-I & I_{yz} \\ I_{zx} & I_{zy} & I_{zz}-I \end{vmatrix} = 0$

$$\Rightarrow \begin{vmatrix} \frac{1}{3}\beta - I & -\frac{1}{4}\beta & 0 \\ -\frac{1}{4}\beta & \frac{1}{3}\beta - I & 0 \\ 0 & 0 & \frac{2}{3}\beta - I \end{vmatrix} = 0$$

$$\Rightarrow I_1 = \frac{2}{3}\beta,\ I_2 = \frac{1}{3}\beta + \frac{\beta}{24}\sqrt{88}, I_3 = \frac{1}{3}\beta - \frac{\beta}{24}\sqrt{88}$$

For Directions of Principal Axes

Directions for first Principal Axes

$$(I_{xx} - I)\omega_x + I_{xy}\omega_y + I_{xz}\omega_z = 0$$

$$I_{xy}\omega_x + \left(I_{yy} - I\right)\omega_y + I_{yz}\omega_z = 0 \qquad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$I_{xz}\omega_x + I_{yz}\omega_y + (I_{zz} - I)\omega_z = 0$$

Using $I = \frac{2}{3}\beta$ in (1) also using $\beta = Ma^2$ in previously find axes

$$\left(\frac{1}{3}Ma^2 - \frac{1}{3}Ma^2\right)\omega_x - \frac{1}{4}Ma^2\omega_y + 0 = 0 \Rightarrow -\frac{1}{4}Ma^2\omega_y = 0 \Rightarrow \omega_y = a \neq 0$$

$$-\frac{1}{4}Ma^2\omega_x + \left(\frac{1}{3}Ma^2 - \frac{1}{3}Ma^2\right)\omega_y + 0 = 0 \Rightarrow -\frac{1}{4}Ma^2\omega_x = 0 \Rightarrow \omega_x = 0$$

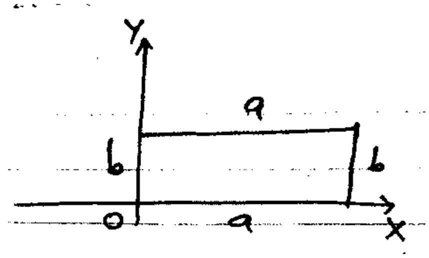
$$0 + 0 + \left(\frac{2}{3}Ma^2 - \frac{1}{3}Ma^2\right)\omega_z = 0 \Rightarrow \frac{1}{3}Ma^2\omega_z = 0 \Rightarrow \omega_z = 0$$

$$\Rightarrow \vec{\omega} = \omega_x\hat{\imath} + \omega_y\hat{\jmath} + \omega_z\hat{k} \Rightarrow \vec{\omega} = a\hat{\jmath}$$

Similarly find Directions for second, third Principal Axes

Question

Find the principal moments and principal axes of inertia matrix for a uniform rectangular plate of sides a, b at its centre.



Solution

M.I. about x axis $= I_{xx} = \int_R (y^2 + z^2) dm$

M.I. about x axis $= I_{xx} = \int_0^b \int_0^a (y^2 + z^2)\, dm$

M.I. about x axis in xy – plane $= I_{xx} = \int_R y^2 dm = \int_0^b \int_0^a y^2\, dm$ ……..(i)

Now by using area mass density $\rho = \frac{dm}{dA} = \frac{dm}{dxdy}$. i.e. $dm = \rho dxdy$

$(i) \Rightarrow I_{xx} = \rho \int_0^b \int_0^a y^2\, dxdy \Rightarrow \boldsymbol{I_{xx} = \frac{1}{3} M b^2}$ using $\rho = \frac{M}{A} = \frac{M}{(a \times b)} = \frac{M}{ab}$

M.I. about y axis $= I_{yy} = \int_R (y^2 + z^2) dm$

M.I. about y axis $= I_{yy} = \int_0^b \int_0^a (y^2 + z^2)\, dm$

M.I. about y axis in xy – plane $= I_{yy} = \int_R x^2 dm = \int_0^b \int_0^a x^2\, dm$ ……..(ii)

Now by using area mass density $\rho = \frac{dm}{dA} = \frac{dm}{dxdy}$. i.e. $dm = \rho dxdy$

$(ii) \Rightarrow I_{yy} = \rho \int_0^b \int_0^a x^2\, dxdy \Rightarrow \boldsymbol{I_{yy} = \frac{1}{3} M a^2}$ using $\rho = \frac{M}{A} = \frac{M}{(a \times b)} = \frac{M}{ab}$

By using Perpendicular axis theorem $\boldsymbol{I_{zz} = I_{xx} + I_{yy} = \frac{1}{3} M (a^2 + b^2)}$

Product of inertia $= I_{xy} = -\int_R xy dm = -\int_0^b \int_0^a xy\, dm$ ……..(iii)

Now by using area mass density $\rho = \frac{dm}{dA} = \frac{dm}{dxdy}$. i.e. $dm = \rho dxdy$

$(iii) \Rightarrow I_{xy} = -\rho \int_0^b \int_0^a xy\, dxdy \Rightarrow \boldsymbol{I_{xy} = -\frac{1}{4} M ab}$ using $\rho = \frac{M}{A} = \frac{M}{(a \times b)} = \frac{M}{ab}$

Here $\boldsymbol{I_{yz} = I_{zx} = 0}$ for xy – plane.

Now inertia matrix will be written as $I = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix}$

$$I = \begin{bmatrix} \frac{1}{3}Mb^2 & -\frac{1}{4}Mab & 0 \\ -\frac{1}{4}Mab & \frac{1}{3}Ma^2 & 0 \\ 0 & 0 & \frac{1}{3}M(a^2+b^2) \end{bmatrix}$$ solve yourself

Now for Principal Moment of Inertia solve yourself

$$\begin{vmatrix} I_{xx}-I & I_{xy} & I_{xz} \\ I_{yx} & I_{yy}-I & I_{yz} \\ I_{zx} & I_{zy} & I_{zz}-I \end{vmatrix} = 0$$

For Directions of Principal Axes solve following equations

$$(I_{xx}-I)\omega_x + I_{xy}\omega_y + I_{xz}\omega_z = 0$$

$$I_{xy}\omega_x + \left(I_{yy}-I\right)\omega_y + I_{yz}\omega_z = 0$$

$$I_{xz}\omega_x + I_{yz}\omega_y + (I_{zz}-I)\omega_z = 0$$

Question

Three uniform rods OA,OB and OC are each of unit length and unit mass relative to coordinate system OXYZ, the coordinates of A,B and C are respectively (1,0,0), (0,0,1) and $\left(-\frac{\sqrt{3}}{2},\frac{1}{2},0\right)$. Show their principal moment of inertia.

Solution

M.I. of rod about x axis $= I_{xx} = \frac{1}{3}\sum_{i=1}^{3} m_i\left(y_i^2 + z_i^2\right)$

$\Rightarrow I_{xx} = \frac{1}{3}(1)(0) + \frac{1}{3}(1)(1) + \frac{1}{3}(1)\left(\frac{1}{4}\right) \Rightarrow \boldsymbol{I_{xx}} = \frac{\mathbf{5}}{\mathbf{12}}$

M.I. of rod about y axis $= I_{yy} = \frac{1}{3}\sum_{i=1}^{3} m_i\left(x_i^2 + z_i^2\right)$

$\Rightarrow I_{yy} = \frac{1}{3}(1)(1) + \frac{1}{3}(1)(1) + \frac{1}{3}(1)\left(\frac{3}{4}\right) \Rightarrow \boldsymbol{I_{yy}} = \frac{\mathbf{11}}{\mathbf{12}}$

M.I. of rod about z axis $= I_{zz} = \frac{1}{3}\sum_{i=1}^{3} m_i\left(x_i^2 + y_i^2\right)$

$\Rightarrow I_{zz} = \frac{1}{3}(1)(1) + \frac{1}{3}(1)(0) + \frac{1}{3}\left[(1)\left(\frac{3}{4}\right) + \left(\frac{1}{4}\right)(1)\right] \Rightarrow \boldsymbol{I_{zz}} = \frac{\mathbf{2}}{\mathbf{3}}$

Product of Inertia $= I_{xy} = -\frac{1}{3}\sum_{i=1}^{3} m_i(x_i y_i)$

$\Rightarrow I_{xy} = -\frac{1}{3}(1)(0) - \frac{1}{3}(1)(0) + \frac{1}{3}(1)\left(\frac{\sqrt{3}}{4}\right) \Rightarrow \boldsymbol{I_{xx}} = \frac{\sqrt{\mathbf{3}}}{\mathbf{12}}$

$\Rightarrow I_{yz} = -\frac{1}{3}\sum_{i=1}^{3} m_i(y_i z_i) \Rightarrow \boldsymbol{I_{yz}} = \mathbf{0} \Rightarrow I_{xz} = -\frac{1}{3}\sum_{i=1}^{3} m_i(x_i z_i) \Rightarrow \boldsymbol{I_{xz}} = \mathbf{0}$

Now inertia matrix will be written as $I = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix}$

$$\Rightarrow I = \begin{bmatrix} \frac{5}{12} & \frac{\sqrt{3}}{12} & 0 \\ \frac{\sqrt{3}}{12} & \frac{11}{12} & 0 \\ 0 & 0 & \frac{2}{3} \end{bmatrix} \Rightarrow I = \begin{bmatrix} 5\beta & \sqrt{3}\beta & 0 \\ \sqrt{3}\beta & 11\beta & 0 \\ 0 & 0 & 8\beta \end{bmatrix} \quad \text{using } \beta = \frac{1}{12}$$

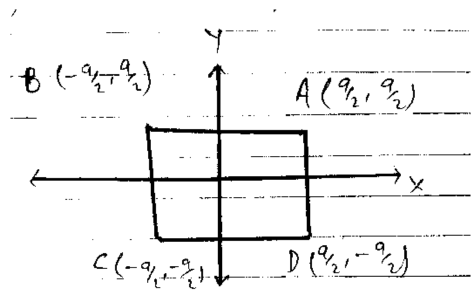
Now for Principal Moment of Inertia we have $\begin{vmatrix} I_{xx} - I & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} - I & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} - I \end{vmatrix} = 0$

$\Rightarrow I_1 = 8\beta + 8\beta\sqrt{3} = \frac{2}{3} + \frac{2}{3}\sqrt{3},\ I_2 = 8\beta - 8\beta\sqrt{3} = \frac{2}{3} - \frac{2}{3}\sqrt{3}$

Question

A square of side a has particles of masses $m, 2m, 3m, 4m$ at its vertices. Calculate Principal M.I. also find direction of principal axes.

Solution



M.I. about x axis = $I_{xx} = \sum_{i=1}^{4} m_i\left(y_i^2 + z_i^2\right)$

$\Rightarrow I_{xx} = \sum_{i=1}^{4} m_i y_i^2$ for xy – plane z = 0

$\Rightarrow I_{xx} = m_1 y_1^2 + m_2 y_2^2 + m_3 y_3^2 + m_4 y_4^2$

$\Rightarrow I_{xx} = m\left(\frac{a^2}{4}\right) + 2m\left(\frac{a^2}{4}\right) + 3m\left(\frac{a^2}{4}\right) + 4m\left(\frac{a^2}{4}\right)$

$\Rightarrow \boldsymbol{I_{xx} = \frac{5}{2}ma^2}$

M.I. about y axis = $I_{yy} = \sum_{i=1}^{4} m_i\left(x_i^2 + z_i^2\right)$

$\Rightarrow I_{yy} = \sum_{i=1}^{4} m_i x_i^2$ for xy – plane z = 0

$\Rightarrow I_{yy} = m_1 x_1^2 + m_2 x_2^2 + m_3 x_3^2 + m_4 x_4^2$

$\Rightarrow I_{yy} = m\left(\frac{a^2}{4}\right) + 2m\left(\frac{a^2}{4}\right) + 3m\left(\frac{a^2}{4}\right) + 4m\left(\frac{a^2}{4}\right)$

$\Rightarrow \boldsymbol{I_{yy} = \frac{5}{2}ma^2}$

Using perpendicular axis theorem $\boldsymbol{I_{xx} + I_{yy} = I_{zz} = 5ma^2}$

Product of Inertia $= I_{xy} = -\sum_{i=1}^{4} m_i(x_i y_i)$

$\Rightarrow I_{xy} = -[m_1(x_1 y_1) + m_2(x_2 y_2) + m_3(x_3 y_3) + m_4(x_4 y_4)]$

$\Rightarrow I_{xy} = -m\left(\frac{a^2}{2}.\frac{a^2}{2}\right) - 2m\left(-\frac{a^2}{2}.\frac{a^2}{2}\right) - 3m\left(-\frac{a^2}{2}, -\frac{a^2}{2}\right) - 4m\left(\frac{a^2}{2}, -\frac{a^2}{2}\right)$

$\Rightarrow I_{xy} = -m\left(\frac{a^2}{4}\right) + 2m\left(\frac{a^2}{4}\right) + 3m\left(\frac{a^2}{4}\right) + 4m\left(\frac{a^2}{4}\right)$

$\Rightarrow \boldsymbol{I_{xy} = \frac{1}{2}ma^2}$

In this case $\boldsymbol{I_{xz} = I_{yz} = 0}$

Now inertia matrix will be written as $I = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix}$

$$\Rightarrow I = \begin{bmatrix} \frac{5}{2}ma^2 & \frac{1}{2}ma^2 & 0 \\ \frac{1}{2}ma^2 & \frac{5}{2}ma^2 & 0 \\ 0 & 0 & 5ma^2 \end{bmatrix} \Rightarrow I = \begin{bmatrix} 5\beta & \beta & 0 \\ \beta & 5\beta & 0 \\ 0 & 0 & 10\beta \end{bmatrix} \qquad \text{using } \beta = \frac{1}{2}ma^2$$

Now for Principal Moment of Inertia we have $\begin{vmatrix} I_{xx} - I & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} - I & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} - I \end{vmatrix} = 0$

$$\Rightarrow \begin{vmatrix} 5\beta - I & \beta & 0 \\ \beta & 5\beta - I & 0 \\ 0 & 0 & 10\beta - I \end{vmatrix} = 0 \Rightarrow (10\beta - I)\,[(5\beta - I)^2 - (\beta)^2] = 0$$

$$\Rightarrow (10\beta - I) = 0, (5\beta - I)^2 - (\beta)^2 = 0$$

$$\Rightarrow (10\beta - I) = 0, (5\beta - I - \beta)(5\beta - I + \beta) = 0$$

$$\Rightarrow (10\beta - I) = 0, (4\beta - I)(6\beta - I) = 0 \Rightarrow I = 10\beta,\; I = 4\beta, I = 6\beta$$

$$\Rightarrow I_1 = 5ma^2,\; I_2 = 2ma^2, I_3 = 3ma^2 \qquad \text{using } \beta = \frac{1}{2}ma^2$$

For Directions of Principal Axes

Directions for first Principal Axes

$$(I_{xx} - I)\omega_x + I_{xy}\omega_y + I_{xz}\omega_z = 0$$

$$I_{xy}\omega_x + \left(I_{yy} - I\right)\omega_y + I_{yz}\omega_z = 0 \qquad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$I_{xz}\omega_x + I_{yz}\omega_y + (I_{zz} - I)\omega_z = 0$$

Using $I = 10\beta$ in (1) also using $\beta = \frac{1}{2}ma^2$ in previously find axes

$$(5\beta - 10\beta)\omega_x + \beta\omega_y + 0 = 0$$

$$\beta\omega_x + (5\beta - 10\beta)\omega_y + 0 = 0$$

$$0 + 0 + (10\beta - 10\beta)\omega_z = 0$$

Put $\omega_z = a \neq 0$ any arbitrary constant we get

$-5\beta\omega_x + \beta\omega_y = 0$; $\beta\omega_x - 5\beta\omega_y = 0$

$\Rightarrow -5\omega_x + \omega_y = 0$; $\omega_x - 5\omega_y = 0$ since $\beta \neq 0$

$\Rightarrow -5\omega_x + \omega_y = 0$; $5\omega_x - 25\omega_y = 0$ multiplying by 5

$\Rightarrow 24\omega_y = 0 \Rightarrow \omega_y = 0 \Rightarrow \omega_x = 0$ subtracting and solving

$\Rightarrow \vec{\omega} = \omega_x\hat{\imath} + \omega_y\hat{\jmath} + \omega_z\hat{k} \Rightarrow \vec{\omega} = a\hat{k}$

Directions for second Principal Axes

Using $I = 4\beta$ in (1) also using $\beta = \frac{1}{2}ma^2$ in previously find axes

$(5\beta - 4\beta)\omega_x + \beta\omega_y + 0 = 0$

$\beta\omega_x + (5\beta - 4\beta)\omega_y + 0 = 0$

$0 + 0 + (10\beta - 4\beta)\omega_z = 0 \Rightarrow \omega_z = 0$

And $\beta\omega_x + \beta\omega_y = 0$; $\beta\omega_x + \beta\omega_y = 0$

$\Rightarrow \omega_x = -\omega_y \Rightarrow \omega_x = -\omega_y = C_1$ arbitrary constant

$\Rightarrow \vec{\omega} = \omega_x\hat{\imath} + \omega_y\hat{\jmath} + \omega_z\hat{k} \Rightarrow \vec{\omega} = C_1\hat{\imath} - C_1\hat{\jmath}$

Directions for third Principal Axes

Using $I = 6\beta$ in (1) also using $\beta = \frac{1}{2}ma^2$ in previously find axes

$(5\beta - 6\beta)\omega_x + \beta\omega_y + 0 = 0$

$\beta\omega_x + (5\beta - 6\beta)\omega_y + 0 = 0$

$0 + 0 + (10\beta - 6\beta)\omega_z = 0 \Rightarrow \omega_z = 0$

And $-\beta\omega_x + \beta\omega_y = 0$; $\beta\omega_x - \beta\omega_y = 0$

$\Rightarrow \omega_x = \omega_y \Rightarrow \omega_x = \omega_y = C_2$ arbitrary constant

$\Rightarrow \vec{\omega} = \omega_x\hat{\imath} + \omega_y\hat{\jmath} + \omega_z\hat{k} \Rightarrow \vec{\omega} = C_2\hat{\imath} + C_2\hat{\jmath}$

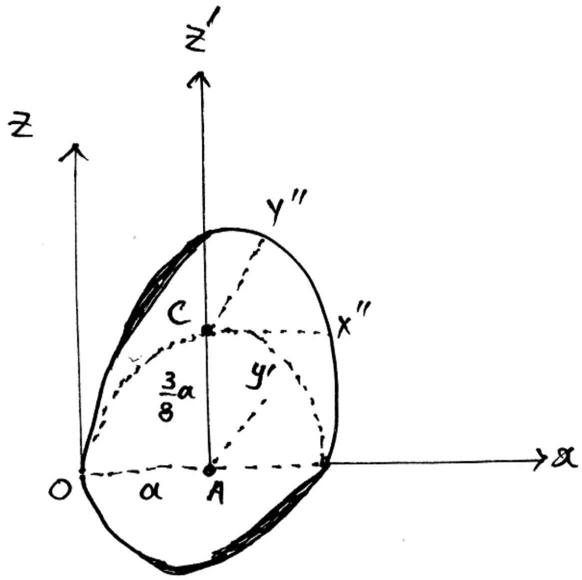
Question

Find the principal moments of inertia and the principal axes of a uniform solid hemisphere about a point on its rim.

Solution

Let M be the mass and a be the radius.

Inertia matrix at A



M.I. at the base A $= I_{Axx} = I_{Ayy} = I_{Azz} = \frac{2}{5}Ma^2$

P.I. at the base A $= I_{Axy} = I_{Ayz} = I_{Azx} = 0$

$$I_{A_{ij}} = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix}$$

$$\Rightarrow I_{A_{ij}} = \begin{bmatrix} \frac{2}{5}Ma^2 & 0 & 0 \\ 0 & \frac{5}{2}Ma^2 & 0 \\ 0 & 0 & \frac{2}{5}Ma^2 \end{bmatrix}$$

$$\Rightarrow I_{A_{ij}} = \begin{bmatrix} \beta & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \beta \end{bmatrix} \qquad \text{using } \beta = \frac{2}{5}Ma^2$$

Inertia matrix at C

Using parallel axis theorem $\quad I_{Cxx} = I_{Axx} - Md_1^2 = I_{Axx} = \frac{2}{5}Ma^2 \qquad \because d_1 = 0$

$$I_{Cyy} = I_{Ayy} - Md_2^2 = \frac{2}{5}Ma^2 - M\left(\frac{3}{8}a\right)^2 = \frac{83}{320}Ma^2 \qquad \because d_2 = AC = \frac{3}{8}a$$

$$I_{Czz} = I_{Azz} - Md_3^2 = I_{Azz} = \frac{2}{5}Ma^2 \qquad \because d_3 = a$$

P.I. at C due to symmetry $= I_{Cxy} = I_{Cyz} = I_{Czx} = 0$

$$\Rightarrow I_{C_{ij}} = \begin{bmatrix} \frac{2}{5}Ma^2 & 0 & 0 \\ 0 & \frac{83}{320}Ma^2 & 0 \\ 0 & 0 & \frac{2}{5}Ma^2 \end{bmatrix}$$

Inertia matrix at O

Using parallel axis theorem $I_{Oxx} = I_{Cxx} + Md'^2_1 = I_{Cxx} = \frac{2}{5}Ma^2 \quad \because d'_1 = 0$

$I_{Oyy} = I_{Cyy} + Md'^2_2 = \frac{83}{320}Ma^2 + \frac{73}{64}Ma^2 = \frac{7}{5}Ma^2 \qquad \because \boldsymbol{d'_2 = OC = OA + AC = a + \frac{3}{8}a}$

$I_{Ozz} = I_{Czz} + Md'^2_3 = \frac{2}{5}Ma^2 + Ma^2 = \frac{7}{5}Ma^2 \quad \because d_3 = 0$

If $\left(\bar{x} = a, \bar{y} = 0, \bar{z} = \frac{3}{8}a\right)$ denote the coordinate of the centroid w.r.to $Oxyz$ then

$I_{Oxy} = I_{Cxy} + M\bar{x}\bar{y} = 0 - 0 = 0$

$I_{Oxz} = I_{Cxz} + M\bar{x}\bar{z} = 0 - \frac{3}{8}Ma^2 = -\frac{3}{8}Ma^2$

$I_{Oyz} = I_{Cyz} - M\bar{y}\bar{z} = 0 - 0 = 0$

$$\Rightarrow I_{O_{ij}} = \begin{bmatrix} \frac{2}{5}Ma^2 & 0 & -\frac{3}{8}Ma^2 \\ 0 & \frac{7}{5}Ma^2 & 0 \\ -\frac{3}{8}Ma^2 & 0 & \frac{7}{5}Ma^2 \end{bmatrix}$$

$$\Rightarrow I_{O_{ij}} = \begin{bmatrix} 76\beta & 0 & -15\beta \\ 0 & 56\beta & 0 \\ -15\beta & 0 & 56\beta \end{bmatrix} \text{ using } \beta = \frac{5}{40}Ma^2$$

Now for Principal Moment of Inertia about O, the rim we have

$$\begin{vmatrix} I_{Oxx} - I & I_{Oxy} & I_{Oxz} \\ I_{Oyx} & I_{Oyy} - I & I_{Oyz} \\ I_{Ozx} & I_{Ozy} & I_{Ozz} - I \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} 16\beta - I & 0 & -15\beta \\ 0 & 56\beta - I & 0 \\ -15\beta & 0 & 56\beta - I \end{vmatrix} = 0$$

$\Rightarrow (16\beta - I)(56\beta - I)^2 + (-15\beta)(-15\beta)(56\beta - I) = 0$

$\Rightarrow (16\beta - I)(56\beta - I)^2 + (15\beta)^2(56\beta - I) = 0$

$\Rightarrow (56\beta - I)[(16\beta - I)(56\beta - I) + 225\beta^2] = 0$

$\Rightarrow (56\beta - I)(I^2 - 72\beta I + 671\beta^2) = 0$

$\Rightarrow 56\beta - I = 0, I^2 - 72\beta I + 671\beta^2 = 0$

$\Rightarrow I_1 = 56\beta,\ I_2 = 11\beta, I_3 = 61\beta$

Eigenvector or Directions for Principal Axes about ring O

Directions for first Principal Axes

$(I_{xx} - I)\omega_x + I_{xy}\omega_y + I_{xz}\omega_z = 0$

$I_{xy}\omega_x + \left(I_{yy} - I\right)\omega_y + I_{yz}\omega_z = 0$ ……………….(1)

$I_{xz}\omega_x + I_{yz}\omega_y + (I_{zz} - I)\omega_z = 0$

Using $I = 56\beta$ in (1) also using previously find axes

$(16\beta - 56\beta)\omega_x + 0 - 15\beta\omega_z = 0 \Rightarrow -40\beta\omega_x + 0\omega_y - 15\beta\omega_z = 0$

$0 + (56\beta - 56\beta)\omega_y + 0 = 0 \Rightarrow 0 = 0$

$-15\beta\omega_x + 0 + (56\beta - 56\beta)\omega_z = 0 \Rightarrow -15\beta\omega_x + 0\omega_z = 0$

Put $\omega_y = a \neq 0$ any arbitrary constant we get

$-40\beta\omega_x - 15\beta\omega_z = 0$; $-15\beta\omega_x + 0\omega_z = 0$

$\Rightarrow -40\omega_x - 15\omega_z = 0$; $-15\omega_x + 0\omega_z = 0$ since $\beta \neq 0$

$\Rightarrow \omega_y = a \Rightarrow \omega_x = 0, \omega_z = 0 \Rightarrow \vec{\omega} = \omega_x\hat{\imath} + \omega_y\hat{\jmath} + \omega_z\hat{k} \Rightarrow \vec{\omega} = a\hat{\jmath} = [0, a, 0]^t$

Directions for second Principal Axes

Using $I = 11\beta$ in (1) also using previously find axes

$(16\beta - 11\beta)\omega_x + 0 - 15\beta\omega_z = 0 \Rightarrow 5\beta\omega_x + 0\omega_y - 15\beta\omega_z = 0$

$0 + (56\beta - 11\beta)\omega_y + 0 = 0 \Rightarrow 45\beta\omega_y = 0$

$-15\beta\omega_x + 0 + (56\beta - 11\beta)\omega_z = 0 \Rightarrow -15\beta\omega_x + 0\omega_y + 45\beta\omega_z = 0$

Put $\omega_y = 0$ then

$5\beta\omega_x - 15\beta\omega_z = 0$; $-15\beta\omega_x + 45\beta\omega_z = 0$

$\Rightarrow 5\omega_x - 15\omega_z = 0$; $\beta\omega_x - 3\beta\omega_z = 0$ since $\beta \neq 0$

$\Rightarrow \frac{\omega_x}{3} = \frac{\omega_z}{1} = C_1 \Rightarrow \omega_x = 3C_1, \omega_z = C_1$

$\Rightarrow \vec{\omega} = \omega_x\hat{\imath} + \omega_y\hat{\jmath} + \omega_z\hat{k} \Rightarrow \vec{\omega} = 3C_1\hat{\imath} + C_1\hat{k} = [3C_1, 0, C_1]^t$

Directions for third Principal Axes

Using $I = 61\beta$ in (1) also using previously find axes

$(16\beta - 61\beta)\omega_x + 0 - 15\beta\omega_z = 0$

$0 + (56\beta - 61\beta)\omega_y + 0 = 0$

$-15\beta\omega_x + 0 + (56\beta - 61\beta)\omega_z = 0$

We get

$\Rightarrow -45\beta\omega_x + 0\omega_y - 15\beta\omega_z = 0$

$\Rightarrow 0\omega_x - 5\beta\omega_y + 0\omega_z = 0$

$\Rightarrow -15\beta\omega_x + 0\omega_y - 5\beta\omega_z = 0$

Put $\omega_y = 0$ then

$-45\beta\omega_x - 15\beta\omega_z = 0$; $-15\beta\omega_x - 5\beta\omega_z = 0$

$\Rightarrow 3\omega_x - \omega_z = 0$; $3\omega_x - \omega_z = 0$ since $\beta \neq 0$

$\Rightarrow \frac{\omega_x}{1} = \frac{\omega_z}{3} = C_2 \Rightarrow \omega_x = C_2, \omega_z = 3C_2$

$\Rightarrow \vec{\omega} = \omega_x\hat{\imath} + \omega_y\hat{\jmath} + \omega_z\hat{k}$

$\Rightarrow \vec{\omega} = C_2\hat{\imath} + 3C_2\hat{k} = [C_2, 0, 3C_2]^t$

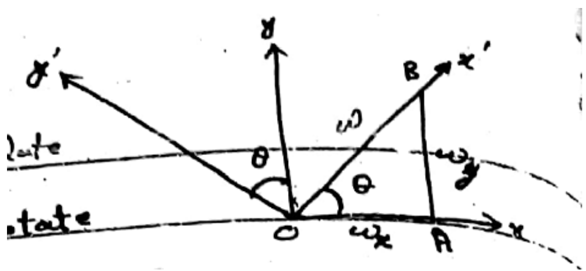
Theorem (Inclination of Principal Axes with Coordinate Axes)

Show that for two dimensional Lamina one of the principal axes is in inclined at an angle θ to the x – axis then $Tan2\theta = \frac{2I_{xy}}{I_{xx}-I_{yy}}$

Solution:

Consider two dimensional plate in xy – plane which rotate with an angle θ.

Then $Tan\theta = \frac{\omega_y}{\omega_x}$ and $\omega_z = 0$



Using the following by Principal Axis theorem

$(I_{xx} - I)\omega_x + I_{xy}\omega_y + I_{xz}\omega_z = 0$ ………..(1)

$I_{yx}\omega_x + \left(I_{yy} - I\right)\omega_y + I_{yz}\omega_z = 0$ ………..(2)

$I_{zx}\omega_x + I_{zy}\omega_y + (I_{zz} - I)\omega_z = 0$ ………..(3)

Using $\omega_z = 0$ and with product of inertia $= I_{xz} = I_{yz} = 0$

$(1) \Rightarrow (I_{xx} - I)\omega_x + I_{xy}\omega_y = 0$ ………..(4)

$(2) \Rightarrow I_{yx}\omega_x + \left(I_{yy} - I\right)\omega_y = 0$ ………..(5)

$(4) \Rightarrow (I_{xx} - I)\omega_x = -I_{xy}\omega_y$

$\Rightarrow I_{xx} - I = -I_{xy}\frac{\omega_y}{\omega_x}$ ………..(6)

$(5) \Rightarrow \left(I_{yy} - I\right)\omega_y = -I_{yx}\omega_x$

$\Rightarrow I_{yy} - I = -I_{yx}\frac{\omega_x}{\omega_y}$ ………..(7)

Subtracting (6) and (7)

$\Rightarrow I_{xx} - I - I_{yy} + I = -I_{xy}\frac{\omega_y}{\omega_x} + I_{yx}\frac{\omega_x}{\omega_y}$

$\Rightarrow I_{xx} - I_{yy} = I_{xy}\left(\frac{\omega_x}{\omega_y} - \frac{\omega_y}{\omega_x}\right) \Rightarrow I_{xx} - I_{yy} = I_{xy}\left(\frac{\omega_x^2 - \omega_y^2}{\omega_x\omega_y}\right)$

$\Rightarrow \frac{I_{xx}-I_{yy}}{I_{xy}} = \left(\frac{\omega^2 cos^2\theta - \omega^2 sin^2\theta}{\omega^2 cos\theta sin\theta}\right)$ using $\omega_x = \omega cos\theta, \omega_y = \omega sin\theta$

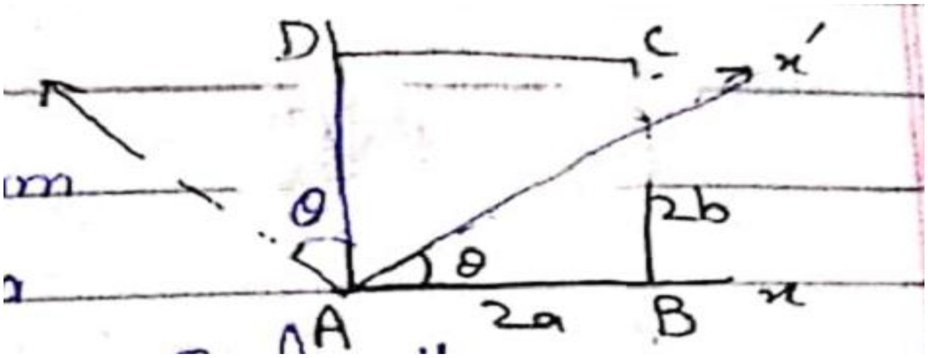
$\Rightarrow \frac{I_{xx}-I_{yy}}{I_{xy}} = 2\left(\frac{cos^2\theta - sin^2\theta}{2cos\theta sin\theta}\right) \Rightarrow \frac{I_{xx}-I_{yy}}{I_{xy}} = 2\left(\frac{cos2\theta}{sin2\theta}\right) \Rightarrow \frac{sin2\theta}{cos2\theta} = \frac{2I_{xy}}{I_{xx}-I_{yy}}$

$\Rightarrow \boldsymbol{Tan2\theta = \frac{2I_{xy}}{I_{xx}-I_{yy}}}$

Question

For a uniform rectangular lamina ABCD with sides of length $2a, 2b$; $b > a$, find the direction of principal axis at the corner A.

Solution



Consider a uniform rectangular lamina ABCD with sides of length $2a, 2b$; $b > a$ as shown in figure. For rectangular plate we have

M.I. about x axis $= I_{xx} = \int_R (y^2 + z^2)dm$

M.I. about x axis in xy – plane $= I_{xx} = \int_R y^2 dm = \int_0^{2b}\int_0^{2a} y^2\, dm$ ……..(i)

Now by using area mass density $\rho = \frac{dm}{dA}$. i.e. $dm = \rho dA \Rightarrow dm = \rho dxdy$

$(i) \Rightarrow I_{xx} = \int_0^{2b}\int_0^{2a} y^2\, \rho dxdy \Rightarrow I_{xx} = \frac{16}{3}\rho ab^3$

For whole mass of the lamina $\rho = \frac{M}{A} = \frac{M}{2a.2b} = \frac{M}{4ab}$. Then

$\Rightarrow I_{xx} = \frac{16}{3}ab^3\left(\frac{M}{4ab}\right) \Rightarrow \boldsymbol{I_{xx} = \frac{4}{3}Mb^2}$

Similarly $\boldsymbol{I_{yy} = \frac{4}{3}Ma^2}$

Product of inertia $= I_{xy} = \int_R xydm = \int_0^{2b}\int_0^{2a} xy\,dm$ ……..(ii)

Now by using area mass density $\rho = \frac{dm}{dA}$. i.e. $dm = \rho dA \Rightarrow dm = \rho dxdy$

$(ii) \Rightarrow I_{xy} = \int_0^{2b}\int_0^{2a} xy\,\rho dxdy \Rightarrow I_{xy} = 4a^2b^2\rho$

For whole mass of the lamina $\rho = \frac{M}{A} = \frac{M}{2a.2b} = \frac{M}{4ab}$. Then

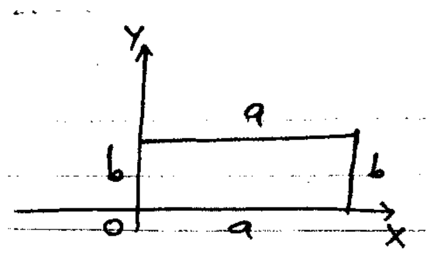
$\Rightarrow I_{xy} = 4a^2b^2.\left(\frac{M}{4ab}\right) \Rightarrow \boldsymbol{I_{xy} = Mab}$. Here $\boldsymbol{I_{yz} = I_{zx} = 0}$

For the direction of principal axis at the corner A we use $Tan2\theta = \frac{2I_{xy}}{I_{xx}-I_{yy}}$

$\Rightarrow Tan2\theta = \frac{2Mab}{\frac{4}{3}Mb^2-\frac{4}{3}Ma^2} \Rightarrow Tan2\theta = \frac{3}{2}\left(\frac{ab}{b^2-a^2}\right) \Rightarrow \theta = \frac{1}{2}Tan^{-1}\left(\frac{3}{2}\left(\frac{ab}{b^2-a^2}\right)\right)$

Question

Show that in a plane rectangular lamina the direction of the principal axes at a corner is given by $Tan2\varphi = \frac{2\left(\frac{Mab}{4}\right)}{\frac{1}{3}Ma^2-\frac{1}{3}Mb^2}$



Solution

M.I. about x axis $= I_{xx} = \int_0^b\int_0^a (y^2+z^2)\,dm$

M.I. about x axis in xy – plane $= I_{xx} = \int_R y^2dm = \int_0^b\int_0^a y^2\,dm$ ……..(i)

Now by using area mass density $\rho = \frac{dm}{dA} = \frac{dm}{dxdy}$. i.e. $dm = \rho dxdy$

$(i) \Rightarrow I_{xx} = \rho\int_0^b\int_0^a y^2\,dxdy \Rightarrow \boldsymbol{I_{xx} = \frac{1}{3}Mb^2}$ using $\rho = \frac{M}{A} = \frac{M}{(a\times b)} = \frac{M}{ab}$

M.I. about y axis $= I_{yy} = \int_0^b\int_0^a (y^2+z^2)\,dm$

M.I. about y axis in xy – plane $= I_{yy} = \int_R x^2dm = \int_0^b\int_0^a x^2\,dm$ ……..(ii)

Now by using area mass density $\rho = \frac{dm}{dA} = \frac{dm}{dxdy}$. i.e. $dm = \rho dxdy$

$(ii) \Rightarrow I_{yy} = \rho\int_0^b\int_0^a x^2\,dxdy \Rightarrow \boldsymbol{I_{yy} = \frac{1}{3}Ma^2}$ using $\rho = \frac{M}{A} = \frac{M}{(a\times b)} = \frac{M}{ab}$

Product of inertia $= I_{xy} = -\int_R xydm = -\int_0^b \int_0^a xy\, dm$ ……..(iii)

Now by using area mass density $\rho = \frac{dm}{dA} = \frac{dm}{dxdy}$. i.e. $dm = \rho dxdy$

$(iii) \Rightarrow I_{xy} = -\rho \int_0^b \int_0^a xy\, dxdy$

$\Rightarrow \boldsymbol{I_{xy} = -\frac{1}{4}Mab}$ using $\rho = \frac{M}{A} = \frac{M}{(a\times b)} = \frac{M}{ab}$

For the direction of principal axis at the corner A we use $Tan2\varphi = \frac{2I_{xy}}{I_{xx}-I_{yy}}$

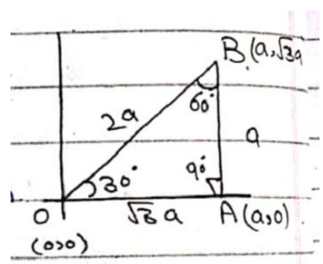
$$\Rightarrow Tan2\varphi = \frac{-2\left(\frac{Mab}{4}\right)}{\frac{1}{3}Mb^2-\frac{1}{3}Ma^2}$$

$$\Rightarrow Tan2\varphi = \frac{2\left(\frac{Mab}{4}\right)}{\frac{1}{3}Ma^2-\frac{1}{3}Mb^2}$$

Question

A triangular plate is made up of uniform material and has sides of lengths $a, 2a, \sqrt{3}a$. Calculate Principal M.I. about the 30° corner and find the direction of the Principal Axis.

Solution



Consider a triangular plate OAB which has sides of length $a, 2a, \sqrt{3}a$ as shown in figure.

M.I. about x axis $= I_{xx} = \int_R (y^2+z^2)dm = \int_0^{\sqrt{3}a}\int_0^a (y^2+z^2)\, dm$

M.I. about x axis in xy – plane $= I_{xx} = \int_R y^2 dm = \int_0^{\sqrt{3}a}\int_0^a y^2\, dm$ ……..(i)

Now by using area mass density $\rho = \frac{dm}{dA} = \frac{dm}{\frac{1}{2}dxdy}$. i.e. $dm = \frac{\rho}{2}dxdy$

$(i) \Rightarrow I_{xx} = \frac{\rho}{2}\int_0^{\sqrt{3}a}\int_0^a y^2\, dxdy \Rightarrow I_{xx} = \frac{\sqrt{3}}{2}\rho a^4$

For whole mass $\rho = \frac{M}{A} = \frac{M}{\frac{1}{2}(\sqrt{3}a\times a)} = \frac{2M}{\sqrt{3}a^2}$. Then

$\Rightarrow I_{xx} = \frac{\sqrt{3}}{2}a^4\left(\frac{2M}{\sqrt{3}a^2}\right) \Rightarrow \boldsymbol{I_{xx} = Ma^2}$

M.I. about y axis $= I_{yy} = \int_R (x^2 + z^2)dm = \int_0^{\sqrt{3}a} \int_0^a (x^2 + z^2)\, dm$

M.I. about y axis in xy – plane $= I_{yy} = \int_R x^2 dm = \int_0^{\sqrt{3}a} \int_0^a x^2\, dm$ ……..(ii)

Now by using area mass density $\rho = \frac{dm}{dA} = \frac{dm}{\frac{1}{2}dxdy}$. i.e. $dm = \frac{\rho}{2} dxdy$

$(ii) \Rightarrow I_{yy} = \frac{\rho}{2} \int_0^{\sqrt{3}a} \int_0^a x^2\, dxdy \Rightarrow I_{yy} = \frac{\sqrt{3}}{6} \rho a^4$

For whole mass $\rho = \frac{M}{A} = \frac{M}{\frac{1}{2}(\sqrt{3}a \times a)} = \frac{2M}{\sqrt{3}a^2}$. Then

$\Rightarrow I_{yy} = \frac{\sqrt{3}}{6} a^4 \left(\frac{2M}{\sqrt{3}a^2}\right) \Rightarrow \boldsymbol{I_{yy} = \frac{1}{3} Ma^2}$

By using Perpendicular axis theorem $\boldsymbol{I_{zz} = I_{xx} + I_{yy} = \frac{4}{3} Ma^2}$

Product of inertia $= I_{xy} = \int_R xydm = \int_0^{\sqrt{3}a} \int_0^a xy\, dm$ ……..(ii)

Now by using area mass density $\rho = \frac{dm}{dA} = \frac{dm}{\frac{1}{2}dxdy}$. i.e. $dm = \frac{\rho}{2} dxdy$

$(ii) \Rightarrow I_{xy} = \frac{\rho}{2} \int_0^{\sqrt{3}a} \int_0^a xy\, dxdy \Rightarrow I_{xy} = \frac{3}{4} a^4 \rho$

For whole mass $\rho = \frac{M}{A} = \frac{M}{\frac{1}{2}(\sqrt{3}a \times a)} = \frac{2M}{\sqrt{3}a^2}$. Then

$\Rightarrow I_{xy} = \frac{3}{4} a^4 . \left(\frac{2M}{\sqrt{3}a^2}\right) \Rightarrow \boldsymbol{I_{xy} = \frac{\sqrt{3}}{2} Ma^2}$. Here $\boldsymbol{I_{yz} = I_{zx} = 0}$

Now inertia matrix will be written as $I = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix}$

$$\Rightarrow I = \begin{bmatrix} Ma^2 & \frac{\sqrt{3}}{2} Ma^2 & 0 \\ \frac{\sqrt{3}}{2} Ma^2 & \frac{1}{3} Ma^2 & 0 \\ 0 & 0 & \frac{4}{3} Ma^2 \end{bmatrix} \Rightarrow I_A = \frac{1}{6} Ma^2 \begin{bmatrix} 6 & 3\sqrt{3} & 0 \\ 3\sqrt{3} & 2 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$

$$\Rightarrow I_A = \beta\begin{bmatrix} 6 & 3\sqrt{3} & 0 \\ 3\sqrt{3} & 2 & 0 \\ 0 & 0 & 8 \end{bmatrix} \Rightarrow I_A = \begin{bmatrix} 6\beta & 3\sqrt{3}\beta & 0 \\ 3\sqrt{3}\beta & 2\beta & 0 \\ 0 & 0 & 8\beta \end{bmatrix} \quad \text{using } \beta = \frac{1}{6}Ma^2$$

Now for Principal Moment of Inertia we have $\begin{vmatrix} I_{xx} - I & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} - I & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} - I \end{vmatrix} = 0$

$$\Rightarrow \begin{vmatrix} 6\beta - I & 3\sqrt{3}\beta & 0 \\ 3\sqrt{3}\beta & 2\beta - I & 0 \\ 0 & 0 & 8\beta - I \end{vmatrix} = 0 \Rightarrow (8\beta - I)\begin{vmatrix} 6\beta - I & 3\sqrt{3}\beta \\ 3\sqrt{3}\beta & 2\beta - I \end{vmatrix} = 0 \text{ exp.by } R_3$$

$$\Rightarrow (8\beta - I) = 0 \ ; \ \begin{vmatrix} 6\beta - I & 3\sqrt{3}\beta \\ 3\sqrt{3}\beta & 2\beta - I \end{vmatrix} = 0 \Rightarrow (8\beta - I) = 0, (I - 5\beta)(I - 3\beta) = 0$$

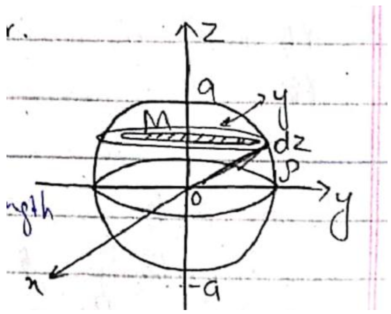
$$\Rightarrow I = 8\beta,\ I = 5\beta, I = 3\beta$$

$$\Rightarrow I_1 = \frac{8}{3}Ma^2,\ I_2 = \frac{5}{6}Ma^2, I_3 = \frac{1}{2}Ma^2 \quad \text{using } \beta = \frac{1}{6}Ma^2$$

For the direction of principal axis at the corner A we use $Tan2\theta = \frac{2I_{xy}}{I_{xx} - I_{yy}}$

$$\Rightarrow Tan2\theta = \frac{\frac{\sqrt{3}}{2}Ma^2}{Ma^2 - \frac{1}{3}Ma^2} \Rightarrow Tan2\theta = \frac{\frac{\sqrt{3}}{2}Ma^2}{\frac{2}{3}Ma^2} \Rightarrow Tan2\theta = \frac{3\sqrt{3}}{2}$$

$$\Rightarrow \theta = \frac{1}{2}Tan^{-1}\left(\frac{3\sqrt{3}}{2}\right) \Rightarrow \theta = 34.5°$$

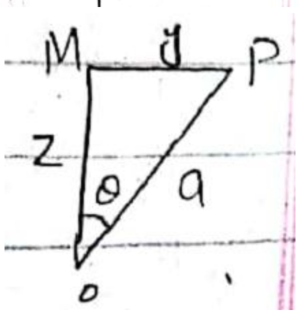


Question

Find the M.I. of solid sphere about its any diameter.

Solution

Consider a sphere of diameter of length $2a$ as shown in figure.



Now consider small disk of thickness $dz$with mass dm at a distance z from the origin and radius of disk is y. then

M.I. about z – axis (diameter) $= I_{zz} = \frac{1}{2}\int y^2 dm$

$$\Rightarrow I_{zz} = \frac{1}{2}Ma^2$$

Theorem

Prove that $\vec{\boldsymbol{L}} = \boldsymbol{\Omega} = \boldsymbol{H} = \boldsymbol{I}\vec{\boldsymbol{\omega}}$

Proof

$$\vec{L} = \vec{r} \times \vec{P} = \vec{r} \times m\vec{v} = \vec{r} \times mr\vec{\omega} = mr^2\vec{\omega} = I\vec{\omega}$$

Theorem

Prove that $\boldsymbol{d\omega} = \boldsymbol{\Omega d\theta}$

Proof

$$d\omega = F.dr = F.\frac{dr}{dt}dt = F.vdt = F.(\omega \times r)dt$$

$$d\omega = (r \times F).\omega dt = \Omega.\omega dt = \Omega.\frac{d\theta}{dt}dt = \Omega d\theta$$

Theorem

Prove that $\boldsymbol{P} = \boldsymbol{\Omega\omega}$

Proof

$$P = \frac{d\omega}{dt} = \frac{d\omega}{dt}\frac{d\theta}{d\theta} = \frac{d\omega}{d\theta}\frac{d\theta}{dt}$$

Using $d\omega = \Omega d\theta$ implies $\frac{d\omega}{d\theta} = \Omega$ and $\omega = \frac{d\theta}{dt}$ we have

$$P = \Omega\omega$$

Equimomental Systems

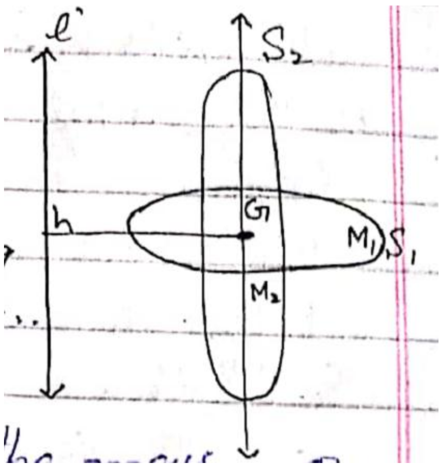
Two systems are said to be Equimomental if they have the same moment of inertia bout any line in space.

Theorem

Two systems are said to be Equimomental iff

i. They have the same mass
ii. They have the same centroid
iii. They have the same moment of inertia at the centre of mass.

Proof



Consider two system satisfy the given conditions. i.e.

i. They have the same mass
ii. They have the same centroid
iii. They have the same moment of inertia at the centre of mass.

Then we have to show these are Equimomental. Let

M = the mass of each system

$l =$ line through common centroid

$l' =$ any line in space parallel to l

$h =$ perpendicular distance between parallel lines

Moment of inertia of first system S_1 about a line l with direction cosines (λ, μ, υ) is

$$I_l = \lambda^2 I_{xx} + \mu^2 I_{yy} + \upsilon^2 I_{zz} + 2\lambda\mu I_{xy} + 2\mu\upsilon I_{yz} + 2\upsilon\lambda I_{zx}$$

For Principal Axis $I_{xy} = I_{yz} = I_{zx} = 0$; So

$$I_l = \lambda^2 I_{xx} + \mu^2 I_{yy} + \upsilon^2 I_{zz}$$

Now M.I. of first system S_1 about a line l by using Parallel Axis Theorem

$$I_{l_1'} = I_l + Mh^2$$

Similarly M.I. of second system S_2 about a line l by using Parallel Axis Theorem

$I_{l'_2} = I_l + Mh^2$

Implies $\quad I_{l'_1} = I_{l'_2} = I_l + Mh^2$

This show that two systems are in Equimomental Condition.

Conversely

Suppose that two systems are in Equimomental. i. e. $I_{l'_1} = I_{l'_2} = I_l$

i. Same mass

Consider M_1 and M_2 are the masses of two systems.

Now M.I. of first system S_1 about a line l by using Parallel Axis Theorem

$I_{l'_1} = I_l + M_1h^2$

Similarly M.I. of second system S_2 about a line l by using Parallel Axis Theorem

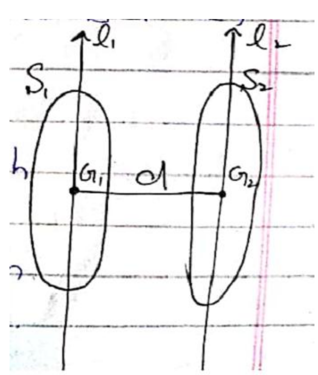
$I_{l'_2} = I_l + M_2h^2$

$\Rightarrow I_{l'_1} = I_{l'_2}$

$\Rightarrow I_l + M_1h^2 = I_2 + M_2h^2$

$\Rightarrow I_l + M_1h^2 = I_l + M_2h^2$ by Supposition $I_{l'_1} = I_{l'_2} = I_l$

$\Rightarrow M_1 = M_2$



ii. Same centroid

Consider G_1 and G_2 be the centroid of two systems.

$l_1 =$ line passes through the1st system at G_1

$l_2 =$ line passes through the 2nd system at G_2

$I_1 =$ M.I. of S_1 about l_1

$I_2 =$ M.I. of S_1 about l_2

By using Parallel Axis Theorem M.I. of S_1 about l_2

$I_2 = I_1 + Md^2$

Now $I_1 =$ M.I. of S_2 about l_1

$I_2 =$ M.I. of S_2 about l_2

By using Parallel Axis Theorem M.I. of S_2 about l_1

$I_1 = I_2 + Md^2$

$\Rightarrow I_1 = I_1 + Md^2 + Md^2$ using $I_2 = I_1 + Md^2$

$\Rightarrow 2Md^2 = 0$

$\Rightarrow d^2 = 0$

$\Rightarrow \left|\overrightarrow{G_1G_2}\right|^2 = 0$

$\Rightarrow \left|\overrightarrow{G_1G_2}\right| = 0$

$\Rightarrow G_1 = G_2$

This shows that systems have same centroid.

iii. Same moment of inertia at the centre of mass.

As both systems are Equimomental and have the same principal axis, therefore principal moment of inertia remains same for both systems.

Momental Ellipsoid

A surface all of whose cross sections are elliptical or circular is called ellipsoid. For momental ellipsoid the moment of inertia about any line L is equal to 1.

In this case direction cosines of line L are $(\lambda, \mu, \upsilon) = (x, y, z)$

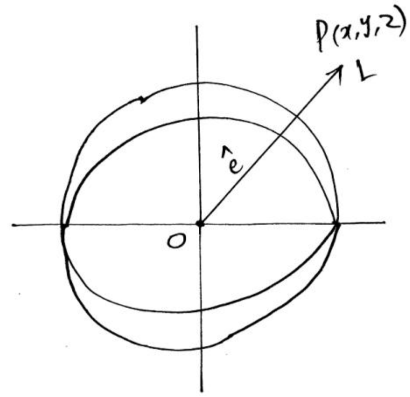
Equation of Momental Ellipsoid

We know that the moment of inertia of a rigid body about line a line L having direction cosines (λ, μ, υ) is given by

$$I = \lambda^2 I_{xx} + \mu^2 I_{yy} + \upsilon^2 I_{zz} + 2\lambda\mu I_{xy} + 2\mu\upsilon I_{yz} + 2\upsilon\lambda I_{zx} \quad \ldots\ldots\ldots\ldots\ldots..(1)$$

Let $\frac{\hat{e}}{\sqrt{I}}$ be a vector along a line L and $P(x, y, z)$ be a point on L such that $\overrightarrow{OP} = \frac{\hat{e}}{\sqrt{I}}$ and $\left|\overrightarrow{OP}\right| = \frac{1}{\sqrt{I}}$ with $\overrightarrow{OP} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$ then

Direction cosines of $\overrightarrow{OP}$ are



$$l_1 = \frac{x}{|\overrightarrow{OP}|} = x\sqrt{I}, l_2 = \frac{y}{|\overrightarrow{OP}|} = y\sqrt{I}, l_3 = \frac{z}{|\overrightarrow{OP}|} = z\sqrt{I}$$

Since the direction cosines of line L and $\overrightarrow{OP}$ are same so

$$\lambda = x\sqrt{I}, \mu = y\sqrt{I}, \upsilon = z\sqrt{I}$$

Then equation (1) becomes

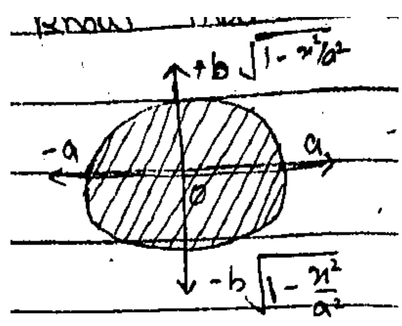
$$I = x^2 I I_{xx} + y^2 I I_{yy} + z^2 I I_{zz} + 2xy I I_{xy} + 2yz I I_{yz} + 2zx I I_{zx}$$

$$\boldsymbol{x^2 I_{xx} + y^2 I_{yy} + z^2 I_{zz} + 2xy I_{xy} + 2yz I_{yz} + 2zx I_{zx} = 1}$$

This is the required equation.

Momental Ellipsoid of the Centre of Elliptical Disk

We know that



$$I_{xx} = \frac{1}{4}Mb^2, I_{yy} = \frac{1}{4}Ma^2$$

$$I_{zz} = \frac{1}{4}M(a^2 + b^2) \qquad \text{by Perpendicular Axis Theorem}$$

For Product of inertia

$$I_{xy} = \int_A xy\,dm = 4\rho \iint xy\,dy\,dx \qquad \text{using area mass density formula}$$

Since $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ is an equation of ellipsoid for elliptical disk, so putting $z = 0$ we get $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$\Rightarrow y^2 = b^2\left(1 - \frac{x^2}{a^2}\right) \Rightarrow y = \pm b\sqrt{\left(1 - \frac{x^2}{a^2}\right)}$$

Then $I_{xy} = 4\rho \int_{-a}^{a} \int_{-b\sqrt{\left(1-\frac{x^2}{a^2}\right)}}^{b\sqrt{\left(1-\frac{x^2}{a^2}\right)}} xy\, dydx$

$$\Rightarrow I_{xy} = 4\rho \int_{-a}^{a} \left|\frac{y^2}{2}\right|_{-b\sqrt{\left(1-\frac{x^2}{a^2}\right)}}^{b\sqrt{\left(1-\frac{x^2}{a^2}\right)}} dx$$

$$\Rightarrow I_{xy} = 2\rho \int_{-a}^{a} \left[b^2\left(1 - \frac{x^2}{a^2}\right) - b^2\left(1 - \frac{x^2}{a^2}\right)\right] dx$$

$$\Rightarrow I_{xy} = 0$$

similarly $I_{yz} = 0, I_{zx} = 0$

Now by using equation of momental ellipsoid

$$x^2 I_{xx} + y^2 I_{yy} + z^2 I_{zz} + 2xyI_{xy} + 2yzI_{yz} + 2zxI_{zx} = 1$$

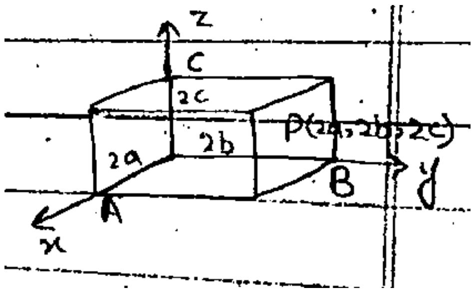
$$\Rightarrow x^2\left(\frac{1}{4}Mb^2\right) + y^2\left(\frac{1}{4}Ma^2\right) + z^2\left(\frac{1}{4}M(a^2 + b^2)\right) + 0 + 0 + 0 = 1$$

$$\Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} + z^2\left(\frac{1}{a^2} + \frac{1}{b^2}\right) = \frac{4}{Ma^2b^2}$$

$$\Rightarrow \boldsymbol{\frac{x^2}{a^2} + \frac{y^2}{b^2} + z^2\left(\frac{1}{a^2} + \frac{1}{b^2}\right) = Constant}$$

Momental Ellipsoid of the Uniform Rectangular Parallalopiped

We know that for a Parallalopiped with dimensions



$0 \le x \le 2a$; $0 \le y \le 2a$; $0 \le z \le 2a$

We have

$I_{xx} = \frac{4}{3}M(b^2 + c^2), I_{yy} = \frac{4}{3}M(a^2 + c^2)$

$I_{zz} = \frac{4}{3}M(a^2 + b^2)$; $I_{xy} = Mab, I_{yz} = Mbc, I_{zx} = Mca$

Now by using equation of momental ellipsoid

$x^2 I_{xx} + y^2 I_{yy} + z^2 I_{zz} + 2xyI_{xy} + 2yzI_{yz} + 2zxI_{zx} = 1$

$\Rightarrow x^2\left(\frac{4}{3}M(b^2 + c^2)\right) + y^2\left(\frac{4}{3}M(a^2 + c^2)\right) + z^2\left(\frac{4}{3}M(a^2 + b^2)\right) + 2xy(Mab) + 2yz(Mbc) + 2zx(Mca) = 1$

$\Rightarrow \boldsymbol{4(b^2 + c^2)x^2 + 4(a^2 + c^2)y^2 + 4(a^2 + b^2)z^2 + 6abxy + 6bcyz + 6zxab = Constant}$

Question

Write Inertia Matrix of Equation of Momental of the form

$2x^2 + 3y^2 + 5z^2 - xy + 2yz + 5zx = 3$

Solution

Given that $2x^2 + 3y^2 + 5z^2 - xy + 2yz + 5zx = 3$

$\Rightarrow \frac{2}{3}x^2 + y^2 + \frac{5}{3}z^2 - \frac{1}{3}xy + \frac{2}{3}yz + \frac{5}{3}zx = 1$

Comparing with $I_{xx} = \frac{2}{3}, I_{yy} = 1, I_{zz} = \frac{5}{3}, I_{xy} = -\frac{1}{6}, I_{yz} = \frac{1}{3}, I_{zx} = \frac{5}{6}$

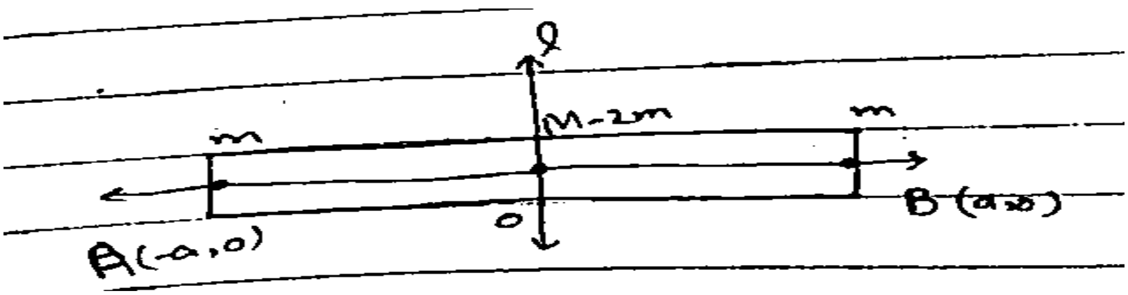
Now inertia matrix will be written as $I = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & -\frac{1}{6} & \frac{5}{6} \\ -\frac{1}{6} & 1 & \frac{1}{3} \\ \frac{5}{6} & \frac{1}{3} & \frac{5}{3} \end{bmatrix}$

Question

Find an Equimomental system of particles for a uniform rod AB of mass M.

Solution

Consider a uniform rod of length $2a$. if a be the centre of mass of the rod then let the mass $m, M-2m, m$ are located at points A,O,B respectively.



The system of particles will be Equimomental with rod if its moment of inertia about any line is equal to the moment of inertia about the same line then M.I. about y – axis (axis passing through the centroid of the rod) is

$I_1 = \frac{1}{3}Ma^2$

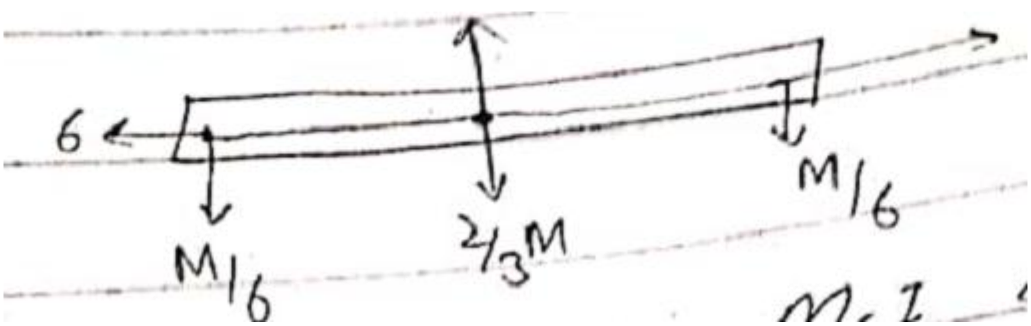
And the moment of inertia of the system of particles about y – axis is

$I_2 = M(-a)^2 + (M-2m)(0)^2 + ma^2 = 2ma^2$

If both systems are Equimomental then $I_1 = I_2$

$\Rightarrow \frac{1}{3}Ma^2 = 2ma^2 \Rightarrow m = \frac{M}{6}$

Hence if we take two particle each of mass $m = \frac{M}{6}$ at end points of rod and particles of mass $M - 2m = M - \frac{M}{3} = \frac{2}{3}M$ at the centre of the rod then this system of three particles will be in the Equimomental with the given rod of mass M.



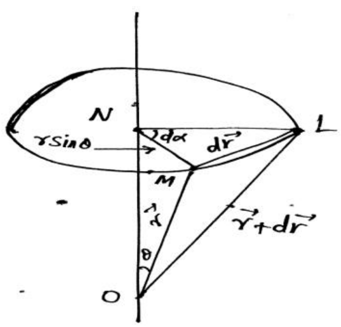
CHAPTER

8 EULER EQUATION OF MOTION OF A RIGID BODY

Coriolis/ Coriolis Force

The Coriolis force is an inertial or fictitious force that acts on objects in motion within a frame of reference that rotates with respect to an inertial frame. In a reference frame with clockwise rotation, the force acts to the left of the motion of the object.

Infinitesimal (So Small) Rotation of a Body



Consider the change in the position vector $\vec{r}$ of the point M produced by an infinitesimal anticlockwise rotation through an angle $d\varphi$ about the axis of rotation as shown in figure.

Since we know that $l = (radius)\theta$ …………..(1)

Therefore from figure $radius = \overrightarrow{NM} = rsin\theta$, $l = dr$, $\theta = d\varphi$

$(1) \Rightarrow dr = rsin\theta d\varphi \Rightarrow dr = |d\varphi \times \vec{r}| \Rightarrow dr\hat{n} = |d\varphi \times \vec{r}|\hat{n} \Rightarrow d\vec{r} = d\varphi \times \vec{r}$

$\Rightarrow \frac{d\vec{r}}{dt} = \frac{d\varphi}{dt} \times \vec{r} \Rightarrow \vec{v} = \vec{\omega} \times \vec{r}$

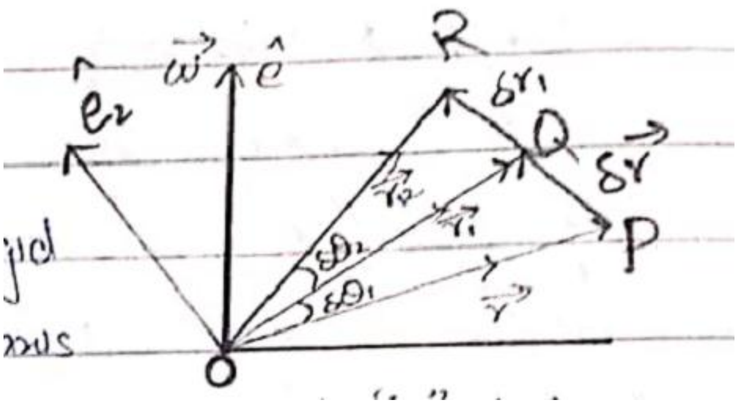
In operator form $\Rightarrow \frac{d}{dt}(\quad) = \frac{d\varphi}{dt} \times (\quad)$

Generalized for a vector $\vec{A}$ we have $\Rightarrow \frac{d\vec{A}}{dt} = \frac{d\varphi}{dt} \times \vec{A}$

Question (Addition of Angular Displacement and Velocities)

Show that finite rotation of the rigid body do not commute but infinite time rotation commute. Also show that sum of angular velocities is an angular velocity.

Proof



Consider the rotation of a rigid body about an axis passes through a common point O. Let a particle P with position vector $\vec{r}$ be displaced through an angle $\delta\theta_1$ about the axis specified by the unit vector $\hat{e}_1$. Then the linear displacement will be
$\vec{r}_1 = \vec{r} + d\vec{r}$ …………………(1)

$\Rightarrow \vec{r}_1 = \vec{r} + (\delta\theta_1\hat{e}_1 \times \vec{r})$ ………………(2) where $d\vec{r} = \delta\theta_1\hat{e}_1 \times \vec{r}$

Let the same particle naming Q with position vector $\vec{r}_1$ be displaced through an angle $\delta\theta_2$ about the axis specified by the unit vector $\hat{e}_2$. Then the linear displacement will be $\vec{r}_{12} = \vec{r}_1 + d\vec{r}_1$ …………………(3)

$\Rightarrow \vec{r}_{12} = \vec{r}_1 + (\delta\theta_2\hat{e}_2 \times \vec{r}_1)$ where $d\vec{r}_1 = \delta\theta_2\hat{e}_2 \times \vec{r}_1$

$\Rightarrow \vec{r}_{12} = \vec{r} + (\delta\theta_1\hat{e}_1 \times \vec{r}) + \left(\delta\theta_2\hat{e}_2 \times \left(\vec{r} + (\delta\theta_1\hat{e}_1 \times \vec{r})\right)\right)$ using (2)

$\Rightarrow \vec{r}_{12} = \vec{r} + \delta\theta_1\hat{e}_1 \times \vec{r} + \delta\theta_2\hat{e}_2 \times r + \delta\theta_2\delta\theta_1\hat{e}_2 \times (\hat{e}_1 \times \vec{r})$ ………………(4)

If we reverse the order of rotation then

$\Rightarrow \vec{r}_{21} = \vec{r} + \delta\theta_1\hat{e}_1 \times \vec{r} + \delta\theta_2\hat{e}_2 \times \vec{r} + \delta\theta_1\delta\theta_2\hat{e}_1 \times (\hat{e}_2 \times \vec{r})$ ………………(5)

Comparing (4) and (5) we have $\vec{r}_{12} \neq \vec{r}_{21}$ (Rotation is not Commute)

If $\delta\theta_1, \delta\theta_2$ are finite then the sum of angular displacement is not same. In other words finite displacement do not satisfy the vector law of addition. i.e.

$\delta\theta\hat{e} \neq \delta\theta_1\hat{e}_1 + \delta\theta_2$

If the angular displacements are infinitesimal (very very small) then $\delta\theta_1\delta\theta_2 \approx 0$ then from (4) and (5) we have $\vec{r}_{12} = \vec{r}_{21}$ (Infinitesimal Rotation is Commute) i.e. Angular displacement satisfy the vector law of addition

When the angular displacements are infinitesimal then we have

$$\vec{r}_{12} = \vec{r}_{21} = \vec{r} + \delta\theta_1\hat{e}_1 \times \vec{r} + \delta\theta_2\hat{e}_2 \times \vec{r}$$

$$\vec{r}_{12} - \vec{r} = \delta\theta_1\hat{e}_1 \times \vec{r} + \delta\theta_2\hat{e}_2 \times \vec{r}$$

$$\delta\vec{r} = \delta\theta_1\hat{e}_1 \times \vec{r} + \delta\theta_2\hat{e}_2 \times \vec{r}$$

$$\lim_{\delta t\to 0}\frac{\delta\vec{r}}{\delta t} = \lim_{\delta t\to 0}\frac{\delta\theta_1}{\delta t}\hat{e}_1 \times \vec{r} + \lim_{\delta t\to 0}\frac{\delta\theta_2}{\delta t}\hat{e}_2 \times \vec{r}$$

$$\frac{d\vec{r}}{dt} = \frac{d\theta_1}{dt}\hat{e}_1 \times \vec{r} + \frac{d\theta_2}{dt}\hat{e}_2 \times \vec{r}$$

$$\vec{v} = \omega_1\hat{e}_1 \times \vec{r} + \omega_2\hat{e}_2 \times \vec{r}$$

$$\vec{\omega} \times \vec{r} = (\vec{\omega}_1 + \vec{\omega}_2) \times \vec{r}$$

$$\boldsymbol{\vec{\omega} = \vec{\omega}_1 + \vec{\omega}_2}$$

Which shows the addition of angular velocities.

In order to derive the relationship between fixed and rotating frames of reference, we will study the following theorem;

Rotating Axes Theorem (Find velocity in a moving coordinate system)

Or Rotate of Change of Vector in a Rotating Frame

Or Transformation Equation of the Time Derivative between the Body Fixed and the Space Fixed coordinates

Or Relationship between the Fixed and the rotating coordinates

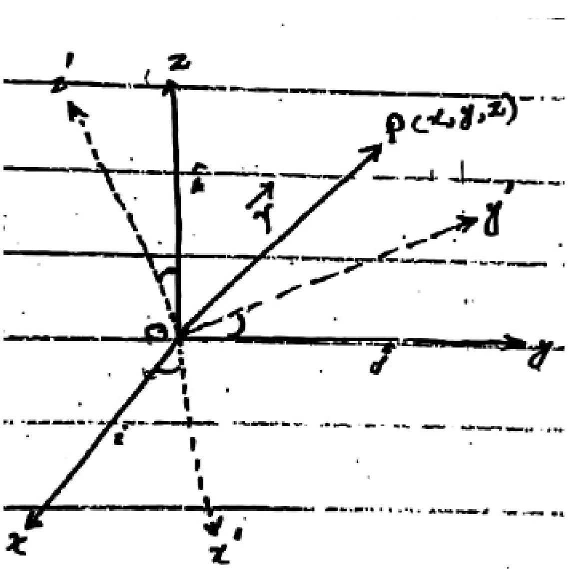
If a time dependent vector function $\vec{A}$ is represented by $\vec{A}_f$ and $\vec{A}_r$ in fixed and rotating coordinate system, then

$$\left(\frac{d\vec{A}}{dt}\right)_f = \left(\frac{d\vec{A}}{dt}\right)_r + \vec{\omega} \times \vec{A}_r$$

Where it is understood that the origins of the two systems coincide at $t = 0$.

Proof

Let $Oxyz$ be a body fixed coordinate system for a rotating body and $Ox'y'z'$ be a space fixed coordinate system. Let $P(x, y, z)$ be a position of particles in both frames.



For body fixed system;

Let $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$ be a position vector in $xyz -$ system with $\hat{\imath}, \hat{\jmath}, \hat{k}$ constant unit vectors then

$$\left(\frac{d\vec{r}}{dt}\right)_r = \frac{dx}{dt}\hat{\imath} + \frac{dy}{dt}\hat{\jmath} + \frac{dz}{dt}\hat{k}$$

For Space fixed system;

Let $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$ be a position vector in $x'y'z' -$ system with $\hat{\imath}, \hat{\jmath}, \hat{k}$ changing unit vectors with respect to time then

$$\left(\frac{d\vec{r}}{dt}\right)_f = \frac{d}{dt}\left(x\hat{\imath} + y\hat{\jmath} + z\hat{k}\right)$$

$$\left(\frac{d\vec{r}}{dt}\right)_f = \left(\frac{dx}{dt}\hat{\imath} + \frac{dy}{dt}\hat{\jmath} + \frac{dz}{dt}\hat{k}\right) + \left(x\frac{d\hat{\imath}}{dt} + y\frac{d\hat{\jmath}}{dt} + z\frac{d\hat{k}}{dt}\right)$$

$$\left(\frac{d\vec{r}}{dt}\right)_f = \left(\frac{d\vec{r}}{dt}\right)_r + \left(x\frac{d\hat{\imath}}{dt} + y\frac{d\hat{\jmath}}{dt} + z\frac{d\hat{k}}{dt}\right)$$

Using operator form $\frac{d\vec{A}}{dt} = \vec{\omega} \times \vec{A}$ implies $\frac{d\hat{\imath}}{dt} = \vec{\omega} \times \hat{\imath}, \frac{d\hat{\jmath}}{dt} = \vec{\omega} \times \hat{\jmath}, \frac{d\hat{k}}{dt} = \vec{\omega} \times \hat{k}$

$$\left(\frac{d\vec{r}}{dt}\right)_f = \left(\frac{d\vec{r}}{dt}\right)_r + \left[x(\vec{\omega} \times \hat{\imath}) + y(\vec{\omega} \times \hat{\jmath}) + z\left(\vec{\omega} \times \hat{k}\right)\right]$$

$$\left(\frac{d\vec{r}}{dt}\right)_f = \left(\frac{d\vec{r}}{dt}\right)_r + \vec{\omega} \times \left(x\hat{\imath} + y\hat{\jmath} + z\hat{k}\right)$$

$$\left(\frac{d\vec{r}}{dt}\right)_f = \left(\frac{d\vec{r}}{dt}\right)_r + \vec{\omega} \times \vec{r}$$

Hence by replacing $\vec{r}$ with $\vec{A}_r$ we have $\left(\frac{d\vec{\boldsymbol{A}}}{\boldsymbol{dt}}\right)_{\boldsymbol{f}} = \left(\frac{d\vec{\boldsymbol{A}}}{\boldsymbol{dt}}\right)_{\boldsymbol{r}} + \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{A}}_{\boldsymbol{r}}$

Question Show that using operators, the fixed and rotating coordinate systems can be related as $\boldsymbol{D_f} = \boldsymbol{D_r} + \vec{\boldsymbol{\omega}}\,\times$, where D_f and D_r stands for $\frac{d}{dt}$ in the fixed and rotating coordinates systems.

Solution: Using rotating axes theorem $\left(\frac{d\vec{A}}{dt}\right)_f = \left(\frac{d\vec{A}}{dt}\right)_r + \vec{\omega} \times \vec{A}$

$$\Rightarrow D_f\vec{A} = D_r\vec{A} + \vec{\omega} \times \vec{A} \Rightarrow D_f\vec{A} = (D_r + \vec{\omega}\,\times)\vec{A} \Rightarrow D_f = D_r + \vec{\omega}\,\times$$

Question

Show that the angular acceleration $\dot{\vec{\omega}}$ is the same in both the coordinate systems.

Solution: Using rotating axes theorem $\left(\frac{d\vec{A}}{dt}\right)_f = \left(\frac{d\vec{A}}{dt}\right)_r + \vec{\omega} \times \vec{A}$

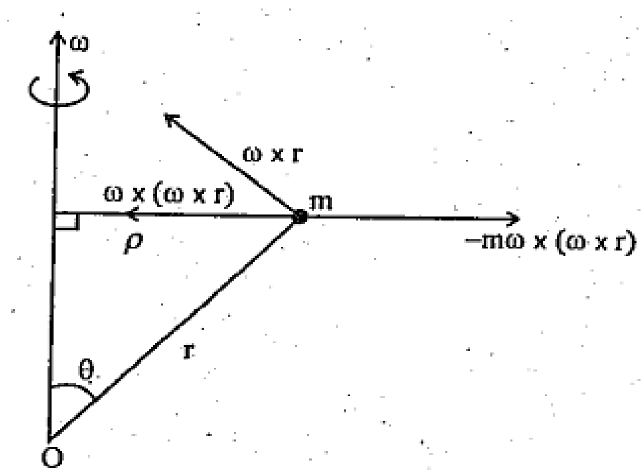
$$\Rightarrow \left(\frac{d\vec{\omega}}{dt}\right)_f = \left(\frac{d\vec{\omega}}{dt}\right)_r + \vec{\omega} \times \vec{\omega} \Rightarrow \left(\frac{d\vec{\omega}}{dt}\right)_f = \left(\frac{d\vec{\omega}}{dt}\right)_r \quad \text{or} \quad \left(\dot{\vec{\omega}}\right)_f = \left(\dot{\vec{\omega}}\right)_r$$

Hence the angular acceleration $\dot{\vec{\omega}}$ is the same in both the coordinate systems.

Question

Show that the centripetal acceleration term $\vec{\omega} \times (\vec{\omega} \times \vec{r})$ can be written as $\omega^2\rho$ where ρ is the distance of the particle from the axis of rotation.

Solution:



$$\vec{\omega} \times \vec{r} = \omega r sin\theta \hat{n}$$

$$\Rightarrow \vec{\omega} \times (\vec{\omega} \times \vec{r}) = \omega r sin\theta(\vec{\omega} \times \hat{n})$$

$$\Rightarrow \vec{\omega} \times (\vec{\omega} \times \vec{r}) = \omega r sin\theta(\omega 1 sin90°)$$

$$\Rightarrow \vec{\omega} \times (\vec{\omega} \times \vec{r}) = \omega^2 r sin\theta$$

$$\Rightarrow \vec{\omega} \times (\vec{\omega} \times \vec{r}) = \omega^2\rho$$

Question

A coordinate system OXYZ is rotating with angular velocity $\vec{\omega} = 5\hat{\imath} - 4\hat{\jmath} - 10\hat{k}$ relative to a fixed coordinate system $OXYZ$ both systems having the same origin. Find the velocity of a particle at rest in the $OX'Y'Z'$ system at the point $(3,1,-2)$ as seen by an observer in the fixed system.

Solution

Given that $\vec{\omega} = 5\hat{\imath} - 4\hat{\jmath} - 10\hat{k}$ and we to find $\vec{v}_s$ at $(3,1,-2)$

Using rotating axes theorem $\left(\frac{d\vec{r}}{dt}\right)_f = \left(\frac{d\vec{r}}{dt}\right)_r + \vec{\omega} \times \vec{r}$

$\Rightarrow \left(\frac{d\vec{r}}{dt}\right)_s = \left(\frac{d\vec{r}}{dt}\right)_b + \vec{\omega} \times \vec{r} \Rightarrow \vec{v}_s = \vec{v}_b + \vec{\omega} \times \vec{r}$

$\Rightarrow \vec{v}_s = \vec{\omega} \times \vec{r}$ $\qquad$ $\vec{v}_b = 0$ as particle at rest in $OX'Y'Z'$ system

$$\Rightarrow \vec{v}_s = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 5 & -4 & -10 \\ x & y & z \end{vmatrix} = (-4z + 10y)\hat{\imath} + (-10x - 5z)\hat{\jmath} + (5y + 4x)\hat{k}$$

$\Rightarrow (\vec{v}_s)_{(3,1,-2)} = 18\hat{\imath} - 20\hat{\jmath} + 17\hat{k}$

Question

A coordinate system OXYZ is rotating with angular velocity $\vec{\omega} = cost\hat{\imath} + sint\hat{\jmath} + \hat{k}$ relative to a fixed coordinate system $OXYZ$ both systems having the same origin. Position vector of the particle is given by $\vec{r} = cost\hat{\imath} + sint\hat{\jmath} + t\hat{k}$. Determine the apparent and true acceleration of the particle.

Solution

Given that $\vec{\omega} = cost\hat{\imath} + sint\hat{\jmath} + \hat{k}$ and $\vec{r} = cost\hat{\imath} + sint\hat{\jmath} + t\hat{k}$

We to find $\vec{a}_b$ and $\vec{a}_s$

For $\vec{a}_b$

$\vec{v}_b = \frac{d\vec{r}}{dt} = cost\hat{\imath} - sint\hat{\jmath} + \hat{k}$ and then $\vec{a}_b = \frac{d\vec{v}_b}{dt} = \frac{d^2\vec{r}}{dt^2} = -sint\hat{\imath} - cost\hat{\jmath} + 0\hat{k}$

For $\vec{a}_s$

Using rotating axes theorem $\left(\frac{dA}{dt}\right)_f = \left(\frac{dA}{dt}\right)_r + \vec{\omega} \times A$

$\Rightarrow \left(\frac{d\vec{r}}{dt}\right)_s = \left(\frac{d\vec{r}}{dt}\right)_b + \vec{\omega} \times \vec{r} \Rightarrow \vec{v}_s = \vec{v}_b + \vec{\omega} \times \vec{r}$ using $A = \vec{r}$

$$\Rightarrow \vec{v}_s = \left(cost\hat{\imath} - sint\hat{\jmath} + \hat{k}\right) + \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ cost & sint & 1 \\ sint & cost & 1 \end{vmatrix}$$

$$\Rightarrow \vec{v}_s = \left(cost\hat{\imath} - sint\hat{\jmath} + \hat{k}\right) + \left[(tsint - cost)\hat{\imath} + (sint - tcost)\hat{\jmath} + (cos^2t - sin^2t)\hat{k}\right]$$

$$\Rightarrow \vec{v}_s = (cost + tsint - cost)\hat{\imath} + (-sint + sint - tcost)\hat{\jmath} + (1 + cos^2t - sin^2t)\hat{k}$$

$$\Rightarrow \vec{v}_s = (tsint)\hat{\imath} + (-tcost)\hat{\jmath} + (cos^2t + cos^2t)\hat{k}$$

$$\Rightarrow \boldsymbol{\vec{v}_s = tsint\hat{\imath} - tcost\hat{\jmath} + 2cos^2t\hat{k}}$$

Again Using rotating axes theorem $\left(\frac{dA}{dt}\right)_f = \left(\frac{dA}{dt}\right)_r + \vec{\omega} \times A$

$\Rightarrow \left(\frac{d\vec{v}_s}{dt}\right)_s = \left(\frac{d\vec{v}_s}{dt}\right)_b + \vec{\omega} \times \vec{v}_s \Rightarrow \vec{a}_s = \vec{a}_b + \vec{\omega} \times \vec{v}_s$ using $A = \vec{v}$

$$\Rightarrow \vec{a}_s = \frac{d}{dt}\left(tsint\hat{\imath} - tcost\hat{\jmath} + 2cos^2t\hat{k}\right) + \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ cost & sint & 1 \\ tsint & -tcost & 2cos^2t \end{vmatrix}$$

$$\Rightarrow \vec{a}_s = \left[(tcost + sint)\hat{\imath} - (-tsint + cost)\hat{\jmath} - 4costsint\hat{k}\right] + \left[(2cos^2tsint + tcost)\hat{\imath} + (2cos^3t - tsint)\hat{\jmath} + (-tcos^2t - tsin^2t)\hat{k}\right]$$

$$\Rightarrow \vec{a}_s = \left[(tcost + sint + 2cos^2tsint + tcost)\hat{\imath} - (-tsint + cost + 2cos^3t - tsint)\hat{\jmath} + (-4costsint - tcos^2t - tsin^2t)\hat{k}\right]$$

$$\Rightarrow \boldsymbol{\vec{a}_s = (2tcost + 2cos^2tsint + sint)\hat{\imath} + (2tsint - 2cos^3t - cost)\hat{\jmath} + (-4costsint - t)\hat{k}}$$

Equation of Motion in terms of a Rotating System

Equation of Motion in Space Body and Fixed Body System

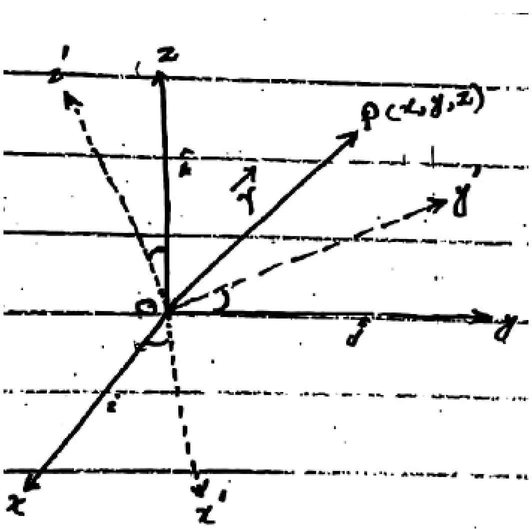
Let $Oxyz$ be a body fixed coordinate system for a rotating body and $Ox'y'z'$ be a space fixed coordinate system. Let $P(x, y, z)$ be a position of particles in both frames then

$$\vec{F}_s = \vec{F}_r + 2m(\vec{\omega} \times \vec{v}_r) + m\big(\vec{\omega} \times (\vec{\omega} \times \vec{r})\big)$$

$$\vec{F}_s = \vec{F}_r + 2(Coriolis\ Force) + (Centrifugal\ Force)$$

Proof

Let $Oxyz$ be a body fixed coordinate system for a rotating body and $Ox'y'z'$ be a space fixed coordinate system. Let $P(x, y, z)$ be a position of particles in both frames.



For body fixed system;

Let $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$ be a position vector in $xyz -$ system with $\hat{\imath}, \hat{\jmath}, \hat{k}$ constant unit vectors then

$$\left(\frac{d\vec{r}}{dt}\right)_r = \frac{dx}{dt}\hat{\imath} + \frac{dy}{dt}\hat{\jmath} + \frac{dz}{dt}\hat{k}$$

For Space fixed system;

Let $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$ be a position vector in $x'y'z'-$ system with $\hat{\imath}, \hat{\jmath}, \hat{k}$ changing unit vectors with respect to time then

$$\left(\frac{d\vec{r}}{dt}\right)_s = \frac{d}{dt}\left(x\hat{\imath} + y\hat{\jmath} + z\hat{k}\right) = \left(\frac{dx}{dt}\hat{\imath} + \frac{dy}{dt}\hat{\jmath} + \frac{dz}{dt}\hat{k}\right) + \left(x\frac{d\hat{\imath}}{dt} + y\frac{d\hat{\jmath}}{dt} + z\frac{d\hat{k}}{dt}\right)$$

$$\left(\frac{d\vec{r}}{dt}\right)_s = \left(\frac{d\vec{r}}{dt}\right)_r + \left(x\frac{d\hat{\imath}}{dt} + y\frac{d\hat{\jmath}}{dt} + z\frac{d\hat{k}}{dt}\right)$$

Using operator form $\frac{d\vec{A}}{dt} = \vec{\omega} \times \vec{A}$ implies $\frac{d\hat{\imath}}{dt} = \vec{\omega} \times \hat{\imath}, \frac{d\hat{\jmath}}{dt} = \vec{\omega} \times \hat{\jmath}, \frac{d\hat{k}}{dt} = \vec{\omega} \times \hat{k}$

$$\left(\frac{d\vec{r}}{dt}\right)_s = \left(\frac{d\vec{r}}{dt}\right)_r + \left[x(\vec{\omega} \times \hat{\imath}) + y(\vec{\omega} \times \hat{\jmath}) + z\left(\vec{\omega} \times \hat{k}\right)\right]$$

$$\left(\frac{d\vec{r}}{dt}\right)_s = \left(\frac{d\vec{r}}{dt}\right)_r + \vec{\omega} \times \left(x\hat{\imath} + y\hat{\jmath} + z\hat{k}\right) = \left(\frac{d\vec{r}}{dt}\right)_r + \vec{\omega} \times \vec{r}$$

$\vec{v}_s = \vec{v}_r + \vec{\omega} \times \vec{r}$ this is the relation between velocities of fixed body and space body system

Generalized for a vector $\vec{A}$ we have $\left(\frac{d\vec{A}}{dt}\right)_s = \left(\frac{d\vec{A}}{dt}\right)_r + \vec{\omega} \times \vec{A}$

$\Rightarrow \left(\frac{d\vec{v}_s}{dt}\right)_s = \left(\frac{d\vec{v}_s}{dt}\right)_r + \vec{\omega} \times \vec{v}_s$ Put $\vec{A} = \vec{v}_s$

$\Rightarrow \left(\frac{d\vec{v}_s}{dt}\right)_s = \left(\frac{d}{dt}(\vec{v}_r + \vec{\omega} \times \vec{r})\right)_r + \omega \times (\vec{v}_r + \vec{\omega} \times \vec{r})$ $\because \vec{v}_s = \vec{v}_r + \vec{\omega} \times \vec{r}$

$$\Rightarrow \vec{a}_s = \left(\frac{d\vec{v}_r}{dt}\right)_r + \left(\frac{d}{dt}(\vec{\omega} \times \vec{r})\right)_r + \vec{\omega} \times \vec{v}_r + \vec{\omega} \times (\vec{\omega} \times \vec{r})$$

$$\Rightarrow \vec{a}_s = \vec{a}_r + \left(\vec{\omega} \times \frac{d\vec{r}}{dt} + \frac{d\vec{\omega}}{dt} \times \vec{r}\right)_r + \vec{\omega} \times \vec{v}_r + \vec{\omega} \times (\vec{\omega} \times \vec{r})$$

$\Rightarrow \vec{a}_s = \vec{a}_r + \vec{\omega} \times \vec{v}_r + \vec{\omega} \times \vec{v}_r + \vec{\omega} \times (\vec{\omega} \times \vec{r})$ $\because \vec{\omega} = Constant\ ;\ \frac{d\vec{\omega}}{dt} = 0$

$$\Rightarrow \vec{a}_s = \vec{a}_r + 2(\vec{\omega} \times \vec{v}_r) + \left(\vec{\omega} \times (\vec{\omega} \times \vec{r})\right)$$

this is the relation between accelerations of fixed body and space body system

$$\Rightarrow m\vec{a}_s = m\vec{a}_r + 2m(\vec{\omega} \times \vec{v}_r) + m\big(\vec{\omega} \times (\vec{\omega} \times \vec{r})\big)$$

$$\Rightarrow \vec{F}_s = \vec{F}_r + 2m(\vec{\omega} \times \vec{v}_r) + m\big(\vec{\omega} \times (\vec{\omega} \times \vec{r})\big)$$

$$\Rightarrow \vec{F}_s = \vec{F}_r + 2(Coriolis\ Force) + (Centrifugal\ Force)$$ **required**

Where Coriolis Forces and Centrifugal Forces are Fictitious/Newtonian forces. Coriolis Force is a negligible force. It moves the body up and down during rotation of a body about its axis.

Centrifugal force is reactive force of the rotating system which produced by increasing the centripetal force. Centrifugal force is directed away from the centre of rotation. Coriolis force is perpendicular to the velocity of moving particles.

Coriolis Theorem

Let $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$ be a position vector in $x'y'z'-$ system with $\hat{\imath}, \hat{\jmath}, \hat{k}$ changing unit vectors with respect to time then

$$\left(\frac{d\vec{r}}{dt}\right)_s = \frac{d}{dt}\left(x\hat{\imath} + y\hat{\jmath} + z\hat{k}\right) = \left(\frac{dx}{dt}\hat{\imath} + \frac{dy}{dt}\hat{\jmath} + \frac{dz}{dt}\hat{k}\right) + \left(x\frac{d\hat{\imath}}{dt} + y\frac{d\hat{\jmath}}{dt} + z\frac{d\hat{k}}{dt}\right)$$

$$\left(\frac{d\vec{r}}{dt}\right)_s = \left(\frac{d\vec{r}}{dt}\right)_r + \left(x\frac{d\hat{\imath}}{dt} + y\frac{d\hat{\jmath}}{dt} + z\frac{d\hat{k}}{dt}\right)$$

Using operator form $\frac{d\vec{A}}{dt} = \vec{\omega} \times \vec{A}$ implies $\frac{d\hat{\imath}}{dt} = \vec{\omega} \times \hat{\imath}, \frac{d\hat{\jmath}}{dt} = \vec{\omega} \times \hat{\jmath}, \frac{d\hat{k}}{dt} = \vec{\omega} \times \hat{k}$

$$\left(\frac{d\vec{r}}{dt}\right)_s = \left(\frac{d\vec{r}}{dt}\right)_r + \left[x(\vec{\omega} \times \hat{\imath}) + y(\vec{\omega} \times \hat{\jmath}) + z\left(\vec{\omega} \times \hat{k}\right)\right]$$

$$\left(\frac{d\vec{r}}{dt}\right)_s = \left(\frac{d\vec{r}}{dt}\right)_r + \vec{\omega} \times \left(x\hat{\imath} + y\hat{\jmath} + z\hat{k}\right) = \left(\frac{d\vec{r}}{dt}\right)_r + \vec{\omega} \times \vec{r}$$

$\vec{v}_s = \vec{v}_r + \vec{\omega} \times \vec{r}$ this is the relation between velocities of fixed body and space body system

Generalized for a vector $\vec{A}$ we have $\left(\frac{d\vec{A}}{dt}\right)_s = \left(\frac{d\vec{A}}{dt}\right)_r + \vec{\omega} \times \vec{A}$

$\Rightarrow \left(\frac{d\vec{v}_s}{dt}\right)_s = \left(\frac{d\vec{v}_s}{dt}\right)_r + \vec{\omega} \times \vec{v}_s$ Put $\vec{A} = \vec{v}_s$

$\Rightarrow \left(\frac{d\vec{v}_s}{dt}\right)_s = \left(\frac{d}{dt}(\vec{v}_r + \vec{\omega} \times \vec{r})\right)_r + \omega \times (\vec{v}_r + \vec{\omega} \times \vec{r})$ $\because \vec{v}_s = \vec{v}_r + \vec{\omega} \times \vec{r}$

$$\Rightarrow \vec{a}_s = \left(\frac{d\vec{v}_r}{dt}\right)_r + \left(\frac{d}{dt}(\vec{\omega} \times \vec{r})\right)_r + \vec{\omega} \times \vec{v}_r + \vec{\omega} \times (\vec{\omega} \times \vec{r})$$

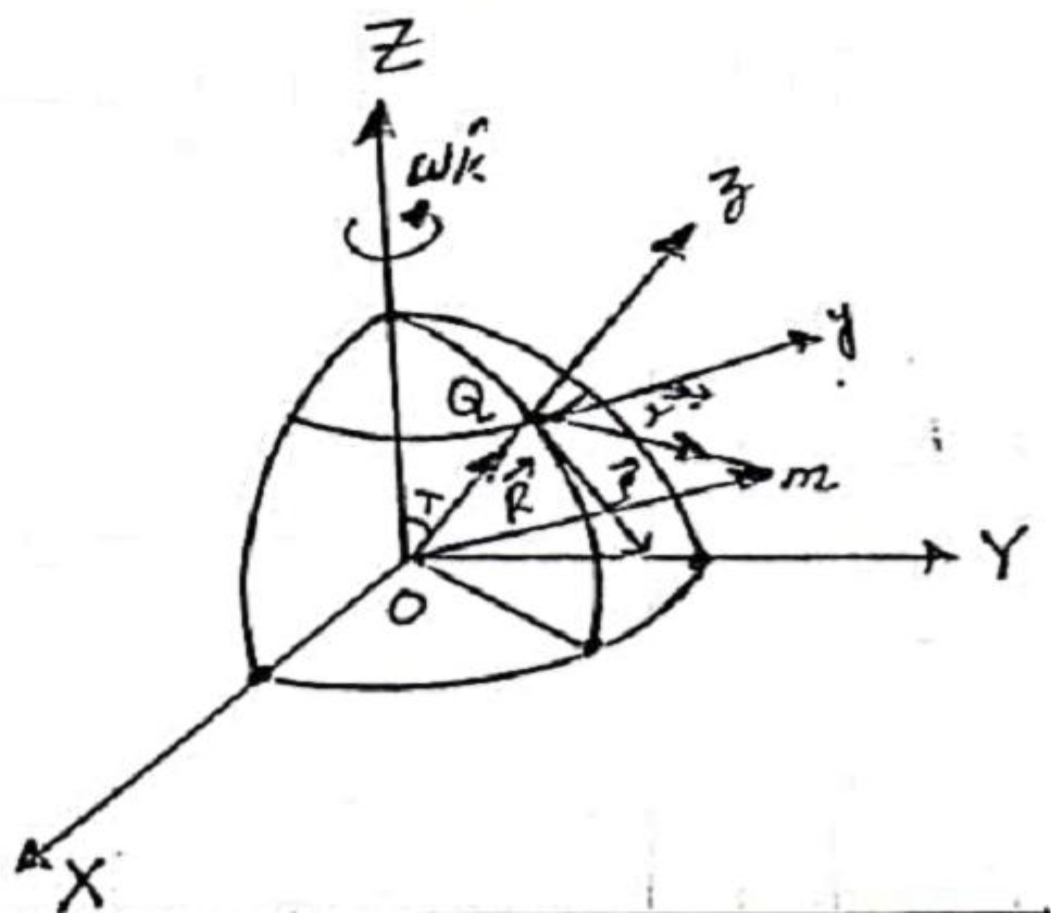
$$\Rightarrow \vec{a}_s = \vec{a}_r + \left(\vec{\omega} \times \frac{d\vec{r}}{dt} + \frac{d\vec{\omega}}{dt} \times \vec{r}\right)_r + \vec{\omega} \times \vec{v}_r + \vec{\omega} \times (\vec{\omega} \times \vec{r})$$

$\Rightarrow \vec{a}_s = \vec{a}_r + \vec{\omega} \times \vec{v}_r + \vec{\omega} \times \vec{v}_r + \vec{\omega} \times (\vec{\omega} \times \vec{r})$ $\because \vec{\omega} = Constant\ ;\ \frac{d\vec{\omega}}{dt} = 0$

$$\Rightarrow \boldsymbol{\vec{a}_s = \vec{a}_r + 2(\vec{\omega} \times \vec{v}_r) + \left(\vec{\omega} \times (\vec{\omega} \times \vec{r})\right)}$$

this is the relation between accelerations of fixed body and space body system

Equation of Motion of a Particle relative to an observer on Earth's surface



Assuming the earth to be sphere with centre at O rotating about Z – axis with a angular velocity $\vec{\omega} = \omega\hat{k}$ neglecting the effect of earth's rotation about sun, XYZ can be taken as inertial frame.

Since the rotation of earth about its axis is with constant angular speed so $\dot{\vec{\omega}} = 0$
The acceleration of Q(origin of moving system) w.r.to O is centripetal, so
$\ddot{\vec{R}} = \vec{\omega} \times (\vec{\omega} \times \vec{r})$

By Newton's Law of Gravitation $\vec{F} = -G\frac{Mm}{\rho^3}\vec{\rho}$

$$\Rightarrow m\frac{d^2\vec{\rho}}{dt^2} = -G\frac{Mm}{\rho^3}\vec{\rho} \Rightarrow \frac{d^2\vec{\rho}}{dt^2} = -G\frac{M}{\rho^3}\vec{\rho}$$

Since $\frac{d^2\vec{\rho}}{dt^2} = \ddot{\vec{R}} + \ddot{\vec{r}} + (\dot{\vec{\omega}} \times \vec{r}) + 2(\vec{\omega} \times \dot{\vec{r}}) + (\vec{\omega} \times (\vec{\omega} \times \vec{r}))$

$\Rightarrow \ddot{\vec{r}} = \frac{d^2\vec{\rho}}{dt^2} - \ddot{\vec{R}} - (\dot{\vec{\omega}} \times \vec{r}) - 2(\vec{\omega} \times \dot{\vec{r}}) - (\vec{\omega} \times (\vec{\omega} \times \vec{r}))$

$\Rightarrow \ddot{\vec{r}} = \frac{d^2\vec{\rho}}{dt^2} - \ddot{\vec{R}} - 2(\vec{\omega} \times \dot{\vec{r}}) - (\vec{\omega} \times (\vec{\omega} \times \vec{r}))$ since $\dot{\vec{\omega}} = 0$

$\Rightarrow \ddot{\vec{r}} = -G\frac{M}{\rho^3}\vec{\rho} - \vec{\omega} \times (\vec{\omega} \times \vec{r}) - 2(\vec{\omega} \times \dot{\vec{r}}) - (\vec{\omega} \times (\vec{\omega} \times \vec{r}))$ using $\ddot{\vec{R}}, \frac{d^2\vec{\rho}}{dt^2}$

Where other forces acting on mass like air resistance etc are neglected

Define $\vec{g} = -G\frac{M}{\rho^3}\vec{\rho} - \vec{\omega} \times (\vec{\omega} \times \vec{r})$ then

$\Rightarrow \ddot{\vec{r}} = \vec{g} - 2(\vec{\omega} \times \dot{\vec{r}}) - (\vec{\omega} \times (\vec{\omega} \times \vec{r}))$

Near earth surface $\vec{\omega} \times (\vec{\omega} \times \vec{r})$ can be neglected, so

$\Rightarrow \ddot{\vec{r}} = \vec{g} - 2(\vec{\omega} \times \dot{\vec{r}})$

Which is required equation to a high degree of approximation.

Acceleration in a Moving Coordinate System

Let $\vec{r}$ be a position vector in $XYZ -$ system (in space) then

$$D_f^2\vec{r} = D_f\left(D_f\vec{r}\right) = D_f(D_m\vec{r} + \vec{\omega} \times \vec{r})$$

$$D_f^2\vec{r} = (D_m + \vec{\omega}\times)(D_m\vec{r} + \vec{\omega} \times \vec{r})$$

$$D_f^2\vec{r} = D_m(D_m\vec{r} + \vec{\omega} \times \vec{r}) + \vec{\omega} \times (D_m\vec{r} + \vec{\omega} \times \vec{r})$$

$$D_f^2\vec{r} = D_m^2\vec{r} + D_m(\vec{\omega} \times \vec{r}) + \vec{\omega} \times D_m\vec{r} + \vec{\omega} \times (\vec{\omega} \times \vec{r}) \qquad \text{.........(1)}$$

$$D_m(\vec{\omega} \times \vec{r}) = D_m\vec{\omega} \times \vec{r} + \vec{\omega} \times D_m\vec{r}$$

$$(1) \Rightarrow D_f^2\vec{r} = D_m^2\vec{r} + D_m\vec{\omega} \times \vec{r} + \vec{\omega} \times D_m\vec{r} + \vec{\omega} \times D_m\vec{r} + \vec{\omega} \times (\vec{\omega} \times \vec{r})$$

$$\Rightarrow D_f^2\vec{r} = D_m^2\vec{r} + D_m\vec{\omega} \times \vec{r} + 2(\vec{\omega} \times D_m\vec{r}) + \vec{\omega} \times (\vec{\omega} \times \vec{r})$$

$$\Rightarrow D_f^2\vec{r} = D_m^2\vec{r} + D_m\vec{\omega} \times \vec{r} + 2\vec{\omega} \times D_m\vec{r} + \vec{\omega} \times (\vec{\omega} \times \vec{r})$$

So the acceleration of particle relative to the moving system is

$$\frac{d^2\vec{r}}{dt^2} = \frac{d^2x}{dt^2}\hat{\imath} + \frac{d^2y}{dt^2}\hat{\jmath} + \frac{d^2z}{dt^2}\hat{k}$$

And the acceleration of particle relative to the fixed system is

$$\left(\frac{d^2\vec{r}}{dt^2}\right)_f = \left(\frac{d^2\vec{r}}{dt^2}\right)_m + \frac{d\vec{\omega}}{dt} \times \vec{r} + 2\vec{\omega} \times \left(\frac{d\vec{r}}{dt}\right)_m + \vec{\omega} \times (\vec{\omega} \times \vec{r})$$

Question: Express the components of equation of motion in terms of rotating coordinate system.

Solution: Rotating coordinate system is a space coordinate system, so we have

$$\vec{F} = m\vec{a}_s \Rightarrow \vec{F} = m\left(\frac{d\vec{v}}{dt}\right)_s \qquad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Using rotating axes theorem we have $\left(\frac{d\vec{A}}{dt}\right)_s = \left(\frac{d\vec{A}}{dt}\right)_r + \vec{\omega} \times \vec{A}_r$

$$\Rightarrow \left(\frac{d\vec{v}}{dt}\right)_s = \left(\frac{d\vec{v}}{dt}\right)_r + \vec{\omega} \times \vec{v}_r$$

$$(1) \Rightarrow \vec{F} = m\left(\left(\frac{d\vec{v}}{dt}\right)_r + \vec{\omega} \times \vec{v}_r\right) \qquad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Now $\vec{F} = F_1\hat{\imath} + F_2\hat{\jmath} + F_3\hat{k}$; $\left(\frac{d\vec{v}}{dt}\right)_r = \frac{dv_1}{dt}\hat{\imath} + \frac{dv_2}{dt}\hat{\jmath} + \frac{dv_3}{dt}\hat{k}$

Also $\vec{\omega} \times \vec{v} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \omega_1 & \omega_2 & \omega_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$

$$\vec{\omega} \times \vec{v} = (\omega_2 v_3 - \omega_3 v_2)\hat{\imath} + (\omega_3 v_1 - \omega_1 v_3)\hat{\jmath} + (\omega_1 v_2 - \omega_2 v_1)\hat{k}$$

$$(2) \Rightarrow F_1\hat{\imath} + F_2\hat{\jmath} + F_3\hat{k} = m\left(\frac{dv_1}{dt}\hat{\imath} + \frac{dv_2}{dt}\hat{\jmath} + \frac{dv_3}{dt}\hat{k} + (\omega_2 v_3 - \omega_3 v_2)\hat{\imath} + (\omega_3 v_1 - \omega_1 v_3)\hat{\jmath} + (\omega_1 v_2 - \omega_2 v_1)\hat{k}\right)$$

$$\Rightarrow F_1\hat{\imath} + F_2\hat{\jmath} + F_3\hat{k} = m\left[\frac{dv_1}{dt} + (\omega_2 v_3 - \omega_3 v_2)\right]\hat{\imath} + m\left[\frac{dv_2}{dt} + (\omega_3 v_1 - \omega_1 v_3)\right]\hat{\jmath} + m\left[\frac{dv_3}{dt} + (\omega_1 v_2 - \omega_2 v_1)\right]\hat{k}$$

$$\boldsymbol{F_1 = m\left[\frac{dv_1}{dt} + (\omega_2 v_3 - \omega_3 v_2)\right]}$$

$$\boldsymbol{F_2 = m\left[\frac{dv_2}{dt} + (\omega_3 v_1 - \omega_1 v_3)\right]} \qquad \text{Required equations}$$

$$\boldsymbol{F_3 = m\left[\frac{dv_3}{dt} + (\omega_1 v_2 - \omega_2 v_1)\right]}$$

Euler's Dynamical Equations of Motion for a Rigid Body Fixed at a Point/General Motion of a Rigid Body

Consider a rotation of a rigid body (earth, sum, moon or other galaxy system) in two systems. i. Body fixed system, ii. Space system

Body rotates in the space system. Then the angular momentum of a rotating body w.r.to the origin is given by

$$\vec{L} = \vec{r} \times \vec{P} = \vec{r} \times m\vec{v}$$

$$\Rightarrow \frac{d\vec{L}}{dt} = \frac{d}{dt}(\vec{r} \times m\vec{v}) = \frac{d\vec{r}}{dt} \times m\vec{v} + \vec{r} \times m\frac{d\vec{v}}{dt} = m(\vec{v} \times \vec{v}) + \vec{r} \times m\vec{a}$$

$$\Rightarrow \frac{d\vec{L}}{dt} = \vec{r} \times \vec{F} \Rightarrow \left(\frac{d\vec{L}}{dt}\right)_s = \vec{\tau} \quad \ldots\ldots\ldots\ldots\ldots(1)$$

By using rotating axes theorem $\left(\frac{d\vec{A}}{dt}\right)_s = \left(\frac{d\vec{A}}{dt}\right)_r + \vec{\omega} \times \vec{A}$

$$\Rightarrow \left(\frac{d\vec{L}}{dt}\right)_s = \left(\frac{d\vec{L}}{dt}\right)_r + \vec{\omega} \times \vec{L} \qquad \ldots\ldots\ldots\ldots\ldots(2)$$

As we know that $\vec{L} = I\vec{\omega}$

$$\Rightarrow \begin{bmatrix} L_1 \\ L_2 \\ L_3 \end{bmatrix} = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} \Rightarrow \begin{bmatrix} L_1 \\ L_2 \\ L_3 \end{bmatrix} = \begin{bmatrix} I_1\omega_1 \\ I_2\omega_2 \\ I_3\omega_3 \end{bmatrix}$$

$$\Rightarrow L_1 = I_1\omega_1 \ ; \ L_2 = I_2\omega_2 \ ; \ L_3 = I_3\omega_3 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

And $\frac{d\vec{L}}{dt} = I\frac{d\vec{\omega}}{dt}$ where I is constant in this case.

$$\left(\frac{d\vec{L}}{dt}\right)_r = I\dot{\vec{\omega}} = I\left(\dot{\omega}_1\hat{\imath} + \dot{\omega}_2\hat{\jmath} + \dot{\omega}_3\hat{k}\right) \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

Also $\vec{\omega} \times \vec{L} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \omega_1 & \omega_2 & \omega_3 \\ L_1 & L_2 & L_3 \end{vmatrix}$

$$\Rightarrow \vec{\omega} \times \vec{L} = (\omega_2 L_3 - \omega_3 L_2)\hat{\imath} + (\omega_3 L_1 - \omega_1 L_3)\hat{\jmath} + (\omega_1 L_2 - \omega_2 L_1)\hat{k}$$

$$\Rightarrow \vec{\omega} \times \vec{L} = (\omega_2 I_3\omega_3 - \omega_3 I_2\omega_2)\hat{\imath} + (\omega_3 I_1\omega_1 - \omega_1 I_3\omega_3)\hat{\jmath} + (\omega_1 I_2\omega_2 - \omega_2 I_1\omega_1)\hat{k}$$

$\Rightarrow \vec{\omega} \times \vec{L} = [\omega_2\omega_3(I_3 - I_2)]\hat{\imath} + [\omega_1\omega_3(I_1 - I_3)]\hat{\jmath} + [\omega_1\omega_2(I_2 - I_1)]\hat{k}$ ………(5)

Using (1), (4),(5) in (2)

$\Rightarrow \left(\frac{d\vec{L}}{dt}\right)_s = \vec{\tau} = I(\dot{\omega}_1\hat{\imath} + \dot{\omega}_2\hat{\jmath} + \dot{\omega}_3\hat{k}) + [\omega_2\omega_3(I_3 - I_2)]\hat{\imath} + [\omega_1\omega_3(I_1 - I_3)]\hat{\jmath} + [\omega_1\omega_2(I_2 - I_1)]\hat{k}$

$\Rightarrow \tau_1\hat{\imath} + \tau_2\hat{\jmath} + \tau_3\hat{k} = [I\dot{\omega}_1 + \omega_2\omega_3(I_3 - I_2)]\hat{\imath} + [I\dot{\omega}_2 + \omega_1\omega_3(I_1 - I_3)]\hat{\jmath} + [I\dot{\omega}_3 + \omega_1\omega_2(I_2 - I_1)]\hat{k}$

On comparing we have

$\boldsymbol{\tau_1 = I\dot{\omega}_1 + \omega_2\omega_3(I_3 - I_2)}$

$\boldsymbol{\tau_2 = I\dot{\omega}_2 + \omega_1\omega_3(I_1 - I_3)}$

$\boldsymbol{\tau_3 = I\dot{\omega}_3 + \omega_1\omega_2(I_2 - I_1)}$

These are called Euler Dynamical equations of motion.

Symmetrical Top

A rigid body is called Symmetrical Top if its two Principal Moment of Inertia are equal. i.e. $I_1 = I_2 \neq I_3$ or $I_1 \neq I_2 = I_3$.

Spherical Top

A rigid body is called Spherical Top if any three mutually perpendicular axes can be selected as the Principal Axes. i.e. $I_1 = I_2 \neq I_3$ or $I_1 \neq I_2 = I_3$.

Remark

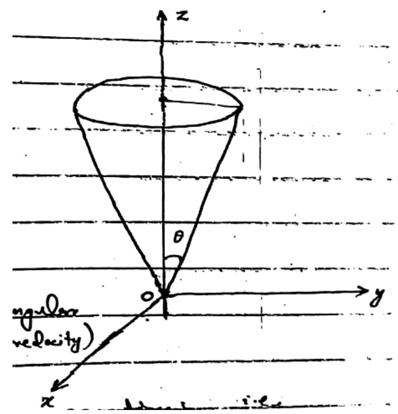
- A rigid body is called **Oblate Symmetrical Top** if $I_1 = I_2 < I_3$.
- A rigid body is called **Prolate Symmetrical Top** if $I_1 < I_2 = I_3$.
- A rigid body is called **Rotator** if $I_1 = I_2 \neq 0$ but $I_3 = 0$.
- The motion of an object in which linear and angular velocities are in the same direction (or Parallel) is called **Screw Motion**.

Force Free Motion of a Symmetrical Top

Free Rotation of a Rigid Body with an axis of Symmetry

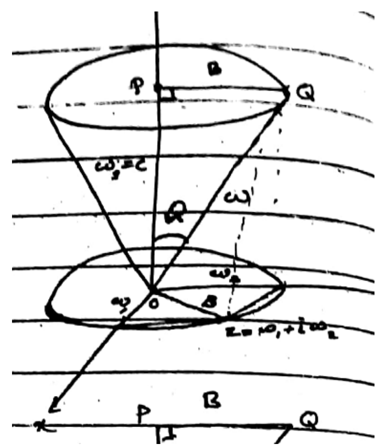
Torque Free Motion of a Symmetrical Top

Euler Equation of Motion for Symmetrical Case



Consider a symmetrical top as shown in figure. Symmetrical Top rotate about z – axis. In the case of principal axis we have a condition for inertia $I_1 = I_2 = I \neq I_3$.

We have to find angular velocity $\vec{\omega}$ by using
Euler Torque Free Equations. i.e.



$\tau_1 = I_1\dot{\omega}_1 + \omega_2\omega_3(I_3 - I_2) = 0$ ……………..(1)

$\tau_2 = I_2\dot{\omega}_2 + \omega_1\omega_3(I_1 - I_3) = 0$ ……………..(2)

$\tau_3 = I_3\dot{\omega}_3 + \omega_1\omega_2(I_2 - I_1) = 0$ ……………..(3)

Put $I_1 = I_2 = I$ in (3) we get $I_3\dot{\omega}_3 = 0$

$\Rightarrow I_3 \neq 0\,;\ \dot{\omega}_3 = 0 \Rightarrow \omega_3 = C$

Put $I_1 = I_2 = I$ in (1), (2)

$(1) \Rightarrow I\dot{\omega}_1 + \omega_2\omega_3(I_3 - I) = 0 \Rightarrow \dot{\omega}_1 + \omega_2\omega_3\left(\frac{I_3 - I}{I}\right) = 0$

$\Rightarrow \dot{\omega}_1 + k\omega_2 = 0$ ……………..(4) using $\omega_3\left(\frac{I_3 - I}{I}\right) = k$

$(2) \Rightarrow I\dot{\omega}_2 + \omega_1\omega_3(I - I_3) = 0 \Rightarrow \dot{\omega}_2 - \omega_1\omega_3\left(\frac{I_3 - I}{I}\right) = 0$

$\Rightarrow \dot{\omega}_2 - k\omega_1 = 0$ ……………..(5) using $\omega_3\left(\frac{I_3 - I}{I}\right) = k$

$\Rightarrow i\dot{\omega}_2 - ik\omega_1 = 0$ ……………..(6) multiplying (5) by i

Adding (4) and (6)

$\Rightarrow \dot{\omega}_1 + i\dot{\omega}_2 + k\omega_2 - ik\omega_1 = 0 \Rightarrow \frac{d}{dt}(\omega_1 + i\omega_2) - ik\omega_1 - i^2k\omega_2 = 0$

$\Rightarrow \frac{d}{dt}(\omega_1 + i\omega_2) - ik(\omega_1 + i\omega_2) = 0$

$\Rightarrow \frac{d\alpha}{dt} - ik\alpha = 0$ using $\omega_1 + i\omega_2 = \alpha$

$\Rightarrow \frac{d\alpha}{dt} = ik\alpha \Rightarrow \int \frac{d\alpha}{\alpha} = ik \int dt \Rightarrow ln\alpha = ikt + A \Rightarrow \alpha = e^{ikt+A} \Rightarrow \alpha = Be^{ikt}$

$\Rightarrow \alpha = B(coskt + isinkt) \Rightarrow \omega_1 + i\omega_2 = Bcoskt + iBsinkt$

$\Rightarrow \omega_1 = Bcoskt$; $\omega_2 = Bsinkt$

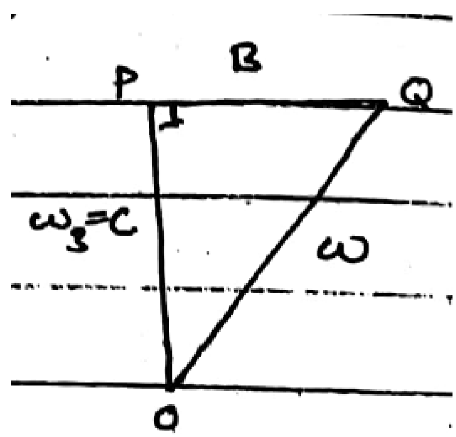
These are parametric equations of circle of radius $B = \sqrt{\omega_1^2 + \omega_2^2}$

Since we know that $\vec{\omega} = \omega_1\hat{\imath} + \omega_2\hat{\jmath} + \omega_3\hat{k}$ therefore $\omega^2 = \omega_1^2 + \omega_2^2 + \omega_3^2$

$\Rightarrow \omega^2 = B^2 + C^2$ using $B = \sqrt{\omega_1^2 + \omega_2^2}$ and $\omega_3 = C$

$\Rightarrow \omega = \sqrt{B^2 + C^2}$ this is the equation of cone (Symmetrical Top)

Results



- **Angle of the Cone/ Symmetrical Top**
  In triangle OPQ
  $tan\theta = \frac{B}{C} = \frac{\sqrt{\omega_1^2+\omega_2^2}}{\omega_3} \Rightarrow \theta = tan^{-1}\left(\frac{\sqrt{\omega_1^2+\omega_2^2}}{\omega_3}\right)$
- **Time Period of the Cone/ Symmetrical Top**
  Since $t = \frac{2\pi}{k} \Rightarrow t = \frac{2\pi}{\omega_3\left(\frac{I_3-I}{I}\right)} \Rightarrow t = \frac{2\pi I}{(I_3-I)\omega_3}$
- **Frequency of the Cone/ Symmetrical Top**
  Since $f = \frac{1}{t} \Rightarrow f = \frac{(I_3-I)\omega_3}{2\pi I}$
- **Kinetic Energy of the Cone/ Symmetrical Top**
  Since $K.E = T = \frac{1}{2}\vec{\omega}.\vec{L}$
  $\Rightarrow T = \frac{1}{2}\left(\omega_1\hat{\imath} + \omega_2\hat{\jmath} + \omega_3\hat{k}\right).\left(I_1\omega_1\hat{\imath} + I_2\omega_2\hat{\jmath} + I_3\omega_3\hat{k}\right)$
  $\Rightarrow T = \frac{1}{2}(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2) = \frac{1}{2}(I(\omega_1^2 + \omega_2^2) + I_3\omega_3^2)$ using $I_1 = I_2 = I$
  $\Rightarrow T = \frac{1}{2}(IB^2 + I_3\omega_3^2)$
  $\Rightarrow 2T = IB^2 + I_3\omega_3^2$

- **Angular Momentum of the Cone/ Symmetrical Top**

  Since $\vec{L} = I_1\omega_1\hat{\imath} + I_2\omega_2\hat{\jmath} + I_3\omega_3\hat{k}$

  $\Rightarrow L^2 = I_1^2\omega_1^2 + I_2^2\omega_2^2 + I_3^2\omega_3^2$

  $\Rightarrow L^2 = I^2(\omega_1^2 + \omega_2^2) + I_3^2\omega_3^2$ using $I_1 = I_2 = I$

  $\Rightarrow L^2 = IB^2 + I_3^2\omega_3^2$

Radius B and Angular Velocity $\vec{\omega}_3$ using Angular Momentum and K.E. of the Cone/ Symmetrical Top

Since we know that

$2T = IB^2 + I_3\omega_3^2$ …………….(1)

$L^2 = IB^2 + I_3^2\omega_3^2$ …………….(2)

Multiplying (1) by I

$2IT = I^2B^2 + II_3\omega_3^2$ …………….(3)

$L^2 = IB^2 + I_3^2\omega_3^2$ …………….(2)

Subtracting (3) and (2) we have

$$\vec{\omega}_3 = \sqrt{\frac{2IT - L^2}{I_3(I - I_3)}}$$

Multiplying (1) by I_3

$2I_3T = II_3B^2 + I_3^2\omega_3^2$ …………….(4)

$L^2 = IB^2 + I_3^2\omega_3^2$ …………….(2)

Subtracting (4) and (2) we have

$$B = \sqrt{\frac{2I_3T - L^2}{I(I_3 - I)}}$$

Example

A body moves about a point O under no force (torque free). The principal moment of inertia at O being 3A,5A,6A. Initially the angular velocity has components $\omega_1 = \omega, \omega_2 = 0, \omega_3 = \omega$ about the corresponding principal axis. Show that at time t, we have $\omega_2 = \frac{3\omega}{\sqrt{5}} tanh \frac{\omega t}{\sqrt{5}}$ if $\int \frac{dx}{a^2 - x^2} = \frac{1}{a} tanh^{-1}\left(\frac{x}{a}\right)$. Also show that the body rotates about the mean axis where $t \to \infty$

Solution:

Given that the principal moment of inertia are $I_1 = 3A, I_2 = 5A, I_3 = 6A$. Initially the angular velocity has components $\omega_1 = \omega, \omega_2 = 0, \omega_3 = \omega$ about the corresponding principal axis. In the torque free case the Euler equations are

$I_1\dot{\omega}_1 + \omega_2\omega_3(I_3 - I_2) = 0$

$I_2\dot{\omega}_2 + \omega_1\omega_3(I_1 - I_3) = 0$ ……………..(1)

$I_3\dot{\omega}_3 + \omega_1\omega_2(I_2 - I_1) = 0$

Put $I_1 = 3A, I_2 = 5A, I_3 = 6A$ in (1)

$3A\dot{\omega}_1 + \omega_2\omega_3(6A - 5A) = 0$

$5A\dot{\omega}_2 + \omega_1\omega_3(3A - 6A) = 0$

$6A\dot{\omega}_3 + \omega_1\omega_2(5A - 3A) = 0$

After simplification we get

$3\dot{\omega}_1 = -\omega_2\omega_3$ ……………..(2)

$5\dot{\omega}_2 = 3\omega_1\omega_3$ ……………..(3)

$3\dot{\omega}_3 = -\omega_1\omega_2$ ……………..(4)

Multiplying (2) by $3\omega_1$ and (3) by ω_2 then adding we have $9\omega_1\dot{\omega}_1 + 5\omega_2\dot{\omega}_2 = 0$

On integrating $\frac{9}{2}\omega_1^2 + \frac{5}{2}\omega_2^2 = C$

$\Rightarrow 9\omega_1^2 + 5\omega_2^2 = 2C \Rightarrow 9\omega_1^2 + 5\omega_2^2 = C_1$ ……………..(5)

Initially using $t = 0, \omega_1 = \omega, \omega_2 = 0$ we get $C_1 = 9\omega^2$

$(5) \Rightarrow 9\omega_1^2 + 5\omega_2^2 = 9\omega^2 \Rightarrow \omega_1^2 + \frac{5}{9}\omega_2^2 = \omega^2$

$\Rightarrow \omega_1^2 = \omega^2 - \frac{5}{9}\omega_2^2$ ……………..(6)

Multiplying (2) by ω_1 and (4) by ω_3 then subtracting we have $\omega_1\dot{\omega}_1 - \omega_3\dot{\omega}_3 = 0$

On integrating $\frac{1}{2}\omega_1^2 - \frac{1}{2}\omega_3^2 = C$

$\Rightarrow \omega_1^2 - \omega_3^2 = 2C \Rightarrow \omega_1^2 - \omega_3^2 = C_2$ ……………..(7)

Initially using $t = 0, \omega_1 = \omega, \omega_3 = \omega$ we get $C_2 = 0$

$(7) \Rightarrow \omega_1^2 - \omega_3^2 = 0 \Rightarrow \omega_1^2 = \omega_3^2$

$\Rightarrow \omega_1 = \omega_3$ ……………..(8)

Using $\omega_1 = \omega_3$ in (3)

$\Rightarrow 5\dot{\omega}_2 = 3\omega_1\omega_3 \Rightarrow 5\dot{\omega}_2 = 3\omega_1^2 \Rightarrow 5\dot{\omega}_2 = 3\omega^2 - \frac{5}{3}\omega_2^2$ using $\omega_1^2 = \omega^2 - \frac{5}{9}\omega_2^2$

$\Rightarrow 15\dot{\omega}_2 = 9\omega^2 - 5\omega_2^2 \Rightarrow \frac{15}{5}\int \frac{d\omega_2}{\frac{9}{5}\omega^2 - \omega_2^2} = \int dt \Rightarrow 3\int \frac{d\omega_2}{\left(\frac{3}{\sqrt{5}}\omega\right)^2 - \omega_2^2} = t + C$

$\Rightarrow 3\frac{1}{\left(\frac{3}{\sqrt{5}}\omega\right)}tanh^{-1}\left(\frac{\omega_2}{\left(\frac{3}{\sqrt{5}}\omega\right)}\right) = t + C$ using $\int \frac{dx}{a^2 - x^2} = \frac{1}{a}tanh^{-1}\left(\frac{x}{a}\right)$

Initially using $t = 0, \omega_2 = 0,$ we get $C = 0$

$\Rightarrow \frac{\sqrt{5}}{\omega}tanh^{-1}\left(\frac{\omega_2}{\left(\frac{3}{\sqrt{5}}\omega\right)}\right) = t \Rightarrow tanh^{-1}\left(\frac{\omega_2}{\left(\frac{3}{\sqrt{5}}\omega\right)}\right) = \frac{\omega t}{\sqrt{5}} \Rightarrow \frac{\omega_2}{\left(\frac{3}{\sqrt{5}}\omega\right)} = tanh\left(\frac{\omega t}{\sqrt{5}}\right)$

$\Rightarrow \omega_2(t) = \frac{3\omega}{\sqrt{5}}tanh\left(\frac{\omega t}{\sqrt{5}}\right)$ after time t

$\Rightarrow \omega_2(\infty) = \frac{3\omega}{\sqrt{5}}tanh\left(\frac{\omega.\infty}{\sqrt{5}}\right)$ when $t \to \infty$

$\Rightarrow \omega_2(\infty) = \frac{3\omega}{\sqrt{5}}(1)$

$$tanh\left(\frac{\omega t}{\sqrt{5}}\right) = \frac{e^{\frac{\omega t}{\sqrt{5}}} - e^{-\frac{\omega t}{\sqrt{5}}}}{e^{\frac{\omega t}{\sqrt{5}}} + e^{-\frac{\omega t}{\sqrt{5}}}}$$

$$tanh\left(\frac{\omega t}{\sqrt{5}}\right) = \frac{1 - e^{-\frac{2\omega t}{\sqrt{5}}}}{1 + e^{-\frac{2\omega t}{\sqrt{5}}}}$$

$$tanh\left(\frac{\omega.\infty}{\sqrt{5}}\right) = \frac{1 - e^{-\infty}}{1 + e^{-\infty}} = 1$$

$\Rightarrow \boldsymbol{\omega_2(\infty) = \frac{3\omega}{\sqrt{5}}}$

Since $\omega_1^2 = \omega^2 - \frac{5}{9}\omega_2^2$

$\Rightarrow \omega_1 = \sqrt{\omega^2 - \frac{5}{9}\omega_2^2} \Rightarrow \omega_1(\infty) = \sqrt{\omega^2 - \frac{5}{9}\left(\frac{3\omega}{\sqrt{5}}\right)^2} \Rightarrow \omega_1(\infty) = \sqrt{\omega^2 - \frac{5}{9} \times \frac{9}{5}\omega^2}$

$\Rightarrow \omega_1(\infty) = \sqrt{\omega^2 - \omega^2} \Rightarrow \boldsymbol{\omega_1(\infty) = 0}$

$\Rightarrow \boldsymbol{\omega_3(\infty) = 0}$ since $\omega_1 = \omega_3$ and $t \to \infty$

Question

A body moves about a point O under no force (torque free). The principal moment of inertia at O being 6A,3A,A. Initially the angular velocity has components $\omega_1 = n, \omega_2 = 0, \omega_3 = 3n$ about the corresponding principal axis. Show that at time t, we have $\omega_2 = \sqrt{5}n\tanh\left(\sqrt{5}nt\right)$. Also show that the body rotates about the mean axis where $t \to \infty$

Solution:

Given that the principal moment of inertia are $I_1 = 6A, I_2 = 3A, I_3 = A$. Initially the angular velocity has components $\omega_1 = n, \omega_2 = 0, \omega_3 = 3n$ about the corresponding principal axis. In the torque free case the Euler equations are

$I_1\dot{\omega}_1 + \omega_2\omega_3(I_3 - I_2) = 0$

$I_2\dot{\omega}_2 + \omega_1\omega_3(I_1 - I_3) = 0$ ……………..(1)

$I_3\dot{\omega}_3 + \omega_1\omega_2(I_2 - I_1) = 0$

Put $I_1 = 6A, I_2 = 3A, I_3 = A$ in (1)

$6A\dot{\omega}_1 + \omega_2\omega_3(A - 3A) = 0$

$3A\dot{\omega}_2 + \omega_1\omega_3(6A - A) = 0$

$A\dot{\omega}_3 + \omega_1\omega_2(3A - 6A) = 0$

After simplification we get

$3\dot{\omega}_1 = \omega_2\omega_3$ ……………..(2)

$3\dot{\omega}_2 = -5\omega_1\omega_3$ ……………..(3)

$\dot{\omega}_3 = 3\omega_1\omega_2$ ……………..(4)

Multiplying (2) by $5\omega_1$ and (3) by ω_2 then adding we have $5\omega_1\dot{\omega}_1 + \omega_2\dot{\omega}_2 = 0$

On integrating $\frac{5}{2}\omega_1^2 + \frac{1}{2}\omega_2^2 = C$

$\Rightarrow 5\omega_1^2 + \omega_2^2 = 2C \Rightarrow 5\omega_1^2 + \omega_2^2 = C_1$ ……………..(5)

Initially using $t = 0, \omega_1 = n, \omega_2 = 0$ we get $C_1 = 5n^2$

$(5) \Rightarrow 5\omega_1^2 + \omega_2^2 = 5n^2 \Rightarrow \omega_1^2 + \frac{1}{5}\omega_2^2 = n^2$

$\Rightarrow \omega_1^2 = n^2 - \frac{1}{5}\omega_2^2$ ……………..(6)

Multiplying (2) by $3\omega_1$ and (4) by ω_3 then subtracting we have

$9\omega_1\dot{\omega}_1 - \omega_3\dot{\omega}_3 = 0$

On integrating $\frac{9}{2}\omega_1^2 - \frac{1}{2}\omega_3^2 = C$

$\Rightarrow 9\omega_1^2 - \omega_3^2 = 2C \Rightarrow 9\omega_1^2 - \omega_3^2 = C_2$ ……………..(7)

Initially using $t = 0, \omega_1 = n, \omega_3 = 3n$ we get $C_2 = 0$

$(7) \Rightarrow 9\omega_1^2 - \omega_3^2 = 0 \Rightarrow 9\omega_1^2 = \omega_3^2$

$\Rightarrow 3\omega_1 = \omega_3$ ……………..(8)

Using $\omega_3 = 3\omega_1$ in (3)

$\Rightarrow 3\dot{\omega}_2 = -5\omega_1\omega_3 \Rightarrow 3\dot{\omega}_2 = -15\omega_1^2 \Rightarrow \dot{\omega}_2 = -5\omega_1^2$

$\Rightarrow \dot{\omega}_2 = -5\left(n^2 - \frac{1}{5}\omega_2^2\right)$ using $\omega_1^2 = n^2 - \frac{1}{5}\omega_2^2$

$\Rightarrow \dot{\omega}_2 = -5n^2 + \omega_2^2 \Rightarrow \dot{\omega}_2 = \frac{d\omega_2}{dt} = -\left[\left(\sqrt{5}n\right)^2 - \omega_2^2\right]$

$\Rightarrow \int \frac{d\omega_2}{\left(\sqrt{5}n\right)^2 - \omega_2^2} = -\int dt$

$\Rightarrow \frac{1}{\sqrt{5}n} tanh^{-1}\left(\frac{\omega_2}{\sqrt{5}n}\right) = -t + C$ using $\int \frac{dx}{a^2-x^2} = \frac{1}{a} tanh^{-1}\left(\frac{x}{a}\right)$

Initially using $t = 0, \omega_2 = 0$, we get $C = 0$

$\Rightarrow \frac{1}{\sqrt{5}n} tanh^{-1}\left(\frac{\omega_2}{\sqrt{5}n}\right) = -t \Rightarrow tanh^{-1}\left(\frac{\omega_2}{\sqrt{5}n}\right) = -\sqrt{5}nt \Rightarrow \frac{\omega_2}{\sqrt{5}n} = tanh\left(-\sqrt{5}nt\right)$

$\Rightarrow \omega_2(t) = \sqrt{5}ntanh\left(-\sqrt{5}nt\right)$ after time t

$\Rightarrow \omega_2(\infty) = \sqrt{5}ntanh\left(-\sqrt{5}n.\infty\right)$ when $t \to \infty$

$\Rightarrow \omega_2(\infty) = \sqrt{5}n(-1)$

$\Rightarrow \boldsymbol{\omega_2}(\infty) = -\sqrt{\mathbf{5}}\boldsymbol{n}$

$$tanh\left(-\sqrt{5}nt\right) = \frac{e^{-\sqrt{5}nt} - e^{\sqrt{5}nt}}{e^{\frac{\omega t}{\sqrt{5}}} + e^{-\frac{\omega t}{\sqrt{5}}}}$$

$$tanh\left(-\sqrt{5}nt\right) = \frac{e^{-2\sqrt{5}nt} - 1}{e^{-2\sqrt{5}nt} + 1}$$

$$tanh(\infty) = \frac{e^{-\infty} - 1}{e^{-\infty} + 1} = -1$$

Since $\omega_1^2 = n^2 - \frac{1}{5}\omega_2^2$

$\Rightarrow \omega_1 = \sqrt{n^2 - \frac{1}{5}\omega_2^2} \Rightarrow \omega_1(\infty) = \sqrt{n^2 - \frac{1}{5}\left(-\sqrt{5}n\right)^2}$

$\Rightarrow \omega_1(\infty) = \sqrt{n^2 - \frac{1}{5} \times 5n^2}$

$\Rightarrow \omega_1(\infty) = \sqrt{n^2 - n^2} \Rightarrow \boldsymbol{\omega_1}(\infty) = \mathbf{0}$

$\Rightarrow \boldsymbol{\omega_3}(\infty) = \mathbf{0}$ since $\omega_3 = 3\omega_1$ and $t \to \infty$

Hence prove that the body rotates about the mean axis where $t \to \infty$

Question

An ellipsoid free to move about its centre is set in rotation at t = 0 with component of angular velocity $(n, 0, 3n)$. The principal M.I. at the centre are 6A,3A,A. Find the component of angular velocity after time 't' and show that for $t \to \infty$ velocity is $n\sqrt{5}$.

Solution:

Given that the principal moment of inertia are $I_1 = 6A, I_2 = 3A, I_3 = A$. Initially the angular velocity has components $\omega_1 = n, \omega_2 = 0, \omega_3 = 3n$ about the corresponding principal axis. In the torque free case the Euler equations are

$I_1\dot{\omega}_1 + \omega_2\omega_3(I_3 - I_2) = 0$

$I_2\dot{\omega}_2 + \omega_1\omega_3(I_1 - I_3) = 0$ ……………..(1)

$I_3\dot{\omega}_3 + \omega_1\omega_2(I_2 - I_1) = 0$

Put $I_1 = 6A, I_2 = 3A, I_3 = A$ in (1)

$6A\dot{\omega}_1 + \omega_2\omega_3(A - 3A) = 0$

$3A\dot{\omega}_2 + \omega_1\omega_3(6A - A) = 0$ …………….. (A*)

$A\dot{\omega}_3 + \omega_1\omega_2(3A - 6A) = 0$

After simplification we get

$3\dot{\omega}_1 = \omega_2\omega_3$ ……………..(2)

$3\dot{\omega}_2 = -5\omega_1\omega_3$ ……………..(3)

$\dot{\omega}_3 = 3\omega_1\omega_2$ ……………..(4)

Multiplying (2) by $5\omega_1$ and (3) by ω_2 then adding we have $5\omega_1\dot{\omega}_1 + \omega_2\dot{\omega}_2 = 0$

On integrating $\frac{5}{2}\omega_1^2 + \frac{1}{2}\omega_2^2 = C$

$\Rightarrow 5\omega_1^2 + \omega_2^2 = 2C \Rightarrow 5\omega_1^2 + \omega_2^2 = C_1$ ……………..(5)

Initially using $t = 0, \omega_1 = n, \omega_2 = 0, \omega_3 = 3n$ we get $C_1 = 5n^2$

$(5) \Rightarrow 5\omega_1^2 + \omega_2^2 = 5n^2 \Rightarrow 5\omega_1^2 = 5n^2 - \omega_2^2$

$\Rightarrow \omega_1 = \frac{1}{5}\sqrt{5n^2 - \omega_2^2}$ ……………..(6)

Multiplying (2) by $3\omega_2$ and (4) by $5\omega_3$ then subtracting we have

$9\omega_2\dot{\omega}_2 + 5\omega_3\dot{\omega}_3 = 0$

On integrating $\frac{9}{2}\omega_2^2 + \frac{5}{2}\omega_3^2 = C$

$\Rightarrow 9\omega_2^2 + 5\omega_3^2 = 2C \Rightarrow 9\omega_2^2 + 5\omega_3^2 = C_2$ ……………..(7)

Initially using $t = 0, \omega_1 = n, \omega_2 = 0, \omega_3 = 3n$ we get $C_2 = 45n^2$

$(7) \Rightarrow 9\omega_2^2 + 5\omega_3^2 = 45n^2 \Rightarrow 5\omega_3^2 = 45n^2 - 9\omega_2^2$

$\Rightarrow \omega_3 = \frac{1}{5}\sqrt{45n^2 - 9\omega_2^2}$ ……………..(8)

Using (7), (8) in (A*)

$\Rightarrow 3A\dot{\omega}_2 + \omega_1\omega_3(6A - A) = 0 \Rightarrow 3A\dot{\omega}_2 + 5A\omega_1\omega_3 = 0$

$\Rightarrow 3\dot{\omega}_2 = -5\omega_1\omega_3 \Rightarrow \dot{\omega}_2 = -\frac{5}{3} \times \frac{1}{5}\sqrt{5n^2 - \omega_2^2} \times \frac{1}{5}\sqrt{45n^2 - 9\omega_2^2}$

$\Rightarrow \dot{\omega}_2 = -5n^2 + \omega_2^2 \Rightarrow \dot{\omega}_2 = \frac{d\omega_2}{dt} = -\left[\left(\sqrt{5}n\right)^2 - \omega_2^2\right] \Rightarrow \int \frac{d\omega_2}{\left(\sqrt{5}n\right)^2 - \omega_2^2} = -\int dt$

$\Rightarrow \frac{1}{\sqrt{5}n} tanh^{-1}\left(\frac{\omega_2}{\sqrt{5}n}\right) = -t + C$ using $\int \frac{dx}{a^2 - x^2} = \frac{1}{a} tanh^{-1}\left(\frac{x}{a}\right)$

Initially using $t = 0, \omega_2 = 0,$ we get $C = 0$

$\Rightarrow \frac{1}{\sqrt{5}n} tanh^{-1}\left(\frac{\omega_2}{\sqrt{5}n}\right) = -t \Rightarrow tanh^{-1}\left(\frac{\omega_2}{\sqrt{5}n}\right) = -\sqrt{5}nt \Rightarrow \frac{\omega_2}{\sqrt{5}n} = tanh\left(-\sqrt{5}nt\right)$

$\Rightarrow \omega_2(t) = \sqrt{5}ntanh\left(-\sqrt{5}nt\right)$ after time t

$\Rightarrow \omega_2(\infty) = \sqrt{5}ntanh\left(-\sqrt{5}n.\infty\right)$ when $t \to \infty$

$\Rightarrow \omega_2(\infty) = \sqrt{5}n(-1)$

$\Rightarrow \boldsymbol{\omega_2(\infty) = -\sqrt{5}n}$

$tanh\left(-\sqrt{5}nt\right) = \frac{e^{-\sqrt{5}nt} - e^{\sqrt{5}nt}}{e^{\frac{\omega t}{\sqrt{5}}} + e^{-\frac{\omega t}{\sqrt{5}}}}$

$tanh\left(-\sqrt{5}nt\right) = \frac{e^{-2\sqrt{5}nt} - 1}{e^{-2\sqrt{5}nt} + 1}$

$tanh(\infty) = \frac{e^{-\infty} - 1}{e^{-\infty} + 1} = -1$

$(6) \Rightarrow \omega_1 = \frac{1}{5}\sqrt{5n^2 - \omega_2^2}$

$\Rightarrow \omega_1 = \frac{1}{5}\sqrt{5n^2 - \left(\sqrt{5}ntanh\left(-\sqrt{5}nt\right)\right)^2} \Rightarrow \omega_1 = \frac{1}{5}\sqrt{5n^2 - \left(\sqrt{5}ntanh\left(\sqrt{5}nt\right)\right)^2}$

$\Rightarrow \omega_1 = nSech\left(\sqrt{5}nt\right)$

$\Rightarrow \omega_1 = n$ when $t \to \infty$

$(8) \Rightarrow \omega_3 = \frac{1}{5}\sqrt{45n^2 - 9\omega_2^2}$

$\Rightarrow \omega_3 = \frac{1}{5}\sqrt{45n^2 - 9\left(\sqrt{5}ntanh\left(-\sqrt{5}nt\right)\right)^2}$

$\Rightarrow \omega_3 = \frac{1}{5}\sqrt{45n^2 - 9\left(\sqrt{5}ntanh\left(\sqrt{5}nt\right)\right)^2}$

$\Rightarrow \omega_3 = 3nSech\left(\sqrt{5}nt\right)$

$\Rightarrow \omega_3 = 3n$ when $t \to \infty$

The components of velocity are

$\omega_1 = n \;\; ; \;\; \omega_2(t) = -\sqrt{5}n \;\; ; \;\; \omega_3 = 3n$

$\Rightarrow \vec{\omega} = \omega_1\hat{\imath} + \omega_2\hat{\jmath} + \omega_3\hat{k}$

$\Rightarrow \vec{\omega} = n\hat{\imath} - \sqrt{5}n\hat{\jmath} + 3n\hat{k}$

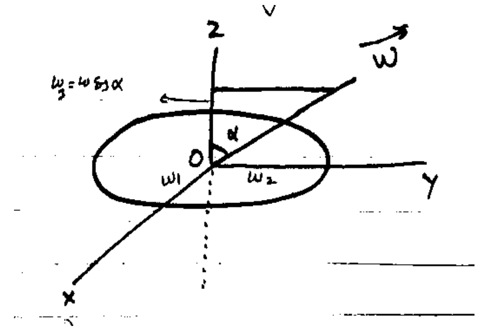
$\Rightarrow |\vec{\omega}| = \omega = \sqrt{n^2 + 5n^2 + 9n^2} = n\sqrt{15}$

Question

A circular disk of radius a and mass m is supported on a needle point at its centre. It is set spinning with angular velocity ω_0 about a line making an angle $\propto$ with the normal to the disk. Find the angular velocity of the disk at any subsequent time.

Solution:

We know that the principal M.I. of circular disk are $I_1 = I_2 = \frac{1}{4}Ma^2, I_3 = \frac{1}{2}Ma^2$. Initially at t = 0 the angular velocity has components $(0, \omega sin \propto, \omega cos \propto)$ about the corresponding principal axis. In the torque free case the Euler equations are



$I_1\dot{\omega}_1 + \omega_2\omega_3(I_3 - I_2) = 0$ ……………..(i)

$I_2\dot{\omega}_2 + \omega_1\omega_3(I_1 - I_3) = 0$ ……………..(ii)

$I_3\dot{\omega}_3 + \omega_1\omega_2(I_2 - I_1) = 0$ ……………..(iii)

For symmetrical torque put $I_1 = I_2$ in (iii)

$\Rightarrow I_3\dot{\omega}_3 = 0 \Rightarrow \omega_3 =$ constant

Initially given that $\omega_3 = \omega cos \propto$

Multiplying (ii) by i and adding in (i) we have

$I_1(\dot{\omega}_1 + i\dot{\omega}_2) - (\omega_2\omega_3 - i\omega_1\omega_3)(I_1 - I_3) = 0$

$\Rightarrow I_1\dot{y} + (i\omega_1\omega_3 + i^2\omega_2\omega_3)(I_1 - I_3) = 0$ where $\dot{y} = \dot{\omega}_1 + i\dot{\omega}_2$

$\Rightarrow I_1\dot{y} + i(\omega_1 + i\omega_2)\omega_3(I_1 - I_3) = 0 \Rightarrow I_1\dot{y} + i\, y\omega_3(I_1 - I_3) = 0$

$\Rightarrow \dot{y} + i\, y\omega_3\left(\frac{I_1 - I_3}{I_1}\right) = 0$

$\Rightarrow \dot{y} + i\mathrm{k}y = 0$ using $k = \omega_3\left(\frac{I_1 - I_3}{I_1}\right)$

$\Rightarrow \dot{y} = -i\mathrm{k}y \Rightarrow \frac{dy}{dt} = -i\mathrm{k}y \Rightarrow \frac{1}{y}dy = -i\mathrm{k}dt \Rightarrow y = Ae^{-i\mathrm{k}t}$

$\Rightarrow y = Aexp\left[-i\left(\omega_3\left(\frac{I_1 - I_3}{I_1}\right)\right)t\right]$

$\Rightarrow y = Aexp\left[i\left(\omega_3\left(\frac{I_3-I_1}{I_1}\right)\right)t\right]$

$\Rightarrow y = Aexp[i\omega_3 t]$ $\because$ if $I_3 = 2I_1$ then $\frac{I_3-I_1}{I_1} = 1$

$\Rightarrow y = Aexp[i\omega cos \propto t]$ ……………..(iv)

$\Rightarrow y = A$ using t = 0

$\Rightarrow \omega_1 + i\omega_2 = A \Rightarrow \omega_1(0) + i\omega_2(0) = A$

$\Rightarrow i\omega sin \propto = A$

$(iv) \Rightarrow \omega_1 + i\omega_2 = i\omega sin \propto exp[i\omega cos \propto t]$

$\Rightarrow \omega_1 + i\omega_2 = i\omega sin \propto [cos(\omega cos \propto t) + isin(\omega cos \propto t)]$

$\Rightarrow \omega_1 + i\omega_2 = -\omega sin \propto sin(\omega cos \propto t) + i\omega sin \propto cos(\omega cos \propto t)$

Comparing real and imaginary parts

$\omega_1 = -\omega sin \propto sin(\omega cos \propto t)$

$\Rightarrow \omega_2 = \omega sin \propto cos(\omega cos \propto t)$

Question

A body moves about a point O under no force (torque free). The principal moment of inertia at O being A,3A,6A. Initially the angular velocity has components $\omega_1 = 3n, \omega_2 = 2n, \omega_3 = n$ about the corresponding principal axis. Show that at time t, we have $\omega_2 = 3n tanh u, \omega_3 = \frac{3n}{\sqrt{5}} sech u, \omega_1 = \frac{9n}{\sqrt{5}} sech u$ where $u = 3nt + ln\sqrt{5}$.

Solution:

Given that the principal moment of inertia are $I_1 = A, I_2 = 3A, I_3 = 6A$. Initially the angular velocity has components $\omega_1 = 3n, \omega_2 = 2n, \omega_3 = n$ about the corresponding principal axis. In the torque free case the Euler equations are

$I_1\dot{\omega}_1 + \omega_2\omega_3(I_3 - I_2) = 0$

$I_2\dot{\omega}_2 + \omega_1\omega_3(I_1 - I_3) = 0$ ……………..(1)

$I_3\dot{\omega}_3 + \omega_1\omega_2(I_2 - I_1) = 0$

Put $I_1 = A, I_2 = 3A, I_3 = 6A$ in (1)

$A\dot{\omega}_1 + \omega_2\omega_3(6A - 3A) = 0$

$3A\dot{\omega}_2 + \omega_1\omega_3(A - 6A) = 0$

$6A\dot{\omega}_3 + \omega_1\omega_2(3A - A) = 0$

After simplification we get

$\dot{\omega}_1 = -3\omega_2\omega_3$ ……………..(2)

$3\dot{\omega}_2 = 5\omega_1\omega_3$ ……………..(3)

$3\dot{\omega}_3 = -\omega_1\omega_2$ ……………..(4)

Multiplying (2) by $5\omega_1$ and (3) by $3\omega_2$ then adding

we have $5\omega_1\dot{\omega}_1 + 9\omega_2\dot{\omega}_2 = 0$

On integrating $\frac{5}{2}\omega_1^2 + \frac{9}{2}\omega_2^2 = C$

$\Rightarrow 5\omega_1^2 + 9\omega_2^2 = 2C \Rightarrow 5\omega_1^2 + 9\omega_2^2 = C_1$ ……………..(5)

Initially using $t = 0, \omega_1 = 3n, \omega_2 = 2n$ we get $C_1 = 81n^2$

(5) $\Rightarrow 5\omega_1^2 + 9\omega_2^2 = 81n^2 \Rightarrow \omega_1^2 + \frac{9}{5}\omega_2^2 = \frac{81}{5}n^2$

$\Rightarrow \omega_1^2 = \frac{81}{5}n^2 - \frac{9}{5}\omega_2^2$ ……………..(6)

Multiplying (2) by ω_1 and (4) by $3\omega_3$ then subtracting we have

$\omega_1\dot{\omega}_1 - 9\omega_3\dot{\omega}_3 = 0$

On integrating $\frac{1}{2}\omega_1^2 - \frac{9}{2}\omega_3^2 = C$

$\Rightarrow \omega_1^2 - 9\omega_3^2 = 2C \Rightarrow \omega_1^2 - 9\omega_3^2 = C_2$ ……………..(7)

Initially using $t = 0, \omega_1 = 3n, \omega_3 = n$ we get $C_2 = 0$

(7) $\Rightarrow \omega_1^2 - 9\omega_3^2 = 0 \Rightarrow \omega_1^2 = 9\omega_3^2$

$\Rightarrow \omega_1 = 3\omega_3$ ……………..(8)

Using $\omega_3 = \frac{1}{3}\omega_1$ in (3)

$\Rightarrow 3\dot{\omega}_2 = 5\omega_1\omega_3 \Rightarrow 3\dot{\omega}_2 = 5\omega_1.\frac{1}{3}\omega_1 \Rightarrow \dot{\omega}_2 = \frac{5}{9}\omega_1^2$

$\Rightarrow \dot{\omega}_2 = \frac{5}{9}\left(\frac{81}{5}n^2 - \frac{9}{5}\omega_2^2\right)$ using $\omega_1^2 = \frac{81}{5}n^2 - \frac{9}{5}\omega_2^2$

$\Rightarrow \dot{\omega}_2 = 9n^2 - \omega_2^2 \Rightarrow \dot{\omega}_2 = \frac{d\omega_2}{dt} = [(3n)^2 - \omega_2^2]$

$\Rightarrow \int \frac{d\omega_2}{(3n)^2 - \omega_2^2} = \int dt$

$$tanh^{-1}\left(\frac{x}{a}\right) = \frac{1}{2}ln\left|\frac{a+x}{a-x}\right|$$

$$tanh^{-1}\left(\frac{2}{3}\right) = \frac{1}{2}ln\left|\frac{3+2}{3-2}\right|$$

$$tanh^{-1}\left(\frac{2}{3}\right) = \frac{1}{2}ln5 = ln\sqrt{5}$$

$\Rightarrow \frac{1}{3n}tanh^{-1}\left(\frac{\omega_2}{3n}\right) = t + C$ using $\int \frac{dx}{a^2 - x^2} = \frac{1}{a}tanh^{-1}\left(\frac{x}{a}\right)$

Initially using $t = 0, \omega_2 = 2n,$ we get $C = \frac{1}{3n}ln\sqrt{5}$

$\Rightarrow \frac{1}{3n}tanh^{-1}\left(\frac{\omega_2}{3n}\right) = t + \frac{1}{3n}ln\sqrt{5} \Rightarrow tanh^{-1}\left(\frac{\omega_2}{3n}\right) = 3nt + ln\sqrt{5}$

$\Rightarrow \frac{\omega_2}{3n} = tanh\left(3nt + ln\sqrt{5}\right) \Rightarrow \omega_2 = 3ntanh\left(3nt + ln\sqrt{5}\right)$

$\Rightarrow \boldsymbol{\omega_2 = 3ntanhu}$ put $u = 3nt + ln\sqrt{5}$

$(6) \Rightarrow \omega_1^2 = \frac{81}{5}n^2 - \frac{9}{5}\omega_2^2 \Rightarrow \omega_1^2 = \frac{81}{5}n^2 - \frac{9}{5}(3ntanhu)^2$

$\Rightarrow \omega_1^2 = \frac{81}{5}n^2 - \frac{81}{5}n^2 tanh^2 u \Rightarrow \omega_1^2 = \frac{81}{5}n^2(1 - tanh^2 u) \Rightarrow \omega_1^2 = \frac{81}{5}n^2 sech^2 u$

$\Rightarrow \boldsymbol{\omega_1 = \frac{9n}{\sqrt{5}} sechu}$

$\Rightarrow \boldsymbol{\omega_3 = \frac{3n}{\sqrt{5}} sechu}$ using $\omega_1 = 3\omega_3$

Question

A body moves about a point O under no force (torque free). The principal moment of inertia at O being 7,25,32. Initially t = 0 and the angular velocity has components $\omega_1 = \frac{4}{5}, \omega_2 = 0, \omega_3 = \frac{3}{5}$ about the corresponding principal axis. Show that $\omega_2 = \frac{4}{5} tanh\left(\frac{3t}{5}\right)$ then find ω_1, ω_3 after time t.

Solution:

Given that the principal moment of inertia are $I_1 = 7, I_2 = 25, I_3 = 32$. Initially t = 0 and the angular velocity has components $\omega_1 = \frac{4}{5}, \omega_2 = 0, \omega_3 = \frac{3}{5}$ about the corresponding principal axis. In the torque free case the Euler equations are

$I_1\dot{\omega}_1 + \omega_2\omega_3(I_3 - I_2) = 0$

$I_2\dot{\omega}_2 + \omega_1\omega_3(I_1 - I_3) = 0$ ……………..(1)

$I_3\dot{\omega}_3 + \omega_1\omega_2(I_2 - I_1) = 0$

Put $I_1 = 7, I_2 = 25, I_3 = 32$ in (1)

$7\dot{\omega}_1 + \omega_2\omega_3(32 - 25) = 0$

$25\dot{\omega}_2 + \omega_1\omega_3(7 - 32) = 0$

$32\dot{\omega}_3 + \omega_1\omega_2(25 - 7) = 0$

After simplification we get

$\dot{\omega}_1 = -\omega_2\omega_3$ ……………..(2)

$\dot{\omega}_2 = \omega_1\omega_3$ ……………..(3)

$16\dot{\omega}_3 = -9\omega_1\omega_2$ ……………..(4)

Multiplying (2) by ω_1 and (3) by ω_2 then adding we have $\omega_1\dot{\omega}_1 + \omega_2\dot{\omega}_2 = 0$

On integrating $\frac{1}{2}\omega_1^2 + \frac{1}{2}\omega_2^2 = C$

$\Rightarrow \omega_1^2 + \omega_2^2 = 2C \Rightarrow \omega_1^2 + \omega_2^2 = C_1$ ……………..(5)

Initially using $t = 0, \omega_1 = \frac{4}{5}, \omega_2 = 0$ we get $C_1 = \frac{16}{25}$

$(5) \Rightarrow \omega_1^2 + \omega_2^2 = \frac{16}{25} \Rightarrow \omega_1^2 = \frac{16}{25} - \omega_2^2$ ……………..(6)

Multiplying (2) by $9\omega_1$ and (4) by ω_3 then subtracting we have

$9\omega_1\dot{\omega}_1 - 16\omega_3\dot{\omega}_3 = 0$

On integrating $\frac{9}{2}\omega_1^2 - \frac{16}{2}\omega_3^2 = C$

$\Rightarrow 9\omega_1^2 - 16\omega_3^2 = 2C \Rightarrow 9\omega_1^2 - 16\omega_3^2 = C_2$ ……………..(7)

Initially using $t = 0, \omega_1 = \frac{4}{5}, \omega_3 = \frac{3}{5}$ we get $C_2 = 0$

$(7) \Rightarrow 9\omega_1^2 - 16\omega_3^2 = 0 \Rightarrow 9\omega_1^2 = 16\omega_3^2$

$\Rightarrow \omega_1 = \frac{4}{3}\omega_3$ ……………..(8)

Using $\omega_3 = \frac{3}{4}\omega_1$ in (3)

$\Rightarrow \dot{\omega}_2 = \omega_1\omega_3 \Rightarrow \dot{\omega}_2 = \omega_1.\frac{3}{4}\omega_1 \Rightarrow \dot{\omega}_2 = \frac{3}{4}\omega_1^2$

$\Rightarrow \dot{\omega}_2 = \frac{3}{4}\left(\frac{16}{25} - \omega_2^2\right)$ using $\omega_1^2 = \frac{16}{25} - \omega_2^2$

$\Rightarrow \dot{\omega}_2 = \frac{d\omega_2}{dt} = \frac{3}{4}\left[\left(\frac{4}{5}\right)^2 - \omega_2^2\right] \Rightarrow \int \frac{d\omega_2}{\left(\frac{4}{5}\right)^2 - \omega_2^2} = \frac{3}{4}\int dt$

$\Rightarrow \frac{1}{\left(\frac{4}{5}\right)} tanh^{-1}\left(\frac{\omega_2}{\left(\frac{4}{5}\right)}\right) = \frac{3}{4}t + C$ using $\int \frac{dx}{a^2-x^2} = \frac{1}{a} tanh^{-1}\left(\frac{x}{a}\right)$

Initially using $t = 0, \omega_2 = 0$, we get $C = 0$

$\Rightarrow \frac{5}{4} tanh^{-1}\left(\frac{\omega_2}{\left(\frac{4}{5}\right)}\right) = \frac{3}{4}t \Rightarrow tanh^{-1}\left(\frac{\omega_2}{\left(\frac{4}{5}\right)}\right) = \frac{3}{5}t$

$\Rightarrow \frac{\omega_2}{\left(\frac{4}{5}\right)} = tanh\left(\frac{3}{5}t\right) \Rightarrow \boldsymbol{\omega_2 = \frac{4}{5} tanh\left(\frac{3t}{5}\right)}$

$(6) \Rightarrow \omega_1^2 = \frac{16}{25} - \omega_2^2 \Rightarrow \omega_1^2 = \frac{16}{25} - \left(\frac{4}{5} tanh\left(\frac{3t}{5}\right)\right)^2$

$\Rightarrow \omega_1^2 = \frac{16}{25} - \frac{16}{25} tanh^2\left(\frac{3t}{5}\right) \Rightarrow \omega_1^2 = \frac{16}{25}\left(1 - tanh^2\left(\frac{3t}{5}\right)\right) \Rightarrow \omega_1^2 = \frac{16}{25} sech^2\left(\frac{3t}{5}\right)$

$\Rightarrow \boldsymbol{\omega_1 = \frac{4}{5} sech\left(\frac{3t}{5}\right)}$

$\Rightarrow \boldsymbol{\omega_3 = \frac{3}{5} sech\left(\frac{3t}{5}\right)}$ using $\omega_3 = \frac{3}{4}\omega_1$

Question

If $\omega_1 = \frac{4}{5} cos\varphi, \omega_2 = \frac{4}{5} Sin\varphi, \omega_3 = \frac{3}{5} cos\varphi$. Show that body rotate about its intermediate principal axes for $t \to \infty$ and $\omega_2 = \frac{4}{5} tanh\left(\frac{3t}{5}\right)$.

Solution:

If $t \to \infty$ then $tanh\left(\frac{3t}{5}\right) \to 1$

As $tanh\left(\frac{3t}{5}\right) = \frac{e^{\left(\frac{3t}{5}\right)} - e^{-\left(\frac{3t}{5}\right)}}{e^{\left(\frac{3t}{5}\right)} + e^{-\left(\frac{3t}{5}\right)}} = \frac{1 - \frac{e^{-\left(\frac{3t}{5}\right)}}{e^{\left(\frac{3t}{5}\right)}}}{1 + \frac{e^{-\left(\frac{3t}{5}\right)}}{e^{\left(\frac{3t}{5}\right)}}} = \frac{1 - \frac{1}{e^{\left(\frac{6t}{5}\right)}}}{1 + \frac{1}{e^{\left(\frac{6t}{5}\right)}}} = \frac{1-0}{1+0} = 1$

As $sin\varphi = tanh\left(\frac{3t}{5}\right) = 1$ then $\varphi = \frac{\pi}{2}$ also $\omega_2 = \frac{4}{5} Sin\varphi = \frac{4}{5} tanh\left(\frac{3t}{5}\right)$

$\Rightarrow \omega_1 = \frac{4}{5} cos\varphi = 0, \omega_2 = \frac{4}{5} Sin\varphi = \frac{4}{5}, \omega_3 = \frac{3}{5} cos\varphi = 0$ using $\varphi = \frac{\pi}{2}$

$\Rightarrow \vec{\omega} = \omega_1 \hat{\imath} + \omega_2 \hat{\jmath} + \omega_3 \hat{k} \Rightarrow \vec{\omega} = \frac{4}{5}\hat{\jmath} \Rightarrow |\vec{\omega}| = \omega = \frac{4}{5}$

Question

In the absence of an external torque in a body prove that

i. K.E. is constant for torque free motion
ii. The magnitude of the square of the angular momentum L^2 is constant.

Solution:

In the torque free case the Euler equations are

$I_1\dot{\omega}_1 + \omega_2\omega_3(I_3 - I_2) = 0$ ……………..(1)

$I_2\dot{\omega}_2 + \omega_1\omega_3(I_1 - I_3) = 0$ ……………..(2)

$I_3\dot{\omega}_3 + \omega_1\omega_2(I_2 - I_1) = 0$ ……………..(3)

Multiplying (1) by ω_1 , (2) by ω_2 , (3) by ω_3 then adding

$I_1\omega_1\dot{\omega}_1 + \omega_1\omega_2\omega_3(I_3 - I_2) + I_2\omega_2\dot{\omega}_2 + \omega_1\omega_2\omega_3(I_1 - I_3) + I_3\omega_3\dot{\omega}_3 + \omega_1\omega_2\omega_3(I_2 - I_1) = 0$

$\Rightarrow I_1\omega_1\dot{\omega}_1 + I_2\omega_2\dot{\omega}_2 + I_3\omega_3\dot{\omega}_3 + \omega_1\omega_2\omega_3(I_3 - I_2 + I_1 - I_3 + I_2 - I_1) = 0$

$\Rightarrow I_1\omega_1\dot{\omega}_1 + I_2\omega_2\dot{\omega}_2 + I_3\omega_3\dot{\omega}_3 = 0$

$\Rightarrow \frac{1}{2}I_1\omega_1^2 + \frac{1}{2}I_2\omega_2^2 + \frac{1}{2}I_3\omega_3^2 = C$

$\Rightarrow \frac{1}{2}[I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2] = C$

$\Rightarrow \frac{1}{2}[\omega_1 I_1\omega_1 + \omega_2 I_2\omega_2 + \omega_3 I_3\omega_3] = C$

$\Rightarrow \frac{1}{2}\left[\left(\omega_1\hat{\imath} + \omega_2\hat{\jmath} + \omega_3\hat{k}\right).\left(I_1\omega_1\hat{\imath} + I_2\omega_2\hat{\jmath} + I_3\omega_3\hat{k}\right)\right] = C$

$\Rightarrow \frac{1}{2}\vec{\omega}.\vec{L} = C$

$\Rightarrow$ $\boldsymbol{K.E = T}$ = **Constant**

Multiplying (1) by $I_1\omega_1$, (2) by $I_2\omega_2$, (3) by $I_3\omega_3$ then adding

$I_1^2\omega_1\dot{\omega}_1 + \omega_1\omega_2\omega_3(I_1I_3 - I_1I_2) + I_1^2\omega_2\dot{\omega}_2 + \omega_1\omega_2\omega_3(I_1I_2 - I_2I_3) + I_1^2\omega_3\dot{\omega}_3 + \omega_1\omega_2\omega_3(I_2I_3 - I_1I_3) = 0$

$\Rightarrow I_1^2\omega_1\dot{\omega}_1 + I_2^2\omega_2\dot{\omega}_2 + I_3^2\omega_3\dot{\omega}_3 + \omega_1\omega_2\omega_3(I_1I_3 - I_1I_2 + I_1I_2 - I_2I_3 + I_2I_3 - I_1I_3) = 0$

$\Rightarrow I_1^2\omega_1\dot{\omega}_1 + I_2^2\omega_2\dot{\omega}_2 + I_3^2\omega_3\dot{\omega}_3 = 0$

$\Rightarrow \frac{1}{2}I_1^2\omega_1^2 + \frac{1}{2}I_2^2\omega_2^2 + \frac{1}{2}I_3^2\omega_3^2 = C_1$

$\Rightarrow \frac{1}{2}[I_1^2\omega_1^2 + I_2^2\omega_2^2 + I_3^2\omega_3^2] = C_1$

$\Rightarrow I_1^2\omega_1^2 + I_2^2\omega_2^2 + I_3^2\omega_3^2 = 2C_1 = C$

$\Rightarrow \boldsymbol{L^2} =$ **Constant**

$\Rightarrow \boldsymbol{L} =$ **Constant** we may write it

Question

Show directly from Euler dynamical equations of motion that if $\vec{N} = 0$ and $I_{xx} = I_{yy}$ then angular velocity ω is constant.

Solution:

Given that $\vec{N} = 0$ (torque is zero). In the torque free case the Euler equations are

$I_{xx}\dot{\omega}_x + \omega_y\omega_z\left(I_{zz} - I_{yy}\right) = 0$ ……………..(1)

$I_{yy}\dot{\omega}_y + \omega_x\omega_z(I_{xx} - I_{zz}) = 0$ ……………..(2)

$I_{zz}\dot{\omega}_z + \omega_x\omega_y\left(I_{yy} - I_{xx}\right) = 0$ ……………..(3)

Using $I_{xx} = I_{yy}$ in (3) we have

$I_{zz}\dot{\omega}_z = 0 \Rightarrow I_{zz} \neq 0;\ \dot{\omega}_z = 0 \Rightarrow \omega_z = C_1$ ……………..(4)

Multiplying (1) by ω_x , (2) by ω_y then adding

$I_{xx}\omega_x\dot{\omega}_x + \omega_x\omega_y\omega_z\left(I_{zz} - I_{yy}\right) + I_{yy}\omega_y\dot{\omega}_y + \omega_x\omega_y\omega_z(I_{xx} - I_{zz}) = 0$

$I_{xx}\omega_x\dot{\omega}_x + I_{yy}\omega_y\dot{\omega}_y + \omega_x\omega_y\omega_z\left(I_{zz} - I_{yy} + I_{xx} - I_{zz}\right) = 0$

$I_{xx}\omega_x\dot{\omega}_x + I_{yy}\omega_y\dot{\omega}_y + \omega_x\omega_y\omega_z\left(-I_{yy} + I_{xx}\right) = 0$

$I_{xx}\omega_x\dot{\omega}_x + I_{xx}\omega_y\dot{\omega}_y = 0$ using $I_{xx} = I_{yy}$

$I_{xx}\left(\omega_x\dot{\omega}_x + \omega_y\dot{\omega}_y\right) = 0 \Rightarrow I_{xx}0\ ;\ \left(\omega_x\dot{\omega}_x + \omega_y\dot{\omega}_y\right) = 0 \Rightarrow \omega_x\dot{\omega}_x + \omega_y\dot{\omega}_y = 0$

$\Rightarrow \frac{1}{2}\omega_x^2 + \frac{1}{2}\omega_y^2 = C_2 \Rightarrow \omega_x^2 + \omega_y^2 = 2C_2 \Rightarrow \omega_x^2 + \omega_y^2 = C_3$ ……………..(5)

Adding (4) and (5)

$\Rightarrow \omega_x^2 + \omega_y^2 + \omega_z^2 = C_3 + C_1^2 \Rightarrow \omega_x^2 + \omega_y^2 + \omega_z^2 = C \Rightarrow \omega^2 = C$

$\Rightarrow \vec{\boldsymbol{\omega}} =$ **Constant**

Question

A rigid body is rotating abut a fixed point with angular velocity $\vec{\omega}$. If coordinate axis coincide with the principal axis then prove that $\frac{dT}{dt} = \vec{G}.\vec{\omega}$ where T is K.E. and $\vec{G}$ is an external torque acting on the body.

Solution:

For a rotating body we have rotational K.E. $T = \frac{1}{2}\vec{\omega}.\vec{L}$

$$\Rightarrow T = \frac{1}{2}\left(\omega_1\hat{\imath} + \omega_2\hat{\jmath} + \omega_3\hat{k}\right).\left(I_1\omega_1\hat{\imath} + I_2\omega_2\hat{\jmath} + I_3\omega_3\hat{k}\right)$$
$$\Rightarrow T = \frac{1}{2}(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2)$$
$$\Rightarrow \frac{dT}{dt} = \frac{1}{2}(I_1.2\omega_1\dot{\omega}_1 + I_2.2\omega_2\dot{\omega}_2 + I_3.2\omega_3\dot{\omega}_3)$$
$$\Rightarrow \frac{dT}{dt} = (I_1\dot{\omega}_1)\omega_1 + (I_2\dot{\omega}_2).\omega_2 + (I_3\dot{\omega}_3).\omega_3 \quad \ldots\ldots\ldots\ldots(1)$$

With external torque the Euler dynamical equations are

$$I_1\dot{\omega}_1 = \omega_2\omega_3(I_2 - I_3) + G_1$$

$$I_2\dot{\omega}_2 = \omega_1\omega_3(I_3 - I_1) + G_2$$

$$I_3\dot{\omega}_3 = \omega_1\omega_2(I_1 - I_2) + G_3$$

Using above values we have

$$(1) \Rightarrow \frac{dT}{dt} = (\omega_2\omega_3(I_2 - I_3) + G_1)\omega_1 + (\omega_1\omega_3(I_3 - I_1) + G_2).\omega_2 + (\omega_1\omega_2(I_1 - I_2) + G_3).\omega_3$$

$$\Rightarrow \frac{dT}{dt} = (I_2 - I_3)\omega_1\omega_2\omega_3 + G_1\omega_1 + (I_3 - I_1)\omega_1\omega_2\omega_3 + G_2\omega_2 + (I_1 - I_2)\omega_1\omega_2\omega_3 + G_3\omega_3$$

$$\Rightarrow \frac{dT}{dt} = (I_2 - I_3 + I_3 - I_1 + I_1 - I_2)\omega_1\omega_2\omega_3 + G_1\omega_1 + G_2\omega_2 + G_3\omega_3$$

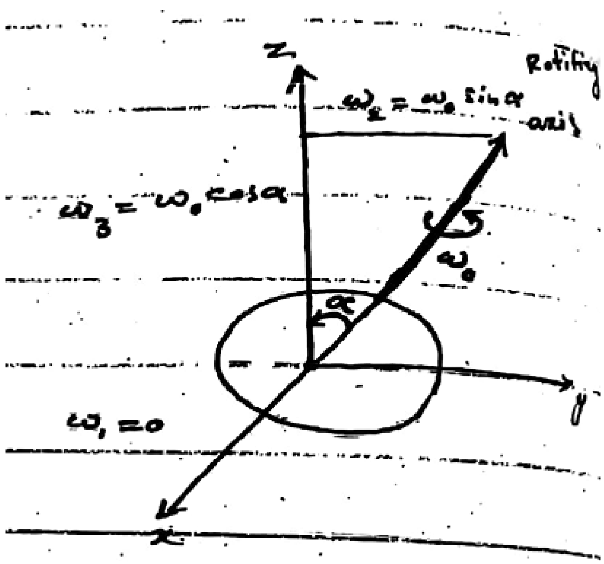
$$\Rightarrow \frac{dT}{dt} = G_1\omega_1 + G_2\omega_2 + G_3\omega_3 = \left(G_1\hat{\imath} + G_2\hat{\jmath} + G_3\hat{k}\right).\left(\omega_1\hat{\imath} + \omega_2\hat{\jmath} + \omega_3\hat{k}\right)$$

$$\Rightarrow \boldsymbol{\frac{dT}{dt} = \vec{G}.\vec{\omega}}$$

Question

A circular disk of radius a and mass m is set spinning motion with angular velocity ω_0 about a line making angle $\propto$ with the normal to the disk in yz – plane. Find angular velocity $\vec{\omega}$ of the disk at any time.

Solution:



Given that initially at $t = 0$; $\omega_1 = 0,\ \omega_2 = \omega_0 sin \propto, \omega_3 = \omega_0 cos \propto$

Also we know that principal moment of inertia of a disk is $I_1 = I_2 = \frac{1}{4}Ma^2$

By perpendicular axis theorem $I_3 = I_1 + I_2 = \frac{1}{2}Ma^2 \Rightarrow I_3 = 2I_1$

Using the torque free case of the Euler equations

$I_1\dot{\omega}_1 + \omega_2\omega_3(I_3 - I_2) = 0$ ……………..(1)

$I_2\dot{\omega}_2 + \omega_1\omega_3(I_1 - I_3) = 0$ ……………..(2)

$I_3\dot{\omega}_3 + \omega_1\omega_2(I_2 - I_1) = 0$ ……………..(3)

After using given values we have

$(3) \Rightarrow I_3\dot{\omega}_3 = 0 \Rightarrow I_3 \neq 0;\ \dot{\omega}_3 = 0 \Rightarrow \omega_3 = Cosntant$

Initially at $t = 0$ we have $\omega_3 = \omega_0 cos \propto$

Using $I_1 = 2$ in (1) and (2)

$I_1\dot{\omega}_1 + \omega_2\omega_3(I_3 - I_1) = 0$ ……………..(4)

$I_1\dot{\omega}_2 + \omega_1\omega_3(I_1 - I_3) = 0$ ……………..(5)

Taking addition in form $(4) + i(5)$

$[I_1\dot{\omega}_1 + \omega_2\omega_3(I_3 - I_1)] + i[I_1\dot{\omega}_2 + \omega_1\omega_3(I_1 - I_3)] = 0$

$\Rightarrow I_1(\dot{\omega}_1 + i\dot{\omega}_2) + \omega_3(I_1 - I_3)(-\omega_2 + i\omega_1) = 0$

$\Rightarrow I_1(\dot{\omega}_1 + i\dot{\omega}_2) + \omega_3(I_1 - 2I_1)(i^2\omega_2 + i\omega_1) = 0$

$\Rightarrow I_1(\dot{\omega}_1 + i\dot{\omega}_2) - I_1\omega_3 i(\omega_1 + i\omega_2) = 0$

$\Rightarrow I_1\dot{P} - iI_1\omega_3 P = 0$ using $\omega_1 + i\omega_2 = P$

$\Rightarrow I_1\left(\dot{P} - i\omega_3 P\right) = 0 \Rightarrow I_1 \neq 0\,;\ \dot{P} - i\omega_0 cos \propto P = 0$

$\Rightarrow \frac{dP}{dt} = i\omega_0 cos \propto P \Rightarrow \int \frac{dP}{P} = i\omega_0 cos \propto \int dt \Rightarrow lnP = i\omega_0 cos \propto t + C$

$\Rightarrow P = e^{i\omega_0 cos\propto t + C} \Rightarrow P = e^{i\omega_0 cos\propto t}.e^{C} \Rightarrow P = Ae^{i\omega_0 cos\propto t}$

$\Rightarrow \omega_1 + i\omega_2 = Ae^{i\omega_0 cos\propto t}$ ……………..(6)

Initially at $t = 0$ using $\omega_1 = 0\,, \omega_2 = \omega_0 sin \propto$ we have $A = i\omega_0 sin \propto$

$(6) \Rightarrow \omega_1 + i\omega_2 = i\omega_0 sin \propto e^{i\omega_0 cos\propto t}$

$\Rightarrow \omega_1 + i\omega_2 = i\omega_0 sin \propto [cos\omega_0 cos \propto t + isin\omega_0 cos \propto t]$

$\Rightarrow \omega_1 + i\omega_2 = i\omega_0 sin \propto cos\omega_0 cos \propto t + i^2\omega_0 sin \propto sin\omega_0 cos \propto t$

$\Rightarrow \omega_1 + i\omega_2 = i\omega_0 sin \propto cos\omega_0 cos \propto t - \omega_0 sin \propto sin\omega_0 cos \propto t$

Comparing real and imaginary parts

$\Rightarrow \omega_1 = -\omega_0 sin \propto sin\omega_0 cos \propto t\,;\ \omega_2 = \omega_0 sin \propto cos\omega_0 cos \propto t$

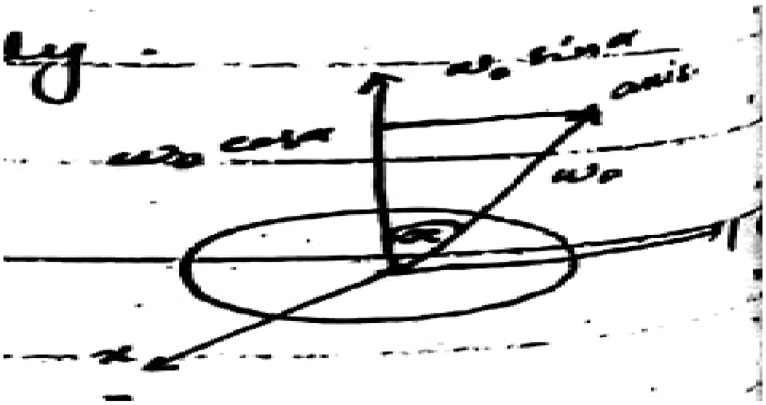
$\Rightarrow \vec{\omega} = \omega_1\hat{\imath} + \omega_2\hat{\jmath} + \omega_3\hat{k}$

$\Rightarrow \boldsymbol{\vec{\omega} = -\omega_0 sin \propto sin\omega_0 cos \propto t\hat{\imath} + \omega_0 sin \propto cos\omega_0 cos \propto t\hat{\jmath} + \omega_0 cos \propto \hat{k}}$

Question

A circular disk of radius a and mass m is set spinning motion with constant angular velocity ω_0 about a line making angle $\propto$ with the normal to the disk in yz – plane. Find torque $\vec{N}$ of the body.

Solution:



Given that initially at $t = 0$; $\quad \omega_1 = 0,\ \omega_2 = \omega_0 sin \propto, \omega_3 = \omega_0 cos \propto$

Also $\vec{\omega} = \text{Constant} \Rightarrow \dot{\vec{\omega}} = 0 \Rightarrow \dot{\omega}_1 = \dot{\omega}_2 = \dot{\omega}_3 = 0$

The principal moment of inertia of a circular disk is $I_1 = I_2 = \frac{1}{4}Ma^2$

By perpendicular axis theorem $I_3 = I_1 + I_2 = \frac{1}{2}Ma^2 \Rightarrow I_3 = 2I_1$

Using the Euler Dynamical equations

$I_1\dot{\omega}_1 + \omega_2\omega_3(I_3 - I_2) = N_1$ ……………..(1)

$I_2\dot{\omega}_2 + \omega_1\omega_3(I_1 - I_3) = N_2$ ……………..(2)

$I_3\dot{\omega}_3 + \omega_1\omega_2(I_2 - I_1) = N_3$ ……………..(3)

After using given values we have

$(1) \Rightarrow 0 + \omega_2\omega_3\left(\frac{1}{2} - \frac{1}{4}\right)Ma^2 = N_1 \Rightarrow N_1 = \frac{1}{4}Ma^2\omega_2\omega_3 = \frac{1}{6}Ma^2\omega_0^2 sin2 \propto$

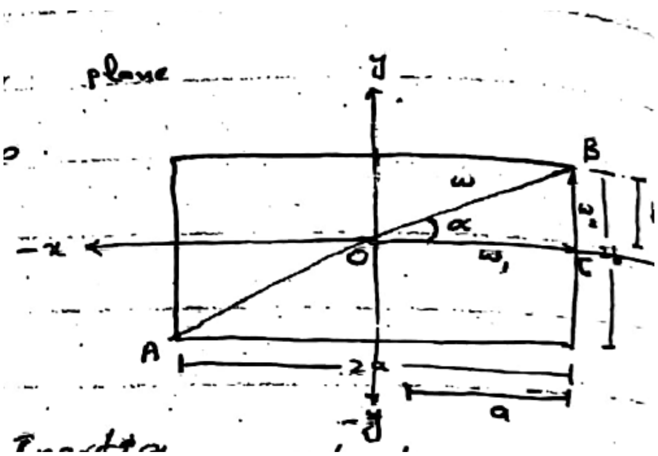
$(2) \Rightarrow 0 + 0 = N_2 \Rightarrow N_2 = 0$ and $(3) \Rightarrow 0 + 0 = N_3 \Rightarrow N_3 = 0$

$\Rightarrow \vec{N} = N_1\hat{\imath} + N_2\hat{\jmath} + N_3\hat{k} \Rightarrow \boldsymbol{\vec{N} = \frac{1}{6}Ma^2\omega_0^2 sin2 \propto \hat{\imath}}$

Question

A rectangular plate spins at its centre with constant angular velocity about diagonal. Find torque which must act on the plane in order to maintain its motion.

Solution:



Consider a rectangular plate of dimensions 2a and 2b. Let diagonal AB makes an angle $\propto$ with x – axis. Then moments of inertia are

About x – axis $\qquad I_1 = I_{xx} = \frac{1}{3}Mb^2$

About y – axis $\qquad I_2 = I_{yy} = \frac{1}{3}Ma^2$

About z – axis $\qquad I_3 = I_{zz} = \frac{1}{3}M(a^2 + b^2)$

From triangle OCB: $\qquad \omega_1 = \omega cos \propto,\ \omega_2 = \omega sin \propto, \omega_3 = 0$

Also $\vec{\omega} = \text{Constant} \Rightarrow \dot{\vec{\omega}} = 0 \Rightarrow \dot{\omega}_1 = \dot{\omega}_2 = \dot{\omega}_3 = 0$

Using the Euler Dynamical equations

$I_1\dot{\omega}_1 + \omega_2\omega_3(I_3 - I_2) = N_1$ ……………..(1)

$I_2\dot{\omega}_2 + \omega_1\omega_3(I_1 - I_3) = N_2$ ……………..(2)

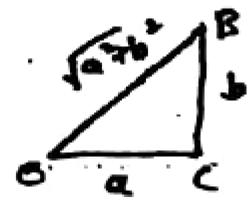
$I_3\dot{\omega}_3 + \omega_1\omega_2(I_2 - I_1) = N_3$ ……………..(3)

After using given values we have

$(1) \Rightarrow 0 + 0 = N_1 \Rightarrow N_1 = 0$ and $(2) \Rightarrow 0 + 0 = N_2 \Rightarrow N_2 = 0$

$(3) \Rightarrow 0 + \omega_1\omega_2\left(\frac{1}{3}Mb^2 - \frac{1}{3}Ma^2\right) = N_3$

$\because$ From fig.



$\because \sin\theta = \frac{b}{\sqrt{a^2+b^2}}$

$\because \cos\theta = \frac{a}{\sqrt{a^2+b^2}}$.

$\Rightarrow N_3 = \omega_1\omega_2\left(\frac{1}{3}Ma^2 - \frac{1}{3}Mb^2\right)$

$\Rightarrow N_3 = \omega cos \propto \omega sin \propto \left(\frac{1}{3}Ma^2 - \frac{1}{3}Mb^2\right)$

$\Rightarrow N_3 = \frac{1}{3}M(a^2 - b^2)\omega^2 sin \propto cos \propto$

$\Rightarrow N_3 = \frac{1}{3}M(a^2 - b^2)\omega^2 . \frac{b}{\sqrt{a^2+b^2}} . \frac{a}{\sqrt{a^2+b^2}}$

$\Rightarrow N_3 = \frac{1}{3}M\omega^2 . \frac{ab(a^2-b^2)}{a^2+b^2}$

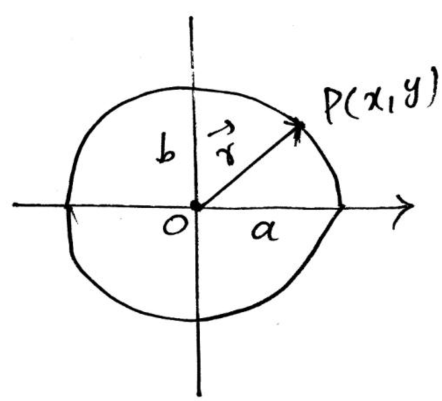
$\Rightarrow \vec{N} = N_1\hat{\imath} + N_2\hat{\jmath} + N_3\hat{k}$

$\Rightarrow \boldsymbol{\vec{N} = \frac{1}{3}M\omega^2 . \frac{ab(a^2-b^2)}{a^2+b^2}}$

Theorem

A particle moves in an elliptical part with constant angular speed. At what points the magnitude of the acceleration (a) maximum and (b) minimum? If the major and minor axes of the elliptical part are 4 and 2 feet respectively determine the magnitude of these accelerations.

Solution:



For elliptical part we have

Length of the major axis $= 2a$

Length of the minor axis $= 2b$

And its parametric equations are

$x = acos\theta = acos\omega t \, ; y = bsin\theta = bsin\omega t$ where $0 \le \theta \le 2\pi$

Let $\vec{r}$ be the position vector of $P(x,y)$ then

$$\vec{r} = x\hat{\imath} + y\hat{\jmath} = acos\omega t\hat{\imath} + bsin\omega t\hat{\jmath}$$

$$\Rightarrow \frac{d\vec{r}}{dt} = \vec{v} = -a\omega sin\omega t\hat{\imath} + b\omega cos\omega t\hat{\jmath}$$

$$\Rightarrow \frac{d^2\vec{r}}{dt^2} = \vec{a} = -a\omega^2 cos\omega t\hat{\imath} - b\omega^2 sin\omega t\hat{\jmath} = -\omega^2(acos\omega t\hat{\imath} + bsin\omega t\hat{\jmath})$$

$$\Rightarrow |\vec{a}| = a = \omega^2\sqrt{(acos\omega t)^2 + (bsin\omega t)^2} = \omega^2\sqrt{a^2\left(\frac{1+cos2\omega t}{2}\right) + b^2\left(\frac{1-cos2\omega t}{2}\right)}$$

$$\Rightarrow |\vec{a}| = \omega^2\sqrt{\left(\frac{a^2+b^2}{2}\right) + \left(\frac{a^2-b^2}{2}\right)cos2\omega t}$$

Maximum Acceleration

Using $cos2\omega t = 1$

$$\Rightarrow 2\omega t = cos^{-1}(1) \Rightarrow 2\omega t = 0,2\pi,4\pi,\ldots \Rightarrow \omega t = 0,\pi,2\pi,\ldots$$

$$\Rightarrow a_{max} = \omega^2\sqrt{\left(\frac{a^2+b^2}{2}\right) + \left(\frac{a^2-b^2}{2}\right)} \Rightarrow a_{max} = \omega^2 a$$

Minimum Acceleration

Using $cos2\omega t = -1$

$$\Rightarrow 2\omega t = cos^{-1}(-1) \Rightarrow 2\omega t = \pi,3\pi,5\pi,\ldots \Rightarrow \omega t = \frac{\pi}{2},\frac{3\pi}{2},\frac{5\pi}{2},\ldots$$

$$\Rightarrow a_{mini} = \omega^2\sqrt{\left(\frac{a^2+b^2}{2}\right) + \left(\frac{a^2-b^2}{2}\right)(-1)} \Rightarrow a_{mini} = \omega^2 b$$

Further given that $2a = 4 \Rightarrow a = 2$ and $2b = 2 \Rightarrow b = 1$ thus

$$\Rightarrow a_{max} = \omega^2 a \Rightarrow \boldsymbol{a_{max} = 2\omega^2}$$

$$\Rightarrow a_{mini} = \omega^2 b \Rightarrow \boldsymbol{a_{mini} = \omega^2}$$

The Eulerian Angles

A rigid body constrained to rotate about a fixed point has only three degrees of freedom. Therefore we require three parameters to specify the configuration of such a body. Euler's angle are three angular coordinates which are used to specify the configuration (orientation) of a rigid body. The Euler angles are usually denoted by θ, ϕ, ψ. Note that there is no universally agreed notation, neither is there agreed convention about their signs.

Let the fixed point about which the body is rotating be O. To define the Euler angles we consider a coordinate system (or a frame of reference) $OX_0Y_0Z_0$ fixed in space, and another coordinate system $OXYZ$ fixed in the body and rotating with it. The first coordinate system is usually referred to as *space* or *fixed* or *inertial* coordinate system, whereas the second coordinate system is referred to as *body* or *moving* or *rotating* coordinate system. We suppose that the two coordinate systems are initially (i.e. at $t = 0$) coincident and define the the Eulerian angles θ, ϕ, ψ in relation to the orientation of the axes of the rotating coordinate system, as follows.

θ = angle between the axes OZ_0 and OZ. It varies from 0 to π.

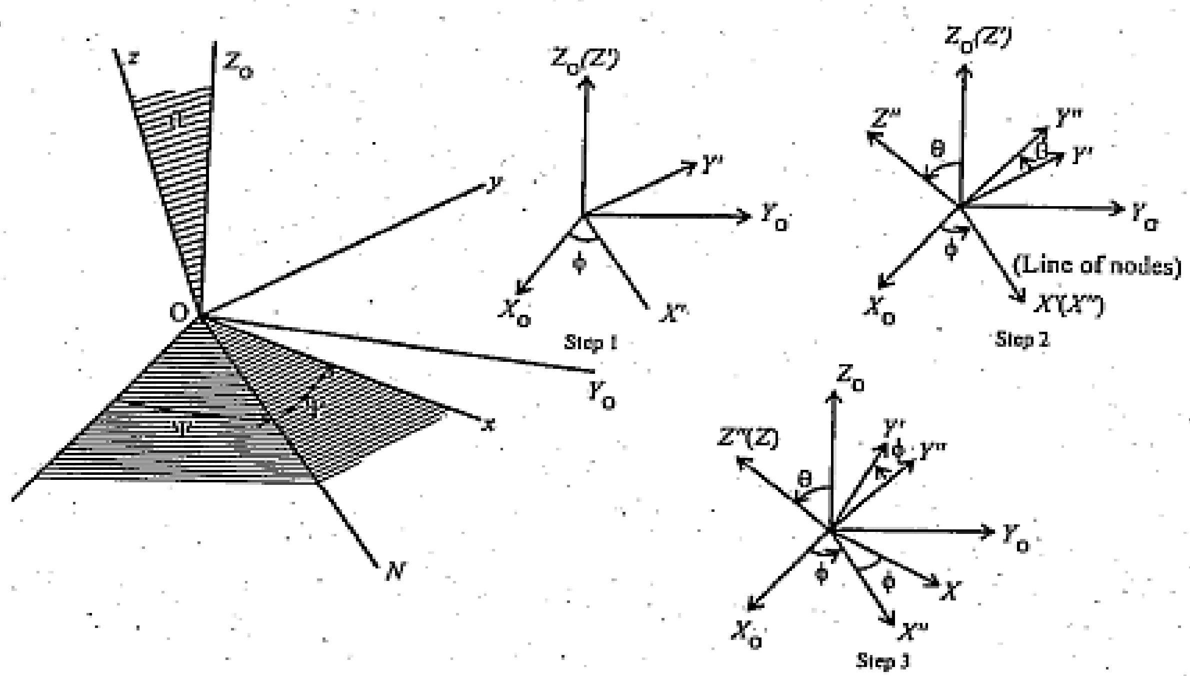


Figure 9.5: Steps in the determination of the Euler angles.

ϕ = angle between the fixed axis OX_0 and the line ON. The line ON is the line of intersection of the planes OX_0Y_0 and OXY, and is called the *line of nodes*. The angle ϕ can also be regarded as the angle between the planes OZ_0Z and OX_0Z_0. It varies from 0 to 2π.

ψ = angle between the body axis OX and the line of nodes ON. It varies from 0 to 2π.

As the body rotates the Euler angles θ, ϕ, ψ vary with time and their derivatives $\dot{\theta}$, $\dot{\phi}$, $\dot{\psi}$ represent angular speeds about certain axes.

Next we discuss the transformation from the space coordinate system $OX_0Y_0Z_0$ to the body coordinate system $OXYZ$, and find the corresponding rotation matrix. In order to obtain the desired rotation matrix, we introduce two other coordinate systems $OX'Y'Z'$ and $OX''Y''Z''$ and perform the following sequence of rotations:

$$(1)\ OX_0Y_0Z_0 \to OX'Y'Z', \quad (2)\ OX'Y'Z' \to OX''Y''Z''$$
$$(3)\ OX''Y''Z'' \to OXYZ$$

1. The first rotation which we perform, through an angle ϕ, is in the counterclockwise direction, in the X_0Y_0-plane (i.e. XY- plane of the fixed coordinate system), about the axis OZ_0. This rotation can be represented by

the matrix

$$R_\phi = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{9.7.1}$$

The angle ϕ is called *precession angle*. After applying this transformation, the new coordinate system is denoted by $OX'Y'Z'$, and the relation between the coordinates is given by

$$\mathbf{X}' = R_\phi \mathbf{X}_0 \tag{9.7.2}$$

where $\mathbf{X}_0$ denotes the column vector of coordinates i.e. $[x_0,\ y_0,\ z_0]^t$. The column vector $\mathbf{X}'$ has a similar definition.

2. The second rotation takes place in the $Y'Z'$-plane, in the counterclockwise direction about the OX'-axis through an angle θ. The rotation matrix in this case is given by

$$R_\theta = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \tag{9.7.3}$$

The angle θ is called *nutation* angle. The new coordinate system is now denoted by $OX''Y''Z''$, and the coordinates are related by

$$\mathbf{X}'' = R_\theta \mathbf{X}' \tag{9.7.4}$$

3. The third rotation takes place in the $OX''Y''$-plane in the counterclockwise direction through an angle ψ about the OZ''-axis. This transformation brings us to the body coordinate system $OXYZ$. The rotation matrix in this case is given by

$$R_\psi = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{9.7.5}$$

and the corresponding coordinate vectors are related by

$$\mathbf{X} = R_\psi \mathbf{X}'' \tag{9.7.6}$$

The angle ψ is called the *body angle*. The transformation from the fixed coordinate system $OX_0Y_0Z_0$ to the body coordinate system $OXYZ$ (see figure 9.5) is given by the rotation matrix $R = R_\psi R_\theta R_\phi$, which when written in full becomes

$$R = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta & 0 \end{bmatrix} \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$= \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\cos\theta\sin\phi & \cos\theta\cos\phi & \sin\theta \\ \sin\theta\sin\phi & -\sin\theta\cos\phi & \cos\theta \end{bmatrix}$$

The elements of the product matrix $R = (r_{ij})$ are given by

$$r_{11} = \begin{bmatrix} \cos\psi & \sin\psi & 0 \end{bmatrix} \begin{bmatrix} \cos\phi \\ -\cos\theta\sin\phi \\ \sin\theta\sin\phi \end{bmatrix} = \cos\psi\cos\phi - \cos\theta\sin\phi\sin\psi$$

$$r_{12} = \begin{bmatrix} \cos\psi & \sin\psi & 0 \end{bmatrix} \begin{bmatrix} \sin\phi \\ -\cos\theta\cos\phi \\ -\sin\theta\cos\phi \end{bmatrix} = \cos\psi\sin\phi + \sin\psi\cos\theta\cos\phi$$

$$r_{13} = \begin{bmatrix} \cos\psi & \sin\psi & 0 \end{bmatrix} \begin{bmatrix} 0 \\ \sin\theta \\ \cos\theta \end{bmatrix} = \sin\psi\sin\theta.$$

$$r_{21} = \begin{bmatrix} -\sin\psi & \cos\psi & 0 \end{bmatrix} \begin{bmatrix} \cos\phi \\ -\cos\theta\sin\phi \\ \sin\theta\sin\phi \end{bmatrix} = -\sin\psi\cos\phi - \cos\psi\cos\theta\sin\phi.$$

$$r_{22} = \begin{bmatrix} -\sin\psi & \cos\psi & 0 \end{bmatrix} \begin{bmatrix} \sin\phi \\ \cos\theta\cos\phi \\ -\sin\theta\sin\phi \end{bmatrix} = -\sin\psi\sin\phi + \cos\psi\cos\theta\cos\phi.$$

$$r_{23} = \begin{bmatrix} -\sin\psi & \cos\psi & 0 \end{bmatrix} \begin{bmatrix} 0 \\ \sin\theta \\ \cos\theta \end{bmatrix} = \cos\psi\ \sin\theta.$$

$$r_{31} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\phi \\ -\cos\theta\sin\phi \\ \sin\theta\sin\phi \end{bmatrix} = \sin\theta\sin\phi.$$

$$r_{32} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sin\phi \\ \cos\theta\cos\phi \\ -\sin\theta\cos\phi \end{bmatrix} = -\sin\theta\cos\phi$$

$$r_{33} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ \sin\theta \\ \cos\theta \end{bmatrix} = \cos\theta$$

An infinitesimal rotation can be represented as a vector. (Recall that such a rotation about an axis specified by the unit vector e is given by $\delta\vec{\theta}=\delta\theta\mathbf{e}$). When the body is rotating about an instantaneous axis through the fixed point O, its angular velocity can be expressed in terms of the time derivatives $\dot{\theta}$, $\dot{\phi}$, $\dot{\psi}$ of the Euler angles. We note that

(i) $\dot{\phi}$ is along the axis OZ_0.

(ii) $\dot{\theta}$ is along the line of nodes, which is the line of intersection of the planes OX_0Y_0 and OXY.

(iii) $\dot{\psi}$ is along the axis OZ.

It is not convenient to use these components of angular velocity $\vec{\omega}$ of the rigid body. Instead we use the body coordinate system $OXYZ$ and express the angular velocity components $(\omega_1, \omega_2, \omega_3)$ w.r.t. this system in terms of $\dot{\theta}$, $\dot{\phi}$, $\dot{\psi}$. For this purpose we consider these components along the three body axes. Remembering that the general infinitesimal rotation associated with ω can be regarded as consisting of three successive infinitesimal rotations with angular velocities $\vec{\omega}_\phi$, $\vec{\omega}_\theta$ and $\vec{\omega}_\psi$ with their magnitudes equal to $\dot{\phi}$, $\dot{\theta}$ and $\dot{\psi}$ respectively. Therefore the vector $\vec{\omega}$ can be expressed as $\vec{\omega} = \vec{\omega}_\phi + \vec{\omega}_\theta + \vec{\omega}_\psi$

We note that $\vec{\omega}_\phi$ is along the OZ_0-axis, $\vec{\omega}_\theta$ along OX'-axis (or along ON, the line of nodes) and $\vec{\omega}_\psi$ along OZ-axis. We will now use the orthogonal transformation given in (9.7.2), (9.7.4), and (9.7.6) to obtain the components of $\vec{\omega}$ along the set of axes we desire.

The body system of axes is the most useful for discussing the equations of motion. Therefore we will obtain components of $\vec{\omega}$ in this system.

Now since $\vec{\omega}_\phi = (0, 0, \dot{\phi}) = [0, 0, \dot{\phi}]^t$ in $OX_0Y_0Z_0$ coordinate system, and the vectors in space and body coordinate systems are connected by $X=RX_0=R_\psi R_\theta R_\phi \mathbf{X}_0$

$$\vec{\omega}_\phi(b) = R_\psi R_\theta R_\phi \vec{\omega}_\theta = R_\psi R_\theta \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \dot{\phi} \end{bmatrix}$$

$$= R_\psi R_\theta \begin{bmatrix} 0 \\ 0 \\ \dot{\phi} \end{bmatrix} = R_\psi \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \dot{\phi} \end{bmatrix}$$

$$= \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ \dot{\phi}\sin\theta \\ \dot{\phi}\cos\theta \end{bmatrix} = \begin{bmatrix} \dot{\phi}\sin\theta\sin\psi \\ \dot{\phi}\sin\theta\cos\psi \\ \dot{\phi}\cos\theta \end{bmatrix}$$

where $\vec{\omega}_\theta(b)$ denotes the contribution to angular velocity in the body system due to rotation through angle ϕ about OZ_0-axis.

The rotation through angle θ is about the axis OX' and therefore the corresponding angular velocity vector $\vec{\omega}_\theta$ is directed along OX' axis. The

vector $\vec{\omega}_\theta$ therefore is represented by the column vector $[\dot{\phi},0,0]^t$ in the coordinate system $OX'Y'Z'$. Its transform in the body coordinate system is $\vec{\omega}_\theta(b)$ and is related by

$$\vec{\omega}_\theta(b) = R_\psi R_\theta \vec{\omega}_\theta = R_\psi \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \dot{\theta} \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{\theta} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \dot{\theta}\cos\psi \\ -\dot{\theta}\sin\psi \\ 0 \end{bmatrix}$$

The rotation about OZ'' axis through angle ψ is the same as rotation about OZ through the same angle. Hence $\vec{\omega} = \dot{\psi}\mathbf{k}'' = \dot{\psi}\mathbf{k}$. Now

$$\vec{\omega} = \vec{\omega}_\theta(b) + \vec{\omega}_\theta(b) + \vec{\omega}_\psi(b)$$

or

$$\begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} = \begin{bmatrix} \dot{\phi}\sin\theta\sin\psi \\ \dot{\phi}\sin\theta\cos\psi \\ \dot{\phi}\cos\theta \end{bmatrix} + \begin{bmatrix} \dot{\theta}\cos\psi \\ -\dot{\theta}\sin\psi \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot{\psi} \end{bmatrix}$$

which gives

$$\omega_1 = \dot{\phi}\sin\theta\sin\phi + \dot{\phi}\cos\psi \qquad (9.7.7a.)$$
$$\omega_2 = \dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi \qquad (9.7.7b.)$$
$$\omega_3 = \dot{\phi}\cos\theta + \dot{\psi} \qquad (9.7.7c.)$$

These equations are called *Euler's geometrical equations.* They describe rigid body motion relative to the body coordinate system.

Tops and Gyroscope

Motions of toy tops are quite frequently seen in everyday life. It's always fascinating to observe the spinning motion of a top along with its precession, its rise, its sleep and finally its death. The theory of spinning

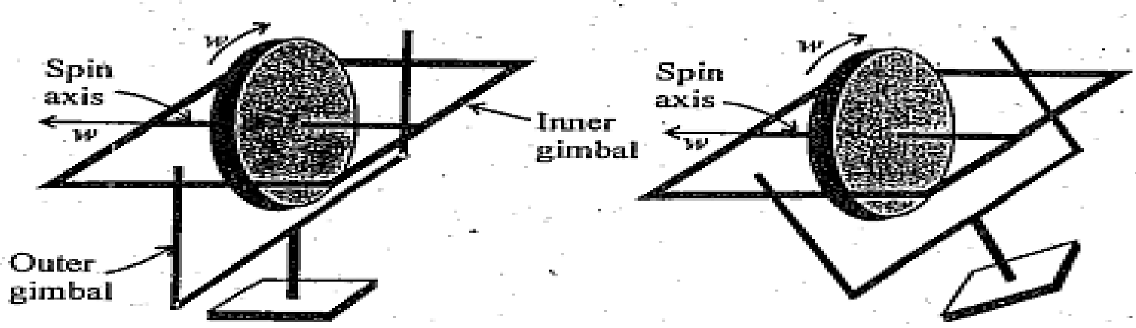
top has relevance in many areas of practical life in applied mechanics (gyroscopic instruments), atomic, molecular and nuclear physics (a whirling molecule/atom or nucleus), and in Astronomy (a spinning planet etc.)

A top is called *sleeping* if it is spinning about its axis of symmetry, which is vertical.

Mathematically gyroscope or top is a rigid body symmetrical about an axis and rotating about that axis. (When the gyroscope rotates about a fixed axis, the angular momentum vector of the gyroscope, about a point on the axis of rotation, is directed along the axis of rotation.) However in applied mechanics gyroscope is a specific device.

Rapidly rotating and heavy bodies are very stable. This fact is the basis of the*gyroscope*. Essentially this consists of a spinning body suspended in such a way that its axis is free to rotate relative to its support. The bearing are designed to be nearly frictionless so that the effect of torque due to the friction is nearly zero. When this is the case, then no matter how use turn the support, the axis of the gyroscope will remain pointing very closely to the same direction in space. More detailed description of the gyroscope is given below.

It consists of a heavy rotating fly wheel, which is mounted in such a way that its axis can freely change direction. This can be achieved by supporting it on a universal joint, or more usually, in what is called*gimbal mounting*. This consists of an outer and an inner ring. The outer ring turns



freely about a vertical axis fixed to an external support, while the inner ring turns freely about a horizontal axis fixed to the outer ring. The flywheel rotates about an axis fixed to the inner ring, which is at right angles to both the other axes. As a result of this arrangement, any torque on the external support does not transfer itself to the flywheel, which continues to point in the same direction in space. Further, if there is a little friction in the bearing, which transfer part of the torque, the gyroscopic effect mentioned above takes care of this decrease in the torque. For this reason, the arrangement is used in inertial guiding systems in ships and aeroplanes.

In a *gyroscope* the inner and outer rings are fixed to each other, and the external casing is arranged to move freely in a horizontal plane.

The stability induced by the spin about the symmetry axis is called the *gyroscopic effect*; since it is this principle on which the working of a gyroscope is based. This principle is used, among other things, in the construction of the barrel of a rifle. The barrel of a rifle has a helical groove cut into it. This makes the bullet move along its axis, which ensures that it continuous to point in the direction of its motion after leaving the barrel.

The importance of the gyroscope as a directional stabilizer arises from the fact that the angular momentum vector **L** remains constant when the torque is applied. The changes in the direction of a well-made gyroscope are small because the applied torques are small and **L** is very large, so that $d\mathbf{L}/dt$ gives no appreciable change in direction. Moreover, a gyroscope only changes direction while a torque is applied. If it shifts slightly due to occasional small frictional torques in its mountings, it stops shifting when the torque stops. A large non-rotating mass, if mounted like a gyroscope, would acquire only small angular velocities due to frictional torques, but once set in motion by a small torque, it would continue to rotate, and the change in position might become large as $t \to \infty$.

Example

Obtain an expression for the kinetic energy of rotation of a rigid body in terms of the Euler angles.

Solution

Since three mutually orthogonally principal axes exist at each point of a rigid body, we choose the principal axes at the point O, as the body coordinate axes. Then in the usual notation

$$T = \frac{1}{2}\vec{\omega}\cdot\mathbf{L}$$

where

$$\vec{\omega} = \omega_1\mathbf{i} + \omega_2\mathbf{j} + \omega_3\mathbf{k}$$

and

$$\mathbf{L} = I_1\omega_1\,\mathbf{i} + I_2\omega_2\,\mathbf{j} + I_3\omega_3\,\mathbf{k}$$

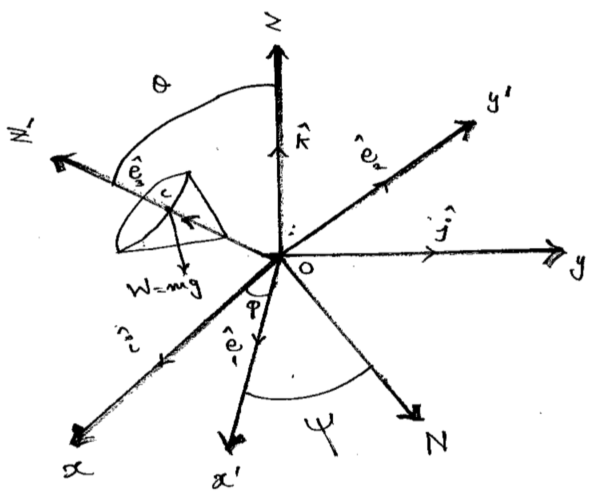
Therefore

$$T = \frac{1}{2}(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2)$$

Substituting the values for ω_1, ω_2 and ω_3 from equations (9.7.7 a, b, c), we have

$$\begin{aligned} T = \frac{1}{2}\{ & I_1(\dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi)^2 \\ + & I_2(\dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi)^2 \\ + & I_3(\dot{\phi}\cos\theta + \dot{\psi})^2\} \end{aligned}$$

Equations of Motion for a Spinning Top having fixed point



Let xyz be a fixed coordinate system in space with origin O. Let $x'y'z'$ be a moving coordinate system (due to rotation of earth) having same origin, which is at earth.

The angular velocity due to rotation of $x'y'z'$ is as follows;

$\vec{\omega} = \omega_1 e_1 + \omega_2 e_2 + \omega_3 e_3$

The angular momentum in component form due to rotation of $x'y'z'$ is as follows;

$\vec{L} = I_1\omega_1 e_1 + I_2\omega_2 e_2 + I_3(\omega_3 + s)e_3$ where $s = se_3 = \dot{\psi}e_3$

$\vec{L} = I_1\omega_1 e_1 + I_2\omega_2 e_2 + (I_3\omega_3 + I_3 s)e_3$

By using rotating axes theorem $\left(\frac{d\vec{A}}{dt}\right)_f = \left(\frac{d\vec{A}}{dt}\right)_m + \vec{\omega} \times \vec{A}$

$$\Rightarrow \left(\frac{d\vec{L}}{dt}\right)_f = \left(\frac{d\vec{L}}{dt}\right)_m + \vec{\omega} \times \vec{L}$$

$$\Rightarrow \left(\frac{d\vec{L}}{dt}\right)_f = \frac{d}{dt}(I_1\omega_1 e_1 + I_2\omega_2 e_2 + (I_3\omega_3 + I_3 s)e_3) + (\omega_1 e_1 + \omega_2 e_2 + \omega_3 e_3) \times (I_1\omega_1 e_1 + I_2\omega_2 e_2 + (I_3\omega_3 + I_3 s)e_3)$$

$$\Rightarrow \left(\frac{d\vec{L}}{dt}\right)_f = I_1\dot{\omega}_1 e_1 + I_2\dot{\omega}_2 e_2 + I_3\dot{\omega}_3 e_3 + I_3\dot{s}e_3 + \begin{vmatrix} e_1 & e_2 & e_3 \\ \omega_1 & \omega_2 & \omega_3 \\ I_1\omega_1 & I_2\omega_2 & I_3\omega_3 + I_3 s \end{vmatrix}$$

$$\Rightarrow \left(\frac{d\vec{L}}{dt}\right)_f = I_1\dot{\omega}_1 e_1 + I_2\dot{\omega}_2 e_2 + I_3\dot{\omega}_3 e_3 + I_3\dot{s}e_3 + e_1(I_3\omega_2\omega_3 + I_3\omega_2 s - I_2\omega_2\omega_3)$$

$$-e_2(I_3\omega_1\omega_3 + I_3\omega_1 s - I_1\omega_1\omega_3) + e_3(I_2\omega_1\omega_2 - I_1\omega_1\omega_2)$$

$$\Rightarrow \vec{\tau} = I_1\dot{\omega}_1 e_1 + I_2\dot{\omega}_2 e_2 + I_3\dot{\omega}_3 e_3 + I_3\dot{s}e_3 + e_1(I_3\omega_2\omega_3 + I_3\omega_2 s - I_2\omega_2\omega_3)$$

$$-e_2(I_3\omega_1\omega_3 + I_3\omega_1 s - I_1\omega_1\omega_3) + e_3(I_2\omega_1\omega_2 - I_1\omega_1\omega_2)$$

$$\Rightarrow \tau_1 e_1 + \tau_2 e_2 + \tau_3 e_3 = I_1\dot{\omega}_1 e_1 + I_2\dot{\omega}_2 e_2 + I_3\dot{\omega}_3 e_3 + I_3\dot{s}e_3 + e_1(I_3\omega_2\omega_3 + I_3\omega_2 s - I_2\omega_2\omega_3) - e_2(I_3\omega_1\omega_3 + I_3\omega_1 s - I_1\omega_1\omega_3) + e_3(I_2\omega_1\omega_2 - I_1\omega_1\omega_2)$$

$$\Rightarrow \tau_1 e_1 + \tau_2 e_2 + \tau_3 e_3 = [I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3 + I_3\omega_2 s]e_1 + [I_2\dot{\omega}_2 + (I_1 - I_3)\omega_1\omega_3 + I_3\omega_1 s]e_2 + [I_3\dot{\omega}_3 + (I_2 - I_1)\omega_1\omega_2 + I_3\dot{s}]e_3$$

………………(1)

Since $\vec{\tau} = \vec{r} \times \vec{F} = le_3 \times mg = le_3 \times \left(-mg\hat{k}\right)$

$$\vec{\tau} = -mgle_3 \times \left((k_1.e_1)e_1 + (k_2.e_2)e_2 + (k_3.e_3)e_3\right)$$

$$\vec{\tau} = -mgle_3 \times \left((|k_1||e_1|cos90^\circ)e_1 + \left(|k_2||e_2|cos(90^\circ - \theta)\right)e_2 + (|k_3||e_3|cos\theta)e_3\right)$$

$$\vec{\tau} = -mgle_3 \times \left((1.1.0)e_1 + (1.1.sin\theta)e_2 + (1.1.cos\theta)e_3\right)$$

$$\vec{\tau} = -mgle_3 \times (sin\theta e_2 + cos\theta e_3) = -mglsin\theta e_3 \times e_2$$

$$\vec{\tau} = \tau_1 e_1 + \tau_2 e_2 + \tau_3 e_3 = mglsin\theta e_1 \qquad \text{………………(2)}$$

Comparing (1) and (2) and using $I_1 = I_2$ we have

$$\boldsymbol{I_1\dot{\omega}_1 + \omega_2\omega_3(I_3 - I_2) + I_3\omega_2 s = mglsin\theta}$$

$$\boldsymbol{I_2\dot{\omega}_2 + \omega_1\omega_3(I_1 - I_3) + I_3\omega_1 s = 0}$$

$$\boldsymbol{I_3(\dot{\omega}_3 + \dot{s}) = 0} \qquad (I_1 = I_2 \Rightarrow I_1 - I_2 = 0)$$

Relationship between the time rate of change of Angular Momentum of a Rigid Body relative to axes Fixed in space and in the body respectively

If the axes of rigid body are choosen as principal axes (rotating) then

The angular velocity due to rotation of $x'y'z'$ is as follows;

$\vec{\omega} = \omega_1 e_1 + \omega_2 e_2 + \omega_3 e_3$

The angular momentum in component form due to rotation of $x'y'z'$ is as follows;

$\vec{L} = I_1\omega_1 e_1 + I_2\omega_2 e_2 + I_3\omega_3 e_3$

By using rotating axes theorem $\left(\frac{d\vec{A}}{dt}\right)_f = \left(\frac{d\vec{A}}{dt}\right)_m + \vec{\omega} \times \vec{A}$

$$\Rightarrow \left(\frac{d\vec{L}}{dt}\right)_f = \left(\frac{d\vec{L}}{dt}\right)_m + \vec{\omega} \times \vec{L}$$

$$\Rightarrow \left(\frac{d\vec{L}}{dt}\right)_f = \frac{d}{dt}(I_1\omega_1 e_1 + I_2\omega_2 e_2 + I_3\omega_3 e_3) + (\omega_1 e_1 + \omega_2 e_2 + \omega_3 e_3)$$

$$\times (I_1\omega_1 e_1 + I_2\omega_2 e_2 + I_3\omega_3 e_3)$$

$$\Rightarrow \left(\frac{d\vec{L}}{dt}\right)_f = I_1\dot{\omega}_1 e_1 + I_2\dot{\omega}_2 e_2 + I_3\dot{\omega}_3 e_3 + \begin{vmatrix} e_1 & e_2 & e_3 \\ \omega_1 & \omega_2 & \omega_3 \\ I_1\omega_1 & I_2\omega_2 & I_3\omega_3 \end{vmatrix}$$

$$\Rightarrow \left(\frac{d\vec{L}}{dt}\right)_f = I_1\dot{\omega}_1 e_1 + I_2\dot{\omega}_2 e_2 + I_3\dot{\omega}_3 e_3 + e_1(I_3\omega_2\omega_3 - I_2\omega_2\omega_3)$$

$$-e_2(I_3\omega_1\omega_3 - I_1\omega_1\omega_3) + e_3(I_2\omega_1\omega_2 - I_1\omega_1\omega_2)$$

$$\Rightarrow \left(\frac{d\vec{L}}{dt}\right)_f = [I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3]e_1 + [I_2\dot{\omega}_2 + (I_1 - I_3)\omega_1\omega_3]e_2 +$$
$$[I_3\dot{\omega}_3 + (I_2 - I_1)\omega_1\omega_2]e_3$$

Equations of Motion of the Gyroscope

$$\omega_{x'} = \dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi$$

$$\omega_{y'} = \dot{\phi}\sin\theta\cos\psi + \dot{\theta}\sin\psi$$

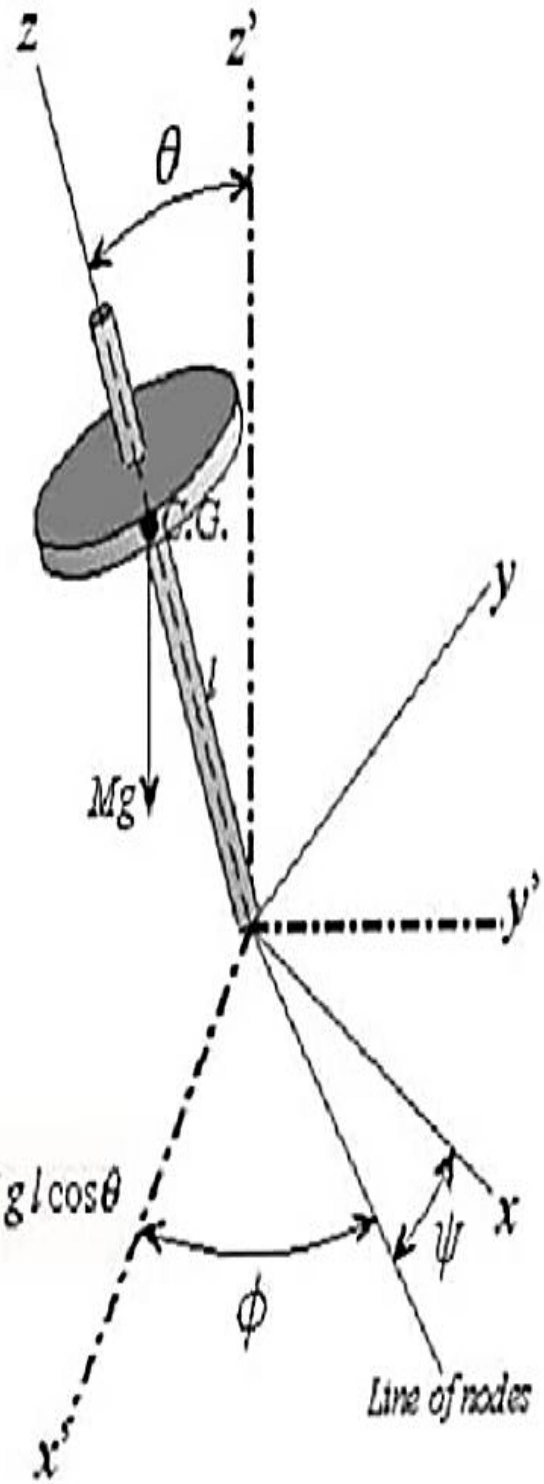
$$\omega_{z'} = \dot{\phi}\cos\theta + \dot{\psi}$$

$$T = \frac{1}{2}I_1(\omega_x^2 + \omega_y^2) + \frac{1}{2}I_3\omega_z^2$$

$$T = \frac{I_1}{2}(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta) + \frac{I_3}{2}(\dot{\psi} + \dot{\phi}\cos\theta)^2$$

$$V = Mgl\cos\theta$$

$$L = \frac{I_1}{2}(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta) + \frac{I_3}{2}(\dot{\psi} + \dot{\phi}\cos\theta)^2 - Mgl\cos\theta$$



حرفِ آخر (2023-08-08)

خوش رہیں خوشیاں بانٹیں اور جہاں تک ہو سکے دوسروں کے لیے آسانیاں پیدا کریں۔

اللہ تعالٰی آپ کو زندگی کے ہر موڑ پر کامیابیوں اور خوشیوں سے نوازے۔ (آمین)

محمد عثمان حامد

چک نمبر 105 شمالی (گودھے والا) سرگودھا

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