FLUID MECHANICS

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INTRODUCTION

Fluid Mechanics or Hydrostatics

As the name implies, fluid mechanics is the study of fluids at rest or in motion. It has traditionally been applied in such areas as the design of canal, levee, and dam systems; the design of pumps, compressors, and piping and ducting used in the water and air conditioning systems of homes and businesses, as well as the piping systems needed in chemical plants; the aerodynamics of automobiles and sub- and supersonic airplanes; and the development of many different flow measurement devices such as gas pump meters.

Or The branch of science which is concerned with the study of motion of fluids or those bodies in contact with fluids is called fluid mechanics or hydrostatics.

Or The study of forces and flows in fluid is called mechanics.

There are three categories of fluid mechanics;

Fluid statistics: Fluid statistics is the study of fluids at rest.

Fluid Kinematics:

Fluid Kinematics is the study of fluids in motion without considering the force which causes the motion. e.g. Speed, Velocity etc.

Fluid Dynamics:

Fluid Dynamics is the study of fluids in motion. It used to analyze flow of air over an aeroplane wing or over a surface of automobile.

Why we study the fluid mechanics?

We casually look around most things seem to be solids but when one thinks of the oceans, the atmosphere and on out into space it becomes rather obvious that a large portion of the earth surface and of the entire universe is in a fluid state. Therefore, it becomes essential for sciences and engineers to know something about fluid mechanics.

Applications (Scope) of Fluid Mechanics:

There are many applications of fluid mechanics make it one of the most important and fundamental in almost all engineering and applied scientific studies such as applied mathematics, plasma physics, geo-physics, bio physics and physical chemistry etc. The experimental aspects of fluid mechanics are the studied through various discipline of engineering. The flow of fluids in pipes and channel makes fluid mechanics of importance to civil engineer. They utilize the results of fluid mechanics to understand the transport of river, irrigation channels, the pollution of air and water & to design pipe line systems, flood control systems and dams etc.

The study of fluid machinery such as pumps, fans, blowers, air compressors heat exchangers, jet and rocket engines, gas turbines, power plants, pollution control equipment etc.

Fluid

A fluid is a substance that deforms continuously under the application of a shear (tangential) stress no matter how small the shear stress may be. Fluids comprise the liquid and gas (or vapor) phases of the physical forms in which matter exists.

Or a fluid as any substance that cannot sustain a shear stress when at rest.

Or a fluid is something which has the property of flowing freely

Note: The *distinction between a fluid and the solid state of matter* is clear if you compare fluid and solid behavior. A solid deforms when a shear stress is applied, but its deformation does not continue to increase with time.

Properties of a fluid are of at least four classes

1. **Kinematic Properties:** Linear Velocity, Angular Velocity, Vorticity, Acceleration and Strain Rate.
2. **Transport Properties:** Viscosity, Thermal Conductivity, Mass Diffusivity.
3. **Thermodynamic Properties:** Pressure, Density, Temperature, Enthalpy, Entropy, Specific Heat, Bulk Modulus, Coefficient of Thermal Expansion.
4. **Other Miscellaneous Properties:** Surface Tension, Vapor Pressure, Surface Accommodation Coefficient.

Motion of fluid particles

A fluid consists of innumerable (countless) whose relative position never fix whenever fluid is in motion the particle moves along certain line depending upon the characteristics of fluid and shape of the passage through which the fluid particle moves. It is necessary to observe the motion of fluid particle at various time and point.

Fluid mechanics have two method of fluid motion

(i) Lagrange’s method (ii) Eulerian Method

Basic Equations (Laws)

Analysis of any problem in fluid mechanics necessarily includes statement of the basic laws governing the fluid motion. The basic laws, which are applicable to any fluid, are:

1. The conservation of mass.

2. Newton's second law of motion.

3. The principle of angular momentum.

4. The first law of thermodynamics,

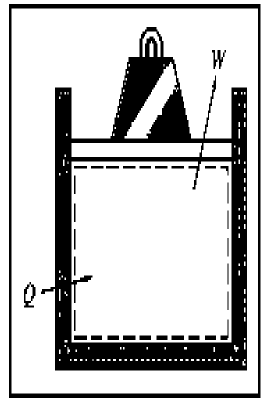
5. The second law of thermodynamics.

Note: All basic laws are always required to solve any one problem. On the other hand, in many problems it is necessary to bring into the analysis additional relations that describe the behavior of physical properties of fluids under given conditions.

Dimensional Analysis of Fluid Flow: It is a mathematical technique used to predict physical parameters that influence the flow in fluid mechanics, heat transfer in thermodynamics and so forth. The analysis involves the fundamental units of dimensions MLt (mass, length and time)

Example First Law Application to Closed System

A piston-cylinder device contains 0.95 kg of oxygen initially at a temperature of 27°C and a pressure due to the weight of 150 kPa (abs). Heat is added to the gas until it reaches a temperature of 627°C. Determine the amount of heat added during the process.



Given: Piston-cylinder containing O_2, $m = 0.95$ kg.

$$T_1 = 27°C \qquad T_2 = 627°C$$

Find: $Q_{1\to 2}$.

Solution: p = constant = 150 kPa (abs)
We are dealing with a system, $m = 0.95$ kg.

Governing equation: First law for the system, $Q_{12} - W_{12} = E_2 - E_1$

Assumptions: (1) $E = U$, since the system is stationary.
(2) Ideal gas with constant specific heats.

Under the above assumptions,

$$E_2 - E_1 = U_2 - U_1 = m(u_2 - u_1) = mc_v(T_2 - T_1)$$

The work done during the process is moving boundary work

$$W_{12} = \int_{V_1}^{V_2} p\,dV = p(V_2 - V_1)$$

For an ideal gas, $pV = mRT$. Hence $W_{12} = mR(T_2 - T_1)$. Then from the first law equation,

$$Q_{12} = E_2 - E_1 + W_{12} = mc_v(T_2 - T_1) + mR(T_2 - T_1)$$
$$Q_{12} = m(T_2 - T_1)(c_v + R)$$
$$Q_{12} = mc_p(T_2 - T_1) \quad \{R = c_p - c_v\}$$

This problem:
- ✓ Was solved using the nine logical steps discussed earlier.
- ✓ Reviewed use of the ideal gas equation and the first law of thermodynamics for a system.

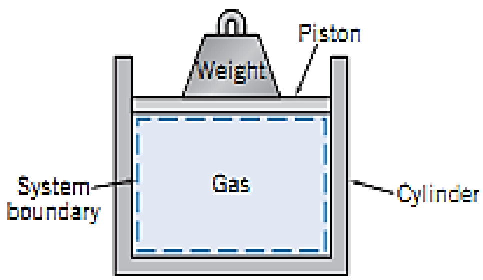
From the Appendix, Table A.6, for O_2, $c_p = 909.4$ J/(kg·K). Solving for Q_{12}, we obtain

$$Q_{12} = 0.95\,\text{kg} \times 909\frac{\text{J}}{\text{kg}\cdot\text{K}} \times 600\,\text{K} = 518\,\text{kJ} \longleftarrow \underline{\quad Q_{12} \quad}$$

System and Surroundings

A **system** is defined as a fixed, identifiable quantity of mass where the region of physical space beyond the system boundaries is called **surroundings**. The system boundaries separate the system from the surroundings. The boundaries of the system may be fixed or movable; however, no mass crosses the system boundaries.

In the familiar **piston-cylinder assembly** from thermodynamics, the gas in the cylinder is the system. If the gas is heated, the piston will lift the weight; the boundary of the system thus moves. Heat and work may cross the boundaries of the system, but the quantity of matter within the system boundaries remains fixed. No mass crosses the system boundaries.



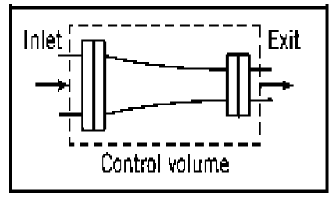
Control Volume and Control Surface

A **control volume** is an arbitrary volume in space through which fluid flows. There are two types of control volume. i.e. Finite size control volume. e.g. Pipe and Differentiable size control volume. e.g. Cube. Both types of control volume are used to drive conservation principles of mass, energy and momentum. Finite size control volume is further divided into two categories. i.e. **Deformable Control Volume** (Such volume in which control surface is allow to change its shape) and **Non - Deformable Control Volume** (Such volume in which control surface is not allow to change its shape)

The geometric boundary of the control volume is called the **control surface**. The control surface may be real or imaginary; it may be at rest or in motion.

Example Mass Conservation Applied To Control Volume

A reducing water pipe section has an inlet diameter of 50 mm and exit diameter of 30 mm. If the steady inlet speed (averaged across the inlet area) is 2.5 m/s, find the exit speed.



Given: Pipe, inlet $D_i = 50$ mm, exit $D_e = 30$ mm.
Inlet speed, $V_i = 2.5$ m/s.

Find: Exit speed, V_e

Solution:

Assumption: Water is incompressible (density ρ = constant).

The physical law we use here is the conservation of mass, which you learned in thermodynamics when studying turbines, boilers, and so on. You may have seen mass flow at an inlet or outlet expressed as either $\dot{m} = VA/v$ or $\dot{m} = \rho VA$ where V, A, v, and ρ are the speed, area, specific volume, and density, respectively. We will use the density form of the equation.

Hence the mass flow is:

$$\dot{m} = \rho VA$$

Applying mass conservation, from our study of thermodynamics,

$$\rho V_i A_i = \rho V_e A_e$$

(Note: $\rho_i = \rho_e = \rho$ by our first assumption.)
(Note: Even though we are already familiar with this equation from thermodynamics, we will derive it in Chapter 4.)

Solving for V_e,

$$V_e = V_i \frac{A_i}{A_e} = V_i \frac{\pi D_i^2/4}{\pi D_e^2/4} = V_i \left(\frac{D_i}{D_e}\right)^2$$

$$V_e = 2.7\frac{\text{m}}{\text{s}}\left(\frac{50}{30}\right)^2 = 7.5\frac{\text{m}}{\text{s}} \longleftarrow V_e$$

This problem:

- ✓ Was solved using the nine logical steps.
- ✓ Demonstrated use of a control volume and the mass conservation law.

Thermodynamic Process:

Thermodynamic Process in fluid flow is a steady state of flow into and out of a vessel with definite wall properties. The internal state of vessel contents is not the primary concern. The quantities of primary concern describe the state of the inflow and outflow materials, and on the side, the transfer of heat, work and kinetic and potential energies for the vessel. Flow processes are of interest in engineering.

Historical development of Fluid Mechanics

Some basic properties of fluids are

Density Mass per unit volume is called density $\rho = \frac{\Delta \mathrm{m}}{\Delta \mathrm{V}}$

$\Delta =$ rate of change And $\rho = F(x, y, z, t)$

Where (x,y,z) are the coordinates of a point and t is the temperature.

Specific Weight

It is defined as the weight per unit volume and is denoted by $\gamma = \rho g$

Specific Volume It is defined as volume per unit mass. Its formula is $\frac{V}{m} = \frac{1}{\rho}$

Pressure Force per unit area is called pressure. Its formula is $P = \lim_{\Delta \mathrm{A} \to 0} \frac{\Delta \mathrm{F}}{\Delta \mathrm{A}}$

Where F is normal force due to fluid in elementary area.

Viscosity It is the property of fluid by which it offers the resistance to sheer (the tangent force per unit area) acting on it i.e. the property of fluid which control the flow of fluid. It is the inertial friction between two layers of a fluid in relative motion. Viscosity of liquids decreases with temperature and viscosity of gases increases with temperature. It is also called **Fluid Friction**.

Bulk modulus and compressibility (bulk compressibility modulus or modulus of elasticity)

It is denoted as $dp \propto \frac{d\rho}{\rho}$ i.e. variation of its density. After arranging $dp = K \frac{d\rho}{\rho}$ where K is called Bulk Modulus. Pressure and density changes in liquids are related by the bulk compressibility modulus, or modulus of elasticity,

If the bulk modulus is independent of temperature, then density is only a function of pressure (the fluid is barotropic).

Buoyancy Force: When a body is immersed in a liquid, or floating on its surface, the net vertical force acting on it due to liquid pressure is called the Buoyancy Force, and is denoted by $F_B = \rho g V$

Methods of Description (Motion of Particles)

A fluid consists of innumerable (countless) whose relative position never fix whenever fluid is in motion the particle moves along certain line depending upon the characteristics of fluid and shape of the passage through which the fluid particle moves. It is necessary to observe the motion of fluid particle at various time and point. Fluid mechanics have two method of fluid motion

(i) Lagrange's Method (ii) Eulerian Method

Lagrange's Method (Flow of Single Particle)

This method deals with the study of flow patterns of the individual particles. In this method the path traced by the particle under consideration with the passage of time is studied in detail.

Eulerian Method This method deals with the study of flow patterns of all particles simultaneously at one section. In this method the path traced by all particles at one section and one time are studied in detail.

The general example for both methods is the study of movement of vehicles on a busy road. The Lagrangian deals with the study of the movement of only one vehicle through a specific distance. And The Eulerian deals with the study of the movement of all vehicles at one section and one time.

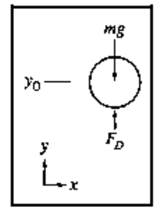
In study of fluid mechanics, Eulerian method is commonly used because of its mathematical simplicity. Moreover in fluid mechanics, movement of individuals is not important.

Rigid Body: Solid body in which deformation is zero. Rigid body is a system of particles whose distance from one another is fixed.

Rigid Body Motion: Solid body in which deformation is zero. Rigid body is a system of particles whose distance from one another is fixed. Motion of rigid body is studied under the influence of forces. The general motion of rigid body consists of a combination of translation and rotation. Its equation of motion can be derived from the equation of motion of its constituent particles.

Example Free Fall Of Ball In Air

The air resistance (drag force) on a 200 g ball in free flight is given by $F_D = 2 \times 10^{-4}\, V^2$, where F_D is in newtons and V is in meters per second. If the ball is dropped from rest 500 m above the ground, determine the speed at which it hits the ground. What percentage of the terminal speed is the result? (The *terminal speed* is the steady speed a falling body eventually attains.)



Given: Ball, $m = 0.2$ kg, released from rest at $y_0 = 500$ m.
Air resistance, $F_D = kV^2$, where $k = 2 \times 10^{-4}\ \mathrm{N \cdot s^2/m^2}$.
Units: $F_D(\mathrm{N})$, $V(\mathrm{m/s})$.

Find: (a) Speed at which the ball hits the ground.
(b) Ratio of speed to terminal speed.

Solution:

Governing equation: $\Sigma \vec{F} = m\vec{a}$

Assumption: Neglect buoyancy force.

The motion of the ball is governed by the equation

$$\Sigma F_y = ma_y = m\frac{dV}{dt}$$

Since $V = V(y)$, we write $\Sigma F_y = m\dfrac{dV}{dy}\dfrac{dy}{dt} = mV\dfrac{dV}{dy}$ Then,

$$\Sigma F_y = F_D - mg = kV^2 - mg = mV\frac{dV}{dy}$$

Separating variables and integrating,

$$\int_{y_0}^{y} dy = \int_0^V \frac{mV\,dV}{kV^2 - mg}$$

$$y - y_0 = \left[\frac{m}{2k}\ln(kV^2 - mg)\right]_0^V = \frac{m}{2k}\ln\frac{kV^2 - mg}{-mg}$$

Taking antilogarithms, we obtain

$$kV^2 - mg = -mg\ e^{[(2k/m)(y - y_0)]}$$

Solving for V gives

$$V = \left\{\frac{mg}{k}\left(1 - e^{[(2k/m)(y - y_0)]}\right)\right\}^{1/2}$$

Substituting numerical values with $y = 0$ yields

$$V = \left\{0.2\ \mathrm{kg} \times 9.81\frac{\mathrm{m}}{\mathrm{s}^2} \times \frac{\mathrm{m}^2}{2 \times 10^{-4}\mathrm{N \cdot s^2}} \times \frac{\mathrm{N \cdot s^2}}{\mathrm{kg \cdot m}}\left(1 - e^{[2 \times 2 \times 10^{-4}/0.2(-500)]}\right)\right\}$$

$$V = 78.7\ \mathrm{m/s} \qquad V$$

At terminal speed, $a_y = 0$ and $\Sigma F_y = 0 = kV_t^2 - mg$.

Then, $V_t = \left[\dfrac{mg}{k}\right]^{1/2} = \left[0.2\ \mathrm{kg} \times 9.81\ \dfrac{\mathrm{m}}{\mathrm{s}^2} \times \dfrac{\mathrm{m}^2}{2 \times 10^{-4}\mathrm{N \cdot s^2}} \times \dfrac{\mathrm{N \cdot s^2}}{\mathrm{kg \cdot m}}\right]^{1/2}$

$= 99.0\ \mathrm{m/s}$

The ratio of actual speed to terminal speed is

$$\frac{V}{V_t} = \frac{78.7}{99.0} = 0.795, \text{ or } 79.5\% \qquad \frac{V}{V_t}$$

This problem:
- ✓ Reviewed the methods used in particle mechanics.
- ✓ Introduced a variable aerodynamic drag force.

Try the *Excel* workbook for this Example for variations on this problem.

DIMENSIONS AND UNITS

Engineering problems are solved to answer specific questions. It goes without saying that the answer must include units (it makes a difference whether a pipe diameter required is one meter or one foot).

Dimension is specific form used to refer any measurable quantity.

If we refer physical quantities such as length, time, mass, and temperature as dimensions in terms of a particular system of dimensions, all measurable quantities are subdivided into two groups—**primary quantities and secondary quantities**.

- **Primary quantities** are those quantities for which we set arbitrary scales of measure. e.g. Mass(Kg), Length(Meter), Time(Second)
- **Secondary quantities** are those quantities whose dimensions are expressible in terms of the dimensions of the primary quantities. e.g. Velocity, Acceleration, Torque, Momentum etc.

Units

Units are the arbitrary names (and magnitudes) assigned to the primary dimensions adopted as standards for measurement. For example, the primary dimension of length may be measured in units of meters, feet, yards, or miles. These units of length are related to each other through unit conversion factors.

(1 mile = 5280 feet = 1609 meters).

Systems of Dimensions

Any valid equation that relates physical quantities must be dimensionally homogeneous; each term in the equation must have the same dimensions. We have three basic systems of dimensions, corresponding to the different ways of specifying the primary dimensions.

a. Mass [M], length [L], time [t], temperature [T].

b. Force [F], length [L], time [t], temperature [T].

c. Force [F], mass [M], length [L], time [t], temperature [T].

In system a, force [F] is a secondary dimension and the constant of proportionality in Newton's second law is dimensionless. In system b, mass [M] is a secondary dimension, and again the constant of proportionality in Newton's second law is dimensionless. In system c, both force [F] and mass [M] have been selected as primary dimensions. In this case the constant of proportionality, g_c, (not to be confused with g, the acceleration of gravity!) in Newton's second law (written $\vec{F} = m\vec{a}/g_c$) is not dimensionless. The dimensions of g_c must in fact be $[ML/Ft^2]$ for the equation to be dimensionally homogeneous. The numerical value of the constant of proportionality depends on the units of measure chosen for each of the primary quantities.

Note

We recognize that Newton's second law (F ∝ ma) relates the four dimensions, F, M, L, and t. Thus force and mass cannot both be selected as primary dimensions without introducing a constant of proportionality that has dimensions (and units). Length and time are primary dimensions in all dimensional systems in common use. In some systems, mass is taken as a primary dimension. In others, force is selected as a primary dimension; a third system chooses both force and mass as primary dimensions

Systems of Units

There is more than one way to select the unit of measure for each primary dimension.

Common Unit Systems

System of Dimensions	Unit System	Force F	Mass M	Length L	Time t	Temperature T
a. MLtT	Système International d'Unités (SI)	(N)	kg	m	s	K
b. FLtT	British Gravitational (BG)	lbf	(slug)	ft	s	°R
c. FMLtT	English Engineering (EE)	lbf	lbm	ft	s	°R

We shall present only the more common engineering systems of units for each of the basic systems of dimensions.

a . MLtT

SI, which is the official abbreviation in all languages for the Systeme International d'Unites, is an extension and refinement of the traditional metric system. More than 30 countries have declared it to be the only legally accepted system.

In the SI system of units, the unit of mass is the kilogram (kg), the unit of length is the meter (m), the unit of time is the second (s), and the unit of temperature is the kelvin (K). Force is a secondary dimension, and its unit, the newton (N), is defined from Newton's second law as

$1\ N = 1\ kg \bullet m/s^2$

In the Absolute Metric system of units, the unit of mass is the gram, the unit of length is the centimeter, the unit of time is the second, and the unit of temperature is the kelvin. Since force is a secondary dimension, the unit of force, the dyne, is defined in terms of Newton's second law as

$1\ dyne = 1\ g \bullet cm/s^2$

b. FLtT

In the British Gravitational system of units, the unit of force is the pound (lbf), the unit of length is the foot (ft), the unit of time is the second, and the unit of temperature is the degree Rankine (1K = 1.8 °R). Since mass is a secondary dimension, the unit of mass, the slug, is defined in terms of Newton's second law as $1\ slug = 1\ lbf \bullet s^2/ft$

c . FMLtT

In the English Engineering system of units, the unit of force is the pound force (lbf), the unit of mass is the pound mass (lbm), the unit of length is the foot, the unit of time is the second, and the unit of temperature is the degree Rankine. Since both force and mass are chosen as primary dimensions, Newton's second law is written as $\vec{F} = m\vec{a}/g_c$

A force of one pound (1 lbf) is the force that gives a pound mass (1 lbm) an acceleration equal to the standard acceleration of gravity on Earth, 32.2 ft/s^2. From Newton's second law we see that

$$1\ lbf = \frac{1\ lbm \times 32.2\ ft/s^2}{g_c} \qquad \text{or} \qquad g_c = \frac{32.2\ ft.lbm}{1\ lbf.s^2}$$

The constant of proportionality, g_c, has both dimensions and units. The dimensions arose because we selected both force and mass as primary dimensions; the units (and the numerical value) are a consequence of our choices for the standards of measurement

Since a force of 1 lbf accelerates 1 lbm at 32.2 ft/s^2, it would accelerate 32.2 lbm at 1 ft/s^2. A slug also is accelerated at 1 ft/s^2 by a force of 1 lbf. Therefore,

1 slug = 32.2 lbm

TABEE

Primary Dimensions	SI units	BG Units	Conversion factors
Mass [M]	Kilogram (kg)	slug	1 slug = 14.59 kg
Length [L]	meter (m)	foot (ft)	1 ft = 0.3048 m
Time [t]	second (sec)	second (sec)	1 s = 1 s
Temp [T]	Kelvine (K)	Rankine (°R)	1 K = 1.8 °R
Force [F]	Newton	lbf	lbf = 4.48 N

Conversion Factors and Definitions

Fundamental Dimension	English Unit	Exact SI Value	Approximate SI Value
Length	1 in.	0.0254 m	—
Mass	1 lbm	0.453 592 37 kg	0.454 kg
Temperature	1°F	5/9 K	—

Definitions:

Acceleration of gravity:	$g = 9.8066\ m/s^2$ (= $32.174\ ft/s^2$)
Energy:	Btu (British thermal unit) ≡ amount of energy required to raise the temperature of 1 lbm of water 1°F (1 Btu = 778.2 ft · lbf)
	kilocalorie ≡ amount of energy required to raise the temperature of 1 kg of water 1 K(1 kcal = 4187 J)
Length:	1 mile = 5280 ft; 1 nautical mile = 6076.1 ft = 1852 m (exact)
Power:	1 horsepower ≡ 550 ft · lbf/s
Pressure:	1 bar ≡ 10^5 Pa
Temperature:	degree Fahrenheit, $T_F = \frac{9}{5}T_C + 32$ (where T_C is degrees Celsius)
	degree Rankine, $T_R = T_F + 459.67$
	Kelvin, $T_K = T_C + 273.15$ (exact)
Viscosity:	1 Poise ≡ 0.1 kg/(m · s)
	1 Stoke ≡ 0.0001 m^2/s
Volume:	1 gal ≡ 231 $in.^3$ (1 ft^3 = 7.48 gal)

Useful Conversion Factors:

Length:	1 ft = 0.3048 m	Power:	1 hp = 745.7 W
	1 in. = 25.4 mm		1 ft · lbf/s = 1.356 W
Mass:	1 lbm = 0.4536 kg		1 Btu/hr = 0.2931 W
	1 slug = 14.59 kg	Area	1 ft^2 = 0.0929 m^2
Force:	1 lbf = 4.448 N		1 acre = 4047 m^2
	1 kgf = 9.807 N	Volume:	1 ft^3 = 0.02832 m^3
Velocity:	1 ft/s = 0.3048 m/s		1 gal (US) = 0.003785 m^3
	1 ft/s = 15/22 mph		1 gal (US) = 3.785 L
	1 mph = 0.447 m/s	Volume flow rate:	1 ft^3/s = 0.02832 m^3/s
Pressure:	1 psi = 6.895 kPa		1 gpm = $6.309 \times 10^{-5}\ m^3/s$
	1 lbf/ft^2 = 47.88 Pa	Viscosity (dynamic)	1 lbf · s/ft^2 = 47.88 N · s/m^2
	1 atm = 101.3 kPa		1 g/(cm · s) = 0.1 N · s/m^2
	1 atm = 14.7 psi		1 Poise = 0.1 N · s/m^2
	1 in. Hg = 3.386 kPa	Viscosity (kinematic)	1 ft^2/s = 0.0929 m^2/s
	1 mm Hg = 133.3 Pa		1 Stoke = 0.0001 m^2/s
Energy:	1 Btu = 1.055 kJ		
	1 ft · lbf = 1.356 J		
	1 cal = 4.187 J		

Example

The label on a jar of peanut butter states its net weight is 510 g. Express its mass and weight in SI, BG, and EE units.

Given: Peanut butter "weight," $m = 510$ g.

Find: Mass and weight in SI, BG, and EE units.

Solution: This problem involves unit conversions and use of the equation relating weight and mass:

$$W = mg$$

The given "weight" is actually the mass because it is expressed in units of mass:

$$m_{\mathrm{SI}} = 0.510\ \mathrm{kg} \xleftarrow{\qquad\qquad m_{\mathrm{SI}}}$$

Using the conversions of Table G.2 (Appendix G),

$$m_{\mathrm{EE}} = m_{\mathrm{SI}}\left(\frac{1\ \mathrm{lbm}}{0.454\ \mathrm{kg}}\right) = 0.510\ \mathrm{kg}\left(\frac{1\ \mathrm{lbm}}{0.454\ \mathrm{kg}}\right) = 1.12\ \mathrm{lbm} \xleftarrow{\qquad\qquad m_{\mathrm{EE}}}$$

Using the fact that 1 slug = 32.2 lbm,

$$m_{\mathrm{BG}} = m_{\mathrm{EE}}\left(\frac{1\ \mathrm{slug}}{32.2\ \mathrm{lbm}}\right) = 1.12\ \mathrm{lbm}\left(\frac{1\ \mathrm{slug}}{32.2\ \mathrm{lbm}}\right)$$

$$= 0.0349\ \mathrm{slug} \xleftarrow{\qquad\qquad m_{\mathrm{BG}}}$$

To find the weight, we use

$$W = mg$$

In SI units, and using the definition of a newton,

$$W_{\mathrm{SI}} = 0.510\ \mathrm{kg} \times 9.81\ \frac{\mathrm{m}}{\mathrm{s}^2} = 5.00\left(\frac{\mathrm{kg}\cdot\mathrm{m}}{\mathrm{s}^2}\right)\left(\frac{\mathrm{N}}{\mathrm{kg}\cdot\mathrm{m/s}^2}\right)$$

$$= 5.00\ \mathrm{N} \xleftarrow{\qquad\qquad W_{\mathrm{SI}}}$$

In BG units, and using the definition of a slug,

$$W_{\mathrm{BG}} = 0.0349\ \mathrm{slug} \times 32.2\ \frac{\mathrm{ft}}{\mathrm{s}^2} = 1.12\,\frac{\mathrm{slug}\cdot\mathrm{ft}}{\mathrm{s}^2}$$

$$= 1.12\left(\frac{\mathrm{slug}\cdot\mathrm{ft}}{\mathrm{s}^2}\right)\left(\frac{\mathrm{s}^2\cdot\mathrm{lbf/ft}}{\mathrm{slug}}\right) = 1.12\ \mathrm{lbf} \xleftarrow{\qquad\qquad W_{\mathrm{BG}}}$$

In EE units, we use the form $W = mg/g_c$, and using the definition of g_c,

$$W_{\mathrm{EE}} = 1.12\ \mathrm{lbm} \times 32.2\ \frac{\mathrm{ft}}{\mathrm{s}^2} \times \frac{1}{g_c} = \frac{36.1}{g_c}\ \frac{\mathrm{lbm}\cdot\mathrm{ft}}{\mathrm{s}^2}$$

$$= 36.1\left(\frac{\mathrm{lbm}\cdot\mathrm{ft}}{\mathrm{s}^2}\right)\left(\frac{\mathrm{lbf}\cdot\mathrm{s}^2}{32.2\ \mathrm{ft}\cdot\mathrm{lbm}}\right) = 1.12\ \mathrm{lbf} \xleftarrow{\qquad\qquad W_{\mathrm{EE}}}$$

Useful Conversions

1 meter = 3.280 ft , 1Slug =14.5939kg = 32.2 lbm , g = 32.2 ft/s^2

lbf = 4.68N , 1K = 1.8 °R , 1gallon = 3.78541 liter

Conversion in MLtT with Dimensions

Quantity	Formula	Unit (SI System)	Dimension
Power	$P=\frac{W}{t}=\frac{Fd}{t}=\frac{ma.d}{t}$	$P=\frac{kg\times ms^{-2}\times m}{s}=\frac{kgm^2}{s^3}$	$\left[\frac{ML^2}{t^3}\right]$
Pressure	$P=\frac{F}{A}=\frac{ma}{A}$	$P=\frac{kg\times ms^{-2}}{m^2}=\frac{kg}{ms^2}$	$\left[\frac{M}{Lt^2}\right]$
Modulus of Elasticity	$\frac{Stress}{Strain}=\frac{F/A}{\frac{Change\ inlayer}{origional\ layer}}=\frac{F}{A}$	$\frac{kgms^{-2}}{m^2}=\frac{kg}{ms^2}$	$\left[\frac{M}{Lt^2}\right]$
Momentum	$P=mv$	$P=\frac{kgm}{s}$	$\left[\frac{ML}{t}\right]$
K.E	$K.E=\frac{1}{2}mv^2$	$K.E=kg(ms^{-1})^2$ $K.E=\frac{kgm^2}{s^2}$	$\left[\frac{ML^2}{t^2}\right]$

Question: Convert the pressure 1Pa to Pounds force per square inches.

$P=\frac{F}{A}\Rightarrow Pa=\frac{N}{m^2}$ ………..(i)

Since $1lbf = 4.68N \Rightarrow 1N=\frac{1}{4.68}lbf$

Also $1\,meter = 3.280\,ft \Rightarrow 1m^2 = 10.7584ft^2$

And $1ft^2=(12in.)^2=144in.^2$

$\Rightarrow 1m^2 = 10.7584\times 144in.^2 = 1549.20in.^2$

$(i)\Rightarrow Pa=\frac{\frac{1}{4.68}lbf}{1549.20in.^2}\Rightarrow 1Pa = 1.44\times 10^{-4}lbf in.^{-2}$

Question: Convert viscosity of 1 $\frac{Ns}{m^2}$ to $\frac{lbf.s}{ft^2}$

Since $1lbf = 4.68N \Rightarrow 1N = \frac{1}{4.68}lbf$

Also $1\ meter = 3.280\ ft \Rightarrow 1m^2 = 10.7584ft^2$

Now $1\frac{Ns}{m^2} = \frac{\frac{1}{4.68}lbf \times s}{10.7584ft^2} \Rightarrow 1\frac{Ns}{m^2} = 0.02074\frac{lbf.s}{ft^2}$

Question: Convert Energy of 1 $\frac{kgm^2}{s^2}$ to $\frac{\text{Slug}.ft^2}{s^2}$

Since $1\text{Slug} = 14.5939\text{kg} \Rightarrow 1kg = \frac{1}{14.5939}\text{Slug}$

Also $1\ meter = 3.280\ ft \Rightarrow 1m^2 = 10.7584ft^2$

Now $\frac{kgm^2}{s^2} = \frac{\frac{1}{14.5939}\text{Slug} \times 10.7584ft^2}{s^2} \Rightarrow 1\frac{kgm^2}{s^2} = 0.7372\frac{\text{Slug}ft^2}{s^2}$

Question: Convert Energy of 1 $\frac{ft^3}{min}$ to $\frac{m^3}{hour}$

Since $1\ meter = 3.280\ ft \Rightarrow 1ft = \frac{1}{3.280}m \Rightarrow 1ft^3 = 0.02833m^3$

Also $1\ h = 60min \Rightarrow 1min = \frac{1}{60}h = 0.0166h$

Now $\frac{ft^3}{min} = \frac{0.02833m^3}{0.0166h} \Rightarrow \frac{ft^3}{min} = 1.6998\frac{m^3}{h}$

Question: Convert Energy of 1 $\frac{gallon}{hour}$ to $\frac{liter}{sec}$

Since 1gallon = 3.78541 liter

Also $1\ h = 3600s$

Now $\frac{gallon}{h} = \frac{3.78541\ \text{liter}}{3600s} \Rightarrow \frac{gallon}{h} = 1.0515 \times 10^{-3}\frac{liter}{sec}$

Question: Convert Stress to lbf

Since $1lbf = 4.68N \Rightarrow 1N = \frac{1}{4.68}lbf$

Also 1 $meter = 3.280\ ft \Rightarrow 1m^2 = 10.7584ft^2$

$1\ meter = 3.280\ ft$ And $1ft^2 = (12in.)^2 = 144in.^2$

$\Rightarrow 1m^2 = 10.7584 \times 144in.^2 = 1549.20in.^2$

$Stress = \frac{N}{m^2} = \frac{\frac{1}{4.68}lbf}{1549.20in.^2} = 1.44 \times 10^{-4} lbf in.^{-2}$

Question: Power into $lb.min.$

Since $P = \frac{Nm}{s} = \frac{lbm.in.}{sec}$ And $P = \frac{W}{t} = \frac{Fd}{t} = \frac{ma.d}{t} = \frac{kg \times ms^{-2} \times m}{s} = \frac{kgm^2}{s^3}$

$1\text{Slug} = 14.5939\text{kg} \Rightarrow 1kg = \frac{1}{14.5939}\text{Slug} = \frac{1}{14.5939} \times 32.2lbm = 2.2064lbm$

Also 1 $meter = 3.280\ ft \Rightarrow 1m^2 = 10.7584ft^2$

And $1ft^2 = (12in.)^2 = 144in.^2 \Rightarrow 1m^2 = 10.7584 \times 144in.^2 = 1549.20in.^2$

Now $P = \frac{kgm^2}{s^3} = \frac{2.2064lbm \times 1549.20in.^2}{s^3} = 3418.15\frac{lbmin.^2}{s^3}$

Question: Modulus of Elasticity into $Slug\ ft$

Since Modulus of Elasticity = $\frac{Stress}{Strain} = \frac{F/A}{\frac{Change\ inlayer}{origional\ layer}} = \frac{F}{A} = \frac{kgms^{-2}}{m^2} = \frac{kg}{ms^2}$

$1\text{Slug} = 14.5939\text{kg} \Rightarrow 1kg = \frac{1}{14.5939}\text{Slug} = 0.06852\text{Slug}$

$1\ meter = 3.280\ ft$ then

Modulus of Elasticity $= \frac{kg}{ms^2} = \frac{0.06852\text{Slug}}{3.280\ ft \times s^2} = 0.0280\frac{\text{Slug}}{fts^2}$

Question: Convert 300kWh in BG.

Since $Watt = P = \frac{W}{t} = \frac{Fd}{t} = 1ft.\frac{lbf}{s}$ and $1hp = 746Watt$

$1hp = 550ft.\frac{lbf}{s}$

Then $300kWh = 300 \times 1000Wh = 300000Wh = \frac{300000hp}{746}h = 402hph$

Question: Convert 40 m²hr in BG.

Since 1 $meter\ =\ 3.280\ ft \Rightarrow 1m^2 = 10.7584ft^2$ Also 1 $h\ = 3600s$

Then $40m^2h = 40 \times 10.7584ft^2 \times 3600s = 1549209.6ft^2Sec$

Question: Convert 50Ns/ m² in BG.

Since 1 $meter\ =\ 3.280\ ft \Rightarrow 1m^2 = 10.7584ft^2$

Also $1lbf\ =\ 4.68N \Rightarrow 1N = \frac{1}{4.68}lbf = 0.223214lbf$

Then $50\frac{Ns}{m^2} = \frac{50 \times 0.223214lbf \times s}{10.7584ft^2} = 1.0373\frac{lbfs}{ft^2}$

Drag Force and Factors upon which it depends

The retarding force experienced by an object moving through a fluid is called a drag force. It increases with the increase of speed of the object.

As the drag force ‘F’ on a sphere of radius ‘r’ moving slowly with speed ‘v’ through a fluid of viscosity is given by Stoke’s Law as under $F = 6\pi\eta rv$.

According to the formula the drag force depends upon the following factors;

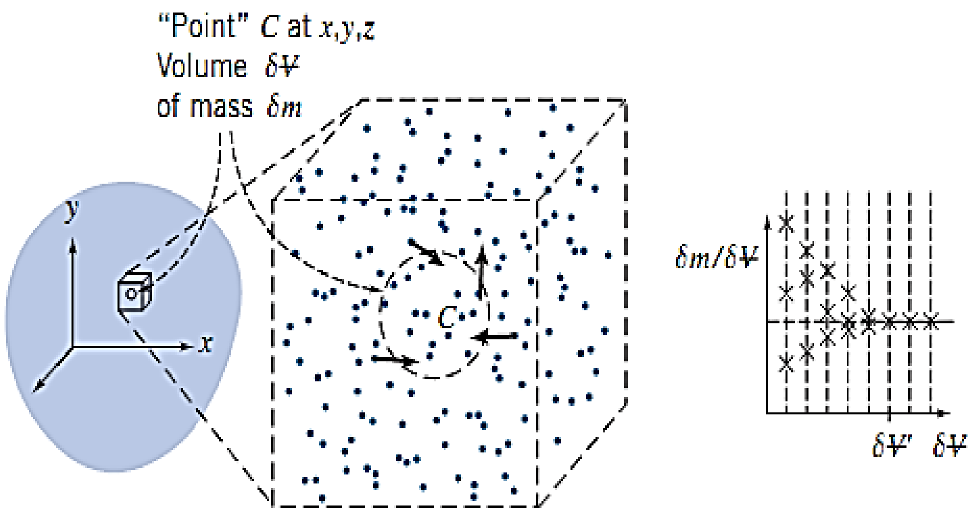
- Speed of the sphere
- Coefficient of viscosity
- Radius of sphere

Stoke’s Law: It states that the drag force ‘F’ on a sphere of radius ‘r’ moving slowly with speed ‘v’ through a fluid of viscosity is given by $F = 6\pi\eta rv$. Remember, at high speed the force is no longer proportional to speed. But $F \propto v^2$, therefore the stoke’s Law is not applicable at high speed.

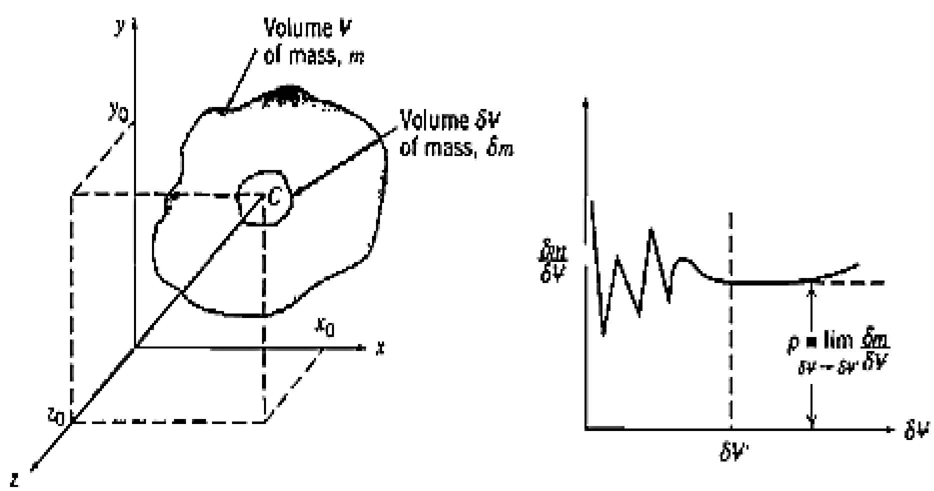
FUNDAMENTAL CONCEPTS

In Chapter 1 we discussed in general terms what fluid mechanics is about, and described some of the approaches we will use in analyzing fluid mechanics problems. In this chapter we will be more specific in defining some important properties of fluids, and ways in which flows can be described and characterized.

Fluid As A Continuum



(a) (b)



(a) (b)

Continuum is smoothly varying and continuously distributed body of matter – no holes or discontinuities. The concept of a continuum is the basis of classical fluid mechanics. The continuum assumption is valid in treating the behavior of fluids under normal conditions. It only breaks down when the mean free path of the molecules becomes the same order of magnitude as the smallest significant characteristic dimension of the problem. This occurs in such specialized problems as rarefied gas flow (e.g., as encountered in flights into the upper reaches of the atmosphere).

As a consequence of the continuum assumption, each fluid property is assumed to have a definite value at every point in space. Thus fluid properties such as density, temperature, velocity, and so on, are considered to be continuous functions of position and time.

To illustrate the concept of a property at a point, consider how we determine the density at a point. A region of fluid is shown in Figure. We are interested in determining the density at the point C, whose coordinates are x_0, y_0, and z_0. **Density** is defined as mass per unit volume. Thus the average density in volume $\bar{V}$ is given by $\rho = m/\bar{V}$.

In general, because the density of the fluid may not be uniform, this will not be equal to the value of the density at point C. To determine the density at point C, we must select a small volume, $\delta\bar{V}$, surrounding point C and then determine the ratio $\frac{\delta m}{\delta \bar{V}}$.

The question is, how small can we make the volume $\delta\bar{V}$? We can answer this question by plotting the ratio $\frac{\delta m}{\delta \bar{V}}$, and allowing the volume to shrink continuously in size.

Assuming that volume $\delta\bar{V}$ is initially relatively large (but still small compared with the volume, V) a typical plot of $\frac{\delta m}{\delta \bar{V}}$ might appear as in Figure (b) In other words, $\delta\bar{V}$ must be sufficiently large to yield a meaningful, reproducible value for the density at a location and yet small enough to be called a point. The average density tends to approach an asymptotic value as the volume is shrunk to enclose only homogeneous fluid in the immediate neighborhood of point C. If $\delta\bar{V}$ becomes so small that it contains only a small number of molecules, it becomes impossible to fix a definite value for $\frac{\delta m}{\delta \bar{V}}$; the value will vary erratically as molecules cross into and out of the volume. Thus there is a lower limiting value of $\delta \bar{V}$, designated $\delta \bar{V}'$ in Figure (b), allowable for use in defining fluid density at a point.

The density at a "point" is then defined as $\rho = \lim_{\delta \bar{V} \to \delta \bar{V}'} \frac{\delta m}{\delta \bar{V}}$

Since point C was arbitrary, the density at any other point in the fluid could be determined in the same manner. If density was measured simultaneously at an infinite number of points in the fluid, we would obtain an expression for the density distribution as a function of the space coordinates, $\rho = \rho$ (x, y, z), at the given instant.

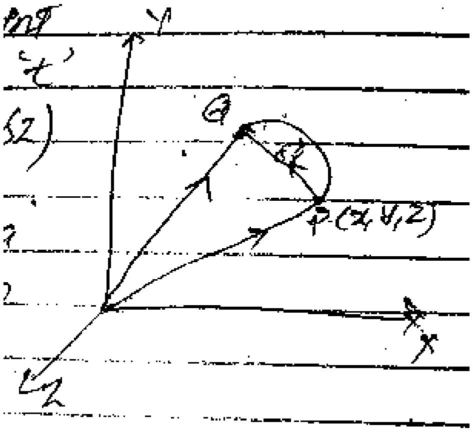
The density at a point may also vary with time (as a result of work done on or by the fluid and/or heat transfer to the fluid). Thus the complete representation of density (the field representation) is given by $\rho = \rho$ (x,y,z,t)

Since density is a scalar quantity, requiring only the specification of a magnitude for a complete description, the field represented by $\rho = \rho$ (x,y,z,t) is a scalar field.

The density of a liquid or solid may also be expressed in dimensionless form as the **specific gravity, SG**, defined as the ratio of material density to the maximum density of water, which is 1000 kg/m^3 at 4°C (1.94 *slug/ft*^{3} *at* 39°F). For example, the SG of mercury is typically 13.6—mercury is 13.6 times as dense as water. The specific gravity of liquids is a function of temperature; for most liquids specific gravity decreases with increasing temperature.

Specific weight, , is defined as weight per unit volume; weight is mass times acceleration of gravity, and density is mass per unit volume, hence $\gamma = \rho g$. For example, the specific weight of water is approximately 9.81 kN/m^3(62.4 *lbf/ft*^{3}).

Local and Partial Rate of Change



Let us suppose that particle of fluid move from point $P(x, y, z)$ at time ‘t’ to point $Q(x+\delta x, y+\delta y, z+\delta z)$ in a time $t+\delta t$. Let $f(x, y, z, t)$ be a scalar function define in a region of a motion of fluid. The motion of particle from P to Q is given by $\quad \delta f=\frac{\partial f}{\partial x}\delta x+\frac{\partial f}{\partial y}\delta y+\frac{\partial f}{\partial z}\delta z+\frac{\partial f}{\partial t}\delta t$

$$\Rightarrow \frac{\delta f}{\delta t}=\frac{\partial f}{\partial x}\frac{\delta x}{\delta t}+\frac{\partial f}{\partial y}\frac{\delta y}{\delta t}+\frac{\partial f}{\partial z}\frac{\delta z}{\delta t}+\frac{\partial f}{\partial t}$$

$$\Rightarrow \lim_{\delta t\to 0}\frac{\delta f}{\delta t}=\frac{\partial f}{\partial x}.\lim_{\delta t\to 0}\frac{\delta x}{\delta t}+\frac{\partial f}{\partial y}.\lim_{\delta t\to 0}\frac{\delta y}{\delta t}+\frac{\partial f}{\partial z}.\lim_{\delta t\to 0}\frac{\delta z}{\delta t}+\frac{\partial f}{\partial t}$$

$$\Rightarrow \frac{df}{dt}=\frac{\partial f}{\partial x}\frac{dx}{dt}+\frac{\partial f}{\partial y}\frac{dy}{dt}+\frac{\partial f}{\partial z}\frac{dz}{dt}+\frac{\partial f}{\partial t}$$

$$\Rightarrow \frac{df}{dt}=\left(\frac{\partial f}{\partial x}\hat{\imath}+\frac{\partial f}{\partial y}\hat{\jmath}+\frac{\partial f}{\partial z}\hat{k}\right).\left(\frac{dx}{dt}\hat{\imath}+\frac{dy}{dt}\hat{\jmath}+\frac{dz}{dt}\hat{k}\right)+\frac{\partial f}{\partial t}$$

$$\Rightarrow \frac{df}{dt}=\nabla f.\frac{d\vec{r}}{dt}+\frac{\partial f}{\partial t}=\nabla f.\vec{V}+\frac{\partial f}{\partial t}=\frac{\partial f}{\partial t}+\vec{V}.\nabla f=\left(\frac{\partial}{\partial t}+\vec{V}.\nabla\right)f$$

$$\Rightarrow \frac{d}{dt}=\frac{\partial}{\partial t}+\vec{V}.\nabla$$

Where $\frac{d}{dt}$ is total or material rate of change, $\frac{\partial}{\partial t}$ is local rate of change and $\vec{V}.\nabla$ is partial or convective rate of change. Above result shows that action of the operator $\frac{d}{dt}$ on the function is same as the action of operator $\frac{\partial}{\partial t}+\vec{V}.\nabla$ on the function.

Example A velocity field $\vec{V} = u\hat{\imath} + v\hat{\jmath} + w\hat{k}$ is given as $u = x + 2y + 3z + t^2$, $v = xyz + t$ and $w = (x + y)z^2 + 2t$ then find

Local acceleration, Convective acceleration and Total acceleration at $P(1,1,1,2)$

Solution

Since $\frac{d}{dt} = \frac{\partial}{\partial t} + \vec{V}.\nabla \quad \Rightarrow \vec{a} = \frac{d\vec{V}}{dt} = \frac{\partial \vec{V}}{\partial t} + \vec{V}.\nabla\vec{V}$

$$\Rightarrow (\vec{a})_{local} = \frac{\partial \vec{V}}{\partial t} = \frac{\partial}{\partial t}\left(u\hat{\imath} + v\hat{\jmath} + w\hat{k}\right) = 2t\hat{\imath} + \hat{\jmath} + 2\hat{k}$$

$$\Rightarrow (\vec{a})_{local}\,|_{P(1,1,1,2)} = 2t\hat{\imath} + \hat{\jmath} + 2\hat{k} = 2(2)\hat{\imath} + \hat{\jmath} + 2\hat{k} = 4\hat{\imath} + \hat{\jmath} + 2\hat{k}$$

Now $(\vec{a})_{convective} = \vec{V}.\nabla\vec{V} = \left(u\hat{\imath} + v\hat{\jmath} + w\hat{k}\right).\left(\frac{d\vec{V}}{dx}\hat{\imath} + \frac{d\vec{V}}{dy}\hat{\jmath} + \frac{d\vec{V}}{dz}\hat{k}\right)$

$$\Rightarrow (\vec{a})_{convective} = u\frac{d\vec{V}}{dx} + v\frac{d\vec{V}}{dy} + w\frac{d\vec{V}}{dz}$$

Then in components form $\Rightarrow (\vec{a}_x)_{convective} = u\frac{du}{dx} + v\frac{du}{dy} + w\frac{du}{dz}$

$$\Rightarrow \left(\vec{a}_y\right)_{convective} = u\frac{dv}{dx} + v\frac{dv}{dy} + w\frac{dv}{dz}$$

$$\Rightarrow (\vec{a}_z)_{convective} = u\frac{dw}{dx} + v\frac{dw}{dy} + w\frac{dw}{dz}$$

Using $\frac{du}{dx} = 1, \frac{du}{dy} = 2, \frac{du}{dz} = 3$

$$\Rightarrow (\vec{a}_x)_{convec} = (x + 2y + 3z + t^2)(1) + (xyz + t)(2) + (w = (x + y)z^2 + 2t)(3)$$

$$\Rightarrow (\vec{a}_x)_{convec}\,|_{P(1,1,1,2)} = 3$$

Using $\frac{dv}{dx} = yz, \frac{dv}{dy} = xz, \frac{dv}{dz} = xy$

$$\Rightarrow \left(\vec{a}_y\right)_{convec} = (x + 2y + 3z + t^2)(yz) + (xyz + t)(xz) + (w = (x + y)z^2 + 2t)(xy)$$

$$\Rightarrow \left(\vec{a}_y\right)_{convec\,|_{P(1,1,1,2)}} = 19$$

Using $\frac{dw}{dx} = z^2, \frac{dw}{dy} = z^2, \frac{dw}{dz} = 2z(x+y)$

$\Rightarrow (\vec{a}_z)_{convec} = (x + 2y + 3z + t^2)(z^2) + (xyz + t)(z^2) + (w = (x+y)z^2 + 2t)\big(2z(x+y)\big)$

$\Rightarrow (\vec{a}_z)_{convec}\,|_{P(1,1,1,2)} = 37$

Then we can get $\qquad (\vec{a})_{convective} = \vec{a}_x\hat{\imath} + \vec{a}_y\hat{\jmath} + \vec{a}_z\hat{k}$

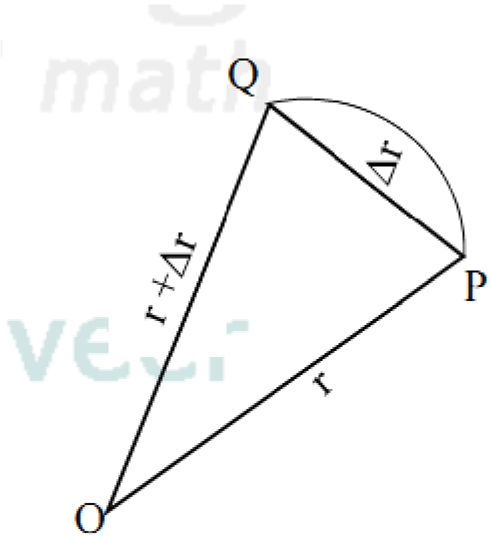
$\Rightarrow (\vec{a})_{convective} = 34\hat{\imath} + 19\hat{\jmath} + 37\hat{k}$

Now

$(\vec{a})_{total} = 4\hat{\imath} + \hat{\jmath} + 2\hat{k} + 34\hat{\imath} + 19\hat{\jmath} + 37\hat{k}$

$(\vec{a})_{total} = 38\hat{\imath} + 20\hat{\jmath} + 39\hat{k}$

Velocity of fluid particle or Velocity of fluid at a point.



$PQ = \Delta r$

And therefore, velocity of a fluid particle denoted as '$\vec{V}$'.

$\vec{V} = \lim_{\Delta t \to 0} \frac{\Delta r}{\Delta t} = \frac{d\vec{r}}{dt}$ ……………(i)

In Cartesian coordinates

$\vec{V} = u\hat{\imath} + v\hat{\jmath} + w\hat{k}$ and $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$

$(i) \Rightarrow \vec{q} = \frac{d}{dt}\left(x\hat{\imath} + y\hat{\jmath} + z\hat{k}\right)$

$\Rightarrow u\hat{\imath} + v\hat{\jmath} + w\hat{k} = \frac{dx}{dt}\hat{\imath} + \frac{dy}{dt}\hat{\jmath} + \frac{dz}{dt}\hat{k}$

On comparing $u = \frac{dx}{dt}, \; v = \frac{dy}{dt}\;, \; w = \frac{dz}{dt}$

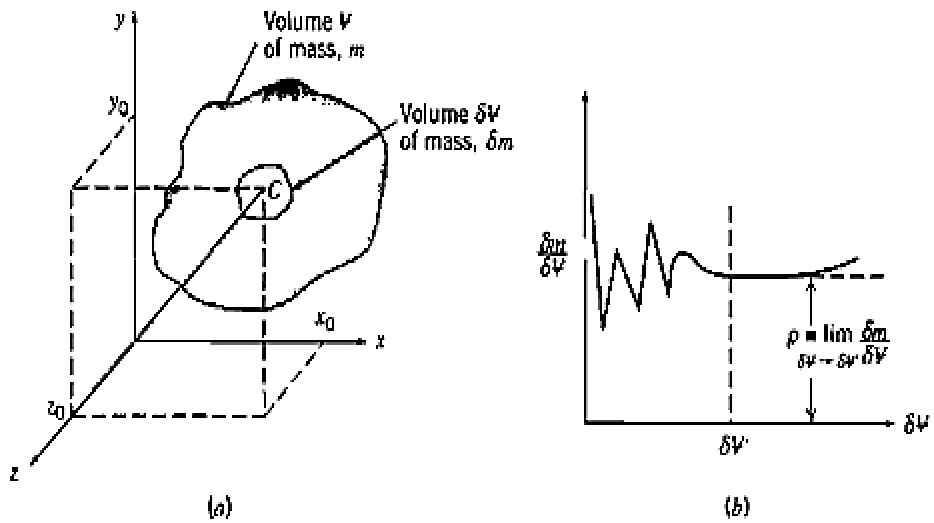
The expression $\vec{V} = \lim_{\Delta t \to 0} \frac{\Delta r}{\Delta t} = \frac{d\vec{r}}{dt}$ gives the velocity at a point P and in time 't' and also shows that velocity depends upon $\vec{r}$ and 't' i.e.

$\vec{V} = \vec{V}(x, y, z, t) = \vec{V}(\vec{r}, t)$

Velocity Field

In continuum mechanics the flow velocity in fluid dynamics, is a vector field used to mathematically describe the motion of a continuum. The length of the flow velocity vector is the **flow speed**, and is a scalar. It is also called velocity field.

In dealing with fluids in motion, we shall be concerned with the description of a velocity field. Consider the following Figure.



Velocity field implies a distribution of velocity in a given region. At a given instant the velocity field, $\vec{V}$, is a function of the space coordinates x, y, z. The velocity at any point in the flow field might vary from one instant to another. Thus the complete representation of velocity (the velocity field) is given by spatial and time coordinates as $\vec{V} = V(x, y, z, t)$

Velocity is a vector quantity, requiring a magnitude and direction for a complete description, so the velocity field $\vec{V} = V(x, y, z, t)$ is a vector field. The velocity vector $\vec{V}$, , also can be written in terms of its three scalar components. Denoting the components in the x, y, and z directions by u, v, and w, then

$\vec{V} = u\hat{\imath} + v\hat{\jmath} + w\hat{k}$ …………(i)

In general, each component , u, v, and w, will be a function of x, y, z, and t.

Since $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$

$(i) \Rightarrow \vec{V} = \frac{d\vec{r}}{dt} = \frac{d}{dt}\left(x\hat{\imath} + y\hat{\jmath} + z\hat{k}\right)$

$\Rightarrow u\hat{\imath} + v\hat{\jmath} + w\hat{k} = \frac{dx}{dt}\hat{\imath} + \frac{dy}{dt}\hat{\jmath} + \frac{dz}{dt}\hat{k}$

On comparing $u = \frac{dx}{dt},\ v = \frac{dy}{dt}\ ,\ w = \frac{dz}{dt}$

If properties at every point in a flow field do not change with time, the flow is termed steady. Stated mathematically, the definition of steady flow is $\frac{\partial \eta}{\partial t} = 0$ where η represents any fluid property. Hence, for steady flow,

$\frac{\partial \rho}{\partial t} = 0$ or $\rho = \rho$ (x, y, z) and $\frac{\partial \vec{V}}{\partial t} = 0$ or $\vec{V} = V(x, y, z)$

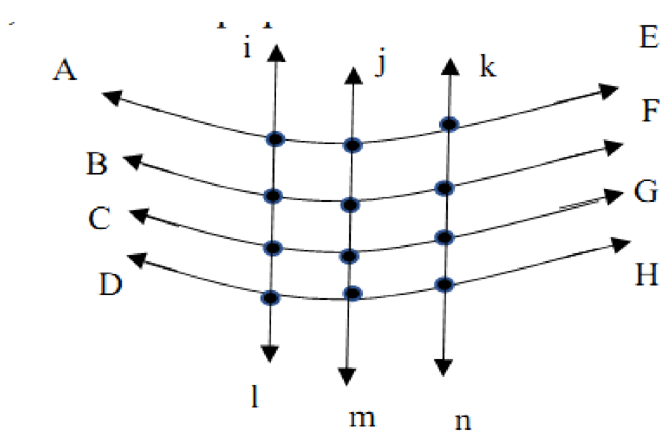
In steady flow, any property may vary from point to point in the field, but all properties remain constant with time at every point.

Flow Field: The term uniform flow field (as opposed to uniform flow at a cross section) is used to describe a flow in which the velocity is constant, i.e., independent of all space coordinates, throughout the entire flow field.

Remark: Sometimes we want a visual representation of a flow. Such a representation is provided by timelines, pathlines, streaklines, and streamlines.

- **Timeline:** <u>If a number of adjacent fluid particles in a flow field are marked at a given instant, they form a line in the fluid at that instant; this line is called a **timeline**</u>. Subsequent observations of the line may provide information about the flow field. For example, in discussing the behavior of a fluid under the action of a constant shear force (Section 1-2) timelines were introduced to demonstrate the deformation of a fluid at successive instants.
- **Pathline:** <u>A pathline is the path or trajectory traced out by a moving fluid particle.</u> To make a pathline visible, we might identify a fluid particle at a given instant, e.g., by the use of dye or smoke, and then take a long exposure photograph of its subsequent motion. The line traced out by the particle is a pathline. This approach might be used to study, for example, the trajectory of a contaminant leaving a smokestack.

- **Streakline:** On the other hand, we might choose to focus our attention on a fixed location in space and identify, again by the use of dye or smoke, all fluid particles passing through this point. After a short period of time we would have a number of identifiable fluid particles in the flow, all of which had, at some time, passed through one fixed location in space. The line joining these fluid particles is defined as a streakline.
- **Streamlines:** The imaginary line drawn in the fluid where the velocity along the tangent. Streamlines are lines drawn in the flow field so that at a given instant they are tangent to the direction of flow at every point in the flow field. Since the streamlines are tangent to the velocity vector at every point in the flow field, there can be no flow across a streamline. Streamlines are the most commonly used visualization technique. For example, they are used to study flow over an automobile in a computer simulation.
- **Stream Tube:** An element of the fluid bounded by the number of a streamlines which confine the flow, is called Stream Tube.
- **Filament lines:** The instantaneous pictures of the position of all particles which have passed through a given point at previous time are called filament line. For example, the line finds by smoke particle exerted from a nozzle of rocket.
- **Potential and Equipotential lines:** We know that there is always a loss of head of fluid particles as we proceed along the flow line. If we draw the line joining the points of equipotential on the adjacent flow lines, we get the potential lines. The points where lines A,B,C,D,E,F,G,H are the potential line and i,j,k,l,m,n are the equipotential line.



- **Flow nets:** The intersection of potential line and stream line of two set of lines are called flow line i.e. intersection with the help of flow nets we can analysis of the behavior of certain phenomenon which cannot be mathematical means. Such a phenomenon is generally analyzed and studied with the joint flow nets.

Difference between stream line and path line

i) The stream line is not in general same as the path line.

ii) Stream line show how each particle is moving at given instant of time whereas path line represents the motion of fluid particle at each instant.

iii) If the flow is steady the stream lines remain unchanged as the time progressed and hence they are also the path line.

iv) For a steady flow stream lines, path lines and streak lines are coincide in a flow field, whereas in general they are quite distinct.

Remark

- **Solid:** Solid has definite shape, which is retain until an external force is applied to after it. In other words, a solid is a substance that deforms when sheer stress is applied, but it does not continue to deform.
- **Liquid:** Liquid takes the shape of a vessel into which it is poured. It is considered to have incompressible flow.
- **Gas:** Gas completely filled up the vessel into which it contains.

Note

- We can determine the state of moving fluid with the help of five quantities. i.e. three components of velocity $\vec{V}(x, y, z)$, Pressure and $\rho(x, y, z)$.
- **Equation of Streamline in Space:** $\frac{dx}{du} = \frac{dy}{dv} = \frac{dz}{dw}$
- **Equation of Streamline in Plane:** $\frac{dx}{du} = \frac{dy}{dv}$

Equation of stream line or stream flow:

As we know that stream line is a curve drawn in the fluid so that tangent at each point is in the direction of motion.

i.e. Fluid velocity at a point.

Let P(r) where r = $x\hat{i}+y\hat{j}+z\hat{k}$ so that position vector of the point P on a stream line and let q = $u\hat{i}+v\hat{j}+w\hat{k}$ be the fluid velocity and point P then q // dr.

Therefore, equation of stream line is

$$q \times dr = 0$$

$$\begin{vmatrix} i & j & k \\ u & v & k \\ dx & dy & dz \end{vmatrix} = 0$$

$$(vdz - wdy)\hat{i} + (wdx - udz)\hat{j} + (udy - vdx)\hat{k} = 0\hat{i} + 0\hat{j} + 0\hat{k}$$

On comparing

$$vdz - wdy = 0 \quad \Rightarrow \quad vdz = wdy \quad \Rightarrow \frac{dz}{w} = \frac{dy}{v} \qquad \_\_\_\_(i)$$

$$wdx - udz = 0 \quad \Rightarrow \quad wdx = udz \quad \Rightarrow \frac{dx}{u} = \frac{dz}{w} \qquad \_\_\_\_(i)$$

$$udy - vdx = 0 \quad \Rightarrow \quad udy = vdx \quad \Rightarrow \frac{dy}{v} = \frac{dx}{u} \qquad \_\_\_\_(i)$$

From (i),(ii) & (iii) we have

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$$ are the equation of stream line.

Example:

Find the equation of stream line $q = (x^2 - y)\hat{i} + (x^2 + yz)\hat{j} + yz^2\hat{k}$

Solution: As we know that equation of stream line

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$$

$$q = (x^2 - y)\hat{i} + (x^2 + yz)\hat{j} + yz^2\hat{k}$$

And $q = u\hat{\imath} + v\hat{\jmath} + w\hat{k}$

On comparing

$$u = x^2 - y \ , \quad v = x^2 + yz \ , \quad w = yz^2$$

$$\frac{dx}{x^2 - y} = \frac{dy}{x^2 + yz} = \frac{dz}{yz^2}$$

Equation of path line:

Path line is a curve are trajectory along which a particle travel during its motion is called path line.

Differential equation of path line is $\frac{dr}{dt} = q$

Where $q = u\hat{\imath} + v\hat{\jmath} + w\hat{k}$ and $r = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$

$$\frac{d\left(x\hat{i} + y\hat{j} + z\hat{k}\right)}{dt} = u\hat{i} + v\hat{j} + w\hat{k}$$

$$\frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k} = u\hat{i} + v\hat{j} + w\hat{k}$$

$$\frac{dx}{dt} = u \ , \quad \frac{dy}{dt} = v \ , \quad \frac{dz}{dt} = w$$

is the required equation of path line.

Conservative Force

A force is said to be conservative if $\vec{\nabla} \times \vec{F} = 0$

Then $\vec{F} = -\vec{\nabla}\varphi$ where φ is scalar function.

In case of velocity field $\vec{\nabla} \times \vec{V} = 0$

We define scalar function (Potential Function) φ such that

$\vec{V} = -\vec{\nabla}\varphi$ then $\vec{\nabla} \times \vec{V} = \vec{\nabla} \times \left(-\vec{\nabla}\varphi\right) = 0$

Now as $\vec{V} = -\vec{\nabla}\varphi$ then this implies $u\hat{\imath} + v\hat{\jmath} + w\hat{k} = -\frac{\partial\varphi}{\partial x}\hat{\imath} - \frac{\partial\varphi}{\partial y}\hat{\jmath} - \frac{\partial\varphi}{\partial z}\hat{k}$

Comparing coefficient $u = -\frac{\partial\varphi}{\partial x}, v = -\frac{\partial\varphi}{\partial y}, w = -\frac{\partial\varphi}{\partial z}$

With potential function defined in this way the Irrotational condition

$\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0$ Is satisfied identically.

In cylindrical coordinates $\vec{\nabla} = \frac{\partial}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial}{\partial\theta}\hat{e}_\theta + \frac{\partial}{\partial z}\hat{e}_z$ then $\vec{V} = -\vec{\nabla}\varphi$ implies

$V_r\hat{e}_r + V_\theta\hat{e}_\theta + V_z\hat{e}_z = -\frac{\partial\varphi}{\partial r}\hat{e}_r - \frac{1}{r}\frac{\partial\varphi}{\partial\theta}\hat{e}_\theta - \frac{\partial\varphi}{\partial z}\hat{e}_z$

Comparing coefficient $V_r = -\frac{\partial\varphi}{\partial r}, V_\theta = -\frac{1}{r}\frac{\partial\varphi}{\partial\theta}, V_z = -\frac{\partial\varphi}{\partial z}$

With potential function defined in this way the Irrotational condition

$\frac{1}{r}\frac{\partial v_z}{\partial\theta} - \frac{\partial v_\theta}{\partial z} = \frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r} = \frac{1}{r}\frac{\partial(rv_\theta)}{\partial r} - \frac{1}{r}\frac{\partial v_r}{\partial\theta}$ Is satisfied identically.

Where negative sign in $\vec{V} = -\vec{\nabla}\varphi$ shows that flow takes place from higher potential to lower potential.

Example If $u = x$ and $v = -y$ then investigate the type of flow.

Solution Since we know that, for $\vec{V} = u\hat{\imath} + v\hat{\jmath} + w\hat{k}$

$$\nabla \times \vec{V} = \begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{bmatrix} = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\hat{\imath} - \left(\frac{\partial w}{\partial x} - \frac{\partial u}{\partial z}\right)\hat{\jmath} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\hat{k}$$

$$\nabla \times \vec{V} = (0 - 0)\hat{\imath} - (0 - 0)\hat{\jmath} + (0 - 0)\hat{k} = \vec{0}$$

Hence flow is Irrotational.

Example

If $u = \frac{-2xy}{(x^2+y^2)^2}, v = \frac{x^2-y^2}{(x^2+y^2)^2}$ and $w = \frac{y}{(x^2+y^2)^2}$ then investigate the type of flow.

Solution Since we know that, for $\vec{V} = u\hat{\imath} + v\hat{\jmath} + w\hat{k}$

$$\nabla \times \vec{V} = \begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{bmatrix} = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\hat{\imath} - \left(\frac{\partial w}{\partial x} - \frac{\partial u}{\partial z}\right)\hat{\jmath} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\hat{k}$$

$$\nabla \times \vec{V} = \left(\frac{x^2-3y^2}{(x^2+y^2)^3}\right)\hat{\imath} + \left(\frac{4xy}{(x^2+y^2)^3}\right)\hat{\jmath} + \left(\frac{6xy^2-2x^3}{(x^2+y^2)^3} + \frac{2y(x^2+y^2)+4y^2(x^2-y^2)}{(x^2+y^2)^3}\right)\hat{k} \neq \vec{0}$$

Hence flow is rotational.

Example If $\vec{V} = \frac{k^2(x\hat{\jmath}-y\hat{\imath})}{x^2+y^2}$ then determine equation of streamline.

Solution Since we know that, for $\vec{V} = u\hat{\imath} + v\hat{\jmath} + w\hat{k}$

Here $u = \frac{-k^2y}{x^2+y^2}, v = \frac{k^2x}{x^2+y^2}$ and $w = 0$

Then using equation of streamlines $\frac{dx}{du} = \frac{dy}{dv} = \frac{dz}{dw} = K$ (say)

We get $x^2 + y^2 = Constant$ equation of circle.

Hence streamlines are circle.

In steady flow, the velocity at each point in the flow field remains constant with time and, consequently, the streamline shapes do not vary from one instant to the next. This implies that a particle located on a given streamline will always move along the same streamline. Furthermore, consecutive particles passing through a fixed point in space will be on the same streamline and, subsequently, will remain on this streamline. Thus in a steady flow, pathlines, streaklines, and streamlines are identical lines in the flow field.

The shapes of the streamlines may vary from instant to instant if the flow is unsteady. In the case of unsteady flow, pathlines, streaklines, and streamlines do not coincide.

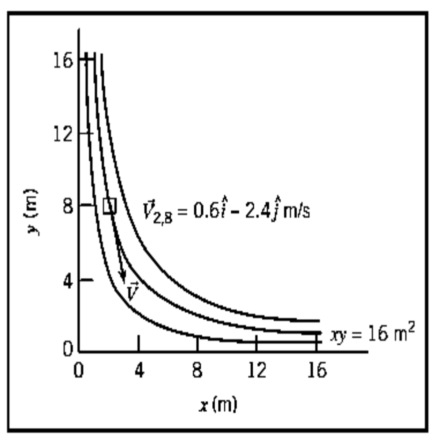
Example: Streamlines And Pathlines In Two-Dimensional Flow

A velocity field is given by $\vec{V} = Ax\hat{i} - Ay\hat{j}$; the units of velocity are m/s; x and y are given in meters; $A = 0.3\ \text{s}^{-1}$.

(a) Obtain an equation for the streamlines in the xy plane.
(b) Plot the streamline passing through the point $(x_0, y_0) = (2, 8)$.
(c) Determine the velocity of a particle at the point (2, 8).
(d) If the particle passing through the point (x_0, y_0) is marked at time $t = 0$, determine the location of the particle at time $t = 6$ s.
(e) What is the velocity of this particle at time $t = 6$ s?
(f) Show that the equation of the particle path (the pathline) is the same as the equation of the streamline.

Given: Velocity field, $\vec{V} = Ax\hat{i} - Ay\hat{j}$; x and y in meters; $A = 0.3\ \text{s}^{-1}$.

Find: (a) Equation of the streamlines in the xy plane.
(b) Streamline plot through point (2, 8).
(c) Velocity of particle at point (2, 8).
(d) Position at $t = 6$ s of particle located at (2, 8) at $t = 0$.
(e) Velocity of particle at position found in (d).
(f) Equation of pathline of particle located at (2, 8) at $t = 0$.



Solution:

(a) Streamlines are lines drawn in the flow field such that, at a given instant, they are tangent to the direction of flow at every point. Consequently,

$$\left.\frac{dy}{dx}\right)_{\text{streamline}} = \frac{v}{u} = \frac{-Ay}{Ax} = \frac{-y}{x}$$

Separating variables and integrating, we obtain

$$\int \frac{dy}{y} = -\int \frac{dx}{x}$$

or

$$\ln y = -\ln x + c_1$$

This can be written as $xy = c$

(b) For the streamline passing through the point $(x_0, y_0) = (2, 8)$ the constant, c, has a value of 16 and the equation of the streamline through the point (2, 8) is

$$xy = x_0 y_0 = 16\,\text{m}^2$$

The plot is as sketched above.

(c) The velocity field is $\vec{V} = Ax\hat{i} - Ay\hat{j}$. At the point (2, 8) the velocity is

$$\vec{V} = A(x\hat{i} - y\hat{j}) = 0.3\text{s}^{-1}(2\hat{i} - 8\hat{j})\text{m} = 0.6\hat{i} - 2.4\hat{j}\,\text{m/s}$$

(d) A particle moving in the flow field will have velocity given by

$$\vec{V} = Ax\hat{i} - Ay\hat{j}$$

Thus

$$u_p = \frac{dx}{dt} = Ax \quad \text{and} \quad v_p = \frac{dy}{dt} = -Ay$$

Separating variables and integrating (in each equation) gives

$$\int_{x_0}^{x} \frac{dx}{x} = \int_0^t A\,dt \quad \text{and} \quad \int_{y_0}^{y} \frac{dy}{y} = \int_0^t -A\,dt$$

Then

$$\ln\frac{x}{x_0} = At \quad \text{and} \quad \ln\frac{y}{y_0} = -At$$

or

$$x = x_0 e^{At} \quad \text{and} \quad y = y_0 e^{-At}$$

At $t = 6$ s,

$$x = 2\,\text{m}\,e^{(0.3)6} = 12.1\,\text{m} \quad \text{and} \quad y = 8\,\text{m}\,e^{-(0.3)6} = 1.32\,\text{m}$$

At $t = 6$ s, particle is at (12.1, 1.32) m

(e) At the point (12.1, 1.32) m,

$$\vec{V} = A(x\hat{i} - y\hat{j}) = 0.3\,\text{s}^{-1}(12.1\hat{i} - 1.32\hat{j})\text{m}$$
$$= 3.63\hat{i} - 0.396\hat{j}\,\text{m/s}$$

(f) To determine the equation of the pathline, we use the parametric equations

$$x = x_0 e^{At} \quad \text{and} \quad y = y_0 e^{-At}$$

and eliminate t. Solving for e^{At} from both equations

$$e^{At} = \frac{y_0}{y} = \frac{x}{x_0}$$

Therefore $xy = x_0 y_0 = 16\ \text{m}^2$

Notes:

- ✓ This problem illustrates the method for computing streamlines and pathlines.
- ✓ Because this is a steady flow, the streamlines and pathlines have the same shape—in an unsteady flow this would not be true.
- ✓ When we follow a particle (the Lagrangian approach), its position (x, y) and velocity ($u_p = dx/dt$ and $v_p = dy/dt$) are functions of time, even though the flow is steady.

Question: A velocity field is given by $\vec{V} = ax\hat{\imath} - bty\hat{\jmath}$, where $a = 1s^{-1}$ and $b = 1s^{-2}$. Find the equation of the streamlines at any time t. Plot several streamlines in the first quadrant at $t = 0s, t = 1\ s$, and $t = 20\ s$.

Solution

For streamline $\frac{v}{u} = \frac{dy}{dx} = -\frac{bty}{ax}$

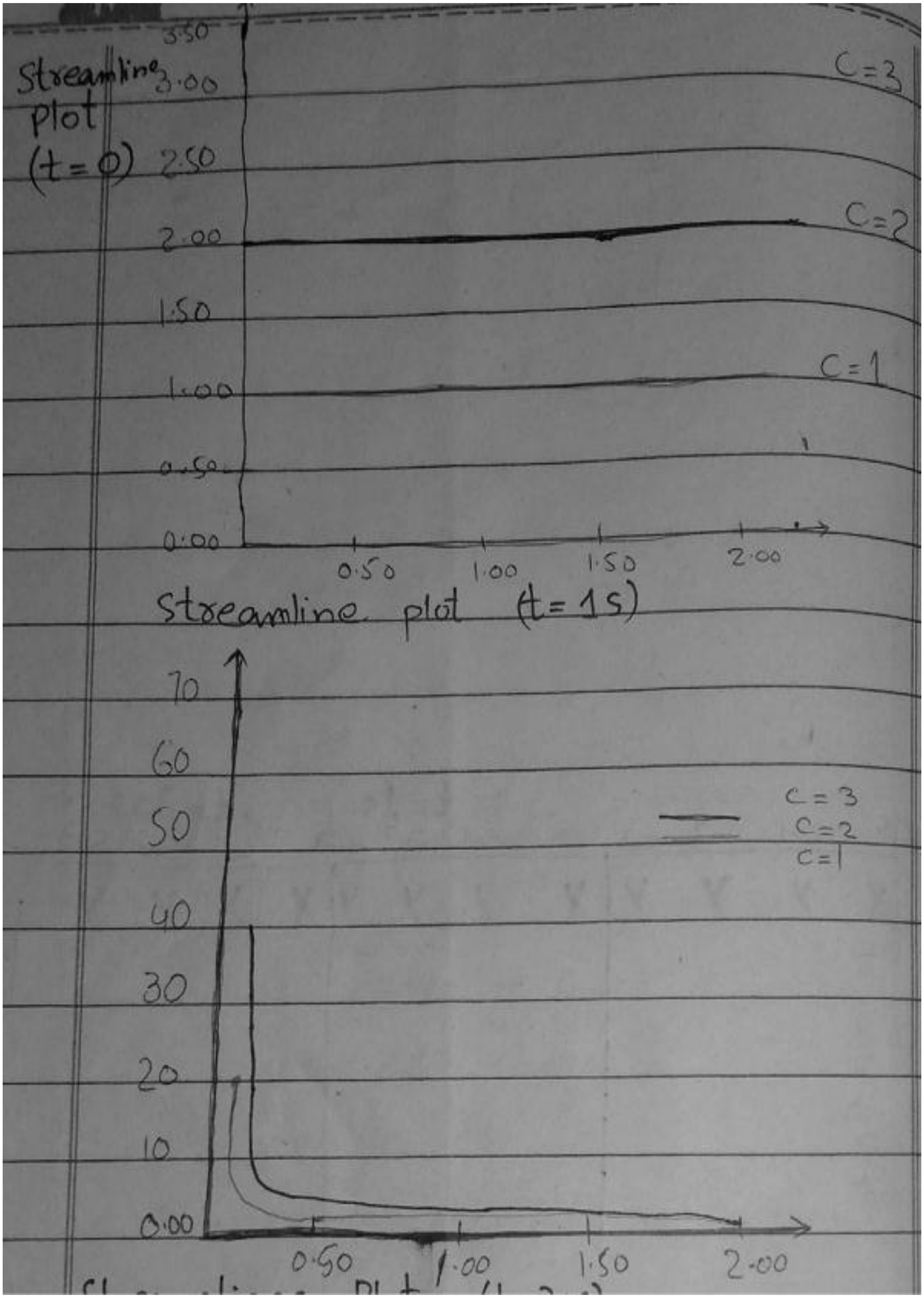
$\Rightarrow \frac{dy}{y} = -\frac{bt}{a}\frac{dx}{x} \Rightarrow lny = -\frac{bt}{a}lnx + c \Rightarrow y = cx^{-\frac{bt}{a}}$

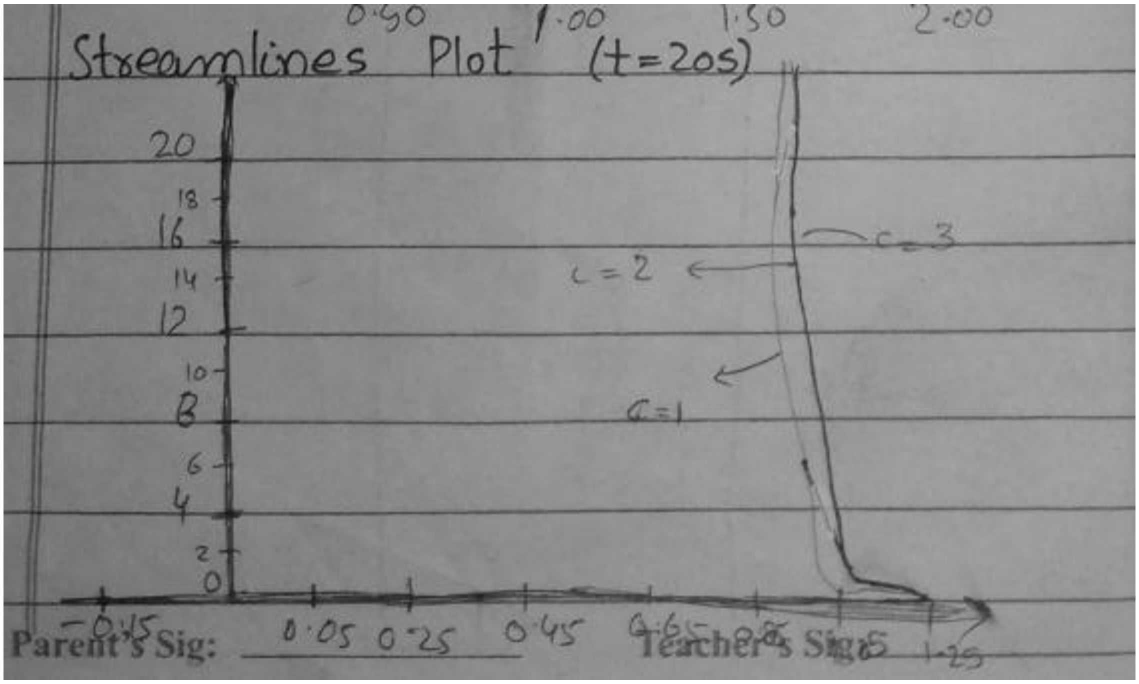
When $t = 0s \Rightarrow y = c$ When $t = 1s \Rightarrow y = \frac{c}{x}$

When $t = 20s \Rightarrow y = cx^{-20}$

t=0s	c=1	c=2	c=3	$y = cx^{-20}$ t=1s	c=1	c=2	c=3	t=20s	c=1	c=2	c=3	
x	y	y	y	x	y	y	y	x	y	y	y	### means too large to view
0.05	1.00	2.00	3.00	0.05	20.00	40.00	60.00	0.05	###	###	###	
0.10	1.00	2.00	3.00	0.10	10.00	20.00	30.00	0.10	"	"	"	
0.20	1.00	2.00	3.00	0.20	5.00	10.00	15.00	0.20	"	"	"	
0.30	"	"	"	0.30	3.33	6.67	10.00	0.30				
0.40	"	"	"	0.40	2.50	5.00	7.50	2.40	"	"	"	
0.50	"	"	"	0.50	2.00	4.00	6.0	2.50	"	"	"	
0.60	"	"	"	0.60	1.67	3.33	5.00	2.60	"	"	"	
0.70	"	"	"	0.70	1.43	2.86	4.29	2.70	"	"	"	
0.80	"	"	"	0.80	1.25	2.50	3.75	2.80	6.14	"	"	
0.90	"	"	"	0.90	1.11	2.22	2.33	2.90	8.23	16.45	24.68	
1.00	"	"	"	1.00	1.00	2.00	3.00	1.00	1.00	2.00	3.00	
1.10	"	"	"	1.10	0.91	1.82	2.73	1.10	0.15	0.30	0.45	
1.20	"	"	"	1.20	0.83	.67	2.50	1.20	0.03	0.05	0.08	
1.30	"	"	"	1.30	0.77	1.54	2.31	1.30	0.01	0.01	0.02	
1.40	"	"	"	1.40	0.71	1.43	2.14	1.40	0.00	0.00	0.00	
1.50	"	"	"	1.50	0.67	1.33	2.00	1.50	"	"	"	
1.60	"	"	"	1.60	0.63	1.25	1.88	1.60	"	"	"	
1.70	"	"	"	1.70	0.59	1.18	1.76	1.70	"	"	"	
1.80	"	"	"	1.80	0.56	1.11	1.67	1.80	"	"	"	
1.90	"	"	"	1.90	0.53	1.05	1.58	1.90	"	"	"	
2.00	"	"	"	2.00	0.50	1.00	1.5	2.00	"	"	"	

Parent's Sig: ______


Streamline plot (t = 0)
3.50
3.00
2.50
2.00
1.50
1.00
0.50
0.00
0.50
1.00
1.50
2.00
C=3
C=2
C=1
Streamline plot (t = 1s)
70
60
50
40
30
20
10
0.00
0.50
1.00
1.50
2.00
C = 3
C=2
C=1



- **Surface Forces:** All forces acting on the boundary of the medium through direct contact: e.g. Pressure and Frictional force.
- **Body Forces:** All forces acting on the boundary of the medium without direct contact: e.g. Gravitational and Electromagnetic force.

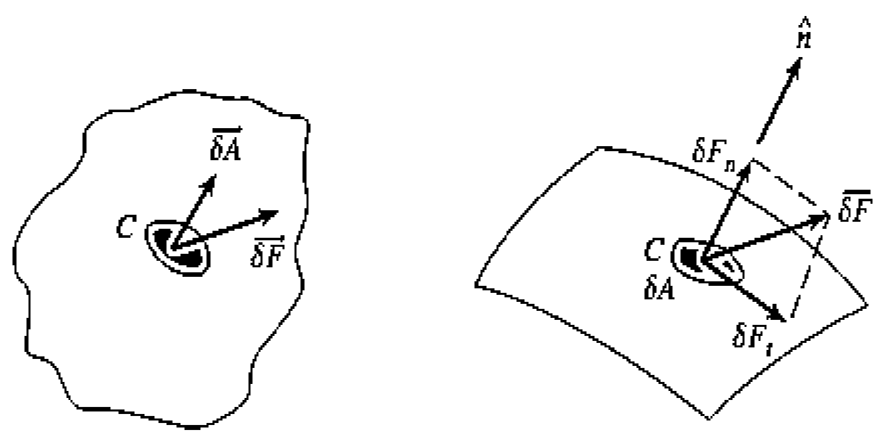
Stress Field **(Surface and body forces are discussed in this phenomenon)**

In our study of fluid mechanics, we will need to understand what kinds of forces act on fluid particles. Each fluid particle can experience: surface forces (pressure, friction) that are generated by contact with other particles or a solid surface; and body forces (such as gravity and electromagnetic) that are experienced throughout the particle.

The gravitational body force acting on an element of volume, d$\bar{V}$, is given by $\rho\vec{g}d\bar{V}$, where ρ is the density (mass per unit volume) and $\vec{g}$ is the local gravitational acceleration. Thus the gravitational body force per unit volume is $\rho\vec{g}$ and the gravitational body force per unit mass $\vec{g}$.

Surface forces on a fluid particle lead to stresses. <u>The concept of stress is useful for describing how forces acting on the boundaries of a medium (fluid or solid) are transmitted throughout the medium</u>. For example, when you stand on a diving board, stresses are generated within the board. On the other hand, when a body moves through a fluid, stresses are developed within the fluid. The difference between a fluid and a solid is, as we've seen, that stresses in a fluid are mostly generated by motion rather than by deflection.

Imagine the surface of a fluid particle in contact with other fluid particles, and consider the contact force being generated between the particles. Consider a portion,$\delta\vec{A}$, of the surface at some point C. The orientation of $\delta\vec{A}$ is given by the unit vector, $\hat{n}$, shown in Figure. The vector $\hat{n}$ is the outwardly drawn unit normal with respect to the particle.



The force, $\delta\vec{F}$, acting on $\delta\vec{A}$ may be resolved into two components, one normal to and the other tangent to the area.

A **normal stress** is then defined as Normal component of force per unit area is called Normal stress. i.e.

$$\sigma_n = \lim_{\delta A_n \to 0} \frac{\delta F_n}{\delta A_n}$$

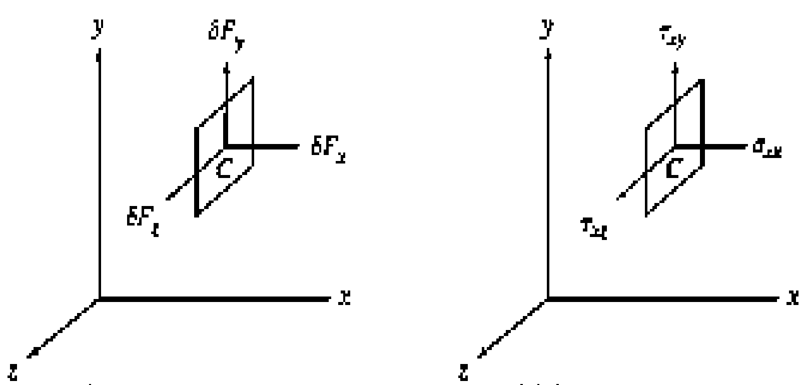
And a **shear stress** is then defined as Tangent component of force per unit area is called sheer stress. i.e.

$$\tau_n = \lim_{\delta A_n \to 0} \frac{\delta F_t}{\delta A_n}$$

Subscript n on the stress is included as a reminder that the stresses are associated with the surface δA through C, having an outward normal in the $\hat{n}$ direction. The fluid is actually a continuum, so we could have imagined breaking it up any number of different ways into fluid particles around point C, and therefore obtained any number of different stresses at point C.

In dealing with vector quantities such as force, we usually consider components in an orthogonal coordinate system. In rectangular coordinates we might consider the stresses acting on planes whose outwardly drawn normals (again with respect to the material acted upon) are in the x, y, or z directions.

Now check the following figure;



(a) Force components (b) Stress components

In Figure we consider the stress on the element δA_x, whose outwardly drawn normal is in the x direction. The force, $\delta \vec{F}$, has been resolved into components along each of the coordinate directions. Dividing the magnitude of each force component by the area, δA_x, and taking the limit as δA_x approaches zero, we define the three stress components shown in Fig.(b):

$$\sigma_{xx} = \lim_{\delta A_x \to 0} \frac{\delta F_x}{\delta A_x}$$

$$\tau_{xy} = \lim_{\delta A_x \to 0} \frac{\delta F_y}{\delta A_x}$$

$$\tau_{xz} = \lim_{\delta A_x \to 0} \frac{\delta F_z}{\delta A_x}$$

We have used a double subscript notation to label the stresses. The first subscript (in this case, x) indicates the *plane* on which the stress acts (in this case, a surface perpendicular to the x axis). The second subscript indicates the *direction* in which the stress acts.

Consideration of area element δA_y would lead to the definitions of the stresses, σ_{yy}, τ_{yx},, and τ_{yz} , use of area element δA_z would similarly lead to the definitions of σ_{zz}, τ_{zx},, and τ_{zy}.

Although we just looked at three orthogonal planes, an infinite number of planes can be passed through point C, resulting in an infinite number of stresses associated with planes through that point. Fortunately, the state of stress at a point can be described completely by specifying the stresses acting on any three mutually perpendicular planes through the point. The stress at a point is specified by the nine components

$$\begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix}$$

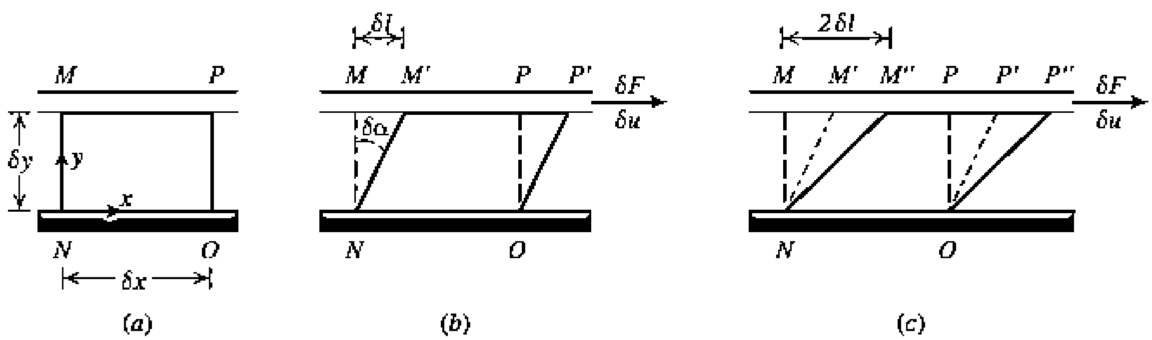
Where σ has been used to denote a normal stress, and τ to denote a shear stress.

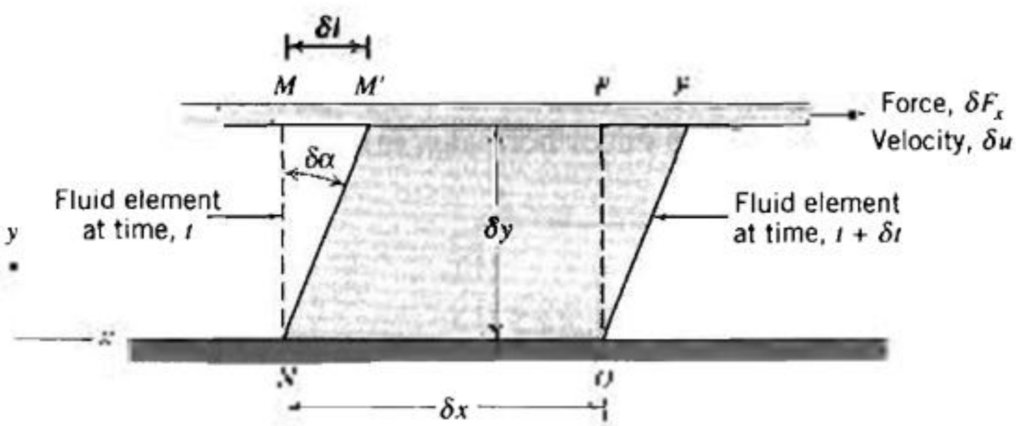
Viscosity

It is the measure of resistance against the motion of fluid. It is denoted by μ. It is also called absolute viscosity and dynamic viscosity.

Explanation

Where do stresses come from? For a solid, stresses develop when the material is elastically deformed or strained; for a fluid, shear stresses arise due to viscous flow (we will discuss a fluid's normal stresses shortly). Hence we say *solids are elastic, and fluids are viscous* (and it's interesting to note that many biological tissues are viscoelastic, meaning they combine features of a solid and a fluid). For a fluid at rest, there will be no shear stresses. We will see that each fluid can be categorized by examining the relation between the applied shear stresses and the flow (specifically the rate of deformation) of the fluid.





Consider the behavior of a fluid element between the two infinite plates shown in Figure. The upper plate moves at constant velocity, δu, under the influence of a constant applied force, δF_x. The shear stress, τ_{yx}, applied to the fluid element is given by $\tau_{yx} = \lim_{\delta A_y \to 0} \frac{\delta F_x}{\delta A_y} = \frac{dF_x}{dA_y}$

Where δA_y is the area of contact of a fluid element with the plate, and δF_x is the force exerted by the plate on that element. During time interval δt, the fluid element is deformed from position MNOP to position M'NOP'. The rate of deformation of the fluid is given by

Deformation rate = $\lim_{\delta t \to 0} \frac{\delta \propto}{\delta t} = \frac{d\propto}{dt}$

We want to express $\frac{d\propto}{dt}$ in terms of readily measurable quantities. This can be done easily. The distance, δl, between the points M and M' is given by $\delta l = \delta u \delta t$

Alternatively, for small angles, $\delta l = \delta y \delta \propto$

Equating these two expressions for δl gives $\frac{\delta \propto}{\delta t} = \frac{\delta u}{\delta y}$

Taking the limits of both sides of the equality, we obtain

$\lim_{\delta t \to 0} \frac{\delta \propto}{\delta t} = \lim_{\delta y \to 0} \frac{\delta u}{\delta y}$

$\frac{d\propto}{dt} = \frac{du}{dy}$

Thus, the fluid element of Figure, when subjected to shear stress, τ_{yx}, experiences a rate of deformation (shear rate) given by $\frac{du}{dy}$.

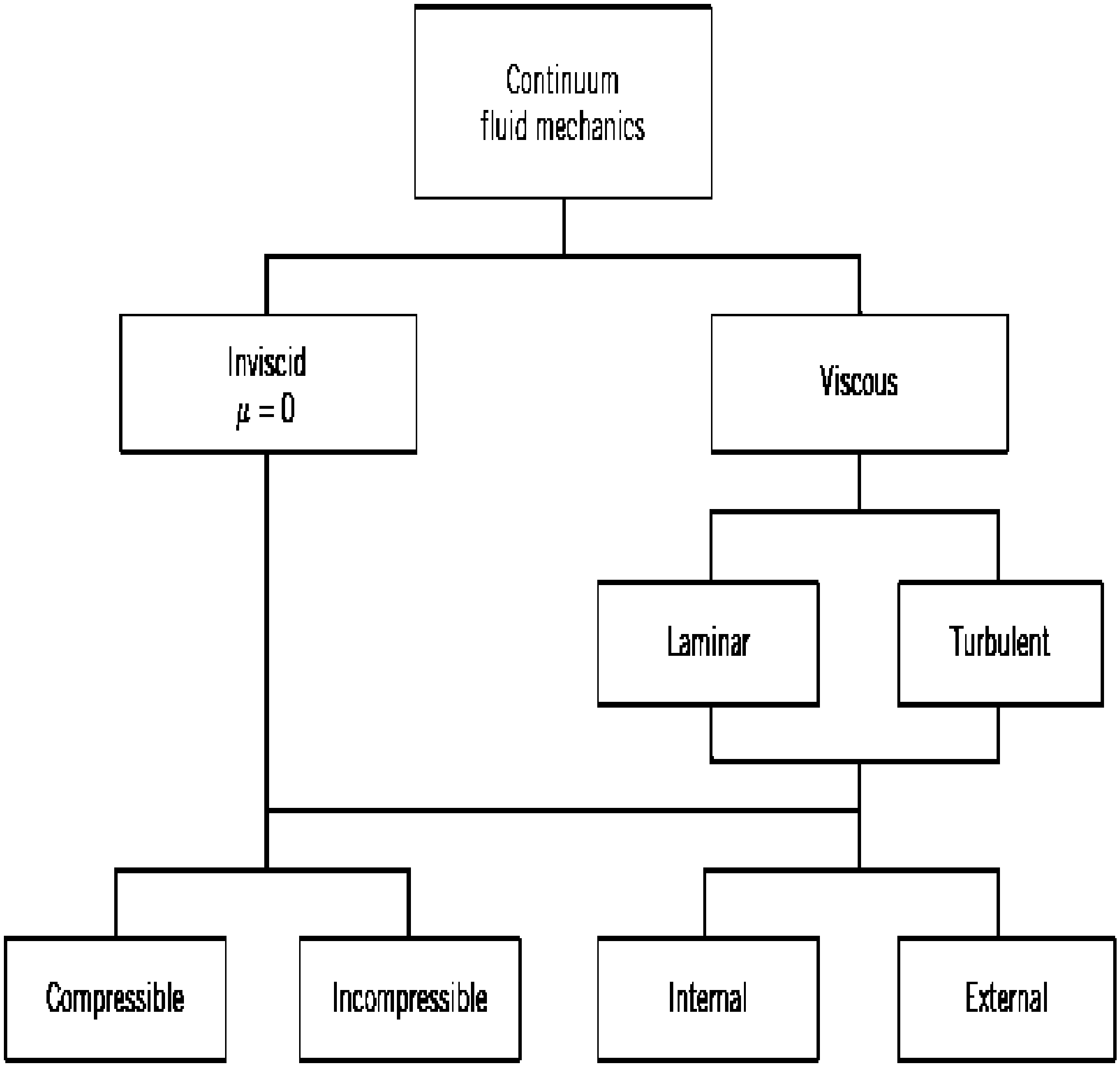


Fig. 2.13 Possible classification of continuum fluid mechanics.

TYPES OF FLUID

Flow

The quantity of fluid passing through a point per unit time is called flow.

Viscous Flow

Fluid that has non-zero viscosity or finite viscosity and can exert sheer stress on the surface is called viscous fluid or real fluid.

Inviscous Flow Fluid having zero viscosity is called inviscid fluid.

Compressibility

Compressibility is the measure of change in fluid w.r.t volume and density under the action of external forces.

Compressible fluid

A type of fluid in which change occur due to volume and density changes by the action of pressure (temperature) is called compressible fluid. Examples: gases.

Incompressible fluid

A type of fluid in which no change occur due to volume and density changes by the action of pressure (temperature) is called incompressible fluid.

Steady and Unsteady Flow

The flow in which the properties and the conditions are associated with motion of fluid particles is independent of time. i.e. $\frac{\partial}{\partial t} = 0$. Such a flow is said to be steady flow. On the other hand, the flow in which the properties and the conditions are associated with motion of fluid particles is independent of time. i.e. $\frac{\partial}{\partial t} \neq 0$. Such a flow is said to be Unsteady flow.

Coquette Flow The flow of viscous fluid in the space between any two surface, one of which is moving tangentially relative to the other. The relative motion of the surface imposes a sheer stress on the fluid and induces flow.

Uniform and Non – Uniform Flow

The flow in which the velocity of the fluid particle at all section of the particle of pipe and channel are equal i.e. constant. On the other hand, the fluid particles are said to be non-uniform it the velocity of the particles is not equal i.e. not constant.

Laminar and Non-Laminar flow or Stream line flow and Turbulent flow

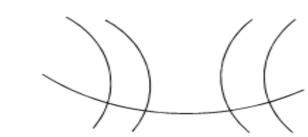
A flow in which fluid particle have definite path of particle and the two paths of two individuals does not cut each other is called Laminar or stream line flow. How a viscous flow can be laminar or turbulent, respectively. A laminar flow is one in which the fluid particles move in smooth layers, or laminas. The velocity of laminar flow is $\bar{v} = u\hat{\imath}$

On the other hand, if the flow of each other particle does not trace out a definite path. The path of individuals particle also crosses each other is called non – laminar or turbulent flow. ; a turbulent flow is one in which the fluid particles rapidly mix as they move along due to random three-dimensional velocity fluctuations. The velocity of turbulent flow is $\vec{v} = \bar{v} + u'\hat{\imath} + v'\hat{\jmath} + w'\hat{k}$

Rotational flow & Irrational flow:

Rotational flow is that flow in which fluid particles rotate about their own axis have the same angular velocity.

On the other hand, the fluid particle does not rotate about their own axis and retain their original orientation is called irrational flow.



Compressible and Non-compressible flow:

A flow in which volume and density of fluid changes during the flow is said to be compressible flow. On the other hand, if volume and density of fluid does not change during the flow is said to be non-compressible flow or incompressible flow. The most common example of compressible flow concerns the flow of gases, while the flow of liquids may frequently be treated as incompressible.

Internal and External Flow System

Internal flows of system are those where fluid flows through confined spaces such as pipe and open channels. Or Flows completely bounded by solid surfaces are called internal or duct flows. **While** External flows of system are those where confining boundaries are at relatively larger or infinite distances, such as atmosphere through which aero planes, missiles and space vehicles travel, or ocean water through which submarines move. Flows over bodies immersed in an unbounded fluid are termed external flows. Both internal and external flows may be laminar or turbulent, compressible or incompressible.

Ideal fluid

Ideal fluid is a fluid in which both inviscid and incompressible fluid is involved is called Ideal fluid or it is a perfect fluid. These are non-Newtonian fluid. In ideal fluid the viscosity is zero. There is no internal resistance between them. It is incompressible, Irrotational and non – viscous (inviscid) fluid.

Real Fluid

A fluid in which the finite viscosity exists and therefore we can exert tangential or sheering stress on a surface with which it is in contact. Real fluid is called viscous fluid. Real fluid can be further divided into Newtonian fluid and Non-Newtonian fluid.

Newton's Law of viscosity According to this law

Shear Stress is directly proportional to deformation rate

i.e. $\tau \propto \frac{du}{dx} \Rightarrow \tau = \mu \frac{du}{dx}$ Where μ is called absolute or dynamic viscosity.

μ is also called **Coefficient of Viscosity** of Newtonian Fluid.

Absolute or dynamic viscosity $\boldsymbol{\mu}$

Force needed by a fluid to overcome its own internal molecular friction so that the fluid will flow.

Or Tangential force per unit area need to move fluid in one horizontal plane with respect to other plane with the unit velocity.

Dimension of Absolute or dynamic viscosity μ

Since the dimensions of force, F, mass, M, length, L, and time, 't', are related by Newton's second law of motion, the dimensions of μ, can also be expressed as [M/Lt].

Since $\tau = \mu \frac{du}{dx} \Rightarrow \mu = \frac{\tau}{\frac{du}{dx}} \Rightarrow \mu = \frac{F/A}{\frac{velocity}{distance}} \Rightarrow \mu = \frac{\left[\frac{M}{Lt^2}\right]}{\left[\frac{L}{tL}\right]} \Rightarrow \mu = \frac{\left[\frac{M}{Lt^2}\right]}{\left[\frac{1}{t}\right]} \Rightarrow \mu = \left[\frac{M}{Lt}\right]$

- In the British Gravitational system, the units of viscosity are lbf • s/ft^2
  Or slug/(ft • s).
- In the Absolute Metric system, the basic unit of viscosity is called a poise [1 poise = 1 g/(cm • s)];
- In the SI system the units of viscosity are kg/(m • s) or Pa • s
  (1 Pa • s = 1 N • s/m^2)

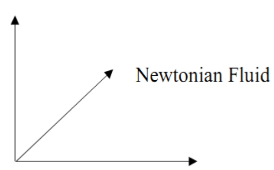
Kinematic viscosity:

In fluid mechanics the ratio of absolute viscosity, μ, to density, ρ, is called **kinematic viscosity** and is represented by the symbol υ. i.e. $\upsilon = \frac{\mu}{\rho}$
Since density has dimensions [M/L^3], the dimensions of υ are [L^2/t]. In the Absolute Metric system of units, the unit for υ is a stoke (1 stoke = 1 cm^2/s).

Newtonian Fluids

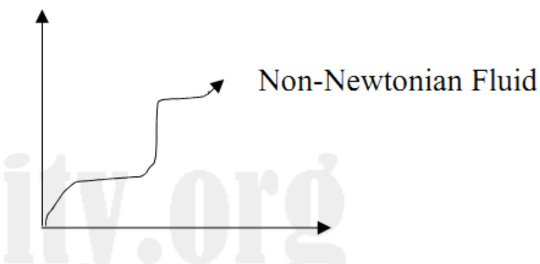
Fluids in which shear stress is directly proportional to rate of deformation are Newtonian fluids. Water and air are Newtonian fluid. This is a fluid in which viscosity is independent of the velocity gradient due to single variable. They obey law of viscosity. i.e. $\tau_{yx} \propto \frac{du}{dx} \Rightarrow \tau_{yx} = \mu \frac{du}{dx}$ where μ is called absolute or dynamic viscosity.

If we draw the graph then we get a straight line. Blood, milk, jellies, butter, water, air, and gasoline are example of Newtonian fluid.



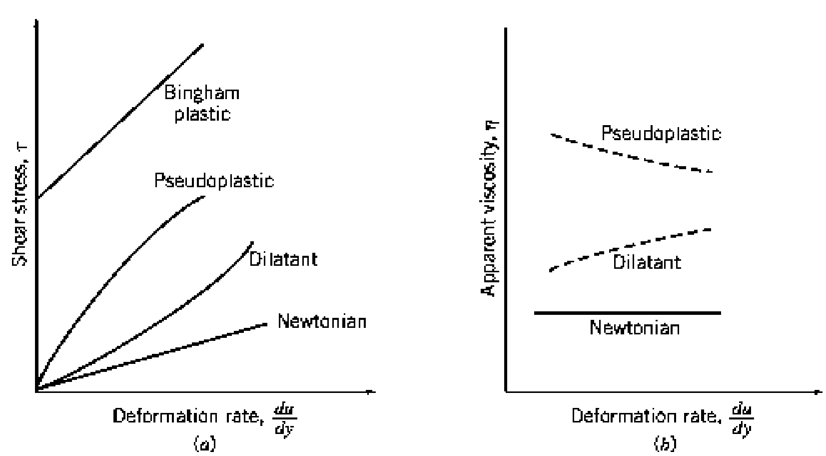
Non-Newtonian Fluids

Fluids in which shear stress is not directly proportional to deformation rate are non-Newtonian. The term non-Newtonian is used to classify all fluids in which shear stress is not directly proportional to shear rate. This is a fluid in which viscosity at a given temperature and pressure for which a viscosity is a function of velocity gradient is not a straight line. Non-Newtonian fluids are very important in fluid mechanics.



Although we will not discuss these much in this text, many common fluids exhibit non-Newtonian behavior. Two familiar examples are toothpaste and Lucite paint. The latter is very "thick" when in the can, but becomes "thin" when sheared by brushing. Toothpaste behaves as a "fluid" when squeezed from the tube. However, it does not run out by itself when the cap is removed. There is a threshold or yield stress below which toothpaste behaves as a solid. Strictly speaking, our definition of a fluid is valid only for materials that have zero yield stress.

Non-Newtonian fluids commonly are classified as having time-independent or time-dependent behavior. Examples of time-independent behavior are shown in the rheological diagram of Figure.



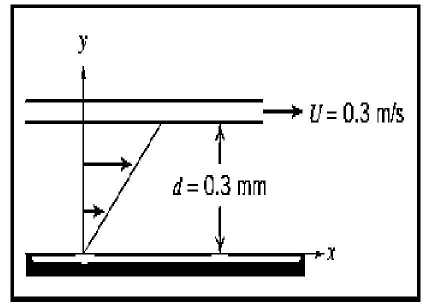
Remark

- To ensure that $\tau_{yx} = \mu \frac{du}{dx}$ has the same sign as $\frac{du}{dx}$, we may write by
  Power's Law $\tau_{yx} = k \left(\frac{du}{dx}\right)^n = k \left|\frac{du}{dx}\right|^{n-1} \frac{du}{dx} = \eta \frac{du}{dx}$
  The term $\eta = k \left|\frac{du}{dx}\right|^{n-1}$ is referred to as the apparent viscosity.
- Fluids in which the apparent viscosity decreases with increasing deformation rate (n < 1) are called **pseudoplastic (or shear thinning) fluids**. Most non-Newtonian fluids fall into this group; examples include polymer solutions, colloidal suspensions, and paper pulp in water.
- Fluids in which the apparent viscosity increases with increasing deformation rate (n > 1) the fluid is termed **dilatant (or shear thickening) fluids**. Suspensions of starch and of sand are examples of dilatant fluids.
- A "fluid" that behaves as a solid until a minimum yield stress, τ_y , is exceeded and subsequently exhibits a linear relation between stress and rate of deformation is referred to as **an ideal or Bingham plastic**. The corresponding shear stress model is $\tau_{yx} = \tau_y + \mu_p \frac{du}{dx}$
  Clay suspensions, drilling muds, and toothpaste are examples of substances exhibiting this behavior.
- The study of non-Newtonian fluids is further complicated by the fact that the apparent viscosity may be time-dependent.
- **Thixotropic fluids** show a decrease in apparent viscosity $\eta = k \left|\frac{du}{dx}\right|^{n-1}$ with time under a constant applied shear stress; many paints are thixotropic.
- **Rheopectic fluids** show an increase in apparent viscosity $\eta = k \left|\frac{du}{dx}\right|^{n-1}$ with time.
- After deformation some fluids partially return to their original shape when the applied stress is released; such fluids are called **viscoelastic**. (Many biological fluids work this way).

Example 2.2 VISCOSITY AND SHEAR STRESS IN NEWTONIAN FLUID

An infinite plate is moved over a second plate on a layer of liquid as shown. For small gap width, d, we assume a linear velocity distribution in the liquid. The liquid viscosity is 0.65 centipoise and its specific gravity is 0.88. Determine:

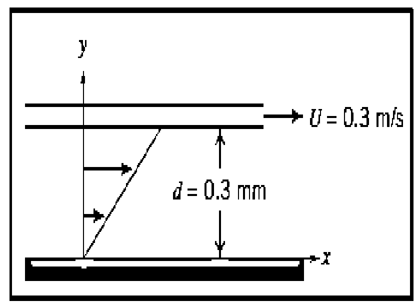
(a) The absolute viscosity of the liquid, in $\mathrm{lbf \cdot s/ft^2}$.
(b) The kinematic viscosity of the liquid, in $\mathrm{m^2/s}$.
(c) The shear stress on the upper plate, in $\mathrm{lbf/ft^2}$.
(d) The shear stress on the lower plate, in Pa.
(e) The direction of each shear stress calculated in parts (c) and (d).



Given: Linear velocity profile in the liquid between infinite parallel plates as shown.

$$\mu = 0.65 \text{ cp}$$
$$\text{SG} = 0.88$$

Find: (a) μ in units of $\mathrm{lbf \cdot s/ft^2}$.
(b) ν in units of $\mathrm{m^2/s}$.
(c) τ on upper plate in units of $\mathrm{lbf/ft^2}$.
(d) τ on lower plate in units of Pa.
(e) Direction of stresses in parts (c) and (d).



Solution:

Governing equation: $\tau_{yx} = \mu \dfrac{du}{dy}$ Definition: $\nu = \dfrac{\mu}{\rho}$

Assumptions: (1) Linear velocity distribution (given)
(2) Steady flow
(3) $\mu = \text{constant}$

$$\text{(a)}\quad \mu = 0.65\,\text{cp} \times \frac{\text{poise}}{100\,\text{cp}} \times \frac{\text{g}}{\text{cm}\cdot\text{s}\cdot\text{poise}} \times \frac{\text{lbm}}{454\,\text{g}} \times \frac{\text{slug}}{32.2\,\text{lbm}} \times 30.5\frac{\text{cm}}{\text{ft}} \times \frac{\text{lbf}\cdot\text{s}^2}{\text{slug}\cdot\text{ft}}$$

$$\mu = 1.36 \times 10^{-5}\ \text{lbf}\cdot\text{s/ft}^2 \longleftarrow \mu$$

(b) $\nu = \dfrac{\mu}{\rho} = \dfrac{\mu}{SG\,\rho_{H_2O}}$

$$= 1.36 \times 10^{-5} \frac{\text{lbf}\cdot\text{s}}{\text{ft}^2} \times \frac{\text{ft}^3}{(0.88)1.94\ \text{slug}} \times \frac{\text{slug}\cdot\text{ft}}{\text{lbf}\cdot\text{s}^2} \times (0.305)^2 \frac{\text{m}^2}{\text{ft}^2}$$

$$\nu = 7.41 \times 10^{-7}\,\text{m}^2/\text{s} \longleftarrow \nu$$

(c) $\tau_{upper} = \tau_{yx,\,upper} = \mu \left.\dfrac{du}{dy}\right)_{y=d}$

Since u varies linearly with y,

$$\frac{du}{dy} = \frac{\Delta u}{\Delta y} = \frac{U-0}{d-0} = \frac{U}{d}$$

$$= 0.3\frac{\text{m}}{\text{s}} \times \frac{1}{0.3\ \text{mm}} \times 1000\frac{\text{mm}}{\text{m}} = 1000\ \text{s}^{-1}$$

Part (c) shows that the shear stress is:

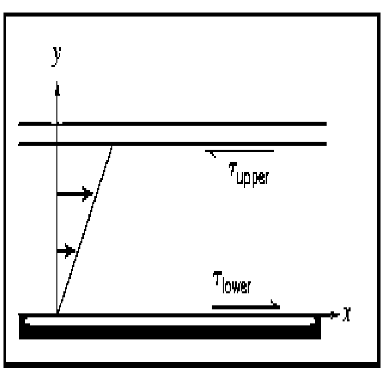
- ✓ **Constant across the gap for a linear velocity profile.**
- ✓ **Directly proportional to the speed of the upper plate (because of the linearity of Newtonian fluids).**
- ✓ **Inversely proportional to the gap between the plates.**

Note that multiplying the shear stress by the plate area in such problems computes the force required to maintain the motion.

$$\tau_{upper} = \mu\frac{U}{d} = 1.36 \times 10^{-5} \frac{\text{lbf}\cdot\text{s}}{\text{ft}^2} \times \frac{1000}{\text{s}} = 0.0136\ \text{lbf/ft}^2 \longleftarrow \tau_{upper}$$

(d) $\tau_{lower} = \mu\dfrac{U}{d} = 0.0136\dfrac{\text{lbf}}{\text{ft}^2} \times 4.45\dfrac{\text{N}}{\text{lbf}} \times \dfrac{\text{ft}^2}{(0.305)^2\text{m}^2} \times \dfrac{\text{Pa}\cdot\text{m}^2}{\text{N}} = 0.651\ \text{Pa} \longleftarrow \tau_{lower}$

(e) Directions of shear stresses on upper and lower plates.



$\left\{\begin{array}{l}\text{The upper plate is a negative } y \text{ surface; so} \\ \text{positive } \tau_{yx} \text{ acts in the negative } x \text{ direction.}\end{array}\right\}$

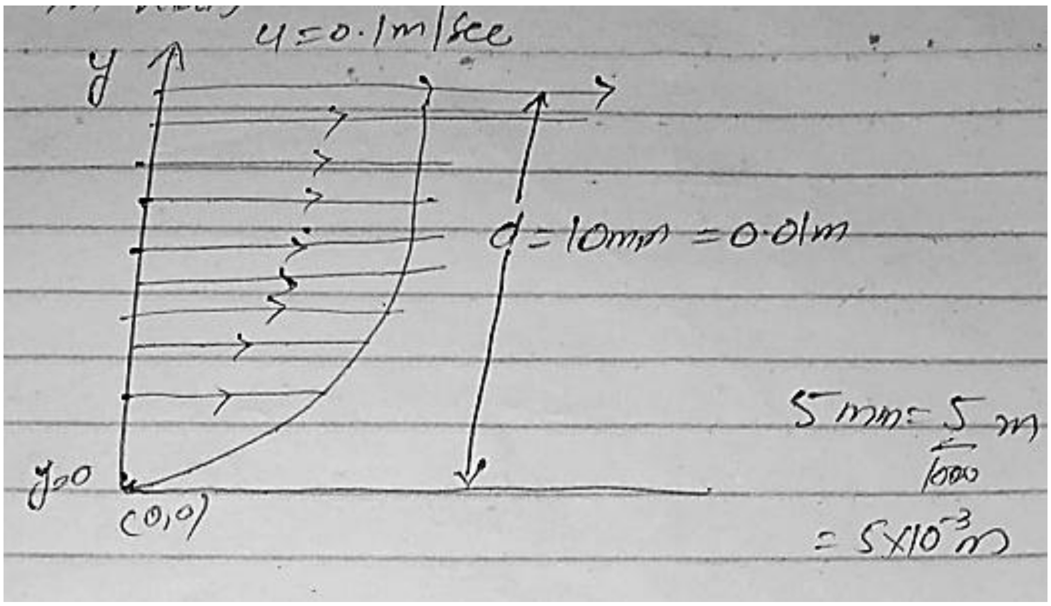
$\left\{\begin{array}{l}\text{The lower plate is a positive } y \text{ surface; so} \\ \text{positive } \tau_{yx} \text{ acts in the positive } x \text{ direction.}\end{array}\right\}$

$\longleftarrow (e)$

Example

Methyl Iodide at a thickness of 10mm and having a viscosity 0.005Pas at a temperature of 20 °C is flowing over a flat plate. The velocity distribution of thin film may be considered as parabolic. Determine the shear stress at y = 0,5 and 10mm from the plate surface.

Solution



Since the velocity distribution is parabolic therefore

$u = A + By + Cy^2$ ……………(1)

Boundary conditions are

i. At y = 0, u = 0 (No slip condition)
ii. At y = 0.01m, u = 0.1m/s
iii. At y = 0.01m (free surface), $\frac{du}{dy} = 0$

Using (i) in (1) we get A = 0 so (1) becomes $u = By + Cy^2$ ……………(2)

Using (ii) in (2) we get $0.1s^{-1} = 0.01B + C(0.01)^2m$ ……………(3)

$(2) \Rightarrow \frac{du}{dy} = B + 2Cy$ ……………(4)

Using (iii) in (4) we get $0 = B + 2C(0.01)m$

$\Rightarrow -\frac{B}{2} = C(0.01)m$ ……………(5)

Using (5) in (3) we get $0.1s^{-1} = 0.01B + (-\frac{B}{2})(0.01)$

$\Rightarrow B = 20s^{-1}$

$(5) \Rightarrow C = -\frac{1000}{ms}$ using $B = 20s^{-1}$ in (5)

Using A,B,C in (1) $u = A + By + Cy^2$

$u = 20s^{-1}y - \frac{1000}{ms}y^2$

$\Rightarrow \frac{du}{dy} = 20s^{-1} - \frac{2000}{ms}y$

Since we know that $\tau_{yx} = \mu\frac{du}{dx}$ therefore

$\tau_{yx})_{y=0} = \mu\frac{du}{dx} = 0.005Pas \times 20s^{-1} = 0.1Pa$

$\tau_{yx})_{y=0.005m} = \mu\frac{du}{dx} = 0.05Pa$

$\tau_{yx})_{y=0.01m} = \mu\frac{du}{dx} = 0$

Question:

Compute the sheer stress in a SAE 30 oil at 20C° if $v = 3ms^{-1}$ and $h = 2cm$.

Solution: Using formula $\tau = \mu\frac{du}{dy} = \mu\frac{v}{h}$

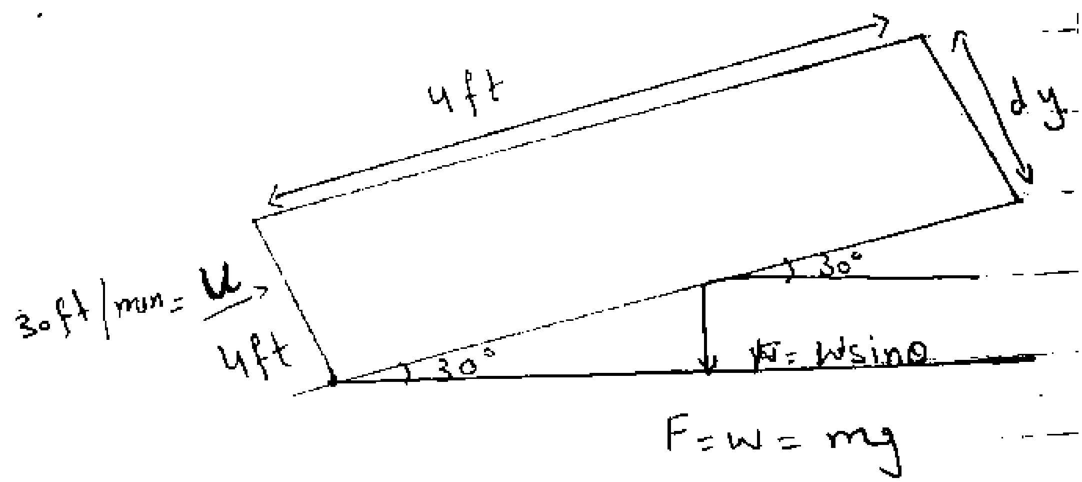
$\Rightarrow \tau = (0.29kg/ms)\frac{3ms^{-1}}{0.02m} = 43\frac{kg}{ms^2} = 43\frac{N}{m^2} = 43Pa$

Here we use $\mu = 0.29kg/ms$ from table for SAE 30 oil at 20C°

Example

A flat plate 4*ft* by 4*ft* slides down an inclined plane at angle of 30° to the horizontal at a velocity of 30*ft* per minute. The inclined plane is lubricated by a thin film of oil having viscosity of 0.001*lbf/ft*^{2}. Find the thickness of the film if the mass of the plate is 1*Slug*.

Solution



Area = $4ft \times 4ft$

Velocity = $u = 30\frac{ft}{min} = \frac{30}{60}\frac{ft}{sec} = \frac{1}{2}\frac{ft}{sec}$ and $\mu = 0.001lbf/ft^2$ also m =1*Slug*

Using $\tau = \mu\frac{du}{dy} = \mu\left(\frac{u}{y}\right)$ and $\tau = \frac{F}{A} \Rightarrow F = \tau A$

We have $F = \mu\left(\frac{u}{y}\right)A$ ....(1)

Since we know that $F = m\vec{g}$

Resolving in components $F = mgSin30°$

$\Rightarrow F = 1 \times 32.2 \times Sin30° = 32.2\frac{ftSlug}{sec^2} \times \frac{1}{2}$

$(1) \Rightarrow 32.2\frac{ftSlug}{sec^2} \times \frac{1}{2} = 0.001lbf/ft^2 \times \left(\frac{\frac{1}{2}\frac{ft}{sec}}{y}\right) \times 16ft^2$

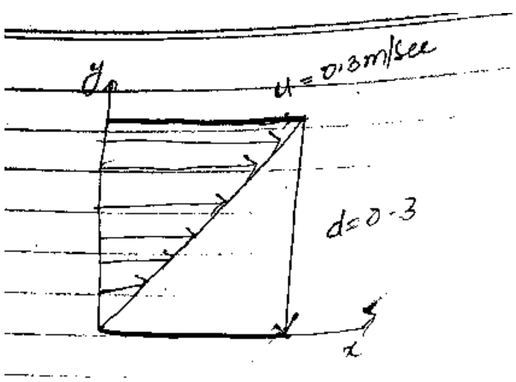
$\Rightarrow Thicknes = y = 4.9844 \times 10^{-4}ftSec$

Example

An infinite plate is move over a second plate on a layer of a liquid. For a small gape width we assume a linear velocity distribution in a liquid. The liquid viscosity is $0.65 \times 10^{-3} kg/ms$ and its specific gravity is 0.88. Then find;

i. Kinematic viscosity of liquid. υ
ii. Shear stress on lower plate. τ_{yx}
iii. Indicate the direction of shear stress.

Solution



$u = 0.3m/s,\ \rho_{H_2O} = 1000kg/m^3,\ d = 0.3 \times 10^{-3}m\ , \mu = 0.65 \times 10^{-3}kg/ms$

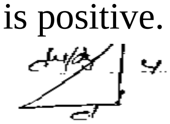
Specific Gravity $= SG = \frac{\rho_{substance}}{\rho_{H_2O}} = 0.88$

$\Rightarrow \rho_{substance} = 0.88 \times \rho_{H_2O} = 0.88 \times \frac{1000kg}{m^3} = 0.88 \times 10^3 kg/m^3$

i. $\upsilon = \frac{\mu}{\rho} = \frac{0.65\times10^{-3}kg/ms}{0.88\times\times10^3 kg/m^3} = m^2/s$

ii. $\tau_{yx} = \mu\frac{du}{dx} = \mu\frac{u}{d} = 0.65 \times \frac{10^{-3}kg}{ms} \times \frac{\frac{0.3m}{s}}{0.3\times10^{-3}m} = 0.65kg/ms^2$

iii. Direction is always positive because plane is positive. And shear stress is positive.



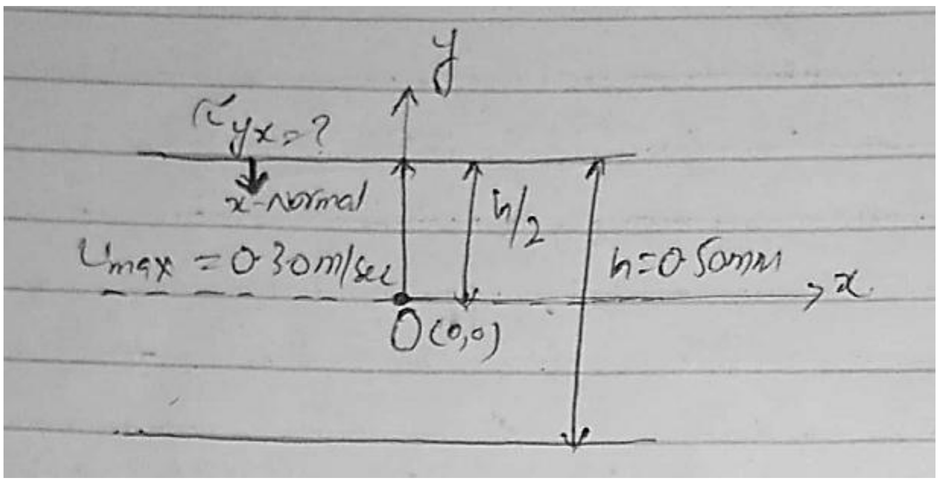
$\tan\theta = \frac{du}{dy} = \frac{u}{d}$

Example

The velocity distribution for laminar flow between parallel plats is given by $\frac{u}{u_{max}} = 1 - \left(\frac{2y}{h}\right)^2$ where 'h' is the distance separating the plates and origin is placed mid – way between the plates. Consider the flow of water at 15 $°C$ with $u_{max} = 0.30m/s$ and $h = 0.50mm$ then

Calculate the shear stress on upper plate and give the direction, sketch the variation of shear stress across the channel.

Solution



Given $\frac{u}{u_{max}} = 1 - \left(\frac{2y}{h}\right)^2$

$$u = u_{max}\left[1 - \left(\frac{2y}{h}\right)^2\right]$$

$$\frac{du}{dy} = u_{max}\left[0 - 2\left(\frac{2y}{h}\right)\left(\frac{2}{h}\right)\right]$$

$$\frac{du}{dy} = -\frac{8y}{h^2}u_{max}$$

Since we know that $\tau_{yx} = \mu\frac{du}{dx}$

Therefore for upper plate $\tau_{yx})_{\frac{h}{2}} = \mu\left(-\frac{8}{h^2}u_{max} \times \frac{h}{2}\right)$

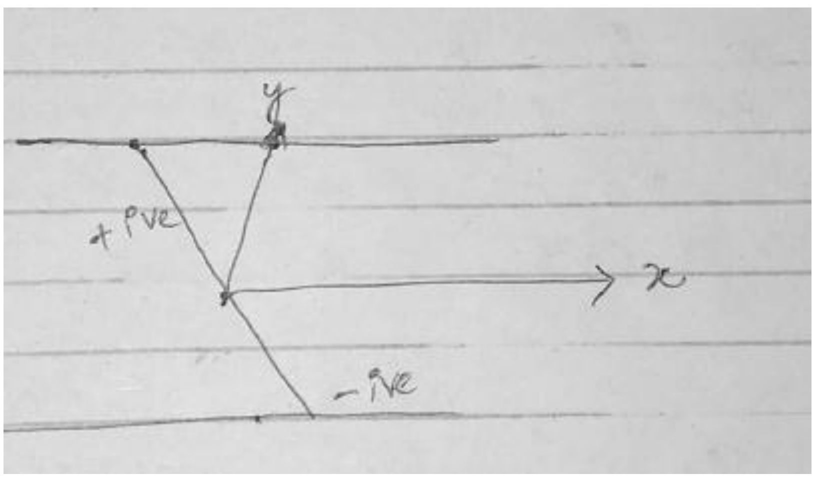
$\tau_{yx})_{\frac{h}{2}} = \mu\left(-\frac{4}{h}u_{max}\right)$

$\tau_{yx})_{\frac{h}{2}} = 1.14 \times 10^{-3}Ns/m^2\left(-\frac{4}{0.50mm} \times 0.30m/s\right)$

$\tau_{yx})_{\frac{h}{2}} = 2.74N/m^2$

Direction of shear stress will be positive and in x – direction.

Change in stress is linear because $\tau_{yx} = -\mu\left(\frac{8y}{h^2}u_{max}\right)$

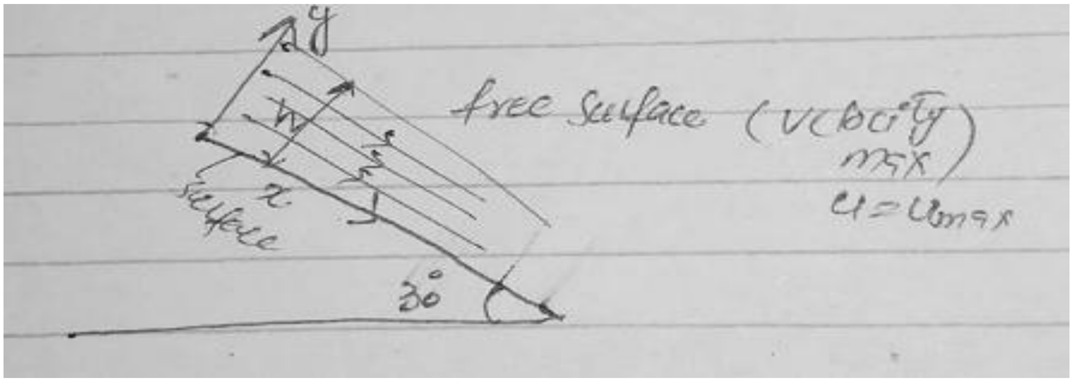


Example

A crude oil with specific gravity $SG = 0.85$ and $\mu = 3.15 \times 10^{-3}lbfsec/ft^2$ flow steadily when a surface inclined at $\theta = 30°$ below the horizontal in the film of thickness $h = 0.125in$.

The velocity profile is given by $u = \frac{\rho g}{\mu}\left(hy - \frac{y^2}{2}\right)Sin\theta$ where the coordinate 'x' along the surface and 'y' is normal to surface. Calculate the magnitude and direction of shear stress that acts on the surface.

Solution



We have to find shear stress on surface y = 0 $\tau_{yx})_{y=0}$ =?

Since we know that $\tau_{yx} = \mu \frac{du}{dx}$

Then $\tau_{yx} = \mu \left[\frac{d}{dx}\left(\frac{\rho g}{\mu}\left(hy - \frac{y^2}{2}\right) Sin\theta\right)\right] = \mu\left(\frac{\rho g}{\mu}(h-y)Sin\theta\right)$

$\tau_{yx})_{y=0} = \rho ghSin\theta = \rho_{substance} ghSin\theta$ …………(i)

Also Specific Gravity = $SG = \frac{\rho_{substance}}{\rho_{H_2O}} = \frac{Soil}{\rho_{H_2O}} = 0.85$

$\Rightarrow Soil = 0.85 \times \rho_{H_2O} = 0.85 \times \frac{1000kg}{m^3} = 0.85 \times 10^3 kg/m^3$

$(i) \Rightarrow \tau_{yx})_{y=0} = 0.85 \times 10^3 kg/m^3 \times 32.2 ft/s^2 \times 0.125 in. \times Sin30°$

$\Rightarrow \tau_{yx})_{y=0} = \frac{0.85\times10^3 slug}{14.5939\left(\frac{ft}{0.3048}\right)^3} \times 32.2 ft/s^2 \times 0.125\left(\frac{ft}{12}\right) \times 0.5$

$\Rightarrow \tau_{yx})_{y=0} = 58.24 \times (0.3048)^5 \frac{slug}{ft^3} \times 32.2 ft/s^2 \times 0.01 ft \times 0.5$

$\Rightarrow \tau_{yx})_{y=0} = 0.265 \frac{slug}{ft^2}$

Direction of Shear Stress is Positive.

Question

The velocity distribution for laminar flow between parallel plates is given by $\frac{u}{u_{max}} = 1 - \left(\frac{2y}{h}\right)^2$ where 'h' is distance separating the plates and origin is placed mid-way between the plates. Consider the flow of water at 15° with maximum speed 0.05*ft/s* and h = 0.1mm. Calculate the force on 1*ft*^{2} section of lower plate.

Solution

We have to find shear stress on surface y = 0 $\tau_{yx} = ?$

Since we know that $\tau_{yx} = \mu \frac{du}{dx}$ and $\frac{u}{u_{max}} = 1 - \left(\frac{2y}{h}\right)^2$

Then $\tau_{yx} = \mu \left[\frac{d}{dx}\left(u_{max}\left(1 - \left(\frac{2y}{h}\right)^2\right)\right)\right] = \mu\left(u_{max}\left(-\frac{8y}{h^2}\right)\right)$

Therefore for lower plate $\tau_{yx})_{y=\frac{h}{2}} = \mu\left(u_{max}\left(-\frac{8\times\frac{h}{2}}{h^2}\right)\right) = -\mu\left(\frac{4u_{max}}{h}\right)$

$$\tau_{yx})_{\frac{h}{2}} = 1.14 \times 10^{-3}\frac{Ns}{m^2}\left(-\frac{4}{0.01mm} \times 0.05ft/s\right)$$

$$\tau_{yx})_{\frac{h}{2}} = 1.14 \times 10^{-3}\frac{0.223lbf \times s}{(3.280ft)^2}\left(-\frac{4}{0.01\times10^{-3}(3.280ft)} \times 0.05\frac{ft}{s}\right)$$

$$\tau_{yx})_{\frac{h}{2}} = -0.01443\frac{lbf}{ft^2}$$

Now $F = PA = \tau_{yx}A = \left(-0.01443\frac{lbf}{ft^2}\right)(1ft^2)$

$$F = -0.01443lbf$$

Surface tension

Whenever a liquid is in contact with other liquids or gases, or in this case a gas/solid surface, an interface develops that acts like a stretched elastic membrane, such phenomenon is known as ***surface tension***.

There are two features to this membrane: the contact angle θ, and the magnitude of the surface tension, σ (N/m or lbf/ft). Both of these depend on the type of liquid and the type of solid surface (or other liquid or gas) with which it shares an interface.

When your car needs waxing: Water droplets tend to appear somewhat flattened out. After waxing, you get a nice "beading" effect. We define a liquid as "wetting" a surface when the contact angle $\theta < 90°$. By this definition, the car's surface was wetted before waxing, and not wetted after. This is an example of effects due to surface tension.

In the car-waxing example, the contact angle changed from being smaller than 90°, to larger than 90°, because, in effect, the waxing changed the nature of the solid surface. Factors that affect the contact angle include the cleanliness of the surface and the purity of the liquid.

Other examples of surface tension effects arise when you are able to place a needle on a water surface and, similarly, when small water insects are able to walk on the surface of the water.

In engineering, probably the most important effect of surface tension is the creation of a curved meniscus that appears in manometers or barometers, leading to a (usually unwanted) capillary rise (or depression).

FLUID STATICS

We defined a fluid as a substance that will continuously deform, or flow, whenever a shear stress is applied to it. It follows that for a fluid at rest the shear stress must be zero. We can conclude that for a static fluid (or one undergoing "rigid-body" motion) only normal stress is present—in other words, pressure. We will study the topic of fluid statics (often called hydrostatics, even though it is not restricted to water) in this chapter.

Although fluid statics problems are the simplest kind of fluid mechanics problems, this is not the only reason we will study them. The pressure generated within a static fluid is an important phenomenon in many practical situations. Using the principles of hydrostatics, we can compute forces on submerged objects, develop instruments for measuring pressures, and deduce properties of the atmosphere and oceans. The principles of hydrostatics also may be used to determine the forces developed by hydraulic systems in applications such as industrial presses or automobile brakes.

In a static, homogeneous fluid, or in a fluid undergoing rigid-body motion, a fluid particle retains its identity for all time, and fluid elements do not deform. We may apply Newton's second law of motion to evaluate the forces acting on the particle.

Static Equilibrium

A fluid body is said to be at rest or in static equilibrium if sum of all components of applied forces acting on it in the direction of arbitrary axis is zero. It is also means that there is no rotation of a fluid body that is sum of moments about that arbitrary axis must also be zero.

Hydrostatics Pressure: Pressure exerted by a fluid at equilibrium at a given point within the fluid, due to the force of gravity.

The Basic Equation Of Fluid Statics/ Pressure Field Equation

According to this equation; A fluid element in static equilibrium under the action of pressure and gravity results a set of equation after applying Newton's second law of motion. Mathematically it can be written as follows;

$$\frac{dp}{dz} = -\rho g = -\gamma \quad \text{Or} \quad dp = -\rho g dz = -\gamma dz$$

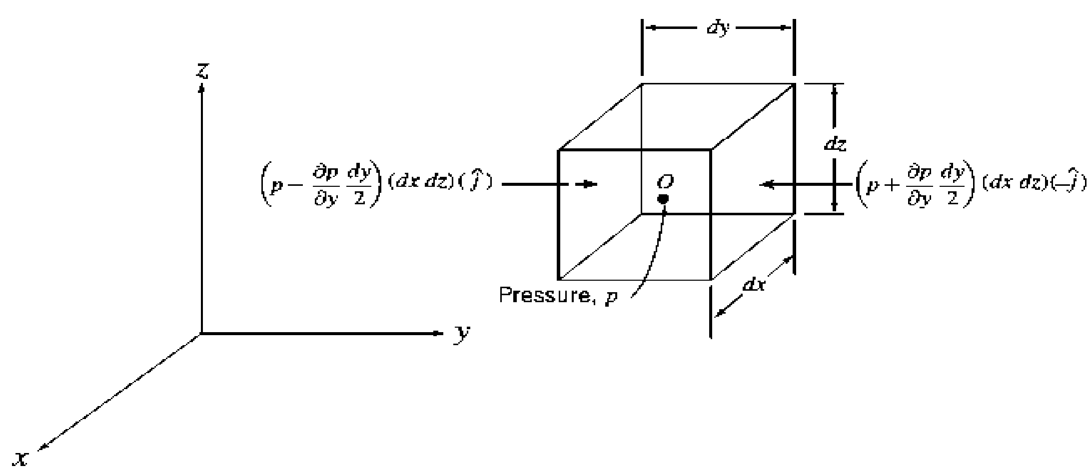
Where γ is the specific weight of the fluid .The **objective of equation** is to obtain an equation for the determination of pressure field within a fluid.

Restrictions

i. Static fluid
ii. Gravity is only body force
iii. z – axis is vertical and upward.

Explanation

Let us consider a fluid element in a large body of a fluid in a static equilibrium having dimensions dx,dy and dz. The fluid element is stationary relative to stationary coordinate axis. Let ‘p’ be the pressure at the center of the element. The pressure at various faces of element can be computed with the help of Taylor’s theorem at point ‘O’. Since area of each face is infinitesimal divisible size. We can assume pressure at the center of face is uniformly distributed. Given figure show differentiable fluid element in x,y and z direction.



Therefore the surface force for each face of the fluid element can be determined by the product of the pressure at center of the face at its area. The unit vector is introduced to indicate the direction. Now we will write different forces as follows;

Pressure at left face of differential element $= p + \frac{\partial p}{\partial y}\left(-\frac{dy}{2}\right) = p - \frac{\partial p}{\partial y}\frac{dy}{2}$

Pressure at right face of differential element $= p + \frac{\partial p}{\partial y}\frac{dy}{2}$

Pressure force at left face $= P_L = \left(p - \frac{\partial p}{\partial y}\frac{dy}{2}\right) dxdz\hat{j}$

Pressure at right face $= P_R = \left(p + \frac{\partial p}{\partial y}\frac{dy}{2}\right) dxdz(-\hat{j})$

$\sum d\vec{F}_S =$ Total Pressure acting at fluid element $= F_x\hat{\imath} + F_y\hat{j} + F_z\hat{k}$

$$\sum d\vec{F}_S = \left[\left(p - \frac{\partial p}{\partial x}\frac{dx}{2}\right) dydz\hat{\imath} + \left(p + \frac{\partial p}{\partial x}\frac{dx}{2}\right) dydz(-\hat{\imath})\right] + \left[\left(p - \frac{\partial p}{\partial y}\frac{dy}{2}\right) dxdz\hat{j} + \left(p + \frac{\partial p}{\partial y}\frac{dy}{2}\right) dxdz(-\hat{j})\right] + \left[\left(p - \frac{\partial p}{\partial z}\frac{dz}{2}\right) dxdy\hat{k} + \left(p + \frac{\partial p}{\partial z}\frac{dz}{2}\right) dydz\left(-\hat{k}\right)\right]$$

$$\sum d\vec{F}_S = -\frac{\partial p}{\partial x} dxdydz\hat{\imath} - \frac{\partial p}{\partial y} dxdydz\hat{j} - \frac{\partial p}{\partial z} dxdydz\hat{k}$$

$$\sum d\vec{F}_S = -\left(\frac{\partial p}{\partial x}\hat{\imath} + \frac{\partial p}{\partial y} + \hat{j} + \frac{\partial p}{\partial z}\hat{k}\right) dxdydz = -\nabla p d\vec{V}$$

$$grad\ p = \nabla p = -\frac{\sum d\vec{F}_S}{d\vec{V}} \qquad \ldots\ldots\ldots\ldots(1)$$

Where ∇p is <u>the rate of change of pressure with distance is called **pressure gradient.**</u>

Physical Significance of Gradient of Pressure/ description of PG.

Physically the gradient of pressure is the negative of the surface force per unit volume due to pressure. Higher the Pressure Gradient, the faster is the fluid flow. For example fresh water has a pressure gradient of 0.433 *ps/ft* which means 0.433 of fluid column acts on 1*ft* of TVG.

Now for body force $d\vec{F}_B = \vec{g}dm = \vec{g}\rho d\vec{V}$

We combine the formulations for surface and body forces that we have developed to obtain the total force acting on a fluid element. Thus

$$d\vec{F} = d\vec{F}_S + d\vec{F}_B = -\nabla p d\vec{V} + \vec{g}\rho d\vec{V} = (-\nabla p + \rho\vec{g})d\vec{V}$$

$$\frac{d\vec{F}}{d\vec{V}} = -\nabla p + \rho\vec{g} \qquad \ldots\ldots\ldots\ldots(2)$$

For a fluid particle, Newton's second law gives $d\vec{F} = \vec{a}dm = \vec{a}\rho d\vec{V}$. Then $\frac{d\vec{F}}{d\vec{V}} = \vec{a}dm$ and For a static fluid, $\vec{a} = 0$. Thus $\frac{d\vec{F}}{d\vec{V}} = 0$ and

$$(2) \Rightarrow -\nabla p + \rho\vec{g} = 0$$

Where $-\nabla p$ is the net pressure per unit volume at a point and $\rho\vec{g}$ is the body force per unit volume at a point.

Above is a vector equation, which means that it is equivalent to three component equations that must be satisfied individually. The component equations are

$-\frac{\partial p}{\partial x} + \rho g_x = 0$ 'x' direction

$-\frac{\partial p}{\partial y} + \rho g_y = 0$ 'y' direction

$-\frac{\partial p}{\partial z} + \rho g_z = 0$ 'z' direction

Above system of Equations describe the pressure variation in each of the three coordinate directions in a static fluid. It is convenient to choose a coordinate system such that the gravity vector is aligned with one of the coordinate axes. If the coordinate system is chosen with the z axis directed vertically upward, then $g_x = g_y = 0$, and $g_z = -g$. Under these conditions, the component equations become $\frac{\partial p}{\partial x} = 0, \frac{\partial p}{\partial y} = 0, \frac{\partial p}{\partial z} = -\rho g$

Implies $dp = -\rho g dz = -\gamma dz$

Where γ is the specific weight of the fluid .The **objective of equation** is to obtain an equation for the determination of pressure field within a fluid.

Incompressible Liquids: Manometers

Hydrostatic Form of an Incompressible ($\boldsymbol{\rho = Constant}$) Fluid

For incompressible fluid $\rho = \rho_0 = Constant$

Since $dp = -\rho g dz = -\rho_0 g dz$ therefore $\int_{p_0}^{p} dp = \int_{z_0}^{z} -\rho_0 g dz$

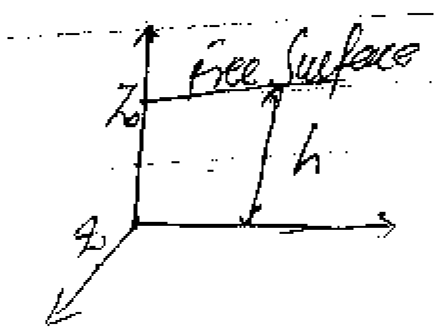
$p - p_0 = \rho_0 g(z_0 - z) = -\rho_0 g h$ where $h = z_0 - z$

$p = p_0 + \rho_0 g h$ where $h = z_0 - z$ then $p - p_0 = \Delta p = \rho_0 g h$

Equation indicates that the pressure difference between two points in a static incompressible fluid can be determined by measuring the elevation difference between the two points. Devices used for this purpose are called manometers.

Special Case: if $z = 0$ is a datum (arbitrary reference point) then

$p - p_0 = -\rho_0 g z_0$ implies $p = p_0 - \rho_0 g z_0$



Remember Since we know that $p - p_0 = \rho_0 g(z_0 - z) = \rho_0 g z_0 - \rho_0 g z$

$p + \rho_0 g z = p_0 + \rho_0 g z_0$

$p_1 + \rho_0 g z_1 = p_2 + \rho_0 g z_2 = \cdots = p_n + \rho_0 g z_n = \cdots =$ Constant

In general $p + \gamma z =$ Constant where $\gamma = \rho g$

$\frac{p}{\gamma} + z =$ Constant $= H$

Where $\frac{p}{\gamma} =$ Pressure energy per unit volume

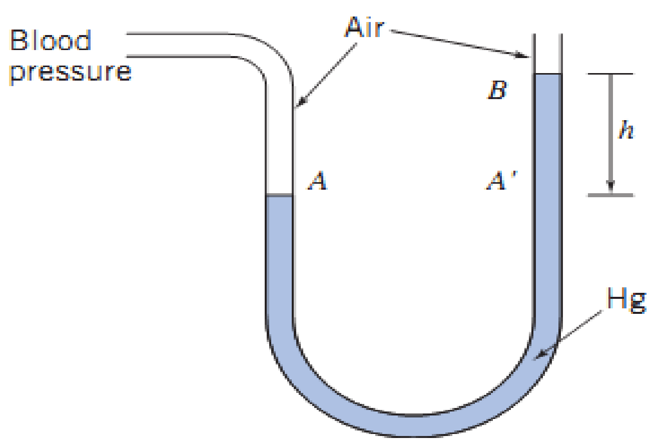
$z =$ Electron Head and $H =$ Static Head

Quantity	Formula	Unit (SI System)	Dimension
Pressure	$P = \frac{F}{A} = \frac{ma}{A}$	$P = \frac{kg \times ms^{-2}}{m^2} = \frac{kg}{ms^2}$	$\left[\frac{M}{Lt^2}\right]$
Specific Weight	$\gamma = \rho g$	$\gamma = kgm^{-3} \times ms^{-2} = kgm^{-2}s^{-2}$	$\left[\frac{M}{L^2t^2}\right]$
Pressure energy per unit volume	$\frac{P}{\gamma}$	$\frac{\frac{kg}{ms^2}}{kgm^{-2}s^{-2}}$	$[L]$

Example **Systolic and Diastolic Pressure**

The normal blood pressure of a human is 120/80 mm Hg. By modeling a sphygmo-manometer pressure gage as a U-tube manometer, convert these pressures to psig.

Solution:



Apply hydrostatic equation to points A, A', and B.

Governing equation:

$$p - p_0 = \Delta p = \rho g h$$

Assumptions: (1) Static fluid.
(2) Incompressible fluids.
(3) Neglect air density (≪ Hg density).

Applying the governing equation between points A' and B (and p_B is atmospheric and therefore zero gage):

$$p_{A'} = p_B + \rho_{Hg} g h = SG_{Hg} \rho_{H_2O} g h$$

In addition, the pressure increases as we go downward from point A' to the bottom of the manometer, and decreases by an equal amount as we return up the left branch to point A. This means points A and A' have the same pressure, so we end up with

$$p_A = p_{A'} = SG_{Hg} \rho_{H_2O} g h$$

Substituting $SG_{Hg} = 13.6$ and $\rho_{H_2O} = 1.94\ \text{slug/ft}^3$ from Appendix A.1 yields for the systolic pressure ($h = 120$ mm Hg)

$$p_{\text{systolic}} = p_A = 13.6 \times 1.94 \frac{\text{slug}}{\text{ft}^3} \times 32.2 \frac{\text{ft}}{\text{s}^2} \times 120\,\text{mm} \times \frac{\text{in.}}{25.4\,\text{mm}} \times \frac{\text{ft}}{12\,\text{in.}} \times \frac{\text{lbf}\cdot\text{s}^2}{\text{slug}\cdot\text{ft}}$$

$$p_{\text{systolic}} = 334\,\text{lbf/ft}^2 = 2.32\,\text{psi} \longleftarrow p_{\text{systolic}}$$

By a similar process, the diastolic pressure ($h = 80$ mm Hg) is

$$p_{\text{diastolic}} = 1.55\,\text{psi} \longleftarrow p_{\text{diastolic}}$$

Notes:

- ✓ Two points at the same level in a continuous single fluid have the same pressure.
- ✓ In manometer problems we neglect change in pressure with depth for a gas: $\rho_{gas} \ll \rho_{liquid}$.
- ✓ This problem shows the conversion from mm Hg to psi, using Eq. 3.7: 120 mm Hg is equivalent to about 2.32 psi. More generally, 1 atm = 14.7 psi = 101 kPa = 760 mm Hg.

Remember

Manometers are simple and inexpensive devices used frequently for pressure measurements. Because the liquid level change is small at low pressure differential, a U-tube manometer may be difficult to read accurately. The **sensitivity of a manometer** is a measure of how sensitive it is compared to a simple water-filled U-tube manometer. Specifically, it is the ratio of the deflection of the manometer to that of a water filled U-tube manometer, due to the same applied pressure difference Δp. Sensitivity can be increased by changing the manometer design or by using two immiscible liquids of slightly different density.

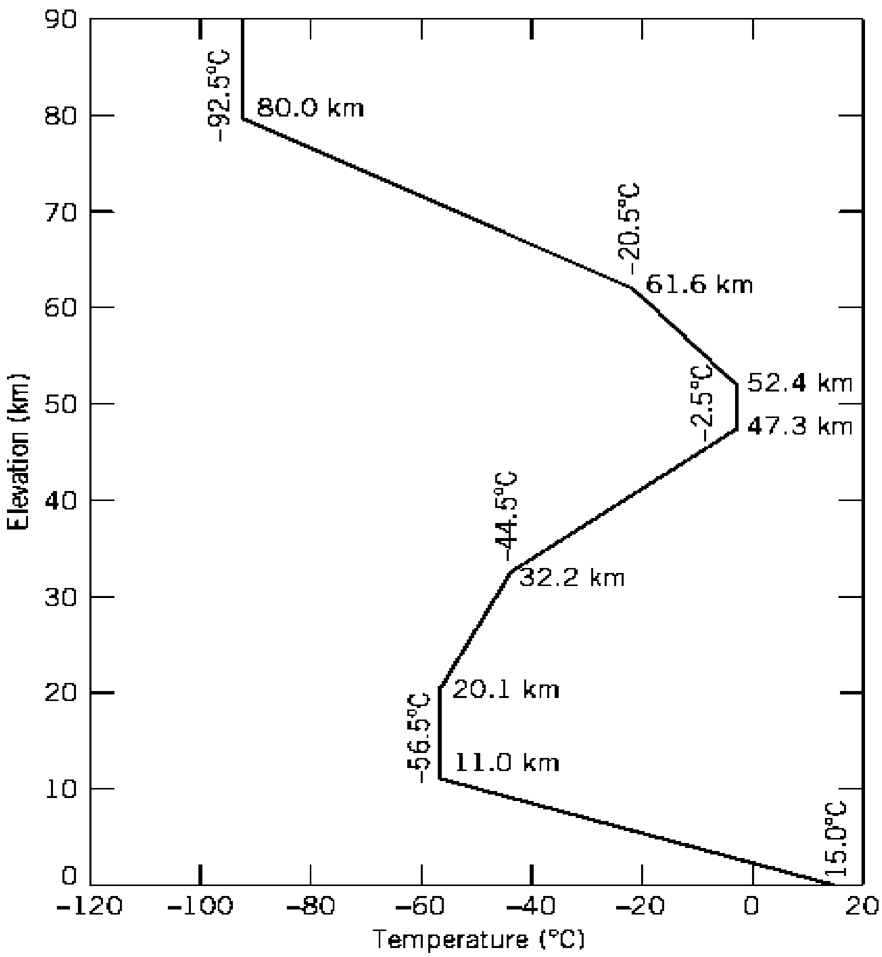
The Standard Atmosphere

Scientists and engineers sometimes need a numerical or analytical model of the

Earth's atmosphere in order to simulate climate variations to study, such type of atmosphere is known as **The Standard Atmosphere** for example, effects of global warming.

There is no single standard model. An International Standard Atmosphere (ISA) has been defined by the International Civil Aviation Organization (ICAO); there is also a similar U.S. Standard Atmosphere.

The temperature profile of the U.S. Standard Atmosphere is shown in Figure.



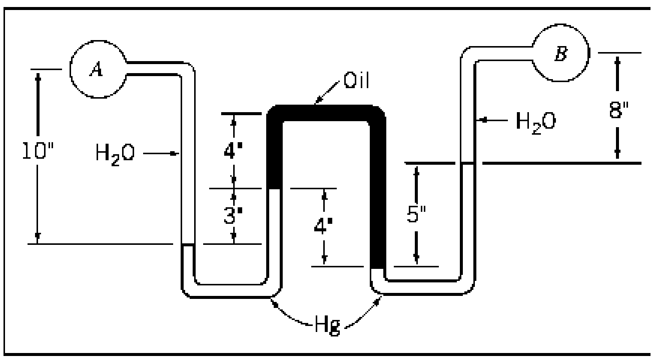
Sea level conditions of the U.S. Standard Atmosphere are summarized in the following Table

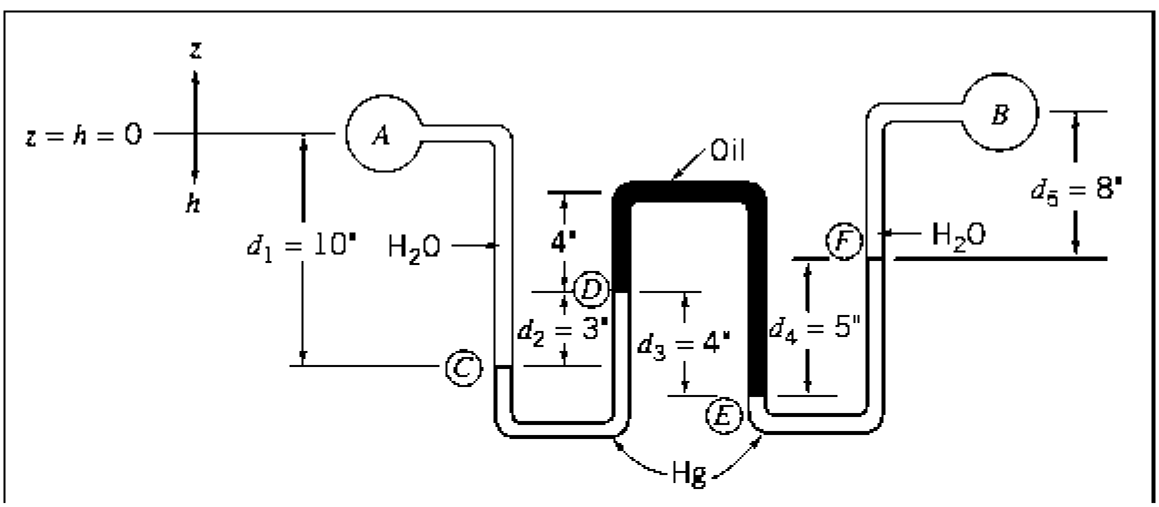
Sea Level Conditions of the U.S. Standard Atmosphere

Property	Symbol	SI	English
Temperature	T	15°C	59°F
Pressure	p	101.3 kPa (abs)	14.696 psia
Density	ρ	1.225 kg/m^3	0.002377 slug/ft^3
Specific weight	γ	—	0.07651 lbf/ft^3
Viscosity	μ	1.789×10^{-5} kg/(m · s) (Pa · s)	3.737×10^{-7} lbf · s/ft^2

Example Multiple-Liquid Manometer

Water flows through pipes A and B. Lubricating oil is in the upper portion of the inverted U. Mercury is in the bottom of the manometer bends. Determine the pressure difference, $p_A - p_B$, in units of lbf/in.^{2}





Solution:

Governing equations: $\Delta p = g\sum_i \rho_i h_i \qquad SG = \dfrac{\rho}{\rho_{H_2O}}$

Assumptions: (1) Static fluid.
(2) Incompressible fluid.

Applying the governing equation, working from point B to A

$$p_A - p_B = \Delta p = g(\rho_{H_2O}d_5 + \rho_{Hg}d_4 - \rho_{oil}d_3 + \rho_{Hg}d_2 - \rho_{H_2O}d_1) \tag{1}$$

This equation can also be derived by repeatedly using Eq. 3.7 in the following form:

$$p_2 - p_1 = \rho g(h_2 - h_1)$$

Beginning at point A and applying the equation between successive points along the manometer gives

$$p_C - p_A = +\rho_{H_2O}gd_1$$
$$p_D - p_C = -\rho_{Hg}gd_2$$
$$p_E - p_D = +\rho_{oil}gd_3$$
$$p_F - p_E = -\rho_{Hg}gd_4$$
$$p_B - p_F = -\rho_{H_2O}gd_5$$

Multiplying each equation by minus one and adding, we obtain Eq. (1)

$$\begin{aligned} p_A - p_B &= (p_A - p_C) + (p_C - p_D) + (p_D - p_E) + (p_E - p_F) + (p_F - p_B) \\ &= -\rho_{H_2O}gd_1 + \rho_{Hg}gd_2 - \rho_{oil}gd_3 + \rho_{Hg}gd_4 + \rho_{H_2O}gd_5 \end{aligned}$$

Substituting $\rho = SG\rho_{H_2O}$ with $SG_{Hg} = 13.6$ and $SG_{oil} = 0.88$ (Table A.2), yields

$$\begin{aligned} p_A - p_B &= g(-\rho_{H_2O}d_1 + 13.6\rho_{H_2O}d_2 - 0.88\rho_{H_2O}d_3 + 13.6\rho_{H_2O}d_4 + \rho_{H_2O}d_5) \\ &= g\rho_{H_2O}(-d_1 + 13.6d_2 - 0.88d_3 + 13.6d_4 + d_5) \end{aligned}$$

$$p_A - p_B = g\rho_{H_2O}(-10 + 40.8 - 3.52 + 68 + 8)\text{ in.}$$

$$p_A - p_B = g\rho_{H_2O} \times 103.3\text{ in.}$$

$$= 32.2\frac{\text{ft}}{\text{s}^2} \times 1.94\frac{\text{slug}}{\text{ft}^3} \times 103.3\text{ in.} \times \frac{\text{ft}}{12\text{ in.}} \times \frac{\text{ft}^2}{144\text{ in.}^2} \times \frac{\text{lbf}\cdot\text{s}^2}{\text{slug}\cdot\text{ft}}$$

This Example shows use of both Eq. 3.7 and Eq. 3.8. Use of either equation is a matter of personal preference.

$$p_A - p_B = 3.73\text{ lbf/in.}^2 \longleftarrow \quad p_A - p_B$$

Gases

In many practical engineering problems density will vary appreciably with altitude, and accurate results will require that this variation be accounted for.

The density of gases generally depends on pressure and temperature. The ideal gas equation of state, $p = \rho RT \Rightarrow \rho = \frac{p}{RT}$ ……………(i)

Where R is the gas constant and T the absolute temperature.

In the U.S. Standard Atmosphere the temperature decreases linearly with altitude up to an elevation of 11.0 km. For a linear temperature variation with altitude given by $T = T_0 - mz$.

As we know that $dp = -\rho g dz \Rightarrow dp = -\frac{p}{RT} g dz$ using (i)

$$\Rightarrow dp = -\frac{p}{R(T_0 - mz)} g dz \Rightarrow \frac{dp}{p} = -\frac{g dz}{R(T_0 - mz)} \Rightarrow \int_{p_0}^{p} \frac{dp}{p} = -\frac{g}{R} \int_0^z \frac{dz}{(T_0 - mz)}$$

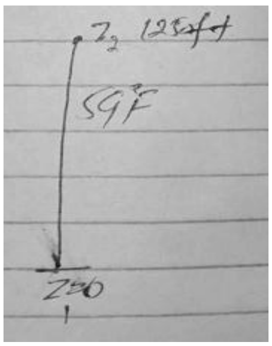
$$\Rightarrow ln|p|_{p_0}^{p} = \frac{g}{mR} ln\left|T_0 - mz\right|_0^z \Rightarrow ln\left(\frac{p}{p_0}\right) = \frac{g}{mR} ln\left(\frac{T_0 - mz}{T_0}\right) = ln\left(\frac{T_0 - mz}{T_0}\right)^{\frac{g}{mR}}$$

$$\Rightarrow \frac{p}{p_0} = \left(\frac{T_0 - mz}{T_0}\right)^{\frac{g}{mR}} \Rightarrow \frac{p}{p_0} = \left(\frac{T}{T_0}\right)^{\frac{g}{mR}} \Rightarrow \boldsymbol{p = p_0 \left(\frac{T}{T_0}\right)^{\frac{g}{mR}}}$$

Example

The Empire State Building in New York city, one of the tallest city in the world, raises the height of approximately 1250*ft*. Estimate ratio of the pressure at its base assuming the air to be at the common temperature 59°F. Compare this result with that obtained assuming the air to be incompressible with γ = 0.0765*lbf/ft*^{3} at 14.7*lbf/m*^{2} (Value of air at standard condition)

Solution



Consider ideal gas equation, $p \;=\; \rho RT \Rightarrow \rho = \frac{p}{RT}$

Where R is the gas constant and T the absolute temperature.

Using basic hydrostatic equation $\frac{dp}{dz} = -\rho g$ where ρ is variable.

$\Rightarrow \frac{dp}{dz} = -\frac{p}{RT}g$ using (i)

$\Rightarrow \frac{dp}{p} = -\frac{gdz}{RT} \Rightarrow \int_{p_1}^{p_2}\frac{dp}{p} = -\frac{g}{RT}\int_{z_1}^{z_2} dz$

Where g and R are assumed to be constant over the elevation change from z_1 to z_2. Since the temperature has constant value T_0 over the range from z_1 to z_2 (isothermal condition) then we have;

$\Rightarrow ln|p|_{p_1}^{p_2} = -\frac{g}{RT}|z|_{z_1}^{z_2} \Rightarrow ln\left(\frac{p_2}{p_1}\right) = -\frac{g}{RT}(z_2 - z_1) = -\frac{g}{RT_0}(z_2 - z_1)$

$\Rightarrow \frac{p_2}{p_1} = e^{-\frac{g}{RT_0}(z_2 - z_1)} \Rightarrow \frac{p_2}{p_1} = e^{-\frac{g}{RT_0}(z_2 - z_1)}$

Where $R = \frac{53.3lbf.ft}{lbm°R} = \frac{1716lbf.ft}{Slug°R}$ since $1Slug = 32.2lbm$

Also $T_0 = (°\mathrm{F} + 460) = (59 + 460)°R$

$$\Rightarrow \frac{p_2}{p_1} = e^{-\left[\frac{32.2 ft/sec^2}{\frac{1716 lbf.ft}{Slug°R}.(59+460)°R}\right](1250-0)ft} = e^{-\left[\frac{32.2 ft/sec^2}{\frac{1716 lbf.ft}{lbf.sec^2/ft.°R}.(59+460)°R}\right](1250-0)ft}$$

$$\Rightarrow \frac{p_2}{p_1} = 0.956$$

Consider air to be incompressible

$$\int_{p_1}^{p_2} dp = -\gamma \int_{z_1}^{z_2} dz$$

$$\Rightarrow |p|_{p_1}^{p_2} = -\gamma\ |z|_{z_1}^{z_2} \Rightarrow p_2 - p_1 = -\gamma\ (z_2 - z_1) \Rightarrow p_2 = p_1 - \gamma\ (z_2 - z_1)$$

$$\Rightarrow \frac{p_2}{p_1} = 1 - \frac{\gamma\ (z_2 - z_1)}{p_1} = 1 - \frac{\frac{0.0765 lbf}{ft^3} \times (1250 ft)}{\frac{14.7 lbf}{m^2}} = 0.095$$

Note that there is a little difference between two results, since pressure difference of top and bottom building is small. It follows that variation in fluid density is small and therefore compressible and incompressible analysis yields essentially the same result.

Example **Pressure and Density Variation in the Atmosphere**

The maximum power output capability of an internal combustion engine decreases with altitude because the air density and hence the mass flow rate of air decrease. A truck leaves Denver (elevation 5280 *ft*) on a day when the local temperature and barometric pressure are 80°F and 24.8 in. of mercury, respectively. It travels through Vail Pass (elevation 10,600 *ft*), where the temperature is 62°F. Determine the local barometric pressure at Vail Pass and the percent change in density.

Given: Truck travels from Denver to Vail Pass.

Denver: $z = 5280$ ft, $p = 24.8$ in. Hg, $T = 80°$F

Vail Pass: $z = 10{,}600$ ft, $T = 62°$F

Find: Atmospheric pressure at Vail Pass.
Percent change in air density between Denver and Vail.

Solution:

Governing equations: $\dfrac{dp}{dz} = -\rho g \qquad p = \rho RT$

Assumptions: (1) Static fluid.
(2) Air behaves as an ideal gas.

We shall consider four assumptions for property variations with altitude.

(a) If we assume temperature varies linearly with altitude, Eq. 3.9 gives

$$\frac{p}{p_0} = \left(\frac{T}{T_0}\right)^{g/mR}$$

Evaluating the constant m gives

$$m = \frac{T_0 - T}{z - z_0} = \frac{(80 - 62)°\text{F}}{(10.6 - 5.28)10^3\ \text{ft}} = 3.38 \times 10^{-3}\ °\text{F/ft}$$

and

$$\frac{g}{mR} = 32.2\frac{\text{ft}}{\text{s}^2} \times \frac{\text{ft}}{3.38 \times 10^{-3}\ °\text{F}} \times \frac{\text{lbm} \cdot °\text{R}}{53.3\ \text{ft} \cdot \text{lbf}} \times \frac{\text{slug}}{32.2\ \text{lbm}} \times \frac{\text{lbf} \cdot \text{s}^2}{\text{slug} \cdot \text{ft}} = 5.55$$

Thus

$$\frac{p}{p_0} = \left(\frac{T}{T_0}\right)^{g/mR} = \left(\frac{460 + 62}{460 + 80}\right)^{5.55} = (0.967)^{5.55} = 0.830$$

and

$$p = 0.830 p_0 = (0.830)24.8\ \text{in. Hg} = 20.6\ \text{in. Hg} \qquad p$$

Note that temperature must be expressed as an absolute temperature in the ideal gas equation of state.
The percent change in density is given by

$$\frac{\rho-\rho_0}{\rho_0}=\frac{\rho}{\rho_0}-1=\frac{p}{p_0}\frac{T_0}{T}-1=\frac{0.830}{0.967}-1=-0.142 \quad \text{or} \quad -14.2\% \longleftarrow \frac{\Delta\rho}{\rho_0}$$

(b) For ρ assumed constant ($=\rho_0$),

$$p=p_0-\rho_0 g(z-z_0)=p_0-\frac{p_0 g(z-z_0)}{RT_0}=p_0\left[1-\frac{g(z-z_0)}{RT_0}\right]$$

$$p=20.2\text{ in. Hg} \quad \text{and} \quad \frac{\Delta\rho}{\rho_0}=0 \longleftarrow p,\frac{\Delta\rho}{\rho_0}$$

(c) If we assume the temperature is constant, then

$$dp=-\rho g\,dz=-\frac{p}{RT}g\,dz$$

and

$$\int_{p_0}^{p}\frac{dp}{p}=-\int_{z_0}^{z}\frac{g}{RT}dz$$

$$p=p_0\exp\left[\frac{-g(z-z_0)}{RT}\right]$$

For $T=\text{constant}=T_0$,

$$p=20.6\text{ in. Hg} \qquad \text{and} \qquad \frac{\Delta\rho}{\rho_0}=-16.9\% \longleftarrow p,\frac{\Delta\rho}{\rho_0}$$

(d) For an adiabatic atmosphere $p/\rho^k=\text{constant}$,

$$p=p_0\left(\frac{T}{T_0}\right)^{k/k-1}=22.0\text{ in. Hg} \quad \text{and} \quad \frac{\Delta\rho}{\rho_0}=-8.2\% \longleftarrow p,\frac{\Delta\rho}{\rho_0}$$

We note that over the modest change in elevation the predicted pressure is not strongly dependent on the assumed property variation; values calculated under four different assumptions vary by a maximum of approximately 9 percent. There is considerably greater variation in the predicted percent change in density. The assumption of a linear temperature variation with altitude is the most reasonable assumption.

This Example shows use of the ideal gas equation with the basic pressure-height relation to obtain the change in pressure with height in the atmosphere under various atmospheric assumptions.

BASIC EQUATIONS IN INTEGRAL FORM FOR A CONTROL VOLUME

We will examine the control volume approach in this chapter. The agenda for this chapter is to review the physical laws as they apply to a system; develop some math to convert from a system to a control volume description; and obtain formulas for the physical laws for control volume analysis.

Basic laws for a system

The basic laws we will apply are conservation of mass, Newton's second law, the angular-momentum principle, and the first and second laws of thermodynamics. For converting these system equations to equivalent control volume formulas, it turns out we want to express each of the laws as a rate equation.

Newton's Second Law

For a system moving relative to an inertial reference frame, Newton's second law states that *the sum of all external forces acting on the system is equal to the time rate of change of linear momentum of the system,* $\vec{F} = \frac{d\vec{P}}{dt})_{system}$

Where the linear momentum of the system is given by

$$\vec{P})_{system} = \int_{(M)_{system}} \vec{V} dm = \int_{(V)_{system}} \vec{V} \rho dV$$

The Angular-Momentum Principle / Angular-Momentum of fluid flow

The angular-momentum of fluid remains constant when the net torque acting on it is zero. The angular-momentum principle for a system states that *the rate of change of angular momentum is equal to the sum of all torques acting on the system,* $\vec{\tau} = \frac{d\vec{H}}{dt})_{system}$

Where the angular momentum of the system is given by

$$\vec{H})_{system} = \int_{(M)_{system}} \vec{r} \times \vec{V} dm = \int_{(V)_{system}} \vec{r} \times \vec{V} \rho dV$$

Torque can be produced by surface and body forces, and also by shafts that cross the system boundary, $\vec{\tau} = \vec{r} \times \vec{F}_s + \int_{(M)_{system}} \vec{r} \times \vec{g} dm + \vec{\tau}_{shaft}$

The First Law of Thermodynamics

The first law of thermodynamics is a statement of conservation of energy for a system, $\delta Q - \delta W = dE$

The equation can be written in rate form as $\dot{Q} - \dot{W} = \frac{dE}{dt})_{system}$

Where the total energy (entropy) of the system is given by

$E)_{system} = \int_{(M)_{system}} e dm = \int_{(V)_{system}} e\rho dV$ And $e = u + \frac{V^2}{2} + gz$

In $\dot{Q} - \dot{W} = \frac{dE}{dt})_{system}$, $\dot{Q}$ (the rate of heat transfer) is positive when heat is added to the system from the surroundings; $\dot{W}$ (the rate of work) is positive when work is done by the system on its surroundings. In $e = u + \frac{V^2}{2} + gz$, u is the specific internal energy, V the speed, and z the height (relative to a convenient datum) of a particle of substance having mass *dm*.

Example: The increase in the energy of a potato in an oven is equals to the amount of heat transferred to it.

The Second Law of Thermodynamics

If an amount of heat, δQ, is transferred to a system at temperature T, the second law of thermodynamics states that the change in entropy, *dS*, of the system satisfies

$$dS \geq \frac{\delta Q}{T}$$

On a rate basis we can write $\frac{dS}{dt})_{system} \geq \frac{1}{T}\dot{Q}$

Where the total energy (entropy) of the system is given by

$S)_{system} = \int_{(M)_{system}} s dm = \int_{(V)_{system}} s\rho dV$

Example: A cold object in contact with a hot one never gets colder, transferring heat to the hot object and making it hotter.

Relation of System Derivatives to the Control Volume Formulation

Intensive and Extensive Properties

Intensive property is a physical property of a system that does not depends on the system size or amount of the material in the system. e.g. Hardness. It is represented by η.

While **Extensive property** is a physical property of a system that depends on the system size or amount of the material in the system. e.g. Mass and Volume. It is represented by N.

And their equation is $N)_{system} = \int_{(M)_{system}} \eta dm = \int_{(V)_{system}} \eta \rho dV$

The basic equation for relation of system derivative to control volume is

$\frac{dN}{dt})_{system} = \frac{\partial}{\partial t}\int_{CV} \eta \rho dV + \int_{CS} \eta \rho \vec{V}.d\vec{A}$

It is the fundamental relation between the rate of change of any arbitrary extensive property, N, of a system and the variations of this property associated with a control volume. Some authors refer to above equation as **the Reynolds Transport Theorem**. Where

$\left.\frac{dN}{dt}\right)_{\text{system}}$	is the rate of change of the system extensive property N. For example, if $N = \vec{P}$, we obtain the rate of change of momentum.
$\frac{\partial}{\partial t}\int_{CV} \eta \rho \, d\forall$	is the rate of change of the amount of property N in the control volume. The term $\int_{CV} \eta \rho \, d\forall$ computes the instantaneous value of N in the control volume ($\int_{CV} \rho \, d\forall$ is the instantaneous mass in the control volume). For example, if $N = \vec{P}$, then $\eta = \vec{V}$ and $\int_{CV} \vec{V} \rho d\forall$ computes the instantaneous amount of momentum in the control volume.
$\int_{CS} \eta \rho \vec{V} \cdot d\vec{A}$	is the rate at which property N is exiting the surface of the control volume. The term $\rho \vec{V} \cdot d\vec{A}$ computes the rate of mass transfer leaving across control surface area element $d\vec{A}$; multiplying by η computes the rate of flux of property N across the element; and integrating therefore computes the net flux of N out of the control volume. For example, if $N = \vec{P}$, then $\eta = \vec{V}$ and $\int_{CS} \vec{V} \rho \vec{V} \cdot d\vec{A}$ computes the net flux of momentum out of the control volume.

Conservation of Mass (Continuity Equation)

The first physical principle to which we apply this conversion from a system to a control volume description is the mass conservation principle:

According to this law; For a system Mass is constant. i.e. Fluid mass can neither created nor destroyed."

$\frac{dM}{dt})_{system} = 0$ …………(A)

Where $M)_{system} = \int_{(M)_{system}} dm = \int_{CV} \rho dV$ …………(1)

Now using $\frac{dN}{dt})_{system} = \frac{\partial}{\partial t}\int_{CV} \eta\rho dV + \int_{CS} \eta\rho\vec{V}.d\vec{A}$ …………(2)

Where $N)_{system} = \int_{(M)_{system}} \eta dm = \int_{(V)_{system}} \eta\rho dV$ …………(3)

To derive the control volume formulation of conservation of mass, we set from (1) & (3) $N = M$ and $\eta = 1$

With this substitution, we obtain $\frac{dM}{dt})_{system} = \frac{\partial}{\partial t}\int_{CV} \rho dV + \int_{CS} \rho\vec{V}.d\vec{A}$

From (A), we arrive (after rearranging) at the control volume formulation of the conservation of mass:

$\boldsymbol{\frac{\partial}{\partial t}\int_{CV} \rho dV + \int_{CS} \rho\vec{V}.d\vec{A} = 0}$ **Integral Form**

In this equation the first term represents the rate of change of mass within the control volume; the second term represents the net rate of mass flux out through the control surface. Equation 4.12 indicates that the rate of change of mass in the control volume plus the net outflow is zero. ***The mass conservation equation is also called the continuity equation.*** In common-sense terms, the rate of increase of mass in the control volume is due to the net inflow of mass.

Remember: In a continuous motion the equation of continuity expresses the fact that increase in the mass of fluid with any closed surface drawn in the fluid at any time must be equal to the access of the mass that flows 'in' over the mass of that flows 'out'. Inward flow is equal to outward flow.

Special Cases

Case – I: Consider first the case of an incompressible fluid, in which density ρ remains constant. When ρ is constant, it is not a function of space or time. Consequently, for incompressible fluids, we may write

$$\rho\frac{\partial}{\partial t}\int_{CV} dV + \rho\int_{CS}\vec{V}.d\vec{A} = 0$$

The integral of dV over the control volume is simply the volume of the control volume. Thus, on dividing through by ρ, we write $\frac{\partial}{\partial t}\int_{CV} dV + \int_{CS}\vec{V}.d\vec{A} = 0$

For a non-deformable control volume of fixed size and shape, V = constant. Then $\frac{\partial}{\partial t}\int_{CV} dV = 0$ and the conservation of mass for incompressible flow through a fixed control volume becomes $\int_{CS}\vec{V}.d\vec{A} = 0$

Where $\vec{V}.d\vec{A}$ is called volume flow rate or volume rate of flow. Then volume rate Q through a section of an area A is given by $Q = \int_{A}\vec{V}.d\vec{A} = 0$ then average velocity could be written as $\vec{V} = \frac{Q}{A} = \frac{\int_A\vec{V}.d\vec{A}}{A}$ = Volume flow rate per unit area.

Case – II: A useful special case is when we have (or can approximate) uniform velocity at each inlet and exit. In this case $\int_{CS}\vec{V}.d\vec{A} = 0$ simplifies to

$$\sum_{CS}\vec{V}.\vec{A} = 0$$

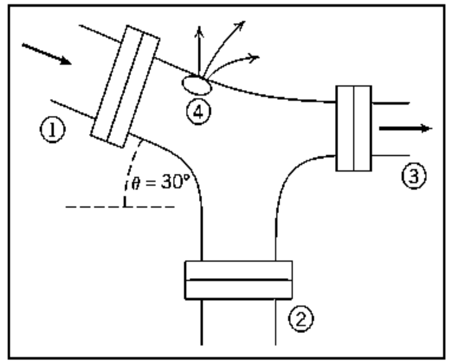
General Case: Consider now the general case of steady, compressible flow through a fixed control volume. Since the flow is steady, this means that at most $\rho = \rho(x, y, z)$. By definition, no fluid property varies with time in a steady flow. Consequently, $\frac{\partial}{\partial t}\int_{CV}\rho dV = 0$ and, hence, for steady flow, the statement of conservation of mass reduces to $\int_{CS}\rho\vec{V}.d\vec{A} = 0$

A useful special case is when we have (or can approximate) uniform velocity at each inlet and exit. In this case, $\int_{CS}\rho\vec{V}.d\vec{A} = 0$ simplifies to $\sum_{CS}\rho\vec{V}.\vec{A} = 0$

Thus, for steady flow, the mass flow rate into a control volume must be equal to the mass flow rate out of the control volume.

Example 4.7 MASS FLOW AT A PIPE JUNCTION

Consider the steady flow in a water pipe joint shown in the diagram. The areas are: $A_1 = 0.2\ \mathrm{m}^2$, $A_2 = 0.2\ \mathrm{m}^2$, and $A_3 = 0.15\ \mathrm{m}^2$. In addition, fluid is lost out of a hole at ④, estimated at a rate of $0.1\ \mathrm{m}^3/\mathrm{s}$. The average speeds at sections ① and ③ are $V_1 = 5$ m/s and $V_3 = 12$ m/s, respectively. Find the velocity at section ②.



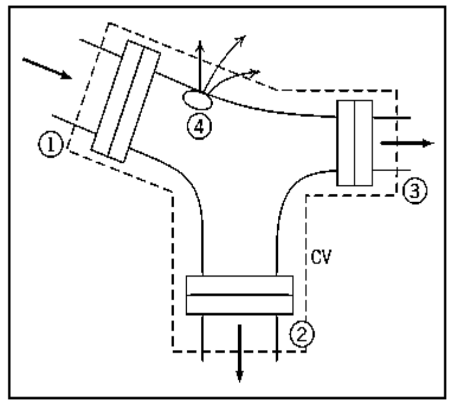
Given: Steady flow of water through the device.

$A_1 = 0.2\,\mathrm{m}^2 \quad A_2 = 0.2\,\mathrm{m}^2 \quad A_3 = 0.15\,\mathrm{m}^2$
$V_1 = 5\,\mathrm{m/s} \quad V_3 = 12\,\mathrm{m/s} \quad \rho = 999\,\mathrm{kg/m}^3$

Volume flow rate at ④ $= 0.1\ \mathrm{m}^3/\mathrm{s}$

Find: Velocity at section ②.

Solution: Choose a fixed control volume as shown. Make an assumption that the flow at section ② is outwards, and label the diagram accordingly (if this assumption is incorrect our final result will tell us).



Governing equation: The general control volume equation is Eq. 4.12, but we can go immediately to Eq. 4.13b because of assumptions (2) and (3) below,

$$\sum_{\mathrm{CS}} \vec{V} \cdot \vec{A} = 0$$

Assumptions:
(1) Steady flow (given).
(2) Incompressible flow.
(3) Uniform properties at each section.

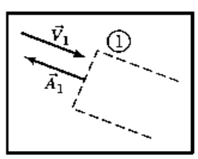
Hence (using Eq. 4.14a for the leak)

$$\vec{V}_1 \cdot \vec{A}_1 + \vec{V}_2 \cdot \vec{A}_2 + \vec{V}_3 \cdot \vec{A}_3 + Q_4 = 0 \tag{1}$$

where Q_4 is the flow rate out of the leak.

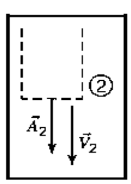
Let us examine the first three terms in Eq. 1 in light of the discussion of Fig. 4.3 and the directions of the velocity vectors:

$\vec{V}_1 \cdot \vec{A}_1 = -V_1 A_1$



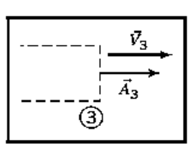
$\left\{ \begin{array}{l} \text{Sign of } \vec{V}_1 \cdot \vec{A}_1 \text{ is} \\ \text{negative at surface ①} \end{array} \right\}$

$\vec{V}_2 \cdot \vec{A}_2 = +V_2 A_2$



$\left\{ \begin{array}{l} \text{Sign of } \vec{V}_2 \cdot \vec{A}_2 \text{ is} \\ \text{positive at surface ②} \end{array} \right\}$

$\vec{V}_3 \cdot \vec{A}_3 = +V_3 A_3$



$\left\{ \begin{array}{l} \text{Sign of } \vec{V}_3 \cdot \vec{A}_3 \text{ is} \\ \text{positive at surface ③} \end{array} \right\}$

Using these results in Eq. 1,

$$-V_1A_1 + V_2A_2 + V_3A_3 + Q_4 = 0$$

or

$$V_2 = \frac{V_1A_1 - V_3A_3 - Q_4}{A_2}$$

$$= \frac{5\frac{\text{m}}{\text{s}} \times 0.2\,\text{m}^2 - 12\frac{\text{m}}{\text{s}} \times 0.15\,\text{m}^2 - \frac{0.1\,\text{m}^3}{\text{s}}}{0.2\,\text{m}^2}$$

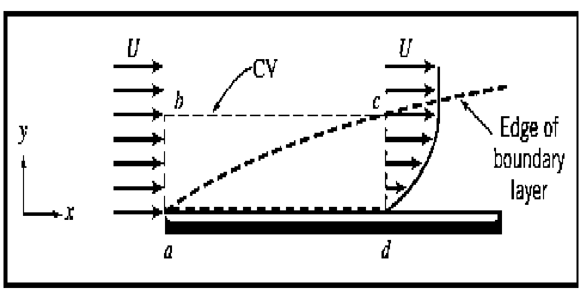
$$= -4.5\,\text{m/s} \longleftarrow V_2$$

Recall that V_2 represents the magnitude of the velocity, which we assumed was outwards from the control volume. The fact that V_2 is negative means that in fact we have an *inflow* at location ②—our initial assumption was invalid.

This problem demonstrates use of the sign convention for evaluating $\int_A \vec{V} \cdot d\vec{A}$ or $\Sigma_{CS} \vec{V} \cdot \vec{A}$. In particular, the area normal is *always* drawn *outwards* from the control surface.

Example 4.2 MASS FLOW RATE IN BOUNDARY LAYER

The fluid in direct contact with a stationary solid boundary has zero velocity; there is no slip at the boundary. Thus the flow over a flat plate adheres to the plate surface and forms a boundary layer, as depicted below. The flow ahead of the plate is uniform with velocity $\vec{V} = U\hat{i}$; $U = 30$ m/s. The velocity distribution within the boundary layer $(0 \le y \le \delta)$ along *cd* is approximated as $u/U = 2(y/\delta) - (y/\delta)^2$.



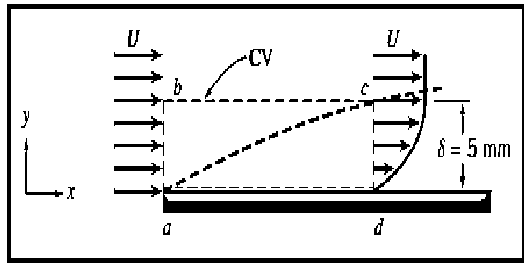
The boundary-layer thickness at location *d* is $\delta = 5$ mm. The fluid is air with density $\rho = 1.24\ \text{kg/m}^3$. Assuming the plate width perpendicular to the paper to be $w = 0.6$ m, calculate the mass flow rate across surface *bc* of control volume *abcd*.

Given: Steady, incompressible flow over a flat plate, $\rho = 1.24$ kg/m^3. Width of plate, $w = 0.6$ m.
Velocity ahead of plate is uniform: $\vec{V} = U\hat{i}$, $U = 30$ m/s.

At $x = x_d$:

$\delta = 5$ mm

$$\frac{u}{U} = 2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2$$



Find: Mass flow rate across surface *bc*.

Solution: The fixed control volume is shown by the dashed lines.

Governing equation: The general control volume equation is Eq. 4.12, but we can go immediately to Eq. 4.15a because of assumption (1) below,

$$\int_{CS} \rho \vec{V} \cdot d\vec{A} = 0$$

Assumptions: (1) Steady flow (given).
(2) Incompressible flow (given).
(3) Two-dimensional flow, given properties are independent of z.

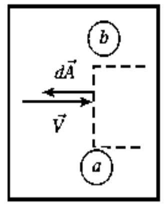
Assuming that there is no flow in the z direction, then

$$\int_{A_{ab}} \rho \vec{V} \cdot d\vec{A} + \int_{A_{bc}} \rho \vec{V} \cdot d\vec{A} + \int_{A_{cd}} \rho \vec{V} \cdot d\vec{A} + \underbrace{\int_{A_{da}} \rho \vec{V} \cdot d\vec{A}}_{\begin{pmatrix}\text{no flow}\\ \text{across } da\end{pmatrix}} = 0$$

$$\therefore \dot{m}_{bc} = \int_{A_{bc}} \rho \vec{V} \cdot d\vec{A} = -\int_{A_{ab}} \rho \vec{V} \cdot d\vec{A} - \int_{A_{cd}} \rho \vec{V} \cdot d\vec{A} \qquad (1)$$

We need to evaluate the integrals on the right side of the equation.

For depth w in the z direction, we obtain



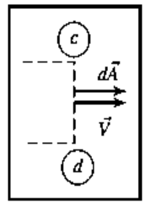
$$\int_{A_{ab}} \rho\vec{V}\cdot d\vec{A} = -\int_{A_{ab}} \rho u\, dA = -\int_{y_a}^{y_b} \rho u w\, dy$$

$$= -\int_0^{\delta} \rho u w\, dy = -\int_0^{\delta} \rho U w\, dy$$

$\left\{ \begin{array}{c} \vec{V}\cdot d\vec{A} \text{ is negative} \\ dA = w dy \end{array} \right\}$

$\{u = U \text{ over area } ab\}$

$$\int_{A_{ab}} \rho\vec{V}\cdot d\vec{A} = -\left[\rho U w y\right]_0^{\delta} = -\rho U w \delta$$



$\left\{ \begin{array}{c} \vec{V}\cdot d\vec{A} \text{ is positive} \\ dA = w dy \end{array} \right\}$

$$\int_{A_{cd}} \rho\vec{V}\cdot d\vec{A} = \int_{A_{cd}} \rho u\, dA = \int_{y_d}^{y_c} \rho u w\, dy$$

$$= \int_0^{\delta} \rho u w\, dy = \int_0^{\delta} \rho w U\left[2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2\right] dy$$

$$\int_{A_{cd}} \rho\vec{V}\cdot d\vec{A} = \rho w U\left[\frac{y^2}{\delta} - \frac{y^3}{3\delta^2}\right]_0^{\delta} = \rho w U \delta\left[1 - \frac{1}{3}\right] = \frac{2\rho U w \delta}{3}$$

Substituting into Eq. 1, we obtain

$$\therefore \dot{m}_{bc} = \rho U w \delta - \frac{2\rho U w \delta}{3} = \frac{\rho U w \delta}{3}$$

$$= \frac{1}{3} \times 1.24\frac{\text{kg}}{\text{m}^3} \times 30\frac{\text{m}}{\text{s}} \times 0.6\,\text{m} \times 5\,\text{mm} \times \frac{\text{m}}{1000\,\text{mm}}$$

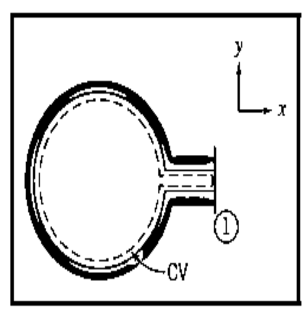
$$\dot{m}_{bc} = 0.0372\,\text{kg/s} \longleftarrow \dot{m}_b$$

$\left\{ \begin{array}{l} \text{Positive sign indicates flow} \\ \text{out across surface } bc. \end{array} \right\}$

This problem demonstrates use of the conservation of mass equation when we have nonuniform flow at a section.

Example 4.3 DENSITY CHANGE IN VENTING TANK

A tank of 0.05 m^3 volume contains air at 800 kPa (absolute) and 15°C. At $t = 0$, air begins escaping from the tank through a valve with a flow area of 65 mm^2. The air passing through the valve has a speed of 300 m/s and a density of 6 kg/m^3. Determine the instantaneous rate of change of density in the tank at $t = 0$.



Given: Tank of volume $\forall = 0.05\,\text{m}^3$ contains air at $p = 800$ kPa (absolute), $T =$ 15°C. At $t = 0$, air escapes through a valve. Air leaves with speed $V = 300$ m/s and density $\rho = 6$ kg/m^3 through area $A = 65$ mm^2.

Find: Rate of change of air density in the tank at $t = 0$.

Solution: Choose a fixed control volume as shown by the dashed line.

Governing equation: $\dfrac{\partial}{\partial t}\displaystyle\int_{\text{CV}} \rho\, d\forall + \int_{\text{CS}} \rho\vec{V}\cdot d\vec{A} = 0$

Assumptions: (1) Properties in the tank are uniform, but time-dependent.
(2) Uniform flow at section (1).

Since properties are assumed uniform in the tank at any instant, we can take ρ out from within the volume integral of the first term,

$$\frac{\partial}{\partial t}\left[\rho_{CV}\int_{CV} d\mathcal{V}\right] + \int_{CS} \rho\vec{V}\cdot d\vec{A} = 0$$

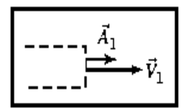
Now, $\int_{CV} d\mathcal{V} = \mathcal{V}$, and hence

$$\frac{\partial}{\partial t}(\rho\mathcal{V})_{CV} + \int_{CS} \rho\vec{V}\cdot d\vec{A} = 0$$

The only place where mass crosses the boundary of the control volume is at surface ①. Hence

$$\int_{CS} \rho\vec{V}\cdot d\vec{A} = \int_{A_1} \rho\vec{V}\cdot d\vec{A} \qquad \text{and} \qquad \frac{\partial}{\partial t}(\rho\mathcal{V}) + \int_{A_1} \rho\vec{V}\cdot d\vec{A} = 0$$

At surface ① the sign of $\rho\vec{V}\cdot d\vec{A}$ is positive, so



$$\frac{\partial}{\partial t}(\rho\mathcal{V}) + \int_{A_1} \rho V\,dA = 0$$

Since flow is assumed uniform over surface ①, then

$$\frac{\partial}{\partial t}(\rho\mathcal{V}) + \rho_1 V_1 A_1 = 0 \qquad \text{or} \qquad \frac{\partial}{\partial t}(\rho\mathcal{V}) = -\rho_1 V_1 A_1$$

Since the volume, $\mathcal{V}$, of the tank is not a function of time,

$$\mathcal{V}\frac{\partial\rho}{\partial t} = -\rho_1 V_1 A_1$$

and

$$\frac{\partial\rho}{\partial t} = -\frac{\rho_1 V_1 A_1}{\mathcal{V}}$$

At $t = 0$,

$$\frac{\partial\rho}{\partial t} = -6\frac{\text{kg}}{\text{m}^3} \times 300\frac{\text{m}}{\text{s}} \times 65\,\text{mm}^2 \times \frac{1}{0.05\,\text{m}^3} \times \frac{\text{m}^2}{10^6\,\text{mm}^2}$$

$$\frac{\partial\rho}{\partial t} = -2.34\,(\text{kg/m}^3)/\text{s} \xleftarrow{\text{\{The density is decreasing.\}}} \frac{\partial\rho}{\partial t}$$

This problem demonstrates use of the conservation of mass equation for unsteady flow problems.

Momentum Equation for Inertial Control Volume

The Control Volume Form of Newton's Second Law

We wish to develop a mathematical formulation of Newton's second law suitable for application to a control volume. In this section our derivation will be restricted to an inertial control volume fixed in space relative to coordinate system xyz that is not accelerating relative to stationary reference frame XYZ.

In deriving the control volume form of Newton's second law, the procedure is analogous to the procedure followed in deriving the mathematical form of the conservation of mass for a control volume.

Recall that for a system moving relative to an inertial reference frame, Newton's second law states that *the sum of all external forces acting on the system is equal to the time rate of change of linear momentum of the system,* $\vec{F} = \frac{d\vec{P}}{dt})_{system}$

Where the linear momentum of the system is given by

$$\vec{P})_{system} = \int_{(M)_{system}} \vec{V} dm = \int_{(V)_{system}} \vec{V} \rho dV$$

And the resultant force, $\vec{F}$, includes all surface and body forces acting on the system, $\vec{F} = \vec{F}_S + \vec{F}_B$

The system and control volume formulations are related using Equation

$$\frac{dN}{dt})_{system} = \frac{\partial}{\partial t} \int_{CV} \eta \rho dV + \int_{CS} \eta \rho \vec{V}. d\vec{A}$$

To derive the control volume formulation of Newton's second law, we set

$N = \vec{P}$ and $\eta = \vec{V}$ then we get

$$\frac{d\vec{P}}{dt})_{system} = \frac{\partial}{\partial t} \int_{CV} \vec{V} \rho dV + \int_{CS} \vec{V} \rho \vec{V}. d\vec{A}$$

Since, the system and the control volume coincided at t_0, then

$$\vec{F})_{on\ system} = \vec{F})_{control\ volume}$$

Also using $\vec{F})_{on\ system} = \frac{d\vec{P}}{dt})_{system}$

In light of above two equations may be combined to yield the control volume formulation of Newton's second law for a nonaccelerating control volume

$$\vec{F} = \vec{F}_S + \vec{F}_B = \frac{\partial}{\partial t}\int_{CV}\vec{V}\rho dV + \int_{CS}\vec{V}\rho\vec{V}.d\vec{A}$$

For cases when we have uniform flow at each inlet and exit, we can use

$$\vec{F} = \vec{F}_S + \vec{F}_B = \frac{\partial}{\partial t}\int_{CV}\vec{V}\rho dV + \sum_{CS}\vec{V}\rho\vec{V}.d\vec{A}$$

This equation states that the sum of all forces (surface and body forces) acting on a nonaccelerating control volume is equal to the sum of the rate of change of momentum inside the control volume and the net rate of flux of momentum out through the control surface.

The momentum equation is a vector equation. As with all vector equations, it may be written as three scalar component equations. The scalar components of Equation, relative to an xyz coordinate system, are

$$F_x = F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV}u\rho dV + \int_{CS}u\rho\vec{V}.d\vec{A}$$

$$F_y = F_{S_y} + F_{B_y} = \frac{\partial}{\partial t}\int_{CV}v\rho dV + \int_{CS}v\rho\vec{V}.d\vec{A}$$

$$F_z = F_{S_z} + F_{B_z} = \frac{\partial}{\partial t}\int_{CV}w\rho dV + \int_{CS}w\rho\vec{V}.d\vec{A}$$

Or, for uniform flow at each inlet and exit,

$$F_x = F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV}u\rho dV + \sum_{CS}u\rho\vec{V}.d\vec{A}$$

$$F_y = F_{S_y} + F_{B_y} = \frac{\partial}{\partial t}\int_{CV}v\rho dV + \sum_{CS}v\rho\vec{V}.d\vec{A}$$

$$F_z = F_{S_z} + F_{B_z} = \frac{\partial}{\partial t}\int_{CV}w\rho dV + \sum_{CS}w\rho\vec{V}.d\vec{A}$$

Note that, for the mass conservation equation, the control surface integrals can be replaced with simple algebraic expressions when we have uniform flow at a each inlet or exit, and that for steady flow the first term on the right side is zero.

This shows that for an ...
in the pipe.

Q. State and derive Bernoulli's equation.

Ans: This fundamental equation developed by **Swiss** scientist **Daniel Bernoulli (1700 – 1782)** in **1738.**

It states that

The sum of pressure , Kinetic and potential energies per unit volume of an ideal fluid flow remains constant at any point of its path is known as Bernoulli's equation.

OR

For an incompressible and non-viscous liquid in steady flow, the sum of pressure, kinetic energy per unit volume and potential energy per unit volume is constant is known as Bernoulli's equation.

The mathematical form of Bernoulli's Equation is given as

$$P + \frac{1}{2}\rho v^2 + \rho g h = \text{Constant}$$

or $$P_1 + \frac{1}{2}\rho v_1^2 + \rho g h_1 = P_2 + \frac{1}{2}\rho v_2^2 + \rho g h_2$$

Derivation:

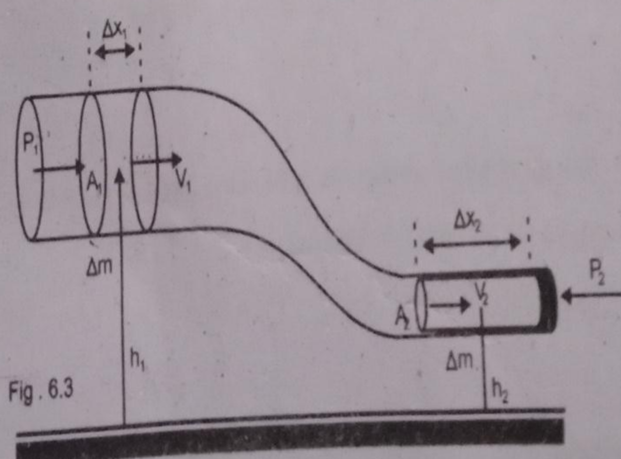


Fig . 6.3

Consider an ideal fluid flow which is flowing through the pipe whose upper end is at height h_1 and the lower end is at height h_2 where $h_1 > h_2$. The area of upper end of the pipe is A_1 and area of lower end of pipe is A_2 as shown in fig.,

FOR PRESSURE AND ENERGY

For upper end of pipe:

The work done on the fluid in moving through distance Δx_1 during time interval Δt is given as

$$\text{Work} = \vec{F}_1 \bullet \Delta\vec{x}_1$$

$$W_1 = F_1 \Delta x_1 Cos\,\theta \quad \text{Where } \theta = 0$$

$$W_1 = F_1 \Delta x_1 Cos\,0$$

$$W_1 = F_1 \Delta x_1 \quad \because \text{as } 0^\circ = 1$$

Due to steady flow

$$\Delta x_1 = v_1 \times \Delta t$$

So $W_1 = F_1 v_1 \Delta t$ ....(1)

If P_1 is the pressure and A_1 is area of cross-section of upper end then force F_1 exerted by the fluid at upper end of the pipe is given as

We know that

$$Pressure = \frac{Force}{Area}$$

$$P_1 = \frac{F_1}{A_1}$$

$$F_1 = P_1 A_1$$

So eq.1 becomes

$$W_1 = P_1 A_1 v_1 \Delta t \quad \text{.................(2)}$$

For lower end of pipe:

The work done by the fluid in moving through distance Δx_2 during same time interval Δt is given as

$$\text{Work} = \vec{F}_2 \bullet \Delta\vec{x}_2$$

As the work done is against an oppositely directed force F_2 So it is taken as negative.

$$W_2 = F_2 \Delta x_2 Cos\theta \quad \text{Where } \theta = 180$$

$$W_2 = F_2 \Delta x_2 Cos180$$

$$W_2 = -F_2 \Delta x_2$$

$\vec{F_1} \cdot \vec{\Delta x_1}$

$W_1 = f_1 \Delta x$

Due to steady flow

$$\Delta x_2 = v_2 \times \Delta t$$

$$W_2 = -F_2 v_2 \Delta t \quad \text{................(3)}$$

If P_2 is the pressure and A_2 is area of cross-section of upper end then force F_2 exerted by the fluid at lower end of the pipe is given as

We know that

$$Pressure = \frac{Force}{Area}$$

$$P_2 = \frac{F_2}{A_2}$$

$$F_2 = P_2 A_2$$

So eq.3 becomes

$$W_2 = -P_2 A_2 v_2 \Delta t \quad \text{.........(4)}$$

Total work done is given as

$$W = W_1 + W_2 \quad \text{.........(5)}$$

From eq.2 and eq.4

$$W = P_1 A_1 v_1 \Delta t - P_2 A_2 v_2 \Delta t \quad \text{.........(6)}$$

According to equation of continuity

$$A_1 v_1 = A_2 v_2 = \frac{V}{\Delta t}$$

or $A_1 v_1 \Delta t = A_2 v_2 \Delta t = V$

Eq.6 becomes

$$W = P_1 V - P_2 V$$

$$W = (P_1 - P_2)V \qquad \text{Where } V = m/\rho$$

So $$W = (P_1 - P_2)\frac{m}{\rho} \quad \text{.........(7)}$$

According to work-energy principle

$$W = \Delta K.E + \Delta P.E \quad \text{.........(8)}$$

Where $\Delta K.E = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2$

$$\Delta P.E = mgh_2 - mgh_1$$

So the eq.8 becomes

$$W = \left(\frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2\right) + (mgh_2 - mgh_1) \quad \text{........(9)}$$

Applying law of conservation of energy

From eq.7 and eq.9

$$(P_1 - P_2)\frac{m}{\rho} = \left(\frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2\right) + (mgh_2 - mgh_1)$$

$$(P_1 - P_2)\frac{m}{\rho} = m\left(\frac{1}{2}v_2^2 - \frac{1}{2}v_1^2 + gh_2 - gh_1\right)$$

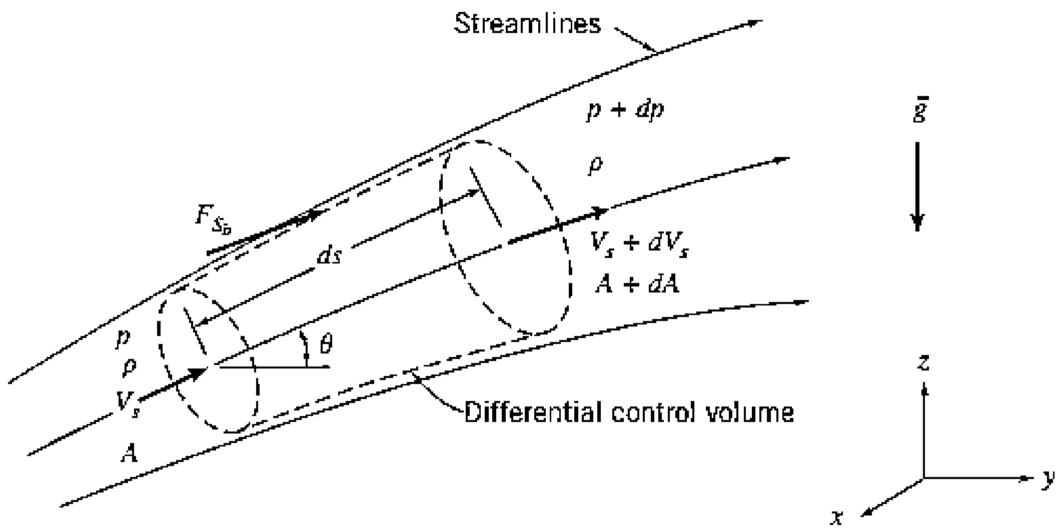
$$(P_1 - P_2) = \rho\left(\frac{1}{2}v_2^2 - \frac{1}{2}v_1^2 + gh_2 - gh_1\right)$$

$$(P_1 - P_2) = \frac{1}{2}\rho v_2^2 - \frac{1}{2}\rho v_1^2 + \rho gh_2 - \rho gh_1$$

$$P_1 + \frac{1}{2}\rho v_1^2 + \rho gh_1 = P_2 + \frac{1}{2}\rho v_2^2 + \rho gh_2$$

$$P + \frac{1}{2}\rho v^2 + \rho gh = Constant$$

Differential Control Volume Analysis (Bernoulli's Equation)



We have considered a number of examples in which conservation of mass and the momentum equation have been applied to finite control volumes. However, the control volume chosen for analysis need not be finite in size.

Application of the basic equations to a differential control volume leads to differential equations describing the relationships among properties in the flow field. In some cases, the differential equations can be solved to give detailed information about property variations in the flow field. For the case of steady, incompressible, frictionless flow along a streamline, integration of one such differential equation leads to a useful (and famous) relationship among speed, pressure, and elevation in a flow field. This case is presented to illustrate the use of differential control volumes.

Let us apply the continuity and momentum equations to a steady incompressible flow without friction, as shown in Figure. The control volume chosen is fixed in space and bounded by flow streamlines, and is thus an element of a stream tube. The length of the control volume is ds.

Because the control volume is bounded by streamlines, flow across the bounding surfaces occurs only at the end sections. These are located at coordinates s and $s + ds$, measured along the central streamline.

Properties at the inlet section are assigned arbitrary symbolic values. Properties at the outlet section are assumed to increase by differential amounts. Thus at $s + ds$, the flow speed is assumed to be $V_s + dV_s$, and so on. The differential changes, dp, dV_s, and dA, all are assumed to be positive in setting up the problem.

Now let us apply the continuity equation and the J - component of the momentum equation to the control volume of Figure.

$$\cancelto{0}{\frac{\partial}{\partial t}\int_{CV}\rho dV} + \int_{CS}\rho\vec{V}.d\vec{A} = 0 \qquad \text{………..(1)}$$

Assumptions: (1) Steady flow. (2) No flow across bounding streamlines.

(3) Incompressible flow, ρ = constant.

Then $(-\rho V_s A) + \{\rho(V_s + dV_s)(A + dA)\} = 0$

So $\quad \rho(V_s + dV_s)(A + dA) = \rho V_s A$

Or $\quad (V_s + dV_s)(A + dA) = V_s A \qquad$ ………..(2)

On expanding the left side and simplifying, we obtain

$$V_s dA + A dV_s + dA dV_s = 0$$

But $dAdV_s$ is a product of differentials, which may be neglected compared with $V_s dA$ or $A dV_s$. Thus $\quad V_s dA + A dV_s = 0 \qquad$ ………..(3)

Streamwise Component of the Momentum Equation

$$F_{S_S} + F_{B_S} = \cancelto{0}{\frac{\partial}{\partial t}\int_{CV}u_s\rho dV} + \int_{CS}u_s\rho\vec{V}.d\vec{A} \qquad \text{………..(4)}$$

Assumption: (4) No friction, so FSB is due to pressure forces only.

The surface force (due only to pressure) will have three terms:

$$F_{S_S} = pA - (p + dp)(A + dA) + \left(p + \frac{dp}{2}\right)dA \qquad \text{………..(5)}$$

The first and second terms in Equation (5) are the pressure forces on the end faces of the control surface. The third term is F_{S_B}, the pressure force acting in the *s* direction on the bounding stream surface of the control volume. Its magnitude is the product of the average pressure acting on the stream surface, $p + \frac{dp}{2}$, times the area component of the stream surface in the s direction, dA.

Equation (5) simplifies to $\quad F_{S_S} = -Adp - \frac{1}{2}dpdA \quad$ ……….(6)

The body force component in the *s* direction is

$$F_{B_s} = \rho g_s dV = \rho(-gSin\theta)\left(A + \frac{dA}{2}\right)ds$$

But $\quad Sin\theta ds = dz$, so that $\quad F_{B_s} = -\rho g\left(A + \frac{dA}{2}\right)dz \quad$ ……….(7)

The momentum flux will be

$$\int_{CS} u_s \rho \vec{V}.d\vec{A} = V_s(-\rho V_s A) + (V_s + dV_s)\{\rho(V_s + dV_s)(A + dA)\}$$

Since there is no mass flux across the bounding stream surfaces. The mass flux factors in parentheses and braces are equal from continuity, so

$$\int_{CS} u_s \rho \vec{V}.d\vec{A} = V_s(-\rho V_s A) + (V_s + dV_s)(\rho V_s A) = \rho V_s A dV_s \quad \text{……….(8)}$$

Substituting 6,7,8 into 4 (the momentum equation) gives

$$-Adp - \frac{1}{2}dpdA - \rho g A dz - \frac{1}{2}\rho g dA dz = \rho V_s A dV_s$$

Dividing by ρA and noting that products of differentials are negligible compared with the remaining terms, we obtain

$$-\frac{dp}{\rho} - gdz = V_s dV_s = d\left(\frac{V_s^2}{2}\right) \quad \text{or} \quad \frac{dp}{\rho} + d\left(\frac{V_s^2}{2}\right) + gdz = 0$$

For incompressible flow, this equation may be integrated to obtain

$$\frac{p}{\rho} + \frac{V_s^2}{2} + gz = constant$$

Or, dropping subscript *s*, $\frac{p}{\rho} + \frac{V^2}{2} + gz = constant$

This Equation is a form of the **Bernoulli equation**. It is such a useful tool for flow analysis and because an alternative derivation will give added insight into the need for care in applying the equation.

This equation is subject to the restrictions:

1. Steady flow.

2. No friction.

3. Flow along a streamline,

4. Incompressible flow.

By applying the momentum equation to an infinitesimal stream tube control volume, for steady incompressible flow without friction, we have derived a relation among pressure, speed, and elevation. This relationship is very powerful and useful.

This equation is widely used in aerodynamics to relate the pressure and velocity in a flow (e.g., it explains the lift of a subsonic wing). It could also be used to find the pressure at the inlet of the reducing elbow analysis or to determine the velocity of water leaving the sluice gate.

Statement of Bernoulli equation (Theorem)

Bernoulli's Equation states that "the sum of pressure, the kinetic and potential energies per unit volume in a steady flow of an incompressible and non – viscous (frictionless) fluid remains constant at every point of its path.

Mathematically expressed as $\frac{p}{\rho} + \frac{V^2}{2} + gz = \ constant$

Momentum Equation for Control Volume with Rectilinear Acceleration

For an inertial control volume (having no acceleration relative to a stationary frame of reference), the appropriate formulation of Newton's second law is given by

$$F = F_S + F_B = \frac{\partial}{\partial t}\int_{CV} \vec{V}_{xyz}\rho dV + \int_{CS} \vec{V}_{xyz}\rho \vec{V}_{xyz}.d\vec{A} \qquad \ldots\ldots\ldots\ldots(1)$$

Relating the system derivatives to the control volume formulation, the flow field, V(x, y, z, t), was specified relative to the control volume's coordinates x, y, and z.

we have $\quad \left.\frac{d\vec{P}}{dt}\right)_{system} = \frac{\partial}{\partial t}\int_{CV} \vec{V}_{xyz}\rho dV + \int_{CS} \vec{V}_{xyz}\rho \vec{V}_{xyz}.d\vec{A} \qquad \ldots\ldots\ldots\ldots(2)$

Also we know $\vec{F} = \left.\frac{d\vec{P}}{dt}\right)_{system} \qquad \ldots\ldots\ldots\ldots(3)$

Where $\quad \vec{P})_{system} = \int_{(M)_{system}} \vec{V} dm = \int_{(V)_{system}} \vec{V}\rho dV \qquad \ldots\ldots\ldots\ldots(4)$

Thus, if we denote the inertial reference frame by XYZ, then Newton's second law states that $\quad \vec{F} = \left.\frac{d\vec{P}_{XYZ}}{dt}\right)_{system} \qquad \ldots\ldots\ldots\ldots(5)$

Since the time derivatives of $\vec{P}_{XYZ}$ and $\vec{P}_{xyz}$ are not equal when the control volume reference frame xyz is accelerating relative to the inertial reference frame, thus

$$\vec{P})_{system} = \int_{(M)_{system}} \vec{V} dm = \int_{(V)_{system}} \vec{V}\rho dV$$

is not valid for an accelerating control volume.

To develop the momentum equation for a linearly accelerating control volume, it is necessary to relate $\vec{P}_{XYZ}$ of the system to $\vec{P}_{xyz}$ of the system. We begin by writing Newton's second law for a system

$$\vec{F} = \left.\frac{d\vec{P}_{XYZ}}{dt}\right)_{system} = \frac{d}{dt}\int_{(M)_{system}} \vec{V}_{XYZ} dm = \int_{(M)_{system}} \frac{d\vec{V}_{XYZ}}{dt} dm \quad \ldots\ldots\ldots\ldots(6)$$

The velocities with respect to the inertial {XYZ) and the control volume coordinates (xyz) are related by the relative-motion equation

$$\vec{V}_{XYZ} = \vec{V}_{xyz} + \vec{V}_{rf} \qquad \ldots\ldots\ldots\ldots(7)$$

where $\vec{V}_{rf}$ is the velocity of the control volume reference frame.

Since we are assuming the motion of xyz is pure translation, without rotation, relative to inertial reference frame XYZ, then

$$\frac{d\vec{V}_{XYZ}}{dt} = \vec{a}_{XYZ} = \frac{d\vec{V}_{xyz}}{dt} + \frac{d\vec{V}_{rf}}{dt} = \vec{a}_{xyz} + \vec{a}_{rf}$$

Where

- $\vec{a}_{XYZ}$ is the rectilinear acceleration of the system relative to inertial reference frame XYZ,
- $\vec{a}_{xyz}$ is the rectilinear acceleration of the system relative to noninertial reference frame xyz (i.e., relative to the control volume), and
- $\vec{a}_{rf}$ is the rectilinear acceleration of noninertial reference frame xyz (i.e., of the control volume) relative to inertial frame XYZ.

Substituting from Eq. 7 into Eq. 6 gives

$$\vec{F} = \frac{d\vec{P}_{XYZ}}{dt}\Big)_{system} = \int_{(M)_{system}} \left(\vec{a}_{xyz} + \vec{a}_{rf}\right) dm$$

$$\vec{F} = \int_{(M)_{system}} \vec{a}_{rf}\, dm + \int_{(M)_{system}} \frac{d\vec{V}_{xyz}}{dt}\, dm$$

Or $\quad \vec{F} - \int_{(M)_{system}} \vec{a}_{rf}\, dm = \frac{d\vec{P}_{xyz}}{dt}\Big)_{system}$

Or $\quad \vec{F} - \int_{CV} \vec{a}_{rf}\,\rho dV = \frac{d\vec{P}_{xyz}}{dt}\Big)_{system}$ ………..(8)

Where the linear momentum of the system is given by

$$\vec{P}_{xyz}\Big)_{system} = \int_{(M)_{system}} \vec{V}_{xyz}\, dm = \int_{(V)_{system}} \vec{V}_{xyz}\,\rho dV$$

And the force, $\vec{F}$, includes all surface and body forces acting on the system.

Since $\quad \frac{dN}{dt}\Big)_{system} = \frac{\partial}{\partial t}\int_{CV} \eta\rho dV + \int_{CS} \eta\rho \vec{V}.d\vec{A}$

To derive the control volume formulation of Newton's second law, we set

$N = \vec{P}_{xyz}$ and $\eta = \vec{V} = \vec{V}_{xyz}$ then

$$\frac{d\vec{P}_{xyz}}{dt}\Big)_{system} = \frac{\partial}{\partial t}\int_{CV} \vec{V}_{xyz}\,\rho dV + \int_{CS} \vec{V}_{xyz}\,\rho \vec{V}_{xyz}.d\vec{A} \qquad \text{………..(9)}$$

Combining Eq. 8 (the linear momentum equation for the system) and Eq. 9 (the system-control volume conversion), and recognizing that at time t_0 the system and control volume coincide, Newton's second law for a control volume accelerating, without rotation, relative to an inertial reference frame is

$$\vec{F} - \int_{CV} \vec{a}_{rf} \rho dV = \frac{\partial}{\partial t} \int_{CV} \vec{V}_{xyz} \rho dV + \int_{CS} \vec{V}_{xyz} \rho \vec{V}_{xyz} . d\vec{A}$$

Since $= F_S + F_B$, this equation becomes

$$F_S + F_B - \int_{CV} \vec{a}_{rf} \rho dV = \frac{\partial}{\partial t} \int_{CV} \vec{V}_{xyz} \rho dV + \int_{CS} \vec{V}_{xyz} \rho \vec{V}_{xyz} . d\vec{A}$$

In this equation F_S represents all surface forces acting on the control volume.

The momentum equation is a vector equation. As with all vector equations, it may be written as three scalar component equations. The scalar components of Equation are

$$F_{S_x} + F_{B_x} - \int_{CV} \vec{a}_{rf_x} \rho dV = \frac{\partial}{\partial t} \int_{CV} u_{xyz} \rho dV + \int_{CS} u_{xyz} \rho \vec{V}_{xyz} . d\vec{A}$$

$$F_{S_y} + F_{B_y} - \int_{CV} \vec{a}_{rf_y} \rho dV = \frac{\partial}{\partial t} \int_{CV} v_{xyz} \rho dV + \int_{CS} v_{xyz} \rho \vec{V}_{xyz} . d\vec{A}$$

$$F_{S_z} + F_{B_z} - \int_{CV} \vec{a}_{rf_y} \rho dV = \frac{\partial}{\partial t} \int_{CV} w_{xyz} \rho dV + \int_{CS} w_{xyz} \rho \vec{V}_{xyz} . d\vec{A}$$

Equation for Fixed Control Volume / The Angular-Momentum Principle

The angular-momentum principle for a system states that *the rate of change of angular momentum is equal to the sum of all torques acting on the system*,

$$\vec{\tau} = \frac{d\vec{H}}{dt}\Big)_{system} \quad ……………(1)$$

Where the angular momentum of the system is given by

$$\vec{H}\Big)_{system} = \int_{(M)_{system}} \vec{r} \times \vec{V} dm = \int_{(V)_{system}} \vec{r} \times \vec{V} \rho dV \quad ……………(2)$$

Torque can be produced by surface and body forces, and also by shafts that cross the system boundary,

$$\vec{\tau} = \vec{r} \times \vec{F}_s + \int_{(M)_{system}} \vec{r} \times \vec{g} dm + \vec{\tau}_{shaft} \quad ……………(3)$$

The position vector, $\vec{r}$, locates each mass or volume element of the system with respect to the coordinate system. Where $\vec{F}_s$ is the surface force exerted on the system. The relation between the system and fixed control volume formulations is

$$\frac{dN}{dt}\Big)_{system} = \frac{\partial}{\partial t}\int_{CV} \eta \rho dV + \int_{CS} \eta \rho \vec{V}.d\vec{A}$$

Where $\quad N)_{system} = \int_{(M)_{system}} \eta dm$

If we set $N = \vec{H}$ and $\eta = \vec{r} \times \vec{V}$ then

$$\frac{d\vec{H}}{dt}\Big)_{system} = \frac{\partial}{\partial t}\int_{CV} \vec{r} \times \vec{V} \rho dV + \int_{CS} \vec{r} \times \vec{V} \rho \vec{V}.d\vec{A} \quad …………(4)$$

Combining Eqs. (1), (3), and (4), we obtain

$$\vec{r} \times \vec{F}_s + \int_{(M)_{system}} \vec{r} \times \vec{g} dm + \vec{\tau}_{shaft} = \frac{\partial}{\partial t}\int_{CV} \vec{r} \times \vec{V} \rho dV + \int_{CS} \vec{r} \times \vec{V} \rho \vec{V}.d\vec{A}$$

Since the system and control volume coincide at time t_0, then $\vec{\tau} = \vec{\tau}_{CV}$ then

$$\vec{r} \times \vec{F}_s + \int_{CV} \vec{r} \times \vec{g} \rho dV + \vec{\tau}_{shaft} = \frac{\partial}{\partial t}\int_{CV} \vec{r} \times \vec{V} \rho dV + \int_{CS} \vec{r} \times \vec{V} \rho \vec{V}.d\vec{A}$$

This is a general formulation of the angular-momentum principle for an inertial control volume.

The First Law of Thermodynamics (Control Volume Form)

The first law of thermodynamics is a statement of conservation of energy for a system, $\delta Q - \delta W = dE$

The equation can be written in rate form as $\dot{Q} - \dot{W} = \frac{dE}{dt})_{system}$

Where the total energy (entropy) of the system is given by

$E)_{system} = \int_{(M)_{system}} e dm = \int_{(V)_{system}} e\rho dV$

And $e = u + \frac{V^2}{2} + gz$

In $\dot{Q} - \dot{W} = \frac{dE}{dt})_{system}$, $\dot{Q}$ (the rate of heat transfer) is positive when heat is added to the system from the surroundings; $\dot{W}$ (the rate of work) is positive when work is done by the system on its surroundings. In $e = u + \frac{V^2}{2} + gz$, u is the specific internal energy, V the speed, and z the height (relative to a convenient datum) of a particle of substance having mass *dm*.

Since $\frac{dN}{dt})_{system} = \frac{\partial}{\partial t}\int_{CV} \eta\rho dV + \int_{CS} \eta\rho\vec{V}.d\vec{A}$

To derive the control volume formulation of 1st law of thermodynamics, we set

$N = E$ and $\eta = e$ then

$\frac{dE}{dt})_{system} = \frac{\partial}{\partial t}\int_{CV} e\rho dV + \int_{CS} e\rho\vec{V}.d\vec{A}$

Since the system and the control volume coincide at t_0,

$\dot{Q} - \dot{W} = \frac{dE}{dt})_{system} = \dot{Q} - \dot{W} = \frac{dE}{dt})_{CV}$

Thus $\frac{dE}{dt})_{CV} = \frac{\partial}{\partial t}\int_{CV} e\rho dV + \int_{CS} e\rho\vec{V}.d\vec{A}$

Note that for steady flow the first term on the right side of above equation is zero.

The Second Law of Thermodynamics (Control Volume Form)

If an amount of heat, δQ, is transferred to a system at temperature T, the second law of thermodynamics states that the change in entropy, dS, of the system satisfies

$$dS \geq \frac{\delta Q}{T}$$

On a rate basis we can write $\left.\frac{dS}{dt}\right)_{system} \geq \frac{1}{T}\dot{Q}$

Where the total energy (entropy) of the system is given by

$$\left.S\right)_{system} = \int_{(M)_{system}} s\,dm = \int_{(V)_{system}} s\rho dV$$

Since $\left.\frac{dN}{dt}\right)_{system} = \frac{\partial}{\partial t}\int_{CV} \eta\rho dV + \int_{CS} \eta\rho\vec{V}.d\vec{A}$

To derive the control volume formulation of 1st law of thermodynamics, we set

$N = S$ and $\eta = s$ then

$$\left.\frac{dS}{dt}\right)_{system} = \frac{\partial}{\partial t}\int_{CV} s\rho dV + \int_{CS} s\rho\vec{V}.d\vec{A}$$

Since the system and the control volume coincide at t_0,

$$\left.\frac{dS}{dt}\right)_{system} = \left.\frac{dS}{dt}\right)_{CV} \geq \frac{1}{T}\dot{Q}$$

Or $\left.\frac{dS}{dt}\right)_{system} = \left.\frac{dS}{dt}\right)_{CV} \geq \int_{CS} \frac{1}{T}\left(\frac{\dot{Q}}{A}\right) dA$

Thus $\left.\frac{dS}{dt}\right)_{CV} = \frac{\partial}{\partial t}\int_{CV} s\rho dV + \int_{CS} s\rho\vec{V}.d\vec{A} \geq \int_{CS} \frac{1}{T}\left(\frac{\dot{Q}}{A}\right) dA$

INTRODUCTION TO DIFFERENTIAL ANALYSIS OF FLUID MOTION

In Previous Chapter, we developed the basic equations in integral form for a control volume. Integral equations are useful when we are interested in the gross behavior of a flow field and its effect on various devices. However, the integral approach does not enable us to obtain detailed point-by-point knowledge of the flow field. For example, the integral approach could provide information on the lift generated by a wing; it could not be used to determine the pressure distribution that produced the lift on the wing.

To obtain detailed knowledge, we must apply the equations of fluid motion in differential form. In this chapter we shall develop differential equations for the conservation of mass and Newton's second law of motion. Since we are interested in formulating differential equations, our analysis will be in terms of infinitesimal systems and control volumes.

Conservation of Mass

In previous Chapters, we developed the field representation of fluid properties. The property fields are defined by continuous functions of the space coordinates and time. The density and velocity fields were related through conservation of mass in integral form. In this chapter we shall derive the differential equation for conservation of mass in rectangular and in cylindrical coordinates. In both cases the derivation is carried out by applying conservation of mass to a differential control volume.

Equation of Continuity or Law of Conservation of Mass:

"Law of conservation of mass state that fluid mass can neither created nor destroyed."

The equation of continuity gives the law of conservation of mass in **analytical form** or **mathematical form** $\frac{\partial\rho}{\partial t} + \nabla.\rho\vec{V} = 0$ where $\vec{V}$ is the velocity of fluid. Therefore, in a continuous motion the equation of continuity expresses the fact that increase in the mass of fluid with any closed surface drawn in the fluid at any time must be equal to the access of the mass that flows 'in' over the mass of that flows 'out'. Inward flow is equal to outward flow.

Q. State and derive equation of continuity of fluid flow:

Ans: The net rate of fluid flow remains constant at any point in the pipe for an ideal fluid is called equation of continuity. OR

The product of cross-sectional area of the pipe and the fluid speed at any point along the pipe remains constant is called equation of continuity. OR

For an ideal fluid flow, the net rate of flow of mass inward across any closed surface of pipe is equal to the net rate of flow of mass outward of the pipe is called equation of continuity.

Net rate of flow of mass inward = Net rate of flow of mass inward Out

$$A_1 v_1 = A_2 v_2$$

or $$Av = Constant$$

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DERIVATION:

Consider an ideal fluid flow which is flowing through a pipe of non-uniform size of pipe. The fluid flowing at the lower end of area A_1 covers a distance Δx_1 with a velocity $\mathbf{V_1}$ for a short interval of time Δt and the fluid flowing at the upper end of area A_2 covers a distance Δx_2 with a velocity $\mathbf{V_2}$ for a short interval of time 'Δt' shown in fig.

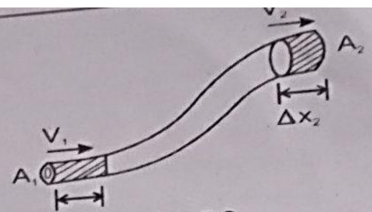


Fig. 6.2

For lower end of pipe:

We know that

$$\rho = \frac{m}{V} \Rightarrow m = \rho \times V$$

OR $\Delta m_1 = \rho_1 \times V_1$ ....(1)

Where volume of fluid $= V_1 = A_1 \Delta x_1$

Putting in eq.1

$$\Delta m_1 = \rho_1 A_1 \Delta x_1 \quad ....(2)$$

Due to steady fluid flow

$$\Delta x_1 = v_1 \Delta t \Rightarrow \because S = vt$$

Put in eq.2

$$\Delta m_1 = \rho_1 A_1 v_1 \Delta t \quad ....(3)$$

For upper end of pipe:

We know that

$$\rho = \frac{m}{V}$$

OR $\Delta m_2 = \rho_2 \times V_2$ ....(4)

Where volume of fluid $= V_2 = A_2 \Delta x_2$

Putting in eq.4

$$\Delta m_2 = \rho_2 A_2 \Delta x_2 \quad ....(5)$$

Due to steady fluid flow

$$\Delta x_2 = v_2 \Delta t$$

Putting in eq. 5

$$\Delta m_2 = \rho_2 A_2 v_2 \Delta t \quad ....(6)$$

Mass

Due to incompressible and non-viscous fluid flow, law of conservation of mass holds.

According to law of conservation of mass

$$\Delta m_1 = \Delta m_2$$

From eq.3 and eq.6

$$\rho_1 A_1 v_1 \Delta t = \rho_2 A_2 v_2 \Delta t$$

$$\rho_1 A_1 v_1 = \rho_2 A_2 v_2$$

Student Physics

Due to incompressible fluid $\rho_1 = \rho_2 = \rho$

So $\rho A_1 v_1 = \rho A_2 v_2$

$A_1 v_1 = A_2 v_2$

or $Av = Constant$

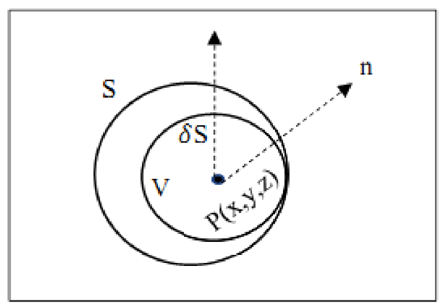
This shows that for an ideal fluid, the net rate of fluid flow remains constant at any point in the pipe.

flow.

★ Equation of Continuity in velocity form or Equation of Continuity by Euler Method or show that the fluid is incompressible or not.

As $$\frac{\partial \rho}{\partial t} + \nabla.(\rho q) = 0$$

Explanation: Let S be an arbitrary small close surface drawn in the compressible fluid and closing volume V. Let P(x,y,z) be any point of S and P(x,y,z,t) be the density of the fluid at point P at any time t and the element of the surface δS and q be the velocity of the fluid. Normal component of q is q.$\hat{n}$



Therefore, the set of mass that flows across $\delta S = (q.\hat{n})\ \rho\ \delta S$. Total rate of mass flows across $\delta S = \int_S \rho(q.n)\,dS$

$$\delta S = \int_V div(\rho q)\,dV \quad \because change\ surface\ integral\ into\ volume\ integral$$

Therefore, the total rate of mass flows into volume V $= -\int_V div(\rho q)\,dV$ ___(i)

*Outward flow is positive and inward flow is negative. Again, the mass of fluid in S at time t is given as $= \int_V \rho\,dV$

Total rate of mass increase with in S is $\frac{\partial}{\partial t}\int_V \rho\,dV = \int_V \frac{\partial \rho}{\partial t}\,dV$ ____(ii)

Let the region V of fluid contains neither source nor sink i.e. not Inlet not outlet. Therefore, from eq (i) & (ii)

$$\int_V \frac{\partial \rho}{\partial t}\,dV = -\int_V \nabla.(\rho q)\,dV$$

$$\int_V \frac{\partial \rho}{\partial t}\,dV + \int_V \nabla.(\rho q)\,dV = 0$$

$$\int_V \left(\frac{\partial \rho}{\partial t} + \nabla.(\rho q)\right)dV = 0 \qquad \text{\_\_\_(iii)}$$

As V is an arbitrary value in which is known as the equation of continuity of Euler form.

$$* \ \frac{\partial \rho}{\partial t} + \nabla.(\rho q) = 0$$

is the equation of Continuity in velocity form.

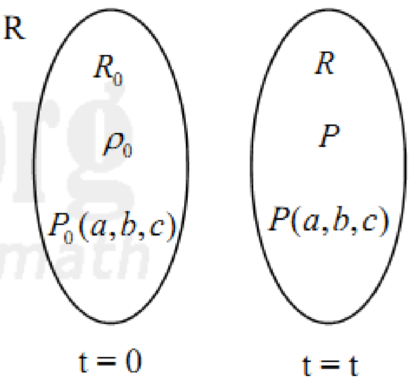
Equation of Continuity by LaGrange method:

Let R_0 be the region occupied by a portion of fluid at time t = 0 and R be the region occupied by the same fluid at time t.

Let (a,b,c) be the initial coordinates of the fluid particle P_0 and ρ_0 is the density at t = 0. Mass of the fluid at t = 0 is $\rho_0 \delta a \delta b \delta c$

Let P be the subsequent position of the fluid (position at P_0 after some time t) at time 't' and ρ is the density of the fluid. Then mass of the fluid at time t is $\rho \delta x \delta y \delta z$. By the law of conservation of mass, the mass contained inside the given volume of the fluid remain same throughout the motion (fluid motion).

Therefore, mass inside R_0 must be equal to mass inside R



$$\iint_{R_0} \rho_0 \delta a \delta b \delta c = \iint_{R} \rho \delta x \delta y \delta z \quad \_\_\_\_\_\_(i)$$

By the advance Calculus

$$J \delta a \delta b \delta c = \delta x \delta y \delta z \quad \_\_\_\_(ii)$$

By equation (i)

$$\iint_{R_0} \rho_0 \delta a \delta b \delta c = \iint_{R} \rho J \delta a \delta b \delta c$$

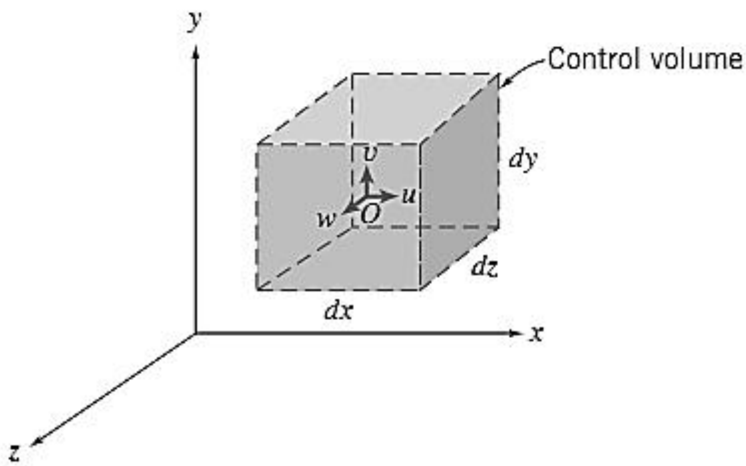
$\because$ *The total mass* R_0 = *Mass inside* $R \quad (R = R_0)$

$$\Rightarrow \quad \iint_{R_0} \rho_0 \delta a \delta b \delta c = \iint_{R_0} \rho J \delta a \delta b \delta c$$

$$\iint_{R_0} (\rho_0 - \rho J) \delta a \delta b \delta c = 0$$

This is called the equation of continuity in Langrangian form.

Equation of Continuity in Cartesian coordinate OR differential form of continuity or Conservation of Mass (Continuity Equation) in Rectangular Coordinate System



In rectangular coordinates, the control volume chosen is an infinitesimal cube with sides of length dx, dy, dz as shown in Figure. The density at the center, O, of the control volume is assumed to be ρ and the velocity there is assumed to be $\vec{V} = u\hat{\imath} + v\hat{\jmath} + w\hat{k}$

To evaluate the properties at each of the six faces of the control surface, we use a Taylor series expansion about point O. For example, at the right face,

$$\rho)_{x+\frac{dx}{2}} = \rho + \left(\frac{\partial \rho}{\partial x}\right)\frac{dx}{2} + \frac{1}{2!}\left(\frac{\partial^2 \rho}{\partial x^2}\right)\left(\frac{dx}{2}\right)^2 + \cdots$$

Neglecting higher-order terms, we can write $\rho)_{x+\frac{dx}{2}} = \rho + \left(\frac{\partial \rho}{\partial x}\right)\frac{dx}{2}$

And $u)_{x+\frac{dx}{2}} = u + \left(\frac{\partial u}{\partial x}\right)\frac{dx}{2}$

Where $\rho, u, \frac{\partial \rho}{\partial x}, \frac{\partial u}{\partial x}$ are all evaluated at point O.

The corresponding terms at the left face are

$$\rho)_{x-\frac{dx}{2}} = \rho + \left(\frac{\partial \rho}{\partial x}\right)\left(-\frac{dx}{2}\right) = \rho - \left(\frac{\partial \rho}{\partial x}\right)\frac{dx}{2}$$

And $\left.u\right)_{x-\frac{dx}{2}} = u + \left(\frac{\partial u}{\partial x}\right)\left(-\frac{dx}{2}\right) = u - \left(\frac{\partial u}{\partial x}\right)\frac{dx}{2}$

We can write similar expressions involving ρ and v for the front and back faces and ρ and w for the top and bottom faces of the infinitesimal cube $dxdydz$. These can then be used to evaluate the surface integral in following Equation

$$\frac{\partial}{\partial t}\int_{CV}\rho dV + \int_{CS}\rho\vec{V}.d\vec{A} = 0 \qquad \ldots\ldots\ldots\ldots(1)$$

Mass Flux Through the Control Surface of a Rectangular Differential Control Volume

Surface	Evaluation of $\int \rho\vec{V}\cdot d\vec{A}$
Left $(-x)$	$= -\left[\rho - \left(\frac{\partial\rho}{\partial x}\right)\frac{dx}{2}\right]\left[u - \left(\frac{\partial u}{\partial x}\right)\frac{dx}{2}\right]dy\,dz = -\rho u\,dy\,dz + \frac{1}{2}\left[u\left(\frac{\partial\rho}{\partial x}\right) + \rho\left(\frac{\partial u}{\partial x}\right)\right]dx\,dy\,dz$
Right $(+x)$	$= \left[\rho + \left(\frac{\partial\rho}{\partial x}\right)\frac{dx}{2}\right]\left[u + \left(\frac{\partial u}{\partial x}\right)\frac{dx}{2}\right]dy\,dz = \rho u\,dy\,dz + \frac{1}{2}\left[u\left(\frac{\partial\rho}{\partial x}\right) + \rho\left(\frac{\partial u}{\partial x}\right)\right]dx\,dy\,dz$
Bottom $(-y)$	$= -\left[\rho - \left(\frac{\partial\rho}{\partial y}\right)\frac{dy}{2}\right]\left[v - \left(\frac{\partial v}{\partial y}\right)\frac{dy}{2}\right]dx\,dz = -\rho v\,dx\,dz + \frac{1}{2}\left[v\left(\frac{\partial\rho}{\partial y}\right) + \rho\left(\frac{\partial v}{\partial y}\right)\right]dx\,dy\,dz$
Top $(+y)$	$= \left[\rho + \left(\frac{\partial\rho}{\partial y}\right)\frac{dy}{2}\right]\left[v + \left(\frac{\partial v}{\partial y}\right)\frac{dy}{2}\right]dx\,dz = \rho v\,dx\,dz + \frac{1}{2}\left[v\left(\frac{\partial\rho}{\partial y}\right) + \rho\left(\frac{\partial v}{\partial y}\right)\right]dx\,dy\,dz$
Back $(-z)$	$= -\left[\rho - \left(\frac{\partial\rho}{\partial z}\right)\frac{dz}{2}\right]\left[w - \left(\frac{\partial w}{\partial z}\right)\frac{dz}{2}\right]dx\,dy = -\rho w\,dx\,dy + \frac{1}{2}\left[w\left(\frac{\partial\rho}{\partial z}\right) + \rho\left(\frac{\partial w}{\partial z}\right)\right]dx\,dy\,dz$
Front $(+z)$	$= \left[\rho + \left(\frac{\partial\rho}{\partial z}\right)\frac{dz}{2}\right]\left[w + \left(\frac{\partial w}{\partial z}\right)\frac{dz}{2}\right]dx\,dy = \rho w\,dx\,dy + \frac{1}{2}\left[w\left(\frac{\partial\rho}{\partial z}\right) + \rho\left(\frac{\partial w}{\partial z}\right)\right]dx\,dy\,dz$

Adding the results for all six faces,

$$\int_{CS}\rho\vec{V}\cdot d\vec{A} = \left[\left\{u\left(\frac{\partial\rho}{\partial x}\right) + \rho\left(\frac{\partial u}{\partial x}\right)\right\} + \left\{v\left(\frac{\partial\rho}{\partial y}\right) + \rho\left(\frac{\partial v}{\partial y}\right)\right\} + \left\{w\left(\frac{\partial\rho}{\partial z}\right) + \rho\left(\frac{\partial w}{\partial z}\right)\right\}\right]dx\,dy\,dz$$

or

$$\int_{CS}\rho\vec{V}\cdot d\vec{A} = \left[\frac{\partial\rho u}{\partial x} + \frac{\partial\rho v}{\partial y} + \frac{\partial\rho w}{\partial z}\right]dx\,dy\,dz$$

The result of all this work is $\int_{CS} \rho\vec{V}.d\vec{A} = \left(\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z}\right) dxdydz$

This expression is the surface integral evaluation for our differential cube.

Now as $\frac{\partial}{\partial t}\int_{CV} \rho dV$ is the rate of change of mass in the control volume therefore

$$\frac{\partial}{\partial t}\int_{CV} \rho dV \to \frac{\partial}{\partial t}(\rho dxdydz) = \frac{\partial \rho}{\partial t}(dxdydz)$$

$$(1) \Rightarrow \frac{\partial \rho}{\partial t}(dxdydz) + \left(\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z}\right) dxdydz = 0$$

Hence, we obtain (after canceling $dxdydz$) a differential form of the mass conservation law

$$\Rightarrow \frac{\partial \rho}{\partial t} + \left(\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z}\right) = 0$$

$$\Rightarrow \frac{\partial \rho}{\partial t} + \left(\frac{\partial \rho}{\partial x}\hat{\imath} + \frac{\partial \rho}{\partial y}\hat{\jmath} + \frac{\partial \rho}{\partial z}\hat{k}\right)\left(u\hat{\imath} + v\hat{\jmath} + w\hat{k}\right) = 0$$

$$\Rightarrow \frac{\partial \rho}{\partial t} + \nabla.\rho\vec{V} = 0$$

Special Cases: Two flow cases for which the differential continuity equation may be simplified are worthy of note.

Case – I: For an **incompressible fluid**, $\rho = constant$ and then $\frac{\partial \rho}{\partial t} = 0$; density is neither a function of space coordinates nor a function of time. For an incompressible fluid, the continuity equation simplifies to $\nabla.\vec{V} = 0$

Thus the velocity field, $\vec{V} = u\hat{\imath} + v\hat{\jmath} + w\hat{k}$, for incompressible flow must satisfy $\nabla.\vec{V} = 0$.

Case – II: For **steady flow**, all fluid properties are, independent of time. Thus $\frac{\partial \rho}{\partial t} = 0$ and at most $\rho = \rho(x, y, z)$. For steady flow, the continuity equation can be written as $\nabla.\rho\vec{V} = 0$.

D' Alembert's Euler's acceleration formula for a fluid flow in plane polar coordinates with $\boldsymbol{\vec{V} = V_r \hat{e}_r + V_\theta \hat{e}_\theta}$

Since we have $\vec{V} = V_r \hat{e}_r + V_\theta \hat{e}_\theta$

Where $V_r = \dot{r},\ V_\theta = r\dot{\theta}$ then $\frac{d}{dt}V_r = \ddot{r}\ ,\frac{d}{dt}V_\theta = \dot{r}\dot{\theta} + r\ddot{\theta}$

$\hat{e}_r = Cos\theta \hat{\imath} + Sin\theta \hat{\jmath}$

$\hat{e}_\theta = -Sin\theta \hat{\imath} + Cos\theta \hat{\jmath}$ then

$\frac{d}{dr}\hat{e}_r = 0,\ \frac{d}{d\theta}\hat{e}_r = -Sin\theta \hat{\imath} + Cos\theta \hat{\jmath} = \hat{e}_\theta$

$\dot{\hat{e}}_r = \frac{d}{dt}\hat{e}_r = \frac{d\hat{e}_r}{d\theta}\frac{d\theta}{dt} = \dot{\theta}\hat{e}_\theta$

$\frac{d}{dr}\hat{e}_\theta = 0,\ \frac{d}{d\theta}\hat{e}_\theta = -Cos\theta \hat{\imath} - Sin\theta \hat{\jmath} = -\hat{e}_r$

$\dot{\hat{e}}_\theta = \frac{d}{d\theta}\hat{e}_\theta = \frac{d\hat{e}_\theta}{d\theta}\frac{d\theta}{dt} = -\dot{\theta}\hat{e}_r$

Now differentiating $\vec{V} = V_r \hat{e}_r + V_\theta \hat{e}_\theta$ with respect to time we get acceleration.

$\frac{d}{dt}\vec{V} = \vec{a} = \frac{d}{dt}(V_r \hat{e}_r + V_\theta \hat{e}_\theta)$

$\vec{a} = \hat{e}_r \frac{d}{dt}(V_r) + V_r \frac{d}{dt}(\hat{e}_r) + \hat{e}_\theta \frac{d}{dt}(V_\theta) + V_\theta \frac{d}{dt}(\hat{e}_\theta)$

$\vec{a} = \hat{e}_r(\ddot{r}) + V_r(\dot{\theta}\hat{e}_\theta) + \hat{e}_\theta(\dot{r}\dot{\theta} + r\ddot{\theta}) + V_\theta(-\dot{\theta}\hat{e}_r)$

$\vec{a} = \hat{e}_r(\ddot{r}) + (\dot{r})(\dot{\theta}\hat{e}_\theta) + \hat{e}_\theta(\dot{r}\dot{\theta} + r\ddot{\theta}) + (r\dot{\theta})(-\dot{\theta}\hat{e}_r)$

$\vec{a} = \ddot{r}\hat{e}_r + \dot{r}\dot{\theta}\hat{e}_\theta + \dot{r}\dot{\theta}\hat{e}_\theta + r\ddot{\theta}\hat{e}_\theta - r\dot{\theta}^2\hat{e}_r$

$\vec{a} = \ddot{r}\hat{e}_r + 2\dot{r}\dot{\theta}\hat{e}_\theta + r\ddot{\theta}\hat{e}_\theta - r\dot{\theta}^2\hat{e}_r$

$\vec{a} = (\ddot{r} - r\dot{\theta}^2)\hat{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{e}_\theta$

$\vec{a} = a_r\hat{e}_r + a_\theta\hat{e}_\theta$

Where $a_r = \ddot{r} - r\dot{\theta}^2$ and $a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}$

Equation of continuity for an incompressible, Irrotational fluid

For an **incompressible fluid** we have $\nabla.\vec{V} = 0$ ………(i)

For an **Irrotational fluid** we have $\nabla = -\nabla\varphi$ and $\nabla \times \vec{V} = 0$ ………(ii)

$(i) \Rightarrow \nabla.(-\nabla\varphi) = 0 \Rightarrow -\nabla.\nabla\varphi = 0 \Rightarrow \nabla^2\varphi = 0$

Example: Integration of Two-Dimensional Differential Continuity Equation

For a two-dimensional flow in the xy plane, the x component of velocity is given by $u = Ax$. Determine a possible y component for incompressible flow. How many y components are possible?

Given: Two-dimensional flow in the xy plane for which $u = Ax$.

Find: (a) Possible y component for incompressible flow.
(b) Number of possible y components.

Solution:

Governing equation: $\nabla \cdot \rho\vec{V} + \frac{\partial \rho}{\partial t} = 0$

For incompressible flow this simplifies to $\nabla \cdot \vec{V} = 0$. In rectangular coordinates

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

For two-dimensional flow in the xy plane, $\vec{V} = \vec{V}(x, y)$. Then partial derivatives with respect to z are zero, and

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Then

$$\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x} = -A$$

which gives an expression for the rate of change of v holding x constant. This equation can be integrated to obtain an expression for v. The result is

$$v = \int \frac{\partial v}{\partial y} dy + f(x, t) = -Ay + f(x, t) \longleftarrow v$$

{The function of x and t appears because we had a partial derivative of v with respect to y.}

Any function $f(x, t)$ is allowable, since $\partial/\partial y\, f(x, t) = 0$. Thus any number of expressions for v could satisfy the differential continuity equation under the given conditions. The simplest expression for v would be obtained by setting $f(x, t) = 0$. Then $v = -Ay$, and

$$\vec{V} = Ax\hat{i} - Ay\hat{j} \longleftarrow \vec{V}$$

This problem:

- ✓ Shows use of the differential continuity equation for obtaining information on a flow field.
- ✓ Demonstrates integration of a partial derivative.
- ✓ Proves that the flow originally discussed in Example 2.1 is indeed incompressible.

Example

Two components of velocity in an incompressible velocity field are given by $u = x^2 - y^2$ and $v = y^2 + logx$ determine third component assuming that origin is at stagnation point (velocity of fluid = 0)

Solution

Since flow is incompressible therefore $\nabla.\vec{V} = 0$

$$\Rightarrow \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \Rightarrow 2x + 2y + \frac{\partial w}{\partial z} = 0 \Rightarrow \frac{\partial w}{\partial z} = -2(x + y)$$

$\Rightarrow w = -2(x + y)z + f(x + y)$ integrating w.r.to 'z' …………(1)

Since velocity is at stagnation point therefore $\vec{V}(u, v, w) = \vec{V}(0,0,0)$

$$\Rightarrow 0 = -2(x + y)z + f(x + y) \Rightarrow f(x + y) = 2(x + y)z$$

$$(i) \Rightarrow w = -2(x + y)z + 2(x + y)z \Rightarrow \mathbf{w = 0}$$

Example

If $u = x$ and $v = -y$ describe a certain flow (compressible or incompressible) field. Determine whether or not equation of continuity is satisfied. Also investigate the type of flow mode (compressible or incompressible).

Solution When ρ not given then it will be constant.

Since the velocity field state in problem is of 2 dimensional characters. Then velocity in third dimension can be safely assumed to be zero. The velocity components are independent of time therefore the flow is steady and ρ is not state in problem hence we can assume flow field is incompressible.

For steady and incompressible flow $\nabla.\vec{V} = 0$

$\Rightarrow \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \Rightarrow 1 - 1 = 0 \Rightarrow$ Equation of Continuity is satisfied.

As velocity components are functions of space coordinates, so flow field is not uniform in character. The velocity gradient in x – direction the flow field is positive and constant and in in y – direction it is negative and constant.

$$\nabla\times\vec{V}=\begin{bmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ u & v & w\end{bmatrix}=0\hat{\imath}+0\hat{\jmath}+0\hat{k}\Rightarrow\nabla\times\vec{V}=0 \qquad \text{after solving}$$

Since $\nabla\times\vec{V}=0$ therefore given field is Irrotational.

Example Show that the velocity components $u=-\frac{2xy}{(x^2+y^2)^2}, v=\frac{x^2-y^2}{(x^2+y^2)^2}$ and $w=\frac{y}{(x^2+y^2)^2}$ represent the possible incompressible fluid.

Solution For incompressible fluid $\nabla.\vec{V}=0$

$$\Rightarrow\nabla.\vec{V}=\left(\frac{\partial}{\partial x}\hat{\imath}+\frac{\partial}{\partial y}\hat{\jmath}+\frac{\partial}{\partial z}\hat{k}\right).\left(-\frac{2xy}{(x^2+y^2)^2}\hat{\imath}+\frac{x^2-y^2}{(x^2+y^2)^2}\hat{\jmath}+\frac{y}{(x^2+y^2)^2}\hat{k}\right)$$

$$\Rightarrow\nabla.\vec{V}=-\frac{\partial}{\partial x}\left(\frac{2xy}{(x^2+y^2)^2}\right)+\frac{\partial}{\partial y}\left(\frac{x^2-y^2}{(x^2+y^2)^2}\right)+\frac{\partial}{\partial z}\left(\frac{y}{(x^2+y^2)^2}\right)$$

$$\Rightarrow\nabla.\vec{V}=0$$

Hence the velocity components $u=-\frac{2xy}{(x^2+y^2)^2}, v=\frac{x^2-y^2}{(x^2+y^2)^2}$ and $w=\frac{y}{(x^2+y^2)^2}$ represent the possible incompressible fluid.

Example

Test whether the motion specified by $\vec{V}=\frac{k^2(xj-y\hat{\imath})}{x^2+y^2}$ is possible motion for incompressible flow, so determine equation of streamline.

Solution For incompressible fluid $\nabla.\vec{V}=0$

$$\Rightarrow\nabla.\vec{V}=\left(\frac{\partial}{\partial x}\hat{\imath}+\frac{\partial}{\partial y}\hat{\jmath}+\frac{\partial}{\partial z}\hat{k}\right).\left(\frac{k^2(xj-y\hat{\imath})}{x^2+y^2}\right)=-\frac{\partial}{\partial x}\left(\frac{k^2y}{x^2+y^2}\right)+\frac{\partial}{\partial y}\left(\frac{k^2x}{x^2+y^2}\right)=0$$

For equation of streamline $u=-\frac{k^2y}{x^2+y^2}, v=\frac{k^2x}{x^2+y^2}, w=0$

$$\frac{dx}{u}=\frac{dy}{v}=\frac{dz}{w}\Rightarrow-\frac{dx}{\frac{k^2y}{x^2+y^2}}=\frac{dy}{\frac{k^2x}{x^2+y^2}}=\frac{dz}{0}\Rightarrow-\frac{dx}{y}=\frac{dy}{x}=\frac{dz}{0}\Rightarrow-xdx=ydy$$

$\Rightarrow x^2+y^2=C$ an equation of circle.

Example

Find a two dimensional in the xy – plane the x component of velocity $u = Ax$. Determine a possible y component for an incompressible flow. How many y components are possible?

Solution For incompressible fluid $\nabla.\vec{V} = 0$

$$\Rightarrow \nabla.\vec{V} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \Rightarrow \frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x} = -A \Rightarrow \frac{\partial v}{\partial y} = -A \Rightarrow v = -Ay + f(x)$$

The simple expression of v can be obtain by putting $f(x) = 0$ $\Rightarrow v = -Ay$

$\Rightarrow \vec{V} = u\hat{\imath} + v\hat{\jmath} = Ax\hat{\imath} - Ay\hat{\jmath}$ and number of expression for $\vec{V}$ could satisfy the differential continuity equation under the given conditions.

Example

Find a two dimensional in the xy – plane the x component of velocity $u = Ax(y - B)$ where $A = 1ft, B = 6ft$, x,y are measured in feet. Determine a possible y component for steady and an incompressible flow.

Solution For steady and incompressible fluid $\nabla.\vec{V} = 0$

$$\Rightarrow \nabla.\vec{V} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \Rightarrow \frac{\partial}{\partial x}(Ax(y - B)) + \frac{\partial v}{\partial y} = 0 \Rightarrow A(y - B) + \frac{\partial v}{\partial y} = 0$$

$$\Rightarrow \frac{\partial v}{\partial y} = -A(y - B) \Rightarrow \frac{\partial v}{\partial y} = -Ay + AB \Rightarrow v = -A\frac{y^2}{2} + ABy + f(x)$$

$$\Rightarrow v = -(1)\frac{y^2}{2} + (1)(6)y + f(x) \Rightarrow v = -\frac{y^2}{2} + 6y + f(x)$$

The simple expression of v can be obtain by putting $f(x) = 0$

$$\Rightarrow v = -\frac{y^2}{2} + 6y$$

$$\Rightarrow \vec{V} = u\hat{\imath} + v\hat{\jmath} = Ax(y - B)\hat{\imath} + \left(-\frac{y^2}{2} + 6y\right)\hat{\jmath}$$

$$\Rightarrow \vec{V} = x(y - 6)\hat{\imath} + \left(-\frac{y^2}{2} + 6y\right)\hat{\jmath}$$ where $A = 1ft, B = 6ft,$

Example

Find a two dimensional in the xy – plane the x component of velocity $u = \frac{A}{x}$ where $A = 2m^2/s$, x, is measured in meter. Determine a possible 'y' component for steady and an incompressible flow.

Solution For steady and incompressible fluid $\nabla.\vec{V} = 0$

$$\Rightarrow \nabla.\vec{V} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \Rightarrow \frac{\partial}{\partial x}\left(\frac{A}{x}\right) + \frac{\partial v}{\partial y} = 0 \Rightarrow Alnx + \frac{\partial v}{\partial y} = 0$$

$$\Rightarrow \frac{\partial v}{\partial y} = -Alnx \Rightarrow v = -Aylnx + f(x) \Rightarrow v = -2ylnx + f(x)$$

The simple expression of v can be obtain by putting $f(x) = 0$

$$\Rightarrow v = -2ylnx \Rightarrow \vec{V} = u\hat{\imath} + v\hat{\jmath} = \frac{A}{x}\hat{\imath} - 2ylnx\hat{\jmath} \Rightarrow \boldsymbol{\vec{V} = \frac{2}{x}\hat{\imath} - 2ylnx\hat{\jmath}}$$

Example

Find a two dimensional in the xy – plane the y component of velocity $v = \frac{2xy}{(x^2+y^2)^2}$. Determine x component of velocity for steady and an incompressible flow.

Solution For steady and incompressible fluid $\nabla.\vec{V} = 0$

$$\Rightarrow \nabla.\vec{V} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \Rightarrow \frac{\partial u}{\partial x} + \frac{\partial}{\partial y}\left(\frac{2xy}{(x^2+y^2)^2}\right) = 0 \Rightarrow \frac{\partial u}{\partial y} = -\frac{\partial}{\partial y}\left(\frac{2xy}{(x^2+y^2)^2}\right)$$

$$\Rightarrow u = \frac{x^2+(x^2+y^2)}{2(x^2+y^2)^2} + f(y)$$

The simple expression of u can be obtain by putting $f(y) = 0$

$$\Rightarrow u = \frac{x^2+(x^2+y^2)}{2(x^2+y^2)^2} \Rightarrow \vec{V} = u\hat{\imath} + v\hat{\jmath} = \frac{x^2+(x^2+y^2)}{2(x^2+y^2)^2}\hat{\imath} + \frac{2xy}{(x^2+y^2)^2}\hat{\jmath}$$

Example

Find a two dimensional in the xy – plane the x component of velocity $u = 3x^2y - y^3$. Determine y component of velocity for steady and an incompressible flow.

Solution For steady and incompressible fluid $\nabla.\vec{V} = 0$

$$\Rightarrow \nabla.\vec{V} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \Rightarrow \frac{\partial}{\partial x}(3x^2y - y^3) + \frac{\partial v}{\partial y} = 0 \Rightarrow 6xy + \frac{\partial v}{\partial y} = 0$$

$$\Rightarrow \frac{\partial v}{\partial y} = -6xy \Rightarrow v = -6x\frac{y^2}{2} + f(x) \Rightarrow v = -3xy^2 + f(x)$$

The simple expression of v can be obtain by putting $f(x) = 0$ $\Rightarrow v = -3xy^2$

$$\Rightarrow \vec{V} = u\hat{\imath} + v\hat{\jmath} = (3x^2y - y^3)\hat{\imath} + (-3xy^2)\hat{\jmath}$$

Example

Show that the velocity components $u = 2x^2 + y^2 - x^2y, v = x^3 + x(y^2 - 4y)$ represent the possible incompressible fluid.

Solution For incompressible fluid $\nabla.\vec{V} = 0$

$$\Rightarrow \nabla.\vec{V} = \left(\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath}\right).\left((2x^2 + y^2 - x^2y)\hat{\imath} + \left(x^3 + x(y^2 - 4y)\right)\hat{\jmath}\right)$$

$$\Rightarrow \nabla.\vec{V} = \frac{\partial}{\partial x}(2x^2 + y^2 - x^2y) + \frac{\partial}{\partial y}\left(x^3 + x(y^2 - 4y)\right)$$

$$\Rightarrow \nabla.\vec{V} = 0$$

Hence the velocity components $u = 2x^2 + y^2 - x^2y, v = x^3 + x(y^2 - 4y)$ represent the possible incompressible fluid.

Example

Show that the velocity components $u = 2xy - x^2y, v = 2xy - y^2 + x^2$ represent the possible incompressible fluid.

Solution For incompressible fluid $\nabla.\vec{V} = 0$

$\Rightarrow \nabla.\vec{V} = \left(\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath}\right).\left((2xy - x^2y)\hat{\imath} + (2xy - y^2 + x^2)\hat{\jmath}\right)$

$\Rightarrow \nabla.\vec{V} = \frac{\partial}{\partial x}(2xy - x^2y) + \frac{\partial}{\partial y}(2xy - y^2 + x^2)$

$\Rightarrow \nabla.\vec{V} = 2x - 2xy \neq 0$

Hence the velocity components $u = 2xy - x^2y, v = 2xy - y^2 + x^2$ do not represent the possible incompressible fluid.

Example

Show that the velocity components $u = 2y^2 + 2xz, v = -2yz + 6x^2yz$ and $w = 3x^2z^2 + x^3y^4$ represent the possible incompressible fluid.

Solution For incompressible fluid $\nabla.\vec{V} = 0$

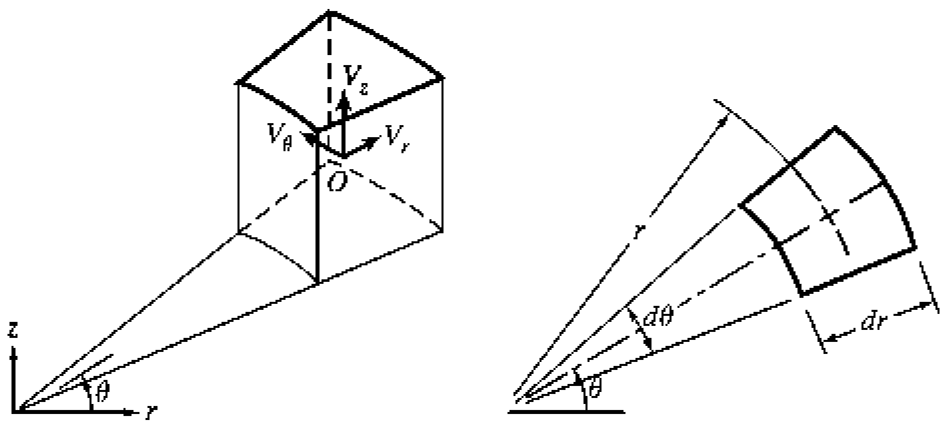
$\Rightarrow \nabla.\vec{V} = \left(\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}\right).\Big((2y^2 + 2xz)\hat{\imath} + (-2yz + 6x^2yz)\hat{\jmath} +$
$(3x^2z^2 + x^3y^4)\hat{k}\Big)$

$\Rightarrow \nabla.\vec{V} = \frac{\partial}{\partial x}(2y^2 + 2xz) + \frac{\partial}{\partial y}(-2yz + 6x^2yz) + \frac{\partial}{\partial z}(3x^2z^2 + x^3y^4)$

$\Rightarrow \nabla.\vec{V} = 12x^2z \neq 0$

Hence the velocity components $u = 2y^2 + 2xz, v = -2yz + 6x^2yz$ and $w = 3x^2z^2 + x^3y^4$ do not represent the possible incompressible fluid.

Equation of Continuity or Conservation of Mass (Continuity Equation) in Cylindrical Coordinate System



(*a*) Isometric view (*b*) Projection on $r\theta$ plane

A suitable differential control volume for cylindrical coordinates is shown in Figure.

The density at the center, O, of the control volume is assumed to be ρ and the velocity there is assumed to be $\vec{V} = \hat{e}_r V_r + \hat{e}_\theta V_\theta + \hat{k} V_z$

Where $\hat{e}_r, \hat{e}_\theta, \hat{k}$ are unit vectors in the $r, \theta,$ and z directions, respectively, and V_r, V_θ, V_z are the velocity components in the $r, \theta,$ and z directions, respectively. To evaluate $\int_{CS} \rho \vec{V}.d\vec{A} = 0$, we must consider the mass flux through each of the six faces of the control surface. The properties at each of the six faces of the control surface are obtained from a Taylor series expansion about point O. The details of the mass flux evaluation are shown in Table (given below). Velocity components V_r, V_θ, V_z are all assumed to be in the positive coordinate directions and we have again used the convention that the area normal is positive outwards on each face, and higher-order terms have been neglected.

We see that the net rate of mass flux out through the control surface is given by (in table)

$$\int_{CS} \rho \vec{V}.d\vec{A} = \left[\rho V_r + r\frac{\partial \rho V_r}{\partial r} + \frac{\partial \rho V_\theta}{\partial \theta} + r\frac{\partial \rho V_z}{\partial z}\right] dr d\theta dz$$

The mass inside the control volume at any instant is the product of the mass per unit volume, ρ, and the volume, $rdrd\theta dz$. Thus the rate of change of mass inside the control volume is given by

$$\frac{\partial}{\partial t}\int_{CV}\rho d\vec{V} = \frac{\partial \rho}{\partial t} rdrd\theta dz$$

In cylindrical coordinates the differential equation for conservation of mass is then

$$\frac{\partial}{\partial t}\int_{CV}\rho d\vec{V} + \int_{CS}\rho\vec{V}.d\vec{A} =$$
$$\frac{\partial \rho}{\partial t} rdrd\theta dz + \left[\rho V_r + r\frac{\partial \rho V_r}{\partial r} + \frac{\partial \rho V_\theta}{\partial \theta} + r\frac{\partial \rho V_z}{\partial z}\right] drd\theta dz = 0$$

$$\Rightarrow \rho V_r + r\frac{\partial \rho V_r}{\partial r} + \frac{\partial \rho V_\theta}{\partial \theta} + r\frac{\partial \rho V_z}{\partial z} + r\frac{\partial \rho}{\partial t} = 0$$

$$\Rightarrow \frac{\partial (r\rho V_r)}{\partial r} + \frac{\partial \rho V_\theta}{\partial \theta} + r\frac{\partial \rho V_z}{\partial z} + r\frac{\partial \rho}{\partial t} = 0$$

$$\Rightarrow \boldsymbol{\frac{1}{r}\frac{\partial (r\rho V_r)}{\partial r} + \frac{1}{r}\frac{\partial \rho V_\theta}{\partial \theta} + \frac{\partial \rho V_z}{\partial z} + \frac{\partial \rho}{\partial t} = 0}$$

$$\Rightarrow \left(\hat{e}_r\frac{\partial}{\partial r} + \hat{e}_\theta\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{k}\frac{\partial}{\partial z}\right).\rho\left(\hat{e}_r V_r + \hat{e}_\theta V_\theta + \hat{k}V_z\right) + \frac{\partial \rho}{\partial t} = 0$$

$$\Rightarrow \nabla.\rho\vec{V} + \frac{\partial \rho}{\partial t} = 0$$

For an **incompressible fluid**, ρ = constant, and Equation reduces to

$$\Rightarrow \nabla.\rho\vec{V} = 0 \Rightarrow \nabla.\vec{V} = 0 \Rightarrow \boldsymbol{\frac{1}{r}\frac{\partial (rV_r)}{\partial r} + \frac{1}{r}\frac{\partial V_\theta}{\partial \theta} + \frac{\partial V_z}{\partial z} = 0}$$

Thus the velocity field, V{x, y, z, t), for incompressible flow must satisfy $\nabla.\vec{V} = 0$

For **steady flow**, Equation reduces to

$$\Rightarrow \boldsymbol{\frac{1}{r}\frac{\partial (r\rho V_r)}{\partial r} + \frac{1}{r}\frac{\partial \rho V_\theta}{\partial \theta} + \frac{\partial \rho V_z}{\partial z} = 0 \Rightarrow \nabla.\rho\vec{V} = 0}$$

Example Differential Continuity Equation in Cylindrical Coordinates

Consider a one-dimensional radial flow in the $r\theta$ plane, given by $V_r = f(r)$ and $V_\theta = 0$. Determine the conditions on $f(r)$ required for the flow to be incompressible.

Given: One-dimensional radial flow in the $r\theta$ plane: $V_r = f(r)$ and $V_\theta = 0$.

Find: Requirements on $f(r)$ for incompressible flow.

Solution:

Governing equation: $\nabla \cdot \rho\vec{V} + \dfrac{\partial \rho}{\partial t} = 0$

For incompressible flow in cylindrical coordinates this reduces to Eq. 5.2b,

$$\frac{1}{r}\frac{\partial}{\partial r}(rV_r) + \frac{1}{r}\frac{\partial}{\partial \theta}V_\theta + \frac{\partial V_z}{\partial z} = 0$$

For the given velocity field, $\vec{V} = \vec{V}(r)$. $V_\theta = 0$ and partial derivatives with respect to z are zero, so

$$\frac{1}{r}\frac{\partial}{\partial r}(rV_r) = 0$$

Integrating with respect to r gives

$$rV_r = \text{constant}$$

Thus the continuity equation shows that the radial velocity must be $V_r = f(r) = C/r$ for one-dimensional radial flow of an incompressible fluid. This is not a surprising result: As the fluid moves outwards from the center, the volume flow rate (per unit depth in the z direction) $Q = 2\pi rV$ at any radius r is constant.

Stream Function for Two-Dimensional Incompressible Flow

It is a function that describe pattern of flow.

Or It is discharge per unit thickness. Denoted by $\psi = \psi(x, y)$

There are various ways to define the stream function. We start with the two dimensional version of the **continuity equation** for incompressible flow.

$$\nabla.\rho\vec{V} = 0 \Rightarrow \frac{\partial\rho u}{\partial x} + \frac{\partial\rho v}{\partial y} + \frac{\partial\rho w}{\partial z} = 0$$

$$\Rightarrow \frac{\partial\rho u}{\partial x} + \frac{\partial\rho v}{\partial y} = 0 \quad \ldots\ldots\ldots(1)$$ For 2 – dimension

If we introduce $\rho u = \frac{\partial\psi}{\partial y}$ and $\rho v = -\frac{\partial\psi}{\partial x}$ ……….(2)

Then (2) satisfied (1) $\Rightarrow \frac{\partial}{\partial x}\left(\frac{\partial\psi}{\partial y}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial\psi}{\partial x}\right) = 0$

$$\Rightarrow \frac{\partial^2\psi}{\partial x\partial y} - \frac{\partial^2\psi}{\partial y\partial x} = 0$$

Thus stream functions are defined as using continuity equation

$$u = \frac{\partial\psi}{\partial y} \quad \text{and} \quad v = -\frac{\partial\psi}{\partial x}$$

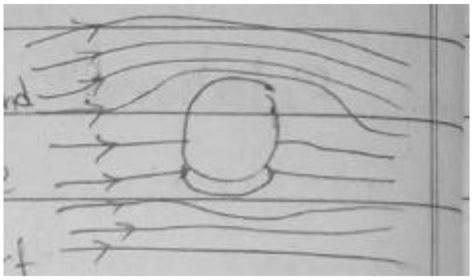
Using equation of streamline $udy - vdx = 0$

$$\Rightarrow u\frac{\partial\psi}{\partial y} + v\frac{\partial\psi}{\partial x} = 0$$ where $d\psi = 0$

Advantages of stream function:

i. Stream function is taken only for a flow in plane.
ii. We have the simplified analysis by having to determine only unknown function $\psi(x, y)$ rather than two functions.
iii. Using stream function we can find stream lines

Description of a stream function describing stream lines in a cylinder



The figure shows that computed streamlines around a cylinder. In this case we move with the object and flow proceeds from left to right. Since the streamline is traced out by moving a particle at every point along the path the velocity is tangent to path. Since there is no normal component of the velocity along the path, mass cannot cross a streamline.

$$\psi(r,\theta) = V_{\infty} r Sin(\theta)\left[1 - \frac{R^2}{r^2}\right]$$

Certain conditions to define stream function

- Stream function is defined for any two or three dimensional flow.
- Stream function is defined for three dimensional axial symmetric flows.
- For two dimensional flow streamlines are perpendicular to equipotential lines.
- Stream function is defined for incompressible (divergence free) flow in 2 dimension.

Example

Compute the velocity components in the fluid flow describing the stream function given as $\psi(x,y) = (aV)^{1/2}xf(\eta)\ ;\eta = (a/V)^{1/2}y$ to verify the velocity component by satisfying the continuity equation of two dimensional flow.

Solution

Given that $\psi(x,y) = (aV)^{1/2}xf(\eta)\ ;\eta = (a/V)^{1/2}y$

The stream functions are defined as using continuity equation

$$u = \frac{\partial\psi}{\partial y} = \frac{\partial}{\partial y}\psi(x,y)$$

$$\Rightarrow u = \frac{\partial}{\partial y}\psi(x,y) = \frac{\partial}{\partial y}\left((aV)^{1/2}xf(\eta)\right) = \frac{\partial}{\partial y}\left((aV)^{1/2}xf\left(\left(\frac{a}{V}\right)^{1/2}y\right)\right)$$

$$\Rightarrow u = (aV)^{1/2}xf'^{\left(\left(\frac{a}{V}\right)^{\frac{1}{2}}y\right)\left(\frac{a}{V}\right)^{\frac{1}{2}}} = a^{\frac{1}{2}}V^{\frac{1}{2}}a^{\frac{1}{2}}V^{-\frac{1}{2}}xf'\left(\left(\frac{a}{V}\right)^{1/2}y\right)$$

$$\Rightarrow u = axf'\left(\left(\frac{a}{V}\right)^{1/2}y\right)$$

Also $v = -\frac{\partial\psi}{\partial x} = -\frac{\partial}{\partial x}\psi(x,y)$

$$\Rightarrow v = -\frac{\partial}{\partial x}\left((aV)^{\frac{1}{2}}xf(\eta)\right) = -\frac{\partial}{\partial x}\left((aV)^{1/2}xf\left(\left(\frac{a}{V}\right)^{1/2}y\right)\right)$$

$$\Rightarrow v = -(aV)^{1/2}f\left(\left(\frac{a}{V}\right)^{1/2}y\right)$$

To verify equation of continuity $\nabla.\vec{V} = 0$

$$\Rightarrow \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{\partial}{\partial y}\left(axf'\left(\left(\frac{a}{V}\right)^{1/2}y\right)\right) - \frac{\partial}{\partial x}\left((aV)^{1/2}f\left(\left(\frac{a}{V}\right)^{1/2}y\right)\right) = 0$$

Hence verified.

Example Stream Function for Flow in a Corner

Given the velocity field for steady, incompressible flow in a corner (Example 2.1), $\vec{V} = Ax\hat{i} - Ay\hat{j}$, with $A = 0.3\,\text{s}^{-1}$, determine the stream function that will yield this velocity field. Plot and interpret the streamline pattern in the first and second quadrants of the xy plane.

Given: Velocity field, $\vec{V} = Ax\hat{i} - Ay\hat{j}$, with $A = 0.3\ \text{s}^{-1}$.

Find: Stream function ψ and plot in first and second quadrants; interpret the results.

Solution:

The flow is incompressible, so the stream function satisfies Eq. 5.4.

From Eq. 5.4, $u = \dfrac{\partial \psi}{\partial y}$ and $v = -\dfrac{\partial \psi}{\partial y}$. From the given velocity field,

$$u = Ax = \frac{\partial \psi}{\partial y}$$

Integrating with respect to y gives

$$\psi = \int \frac{\partial \psi}{\partial y} dy + f(x) = Axy + f(x) \qquad (1)$$

where $f(x)$ is arbitrary. The function $f(x)$ may be evaluated using the equation for v. Thus, from Eq. 1,

$$v = -\frac{\partial \psi}{\partial x} = -Ay - \frac{df}{dx} \qquad (2)$$

From the given velocity field, $v = -Ay$. Comparing this with Eq. 2 shows that $\dfrac{df}{dx} = 0$, or $f(x) = \text{constant}$. Therefore, Eq. 1 becomes

$$\psi = Axy + c \longleftarrow \psi$$

Lines of constant ψ represent streamlines in the flow field. The constant c may be chosen as any convenient value for plotting purposes. The constant is chosen as zero in order that the streamline through the origin be designated as $\psi = \psi_1 = 0$. Then the value for any other streamline represents the flow between the origin and that streamline. With $c = 0$ and $A = 0.3\ \text{s}^{-1}$, then

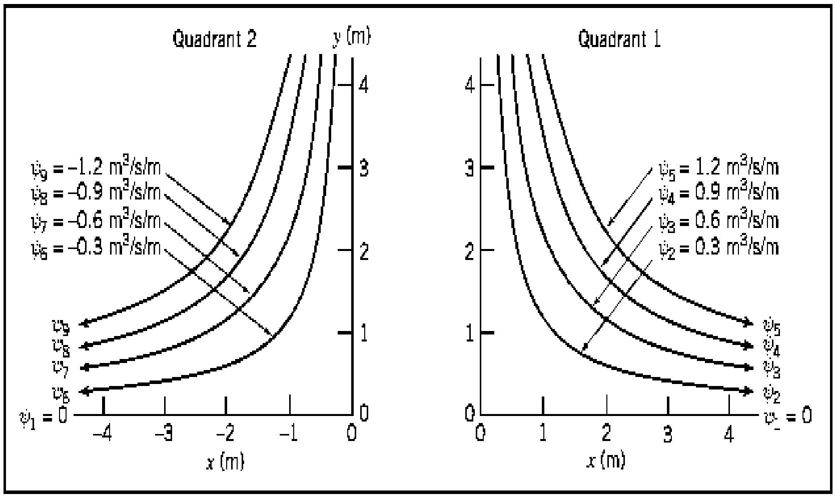
$$\psi = 0.3xy \quad (\text{m}^3/\text{s/m})$$

{This equation of a streamline is identical to the result ($xy = \text{constant}$) obtained in Example 2.1.}

Separate plots of the streamlines in the first and second quadrants are presented below. Note that in quadrant 1, $u > 0$, so ψ values are positive. In quadrant 2, $u < 0$, so ψ values are negative.

In the first quadrant, since $u > 0$ and $v < 0$, the flow is from left to right and down. The volume flow rate between the streamline $\psi = \psi_1$ through the origin and the streamline $\psi = \psi_2$ is

$$Q_{12} = \psi_2 - \psi_1 = 0.3\,\text{m}^3/\text{s/m}$$



In the second quadrant, since $u < 0$ and $v < 0$, the flow is from right to left and down. The volume flow rate between streamlines ψ_7 and ψ_9 is

$$Q_{79} = \psi_9 - \psi_7 = [-1.2 - (-0.6)]\,\mathrm{m}^3/\mathrm{s}/\mathrm{m} = -0.6\,\mathrm{m}^3/\mathrm{s}/\mathrm{m}$$

The negative sign is consistent with flow having $u < 0$.

As both the streamline spacing in the graphs and the equation for $\vec{V}$ indicate, the velocity is smallest near the origin (a "corner").

There is an *Excel* workbook for this problem that can be used to generate streamlines for this and many other stream functions.

Volume Flow Rate

Volume flow rate measure the amount of volume that passes through an area per unit time. The volume flow rate equation is $Q = AV$ where

$Q =$ Volume Flow Rate

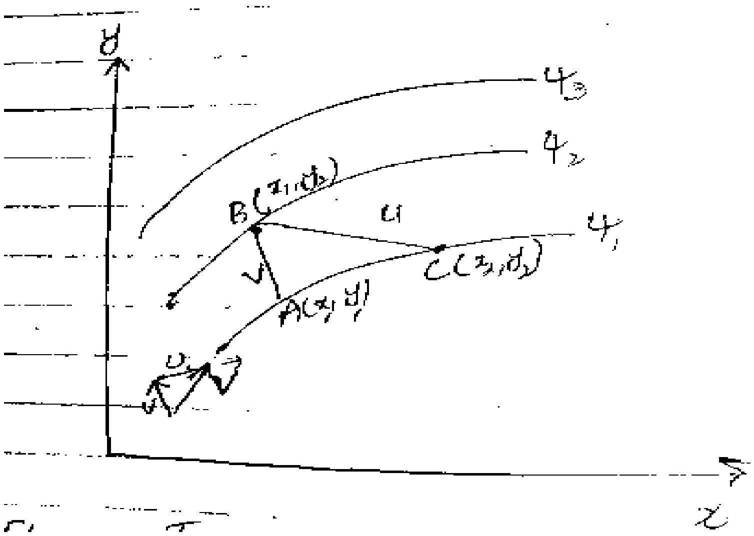
$A =$ Area Occupied By Flow Material

$V =$ Average Velocity

Question

Show that volume flow rate between two streamlines can be written as the difference between the constant values of the stream function ψ defining the two streamlines

Solution



Flow rate across AB $\quad Q = \int_{y_1}^{y_2} u\, dy = \int_{\psi_1}^{\psi_2} \frac{\partial \psi}{\partial y} dy \quad$ ....(1)

Since $\psi = \psi(x, y)$ therefore $d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy$

Along AB 'x' is constant therefore $dx = 0$ and we have $d\psi = \frac{\partial \psi}{\partial y} dy$

(1) Implies Flow rate across AB $\quad Q = \int_{\psi_1}^{\psi_2} d\psi = \psi_2 - \psi_1$

Flow rate across BC $\quad Q = \int_{x_1}^{x_2} v\, dx = -\int_{\psi_1}^{\psi_2} \frac{\partial \psi}{\partial x} dx \quad$ ....(2)

Since $\psi = \psi(x, y)$ therefore $d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy$

Along BC 'y' is constant therefore $dy = 0$ and we have $d\psi = \frac{\partial \psi}{\partial x} dx$

(2) Implies Flow rate across BC $\quad Q = -\int_{\psi_2}^{\psi_1} d\psi = \int_{\psi_1}^{\psi_2} d\psi = \psi_2 - \psi_1$

Question

A 2 Dimensional incompressible flow field is given as $u = 2x, v = -2y$ determine the stream function ψ, also indicate that whether or not the given flow field is Irrotational, if it is Irrotational determine velocity potential.

Solution: Given that $u = 2x, v = -2y$ then

$\frac{\partial\psi}{\partial y} = u = 2x$ …….(1) and $-\frac{\partial\psi}{\partial x} = v = -2y$ …….(2)

Integrating (1) with respect to 'y' $\psi = 2xy + f(x)$ …….(3)

Differentiating (3) with respect to 'x' $\frac{\partial\psi}{\partial x} = 2y + f'(x)$

$2y = 2y + f'(x) \Rightarrow f'(x) = 0 \Rightarrow f(x) = c$ using (2)

$(3) \Rightarrow \psi = 2xy + c$

Now we have to show that $\nabla \times \vec{V} = 0$ with $\vec{V} = 2x\hat{\imath} - 2y\hat{\jmath}$

$$\Rightarrow \nabla \times \vec{V} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x & -2y & 0 \end{vmatrix} = 0 \quad \text{after Simplification}$$

Hence flow is Irrotational.

Now $\vec{V} = -\nabla\varphi \Rightarrow 2x\hat{\imath} - 2y\hat{\jmath} = -\frac{\partial\varphi}{\partial x}\hat{\imath} - \frac{\partial\varphi}{\partial y}\hat{\jmath}$ gives

$\frac{\partial\varphi}{\partial x} = -2x$ …….(1) and $\frac{\partial\varphi}{\partial y} = 2y$ …….(2)

Integrating (1) with respect to 'x' $\varphi = -x^2 + f(y)$ …….(3)

Differentiating (3) with respect to 'y' $\frac{\partial\varphi}{\partial y} = f'(y)$

$2y = f'(x) \Rightarrow f(x) = y^2 + c$ using (2)

$(3) \Rightarrow \varphi = -x^2 + y^2 + c$

Question

Whether $\vec{V} = \frac{k^2(x\hat{j} - y\hat{\imath})}{x^2+y^2}$ is of potential kind? If so determine velocity potential.

Solution:

We have to show that $\nabla \times \vec{V} = 0$ with $\vec{V} = \frac{k^2(x\hat{j} - y\hat{\imath})}{x^2+y^2} = -\frac{k^2 y}{x^2+y^2}\hat{\imath} + \frac{k^2 x}{x^2+y^2}\hat{j}$

$$\Rightarrow \nabla \times \vec{V} = \begin{vmatrix} \hat{\imath} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\frac{k^2 y}{x^2+y^2} & \frac{k^2 x}{x^2+y^2} & 0 \end{vmatrix} = 0 \qquad \text{after Simplification}$$

Hence flow is Irrotational. And the given velocity is of potential kind.

Now $\vec{V} = -\nabla\varphi \Rightarrow -\frac{k^2 y}{x^2+y^2}\hat{\imath} + \frac{k^2 x}{x^2+y^2}\hat{j} = -\frac{\partial\varphi}{\partial x}\hat{\imath} - \frac{\partial\varphi}{\partial y}\hat{j}$ gives

$\frac{\partial\varphi}{\partial x} = \frac{k^2 y}{x^2+y^2}$ …….(1) and $\frac{\partial\varphi}{\partial y} = -\frac{k^2 x}{x^2+y^2}$ …….(2)

Integrating (1) with respect to 'x' $\varphi = k^2 y\tan^{-1}\left(\frac{x}{y}\right) + f(y)$ …….(3)

Differentiating (3) with respect to 'y' $\frac{\partial\varphi}{\partial y} = \cdots + f'(y)$

$\ldots\ldots. = f'(y) \Rightarrow f(y) = \cdots$ ……….. using (2)

$(3) \Rightarrow \varphi = \cdots$ ………..

Motion of a Fluid Particle (Kinematics)

There are four types of motion of a fluid element;

i. **Translation:** Motion in which the particle moves from one point to another. Orientation not changed.

ii. **Rotation:** Motion which can occur about any or all of the x, y or z axes. In this orientation changed.

iii. **Linear Deformation:** Motion in which the particle's sides stretch (expansion/dilation) or contract (contraction/reduction).

- Linear deformation of a fluid element occurs when it passes through the zone of accelerated or decelerated.
- Dilation or expansion of fluid element occurs when it passes through deaccelerated flow region.
- Contraction or reduction of fluid element occurs when it passes through an accelerated flow region.
- Decrease or increase in magnitude of normal stress is responsible for dilation or contraction of the fluid element.

iv. **Angular Deformation:** Motion in which the angles (which were initially 90° for our particle) between the sides change. It is also called stress deformation.

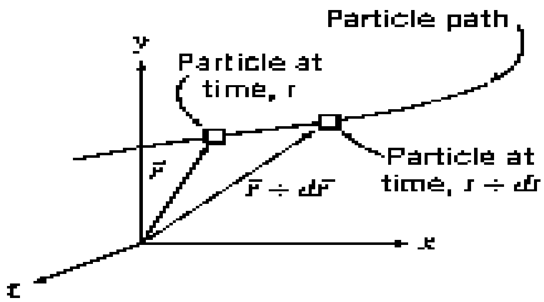
Fluid Translation: Acceleration of a Fluid Particle in a Velocity Field

The translation of a fluid particle is obviously connected with the velocity field $\vec{V} = \vec{V}(x, y, z, t)$. We will need the acceleration of a fluid particle for use in Newton's second law. It might seem that we could simply compute this as $\vec{a} = \frac{\partial \vec{V}}{\partial t}$. This is incorrect, because $\vec{V}$ is a field, i.e., it describes the whole flow and not just the motion of an individual particle.

The problem, then, is to retain the field description for fluid properties and obtain an expression for the acceleration of a fluid particle as it moves in a flow field. Stated simply, the problem is:

Given the velocity field, $\vec{V} = \vec{V}(x, y, z, t)$, find the acceleration of a fluid particle, $\vec{a}_p$.

Consider a particle moving in a velocity field. At time *t*, the particle is at the position x, y, z and has a velocity corresponding to the velocity at that point in space at time t, $\vec{V}_p \mid_t = \vec{V}(x, y, z, t)$



At $t + dt$, the particle has moved to a new position, with coordinates $x + dx,$ $y + dy, z + dz$, and has a velocity given by

$$\vec{V}_p \mid_{t+dt} = \vec{V}(x + dx, y + dy, z + dz, t + dt)$$

Now $d\vec{V}_p = d\vec{V}(x, y, z, t)$

$$d\vec{V}_p = \frac{\partial \vec{V}}{\partial x} dx_p + \frac{\partial \vec{V}}{\partial y} dy_p + \frac{\partial \vec{V}}{\partial z} dz_p + \frac{\partial \vec{V}}{\partial t} dt$$

$$\vec{a}_p = \frac{d\vec{V}_p}{dt} = \frac{\partial \vec{V}}{\partial x}\frac{dx_p}{dt} + \frac{\partial \vec{V}}{\partial y}\frac{dy_p}{dt} + \frac{\partial \vec{V}}{\partial z}\frac{dz_p}{dt} + \frac{\partial \vec{V}}{\partial t}\frac{dt}{dt}$$

$\vec{a}_p = \frac{d\vec{V}_p}{dt} = u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z} + \frac{\partial \vec{V}}{\partial t}$ where $u = \frac{dx_p}{dt}, v = \frac{dy_p}{dt}, w = \frac{dz_p}{dt}$

We may write $\vec{a}_p \equiv \frac{D\vec{V}_p}{Dt} = u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z} + \frac{\partial \vec{V}}{\partial t}$

The derivative, $\frac{D}{Dt} = \frac{dx_p}{dt}\frac{\partial}{\partial x} + \frac{dy_p}{dt}\frac{\partial}{\partial y} + \frac{dz_p}{dt}\frac{\partial}{\partial z} + \frac{\partial}{\partial t}$, is commonly called the **substantial derivative** to remind us that it is computed for a particle of "substance." It often is called the **material derivative** or **particle derivative**.

The **physical significance** of the terms in $\vec{a}_p \equiv \frac{D\vec{V}}{Dt} = u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z} + \frac{\partial \vec{V}}{\partial t}$ is

$$\vec{a}_p = \underset{\substack{\text{total}\\ \text{acceleration}\\ \text{of a particle}}}{\frac{D\vec{V}}{Dt}} = \underbrace{u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}}_{\substack{\text{convective}\\ \text{acceleration}}} + \underset{\substack{\text{local}\\ \text{acceleration}}}{\frac{\partial \vec{V}}{\partial t}}$$

It recognizes that a fluid particle moving in a flow field may undergo acceleration for either of two reasons. This is a steady flow in which particles are convected toward the low-velocity region (near the“corner”), and then away to a high velocity region. If a flow field is unsteady a fluid particle will undergo an additional local acceleration, because the velocity field is a function of time.

The convective acceleration may be written as a single vector expression using the gradient operator ∇. Thus

$$u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z} = \vec{V}.\left(\nabla\vec{V}\right)$$

Hence $$\vec{a}_p \equiv \frac{D\vec{V}}{Dt} = \vec{V}.\left(\nabla\vec{V}\right) + \frac{\partial \vec{V}}{\partial t}$$

For a two-dimensional flow, say $\vec{V} = \vec{V}(x, y, t)$, Equation reduces to

$$\vec{a}_p \equiv \frac{D\vec{V}}{Dt} = u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + \frac{\partial \vec{V}}{\partial t}$$

For a one-dimensional flow, say $\vec{V} = \vec{V}(x, t)$, Equation reduces to

$$\vec{a}_p \equiv \frac{D\vec{V}}{Dt} = u\frac{\partial \vec{V}}{\partial x} + \frac{\partial \vec{V}}{\partial t}$$

Finally, for a steady flow in three dimensions, Equation becomes

$$\vec{a}_p \equiv \frac{D\vec{V}}{Dt} = u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}$$

Which, as we have seen, is not necessarily zero. Thus a fluid particle may undergo a convective acceleration due to its motion, even in a steady velocity field.

Equation $\vec{a}_p \equiv \frac{D\vec{V}}{Dt} = u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z} + \frac{\partial \vec{V}}{\partial t}$ is a vector equation. As with all vector equations, it may be written in scalar component equations. Relative to an xyz coordinate system, the scalar components of Equation are written

$$a_{x_p} = \frac{Du}{Dt} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t}$$

$$a_{y_p} = \frac{Dv}{Dt} = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + \frac{\partial v}{\partial t}$$

$$a_{z_p} = \frac{Dw}{Dt} = u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} + \frac{\partial w}{\partial t}$$

The components of acceleration in cylindrical coordinates may be obtained by expressing the velocity, $\vec{V}$, in cylindrical coordinates. Thus,

$$a_{r_p} = V_r\frac{\partial V_r}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_r}{\partial \theta} - \frac{V_\theta^2}{r} + V_z\frac{\partial V_r}{\partial z} + \frac{\partial V_r}{\partial t}$$

$$a_{\theta_p} = V_r\frac{\partial V_\theta}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_\theta}{\partial \theta} + \frac{V_r V_\theta}{r} + V_z\frac{\partial V_\theta}{\partial z} + \frac{\partial V_\theta}{\partial t}$$

$$a_{z_p} = V_r\frac{\partial V_z}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_z}{\partial \theta} + V_z\frac{\partial V_z}{\partial z} + \frac{\partial V_z}{\partial t}$$

Example

Following the fluid particle, calculate the y component of the acceleration for a particle where velocity vector is given by $\vec{V} = (3z - x^2, yt^2, xz^2)$ in *ft/Sec* at a point $x = 1ft, y = 1ft, z = 9ft, t = 2Sec$

Solution

The y component of the acceleration for given fluid is

$$a_y(x, y, z, t) = \frac{DV_y}{Dt} = V_x\frac{\partial V_y}{\partial x} + V_y\frac{\partial V_y}{\partial y} + V_z\frac{\partial V_y}{\partial z} + \frac{\partial V_y}{\partial t}$$

Using $\vec{V} = 3z - x^2\hat{\imath} + yt^2\hat{\jmath} + xz^2\hat{k}$

$$a_y(x, y, z, t) = (3z - x^2)\frac{\partial}{\partial x}(yt^2) + (yt^2)\frac{\partial}{\partial y}(yt^2) + (xz^2)\frac{\partial}{\partial z}(yt^2) + \frac{\partial}{\partial t}(yt^2)$$

$$a_y(x, y, z, t) = (3z - x^2)(0) + (yt^2)(t^2) + (xz^2)(0) + (2yt)$$

$$a_y(x, y, z, t) = yt^4 + 2yt$$

$$a_y(1,1,9,2) = 20ft/Sec$$

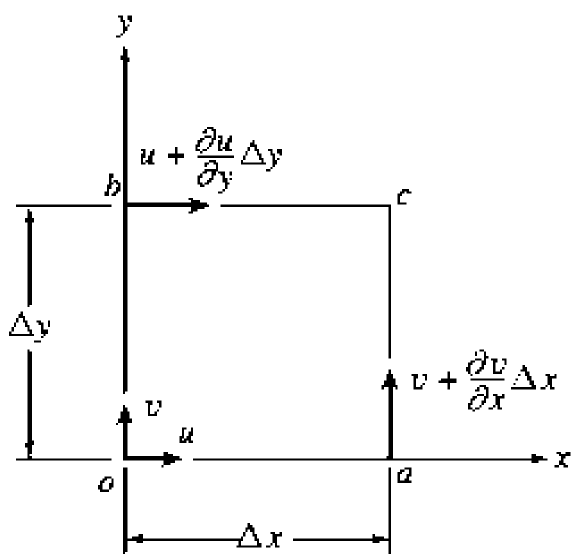
Fluid Rotation

The average angular velocity of any two mutually perpendicular linear elements of the particle is called rotation. It is denoted by $\vec{\omega}$.

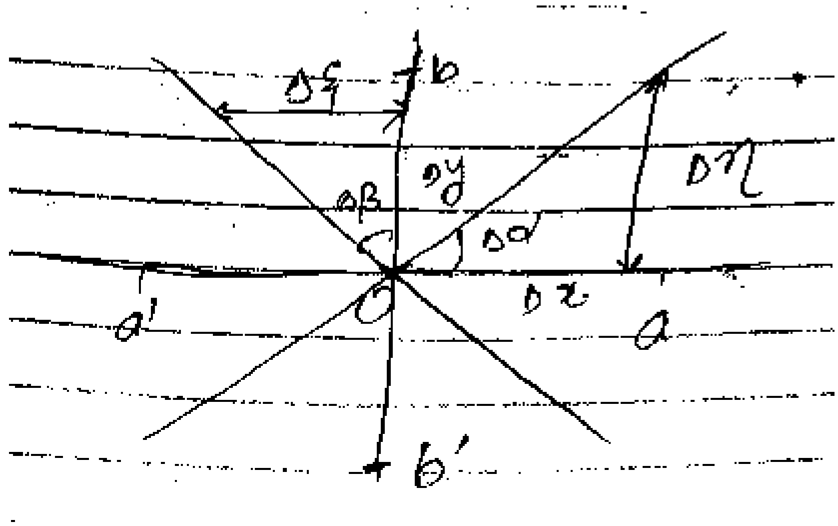
A fluid particle moving in a general three-dimensional flow field may rotate about all three coordinate axes. Thus particle rotation is a vector quantity and, in general, $\vec{\omega} = \omega_x \hat{\imath} + \omega_y \hat{\jmath} + \omega_z \hat{k}$ where ω_x is the rotation about the x axis, ω_y is the rotation about the y axis, and ω_z, is the rotation about the z axis. The positive sense of rotation is given by the right-hand rule.

Mathematical expression for Fluid Rotation

To demonstrate the development of rotational character of flow field, let us consider a fluid element of rectangular shapes with sides $\Delta x, \Delta y$ in 2D flow field. It is assume that velocity vector increases within the direction of increasing coordinate axis.



Velocity components on the boundaries of a fluid element.



Rotation of fluid element in 2 dimension

Consider the motion of fluid element in *xy* plane view of the particle at time *t*. the component of velocity at every point in the flow field are given by $u(x, y), v(x, y)$ Now if velocities at points 'a' and 'b' are different from O then in time interval Δt the two mutually perpendicular lines 'oa' and 'ob' will rotate as show in figure '2'.

Rotation of line 'oa'

The rotation of the line 'oa' of length Δx is due to the variation of 'y' component. If $\vec{V}_0$ is the y – component of the velocity at point O then by Taylor Series expansion of velocities at point 'a' can be written as;

$$\vec{V} = \vec{V}_0 + \frac{\partial v}{\partial x}\Delta x$$

The angular velocity of line 'oa' is given by

$$\omega_{oa} = \lim_{\Delta t \to 0} \frac{\Delta \alpha}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta \eta / \Delta x}{\Delta t} \quad \ldots\ldots\ldots\ldots.(1)$$

Since $\frac{\Delta \eta}{\Delta t} = \Delta \vec{V} = \vec{V} - \vec{V}_0 = \vec{V}_0 + \frac{\partial v}{\partial x}\Delta x - \vec{V}_0$

$$\Rightarrow \frac{\Delta \eta}{\Delta t} = \frac{\partial v}{\partial x}\Delta x \Rightarrow \Delta \eta = \frac{\partial v}{\partial x}\Delta x \Delta t \Rightarrow \frac{\Delta \eta}{\Delta x} = \frac{\partial v}{\partial x}\Delta t$$

$$(1) \Rightarrow \omega_{oa} = \lim_{\Delta t \to 0} \frac{\Delta \eta / \Delta x}{\Delta t} = \lim_{\Delta t \to 0} \frac{\frac{\partial v}{\partial x}\Delta t}{\Delta t} \Rightarrow \omega_{oa} = \frac{\partial v}{\partial x} \quad \ldots\ldots\ldots\ldots.(2)$$

$$\omega = \frac{\theta}{t}$$

$$s = r\theta \Rightarrow \theta = \frac{s}{r}$$

Then $\Delta \alpha = \frac{\Delta \eta}{\Delta x}$

Rotation of line 'ob'

The rotation of the line 'ob' of length Δy is due to the variation of 'x' component. If $\vec{U}_0$ is the x – component of the velocity at point O then by Taylor Series expansion of velocities at point 'b' can be written as;

$$\vec{U} = \vec{U}_0 + \frac{\partial u}{\partial y}\Delta y$$

$\omega = \dfrac{\theta}{t}$
$s = r\theta \Rightarrow \theta = \dfrac{s}{r}$
Then $\Delta\beta = \frac{\Delta\xi}{\Delta y}$

The angular velocity of line 'ob' is given by

$$\omega_{ob} = \lim_{\Delta t\to 0}\frac{\Delta\beta}{\Delta t} = \lim_{\Delta t\to 0}\frac{\Delta\xi/\Delta y}{\Delta t} \qquad \ldots\ldots\ldots\ldots.(3)$$

Since $\frac{\Delta\xi}{\Delta t} = \Delta\vec{U} = \vec{U} - \vec{U}_0 = \vec{U}_0 + \frac{\partial v}{\partial y}\Delta y - \vec{U}_0$

$$\Rightarrow \frac{\Delta\xi}{\Delta t} = \frac{\partial u}{\partial y}\Delta y \Rightarrow \Delta\xi = \frac{\partial u}{\partial y}\Delta y\Delta t \Rightarrow \frac{\Delta\xi}{\Delta y} = \frac{\partial u}{\partial y}\Delta t$$

$$(3) \Rightarrow \omega_{ob} = \lim_{\Delta t\to 0}\frac{\Delta\xi/\Delta y}{\Delta t} = \lim_{\Delta t\to 0}\frac{\frac{\partial v}{\partial y}\Delta t}{\Delta t} \Rightarrow \omega_{ob} = \frac{\partial u}{\partial y} = -\frac{\partial u}{\partial y} \qquad \ldots\ldots\ldots\ldots.(4)$$

Here negative sign shows that ob rotates in clockwise direction.

The rotation of fluid elements about z – axis is the average angular velocity of two mutually perpendicular line elements oa & ob in xy – plane.

$$\omega_z = \frac{1}{2}(\omega_{oa} + \omega_{ob}) = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

In the similar manner we can find the rotation of fluid element in x – axis or y – axis (yz – plane, xz – plane) respectively as follows;

$$\omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) \qquad \text{and} \qquad \omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)$$

Then we have from the relation $\vec{\omega} = \omega_x\hat{\imath} + \omega_y\hat{\jmath} + \omega_z\hat{k}$

$$\vec{\omega} = \frac{1}{2}\left(\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\hat{\imath} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\hat{\jmath} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\hat{k}\right)$$

$\boldsymbol{\vec{\omega} = \frac{1}{2}(\nabla \times \vec{V})}$ **Required Mathematical expression for Fluid Rotation**

Vorticity: The vorticity is a measure of the rotation of a fluid element as it moves in the flow field. It is represented by ζ or Ω and $\zeta \;= 2\vec{\omega} = \nabla \times \vec{V}$

In cylindrical coordinates the vorticity is

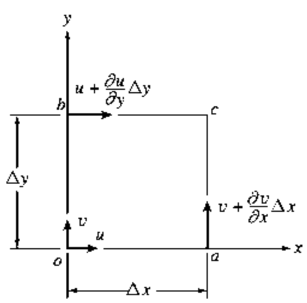
$$\nabla \times \vec{V} = \hat{e}_r \left(\frac{1}{r}\frac{\partial V_z}{\partial \theta} - \frac{\partial V_\theta}{\partial z}\right) + \hat{e}_\theta \left(\frac{\partial V_r}{\partial z} - \frac{\partial V_z}{\partial r}\right) + \hat{k}\left(\frac{1}{r}\frac{\partial V_\theta}{\partial r} - \frac{1}{r}\frac{\partial V_r}{\partial \theta}\right)$$

Circulation: The circulation, Γ , is defined as the line integral of the tangential velocity component about any closed curve fixed in the flow, $\Gamma = \oint_c \vec{V}.d\vec{s}$

Where $d\vec{s}$ is an elemental vector tangent to the curve and having length ds of the element of arc; a positive sense corresponds to a counterclockwise path of integration around the curve.

Relationship between Circulation and Vorticity

We can develop a relationship between circulation and vorticity by considering the rectangular circuit shown in Figure,



Where, the velocity components at O are assumed to be (u, v), and the velocities along segments bc and ac can be derived using Taylor series approximations.

For the closed curve oacb, $\Delta\Gamma = u\Delta x + \left(v + \frac{\partial v}{\partial x}\Delta x\right)\Delta y - \left(u + \frac{\partial u}{\partial y}\Delta y\right) - v\Delta y$

$$\Delta\Gamma = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\Delta x\Delta y = 2\omega_z \Delta x\Delta y$$

Then $\Gamma = \oint_c \vec{V}.d\vec{s} \Rightarrow \Gamma = \int_c 2\omega_z dA \Rightarrow \boldsymbol{\Gamma = \int_c 2\left(\nabla \times \vec{V}\right)_z dA}$

This Equation is a statement of the Stokes Theorem in two dimensions. Thus the circulation around a closed contour is equal to the total vorticity enclosed within it.

Kelvins Theorem: (For rotation or circulation) or State and prove Kelvins theorem for circulation:

Statement:

For an inviscid (non-viscous) incompressible fluid circulation around any closed curve C moving fluid constants at all times provided that the central forces remain conserved.

Proof:

Let C be the closed curve in fluid such that the curve moves with the fluid so that at all instant circulation consist of same fluid particle. Circulation is defined as

$$\Gamma = \oint \overline{V}.dr$$

To prove that circulation is constant it is sufficient to show $\frac{D\Gamma}{Dt} = 0$

Now $$\frac{D\Gamma}{Dt} = \frac{D}{Dt}\oint \overline{V}.dr = \oint \frac{D}{Dt}\left(\overline{V}.dr\right)$$

$$\frac{D\Gamma}{Dt} = \oint \overline{V}.\frac{D}{Dt}(dt) + dr.\frac{D\overline{V}}{Dt} \quad \_\_\_\_\_(i)$$

$$Since \quad \frac{D}{Dt}(dr) = d\left(\frac{Dr}{Dt}\right) = dV \quad \because (Bernoulli\ equation)$$

$$Similarly\ V.\frac{D}{Dt}(dr) = V.dr = \frac{1}{2}d\left(\overline{V}.\overline{V}\right) = d\left(\frac{1}{2}V^2\right) \quad \_\_\_\_(ii)$$

Using equation (ii) in (i)

$$\frac{D\Gamma}{Dt} = \int d\left(\frac{1}{2}V^2\right) + dr.\frac{D\overline{V}}{Dt} \quad \_\_\_\_(iii)$$

From Euler's equation of motion

$$\frac{DV}{Dt} = F - \frac{1}{\rho}\nabla P \quad \_\_\_\_(iv)$$

As we know forces are conservative.

$F = -\nabla\Omega$ _______ (v) Where Ω is force potential.

Using (v) in (iv)

$$\frac{DV}{Dt} = -\nabla\Omega - \frac{1}{\rho}\nabla P \qquad \_\_\_\_(vi)$$

By taking dot product of equation (vi) with dr

$$\frac{DV}{Dt}.dr = -\nabla\Omega.dr - \frac{1}{\rho}\nabla P.dr \qquad \_\_\_\_(vii)$$

$$\Rightarrow \nabla\Omega.dr = \left(\frac{\partial\Omega}{\partial x}\hat{i} + \frac{\partial\Omega}{\partial y}\hat{j} + \frac{\partial\Omega}{\partial z}\hat{k}\right).\left(dx\hat{i} + dy\hat{j} + dz\hat{k}\right)$$

$$\nabla\Omega.dr = \frac{\partial\Omega}{\partial x}dx + \frac{\partial\Omega}{\partial y}dy + \frac{\partial\Omega}{\partial z}dz$$

$$\nabla\Omega.dr = d\Omega$$

$$\textit{Similarly} \;\; \nabla P.\mathrm{dr} = \mathrm{dP}$$

Equation (vii) becomes

$$dr.\frac{DV}{Dt} = -d\Omega - \frac{1}{\rho}dPd = -d\Omega - d\left(\frac{P}{\rho}\right) \qquad \_\_\_\_(viii)$$

Since fluid is incompressible i.e. ρ = constant

Using equation (viii) in (iii)

$$\frac{D\Gamma}{Dt} = \int\left(d\left(\frac{1}{2}V^2\right) - d\Omega - d\left(\frac{P}{\rho}\right)\right)$$

$$\frac{D\Gamma}{Dt} = \int d\left(\frac{1}{2}V^2 - \Omega - \frac{P}{\rho}\right)$$

Since V, P and ρ are constant. Therefore, their derivative will also zero.

$$\frac{D\Gamma}{Dt} = \oint d\left(\text{constant}\right) = \oint 0 = 0$$

$\Rightarrow$ Γ is constant. Hence circulation remains constant.

Example **Free and Forced Vortex Flows**

Consider flow fields with purely tangential motion (circular streamlines): $V_r = 0$ and $V_\theta = f(r)$. Evaluate the rotation, vorticity, and circulation for rigid-body rotation, a *forced vortex*. Show that it is possible to choose $f(r)$ so that flow is irrotational, i.e., to produce a *free vortex*.

Given: Flow fields with tangential motion, $V_r = 0$ and $V_\theta = f(r)$.

Find: (a) Rotation, vorticity, and circulation for rigid-body motion (a *forced vortex*).
(b) $V_\theta = f(r)$ for irrotational motion (a *free vortex*).

Solution:

Governing equation: $\vec{\zeta} = 2\vec{\omega} = \nabla \times \vec{V}$ (5.15)

For motion in the $r\theta$ plane, the only components of rotation and vorticity are in the z direction,

$$\zeta_z = 2\omega_z = \frac{1}{r}\frac{\partial rV_\theta}{\partial r} - \frac{1}{r}\frac{\partial V_r}{\partial \theta}$$

Because $V_r = 0$ everywhere in these fields, this reduces to $\zeta_z = 2\omega_z = \dfrac{1}{r}\dfrac{\partial rV_\theta}{\partial r}$.

(a) For rigid-body rotation, $V_\theta = \omega r$.

Then $\omega_z = \dfrac{1}{2}\dfrac{1}{r}\dfrac{\partial rV_\theta}{\partial r} = \dfrac{1}{2}\dfrac{1}{r}\dfrac{\partial}{\partial r}\omega r^2 = \dfrac{1}{2r}(2\omega r) = \omega$ and $\zeta_z = 2\omega$.

The circulation is $\Gamma = \displaystyle\oint_c \vec{V}\cdot d\vec{s} = \int_A 2\omega_z dA.$ (5.18)

Since $\omega_z = \omega =$ constant, the circulation about any closed contour is given by $\Gamma = 2\omega A$, where A is the area enclosed by the contour. Thus for rigid-body motion (a forced vortex), the rotation and vorticity are constants; the circulation depends on the area enclosed by the contour.

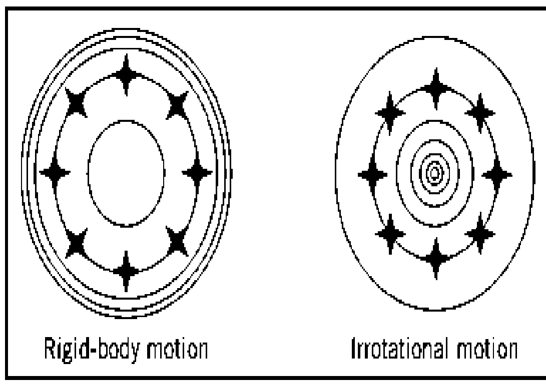
(b) For irrotational flow, $\omega_z = \dfrac{1}{r}\dfrac{\partial}{\partial r}rV_\theta = 0$. Integrating, we find

$$rV_\theta = \text{constant} \quad \text{or} \quad V_\theta = f(r) = \frac{C}{r}$$

For this flow, the origin is a singular point where $V_\theta \to \infty$. The circulation for any contour enclosing the origin is

$$\Gamma = \oint_c \vec{V} \cdot d\vec{s} = \int_0^{2\pi} \frac{C}{r} r \, d\theta = 2\pi C$$

It turns out that the circulation around any contour *not* enclosing the singular point at the origin is zero. Streamlines for the two vortex flows are shown below, along with the location and orientation at different instants of a cross marked in the fluid that was initially at the 12 o'clock position. For the rigid-body motion (which occurs, for example, at the eye of a tornado, creating the "dead" region at the very center), the cross rotates as it moves in a circular motion; also, the streamlines are closer together as we move away from the origin. For the irrotational motion (which occurs, for example, outside the eye of a tornado—in such a large region viscous effects are negligible), the cross does not rotate as it moves in a circular motion; also, the streamlines are farther apart as we move away from the origin.



Momentum Equation/ Law of Conservation of Momentum

A dynamic equation describing fluid motion may be obtained by applying Newton's second law to a particle. To derive the differential form of the momentum equation, we shall apply Newton's second law to an infinitesimal fluid particle of mass dm.

Recall that For a system moving relative to an inertial reference frame, Newton's second law states that *the sum of all external forces acting on the system is equal to the time rate of change of linear momentum of the system,* $\vec{F} = \frac{d\vec{P}}{dt}\Big)_{system}$

Where the linear momentum of the system is given by

$$\vec{P}\Big)_{system} = \int_{(M)_{system}} \vec{V} dm = \int_{(V)_{system}} \vec{V} \rho dV$$

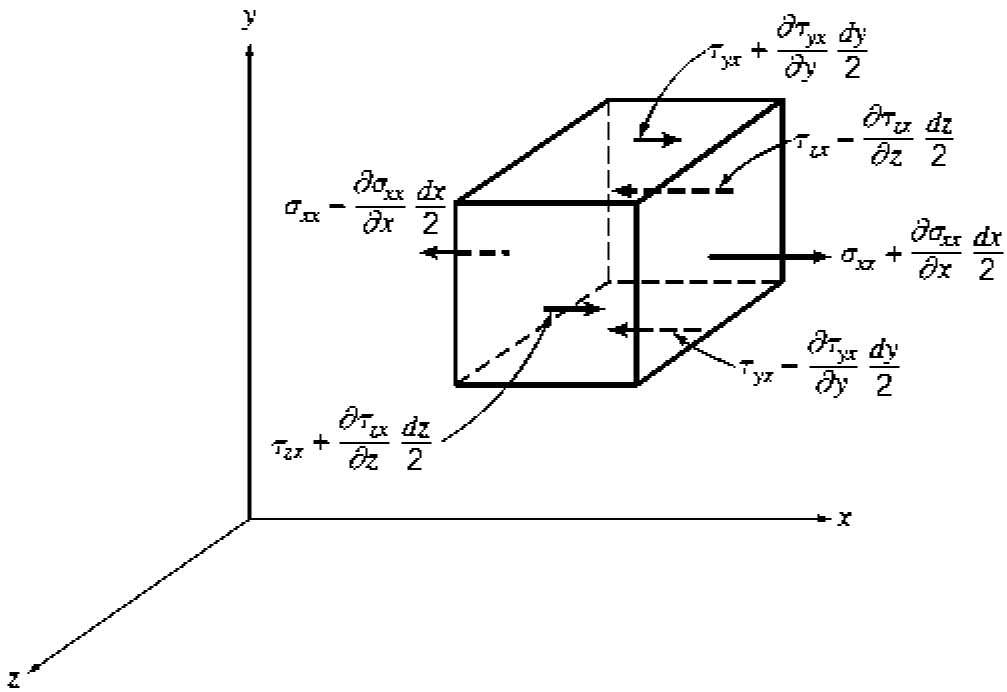
Consider a differential fluid element of mass dm, using Newton's second law

$$d\vec{F} = dm \frac{d\vec{V}}{dt}$$

$$\Rightarrow d\vec{F} = dm \frac{D\vec{V}}{Dt} = dm \left[u \frac{\partial \vec{V}}{\partial x} + v \frac{\partial \vec{V}}{\partial y} + w \frac{\partial \vec{V}}{\partial z} + \frac{\partial \vec{V}}{\partial t}\right] \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

We now need to obtain a suitable formulation for the force, $d\vec{F}$, or its components, $dF_x, dF_y, and\ dF_z$, acting on the element.

We shall consider the *x* component of the force acting on a differential element of mass dm and volume $d\vec{V} = dx\ dy\ dz$. Only those stresses that act in the x direction will give rise to surface forces in the x direction. If the stresses at the center of the differential element are taken to be σ_{xx}, τ_{yx}, and τ_{zx}, then the stresses acting in the *x* direction on all faces of the element (obtained by a Taylor series expansion about the center of the element) are as shown in Figure.



To obtain the net surface force in the x direction, dF_{S_x}, we must sum the forces in the *x* direction. Thus,

$$dF_{S_x} = \left(\sigma_{xx} + \frac{\partial \sigma_{xx}}{\partial x}\frac{dx}{2}\right)dy\,dz - \left(\sigma_{xx} - \frac{\partial \sigma_{xx}}{\partial x}\frac{dx}{2}\right)dy\,dz$$

$$+ \left(\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y}\frac{dy}{2}\right)dx\,dz - \left(\tau_{yx} - \frac{\partial \tau_{yx}}{\partial y}\frac{dy}{2}\right)dx\,dz$$

$$+ \left(\tau_{zx} + \frac{\partial \tau_{zx}}{\partial z}\frac{dz}{2}\right)dx\,dy - \left(\tau_{zx} - \frac{\partial \tau_{zx}}{\partial z}\frac{dz}{2}\right)dx\,dy$$

On simplifying, we obtain $\quad dF_{S_x} = \left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right)dxdydz$

When the force of gravity is the only body force acting, then the body force per unit mass is $\vec{g}$. The net force in the x direction, dF_x, is given by

$$dF_x = dF_{B_x} + dF_{S_x} = \left(\rho g_x + \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right)dxdydz \quad \text{……………….(2)}$$

We can derive similar expressions for the force components in the y and z directions:

$$dF_y = dF_{B_y} + dF_{S_y} = \left(\rho g_y + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z}\right) dxdydz \qquad \ldots\ldots\ldots\ldots\ldots\ldots..(3)$$

$$dF_z = dF_{B_z} + dF_{S_z} = \left(\rho g_z + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z}\right) dxdydz \qquad \ldots\ldots\ldots\ldots\ldots\ldots..(4)$$

These are the required forms of **momentum equation** in the x, y and z directions.

Differential form of Momentum Equation

To find differential momentum equation along x – axis

$$\left(\rho g_x + \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right) dxdydz = \rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t}\right) dxdydz$$

we obtain the **differential equations of motion**, for any fluid satisfying the continuum assumption.

$$\rho g_x + \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = \rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t}\right)$$

Similarly

$$\rho g_y + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} = \rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right)$$

$$\rho g_z + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} = \rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right)$$

Above Equations are the differential equations of motion for any fluid satisfying the continuum assumption. Before the equations can be used to solve for u, v, and w, suitable expressions for the stresses must be obtained in terms of the velocity and pressure fields.

Newtonian Fluid: Navier - Stokes Equations

For a Newtonian (viscous) fluid the viscous stress is directly proportional to the rate of shearing strain (angular deformation rate). The stresses may be expressed in terms of velocity gradients and fluid properties in rectangular coordinates as follows:

$$\tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)$$

$$\tau_{yz} = \tau_{zy} = \mu\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)$$

$$\tau_{zx} = \tau_{xz} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)$$

$$\sigma_{xx} = -p - \frac{2}{3}\mu\nabla\cdot\vec{V} + 2\mu\frac{\partial u}{\partial x}$$

$$\sigma_{yy} = -p - \frac{2}{3}\mu\nabla\cdot\vec{V} + 2\mu\frac{\partial v}{\partial y}$$

$$\sigma_{zz} = -p - \frac{2}{3}\mu\nabla\cdot\vec{V} + 2\mu\frac{\partial w}{\partial z}$$

Where ρ is the local thermodynamic pressure. Above equations are called constitutive equations. Thermodynamic pressure is related to the density and temperature by the thermodynamic relation usually called the equation of state.

If these expressions for the stresses are introduced into the differential equations of motion

$$\rho g_x + \frac{\partial\sigma_{xx}}{\partial x} + \frac{\partial\tau_{yx}}{\partial y} + \frac{\partial\tau_{zx}}{\partial z} = \rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t}\right)$$

Similarly

$$\rho g_y + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} = \rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right)$$

$$\rho g_z + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} = \rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right)$$

We obtain

$$\rho\frac{Du}{Dt} = \rho g_x - \frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left[\mu\left(2\frac{\partial u}{\partial x} - \frac{2}{3}\nabla\cdot\vec{V}\right)\right] + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right]$$
$$+ \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)\right]$$

$$\rho\frac{Dv}{Dt} = \rho g_y - \frac{\partial p}{\partial y} + \frac{\partial}{\partial x}\left[\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right] + \frac{\partial}{\partial y}\left[\mu\left(2\frac{\partial v}{\partial y} - \frac{2}{3}\nabla\cdot\vec{V}\right)\right]$$
$$+ \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\right]$$

$$\rho\frac{Dw}{Dt} = \rho g_z - \frac{\partial p}{\partial z} + \frac{\partial}{\partial x}\left[\mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)\right] + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial w}{\partial z} + \frac{\partial w}{\partial y}\right)\right]$$
$$+ \frac{\partial}{\partial z}\left[\mu\left(2\frac{\partial w}{\partial z} - \frac{2}{3}\nabla\cdot\vec{V}\right)\right]$$

These equations of motion are called the **NavierStokes equations**. The equations are greatly simplified when applied to incompressible flow with constant viscosity. Under these conditions the equations reduce to other forms as follows;

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = \rho g_x - \frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) = \rho g_y - \frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right)$$

$$\rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) = \rho g_z - \frac{\partial p}{\partial z} + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right)$$

This form of the NavierStokes equations is probably most famous set of equations in fluid mechanics, and has been widely studied. These equations form a set of four coupled nonlinear partial differential equations for u, v, w, and ρ. In principle, these four equations describe many common flows; the only restrictions are that the fluid be Newtonian (with a constant viscosity) and incompressible.

Cases for Newtonian Fluid: Navier - Stokes Equations

Write Navier Stokes equation for

i. Compressible flow ($\mu = constant$)
ii. Incompressible flow $\vec{\nabla}.\vec{V} = 0$
iii. Inviscous flow ($\mu = 0$)

Case – I: Navier - Stokes Equation for Compressible flow ($\boldsymbol{\mu = constant}$)

Since we have

$$\rho\frac{Du}{Dt} = \rho g_x - \frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left[\mu\left(2\frac{\partial u}{\partial x} - \frac{2}{3}\nabla\cdot\vec{V}\right)\right] + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right] + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)\right]$$

$\Rightarrow \rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t}\right) = \rho g_x - \frac{\partial p}{\partial x} + 2\mu\frac{\partial^2 u}{\partial x^2} - \frac{2}{3}\mu\frac{\partial}{\partial x}\left(\vec{\nabla}.\vec{V}\right) + \mu\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + \mu\frac{\partial}{\partial z}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)$ …………………(1)

Also

$$\rho\frac{Dv}{Dt} = \rho g_y - \frac{\partial p}{\partial y} + \frac{\partial}{\partial x}\left[\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right] + \frac{\partial}{\partial y}\left[\mu\left(2\frac{\partial v}{\partial y} - \frac{2}{3}\nabla\cdot\vec{V}\right)\right]$$
$$+ \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\right]$$

$\Rightarrow \rho\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + \frac{\partial v}{\partial t}\right) = \rho g_y - \frac{\partial p}{\partial y} + 2\mu\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial y}\right) - \frac{2}{3}\mu\frac{\partial}{\partial y}(\vec{\nabla}.\vec{V}) +$
$\mu\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + \mu\frac{\partial}{\partial z}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)$ …………………(2)

And

$$\rho\frac{Dw}{Dt} = \rho g_z - \frac{\partial p}{\partial z} + \frac{\partial}{\partial x}\left[\mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)\right] + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial w}{\partial z} + \frac{\partial w}{\partial y}\right)\right]$$
$$+ \frac{\partial}{\partial z}\left[\mu\left(2\frac{\partial w}{\partial z} - \frac{2}{3}\nabla\cdot\vec{V}\right)\right]$$

$\Rightarrow \rho\left(u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} + \frac{\partial w}{\partial t}\right) = \rho g_z - \frac{\partial p}{\partial z} + 2\mu\frac{\partial}{\partial z}\left(\frac{\partial w}{\partial z}\right) - \frac{2}{3}\mu\frac{\partial}{\partial z}(\vec{\nabla}.\vec{V}) +$
$\mu\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) + \mu\frac{\partial}{\partial y}\left(\frac{\partial w}{\partial z} + \frac{\partial w}{\partial y}\right)$ …………………(3)

Case – II: Navier - Stokes Equation for Incompressible flow ($\vec{\nabla}.\vec{V} = 0$)

Since we have

$$\rho\frac{Du}{Dt} = \rho g_x - \frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left[\mu\left(2\frac{\partial u}{\partial x} - \frac{2}{3}\nabla\cdot\vec{V}\right)\right] + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right]$$
$$+ \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)\right]$$

$\Rightarrow \rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t}\right) = \rho g_x - \frac{\partial p}{\partial x} + 2\mu\frac{\partial^2 u}{\partial x^2} - \frac{2}{3}\mu\frac{\partial}{\partial x}(\vec{\nabla}.\vec{V}) +$
$\mu\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + \mu\frac{\partial}{\partial z}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)$

$$\Rightarrow \rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}+\frac{\partial u}{\partial t}\right)=\rho g_x-\frac{\partial p}{\partial x}+2\mu\frac{\partial^2 u}{\partial x^2}+\mu\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)+$$
$$\mu\frac{\partial}{\partial z}\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right) \qquad \text{after putting } \vec{\nabla}.\vec{V}=0$$

$$\Rightarrow \rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}+\frac{\partial u}{\partial t}\right)=\rho g_x-\frac{\partial p}{\partial x}+2\mu\frac{\partial^2 u}{\partial x^2}+\mu\left(\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 v}{\partial y\partial x}\right)+$$
$$\mu\left(\frac{\partial^2 w}{\partial z\partial x}+\frac{\partial^2 u}{\partial z^2}\right)$$

$$\Rightarrow \rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}+\frac{\partial u}{\partial t}\right)=\rho g_x-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)+\mu\frac{\partial^2 u}{\partial x^2}+$$
$$\mu\left(\frac{\partial^2 w}{\partial z\partial x}+\frac{\partial^2 v}{\partial y\partial x}\right)$$

$$\Rightarrow \rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}+\frac{\partial u}{\partial t}\right)=\rho g_x-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)+$$
$$\mu\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)$$

$$\Rightarrow \rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}+\frac{\partial u}{\partial t}\right)=\rho g_x-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)+\mu\frac{\partial}{\partial x}\left(\vec{\nabla}.\vec{V}\right)$$

$$\Rightarrow \rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}+\frac{\partial u}{\partial t}\right)=\rho g_x-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right) \qquad \because \vec{\nabla}.\vec{V}=0$$

Similarly

$$\Rightarrow \rho\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}+\frac{\partial v}{\partial t}\right)=\rho g_y-\frac{\partial p}{\partial y}+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)$$

$$\Rightarrow \rho\left(u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}+\frac{\partial w}{\partial t}\right)=\rho g_z-\frac{\partial p}{\partial z}+\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)$$

Case – III: Navier - Stokes Equation for Inviscid flow ($\mu = 0$) (Euler's Equation of motion)

Since we have

$$\rho\frac{Du}{Dt} = \rho g_x-\frac{\partial p}{\partial x}+\frac{\partial}{\partial x}\left[\mu\left(2\frac{\partial u}{\partial x}-\frac{2}{3}\nabla\cdot\vec{V}\right)\right]+\frac{\partial}{\partial y}\left[\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right]$$
$$+\frac{\partial}{\partial z}\left[\mu\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)\right]$$

$$\Rightarrow \rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}+\frac{\partial u}{\partial t}\right)=\rho g_x-\frac{\partial p}{\partial x}+2\mu\frac{\partial^2 u}{\partial x^2}-\frac{2}{3}\mu\frac{\partial}{\partial x}\left(\vec{\nabla}.\vec{V}\right)+$$
$$\mu\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)+\mu\frac{\partial}{\partial z}\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)$$

$$\Rightarrow \rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}+\frac{\partial u}{\partial t}\right)=\rho g_x-\frac{\partial p}{\partial x} \qquad \text{using } \mu=0$$

Similarly

$$\Rightarrow \rho\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}+\frac{\partial v}{\partial t}\right)=\rho g_y-\frac{\partial p}{\partial y}$$

$$\Rightarrow \rho\left(u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}+\frac{\partial w}{\partial t}\right)=\rho g_z-\frac{\partial p}{\partial z}$$

And in vector form we get Euler equation of motion

$$\rho\frac{d\vec{V}}{dt}=\rho\vec{g}-\vec{\nabla}p$$

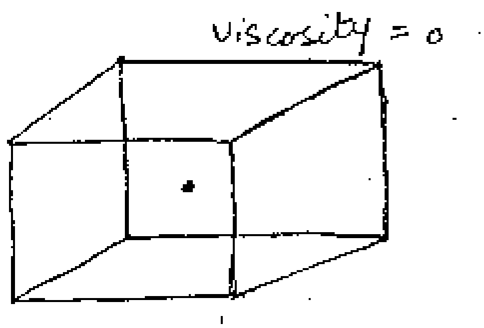
Momentum Equation For Frictionless Flow: Euler's Equation of Motion

Euler's equation after neglecting the viscous terms is given as follows;

$$\rho\frac{d\bar{V}}{dt} = \rho\vec{g} - \nabla p$$

This equation states that for an inviscid fluid the change in momentum of a fluid particle is caused by the body force (assumed to be gravity only) and the net pressure force.

Derivation



Consider finite size control volume through which inviscid fluid is flowing. Applying Newton's Second Law of Motion in x,y,z coordinates

For x axis; $\sum F_x = ma_x$ ……………(i)

Now using the fact Total force = Surface (Pressure Force) + Body Force

The sum of force acting on the fluid particle in x direction is

$$\sum F_x = \left[\left(p - \frac{\partial p}{\partial x}\frac{dx}{2}\right) - \left(p + \frac{\partial p}{\partial x}\frac{dx}{2}\right)\right] dydz + \rho g_x dxdydz$$

$$\sum F_x = -\frac{\partial p}{\partial x}\left(\frac{1}{2} + \frac{1}{2}\right) dxdydz + \rho g_x dxdydz$$

$$\sum F_x = -\frac{\partial p}{\partial x} dxdydz + \rho g_x dxdydz \quad \text{……………(ii)}$$

Since $\rho = \frac{m}{v} \Rightarrow m = \rho dv \Rightarrow m =$ mass of fluid particle $= \rho dxdydz$

$(i) \Rightarrow \sum F_x = a_x \rho dxdydz$ ……………(iii)

Comparing (ii) and (iii)

$$-\frac{\partial p}{\partial x}dxdydz + \rho g_x dxdydz = a_x \rho dxdydz$$

$$-\frac{\partial p}{\partial x} + \rho g_x = a_x \rho \qquad \text{……………(iv)}$$

Similarly

$$-\frac{\partial p}{\partial y} + \rho g_y = a_y \rho \qquad \text{……………(v)}$$

$$-\frac{\partial p}{\partial z} + \rho g_z = a_z \rho \qquad \text{……………(vi)}$$

Now adding (iv),(v),(vi) we get total pressure

$$-\frac{\partial p}{\partial x}\hat{\imath} - \frac{\partial p}{\partial y}\hat{\jmath} - \frac{\partial p}{\partial z}\hat{k} + \rho\left(g_x\hat{\imath} + g_y\hat{\jmath} + g_z\hat{k}\right) = \rho\left(a_x\hat{\imath} + a_y\hat{\jmath} + a_z\hat{k}\right)$$

$$-\left(\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}\right)p + \rho\vec{g} = \rho\vec{a}$$

$$\boldsymbol{-\nabla p + \rho\vec{g} = \rho\vec{a}}$$

Or $$\boldsymbol{\rho\frac{d\overline{V}}{dt} = \rho\vec{g} - \nabla p} \qquad \text{……………..(A)}$$

We may write above as $-\frac{1}{\rho}\nabla p + \vec{g} = \vec{a}$ ……………..(B)

Then using $\vec{a} = \frac{d\vec{V}}{dt} = \frac{\partial\vec{V}}{\partial t} + V.\nabla\vec{V}$

We get $-\frac{1}{\rho}\nabla p + \vec{g} = \frac{\partial\vec{V}}{\partial t} + V.\nabla\vec{V}$ ……………..(C)

Hence (A),(B),(C) are different forms of Euler Equation of motion in Vector notation.

Example Analysis Of Fully Developed Laminar Flow Down An Inclined Plane Surface

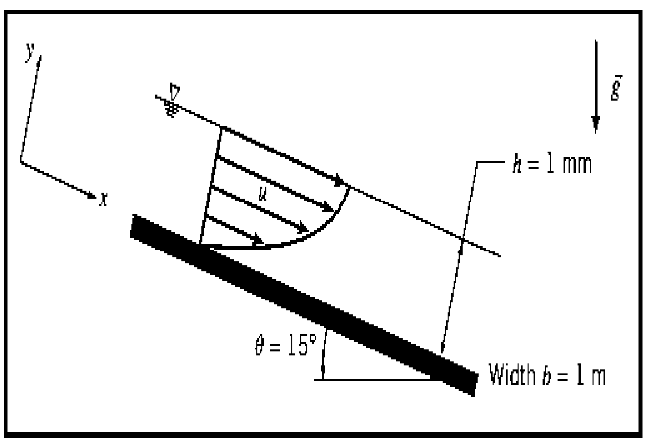
A liquid flows down an inclined plane surface in a steady, fully developed laminar film of thickness h. Simplify the continuity and Navier–Stokes equations to model this flow field. Obtain expressions for the liquid velocity profile, the shear stress distribution, the volume flow rate, and the average velocity. Relate the liquid film thickness to the volume flow rate per unit depth of surface normal to the flow. Calculate the volume flow rate in a film of water $h = 1$ mm thick, flowing on a surface $b = 1$ m wide, inclined at $\theta = 15°$ to the horizontal.

Given: Liquid flow down an inclined plane surface in a steady, fully developed laminar film of thickness h.

Find: (a) Continuity and Navier–Stokes equations simplified to model this flow field.
(b) Velocity profile.
(c) Shear stress distribution.
(d) Volume flow rate per unit depth of surface normal to diagram.
(e) Average flow velocity.
(f) Film thickness in terms of volume flow rate per unit depth of surface normal to diagram.
(g) Volume flow rate in a film of water 1 mm thick on a surface 1 m wide, inclined at 15° to the horizontal.

Solution:

The geometry and coordinate system used to model the flow field are shown. (It is convenient to align one coordinate with the flow down the plane surface.)



The governing equations written for incompressible flow with constant viscosity are

$$\cancelto{4}{\frac{\partial u}{\partial x}} + \frac{\partial v}{\partial y} + \cancelto{3}{\frac{\partial w}{\partial z}} = 0$$

$$\rho\left(\cancelto{1}{\frac{\partial u}{\partial t}} + u\cancelto{4}{\frac{\partial u}{\partial x}} + \cancelto{5}{v\frac{\partial u}{\partial y}} + \cancelto{3}{w\frac{\partial u}{\partial z}}\right) = \rho g_x - \cancelto{4}{\frac{\partial p}{\partial x}} + \mu\left(\cancelto{4}{\frac{\partial^2 u}{\partial x^2}} + \frac{\partial^2 u}{\partial y^2} + \cancelto{3}{\frac{\partial^2 u}{\partial z^2}}\right)$$

$$\rho\left(\cancelto{1}{\frac{\partial v}{\partial t}} + u\cancelto{4}{\frac{\partial v}{\partial x}} + v\cancelto{5}{\frac{\partial v}{\partial y}} + w\cancelto{3}{\frac{\partial v}{\partial z}}\right) = \rho g_y - \frac{\partial p}{\partial y} + \mu\left(\cancelto{4}{\frac{\partial^2 v}{\partial x^2}} + \cancelto{5}{\frac{\partial^2 v}{\partial y^2}} + \cancelto{3}{\frac{\partial^2 v}{\partial z^2}}\right)$$

$$\rho\left(\cancelto{1}{\frac{\partial w}{\partial t}} + u\cancelto{3}{\frac{\partial w}{\partial x}} + v\cancelto{3}{\frac{\partial w}{\partial y}} + w\cancelto{3}{\frac{\partial w}{\partial z}}\right) = \cancelto{3}{\rho g_z} - \frac{\partial p}{\partial z} + \mu\left(\cancelto{3}{\frac{\partial^2 w}{\partial x^2}} + \cancelto{3}{\frac{\partial^2 w}{\partial y^2}} + \cancelto{3}{\frac{\partial^2 w}{\partial z^2}}\right)$$

The terms canceled to simplify the basic equations are keyed by number to the assumptions listed below. The assumptions are discussed in the order in which they are applied to simplify the equations.

Assumptions: (1) Steady flow (given).
(2) Incompressible flow; $\rho = \text{constant}$.
(3) No flow or variation of properties in the z direction; $w = 0$ and $\partial/\partial z = 0$.
(4) Fully developed flow, so no properties vary in the x direction; $\partial/\partial x = 0$.

Assumption (1) eliminates time variations in any fluid property.

Assumption (2) eliminates space variations in density.

Assumption (3) states that there is no z component of velocity and no property variations in the z direction. All terms in the z component of the Navier–Stokes equation cancel.

After assumption (4) is applied, the continuity equation reduces to $\partial v/\partial y = 0$. Assumptions (3) and (4) also indicate that $\partial v/\partial z = 0$ and $\partial v/\partial x = 0$. Therefore v must be constant. Since v is zero at the solid surface, then v must be zero everywhere.

The fact that $v = 0$ reduces the Navier–Stokes equations further, as indicated by (5) in Eqs 5.27a and 5.27b. The final simplified equations are

$$0 = \rho g_x + \mu\frac{\partial^2 u}{\partial y^2} \qquad (1)$$

$$0 = \rho g_y - \frac{\partial p}{\partial y} \qquad (2)$$

Since $\partial u/\partial z = 0$ (assumption 3) and $\partial u/\partial x = 0$ (assumption 4), then u is at most a function of y, and $\partial^2 u/\partial y^2 = d^2u/dy^2$, and from Eq. 1, then

$$\frac{d^2u}{dy^2} = -\frac{\rho g_x}{\mu} = -\rho g\frac{\sin\theta}{\mu}$$

Integrating,

$$\frac{du}{dy} = -\rho g \frac{\sin\theta}{\mu} y + c_1 \tag{3}$$

and integrating again,

$$u = -\rho g \frac{\sin\theta}{\mu} \frac{y^2}{2} + c_1 y + c_2 \tag{4}$$

The boundary conditions needed to evaluate the constants are the no-slip condition at the solid surface ($u = 0$ at $y = 0$) and the zero-shear-stress condition at the liquid free surface ($du/dy = 0$ at $y = h$).

Evaluating Eq. 4 at $y = 0$ gives $c_2 = 0$. From Eq. 3 at $y = h$,

$$0 = -\rho g \frac{\sin\theta}{\mu} h + c_1$$

or

$$c_1 = \rho g \frac{\sin\theta}{\mu} h$$

Substituting into Eq. 4 we obtain the velocity profile

$$u = -\rho g \frac{\sin\theta}{\mu} \frac{y^2}{2} + \rho g \frac{\sin\theta}{\mu} hy$$

or

$$u = \rho g \frac{\sin\theta}{\mu} \left(hy - \frac{y^2}{2} \right) \qquad u(y)$$

The shear stress distribution is (from Eq. 5.25a after setting $\partial v/\partial x$ to zero, or alternatively, for one-dimensional flow, from Eq. 2.15)

$$\tau_{yx} = \mu \frac{du}{dy} = \rho g \sin\theta\,(h - y) \qquad \tau_{yx}(y)$$

The shear stress in the fluid reaches its maximum value at the wall ($y = 0$); as we expect, it is zero at the free surface ($y = h$). At the wall the shear stress τ_{yx} is positive but the surface normal *for the fluid* is in the negative y direction; hence the shear force acts in the negative x direction, and just balances the x component of the body force acting on the fluid. The volume flow rate is

$$Q = \int_A u\,dA = \int_0^h u\,b\,dy$$

where b is the surface width in the z direction. Substituting,

$$Q = \int_0^h \frac{\rho g \sin\theta}{\mu} \left(hy - \frac{y^2}{2} \right) b\,dy = \rho g \frac{\sin\theta\, b}{\mu} \left[\frac{hy^2}{2} - \frac{y^3}{6} \right]_0^h$$

$$Q = \frac{\rho g \sin\theta\, b}{\mu} \frac{h^3}{3} \tag{5}$$

Q

The average flow velocity is $\overline{V} = Q/A = Q/bh$. Thus

$$\overline{V} = \frac{Q}{bh} = \frac{\rho g \sin\theta}{\mu} \frac{h^2}{3} \qquad \overline{V}$$

Solving for film thickness gives

$$h = \left[\frac{3\mu Q}{\rho g \sin\theta\, b}\right]^{1/3} \qquad (6) \quad h$$

A film of water $h = 1$ mm thick on a plane $b = 1$ m wide, inclined at $\theta = 15°$, would carry

$$Q = 999\frac{\text{kg}}{\text{m}^3} \times 9.81\frac{\text{m}}{\text{s}^2} \times \sin(15°) \times 1\text{ m} \times \frac{\text{m}\cdot\text{s}}{1.00\times10^{-3}\text{ kg}}$$

$$\times\ \frac{(0.001)^3\text{ m}^3}{3} \times 1000\frac{\text{L}}{\text{m}^3}$$

$$Q = 0.846\text{ L/s} \qquad Q$$

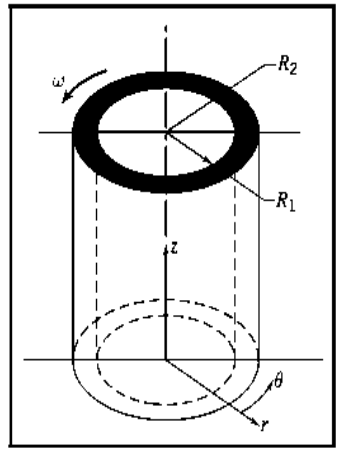
Example

Analysis Of Laminar Viscometric Flow Between Coaxial Cylinders

A viscous liquid fills the annular gap between vertical concentric cylinders. The inner cylinder is stationary, and the outer cylinder rotates at constant speed. The flow is laminar. Simplify the continuity, Navier Stokes, and tangential shear stress equations to model this flow field. Obtain expressions for the liquid velocity profile and the shear stress distribution. Compare the shear stress at the surface of the inner cylinder with that computed from a planar approximation obtained by “unwrapping” the annulus into a plane and assuming a linear velocity profile across the gap. Determine the ratio of cylinder radii for which the planar approximation predicts the correct shear stress at the surface of the inner cylinder within 1 percent.

Given: Laminar viscometric flow of liquid in annular gap between vertical concentric cylinders. The inner cylinder is stationary, and the outer cylinder rotates at constant speed.

Find:
(a) Continuity and Navier–Stokes equations simplified to model this flow field.
(b) Velocity profile in the annular gap.
(c) Shear stress distribution in the annular gap.
(d) Shear stress at the surface of the inner cylinder.
(e) Comparison with "planar" approximation for constant shear stress in the narrow gap between cylinders.
(f) Ratio of cylinder radii for which the planar approximation predicts shear stress within 1 percent of the correct value.



Solution:
The geometry and coordinate system used to model the flow field are shown. (The z coordinate is directed vertically upward; as a consequence, $g_r = g_\theta = 0$ and $g_z = -g$.)

The continuity, Navier–Stokes, and tangential shear stress equations (from Appendix B) written for incompressible flow with constant viscosity are

$$\frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \cancelto{4}{\frac{1}{r}\frac{\partial}{\partial \theta}(v_\theta)} + \cancelto{3}{\frac{\partial}{\partial z}(v_z)} = 0 \qquad \text{(B.1)}$$

r component:

$$\rho\left(\cancelto{1}{\frac{\partial v_r}{\partial t}} + \cancelto{5}{v_r\frac{\partial v_r}{\partial r}} + \cancelto{4}{\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}} - \frac{v_\theta^2}{r} + \cancelto{3}{v_z\frac{\partial v_r}{\partial z}}\right)$$
$$= \cancelto{0}{\rho g_r} - \frac{\partial p}{\partial r} + \mu\left\{\cancelto{5}{\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}[r v_r]\right)} + \cancelto{4}{\frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2}} - \cancelto{4}{\frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}} + \cancelto{3}{\frac{\partial^2 v_r}{\partial z^2}}\right\} \qquad \text{(B.3a)}$$

θ component:

$$\rho\left(\cancelto{1}{\frac{\partial v_\theta}{\partial t}} + \cancelto{5}{v_r\frac{\partial v_\theta}{\partial r}} + \cancelto{4}{\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}} + \cancelto{5}{\frac{v_r v_\theta}{r}} + \cancelto{3}{v_z\frac{\partial v_\theta}{\partial z}}\right)$$
$$= \cancelto{0}{\rho g_\theta} - \cancelto{4}{\frac{1}{r}\frac{\partial p}{\partial \theta}} + \mu\left\{\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}[r v_\theta]\right) + \cancelto{4}{\frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2}} + \cancelto{4}{\frac{2}{r^2}\frac{\partial v_r}{\partial \theta}} + \cancelto{3}{\frac{\partial^2 v_\theta}{\partial z^2}}\right\} \qquad \text{(B.3b)}$$

z component:

$$\rho\left(\cancelto{1}{\frac{\partial v_z}{\partial t}} + \cancelto{5}{v_r\frac{\partial v_z}{\partial r}} + \cancelto{4}{\frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta}} + \cancelto{3}{v_z\frac{\partial v_z}{\partial z}}\right) = \rho g_z - \frac{\partial p}{\partial z} + \mu\left\{\cancelto{3}{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)} + \cancelto{3}{\frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2}} + \cancelto{3}{\frac{\partial^2 v_z}{\partial z^2}}\right\} \qquad \text{(B.3c)}$$

$$\tau_{r\theta} = \mu\left[r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \cancelto{4}{\frac{1}{r}\frac{\partial v_r}{\partial \theta}}\right] \qquad \text{(B.2)}$$

The terms canceled to simplify the basic equations are keyed by number to the assumptions listed below. The assumptions are discussed in the order in which they are applied to simplify the equations.

Assumptions: (1) Steady flow; angular speed of outer cylinder is constant.
(2) Incompressible flow; $\rho = \text{constant}$.
(3) No flow or variation of properties in the z direction; $v_z = 0$ and $\partial/\partial z = 0$.
(4) Circumferentially symmetric flow, so properties do not vary with θ, so $\partial/\partial\theta = 0$.

Assumption (1) eliminates time variations in fluid properties.

Assumption (2) eliminates space variations in density.

Assumption (3) causes all terms in the z component of the Navier–Stokes equation (Eq. B.3c) to cancel, except for the hydrostatic pressure distribution.

After assumptions (3) and (4) are applied, the continuity equation (Eq. B.1) reduces to

$$\frac{1}{r}\frac{\partial}{\partial r}(rv_r) = 0$$

Because $\partial/\partial\theta = 0$ and $\partial/\partial z = 0$ by assumptions (3) and (4), then $\frac{\partial}{\partial r} \to \frac{d}{dr}$, so integrating gives

$$rv_r = \text{constant}$$

Since v_r is zero at the solid surface of each cylinder, then v_r must be zero everywhere.

The fact that $v_r = 0$ reduces the Navier–Stokes equations further, as indicated by cancellations (5). The final equations (Eqs. B.3a and B.3b) reduce to

$$-\rho\frac{v_\theta^2}{r} = -\frac{\partial p}{\partial r}$$

$$0 = \mu\left\{\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}[rv_\theta]\right)\right\}$$

But since $\partial/\partial\theta = 0$ and $\partial/\partial z = 0$ by assumptions (3) and (4), then v_θ is a function of radius only, and

$$\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}[rv_\theta]\right) = 0$$

Integrating once,

$$\frac{1}{r}\frac{d}{dr}[rv_\theta] = c_1$$

or

$$\frac{d}{dr}[rv_\theta] = c_1 r$$

Integrating again,

$$rv_\theta = c_1\frac{r^2}{2} + c_2 \qquad \text{or} \qquad v_\theta = c_1\frac{r}{2} + c_2\frac{1}{r}$$

Two boundary conditions are needed to evaluate constants c_1 and c_2. The boundary conditions are

$$v_\theta = \omega R_2 \quad \text{at} \quad r = R_2 \quad \text{and}$$
$$v_\theta = 0 \quad \text{at} \quad r = R_1$$

Substituting

$$\omega R_2 = c_1 \frac{R_2}{2} + c_2 \frac{1}{R_2}$$

$$0 = c_1 \frac{R_1}{2} + c_2 \frac{1}{R_1}$$

After considerable algebra

$$c_1 = \frac{2\omega}{1 - \left(\frac{R_1}{R_2}\right)^2} \quad \text{and} \quad c_2 = \frac{-\omega R_1^2}{1 - \left(\frac{R_1}{R_2}\right)^2}$$

Substituting into the expression for v_θ,

$$v_\theta = \frac{\omega r}{1 - \left(\frac{R_1}{R_2}\right)^2} - \frac{\omega R_1^2 / r}{1 - \left(\frac{R_1}{R_2}\right)^2} = \frac{\omega R_1}{1 - \left(\frac{R_1}{R_2}\right)^2}\left[\frac{r}{R_1} - \frac{R_1}{r}\right] \longleftarrow \quad v_\theta(r)$$

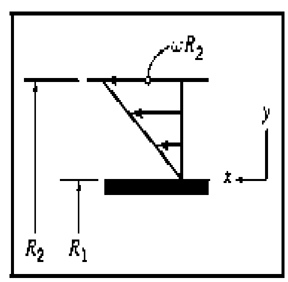
The shear stress distribution is obtained from Eq. B.2 after using assumption (4):

$$\tau_{r\theta} = \mu r \frac{d}{dr}\left(\frac{v_\theta}{r}\right) = \mu r \frac{d}{dr}\left\{\frac{\omega R_1}{1 - \left(\frac{R_1}{R_2}\right)^2}\left[\frac{1}{R_1} - \frac{R_1}{r^2}\right]\right\} = \mu r \frac{\omega R_1}{1 - \left(\frac{R_1}{R_2}\right)^2}(-2)\left(-\frac{R_1}{r^3}\right)$$

$$\tau_{r\theta} = \mu \frac{2\omega R_1^2}{1 - \left(\frac{R_1}{R_2}\right)^2}\frac{1}{r^2} \longleftarrow \quad \tau_{r\theta}$$

At the surface of the inner cylinder, $r = R_1$, so

$$\tau_{\text{surface}} = \mu \frac{2\omega}{1 - \left(\frac{R_1}{R_2}\right)^2} \longleftarrow \quad \tau_{\text{surface}}$$



For a "planar" gap

$$\tau_{\text{planar}} = \mu \frac{\Delta v}{\Delta y} = \mu \frac{\omega R_2}{R_2 - R_1}$$

or

$$\tau_{\text{planar}} = \mu \frac{\omega}{1 - \frac{R_1}{R_2}} \longleftarrow \quad \tau_{\text{planar}}$$

Factoring the denominator of the exact expression for shear stress at the surface gives

$$\tau_{\text{surface}} = \mu \frac{2\omega}{\left(1-\dfrac{R_1}{R_2}\right)\left(1+\dfrac{R_1}{R_2}\right)} = \mu \frac{\omega}{1-\dfrac{R_1}{R_2}} \cdot \frac{2}{1+\dfrac{R_1}{R_2}}$$

Thus

$$\frac{\tau_{\text{surface}}}{\tau_{\text{planar}}} = \frac{2}{1+\dfrac{R_1}{R_2}}$$

For 1 percent accuracy,

$$1.01 = \frac{2}{1+\dfrac{R_1}{R_2}}$$

or

$$\frac{R_1}{R_2} = \frac{1}{1.01}(2-1.01) = 0.980 \xleftarrow{\qquad\qquad\qquad \frac{R_1}{R_2}}$$

The accuracy criterion is met when the gap width is less than 2 percent of the cylinder radius.

Reynolds Number

It is the ratio of inertial force to the viscous force. It tells us whether the flow is laminar or turbulent. If the inertial force that resists the change in velocity is dominant then the flow is turbulent. And if the viscous force that resists the flow is dominant then the flow is laminar.

Its formula is $Re = \frac{\rho V L}{\mu}$

Or $Re = \frac{VL}{\upsilon}$ where $\upsilon = \frac{\mu}{\rho}$

Where ρ is fluid density, V is characteristic velocity, L is characteristic length or size scale of flow, μ is dynamic fluid viscosity and υ is kinetic viscosity.

If the Reynolds number is "large," viscous effects will be negligible, at least in most of the flow; if the Reynolds number is small, viscous effects will be dominant. Finally, if the Reynolds number is neither large nor small, no general conclusions can be drawn.

Remember

Prandtl number: $Pr = \frac{\nu}{\alpha} = \frac{\text{viscous diffusion rate}}{\text{thermal diffusion rate}}$

Schmidt number: $Sc = \frac{\nu}{D} = \frac{\text{viscous diffusion rate}}{\text{mass diffusion rate}}$

Lewis number: $Le = \frac{D}{\alpha} = \frac{\text{mass diffusion rate}}{\text{thermal diffusion rate}}$

Reynolds number: $Re = \frac{\rho_0 U L}{\mu_0}$

Froude number: $Fr = \frac{U^2}{gL}$

Prandtl number: $$Pr = \frac{\mu_0 c_p}{k_0}$$

Eckert number: $$Ec = \frac{U^2}{c_p T_0}$$

Cavitation number: $$Ca = \frac{p_a - p_0}{\rho U^2}$$

Froude number: $$Fr = \frac{U^2}{gL}$$

Weber number: $$We = \frac{\rho_0 U^2 L}{\mathcal{T}}$$

To summarize, the following parameters are important for any particular flow:

1. All viscous flows: Reynolds number
2. Variable-temperature problems: Prandtl and Eckert (or Mach) numbers
3. Flow with free convection: Grashof and Prandtl numbers
4. Wall heat transfer: temperature ratio or Nusselt number
5. Slip flow: Knudsen number and specific-heat ratio
6. Free-surface conditions: Froude number (always); Weber number (sometimes), and cavitation number (sometimes)

INCOMPRESSIBLE INVISCID FLOW

In this chapter, instead of the Navier-Stokes equations, we will study Euler's equation, which applies to an inviscid fluid. Although truly inviscid fluids do not exist, many flow problems (especially in aerodynamics) can be successfully analyzed with the approximation that $\mu = 0$.

Basic Definitions

- **Fluid:** A fluid is a substance that deform continuously stress (tangential stress). No matter how small or large the sheer stress. It is a substance that has no fixed shape and yields easily to external pressure;
  Examples: a gas or (especially) a liquid, Water, milk, oil, jam, lipstick etc.
- **Fluid Dynamics:** It is the branch of mechanics that deals with the study of fluid which are in motion.
- **Stress:** Forcer per unit area (F/A) is called stress. It is denoted by τ. It has two types (i) Sheer stress / Tangential stress (ii) Normal Stress Flow
- **Sheer stress:** Tangent component of force per unit area is called sheer stress.
- **Normal stress:** Normal component of force per unit area is called Normal stress.
- **Flow:** The quantity of fluid passing through a point per unit time is called flow.
- **Viscosity:** It is the measure of resistance against the motion of fluid. It is denoted by μ. It is also called absolute viscosity and dynamic viscosity.
- **Viscous Flow:** Fluid that has non-zero viscosity or finite viscosity and can exert sheer stress on the surface is called viscous fluid or real fluid.
- **Inviscous Flow:** Fluid having zero viscosity is called inviscid fluid.
- **Compressibility:** Compressibility is the measure of change in fluid w.r.t volume and density under the action of external forces.
- **Compressible fluid:** A type of fluid in which change occur due to volume and density changes by the action of pressure (temperature) is called compressible fluid. Examples: gases.
- **Incompressible fluid:** A type of fluid in which no change occur due to volume and density changes by the action of pressure (temperature) is called incompressible fluid.

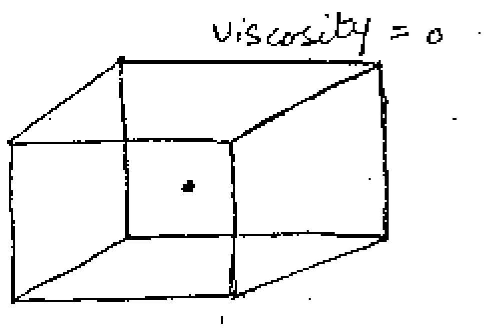
Momentum Equation For Frictionless Flow: Euler's Equation

Euler's equation after neglecting the viscous terms is given as follows;

$$\rho \frac{d\bar{V}}{dt} = \rho \vec{g} - \nabla p$$

This equation states that for an inviscid fluid the change in momentum of a fluid particle is caused by the body force (assumed to be gravity only) and the net pressure force.

Derivation



Consider finite size control volume through which inviscid fluid is flowing. Applying Newton's Second Law of Motion in x,y,z coordinates

For x axis; $\sum F_x = ma_x$ ……………(i)

Now using the fact Total force = Surface (Pressure Force) + Body Force

The sum of force acting on the fluid particle in x direction is

$$\sum F_x = \left[\left(p - \frac{\partial p}{\partial x}\frac{dx}{2}\right) - \left(p + \frac{\partial p}{\partial x}\frac{dx}{2}\right)\right] dydz + \rho g_x dxdydz$$

$$\sum F_x = -\frac{\partial p}{\partial x}\left(\frac{1}{2} + \frac{1}{2}\right) dxdydz + \rho g_x dxdydz$$

$$\sum F_x = -\frac{\partial p}{\partial x} dxdydz + \rho g_x dxdydz \quad \text{……………(ii)}$$

Since $\rho = \frac{m}{v} \Rightarrow m = \rho dv \Rightarrow m =$ mass of fluid particle $= \rho dxdydz$

$$(i) \Rightarrow \sum F_x = a_x \rho dxdydz \quad \text{……………(iii)}$$

Comparing (ii) and (iii)

$$-\frac{\partial p}{\partial x}dxdydz + \rho g_x dxdydz = a_x \rho dxdydz$$

$$-\frac{\partial p}{\partial x} + \rho g_x = a_x \rho \qquad \ldots\ldots\ldots\ldots\ldots(iv)$$

Similarly

$$-\frac{\partial p}{\partial y} + \rho g_y = a_y \rho \qquad \ldots\ldots\ldots\ldots\ldots(v)$$

$$-\frac{\partial p}{\partial z} + \rho g_z = a_z \rho \qquad \ldots\ldots\ldots\ldots\ldots(vi)$$

Now adding (iv),(v),(vi) we get total pressure

$$-\frac{\partial p}{\partial x}\hat{\imath} - \frac{\partial p}{\partial y}\hat{\jmath} - \frac{\partial p}{\partial z}\hat{k} + \rho\left(g_x\hat{\imath} + g_y\hat{\jmath} + g_z\hat{k}\right) = \rho\left(a_x\hat{\imath} + a_y\hat{\jmath} + a_z\hat{k}\right)$$

$$-\left(\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}\right)p + \rho\vec{g} = \rho\vec{a}$$

$$\boldsymbol{-\nabla p + \rho\vec{g} = \rho\vec{a}}$$

Or $$\boldsymbol{\rho\frac{d\bar{V}}{dt} = \rho\vec{g} - \nabla p} \qquad \ldots\ldots\ldots\ldots\ldots..(A)$$

We may write above as $$-\frac{1}{\rho}\nabla p + \vec{g} = \vec{a} \qquad \ldots\ldots\ldots\ldots\ldots..(B)$$

Then using $\vec{a} = \frac{d\vec{V}}{dt} = \frac{\partial\vec{V}}{\partial t} + V.\nabla\vec{V}$

We get $$-\frac{1}{\rho}\nabla p + \vec{g} = \frac{\partial\vec{V}}{\partial t} + V.\nabla\vec{V} \qquad \ldots\ldots\ldots\ldots\ldots..(C)$$

Hence (A),(B),(C) are different forms of Euler Equation of motion in Vector notation.

Momentum Equation (Euler's Equation) In Rectangular Coordinates For Frictionless Flow

Then Equation $\rho \frac{d\bar{V}}{dt} = \rho \vec{g} - \nabla p$ expressed in **rectangular coordinates** is as

$$\rho \frac{du}{dt} = \rho \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right) = \rho g_x - \frac{\partial p}{\partial x}$$

$$\rho \frac{dv}{dt} = \rho \left(\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} \right) = \rho g_y - \frac{\partial p}{\partial y}$$

$$\rho \frac{dw}{dt} = \rho \left(\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} \right) = \rho g_z - \frac{\partial p}{\partial z}$$

If the z axis is assumed vertical, then $g_x = g_y = 0$ and $g_z = -g$, so $\vec{g} = -\rho g \hat{k}$.

This system is applicable only on ideal compressible, rotational, unsteady and baratropic flow (when ρ is function of pressure only)

Momentum Equation (Euler's Equation) In Cylindrical Coordinates For Frictionless Flow

In **cylindrical coordinates**, the equations $\rho \vec{a} = \rho \frac{d\bar{V}}{dt} = \rho \vec{g} - \nabla p$ in component form, with gravity the only body force, are

$$\rho a_r = \rho \left(\frac{\partial V_r}{\partial t} + V_r \frac{\partial V_r}{\partial r} + \frac{V_\theta}{r} \frac{\partial V_r}{\partial \theta} + V_z \frac{\partial V_r}{\partial z} - \frac{V_\theta^2}{r} \right) = \rho g_r - \frac{\partial p}{\partial r}$$

$$\rho a_\theta = \rho \left(\frac{\partial V_\theta}{\partial t} + V_r \frac{\partial V_\theta}{\partial r} + \frac{V_\theta}{r} \frac{\partial V_\theta}{\partial \theta} + V_z \frac{\partial V_\theta}{\partial z} + \frac{V_r V_\theta}{r} \right) = \rho g_\theta - \frac{1}{r} \frac{\partial p}{\partial \theta}$$

$$\rho a_z = \rho \left(\frac{\partial V_z}{\partial t} + V_r \frac{\partial V_z}{\partial r} + \frac{V_\theta}{r} \frac{\partial V_z}{\partial \theta} + V_z \frac{\partial V_z}{\partial z} \right) = \rho g_z - \frac{\partial p}{\partial z}$$

If the z axis is directed vertically upward, then $g_r = g_\theta = 0$ and $g_z = -g$

Question:

The velocity field describes the motion of incompressible fluid steady state two dimensional flow $\vec{V} = (x^2y + y^2)\hat{\imath} - xy^2\hat{\jmath}$ then find pressure gradient in x, y directions (Neglecting Viscous Effect).

Solution: using Euler's Equation because viscous effects are negligible

$\rho \frac{d\bar{V}}{dt} = \rho\vec{g} - \nabla p$ ………………(i)

$\vec{a} = \frac{d\vec{V}}{dt} = \frac{\partial\vec{V}}{\partial t} + V.\nabla\vec{V} \Rightarrow \vec{a} = \frac{d\vec{V}}{dt} = (V.\nabla)\vec{V}$ $\because \frac{\partial\vec{V}}{\partial t} = 0$ for steady state flow

$\Rightarrow a_x = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}$

Since there is no idea about body force, thus we consider $\vec{g} = 0$

$(i) \Rightarrow \rho\frac{d\bar{V}}{dt} = -\nabla p$

$\Rightarrow \rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x}$

$\Rightarrow \frac{\partial p}{\partial x} = -\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right)$ ………(ii) and $\frac{\partial p}{\partial y} = -\rho\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right)$ …………(iii)

Since $u = x^2y + y^2$ and $v = -xy^2$

Therefore $\frac{\partial u}{\partial x} = 2xy, \frac{\partial u}{\partial y} = x^2 + 2y, \frac{\partial v}{\partial x} = -y^2, \frac{\partial v}{\partial y} = -2xy$

$(ii) \Rightarrow \frac{\partial p}{\partial x} = -\rho\big((x^2y + y^2)(2xy) + (-xy^2)(x^2 + 2y)\big) \Rightarrow \frac{\partial p}{\partial x} = -x^3y^2\rho$

$(iii) \Rightarrow \frac{\partial p}{\partial y} = -\rho\big((x^2y + y^2)(-y^2) + (-xy^2)(-2xy)\big) \Rightarrow \frac{\partial p}{\partial y} = -(x^2y^3 - y^4)\rho$

Gradient = $\nabla p = \frac{\partial p}{\partial x}\hat{\imath} + \frac{\partial p}{\partial y}\hat{\jmath}$

Gradient = $\nabla p = (-x^3y^2\rho)\hat{\imath} + (-(x^2y^3 - y^4)\rho)\hat{\jmath}$

Gradient = $\nabla p = -\rho[x^3y^2\hat{\imath} + (x^2y^3 - y^4)\hat{\jmath}]$

Question: The velocity field describes the motion of incompressible fluid steady state two dimensional flow $\vec{V} = (x^2y + y^2)\hat{\imath} - xy^2\hat{\jmath}$ then find value of pressure gradient at $(2,1)$ if fluid is water.

Solution: using Euler's Equation because viscous effects are negligible

$\rho \frac{d\bar{V}}{dt} = \rho\vec{g} - \nabla p$ ………………(i)

$\vec{a} = \frac{d\vec{V}}{dt} = \frac{\partial\vec{V}}{\partial t} + V.\nabla\vec{V} \Rightarrow \vec{a} = \frac{d\vec{V}}{dt} = (V.\nabla)\vec{V}$ $\because \frac{\partial\vec{V}}{\partial t} = 0$ for steady state flow

$\Rightarrow a_x = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}$

Since there is no idea about body force, thus we consider $\vec{g} = 0$

$(i) \Rightarrow \rho\frac{d\bar{V}}{dt} = -\nabla p \Rightarrow \rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x}$

$\Rightarrow \frac{\partial p}{\partial x} = -\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right)$ ………(ii) and $\frac{\partial p}{\partial y} = -\rho\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right)$ …………(iii)

Since $u = x^2y + y^2$ and $v = -xy^2$

Therefore $\frac{\partial u}{\partial x} = 2xy, \frac{\partial u}{\partial y} = x^2 + 2y, \frac{\partial v}{\partial x} = -y^2, \frac{\partial v}{\partial y} = -2xy$

$(ii) \Rightarrow \frac{\partial p}{\partial x} = -\rho\big((x^2y + y^2)(2xy) + (-xy^2)(x^2 + 2y)\big) \Rightarrow \frac{\partial p}{\partial x} = -x^3y^2\rho$

$(iii) \Rightarrow \frac{\partial p}{\partial y} = -\rho\big((x^2y + y^2)(-y^2) + (-xy^2)(-2xy)\big) \Rightarrow \frac{\partial p}{\partial y} = -(x^2y^3 - y^4)\rho$

Gradient $= \nabla p = \frac{\partial p}{\partial x}\hat{\imath} + \frac{\partial p}{\partial y}\hat{\jmath} = (-x^3y^2\rho)\hat{\imath} + (-(x^2y^3 - y^4)\rho)\hat{\jmath}$

Gradient $= \nabla p = -\rho[x^3y^2\hat{\imath} + (x^2y^3 - y^4)\hat{\jmath}]$

If fluid is water then $\rho = 1000kgm^{-3}$

Gradient at $(2,1)$ $= \nabla p\ |_{(2,1)} = -1000[2^3 1^2\hat{\imath} + (2^2 1^3 - 1^4)\hat{\jmath}]$

Gradient at $(2,1)$ $= \nabla p\ |_{(2,1)} = -8000\hat{\imath} - 3000\hat{\jmath}$

Question: The Pressure field in a steady state flow of an ideal fluid is given by $p = (10 - 6x^2 - 3yz^2)Pa$ (Pascal), if the fluid has mass density $1000kgm^{-3}$ then evaluate the acceleration of fluid at location $\vec{r} = 6\hat{\imath} + 2\hat{\jmath} + \hat{k}$ assuming body force is absent i.e. $\vec{g} = 0$.

Solution: Using Euler's Equation $\rho \frac{d\overline{V}}{dt} = \rho\vec{a} = \rho\vec{g} - \nabla p$

$\rho\vec{a} = -\nabla p \qquad \because \vec{g} = 0$

$\vec{a} = -\frac{1}{\rho}\nabla p = -\frac{1}{\rho}\left(\frac{\partial p}{\partial x}\hat{\imath} + \frac{\partial p}{\partial y}\hat{\jmath} + \frac{\partial p}{\partial z}\hat{k}\right)$

$\vec{a} = -\frac{1}{\rho}\left(\frac{\partial}{\partial x}(10 - 6x^2 - 3yz^2)\hat{\imath} + \frac{\partial}{\partial y}(10 - 6x^2 - 3yz^2)\hat{\jmath} + \frac{\partial}{\partial z}(10 - 6x^2 - 3yz^2)\hat{k}\right)$

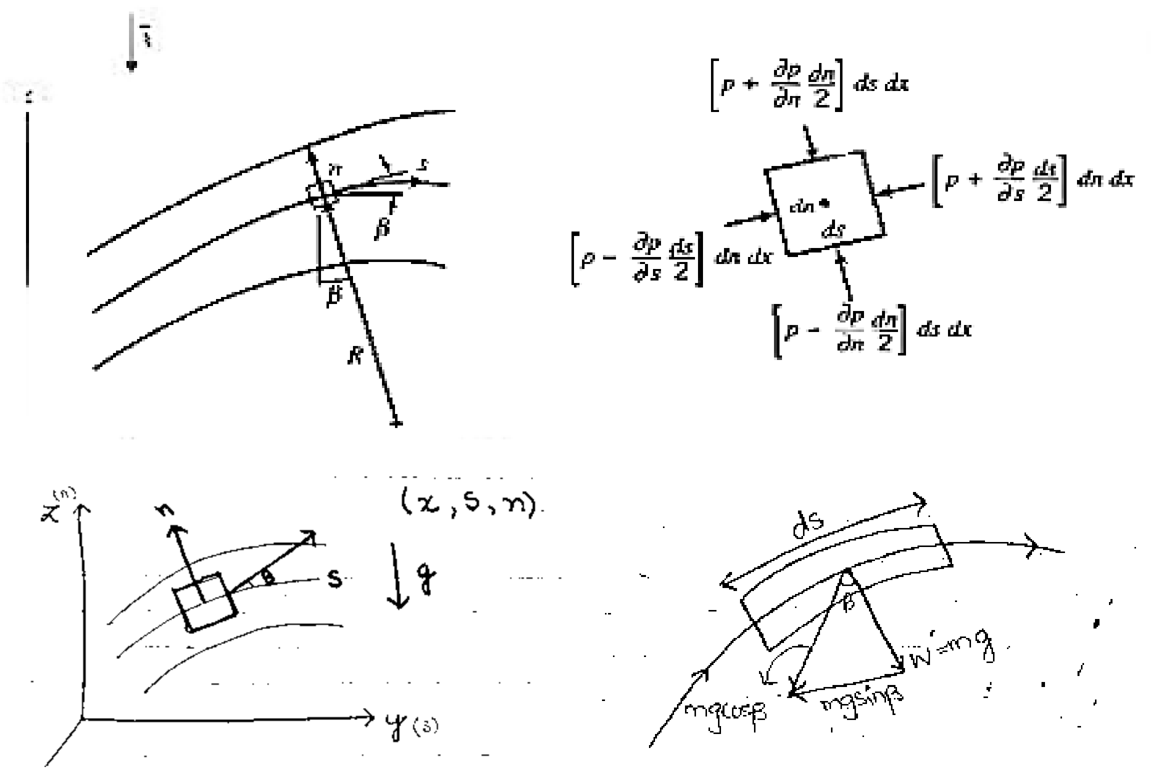
$\vec{a} = -\frac{1}{\rho}\left(-12x\hat{\imath} - 3z^2\hat{\jmath} + 6yz\hat{k}\right)$

Since $\vec{r} = 6\hat{\imath} + 2\hat{\jmath} + \hat{k}$ therefore $x = 6, y = 2, z = 1$ and $\rho = 1000kgm^{-3}$ then

$\vec{a} = -\frac{1}{1000}\left(-72\hat{\imath} - 3\hat{\jmath} - 12\hat{k}\right)$

Euler's Equations In Streamline Coordinates

Consider the flow in the yz plane shown in Figure;



Consider yz – plane. We wish to write the equations of motion in terms of the coordinate s, distance along a streamline, and the coordinate n, distance normal to the streamline. The pressure at the center of the fluid element or the center of the streamline is p. The volume of fluid is $ds\ dn\ dx$. Also using weight always acting downward. If we apply Newton's second law in the streamwise (the s) direction to the fluid element of volume $ds\ dn\ dx$, then neglecting viscous forces we obtain

F = Net Force acting on Fluid element = ma

$$\left(p - \frac{\partial p}{\partial s}\frac{ds}{2}\right) dndx - \left(p + \frac{\partial p}{\partial s}\frac{ds}{2}\right) dndx - \ mgSin\beta = ma_s$$

$$\left(p - \frac{\partial p}{\partial s}\frac{ds}{2}\right) dndx - \left(p + \frac{\partial p}{\partial s}\frac{ds}{2}\right) dndx - \ \rho g Sin\beta dsdndx = \rho a_s dsdndx$$

Where β is the angle between the tangent to the streamline and the horizontal, and a_s is the acceleration of the fluid particle along the streamline. Simplifying the equation, we obtain

$$pdndx - \frac{\partial p}{\partial s}\frac{ds}{2}dndx - pdndx - \frac{\partial p}{\partial s}\frac{ds}{2}dndx - \ \rho g Sin\beta dsdndx = \rho a_s dsdndx$$

$$-\frac{\partial p}{\partial s}\left(\frac{1}{2}+\frac{1}{2}\right)dsdndx - \ \rho g Sin\beta dsdndx = \rho a_s dsdndx$$

$$\left(-\frac{\partial p}{\partial s} - \ \rho g Sin\beta\right) dsdndx = \rho a_s dsdndx$$

$$-\frac{\partial p}{\partial s} - \ \rho g Sin\beta = \rho a_s$$

$$-\frac{\partial p}{\partial s} - \ \rho g \frac{\partial z}{\partial s} = \rho a_s \qquad \because Sin\beta = \frac{\partial z}{\partial s}$$

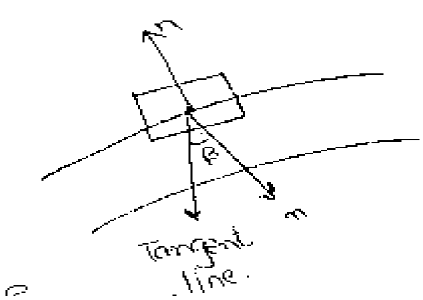
$$-\frac{1}{\rho}\frac{\partial p}{\partial s} - \ g\frac{\partial z}{\partial s} = a_s$$

Along any streamline $V = V(s,t)$, and the material or total acceleration of a fluid particle in the streamwise direction is given by $a_s = \frac{dV}{dt} = \frac{\partial V}{\partial t} + V\frac{\partial V}{\partial s}$

- Euler's equation in the streamwise direction with the z axis directed vertically upward is then $\frac{1}{\rho}\frac{\partial p}{\partial s} - \ g\frac{\partial z}{\partial s} = \frac{\partial V}{\partial t} + V\frac{\partial V}{\partial s}$
- For steady flow, and neglecting body forces, Euler's equation in the streamwise direction reduces to $\frac{1}{\rho}\frac{\partial p}{\partial s} = -V\frac{\partial V}{\partial s}$

  This equation indicates that a *decrease in velocity is accompanied* by an increase in pressure and conversely. This makes sense: The only force experienced by the particle is the net pressure force, so the particle accelerates toward low-pressure regions and decelerates when approaching high-pressure regions.

To obtain Euler's equation in a direction normal to the streamlines, we apply Newton's second law in the n direction to the fluid element.



Again, neglecting viscous forces, we obtain

$$\left(p - \frac{\partial p}{\partial n}\frac{dn}{2}\right) dsdx - \left(p + \frac{\partial p}{\partial n}\frac{dn}{2}\right) dsdx - \; mgCos\beta = ma_n$$

$$\left(p - \frac{\partial p}{\partial n}\frac{dn}{2}\right) dsdx - \left(p + \frac{\partial p}{\partial n}\frac{dn}{2}\right) dsdx - \; \rho gCos\beta dndxds = \rho a_n dndxds$$

where β is the angle between the n direction and the vertical, and a_n is the acceleration of the fluid particle in the *n* direction. Simplifying the equation, we obtain

$$pdsdx - \frac{\partial p}{\partial n}\frac{dn}{2} dsdx - pdsdx - \frac{\partial p}{\partial n}\frac{dn}{2} dsdx - \; \rho gCos\beta dndxds = \rho a_n dndxds$$

$$-\frac{\partial p}{\partial n}\left(\frac{1}{2} + \frac{1}{2}\right) dndsdx - \; \rho gCos\beta dndxds = \rho a_n dndxds$$

$$\left(-\frac{\partial p}{\partial n} - \; \rho gCos\beta\right) dndsdx = \rho a_n dndxds \Rightarrow -\frac{\partial p}{\partial n} - \; \rho gCos\beta = \rho a_n$$

$$-\frac{1}{\rho}\frac{\partial p}{\partial n} - \; gCos\frac{\partial z}{\partial n} = a_n \qquad\qquad \because Cos\beta = \frac{\partial z}{\partial n}$$

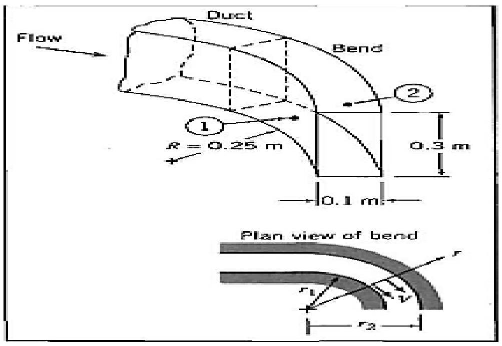
The normal acceleration of the fluid element is toward the center of curvature of the streamline, in the minus n direction; thus in the coordinate system of Figure, the familiar centripetal acceleration is written $a_n = \frac{V^2}{R}$

For steady flow, where R is the radius of curvature of the streamline. Then,

- Euler's equation normal to the streamline is written for steady flow as $\frac{1}{\rho}\frac{\partial p}{\partial n} - \; gCos\frac{\partial z}{\partial n} = \frac{V^2}{R}$
- For steady flow in a horizontal plane, Euler's equation normal to a streamline becomes $\frac{1}{\rho}\frac{\partial p}{\partial n} = \frac{V^2}{R}$

  Equation $\frac{1}{\rho}\frac{\partial p}{\partial n} = \frac{V^2}{R}$ indicates that pressure increases in the direction outward from the center of curvature of the streamlines. This also makes sense: Because the only force experienced by the particle is the net pressure force, the pressure field creates the centripetal acceleration. In regions where the streamlines are straight, the radius of curvature, R, is infinite so there is no pressure variation normal to straight streamlines.

Example: Flow in A Bend: The flow rate of air at standard conditions in a flat duct is to be determined by installing pressure taps across a bend. The duct is 0.3 m deep and 0.l m wide. The inner radius of the bend is 0.25 m. If the measured pressure difference between the taps is 40 mm of water, compute the approximate flow rate.



Flow through duct bend as shown. $p_2 - p_1 = \rho_{H_2O} g \Delta h$ where $\Delta h = 40mm = 0.04m$. Air is at STP. Then find Volume flow rate, Q.

Solution: Apply Euler's *n* component equation across flow streamlines.

$\frac{\partial p}{\partial r} = \rho \frac{V^2}{r}$ Assuming Frictionless, Incompressible, Uniform flow at measurement section. For this flow, $p = p(r)$, so $\frac{\partial p}{\partial r} = \frac{dp}{dr} = \rho \frac{V^2}{r}$ or $dp = \rho V^2 \frac{dr}{r}$

Integrating $\int_{p_1}^{p_2} dp = \rho V^2 \int_{r_1}^{r_2} \frac{dr}{r}$ we get $|p|_{p_1}^{p_2} = \rho V^2 |lnr|_{r_1}^{r_2}$

$$p_2 - p_1 = \rho V^2 ln\frac{r_2}{r_1} \Rightarrow V^2 = \frac{p_2 - p_1}{\rho ln\frac{r_2}{r_1}} \Rightarrow V = \sqrt{\frac{p_2 - p_1}{\rho ln\frac{r_2}{r_1}}} = \sqrt{\frac{\rho_{H_2O} g \Delta h}{\rho ln\frac{r_2}{r_1}}}$$

Substituting numerical values, $V = \sqrt{\frac{999kgm^{-3} \times 9.8ms^{-2} \times 0.04m}{1.23kgm^{-3} ln(0.35m/0.25m)}} = 30.8ms^{-1}$

For uniform flow $Q = VA = 30.8ms^{-1} \times 0.1m \times 0.3m = 0.924m^3s^{-1}$

In this problem we assumed that the velocity is uniform across the section. In fact, the velocity in the bend approximates a free vortex (irrotational) profile in which $V \propto \frac{1}{r}$ (where r is the radius) instead of V = const. Hence, this flow-measurement device could only be used to obtain approximate values of the flow rate.

Bernoulli Equation: Integration of Euler's Equation along a Streamline for Steady Flow

Compared to the viscous-flow equivalents, the momentum or Euler's equation for incompressible, inviscid flow is simpler mathematically, but solution (in conjunction with the mass conservation equation) still presents formidable difficulties in all but the most basic flow problems. One convenient approach for a steady flow is to integrate Euler's equation along a streamline. We will do this below using two different mathematical approaches, and each will result in the Bernoulli equation. Recall that we derived the Bernoulli equation by starting with a differential control volume; these two additional derivations will give us more insight into the restrictions inherent in use of the Bernoulli equation.

Derivation of (Bernoulli Equation: Integration of Euler's Equation along a Streamline for Steady Flow) Using <u>Streamline Coordinates</u>

For Steady, Incompressible, Frictionless flow along a streamline we will have

$$\frac{p}{\rho} - \frac{V^2}{2} + gz = Constant$$

Derivation

Using Euler's equation for steady flow along a streamline

$$-\frac{1}{\rho}\frac{\partial p}{\partial s} - g\frac{\partial z}{\partial s} = V\frac{\partial V}{\partial s} \qquad \ldots\ldots\ldots\ldots\ldots\text{(i)}$$

If a fluid particle moves a distance, ds, along a streamline, then

$\frac{\partial p}{\partial s}ds = dp$ (the change in pressure along s)

$\frac{\partial z}{\partial s}ds = dz$ (the change in elevation along s)

$\frac{\partial V}{\partial s}ds = dV$ (the change in speed along s)

Thus, after multiplying (i) by ds, we can write $-\frac{1}{\rho}\frac{\partial p}{\partial s}ds - g\frac{\partial z}{\partial s}ds = V\frac{\partial V}{\partial s}ds$

$-\frac{dp}{\rho} - gdz = VdV$ or $\frac{dp}{\rho} + VdV + gdz = 0$ (along s)

Integration of this equation gives $\frac{1}{\rho}\int dp + \int VdV + \ g\int dz = \int 0\, ds$

$$\frac{p}{\rho} + \frac{V^2}{2} + gz = Constant \qquad \text{(along s)}$$

Before above can be applied, we must specify the relation between pressure and density. For the special case of *incompressible flow*, ρ = constant, and Equation becomes the Bernoulli equation,

Restrictions:

(1) Steady flow.
(2) Incompressible flow.
(3) Frictionless flow.
(4) Flow along a streamline.

Importance:

The Bernoulli equation is a powerful and useful equation because it relates pressure changes to velocity and elevation changes along a streamline. However, it gives correct results only when applied to a flow situation where all four of the restriction are reasonable. Keep the restrictions firmly in mind whenever you consider using the Bernoulli equation. (In general, the Bernoulli constant in $\frac{p}{\rho} + \frac{V^2}{2} + gz = Constant$ has different values along different streamlines)

Remember

- **The streamlines:** The streamlines are lines drawn in the flow field tangent to the velocity vector at every point. **Or** A curve drawn in the fluid such that tangent to every point of it is in the direction of fluid velocity
- For steady flow, streamlines, pathlines, and streaklines coincide. The motion of a particle along a streamline is governed by equation $(V.\nabla)\vec{V} = -\frac{1}{\rho}\nabla p - \hat{k}g$

Derivation of (Bernoulli Equation: Integration of Euler's Equation along a Streamline for Steady Flow) Using <u>Rectangular Coordinates</u>

For Steady, Incompressible, Frictionless flow along a rectangular coordinates we will have $\frac{p}{\rho}+\frac{V^2}{2}+gz=Constant$

Derivation

The vector form of Euler's equation, $\rho\frac{d\overline{V}}{dt}=\rho\vec{a}=\rho\vec{g}-\nabla p$ ....(i)

$\vec{a}=\frac{d\vec{V}}{dt}=\frac{\partial\vec{V}}{\partial t}+V.\nabla\vec{V}\Rightarrow\vec{a}=\frac{d\vec{V}}{dt}=(V.\nabla)\vec{V}$ $\because\frac{\partial\vec{V}}{\partial t}=0$ for steady state flow

$\rho(V.\nabla)\vec{V}=\rho\vec{g}-\nabla p$ where $\vec{g}$ is acting downward.

Using $\hat{k}=\frac{\vec{g}}{g}$ we get $\vec{g}=-\hat{k}g$ and $\rho(V.\nabla)\vec{V}=-\rho\hat{k}g-\nabla p$

$(V.\nabla)\vec{V}=-\frac{1}{\rho}\nabla p-\hat{k}g$ ....(ii)

For steady flow the velocity field is given by V = V(x, y, z). During time interval dt the particle covers displacement $d\vec{s}$ along the streamline.

Taking the dot product of equation (ii) with displacement $d\vec{s}$ along the streamline

$(\bar{V}.\nabla)\bar{V}.d\vec{s}=-\frac{1}{\rho}\nabla p.d\vec{s}-\vec{g}\hat{k}.d\vec{s}$ ....(iii)

where $d\vec{s}=dx\hat{\imath}+dy\hat{\jmath}+dz\hat{k}$ (along s)

Now we evaluate each of the three terms, starting on the right,

$$-\frac{1}{\rho}\nabla p.d\vec{s}=-\frac{1}{\rho}\left(\frac{\partial p}{\partial x}\hat{\imath}+\frac{\partial p}{\partial y}\hat{\jmath}+\frac{\partial p}{\partial z}\hat{k}\right).\left(dx\hat{\imath}+dy\hat{\jmath}+dz\hat{k}\right)$$

$$-\frac{1}{\rho}\nabla p.d\vec{s}=-\frac{1}{\rho}\left(\frac{\partial p}{\partial x}dx+\frac{\partial p}{\partial y}dy+\frac{\partial p}{\partial z}dz\right)$$

$-\frac{1}{\rho}\nabla p.d\vec{s}=-\frac{1}{\rho}dp$ (along s {x,y,z – axis})

Now $-\vec{g}\hat{k}.d\vec{s}=-\vec{g}\hat{k}.\left(dx\hat{\imath}+dy\hat{\jmath}+dz\hat{k}\right)=-gdz$ (along z)

Now using the following result

$\nabla\left(\vec{U}.\vec{V}\right) = \vec{U} \times \left(\vec{\nabla} \times \vec{V}\right) + \vec{V} \times \left(\vec{\nabla} \times \vec{U}\right) + \vec{U}.\vec{\nabla}\vec{V} + \vec{V}.\vec{\nabla}\vec{U}$

$\nabla\left(\vec{V}.\vec{V}\right) = \vec{V} \times \left(\vec{\nabla} \times \vec{V}\right) + \vec{V} \times \left(\vec{\nabla} \times \vec{V}\right) + \vec{V}.\vec{\nabla}\vec{V} + \vec{V}.\vec{\nabla}\vec{V} \qquad \because \vec{U} = \vec{V}$

$\nabla\vec{V}^2 = 2\vec{V} \times \left(\vec{\nabla} \times \vec{V}\right) + 2\vec{V}.\vec{\nabla}\vec{V}$

$\frac{1}{2}\nabla\vec{V}^2 = \vec{V} \times \left(\vec{\nabla} \times \vec{V}\right) + \vec{V}.\vec{\nabla}\vec{V}$

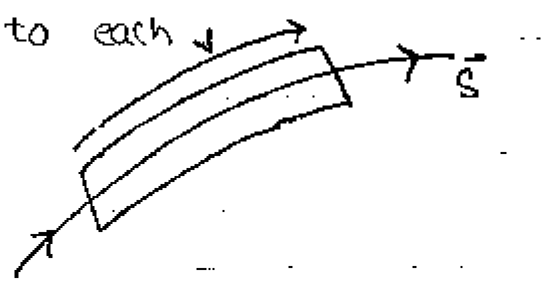
$\frac{1}{2}\nabla\vec{V}^2 - \vec{V} \times \left(\vec{\nabla} \times \vec{V}\right) = \vec{V}.\vec{\nabla}\vec{V} = (\bar{V}.\nabla)\bar{V}$

Now $(\bar{V}.\nabla)\bar{V}.d\vec{s} = \left(\frac{1}{2}\nabla\vec{V}^2 - \vec{V} \times \left(\vec{\nabla} \times \vec{V}\right)\right).d\vec{s}$

$(\bar{V}.\nabla)\bar{V}.d\vec{s} = \frac{1}{2}\nabla\vec{V}^2.d\vec{s} - \vec{V} \times \left(\vec{\nabla} \times \vec{V}\right).d\vec{s}$ ……………(A)

Since $\vec{V} \times \left(\vec{\nabla} \times \vec{V}\right).d\vec{s} = -\left(\vec{\nabla} \times \vec{V}\right) \times \vec{V}.d\vec{s} = -\left(\vec{\nabla} \times \vec{V}\right).\vec{V} \times d\vec{s}$

Since $\vec{V}$ and $d\vec{s}$ are parallel therefore $\vec{V} \times d\vec{s} = 0$



Thus $\vec{V} \times \left(\vec{\nabla} \times \vec{V}\right).d\vec{s} = 0$

$(A) \Rightarrow (\bar{V}.\nabla)\bar{V}.d\vec{s} = \frac{1}{2}\nabla\vec{V}^2.d\vec{s}$

$\Rightarrow (\bar{V}.\nabla)\bar{V}.d\vec{s} = \frac{1}{2}\left(\frac{\partial\vec{V}^2}{\partial x}\hat{\imath} + \frac{\partial\vec{V}^2}{\partial y}\hat{\jmath} + \frac{\partial\vec{V}^2}{\partial z}\hat{k}\right).\left(dx\hat{\imath} + dy\hat{\jmath} + dz\hat{k}\right)$

$\Rightarrow (\bar{V}.\nabla)\bar{V}.d\vec{s} = \frac{1}{2}\left(\frac{\partial\vec{V}^2}{\partial x}dx + \frac{\partial\vec{V}^2}{\partial y}dy + \frac{\partial\vec{V}^2}{\partial z}dz\right)$

$\Rightarrow (\bar{V}.\nabla)\bar{V}.d\vec{s} = \frac{1}{2}d\vec{V}^2$ (along s {x,y,z – axis})

$(iii) \Rightarrow \frac{1}{2} d\vec{V}^2 = -\frac{1}{\rho} \mathrm{dp} - gdz \Rightarrow \frac{1}{\rho} \mathrm{dp} + \frac{1}{2} d\vec{V}^2 + gdz = 0$

Integration of this equation gives $\frac{1}{\rho} \int dp + \frac{1}{2} \int d\vec{V}^2 + g \int dz = \int 0\, ds$

$\frac{p}{\rho} + \frac{V^2}{2} + gz = Constant$

The Bernoulli equation, derived using rectangular coordinates, is still subject to the restrictions: (1) steady flow, (2) incompressible flow, (3) frictionless flow, and (4) flow along a streamline.

Static Pressures

The pressure, p, which we have used in deriving the Bernoulli equation, $\frac{p}{\rho} + \frac{V^2}{2} + gz = Constant,$ is the thermodynamic pressure; it is commonly called the **static pressure**. The static pressure is the pressure seen by the fluid particle as it moves (so it is something of a misnomer)

Stagnation Pressure وہ پریشر جو فلیوڈ کو زیرو ولاسٹی لانے کیلیے لگایا جائے

The stagnation pressure is obtained when a flowing fluid is decelerated to zero speed by a frictionless process. For incompressible flow, the Bernoulli equation can be used to relate changes in speed and pressure along a streamline for such a process, Neglecting elevation differences, Equation $\frac{p}{\rho} + \frac{V^2}{2} + gz = Constant$ becomes $\frac{p}{\rho} + \frac{V^2}{2} = Constant$

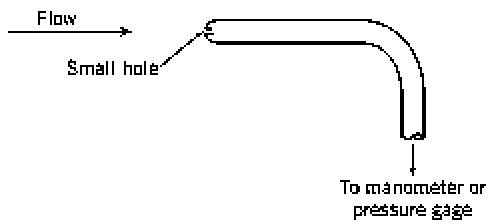
If the static pressure is p at a point in the flow where the speed is V, then the stagnation pressure, p_0 where the stagnation speed, V_0, is zero, may be computed from $\frac{p_0}{\rho} + \frac{{V_0}^2}{2} = \frac{p}{\rho} + \frac{V^2}{2}$ Or $p_0 = p + \frac{1}{2} \rho V^2$ where $\frac{{V_0}^2}{2} = 0$

This Equation is a mathematical statement of the definition of stagnation pressure valid for incompressible flow. Equation states that the stagnation (or total) pressure equals the static pressure plus the dynamic pressure.

$p_{stangation} = p_{static} + p_{dynamic}$

Stagnation Pressure Probe, Or Pitot

Stagnation pressure is measured in the laboratory using a probe with a hole that faces directly upstream as shown in Figure. Such a probe is called a **stagnation pressure probe**, or **pitot** (pronounced pea-toe) tube. Pitot (or pitot-static) tubes are often placed on the exterior of aircraft to indicate air speed relative to the aircraft, and hence aircraft speed relative to the air.

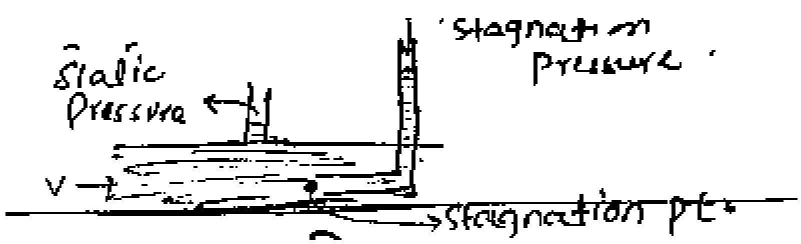


Dynamic Pressure

In equation $p_0 = p + \frac{1}{2}\rho V^2$ the term $\frac{1}{2}\rho V^2$ generally is called the dynamic pressure. i.e.

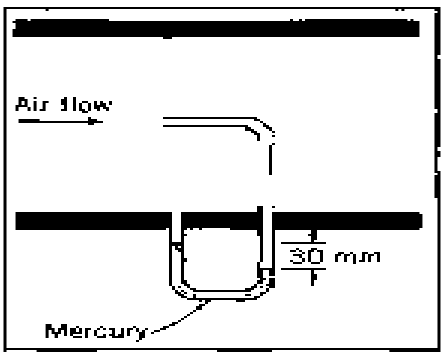
$$p_{dynamic} = p_0 - p = \frac{1}{2}\rho V^2$$

Also remember $V = \sqrt{\frac{2(p_0 - p)}{\rho}}$ is speed for fluid particle.



Example Pitot Tube

A pitot tube is inserted in an air flow (at STP) to measure the flow speed. The tube is inserted so that it points upstream into the flow and the pressure sensed by the tube is the stagnation pressure. The static pressure is measured at the same location in the flow, using a wall pressure tap. If the pressure difference is 30 mm of mercury, determine the flow speed.



Solution: Given a pitot tube inserted in a flow as shown. The flowing fluid is air and the manometer liquid is mercury. We have to find the flow speed. Assume Steady, Incompressible flow along a streamline. Also frictionless deceleration along stagnation streamline.

Writing Bernoulli's equation along the stagnation streamline (with $\Delta z = 0$) yields

$$\frac{p_0}{\rho} = \frac{p}{\rho} + \frac{1}{2}V^2$$

p_0 is the stagnation pressure at the tube opening where the speed has been reduced, without friction, to zero. Solving for V gives $V = \sqrt{\frac{2(p_0 - p)}{\rho_{air}}}$

From the diagram, $p_0 - p = \rho_{Hg} gh = \rho_{H_2O} ghSG_{Hg}$

$$V = \sqrt{\frac{2(\rho_{H_2O} ghSG_{Hg})}{\rho_{air}}} = \sqrt{\frac{2 \times 1000 kgm^{-3} \times 9.8 ms^{-2} \times 30mm \times 13.6 \times 10^{-3} m}{1.23 kgm^{-3}}} = 80.8 ms^{-1}$$

At T = 20°C, the speed of sound in air is 343 m/s. Hence, M = 0.236 and the assumption of incompressible flow is valid.

Applications of Bernoulli Equation

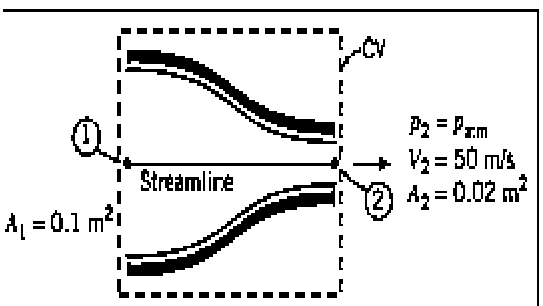
The Bernoulli equation can be applied between any two points on a streamline (in flow field) with Steady, Incompressible, and Frictionless flow. The result is

$$\frac{p_1}{\rho} + \frac{V_1^2}{2} + gz_1 = \frac{p_2}{\rho} + \frac{V_2^2}{2} + gz_2$$

Where subscripts 1 and 2 represent any two points on a streamline.

Example: Nozzle Flow

Air flows steadily at low speed through a horizontal **nozzle** (by definition a device for accelerating a flow), discharging to atmosphere. The area at the nozzle inlet is $0.1m^2$. At the nozzle exit, the area is $0.02m^2$. Determine the gage pressure required at the nozzle inlet to produce an outlet speed of 50 m/s.



Solution: For a Flow through a nozzle, we have to find $p_1 - p_{atm}$

Since we have $\frac{p_1}{\rho} + \frac{V_1^2}{2} + gz_1 = \frac{p_2}{\rho} + \frac{V_2^2}{2} + gz_2$

And Continuity for incompressible and uniform flow is as follows;

$$\sum \vec{V}.\vec{A} = \sum_{CS} \vec{V}.\vec{A} + \sum_{CV} \vec{V}.\vec{A} = \sum_{CV} \vec{V}.\vec{A} = 0$$

Assumptions: (1) Steady flow. (2) Incompressible flow. (3) Frictionless flow.

(4) Flow along a streamline. (5) $z_1 = z_2$ (6) Uniform flow at sections (1) and (2).

The maximum speed of 50 m/s is well below 100 m/s, which corresponds to Mach number $M \approx 0.3$ in standard air. Hence, the flow may be treated as incompressible.

Apply the Bernoulli equation along a streamline between points (1) and (2) to evaluate p_1

We have $\frac{p_1}{\rho} + \frac{V_1^2}{2} + gz_1 = \frac{p_2}{\rho} + \frac{V_2^2}{2} + gz_1$ as $z_1 = z_2$

$p_1 - p_{atm} = p_1 - p_2 = \frac{\rho}{2}(V_2^2 - V_1^2)$ ……………….(i)

Apply the continuity equation to determine V_1 ;

$(-\rho V_1 A_1) + (\rho V_2 A_2) = 0$ or $V_1 A_1 = V_2 A_2 \Rightarrow V_1 = V_2 \frac{A_2}{A_1}$

$\Rightarrow V_1 = 50ms^{-1} \frac{0.02m^2}{0.1m^2} = 10ms^{-1}$

For air at standard conditions, $\rho = 1.23kgm^{-3}$. Then

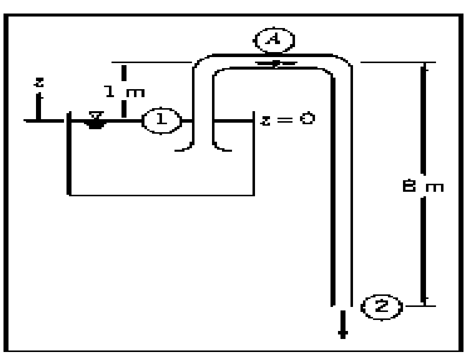
$(i) \Rightarrow p_1 - p_{atm} = \frac{1.23kgm^{-3}}{2}[(50ms^{-1})^2 - (10ms^{-1})^2] = 1.48kPa$

Notes: This problem illustrates a typical application of the Bernoulli equation.

The streamlines must be straight at the inlet and exit in order to have uniform pressures at those locations.

Example: Flow through a Siphon

A U-tube acts as a water siphon. The bend in the tube is 1 m above the water surface; the tube outlet is 7 m below the water surface. The water issues from the bottom of the siphon as a free jet at atmospheric pressure. Determine (after listing the necessary assumptions. The speed of the free jet and the minimum absolute pressure of the water in the bend.



Solution: Given that Water flowing through a siphon as shown. We have to find (a) Speed of water leaving as a free jet. (b) Pressure at point (A) in the flow.

Since we have $\frac{p}{\rho} + \frac{V^2}{2} + gz = Constant$

Assumptions: (1) Neglect friction. (2) Steady flow. (3) Incompressible flow.

(4) Flow along a streamline. (5) Reservoir is large compared with pipe.

Apply the Bernoulli equation between points (1) and (2).

$$\frac{p_1}{\rho} + \frac{V_1^2}{2} + gz_1 = \frac{p_2}{\rho} + \frac{V_2^2}{2} + gz_2$$

Since $area_{reservior} \gg area_{pipe}$ then $V_1 \approx 0$. Also $p_1 = p_2 = p_{atm}$, so

$$\frac{p_{atm}}{\rho} + \frac{V_1^2=0}{2} + gz_1 = \frac{p_{atm}}{\rho} + \frac{V_2^2}{2} + gz_2$$

$gz_1 = \frac{V_2^2}{2} + gz_2$ and $V_2^2 = 2g(z_1 - z_2)$

$V_2 = \sqrt{2g(z_1 - z_2)} = \sqrt{2 \times 9.8ms^{-1} \times 7m} = 11.7ms^{-1}$ where $z_1 = 0, z_2 = -7$

To determine the pressure at location (A), we write the Bernoulli equation between (1) and (A). $\frac{p_1}{\rho} + \frac{V_1^2}{2} + gz_1 = \frac{p_A}{\rho} + \frac{V_A^2}{2} + gz_A$

Again $V_1 \approx 0$ and from conservation of mass $V_A = V_2$. Hence

$$\frac{p_A}{\rho} = \frac{p_1}{\rho} + \frac{V_1^2=0}{2} + gz_1 - \frac{V_2^2}{2} - gz_A$$

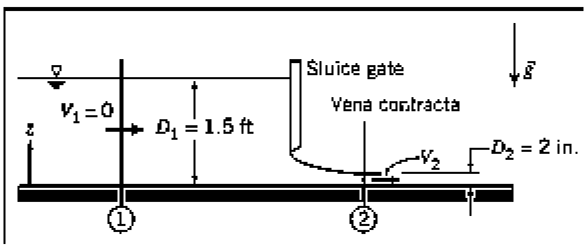
$$\frac{p_A}{\rho} = \frac{p_1}{\rho} + gz_1 - gz_A - \frac{V_2^2}{2} \Rightarrow p_A = p_1 + \rho g(z_1 - z_A) - \rho\frac{V_2^2}{2}$$

$p_A = 1.01 \times 10^5 Nm^{-2} + 999kgm^{-3} \times 9.81ms^{-2} \times (-1Nkg^{-1}s^2) - \frac{1}{2} \times 999kgm^{-3} \times (11.7)^2 Nkg^{-1}m = 22.8kPa(abs)$ or $-78.5kPa\ (gage)$

Notes: This problem illustrates an application of the Bernoulli equation that includes elevation changes. Always take care when neglecting friction in any internal flow. In this problem, neglecting friction is reasonable if the pipe is smooth-surfaced and is relatively short.

Example: Flow under a Sluice Gate

Water flows under a sluice gate on a horizontal bed at the inlet to a flume. Upstream from the gate, the water depth is 1.5 ft and the speed is negligible. At the vena contracta downstream from the gate, the flow streamlines are straight and the depth is 2 in. Determine the flow speed downstream from the gate and the discharge in cubic feet per second per foot of width.



Solution: Given that Row of water under a sluice gate. We have to find

(a) V_2. (b) Q in $ft^3 s^{-1} ft^{-1}$ of width.

Since we have $\frac{p_1}{\rho} + \frac{V_1^2}{2} + gz_1 = \frac{p_2}{\rho} + \frac{V_2^2}{2} + gz_2$

Assumptions: (1) Steady flow. (2) Incompressible flow. (3) Frictionless flow.

(4) Flow along a streamline. (5) Uniform flow at each section.

(6) Hydrostatic pressure distribution.

If we consider the streamline that runs along the bottom of the channel (z = 0) because of assumption 6 the pressures at (1) and (2) are

$p_1 = p_{atm} + \rho g D_1$ and $p_2 = p_{atm} + \rho g D_2$

So that the Bernoulli equation for this streamline is

$$\frac{p_{atm} + \rho g D_1}{\rho} + \frac{V_1^2}{2} = \frac{p_{atm} + \rho g D_2}{\rho} + \frac{V_2^2}{2} \qquad \text{with } (z = 0)$$

Or $$\frac{V_1^2}{2} + gD_1 = \frac{V_2^2}{2} + gD_2 \qquad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

On the other hand, consider die streamline that runs along the free surface on both sides of the gate. For this streamline

$$\frac{p_{atm}}{\rho} + gD_1 + \frac{V_1^2}{2} = \frac{p_{atm}}{\rho} + gD_2 + \frac{V_2^2}{2}$$

Or $\quad \frac{V_1^2}{2} + gD_1 = \frac{V_2^2}{2} + gD_2 \qquad$ ……………..(1)

We have arrived at the same equation (Eq. 1) for the streamline at the bottom and the streamline at the free surface, implying the Bernoulli constant is the same for both streamlines. Solving for V_2 yields

$$V_2 = \sqrt{2g(D_1 - D_2) + V_1^2}$$

But $V_1^2 \approx 0$, so

$$V_2 = \sqrt{2g(D_1 - D_2)} = \sqrt{2 \times 32.2 fts^{-2} \times \left(1.5ft - 2in. \times \frac{ft}{12in.}\right)} = 9.27fts^{-1}$$

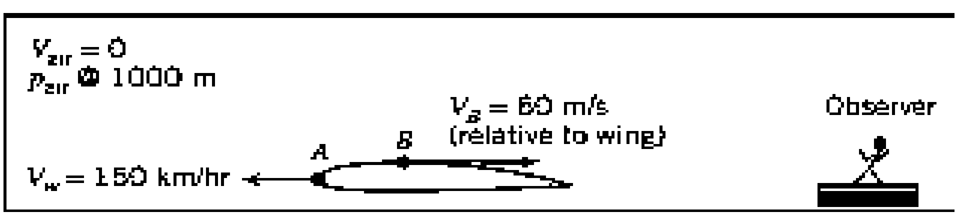
For uniform flow, $Q = VA = VDw$, or $\frac{Q}{w} = VD = V_2D_2 = 9.27fts^{-1} \times 2in. \times \frac{ft}{12in.}$

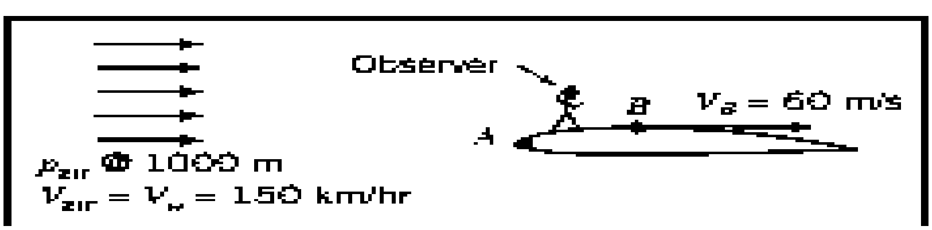
$\frac{Q}{w} = 1.55ft^2s^{-1} = 1.55ft^3s^{-1}$ per foot of width

…………………………………………………

Example: Bernoulli Equation In Translating Reference Frame

A light plane flies at 150 km/hr in standard air at an altitude of 1000 m. Determine the stagnation pressure at the leading edge of the wing. At a certain point close to the wing, the air speed relative to the wing is 60 m/s. Compute the pressure at this point.





Solution: Given that Aircraft in flight at 150 km/hr at 1000 m altitude is in standard air. We have to find Stagnation pressure, p_{0_A}, at point A and static pressure, p_B, at point B. Flow is unsteady when observed from a fixed frame, that is, by an observer on the ground. However, an observer on the wing sees the above (figure 2) steady flow. At z = 1000 m in standard air, the temperature is 281 K and the speed of sound is 336 m/s. Hence at point B, $M_B = \frac{V_B}{c} = 0.178$. This is less than 0.3, so the flow may be treated as incompressible. Thus the Bernoulli equation can be applied along a streamline in the moving observer's inertial reference frame.

Since $\frac{p_{air}}{\rho} + \frac{V_{air}^2}{2} + gz_{air} = \frac{p_A}{\rho} + \frac{V_A^2}{2} + gz_A = \frac{p_B}{\rho} + \frac{V_B^2}{2} + gz_B$

Assumptions: (1) Steady flow. (2) Incompressible flow (V < 100 m/s).

(3) Frictionless flow. (4) Flow along a streamline. (5) Neglect Δz.

Values for pressure and density may be found from Table. Thus, at 1000 m,

$\frac{p}{p_{SL}} = 0.8870$ and $\frac{\rho}{\rho_{SL}} = 0.9075$

Consequently, $p = 0.8870 p_{SL} = 0.8870 \times 1.01 \times 10^5 Nm^{-2} = 8.96 \times 10^4 Nm^{-2}$

And $\rho = 0.9075 \rho_{SL} = 0.9075 \times 1.23 kgm^{-3} = 1.12 kgm^{-3}$

Since the speed is $V_A = 0$ at the stagnation point, and $p_{0_A} = p_{air} + \frac{1}{2} V_{air}^2$

$$p_{0_A} = 8.96 \times 10^4 Nm^{-2} + \frac{1}{2} \times 1.12 kgm^{-3} \left(150 \frac{km}{hr} \times 1000 \frac{m}{km} \times \frac{hr}{3600s}\right)^2 \times \frac{Ns^2}{kg.m}$$

$p_{0_A} = 90.6\text{kPa}$

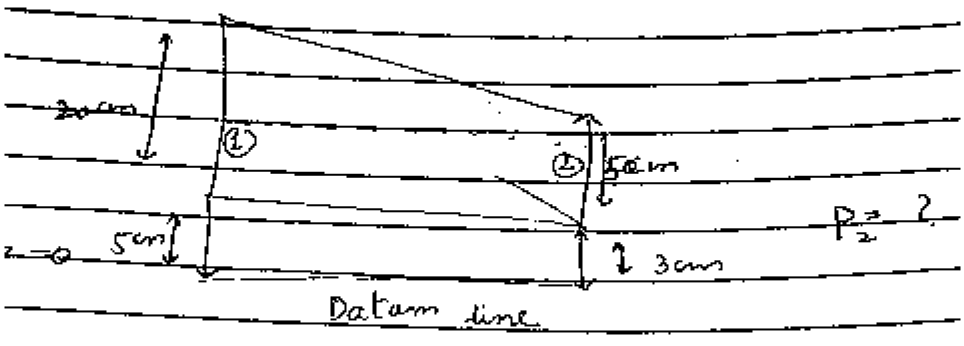
Solving for the static pressure at B, we obtain

$$p_B = p_{air} + \frac{1}{2}\rho\left(V_{air}^2 - V_B^2\right) = 8.96 \times 10^4 Nm^{-2} + \frac{1}{2} \times 1.12 kgm^{-3} \left[\left(150 \frac{km}{hr} \times 1000 \frac{m}{km} \times \frac{hr}{3600s}\right)^2 - (60 ms^{-1})^2\right] \times \frac{Ns^2}{kg.m}$$

$p_{0_A} = 88.6\text{kPa}$

Example: The diameter of a pipe changes from 20cm at a section, 5cm above datum (reference line) to 5cm at a section 3cm above the datum. The pressure of water at 1st section is 5kg/cm^2. If velocity of flow at 1st section is 1m/s then determine pressure at 2nd section.

Solution:



Diameter of larger end = d_1 = 20cm = 0.2m

Area of larger end = A_1 = $\frac{1}{4}\pi d_1^2$ = 0.0314m^2

Diameter of smaller end = d_2 = 5cm = 0.05m

Area of smaller end = A_2 = $\frac{1}{4}\pi d_2^2$ = 0.00196m^2

Velocity of larger end = V_1 = 1m/s

Discharge rate = A_1V_1 = A_2V_2

$\Rightarrow 0.0314 \times 1 = 0.00196V_2 \Rightarrow V_2 = \frac{0.0314\times 1}{0.00196} \Rightarrow V_2 = 16\text{m/s}$

To determine pressure at 2nd section; $\frac{p_1}{\rho} + \frac{V_1^2}{2} + gz_1 = \frac{p_2}{\rho} + \frac{V_2^2}{2} + gz_2$

$\Rightarrow \frac{p_1}{\rho g} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{V_2^2}{2g} + z_2 \Rightarrow \frac{p_1}{\rho g} + \frac{V_1^2}{2g} + z_1 - \frac{V_2^2}{2g} - z_2 = \frac{p_2}{\rho g}$

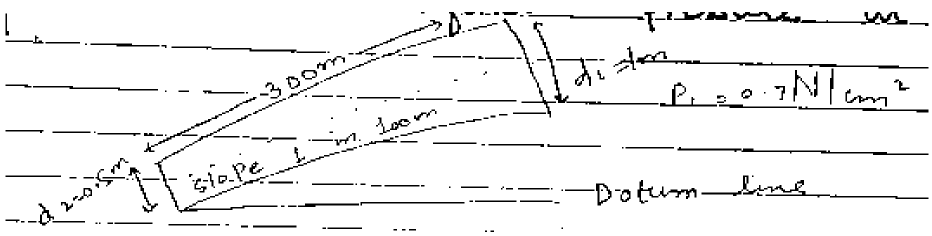
$\Rightarrow p_2 = \rho g\left(\frac{p_1}{\rho g} + \frac{V_1^2}{2g} + z_1 - \frac{V_2^2}{2g} - z_2\right)$

Since $p_1 = 5kgcm^{-2} = 5000kgm^{-2}, \rho = 1000kgm^{-2}, z_1 = 5m, z_2 = 3m$

$\Rightarrow p_2 = 1000 \times 9.8\left(\frac{5000}{1000\times 9.8} + \frac{1}{2\times 9.8} + 5 - \frac{(16)^2}{2\times 9.8} - 3\right) \Rightarrow p_2 = -57898.4kgm^{-2}$

Example: A pipe 300m long has a slope of 1 in 100m and tappers from 1m diameter at a higher end and 0.5m at lower end. Quantity of water flowing is 0.09m^3/s. Then fined pressure at lower end.

Solution:



Length of pipe at lower end = l = 300m

Diameter of pipe at lower end = d_2 = 0.5m

Area of pipe at lower end = $A_2 = \frac{1}{4}\pi d_2^2$ = 0.1963m^2

Diameter of pipe at higher end = d_1 = 1 m

Area of pipe at higher end = $A_1 = \frac{1}{4}\pi d_1^2$ = 0.7854m^2

Height of lower end from datum = z_2 = 0

Height of higher end from datum = $z_1 = \frac{1}{100} \times 300$ = 3m

Quantity of water flowing = Q = 0.09m^3/s

Slope after 100m = 1

Slope after 1m = 1/100

Slope after 100m = 1

Slope after 300m = (1/100)*300

$p_1 = 0.7Ncm^{-2} = 0.7 \times 10^4 Nm^{-2}$

$Q = A_2V_2 \Rightarrow V_2 = \frac{Q}{A_2} = \frac{0.09}{0.19} = 0.474ms^{-1}$

$Q = A_1V_1 \Rightarrow V_1 = \frac{Q}{A_1} = \frac{0.09}{0.19} = 0.7854ms^{-1}$

To determine pressure at lower end ; $\frac{p_1}{\rho} + \frac{V_1^2}{2} + gz_1 = \frac{p_2}{\rho} + \frac{V_2^2}{2} + gz_2$

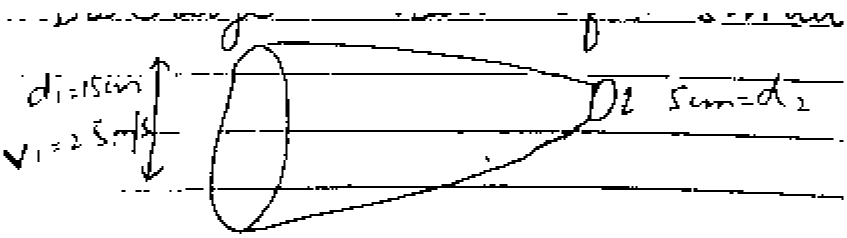
$\Rightarrow \frac{p_1}{\rho g} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{V_2^2}{2g} + z_2 \Rightarrow \frac{p_1}{\rho g} + \frac{V_1^2}{2g} + z_1 - \frac{V_2^2}{2g} - z_2 = \frac{p_2}{\rho g}$

$\Rightarrow p_2 = \rho g \left(\frac{p_1}{\rho g} + \frac{V_1^2}{2g} + z_1 - \frac{V_2^2}{2g} - z_2\right)$

$\Rightarrow p_2 = 1000 \times 9.8 \left(\frac{0.7\times10^4}{1000\times9.8} + \frac{(0.115)^2}{2\times9.8} + 3 - \frac{(0.46)^2}{2\times9.8}\right) \Rightarrow p_2 = 36330.7Nm^{-2}$

Example: Water is flowing through tapered pipe (piper having different section) having diameter 15cm and 5cm at the larger end and smaller respectively. Determine the velocity head and the rate of discharge in litters per second, at the smaller section, if velocity of the water at large end is 2.5cm/s.

Solution:



Diameter of higher end = d_1 = 15cm = 0.15m

Area of higher end = A_1 = $\frac{1}{4}\pi d_1^2$ = 0.01767m^2

Diameter of smaller end = d_2 = 5cm = 0.05m

Area of smaller end = A_2 = $\frac{1}{4}\pi d_2^2$ = 0.00196m^2

Velocity of higher end = V_1 = 2.5m/s

Discharge rate of larger end = Q= A_1V_1

Discharge rate of smaller end = Q = A_2V_2

Since discharge rate is constant therefore Q = A_1V_1= A_2V_2

$\Rightarrow 0.01767 \times 2.5 = 0.00196V_2 \Rightarrow V_2 = \frac{0.01767\times2.5}{0.00196} \Rightarrow V_2 = 32.5\text{m/s}$

Velocity head for smaller region = $\frac{V_2^2}{2g} = \frac{(32.5)^2}{2\times9.8} = 25.81m/s$

Discharge rate of smaller end = Q = A_2V_2 = $0.00196 \times 32.5 = 0.0441m^3s^{-1}$

Discharge rate of smaller end = Q = A_2V_2 = $= 44.1ls^{-1}$

Since $1m^3 = 1000liters$

Cautions on Use of the Bernoulli Equation

In Previous Examples Problems we have seen several situations where the Bernoulli equation may be applied because the restrictions on its use led to a reasonable flow model. However, in some situations you might be tempted to apply the Bernoulli equation where the restrictions are not satisfied. Some subtle cases that violate the restrictions are discussed briefly in this section.

- In a subsonic nozzle (a converging section) the pressure drops, accelerating a flow. Because the pressure drops and the walls of the nozzle converge, there is no flow separation from the walls and the boundary layer remains thin. In addition, a nozzle is usually relatively short so frictional effects are not significant. All of this leads to the conclusion that the Bernoulli equation is suitable for use for subsonic nozzles.
- Sometimes we need to decelerate a flow. This can be accomplished using a subsonic diffuser (a diverging section), or by using a sudden expansion (e.g., from a pipe into a reservoir). In these devices the flow decelerates because of an adverse pressure gradient. As we discussed an adverse pressure gradient tends to lead to rapid growth of the boundary layer and its separation. Hence, we should be care-ful in applying the Bernoulli equation in such devices—at best, it will be an approximation. Because of area blockage caused by boundary-layer growth, pressure rise in actual diffusers always is less than that predicted for inviscid one-dimensional flow.
- The Bernoulli equation is a reasonable model for the siphon because the entrance is well rounded, the bends are gentle, and the overall length is short. Flow separation, which can occur at inlets with sharp corners and in abrupt bends, causes the flow to depart from that predicted by a one dimensional model and the Bernoulli equation. Frictional effects will not be negligible if the tube is long.
- Thus the Bernoulli equation cannot be used to model flow through a hydraulic jump.
- The Bernoulli equation cannot be applied through a machine such as a propeller, pump, turbine, or windmill. The equation was derived by integrating along a stream tube or a streamline in the absence of moving surfaces such as blades or vanes. It is impossible to have locally steady flow or to identify streamlines during flow through a machine. Hence, while the

Bernoulli equation may be applied between points before a machine, or between points after a machine (assuming its restrictions are satisfied), it cannot be applied through the machine. (In effect, a machine will change the value of the Bernoulli constant.)

- Finally, compressibility must be considered for flow of gases.

Unsteady Bernoulli Equation

Integration of Euler's Equation along a Streamline for Unsteady flow

Using Euler's equation in streamlines direction with z – axis directed vertically upward $-\frac{1}{\rho}\frac{\partial p}{\partial s} - g\frac{\partial z}{\partial s} = \frac{\partial V}{\partial t} + V\frac{\partial V}{\partial s}$

$$\left(-\frac{1}{\rho}\frac{\partial p}{\partial s} - g\frac{\partial z}{\partial s}\right)ds = \left(\frac{\partial V}{\partial t} + V\frac{\partial V}{\partial s}\right)ds$$

$$-\frac{1}{\rho}\frac{\partial p}{\partial s}ds - g\frac{\partial z}{\partial s}ds = \frac{\partial V}{\partial t}ds + V\frac{\partial V}{\partial s}ds \qquad \ldots\ldots\ldots\ldots\ldots(i)$$

$\frac{\partial p}{\partial s}ds = dp$ (Change in Pressure along ds)

$\frac{\partial z}{\partial s}ds = dz$ (Change in Elevation along ds)

$\frac{\partial V}{\partial s}ds = dV$ (Change in Velocity along ds)

$$(i) \Rightarrow -\frac{dp}{\rho} - gdz = \frac{\partial V}{\partial t}ds + VdV$$

Integrating form streamline 1 to 2 for incompressible flow

$$\Rightarrow \frac{p_2 - p_1}{\rho} + g(z_2 - z_1) + \frac{V_2^2 - V_1^2}{2} + \int_1^2 \frac{\partial V}{\partial t}d\,s = 0$$

$$\Rightarrow \frac{p_1}{\rho} + \frac{V_1^2}{2} + gz_1 = \frac{p_2}{\rho} + \frac{V_2^2}{2} + gz_2 + \int_1^2 \frac{\partial V}{\partial t}d\,s$$

To evaluate integral form of this equation the variation $\frac{\partial V}{\partial t}$ must be function of 's' where we assume to evaluate required equation the restriction as <u>incompressible, frictionless flow along a streamline</u>.

Irrotational Flow

<u>These are flows in which the fluid particles do not rotate.</u>

In case of irrotational flow $\vec{w} = 0$ and we get irrotationality condition as follows

$\Rightarrow \vec{\nabla} \times \vec{V} = 0 \qquad\qquad \because \vec{w} = \frac{1}{2}\vec{\nabla} \times \vec{V}$

$$\Rightarrow \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix} = 0 \Rightarrow \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\hat{\imath} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\hat{\jmath} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\hat{k} = 0\hat{\imath} + 0\hat{\jmath} + 0\hat{k}$$

$$\Rightarrow \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0$$

In cylindrical coordinate system the irrotationality condition requires that

$$\vec{\nabla} \times \vec{V} = \hat{e}_r\left(\frac{1}{r}\frac{\partial v_z}{\partial \theta} - \frac{\partial v_\theta}{\partial z}\right) + \hat{e}_\theta\left(\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}\right) + \hat{e}_z\left(\frac{1}{r}\frac{\partial (r v_\theta)}{\partial r} - \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right) = 0$$

Bernoulli Equation Applied to Irrotational Flow

Using Bernoulli Equation $\frac{p}{\rho} + \frac{V^2}{2} + gz = constant$ ……………(i)

For irrotational flow $\vec{\nabla} \times \vec{V} = 0$

We show that the constant which is used in the Bernoulli Equation have same values for all streamlines, for this purpose suppose

$-\frac{1}{\rho}\nabla p - g\hat{k} = (\vec{V}.\vec{\nabla})\vec{V}$ ……………(ii)

Using result $(\bar{V}.\nabla)\bar{V} = \frac{1}{2}\nabla\vec{V}^2 - \vec{V} \times (\vec{\nabla} \times \vec{V})$

$(\bar{V}.\nabla)\bar{V} = \frac{1}{2}\nabla\vec{V}^2 \qquad\qquad \because \vec{\nabla} \times \vec{V} = 0$

$(ii) \Rightarrow -\frac{1}{\rho}\nabla p - g\hat{k} = \frac{1}{2}\nabla\vec{V}^2$

Consider displacement in the flow field from $\vec{r}$ to $\vec{r} + d\vec{r}$ then $d\vec{r}$ is small displacement taking dot product of above equation with $d\vec{r}$

$\Rightarrow -\frac{1}{\rho}\nabla p.\, d\vec{r} - g\hat{k}.\, d\vec{r} = \frac{1}{2}\nabla\vec{V}^2.\, d\vec{r}$

$\Rightarrow -\frac{dp}{\rho} - gdz = \frac{1}{2}\mathrm{d}V^2 \Rightarrow \frac{dp}{\rho} + \frac{1}{2}\mathrm{d}\vec{V}^2 + gdz = 0$

Integration of this equation gives $\frac{1}{\rho}\int dp + \frac{1}{2}\int d\vec{V}^2 + \ g\int dz = \int 0\, ds$

$\frac{p}{\rho} + \frac{V^2}{2} + gz = Constant$

Since $d\vec{r}$ is small displacement, so this equation is valid between any two streamlines in steady, incompressible, irrotational and inviscid flow.

Velocity Potential

Since we know that for irrotational flow $\vec{\nabla} \times \vec{V} = 0$ ………..(i)

We define scalar function φ such that $\vec{V} = -\vec{\nabla}\varphi \quad (i) \Rightarrow \vec{\nabla} \times \vec{V} = \vec{\nabla} \times \left(-\vec{\nabla}\varphi\right) = 0$

Now as $\vec{V} = -\vec{\nabla}\varphi$ then this implies $u\hat{\imath} + v\hat{\jmath} + w\hat{k} = -\frac{\partial\varphi}{\partial x}\hat{\imath} - \frac{\partial\varphi}{\partial y}\hat{\jmath} - \frac{\partial\varphi}{\partial z}\hat{k}$

Comparing coefficient $u = -\frac{\partial\varphi}{\partial x}, v = -\frac{\partial\varphi}{\partial y}, w = -\frac{\partial\varphi}{\partial z}$

With potential function defined in this way the irritation condition

$\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0$ Is satisfied identically.

In cylindrical coordinates $\vec{\nabla} = \frac{\partial}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial}{\partial\theta}\hat{e}_\theta + \frac{\partial}{\partial z}\hat{e}_z$ then $\vec{V} = -\vec{\nabla}\varphi$ implies

$V_r\hat{e}_r + V_\theta\hat{e}_\theta + V_z\hat{e}_z = -\frac{\partial\varphi}{\partial r}\hat{e}_r - \frac{1}{r}\frac{\partial\varphi}{\partial\theta}\hat{e}_\theta - \frac{\partial\varphi}{\partial z}\hat{e}_z$

Comparing coefficient $V_r = -\frac{\partial\varphi}{\partial r}, V_\theta = -\frac{1}{r}\frac{\partial\varphi}{\partial\theta}, V_z = -\frac{\partial\varphi}{\partial z}$

With potential function defined in this way the irritation condition

$\frac{1}{r}\frac{\partial v_z}{\partial\theta} - \frac{\partial v_\theta}{\partial z} = \frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r} = \frac{1}{r}\frac{\partial(rv_\theta)}{\partial r} - \frac{1}{r}\frac{\partial v_r}{\partial\theta}$ Is satisfied identically.

Stream Function and Velocity Potential for Two-Dimensional, Irrotational, Incompressible Flow: Laplace's Equation

For a two-dimensional, incompressible, irrotational flow we have expressions for the velocity components, u and v, in terms of both the stream function ψ, and the velocity potential φ,

$u = \frac{\partial \psi}{\partial y}, v = -\frac{\partial \psi}{\partial x}$ ....(i) and $u = -\frac{\partial \varphi}{\partial x}, v = -\frac{\partial \varphi}{\partial y}$ ....(ii)

Since we have irrotationality condition $\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0$

Then $\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0$

$\Rightarrow \frac{\partial}{\partial x}\left(-\frac{\partial \psi}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial y}\right) = 0$ using (i)

$\Rightarrow \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = \vec{\nabla}^2 \psi = 0$

Since we have continuity equation $\vec{\nabla}.\vec{V} = 0$

Then $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$

$\Rightarrow \frac{\partial}{\partial x}\left(-\frac{\partial \varphi}{\partial x}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial \varphi}{\partial y}\right) = 0$ using (ii)

$\Rightarrow \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = \vec{\nabla}^2 \varphi = 0$

Above both Equations are forms of Laplace's equation—an equation that arises in many areas of the physical sciences and engineering. Any function ψ or φ that satisfies Laplace's equation represents a possible two-dimensional, incompressible, irrotational flow field.

Example

Consider the flow field given by $\psi = ax^2 - ay^2$, where $a = 3s^{-1}$. Show that the flow is irrotational. Determine the velocity potential for this flow.

Solution: If the flow is irrotational, then $\vec{\nabla} \times \vec{V} = 0$

Using $\vec{V} = \vec{\nabla}\psi = \vec{\nabla}\varphi$ then $\vec{\nabla} \times \vec{\nabla}\psi = 0$. Hence $\vec{\nabla}^2\psi = 0$

$$\Rightarrow \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 0 \Rightarrow \frac{\partial^2}{\partial x^2}(ax^2 - ay^2) + \frac{\partial^2}{\partial y^2}(ax^2 - ay^2) = 0 \Rightarrow \vec{\nabla}^2\psi = 0$$

Since flow is irrotational therefore

$$u = \frac{\partial\psi}{\partial y}, v = -\frac{\partial\psi}{\partial x} \qquad \text{and} \qquad u = -\frac{\partial\varphi}{\partial x}, v = -\frac{\partial\varphi}{\partial y}$$

Then $u = -\frac{\partial\varphi}{\partial x} = \frac{\partial\psi}{\partial y} \Rightarrow \frac{\partial\varphi}{\partial x} = -\frac{\partial\psi}{\partial y}$

$$\Rightarrow \frac{\partial\varphi}{\partial x} = -\frac{\partial}{\partial y}(ax^2 - ay^2) \Rightarrow \frac{\partial\varphi}{\partial x} = 2ay$$

$\Rightarrow \varphi = 2axy + f(y)$ ....(I) integrating w.r.to 'x'

Now Then $v = -\frac{\partial\varphi}{\partial y} = -\frac{\partial\psi}{\partial x} \Rightarrow \frac{\partial\varphi}{\partial y} = \frac{\partial\psi}{\partial x}$

$$\Rightarrow \frac{\partial\varphi}{\partial y} = \frac{\partial}{\partial x}(ax^2 - ay^2) \Rightarrow \frac{\partial\varphi}{\partial y} = 2ax$$

$$(I) \Rightarrow \frac{\partial}{\partial y}\big(2axy + f(y)\big) = 2ax \Rightarrow 2ax + \frac{\partial f}{\partial y} = 2ax$$

$\Rightarrow \frac{\partial f}{\partial y} = 0 \Rightarrow f = c$ (Constant)

Thus $\varphi = 2axy + c$

This problem illustrates the relations among the stream function, velocity potential, and velocity field.

Potential function:

For potential function we use $\phi(x,y)$ or $\phi(x,y,z)$

$$u=\frac{\partial\phi}{\partial x}\ ,\ v=\frac{\partial\phi}{\partial y}\ ,\ w=\frac{\partial\phi}{\partial z}$$

Theorem: Prove that stream function and potential function are orthogonal.

Proof:

Consider a stream function $\psi=\psi(x,y)$

$$\frac{\partial\psi}{\partial x}dx+\frac{\partial\psi}{\partial y}dy=d\psi$$

For stream function ψ is constant , $d\psi=0$

$$\frac{\partial\psi}{\partial x}dx+\frac{\partial\psi}{\partial y}dy=0$$

$$-vdx+udy=0 \qquad \because \frac{\partial\psi}{\partial y}=u\ ,\ -\frac{\partial\psi}{\partial x}=v$$

$$vdx=udy$$

$$\frac{dy}{dx}=\frac{v}{u}=m_1(say)$$

For potential function $\phi=\phi(x,y)$

$$\frac{\partial\phi}{\partial x}dx+\frac{\partial\phi}{\partial y}dy=d\phi$$

For potential function ϕ is constant, $d\phi=0$

$$\frac{\partial\phi}{\partial x}dx+\frac{\partial\phi}{\partial y}dy=0$$

$$udx+vdy=0 \qquad \because \frac{\partial\psi}{\partial x}=u\ ,\ \frac{\partial\psi}{\partial y}=v$$

$$-udx=vdy$$

$$\frac{dy}{dx}=-\frac{u}{v}=m_2\ (\text{say})$$

Now
$$m_1.m_2=\frac{v}{u}.\left(\frac{-u}{v}\right)=-1$$

Which show that stream function and potential function is orthogonal.

(i) Show that it satisfies law of conservation of mass.
(ii) Obtain the expression for stream function.
(iii) Show that the flow is potential.
(iv) Find the potential function or obtain the expression for potential function.

Solution: (i)

Given that $u = x - 4y$, $v = -y - 4x$

$$\frac{\partial u}{\partial x} = 1 \quad , \quad \frac{\partial v}{\partial y} = -1$$

We know the law of conservation of mass

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

(ii). Now the expression for stream function

$$u = \frac{\partial \psi}{\partial y} = x - 4y \qquad \_\_\_\_\_(i) \qquad \because u = \frac{\partial \psi}{\partial y}$$

$$v = -\frac{\partial \psi}{\partial x} = -y - 4x$$

$$v = \frac{\partial \psi}{\partial x} = y + 4x \qquad \_\_\_\_(ii)$$

From (i) $\Rightarrow$ $\frac{\partial \psi}{\partial y} = x - 4y$

$$d\psi = (x - 4y)\,dy$$

Integrating both sides

$$\int d\psi = \int (x - 4y)\,dy$$

$$\psi = xy - 2y^2 + f(x) \qquad \_\_\_(iii)$$

Diff. w.r.t x we have

$$\frac{\partial \psi}{\partial x} = y + f'(x) \qquad compare\ \ with\ (ii)$$

$$y+4x=y+f'(x)$$

$$\Rightarrow \quad f'(x)=4x$$

Integrating both sides

$$\Rightarrow \quad \int f'(x)=\int 4x\,dx$$

$$\Rightarrow \quad f(x)=2x^2 \qquad put \;\; in \;\; (iii)$$

$$\psi=xy-2y^2+2x^2 \qquad \text{required result}$$

It will satisfy the Laplace equation

$$\frac{\partial\psi}{\partial x}=y+4x \quad , \quad \frac{\partial\psi}{\partial v}=x-4y$$

$$\frac{\partial^2\psi}{\partial x^2}=4 \quad , \quad \frac{\partial^2\psi}{\partial y^2}=-4$$

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}=4+(-4)=0$$

Laplace equation satisfied.

(iii) For potential function

Given that $\quad u=y+4x \quad \Rightarrow \quad \dfrac{\partial u}{\partial y}=-4$

$\quad v=-y-4x \quad \Rightarrow \quad \dfrac{\partial v}{\partial x}=-4$

For potential flow

$$\frac{\partial u}{\partial y}=\frac{\partial v}{\partial x}$$

$$-4=-4 \quad \text{satisfied}$$

(iv) Now the expression for potential function

$$u = \frac{\partial \phi}{\partial y} = x - 4y \qquad \_\_\_\_\_(i) \qquad \because u = \frac{\partial \phi}{\partial y}$$

$$v = \frac{\partial \phi}{\partial x} = -y - 4x \qquad \_\_\_\_(ii) \qquad \because v = \frac{\partial \phi}{\partial y}$$

From (i) $\Rightarrow$ $\frac{\partial \phi}{\partial y} = x - 4y$

$$d\phi = (x - 4y)dx$$

Integrating both sides

$$\int d\phi = \int (x - 4y)dx$$

$$\phi = \frac{x^2}{2} - 4xy + f(y) \qquad \_\_\_(iii)$$

Diff. w.r.t y we have

$$\frac{\partial \phi}{\partial y} = -4x + f'(y) \qquad \textit{compare with (ii)}$$

$$-y - 4x = -4x + f'(y)$$

$$\Rightarrow \quad f'(y) = -y$$

Integrating both sides

$$\Rightarrow \quad \int f'(y) = \int -y dy$$

$$\Rightarrow \quad f(y) = \frac{-y^2}{2} \qquad \textit{put in (iii)}$$

$$\phi = \frac{x^2}{2} - 4xy - \frac{y^2}{2} \qquad \text{required result}$$

It will satisfy the Laplace equation

$$\frac{\partial \phi}{\partial x} = x - 4y \quad , \quad \frac{\partial \phi}{\partial y} = -4x - y$$

$$\frac{\partial^2 \psi}{\partial x^2} = 1 \quad , \quad \frac{\partial^2 \psi}{\partial y^2} = -1$$

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 1 + (-1) = 0$$

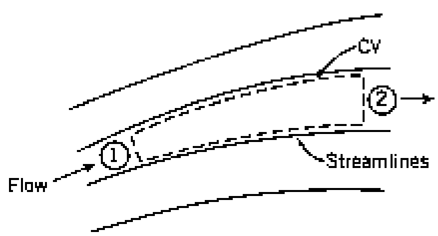
Laplace equation satisfied.

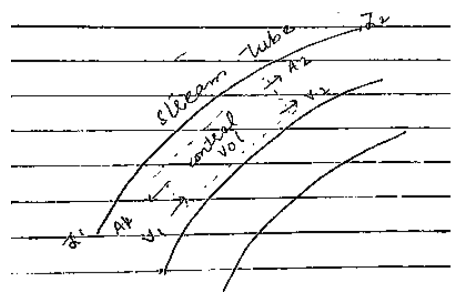
Relationship between 1st law of thermodynamics and Bernoulli Equation

The Bernoulli Equation Interpreted as an Energy Equation

Bernoulli equation cannot deduced from 1st law of thermodynamics. 1st law of thermodynamics only state conservation of energy in isolated system.

An equation identical in form of Bernoulli equation (although requiring very different restrictions) may be obtained from the first law of thermodynamics. Consider steady flow in the absence of shear forces. We choose a control volume bounded by streamlines along its periphery. Such a boundary, shown in Figure, often called a stream tube.





By the 1st law of thermodynamics

$$\dot{Q} - \cancelto{0}{\dot{W}_s} - \cancelto{0}{\dot{W}_{shear}} - \cancelto{0}{\dot{W}_{other}} = \cancelto{0}{\frac{\partial}{\partial t}\int_{cv} e\rho dv} + \int_{cs}(e + pv)\rho\vec{V}.d\vec{A} \quad \ldots\ldots\ldots\ldots\text{(i)}$$

$e = u + \frac{V^2}{2} + gz$ and $\dot{W}_s = \dot{W}_{shaft}$ = Shaft work = Any mechanical energy to flow

Restrictions: (1) $\dot{W}_s = 0$. (2) $\dot{W}_{shear} = 0$. (3) $\dot{W}_{other} = 0$.

(4) Steady flow. (5) Uniform flow and proportional at each section.

(Remember that here *v* represents the specific volume, and *u* represents the specific internal energy, not velocity i.e. $v = \frac{1}{\rho}$)

Under these restrictions, equation (i) becomes

$$\dot{Q} = \int_{cs}\left(u + \frac{V^2}{2} + gz + pv\right)\rho\vec{V}.d\vec{A}$$

$$\left(u_1 + p_1v_1 + \frac{V_1^2}{2} + gz_1\right)(-\rho_1V_1A_1) + \left(u_2 + p_2v_2 + \frac{V_2^2}{2} + gz_2\right)(\rho_2V_2A_2) - \dot{Q} = 0 \quad \ldots\ldots\text{(ii)}$$

By using continuity equation in integral form

$$\cancelto{0}{\frac{\partial}{\partial t}\int_{cv}\rho dv} + \int_{cs}\rho\vec{V}.d\vec{A} = 0 \qquad \text{or} \qquad \sum_{cs}\rho\vec{V}.\vec{A} = 0$$

$$-\rho_1 V_1 A_1 + \rho_2 V_2 A_2 = 0 \Rightarrow \rho_1 V_1 A_1 = \rho_2 V_2 A_2 = \dot{m} \qquad \because Q = \frac{V}{A}, \dot{m} = m\rho$$

Also $\dot{Q}$ = Rate of Heat Transfer

$$\dot{Q} = \frac{dQ}{dt} = \frac{dQ}{dm}\frac{dm}{dt} = \frac{dQ}{dm}\dot{m}$$

$$(ii) \Rightarrow -\left(u_1 + p_1 v_1 + \frac{V_1^2}{2} + gz_1\right)\dot{m} + \left(u_2 + p_2 v_2 + \frac{V_2^2}{2} + gz_2\right)\dot{m} - \frac{dQ}{dm}\dot{m} = 0$$

$$\Rightarrow \left[\left(p_2 v_2 + \frac{V_2^2}{2} + gz_2\right) - \left(p_1 v_1 + \frac{V_1^2}{2} + gz_1\right)\right]\dot{m} + \left(u_2 - u_1 - \frac{dQ}{dm}\right)\dot{m} = 0$$

$$\Rightarrow p_1 v_1 + \frac{V_1^2}{2} + gz_1 = p_2 v_2 + \frac{V_2^2}{2} + gz_2 + u_2 - u_1 - \frac{dQ}{dm}$$

$$\Rightarrow \frac{p_1}{\rho} + \frac{V_1^2}{2} + gz_1 = \frac{p_2}{\rho} + \frac{V_2^2}{2} + gz_2 + u_2 - u_1 - \frac{dQ}{dm} \qquad \because v_1 = v_2 = \frac{1}{\rho}$$

This equation reduces to Bernoulli equation if $u_2 - u_1 - \frac{dQ}{dm} = 0$

$$\Rightarrow \frac{p}{\rho} + \frac{V^2}{2} + gz = \text{Constant}$$

This is Bernoulli equation for steady, incompressible and frictionless flow.

$$\Rightarrow \frac{p}{\rho g} + \frac{V^2}{2g} + z = \frac{constant}{g} = H \qquad \text{dividing by 'g'}$$

Here $\frac{V^2}{2g}$ is Velocity Head (K.E.), $\frac{p}{\rho g}$ is pressure Head (Pressure Energy), z is elevation (Gravitational Energy) and H is total Head (Energy)

Internal Energy And Heat Transfer In Frictionless Incompressible Flow

Consider frictionless, incompressible flow with heat transfer. Show that

$$u_2 - u_1 = \frac{dQ}{dm}$$

Solution: Consider internal energy as $u = u(T, v)$.

For incompressible flow, $v = constant$, and $u = u(T)$.

From the Gibbs equation, $Tds = du + \rho dv$,

But since $dv = 0$ Therefore we obtain $Tds = du$ for incompressible flow

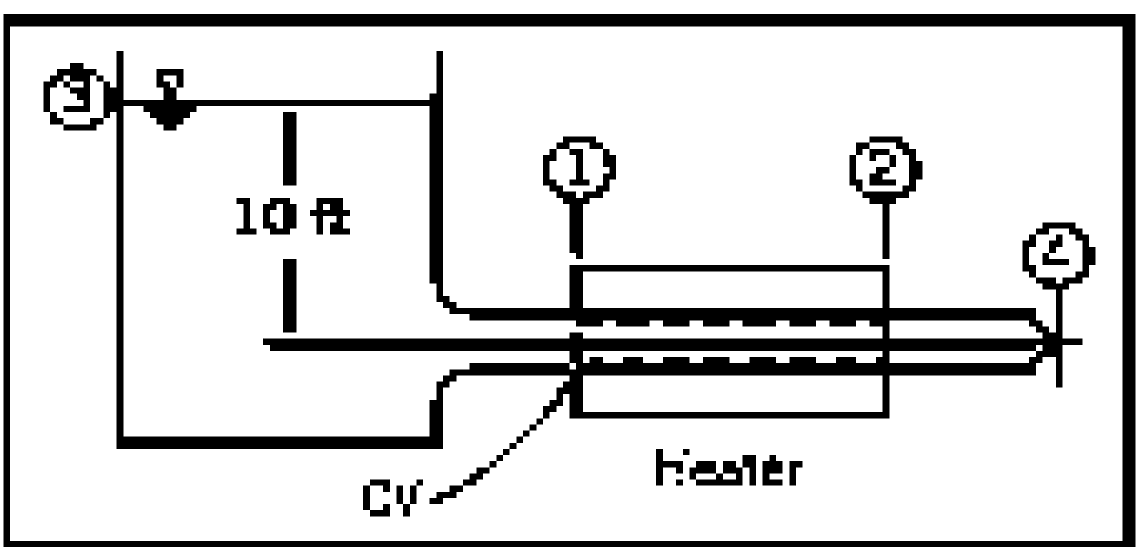
Since the internal energy change, du, between specified end states, is independent of the process, we take a reversible process, for which $Tds = d\left(\frac{dQ}{dm}\right) = du$.

Therefore, on integrating we get $u_2 - u_1 = \frac{dQ}{dm}$

……………………………………….

Example: Frictionless Flow With Heat Transfer

Water flows steadily from a large open reservoir through a short length of pipe and a nozzle with cross-sectional area A = 0.864 in.^{2} A well-insulated 10 kW heater surrounds the pipe. Find the temperature rise of the water.



Solution:

Since we have $\frac{p}{\rho}+\frac{V^2}{2}+gz=$ Constant

Also By using continuity equation in integral form

$\cancelto{0}{\frac{\partial}{\partial t}\int_{cv}\rho dv} + \int_{cs}\rho\vec{V}.d\vec{A}=0$ or $\sum_{cs}\rho\vec{V}.\vec{A}=0$

By the 1st law of thermodynamics

$\cancelto{0}{\dot{Q}} - \cancelto{0}{\dot{W}_s} - \cancelto{0}{\dot{W}_{shear}} = \cancelto{0}{\frac{\partial}{\partial t}\int_{cv}e\rho dv} + \int_{cs}\left(u+pv+\frac{V^2}{2}+gz\right)\rho\vec{V}.d\vec{A}$ …………(i)

Assumptions: (1) Steady flow.(2) Frictionless flow.(3) Incompressible flow.

(4) No shaft work, no shear work.(5) Flow along a streamline.

(6) Uniform flow at each section.

Under the assumptions listed, the first law of thermodynamics for the CV shown becomes $\dot{Q}=\int_{cs}\left(u+pv+\frac{V^2}{2}+gz\right)\rho\vec{V}.d\vec{A}$

$$\dot{Q}=\int_{A_1}\left(u+pv+\frac{V^2}{2}+gz\right)\rho\vec{V}.d\vec{A}+\int_{A_2}\left(u+pv+\frac{V^2}{2}+gz\right)\rho\vec{V}.d\vec{A}$$

For uniform properties at (1) and (2)

$$\dot{Q}=\left(u_1+p_1v_1+\frac{V_1^2}{2}+gz_1\right)(-\rho V_1A_1)+\left(u_2+p_2v_2+\frac{V_2^2}{2}+gz_2\right)(\rho V_2A_2)$$

From conservation of mass $\rho V_1A_1=\rho V_2A_2=\dot{m}$ so

$$\dot{Q}=\dot{m}\left[u_2-u_1+\left(p_2v_2+\frac{V_2^2}{2}+gz_2\right)-\left(p_1v_1+\frac{V_1^2}{2}+gz_1\right)\right]$$

For frictionless, incompressible, steady flow, along a streamline,

$\frac{p}{\rho}+\frac{V^2}{2}+gz=$ Constant Therefore $\dot{Q}=\dot{m}(u_2-u_1)$

Since, for an incompressible fluid, $u_2-u_1=c(T_2-T_1)$, then $T_2-T_1=\frac{\dot{Q}}{\dot{m}c}$

From continuity, $\dot{m}=\rho V_4A_4$

To find V_4, write the Bernoulli equation between the free surface at 3 and point 4.

$$\frac{p_3}{\rho} + \frac{V_3^2}{2} + gz_3 = \frac{p_4}{\rho} + \frac{V_4^2}{2} + gz_4$$

Since $p_3 = p_4$ and $V_3 \approx 0$, then

$$V_4 = \sqrt{2g(z_3 - z_4)} = \sqrt{2 \times 32.2\frac{ft}{s^2} \times 10ft} = 25.4\frac{ft}{s}$$

And $\dot{m} = \rho V_4 A_4 = 1.94\frac{slug}{ft^3} \times 25.4\frac{ft}{s} \times 0.864in.^2 \times \frac{ft^2}{144in.^2} = 0.296\frac{slug}{s}$

Assuming no heat loss to the surroundings, we obtain

$$T_2 - T_1 = \frac{\dot{Q}}{\dot{m}c} = 10kW \times 3413\frac{Btu}{kW.hr} \times \frac{hr}{3600s} \times \frac{s}{0.296slug} \times \frac{slug}{32.2lbm} \times \frac{lbm.°R}{1Btu}$$

$$T_2 - T_1 = 0.995°R$$

This problem illustrates that:

- In general, the first law of thermodynamics and the Bernoulli equation are independent equations.
- For an incompressible, inviscid flow the internal thermal energy is only changed by a heat transfer process, and is independent of the fluid mechanics.

INTERNAL INCOMPRESSIBLE VISCOUS FLOW

Internal Flow: Flows completely bounded by solid surfaces are called internal flows. Internal flows include flows through pipes, ducts, nozzles, diffusers, sudden contractions and expansions, valves, and fittings.

Fully Developed Flow: When the profile shape no longer changes with increasing distance. i.e. no change in momentum, the flow is called fully developed.

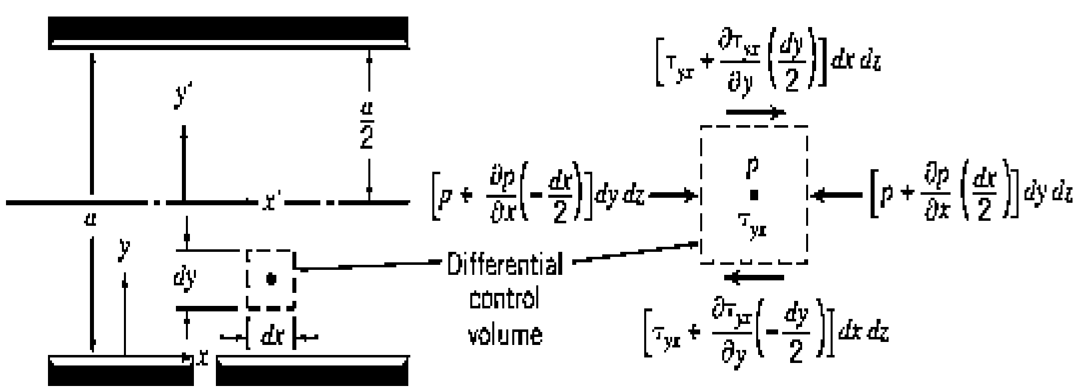
Entrance Length: The distance downstream from the entrance to the location at which fully developed flow begins is called the entrance length.

The flow between parallel plates is appealing because the geometry is the simplest possible, but why would there be a flow at all?

The answer is that flow could be generated by applying a pressure gradient parallel to the plates, or by moving one plate parallel with respect to the other, or by having a body force (e.g., gravity) parallel to the plates, or by a combination of these driving mechanisms.

Fully Developed Laminar Flow between Infinite Parallel Plates

When Both Plates are Stationary



(a) Geometry of CV (b) Forces acting on CV

Let us consider the fully developed laminar flow between horizontal infinite parallel plates. The plates are separated by distance a, as shown in Figure. The plates are considered infinite in the z direction, with no variation of any fluid property in this direction. The flow is also assumed to be steady, fully developed i.e. v = 0, momentum equation = 0, having no body force $F_{B_x} = 0$ and incompressible. The x component of velocity must be zero at both the upper and lower plates as a result of the no-slip condition at the wall. The boundary conditions are at y = 0 u = 0 at y = a u = 0

Since the flow is fully developed, the velocity cannot vary with x and, hence, depends on y only, so that u = u(y). Furthermore, there is no component of velocity in either the y or z direction (v = w = 0). In fact, for fully developed flow only the pressure can and will change (in a manner to be determined from the analysis) in the x direction.

For our analysis we select a differential control volume of size $dV = dxdydz$, and apply the x component of the momentum equation.

$$F_{S_x} + \cancelto{0}{F_{B_x}} = \cancelto{0}{\frac{\partial}{\partial t}\int_{cv}} u\rho dv + \int_{cs} u\rho\vec{V}.d\vec{A}$$

$$F_{S_x} = \int_{cs} u\rho\vec{V}.d\vec{A} = u\rho\int_{cs_1}\vec{V}.d\vec{A} + u\rho\int_{cs_2}\vec{V}.d\vec{A}$$

$$F_{S_x} = u\rho VAcos180° + u\rho VAcos0° = 0$$

Assumptions:

(1) Steady flow (2) Fully developed flow (3) $F_{B_x} = 0$

For fully developed flow the velocity is not changing with x, so the net momentum flux through the control surface is zero. (The momentum flux through the right face of the control surface is equal in magnitude but opposite in sign to the momentum flux through the left face; there is no momentum flux through any of the remaining faces of the control volume.) Since there are no body forces in the x direction, the momentum equation reduces to

F_{S_x} =Sum of Surface Force acting along x direction $= 0$ …………….(i)

The next step is to sum the forces acting on the control volume in the x direction. We recognize that normal forces (pressure forces) act on the left and right faces and tangential forces (shear forces) act on the top and bottom faces.

If the pressure at the center of the element is , then the pressure force on the left face is $dF_L = \left(p - \frac{\partial p}{\partial x}\frac{dx}{2}\right) dydz$

And the pressure force on the right face is $dF_R = -\left(p + \frac{\partial p}{\partial x}\frac{dx}{2}\right) dydz$

If the shear stress at the center of the element is τ_{yx}, then the shear force on bottom face is $dF_B = -\left(\tau_{yx} - \frac{d\tau_{yx}}{dy}\frac{dy}{2}\right) dxdz$

And the shear force on the top face is $dF_T = \left(\tau_{yx} + \frac{d\tau_{yx}}{dy}\frac{dy}{2}\right) dxdz$

Using the four surface forces dF_L, dF_R, dF_B, and dF_T in Eq. (i),

$$F_{S_x} = dF_L + dF_R + dF_B + dF_T = 0$$

$$\left(p - \frac{\partial p}{\partial x}\frac{dx}{2}\right) dydz - \left(p + \frac{\partial p}{\partial x}\frac{dx}{2}\right) dydz - \left(\tau_{yx} - \frac{d\tau_{yx}}{dy}\frac{dy}{2}\right) dxdz + \left(\tau_{yx} + \frac{d\tau_{yx}}{dy}\frac{dy}{2}\right) dxdz = 0$$

This equation simplifies to $$\frac{\partial p}{\partial x} = \frac{d\tau_{yx}}{dy}$$

This equation states that because there is no change in momentum, the net pressure force (which is actually $-\frac{\partial p}{\partial x}$) balances the net friction force (which is actually $-\frac{d\tau_{yx}}{dy}$). Equation (ii) has an interesting feature: The left side is at most a function of x only (this follows immediately from writing the y component of the momentum equation); the right side is at most a function of y only (the flow is fully developed, so it does not change with x). Hence, the only way the equation can be valid for all x and y is for each side to in fact be constant:

$$\frac{d\tau_{yx}}{dy} = \frac{\partial p}{\partial x} = \text{Cosntant} \qquad \text{(Suppose)}$$

Integrating this equation, we obtain $$\tau_{yx} = \left(\frac{\partial p}{\partial x}\right) y + c_1$$

Which indicates that the shear stress varies linearly with y. We wish to find the velocity distribution. To do so, we need to relate the shear stress to the velocity field. For a Newtonian fluid we can use following equation because we have a one-dimensional flow. $\tau_{yx} = \mu \frac{du}{dy}$ So we get $\mu \frac{du}{dy} = \left(\frac{\partial p}{\partial x}\right) y + c_1$

Integrating again $u = \frac{1}{2\mu}\left(\frac{\partial p}{\partial x}\right) y^2 + \frac{c_1}{\mu} y + c_2$

To evaluate the constants, c_1 and c_2, we must apply the boundary conditions.

At $y = 0, u = 0$. Consequently, $c_2 = 0$.

At $y = a, u = 0$. We get $0 = \frac{1}{2\mu}\left(\frac{\partial p}{\partial x}\right) a^2 + \frac{c_1}{\mu} a$ This gives $c_1 = -\frac{1}{2}\left(\frac{\partial p}{\partial x}\right) a$

And hence, $u = \frac{1}{2\mu}\left(\frac{\partial p}{\partial x}\right) y^2 - \frac{1}{2}\left(\frac{\partial p}{\partial x}\right) ay = \frac{a^2}{2\mu}\left(\frac{\partial p}{\partial x}\right)\left[\left(\frac{y}{a}\right)^2 - \left(\frac{y}{a}\right)\right]$

This is the velocity profile for flow. This is key to finding other flow properties, as we next discuss.

Shear Stress Distribution of Fully Developed Laminar Flow between Infinite Parallel Plates When Both Plates are Stationary

The shear stress distribution is given by

$$\tau_{yx} = \left(\frac{\partial p}{\partial x}\right) y + c_1 = \left(\frac{\partial p}{\partial x}\right) y - \frac{1}{2}\left(\frac{\partial p}{\partial x}\right) a = a\left(\frac{\partial p}{\partial x}\right)\left[\left(\frac{y}{a}\right) - \left(\frac{1}{2}\right)\right]$$

Volume Flow Rate of Fully Developed Laminar Flow between Infinite Parallel Plates When Both Plates are Stationary

The volume flow rate is given by $Q = \int_A \vec{V}.d\vec{A} = \int_A VdACos\theta$

$Q = \int_A lxdy$ $\because Cos\theta = Cos0° = 1, dA = lxdy$

For a depth *l* in the z direction, $dA = lxdy \Rightarrow VdA = udA = uldyCos0° = uldy$

$Q = \int_0^a uldy$ Or $\frac{Q}{l} = \int_0^a udy \Rightarrow$ $\frac{Q}{l} = \int_0^a \frac{1}{2\mu}\left(\frac{\partial p}{\partial x}\right)[y^2 - ay]dy$

$\Rightarrow \frac{Q}{l} = \frac{1}{2\mu}\left(\frac{\partial p}{\partial x}\right)\left|\frac{y^3}{3} - \frac{ay^2}{2}\right|_0^a = -\frac{1}{12\mu}\left(\frac{\partial p}{\partial x}\right) a^3$ The volume flow rate per unit depth.

Average Velocity of Fully Developed Laminar Flow between Infinite Parallel Plates When Both Plates are Stationary

The average velocity magnitude, $\bar{V}$, is given by

$$\bar{V} = \frac{Q}{A} = -\frac{1}{12\mu}\left(\frac{\partial p}{\partial x}\right)\frac{a^3 l}{la} = -\frac{1}{12\mu}\left(\frac{\partial p}{\partial x}\right)a^2$$

Flow Rate as a Function of Pressure Drop of Fully Developed Laminar Flow between Infinite Parallel Plates When Both Plates are Stationary

Since $\frac{\partial p}{\partial x}$ is constant, the pressure varies linearly with x and

$$\frac{\partial p}{\partial x} = \frac{p_2 - p_1}{L} = -\frac{\Delta p}{L}$$

Since the volume flow rate is $\frac{Q}{l} = -\frac{1}{12\mu}\left(\frac{\partial p}{\partial x}\right)a^3$

Therefore $\frac{Q}{l} = -\frac{1}{12\mu}\left(-\frac{\Delta p}{L}\right)a^3 = \frac{a^3 \Delta p}{12\mu L}$

Point of Maximum Velocity of Fully Developed Laminar Flow between Infinite Parallel Plates When Both Plates are Stationary

To find the point of maximum velocity, we set $\frac{du}{dy} = 0$ and solve for corresponding y. And From Equation $u = \frac{a^2}{2\mu}\left(\frac{\partial p}{\partial x}\right)\left[\left(\frac{y}{a}\right)^2 - \left(\frac{y}{a}\right)\right]$

We have $\frac{du}{dy} = \frac{a^2}{2\mu}\left(\frac{\partial p}{\partial x}\right)\left[\frac{2y}{a^2} - \frac{1}{a}\right]$

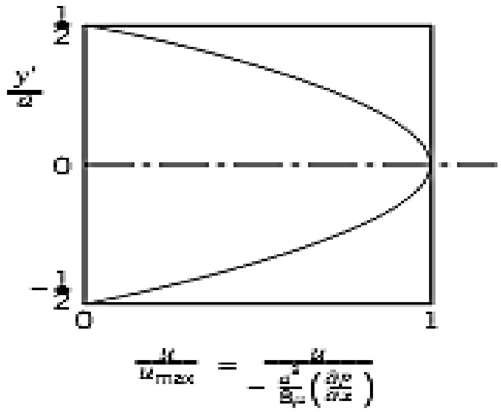
Thus, $\frac{du}{dy} = 0$ we get $y = \frac{a}{2}$

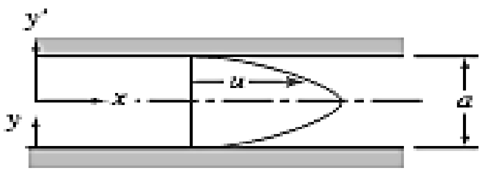
$$\Rightarrow \int \frac{du}{dy} = \frac{a^2}{2\mu}\left(\frac{\partial p}{\partial x}\right)\left|\frac{2}{a^2}\frac{y^2}{2} - \frac{1}{a}y\right|_0^{\frac{a}{2}} \Rightarrow u = \frac{a^2}{2\mu}\left(\frac{\partial p}{\partial x}\right)\left(\frac{1}{4} - \frac{1}{2}\right) = -\frac{a^2}{8\mu}\left(\frac{\partial p}{\partial x}\right)$$

$$u = u_{max} = -\frac{3}{3\times 8\mu}\left(\frac{\partial p}{\partial x}\right)a^2 = -\frac{3}{2\times 12\mu}\left(\frac{\partial p}{\partial x}\right)a^2 = \frac{3}{2}\left(\frac{-1}{12\mu}\left(\frac{\partial p}{\partial x}\right)a^2\right)$$

At $y = \frac{a}{2}$ we have $u = u_{max} = -\frac{1}{8\mu}\left(\frac{\partial p}{\partial x}\right)a^2 = \frac{3}{2}\bar{V}$

Transformation of Coordinates of Fully Developed Laminar Flow between Infinite Parallel Plates When Both Plates are Stationary





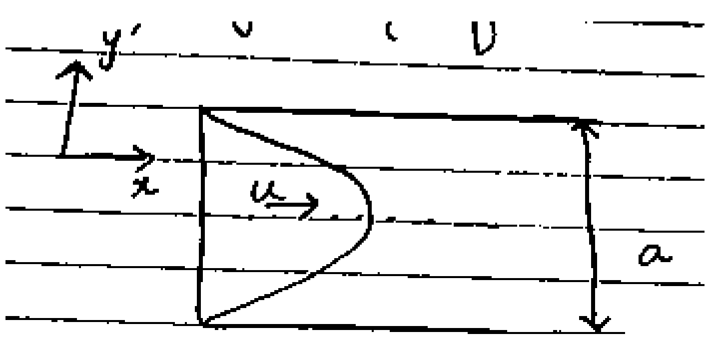
If we denote the coordinates with origin at the channel centerline as x, y', the boundary conditions are u = 0 at $y' = \pm a/2$.

To obtain the velocity profile in terms of x, y', we substitute $y = y' + a/2$ into Equation $u = \frac{a^2}{2\mu}\left(\frac{\partial p}{\partial x}\right)\left[\left(\frac{y}{a}\right)^2 - \left(\frac{y}{a}\right)\right]$

$$\Rightarrow u = \frac{a^2}{2\mu}\left(\frac{\partial p}{\partial x}\right)\left[\left(\frac{y'+\frac{a}{2}}{a}\right)^2 - \left(\frac{y'+\frac{a}{2}}{a}\right)\right] = \frac{a^2}{2\mu}\left(\frac{\partial p}{\partial x}\right)\left[\frac{(y')^2-\frac{a^2}{4}}{a^2}\right]$$

The result is $$u = \frac{a^2}{2\mu}\left(\frac{\partial p}{\partial x}\right)\left[\left(\frac{y'}{a}\right)^2 - \left(\frac{1}{4}\right)\right]$$

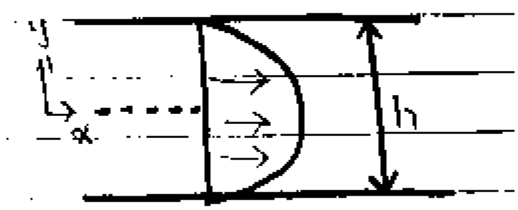
This Equation shows that the velocity profile for laminar flow between stationary parallel plates is parabolic, as shown in Figure.



Example

The velocity profile for fully developed flow between stationary parallel plates is given by $u = ay(h-y)$ where 'a' is a constant and 'h' is a total gap between the plates, 'y' is the distance measure from center of the gap, then find ratio of $\frac{\bar{V}}{u_{max}}$

Solution



Given that $u = ay(h-y)$ and we have to find $\frac{\bar{V}}{u_{max}}$

To find u_{max} we set $\frac{du}{dy} = 0$ implies $ay(-1) + ay(h-y) = 0$ gives $y = \frac{h}{2}$

Then $u_{max} = u = a\left(\frac{h}{2}\right)\left(h - \left(\frac{h}{2}\right)\right) = \frac{ah^2}{4}$

Now $\bar{V} = \frac{Q}{A} = \frac{\int_A \vec{V}.d\vec{A}}{A} = \frac{\int_A u dA}{A} = \frac{\int_A ay(h-y)dA}{A}$

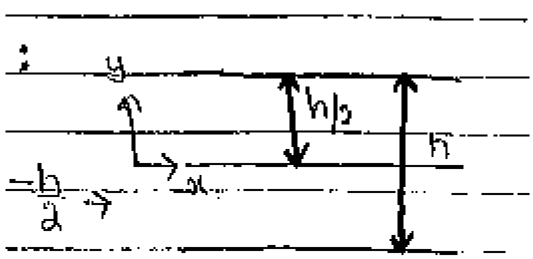
$\bar{V} = \frac{\int_0^h ay(h-y)bdy}{bh} = \frac{\int_0^h ay(h-y)dy}{h} = \frac{a}{h}\int_0^h y(h-y)\,dy = \frac{ah^2}{6}$

Then $\frac{\bar{V}}{u_{max}} = \frac{\frac{ah^2}{6}}{\frac{ah^2}{4}} = \frac{2}{3}$

Example

An incompressible fluid flows between two infinite stationary parallel plates. The velocity profile is given by u $u = u_{max}(Ay^2 + By + C)$ where A, B, and C are constants and y is measured upward from the lower plate. The total gap width is h units. Use appropriate boundary conditions to express the magnitude and units of the constants in terms of h. Develop an expression for volume flow rate per unit depth and evaluate the ratio $\frac{\bar{V}}{u_{max}}$

Solution



Given that $u = u_{max}(Ay^2 + By + C)$ ……………(1)

Boundary conditions are;

(i) $y = \frac{h}{2}, u = 0,$ (ii) $y = -\frac{h}{2}, u = 0,$ (iii) $y = 0, u = u_{max}$

From (i) (1) $\Rightarrow 0 = u_{max}\left(A\frac{h^2}{4} + B\frac{h}{2} + C\right)$ ……………(2)

From (ii) (1) $\Rightarrow u_{max} = u_{max}(A(0) - B(0) - C) \Rightarrow C = 1$

From (iii) (1) $\Rightarrow 0 = u_{max}\left(A\frac{h^2}{4} - B\frac{h}{2} + 1\right)$ ……………(3)

Subtracting (3) from (2)

$$0 - 0 = u_{max}\left(A\frac{h^2}{4} + B\frac{h}{2} + C\right) - u_{max}\left(A\frac{h^2}{4} - B\frac{h}{2} + 1\right)$$

$$\Rightarrow 0 = u_{max}\left(A\frac{h^2}{4} + B\frac{h}{2} + C - A\frac{h^2}{4} + B\frac{h}{2} - 1\right) = u_{max}(Bh)$$

$$\Rightarrow 0 = u_{max}(Bh) \Rightarrow B = 0$$

Adding (2) and (3)

$$0+0=u_{max}\left(A\frac{h^2}{4}+B\frac{h}{2}+C\right)+u_{max}\left(A\frac{h^2}{4}-B\frac{h}{2}+1\right)$$

$$\Rightarrow 0=u_{max}\left(A\frac{h^2}{4}+B\frac{h}{2}+C+A\frac{h^2}{4}-B\frac{h}{2}+1\right)$$

$$\Rightarrow A=-\frac{4}{h^2} \qquad \text{using } B=0, C=1$$

Then (1) $\Rightarrow u=u_{max}\left(-\frac{4}{h^2}y^2+1\right)$

Now the volume flow rate per unit depth is given by $Q=\int_A \vec{V}.d\vec{A}=\int_A udA$

$$Q=\int_{-\frac{h}{2}}^{\frac{h}{2}}ubdy=2\int_0^{\frac{h}{2}}ubdy=2b\int_0^{\frac{h}{2}}u_{max}\left(-\frac{4}{h^2}y^2+1\right)dy=\frac{2}{3}u_{max}bh$$

Now as we know that $Q=\bar{V}A$

$$Q=\bar{V}bh \Rightarrow \frac{Q}{b}=\bar{V}h \Rightarrow \frac{2}{3}u_{max}h=\bar{V}h \Rightarrow \frac{\bar{V}}{u_{max}}=\frac{2}{3}$$

Reynolds Number

It is the ratio of inertial force to the viscous force. It tells us whether the flow is laminar or turbulent. If the inertial force that resists the change in velocity is dominant then the flow is turbulent. And if the viscous force that resists the flow is dominant then the flow is laminar.

Its formula is $Re=\frac{\rho VL}{\mu}$

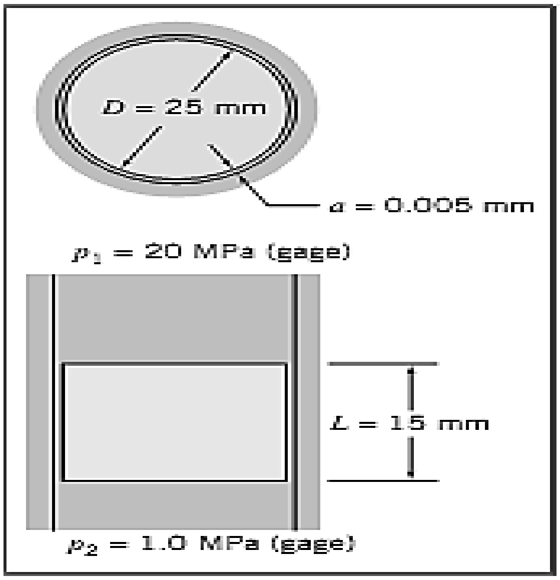
Or $\quad Re=\frac{VL}{\upsilon} \quad$ where $\upsilon=\frac{\mu}{\rho}$

Where ρ is fluid density, V is characteristic velocity, L is characteristic length or size scale of flow, μ is dynamic fluid viscosity and υ is kinetic viscosity.

If the Reynolds number is “large,” viscous effects will be negligible, at least in most of the flow; if the Reynolds number is small, viscous effects will be dominant. Finally, if the Reynolds number is neither large nor small, no general conclusions can be drawn.

Example Leakage Flow Past a Piston

A hydraulic system operates at a gage pressure of 20 MPa and 55°C. The hydraulic fluid is SAE 10W oil. A control valve consists of a piston 25 mm in diameter, fitted to a cylinder with a mean radial clearance of 0.005 mm. Determine the leakage flow rate if the gage pressure on the low-pressure side of the piston is 1.0 MPa. (The piston is 15 mm long.)



Solution: The gap width is very small, so the flow may be modeled as flow between parallel plates. Equation given below may be applied.

$$\frac{Q}{l} = -\frac{1}{12\mu}\left(-\frac{\Delta p}{L}\right)a^3 = \frac{a^3\Delta p}{12\mu L}$$

Assumptions: (1) Laminar flow. (2) Steady flow. (3) Incompressible flow. (4) Fully developed flow. (Note $\frac{L}{a}$ = 15/0.005 = 3000!)

The plate width, l, is approximated as $l = \pi D$. Thus $Q = \frac{\pi D a^3 \Delta p}{12\mu L}$

For SAE 10W oil at 55°C, μ = 0.018 kg/(m • s), from Figure. Thus

$$Q = \frac{\pi}{12} \times 25mm \times (0.005)^3 mm^3 \times (20-1)10^6 Nm^{-1} \times \frac{m.s}{0.018kg} \times \frac{1}{15mm} \times \frac{kg.m}{N.s^2}$$

$$Q = 57.6 mm^3/s$$

To ensure that flow is laminar, we also should check the Reynolds number.

$$\bar{V} = \frac{Q}{A} = \frac{Q}{\pi d A}$$

$$\bar{V} = \frac{57.6mm^3}{s} \times \frac{1}{\pi} \times \frac{1}{25mm} \times \frac{1}{0.005mm} \times \frac{m}{10^3mm} = 0.147m/s$$

And $Re = \frac{\rho \bar{V} a}{\mu} = \frac{SG\rho_{H_2O}\bar{V}a}{\mu}$

For SAE (Society of Automotive Engineers) 10W oil, SG = 0.92. Thus

$$Re = 0.92 \times 10^3 \frac{kg}{m^3} \times 0.147 \frac{m}{s} \times 0.005mm \times \frac{m.s}{0.018kg} \times \frac{m}{10^3mm}$$

$$Re = 0.0375$$

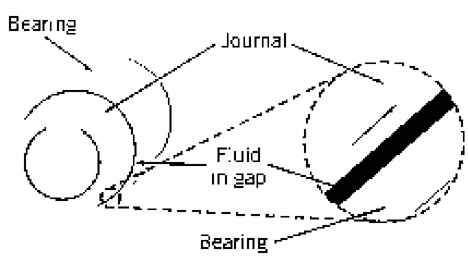
Thus flow is surely laminar, since $Re \ll 1400$.

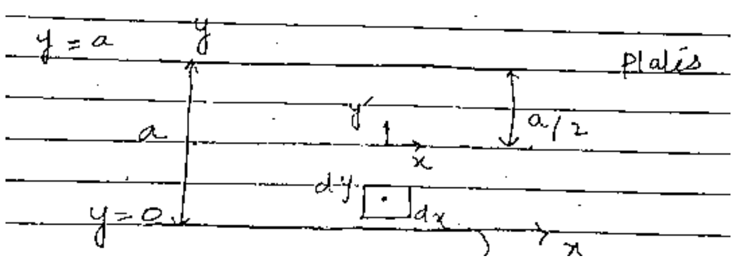
Upper Plate Moving with Constant Speed (Velocity), U

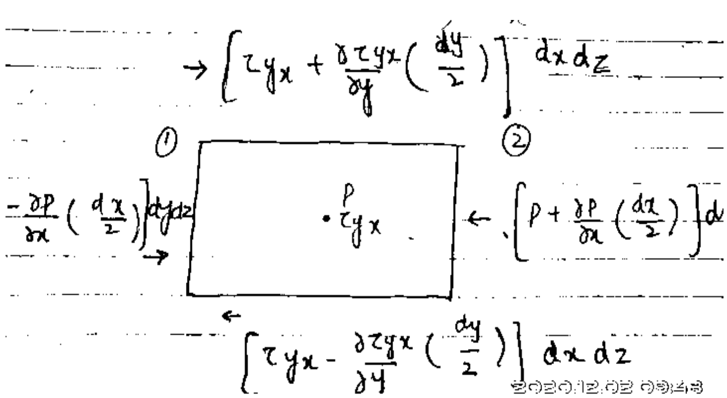
A second laminar flow case of practical importance is flow in a journal bearing (a commonly used type of bearing, e.g., the main crankshaft bearings in the engine of an automobile). In such a bearing, an inner cylinder, the journal, rotates inside a stationary member. At light loads, the centers of the two members essentially coincide, and the small clearance gap is symmetric. Since the gap is small, it is reasonable to "un-fold" the bearing and to model the flow field as flow between infinite parallel plates.

Let us now consider a case where the upper plate is moving to the right with constant speed, U. All we have done in going from a stationary upper plate to a moving upper plate is to change one of the boundary conditions. The boundary conditions for the moving plate case are

$u = 0$ at $y = 0$ and $u = U$ at $y = a$







Since the flow is fully developed, the velocity cannot vary with x and, hence, depends on y only, so that u = u(y). Furthermore, there is no component of velocity in either the y or z direction (v = w = 0). In fact, for fully developed flow only the pressure can and will change (in a manner to be determined from the analysis) in the x direction.

For our analysis we select a differential control volume of size $dV = dxdydz$, and apply the x component of the momentum equation.

$$F_{S_x} + \cancel{F_{B_x}}^{\,0} = \cancel{\frac{\partial}{\partial t}\int_{cv}}^{\,0} u\rho dv + \int_{cs} u\rho\vec{V}.d\vec{A}$$

$$F_{S_x} = \int_{cs} u\rho\vec{V}.d\vec{A} = u\rho\int_{cs_1}\vec{V}.d\vec{A} + u\rho\int_{cs_2}\vec{V}.d\vec{A}$$

$$F_{S_x} = u\rho VA\cos 180° + u\rho VA\cos 0° = 0$$

Assumptions: Steady, Fully developed, incompressible flow with $F_{B_x} = 0$

For fully developed flow the velocity is not changing with x, so the net momentum flux through the control surface is zero. (The momentum flux through the right face of the control surface is equal in magnitude but opposite in sign to the momentum flux through the left face; there is no momentum flux through any of the remaining faces of the control volume.) Since there are no body forces in the x direction, the momentum equation reduces to

F_{S_x} =Sum of Surface Force acting along x direction $= 0$ ……………(i)

The next step is to sum the forces acting on the control volume in the x direction. We recognize that normal forces (pressure forces) act on the left and right faces and tangential forces (shear forces) act on the top and bottom faces.

If the pressure at the center of the element is , then the pressure force on the left face is $dF_L = \left(p - \frac{\partial p}{\partial x}\frac{dx}{2}\right)dydz$

And the pressure force on the right face is $dF_R = -\left(p + \frac{\partial p}{\partial x}\frac{dx}{2}\right)dydz$

If the shear stress at the center of the element is τ_{yx}, then the shear force on bottom face is $dF_B = -\left(\tau_{yx} - \frac{d\tau_{yx}}{dy}\frac{dy}{2}\right)dxdz$

And the shear force on the top face is $dF_T = \left(\tau_{yx} + \frac{d\tau_{yx}}{dy}\frac{dy}{2}\right) dxdz$

Using the four surface forces dF_L, dF_R, dF_B, and dF_T in Eq. (i),

$F_{S_x} = dF_L + dF_R + dF_B + dF_T = 0$

$\left(p - \frac{\partial p}{\partial x}\frac{dx}{2}\right) dydz - \left(p + \frac{\partial p}{\partial x}\frac{dx}{2}\right) dydz - \left(\tau_{yx} - \frac{d\tau_{yx}}{dy}\frac{dy}{2}\right) dxdz + \left(\tau_{yx} + \frac{d\tau_{yx}}{dy}\frac{dy}{2}\right) dxdz = 0$

This equation simplifies to $\quad -\frac{\partial p}{\partial x} + \frac{d\tau_{yx}}{dy} = 0$

Since τ_{yx} is the function of 'y' only and $\frac{\partial p}{\partial x}$ is independent of 'y' thus

$\frac{d\tau_{yx}}{dy} = \frac{\partial p}{\partial x} =$ Constant $\quad$ implies $\quad \tau_{yx} = \left(\frac{\partial p}{\partial x}\right) y + c_1$

Using $\quad \tau_{yx} = \mu \frac{du}{dy} \quad$ we get $\quad \mu \frac{du}{dy} = \left(\frac{\partial p}{\partial x}\right) y + c_1$

Integrating $\quad u = \frac{1}{2\mu}\left(\frac{\partial p}{\partial x}\right) y^2 + \frac{c_1}{\mu} y + c_2$

To evaluate the constants, c_1 and c_2, we must apply the boundary conditions.

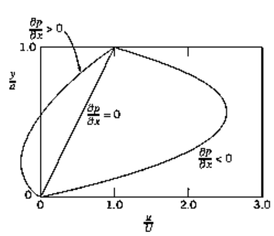
At y = 0, u = 0. Consequently, $c_2 = 0$.

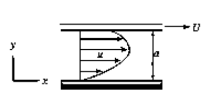
At y = a, u = U. Consequently, $U = \frac{1}{2\mu}\left(\frac{\partial p}{\partial x}\right) a^2 + \frac{c_1}{\mu} a$ thus $c_1 = \frac{U\mu}{a} - \frac{1}{2}\left(\frac{\partial p}{\partial x}\right) a$

$u = \frac{1}{2\mu}\left(\frac{\partial p}{\partial x}\right) y^2 + \frac{y}{\mu}\left(\frac{U\mu}{a} - \frac{1}{2}\left(\frac{\partial p}{\partial x}\right) a\right) = \frac{1}{2\mu}\left(\frac{\partial p}{\partial x}\right) y^2 + \frac{Uy}{a} - \frac{y}{2\mu}\left(\frac{\partial p}{\partial x}\right) a$

Hence, $\quad u = \frac{Uy}{a} + \frac{a^2}{2\mu}\left(\frac{\partial p}{\partial x}\right)\left[\left(\frac{y}{a}\right)^2 - \left(\frac{y}{a}\right)\right]$

This Equation suggests that the velocity profile may be treated as a combination of a linear and a parabolic velocity profile.





Shear Stress Distribution of Fully Developed Laminar Flow between Infinite Parallel Plates When Upper Plate Moving with Constant Speed (Velocity),

The shear stress distribution is given by $\tau_{yx} = \mu \frac{du}{dy}$,

$$\tau_{yx} = \mu \frac{U}{a} + \frac{a^2}{2}\left(\frac{\partial p}{\partial x}\right)\left[\frac{2y}{a^2} - \left(\frac{1}{a}\right)\right] = \mu \frac{U}{a} + a\left(\frac{\partial p}{\partial x}\right)\left[\frac{y}{a} - \frac{1}{2}\right]$$

Volume Flow Rate of Fully Developed Laminar Flow between Infinite Parallel Plates When Upper Plate Moving with Constant Speed (Velocity)

The volume flow rate is given by $Q = \int_A \vec{V}.d\vec{A} = \int_A VdACos\theta$

$Q = \int_A lxdy \qquad\qquad \because Cos\theta = Cos0° = 1, dA = lxdy$

For a depth l in the z direction, $dA = lxdy \Rightarrow VdA = udA = uldyCos0° = uldy$

$Q = \int_0^a uldy \qquad$ Or $\qquad \frac{Q}{l} = \int_0^a udy \Rightarrow \qquad \frac{Q}{l} = \int_0^a \frac{Uy}{a} + \frac{1}{2\mu}\left(\frac{\partial p}{\partial x}\right)[y^2 - ay]dy$

$\Rightarrow \frac{Q}{l} = \frac{Ua}{2} - \frac{1}{12\mu}\left(\frac{\partial p}{\partial x}\right)a^3$ The volume flow rate per unit depth.

Average Velocity of Fully Developed Laminar Flow between Infinite Parallel Plates When Upper Plate Moving with Constant Speed (Velocity)

The average velocity magnitude, $\bar{V}$, is given by

$$\bar{V} = \frac{Q}{A} = \frac{l\left(\frac{Ua}{2} - \frac{1}{12\mu}\left(\frac{\partial p}{\partial x}\right)a^3\right)}{la} = \frac{U}{2} - \frac{1}{12\mu}\left(\frac{\partial p}{\partial x}\right)a^2$$

Point of Maximum Velocity of Fully Developed Laminar Flow between Infinite Parallel Plates When Upper Plate Moving with Constant Speed (Velocity) To find the point of maximum velocity, we set $\frac{du}{dy} = 0$ and solve for corresponding y. And From Equation $u = \frac{Uy}{a} + \frac{a^2}{2\mu}\left(\frac{\partial p}{\partial x}\right)\left[\left(\frac{y}{a}\right)^2 - \left(\frac{y}{a}\right)\right]$

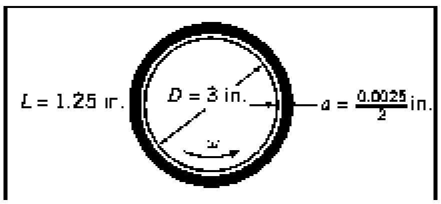
We have $\quad \frac{du}{dy} = \frac{U}{a} + \frac{a^2}{2}\left(\frac{\partial p}{\partial x}\right)\left[\frac{2y}{a^2} - \left(\frac{1}{a}\right)\right]$

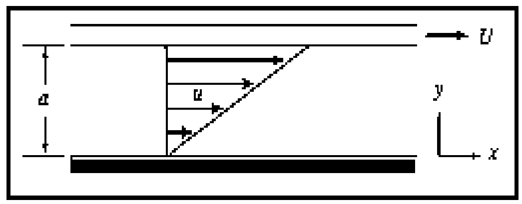
Thus, $\frac{du}{dy} = 0 \qquad$ at $\qquad y = \frac{a}{2} - \frac{U/a}{\frac{1}{\mu}\left(\frac{\partial p}{\partial x}\right)}$

There is no simple relation between the maximum velocity, u_{max}, and the mean velocity, $\bar{V}$, for this flow case.

Example: Torque And Power In A Journal Bearing

A crankshaft journal bearing in an automobile engine is lubricated by SAE 30 oil at 210°F. The bearing diameter is 3 in., the diametral clearance is 0.0025 in., and the shaft rotates at 3600 rpm; it is 1.25 in. long. The bearing is under no load, so the clearance is symmetric. Determine the torque required to turn the journal and the power dissipated.





Solution

Using equation: $\tau_{yx} = \mu \frac{U}{a} + a\left(\frac{\partial p}{\partial x}\right)\left[\frac{y}{a} - \frac{1}{2}\right]$

$\tau_{yx} = \mu \frac{U}{a}$ When $\frac{\partial p}{\partial x} = 0$

Assumptions: (1) Laminar flow. (2) Steady flow. (3) Incompressible flow.

(4) Fully developed flow. (5) Infinite width (L/a = 1.25/0.00125 = 1000, so this is a reasonable assumption). (6) $\frac{\partial p}{\partial x} = 0$ (flow is symmetric in the actual bearing at no load). Then

$\tau_{yx} = \mu \frac{U}{a} = \mu \frac{U}{a} = \mu \frac{\omega R}{a} = \mu \frac{\omega D}{2a}$ when $\frac{\partial p}{\partial x} = 0$

For SAE 30 oil at 210°F (99°C), $\mu = 9.6 \times 10^{-3} Nsm^{-2} (2.01 \times 10^{-4} lbfsft^{-2})$

$\Rightarrow \tau_{yx} = 2.01 \times 10^{-4} \frac{lbfs}{ft^2} \times 3600 \frac{rev}{min} \times 2\pi \frac{rad}{rev} \times \frac{min}{60s} \times 3in. \times \frac{1}{2} \times \frac{1}{0.00125in.}$

$\Rightarrow \tau_{yx} = 90.9 \frac{lbf}{ft^2}$

The total shear force is given by die shear stress times the area. It is applied to the journal surface. Therefore, for the torque

$T = FR = \tau_{yx}\pi DLR = \frac{\pi}{2}\tau_{yx}D^2 L$

$T = \frac{\pi}{2} \times 90.9 \frac{lbf}{ft^2} \times (3in.)^2 \times \frac{ft^2}{144(in.)^2} \times 1.25in. = 11.2$ in.• lbf

The power dissipated in the bearing is $\dot{W} = FU = FR\omega = T\omega$

$$\dot{W} = 11.2 \text{ in.• lbf} \times 3600 \frac{rev}{min} \times 2\pi \frac{rad}{rev} \times \frac{ft}{12in.} \times \frac{hp.s}{550ft.lbf} = 0.640\ hp$$

In this problem we approximated the circular-streamline flow in a small annular gap as a linear flow between infinite parallel plates.

To ensure that flow is laminar, we also should check the Reynolds number.

$$Re = \frac{\rho Ua}{\mu} = \frac{SG\rho_{H_2O}Ua}{\mu} = \frac{SG\rho_{H_2O}\omega Ra}{\mu}$$

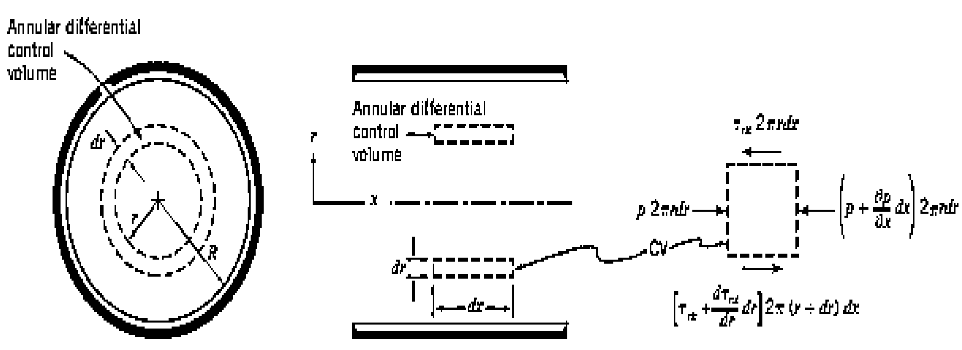
For SAE 30W oil same as 10W oil, SG = 0.92. Thus

$$Re = 0.92 \times 1.94 \frac{slug}{ft^3} \times \frac{(3600)2\pi}{60} \frac{rad}{s} \times 1.5in. \times 0.00125in. \times \frac{ft^2}{2.01\times10^{-4} lbf.s} \times \frac{ft^2}{144in.^2} \times \frac{lbf.s^2}{slug.ft}$$

$Re = 43.6$ Thus flow is surely laminar, since $Re \ll 1500$.

Fully Developed Laminar Flow In A Pipe

Let us consider fully developed laminar flow in a pipe. Here the flow is axisymmetric. Consequently it is most convenient to work in cylindrical coordinates. Since the flow is axisymmetric, the control volume will be a differential annulus, as shown in Figure. The control volume length is dx and its thickness is dr.



(a) End view of CV (b) Side view of CV (c) Forces on CV

For a fully developed steady flow, the x component of the momentum equation, when applied to the differential control volume, reduces to $F_{S_x} = 0$

The next step is to sum the forces acting on the control volume in the x direction. We know that normal forces (pressure forces) act on the left and right ends of the control volume, and that tangential forces (shear forces) act on the inner and outer cylindrical surfaces.

If the pressure at the left face of the control volume is p, then the pressure force on the left end is $dF_L = p2\pi r dr$

The pressure force on the right end is $dF_R = -\left(p + \frac{\partial p}{\partial x}dx\right)2\pi r dr$

If the shear stress at the inner surface of the annular control volume is τ_{rx}, then the shear force on the inner cylindrical surface is $dF_I = -\tau_{rx}2\pi r dx$

The shear force on the outer cylindrical surface is

$$dF_O = \left(\tau_{rx} + \frac{d\tau_{rx}}{dr}dr\right)2\pi(r + dr)dx$$

The sum of the *x* components of force acting on the control volume must be zero.
i.e. $dF_L + dF_R + dF_I + dF_O = 0$

$$\Rightarrow p2\pi r dr - \left(p + \frac{\partial p}{\partial x}dx\right)2\pi r dr - \tau_{rx}2\pi r dx + \left(\tau_{rx} + \frac{d\tau_{rx}}{dr}dr\right)2\pi(r + dr)dx = 0$$

$$\Rightarrow -\frac{\partial p}{\partial x}2\pi r dr dx + \tau_{rx}2\pi r dx + \frac{d\tau_{rx}}{dr}dr2\pi r dr dx = 0$$

$$\Rightarrow \frac{\partial p}{\partial x} = \frac{\tau_{rx}}{r} + \frac{d\tau_{rx}}{dr} = \frac{1}{r}\frac{d(r\tau_{rx})}{dr} \qquad \div 2\pi r dr dx$$

The left side of the equation is at most a function of x only (the pressure is uniform at each section); the right side is at most a function of r only (because the flow is fully developed). Hence, the only way the equation can be valid for all x and r is for both sides to in fact be constant:

$$\frac{1}{r}\frac{d(r\tau_{rx})}{dr} = \frac{\partial p}{\partial x} = \text{Constant} \qquad \text{or} \qquad \frac{d(r\tau_{rx})}{dr} = r\frac{\partial p}{\partial x}$$

We are not quite finished, but already we have an important result: In a constant diameter pipe, the pressure drops uniformly along the pipe length (except for the entrance region).

Integrating this equation, we obtain $\quad r\tau_{rx} = \frac{r^2}{2}\left(\frac{\partial p}{\partial x}\right) + c_1$

Or $\quad \tau_{rx} = \frac{r}{2}\left(\frac{\partial p}{\partial x}\right) + \frac{c_1}{r}$

$\Rightarrow \mu \frac{du}{dr} = \frac{r}{2}\left(\frac{\partial p}{\partial x}\right) + \frac{c_1}{r}$

$\Rightarrow u = \frac{r^2}{4\mu}\left(\frac{\partial p}{\partial x}\right) + \frac{c_1}{\mu} lnr + c_2 \qquad \ldots\ldots\ldots\ldots\ldots(i)$

We need to evaluate constants c_1 and c_2.

Put u as finite at $r = 0$

$(i) \Rightarrow finite = \frac{0^2}{4\mu}\left(\frac{\partial p}{\partial x}\right) + \frac{c_1}{\mu} ln0 + c_2$

$\Rightarrow finite = \frac{c_1}{\mu}\infty + c_2 \Rightarrow finite - c_2 = \frac{c_1}{\mu}\infty \Rightarrow \frac{(finite - c_2)\mu}{\infty} = c_1 \Rightarrow c_1 = 0$

Hence $u = \frac{r^2}{4\mu}\left(\frac{\partial p}{\partial x}\right) + c_2$

The constant, c_2 , is evaluated by using the available boundary condition at the pipe wall: at $r = R$, u = 0. Consequently, $\quad 0 = \frac{R^2}{4\mu}\left(\frac{\partial p}{\partial x}\right) + c_2$

This gives $\quad c_2 = -\frac{R^2}{4\mu}\left(\frac{\partial p}{\partial x}\right)$

And hence $\quad u = \frac{r^2}{4\mu}\left(\frac{\partial p}{\partial x}\right) - \frac{R^2}{4\mu}\left(\frac{\partial p}{\partial x}\right) = \frac{1}{4\mu}\left(\frac{\partial p}{\partial x}\right)[r^2 - R^2]$

Or $\quad u = \frac{r^2}{4\mu}\left(\frac{\partial p}{\partial x}\right) + \frac{1}{4\mu}\left(\frac{\partial p}{\partial x}\right)[r^2 - R^2]$

Hence $\quad u = -\frac{R^2}{4\mu}\left(\frac{\partial p}{\partial x}\right)\left[1 - \left(\frac{r}{R}\right)^2\right]$

Since we have the velocity profile, we can obtain a number of additional features of the flow.

Shear Stress Distribution of Fully Developed Laminar Flow In A Pipe

The shear stress is $\tau_{rx} = \mu \frac{du}{dr} = \frac{r}{2}\left(\frac{\partial p}{\partial x}\right)$

Volume Flow Rate of Fully Developed Laminar Flow In A Pipe

The volume flow rate is given by

$$Q = \int_A \vec{V}.d\vec{A} = \int_0^R u2\pi r dr = \int_0^R \frac{1}{4\mu}\left(\frac{\partial p}{\partial x}\right)(r^2 - R^2)2\pi r dr = -\frac{\pi R^4}{8\mu}\left(\frac{\partial p}{\partial x}\right)$$

Flow Rate as a Function of Pressure Drop of Fully Developed Laminar Flow In A Pipe The pressure gradient, $\frac{\partial p}{\partial x}$, is constant for fully developed flow.

Therefore, $\frac{\partial p}{\partial x} = \frac{p_2 - p_1}{L} = -\frac{\Delta p}{L}$. Substituting into $Q = -\frac{\pi R^4}{8\mu}\left(\frac{\partial p}{\partial x}\right)$ for the volume flow rate gives $Q = -\frac{\pi R^4}{8\mu}\left(-\frac{\Delta p}{L}\right) = \frac{\Delta p \pi R^4}{8\mu L} = \frac{\Delta p \pi D^4}{128\mu L}$ $\quad \because R = \frac{D}{2}$

Average Velocity of Fully Developed Laminar Flow In A Pipe

The average velocity magnitude, $\bar{V}$, is given by

$$\bar{V} = \frac{Q}{A} = \frac{Q}{\pi R^2} = \frac{-\frac{\pi R^4}{8\mu}\left(\frac{\partial p}{\partial x}\right)}{\pi R^2} = -\frac{R^2}{8\mu}\left(\frac{\partial p}{\partial x}\right)$$

Point of Maximum Velocity of Fully Developed Laminar Flow In A Pipe

To find the point of maximum velocity, we set $\frac{du}{dr} = 0$ and solve for corresponding y. And From Equation $u = -\frac{R^2}{4\mu}\left(\frac{\partial p}{\partial x}\right)\left[1 - \left(\frac{r}{R}\right)^2\right]$ We have $\frac{du}{dr} = \frac{1}{2\mu}\left(\frac{\partial p}{\partial x}\right)r$

Thus, $\frac{du}{dy} = 0 \Rightarrow \frac{1}{2\mu}\left(\frac{\partial p}{\partial x}\right)r = 0$ gives $r = 0$

At $r = 0$ we have $u = u_{max} = U = -\frac{R^2}{4\mu}\left(\frac{\partial p}{\partial x}\right) = 2\bar{V}$ $\quad \because \bar{V} = -\frac{R^2}{8\mu}\left(\frac{\partial p}{\partial x}\right)$

The velocity profile $u = -\frac{R^2}{4\mu}\left(\frac{\partial p}{\partial x}\right)\left[1 - \left(\frac{r}{R}\right)^2\right]$ may be written in terms of the maximum (centerline) velocity as $\frac{u}{U} = 1 - \left(\frac{r}{R}\right)^2$

This is the parabolic velocity profile, for fully developed laminar pipe flow.

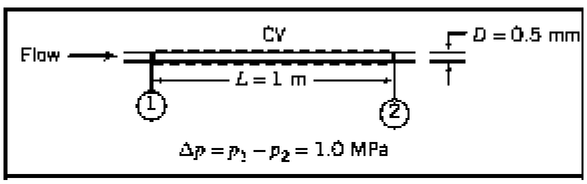
Example: Capillary Viscometer

A simple and accurate viscometer can be made from a length of capillary tubing. If the flow rate and pressure drop are measured, and the tube geometry is known, the viscosity of a Newtonian liquid. A test of a certain liquid in a capillary viscometer gave the following data:

Flow rate: 880 mm^3/s Tube length: 1 m

Tube diameter: 0.50 mm Pressure drop: 1.0 MPa

Determine the viscosity of the liquid.



Solution: Using equation: $Q = \frac{\Delta p \pi D^4}{128 \mu L}$

Assumptions: (1) Laminar flow. (2) Steady flow. (3) Incompressible flow.

(4) Fully developed flow. (5) Horizontal tube.

Then $\mu = \frac{\Delta p \pi D^4}{128 Q L} = \frac{\pi}{128} \times 1.0 \times 10^6 N m^{-2} \times (0.50)^4 mm^4 \times \frac{s}{80 mm^3} \times \frac{1}{1m} \times \frac{m}{10^3 mm}$

$\mu = 1.74 \times 10^{-3} N s m^{-2}$

Check the Reynolds number. Assume the fluid density is similar to that of water, 999 kg/m^3. Then

To ensure that flow is laminar, we also should check the Reynolds number.

$\bar{V} = \frac{Q}{A} = \frac{4Q}{\pi d A} = \frac{80 mm^3}{s} \times \frac{4}{\pi} \times \frac{1}{(0.50)^2 mm^2} \times \frac{m}{10^3 mm} = 4.48 m/s$

And $Re = \frac{\rho \bar{V} D}{\mu} = 999 \frac{kg}{m^3} \times 4.48 \frac{m}{s} \times 0.50 mm \times \frac{m^2}{1.74 \times 10^{-3} Ns} \times \frac{m}{10^3 mm} \times \frac{Ns^2}{kg.m}$

$Re = 1290$

Thus flow is surely laminar, since $Re \ll 2300$.

Flow In Pipes And Ducts

There are three factors that tend to reduce the pressure in a pipe flow:

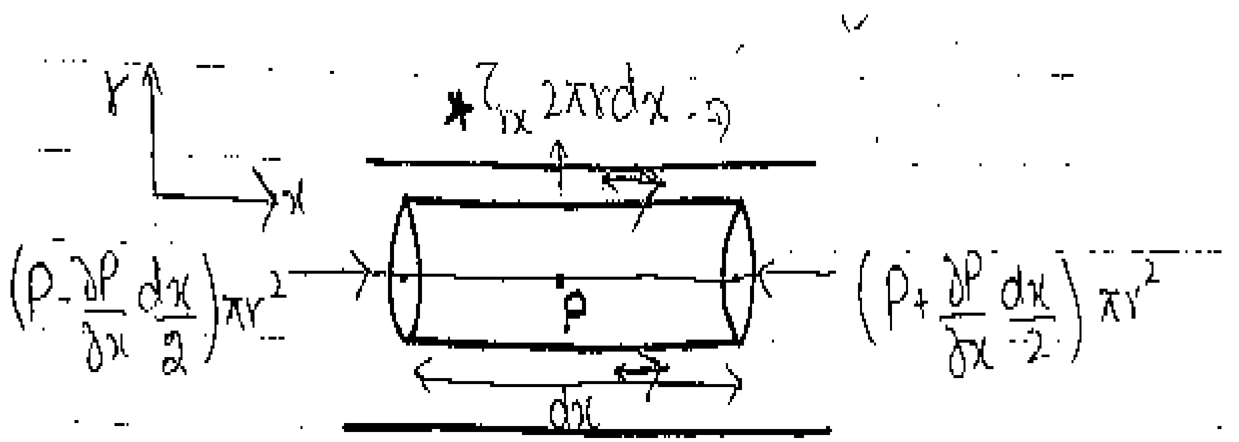
- A decrease in pipe area
- An upward slope
- Friction.

Losses due to friction are of two categories:

- **Major losses:** those losses which are losses due to friction in the constant-area sections of the pipe.
- **Minor losses:** (sometimes larger than “major” losses), those losses which are losses due to valves, elbows, and so on.

Shear Stress Distribution In Fully Developed Pipe Flow

In fully developed steady flow (Laminar or Turbulent) in a horizontal pipe the pressure drop is balanced only by shear forces act at the pipe wall. This can be seen by applying momentum equation to cylindrical control volume in the flow.



Applying the x component of the momentum equation.

$$F_{S_x} + \cancel{F_{B_x}}^{0} = \cancel{\frac{\partial}{\partial t}\int_{cv}}^{0} u\rho dv + \int_{cs} \cancel{u\rho \vec{V}.d\vec{A}}^{0}$$

$F_{S_x} = 0$ …………….(i)

Assumptions:

(1) Steady flow (2) Incompressible Fully developed flow 3) $F_{B_x} = 0$

If the pressure at the center of the element is , then the pressure at each end point is obtained by applying Taylor's series expansion of 'p' about the center of the element. The shear stress acts on the circumferential surface of the element.

Now the pressure force on the left face is $dF_L = \left(p - \frac{\partial p}{\partial x}\frac{dx}{2}\right)\pi r^2$

And the pressure force on the right face is $dF_R = -\left(p + \frac{\partial p}{\partial x}\frac{dx}{2}\right)\pi r^2$

If the shear stress at the center of the element is τ_{rx}, then the shear force on the top face is $dF_T = \tau_{rx} 2\pi r dx$ and there is no bottom force.

Using the four surface forces dF_L, dF_R, and dF_T in Eq. (i),

$$F_{S_x} = dF_L + dF_R + dF_T = 0$$

$$\left(p - \frac{\partial p}{\partial x}\frac{dx}{2}\right)\pi r^2 - \left(p + \frac{\partial p}{\partial x}\frac{dx}{2}\right)\pi r^2 + \tau_{rx} 2\pi r dx = 0$$

This equation simplifies to $\tau_{rx} = \frac{r}{2}\left(\frac{\partial p}{\partial x}\right)$

This equation indicates that the shear stress varies linearly across the pipe from zero at central line to the maximum at the pipe wall. If we denote the shear stress at the pipe wall by τ_w (equal and opposite to the stress in the fluid at the wall) then

$$\tau_w = -(\tau_{rx})_{r=R} = \frac{R}{2}\left(\frac{\partial p}{\partial x}\right)$$

And $\tau_w > 0$ if $\frac{\partial p}{\partial x} < 0$

Turbulent Velocity Profiles In Fully Developed Pipe Flow

For fully developed turbulent pipe flow, the total shear stress is;

$$\tau = \tau_{laminar} + \tau_{turbulent} \qquad \ldots\ldots\ldots\ldots.(i)$$

$$\tau = \mu \frac{d\bar{u}}{dy} - \rho\overline{u'v'}$$

Where y is distance from pipe wall, $\bar{u}$ is mean velocity, u', v' are fluctuating components of velocity in x and y directions , $\overline{u'v'}$ is average of products of u' and v' and the term $-\rho\overline{u'v'}$ is referred to as Reynolds Stress.

The turbulent stress must go the zero at the wall because no slip condition requires that the velocity at the wall is zero. Since the Reynolds Stress is zero at the wall therefore;

$(i) \Rightarrow \tau_w = \left(\mu \frac{d\bar{u}}{dy}\right)_{y=0}$ i.e. viscous effect are dominant

Now $\tau = \mu \frac{d\bar{u}}{dy} - \rho\overline{u'v'} \Rightarrow \frac{\tau}{\rho} = \frac{\mu}{\rho}\frac{d\bar{u}}{dy} - \overline{u'v'}$

$\frac{\tau}{\rho} = \upsilon \frac{d\bar{u}}{dy} - \overline{u'v'}$ where $\upsilon = \frac{\mu}{\rho}$ is kinetic viscosity

The term $\frac{\tau}{\rho}$ frequently in the direction of turbulent flow.

Its dimension is $\tau = \mu \frac{du}{dy}$

As unit of μ is kg/ms or Pa.s therefore

$$\tau = \frac{[M][L]}{[T]} \times \frac{[L]}{[T]} \times \frac{1}{[L]} = \frac{[M]}{[L][T]^2}$$

And $\frac{\tau}{\rho} = \frac{[M]}{[L][T]^2} \times \frac{[L]^3}{[M]} = \frac{[L]^2}{[T]^2} = v^2$

Thus dimension of τ is square of velocity.

We may write $\sqrt{\frac{\tau}{\rho}} = v = u_*$ This quantity is called **friction velocity**.

In a region very close to wall, where viscous forces are dominant the velocity profile is;

$$velocity\ profile = \frac{mean\ velocity}{friction\ velocity}$$

$$u^+ = \frac{\bar{u}}{u_*}$$

$y^+ = \frac{yu_*}{v}$ where $\frac{1}{u^+} = \frac{u_*}{\bar{u}} \Rightarrow y^+ = \frac{u_* y}{v}$ with $\bar{u} = \frac{v}{y}$

Where 'y' is distance measured from wall.

The velocity profile for turbulent flow by empirical power law is

$$\frac{\bar{u}}{u} = \left(\frac{y}{R}\right)^{\frac{1}{n}} = \left(1 - \frac{r}{R}\right)^{\frac{1}{n}}$$

Where u is control line velocity and 'n' vary with Reynolds number.

…………………………………………………

Prove that $\boldsymbol{\frac{\bar{u}}{u_*} = \frac{yu_*}{v}}$

Since $\tau = \mu \frac{d\bar{u}}{dy} = \mu \frac{u}{y}$

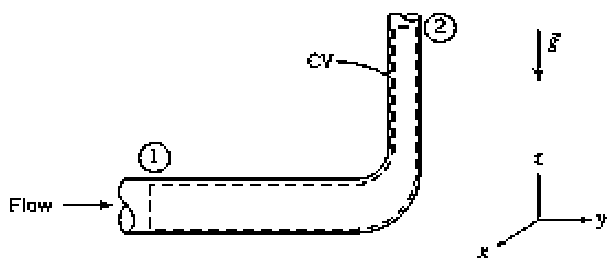
$$u = \frac{\tau y}{\mu}$$

$$\frac{\bar{u}}{u_*} = \frac{\tau y}{\mu} \times \frac{\rho^{\frac{1}{2}}}{\tau^{\frac{1}{2}}}$$

$\frac{\bar{u}}{u_*} = \frac{\tau y}{v\rho} \times \frac{\rho^{\frac{1}{2}}}{\tau^{\frac{1}{2}}} = \frac{\tau^{\frac{1}{2}} y}{v\rho^{\frac{1}{2}}} = \frac{u_* y}{v}$ where $\sqrt{\frac{\tau}{\rho}} = u_*$

Also remember $\frac{\bar{u}}{u} = \frac{2n^2}{(n+1)(2n+1)}$ and 'n' increases the ratio of the average velocity to the central line increases.

Energy Considerations In Pipe Flow



Consider a steady flow through a piping system. The controlled volume boundaries are shown by dashes line. They are normal to flow at section (1) and (2) and consider with inside surface of pipe wall. The basic equation is given as follows

By the 1st law of thermodynamics

$$\dot{Q} - \cancelto{0}{\dot{W}_s} - \cancelto{0}{\dot{W}_{shear}} - \cancelto{0}{\dot{W}_{other}} = \cancelto{0}{\frac{\partial}{\partial t}\int_{cv} e\rho dv} + \int_{cs}(e + pv)\rho\vec{V}.d\vec{A} \quad \ldots\ldots\ldots\ldots\text{(i)}$$

$e = u + \frac{V^2}{2} + gz$ and $\dot{W}_s = \dot{W}_{shaft}$ = Shaft work = Any mechanical energy to flow

Restrictions: (1) $\dot{W}_s = 0$ (2) $\dot{W}_{shear} = 0$ (Although shear stresses are present at the walls of the elbow, the velocities are zero there, so there is no possibility of work)

(3) $\dot{W}_{other} = 0$. (4) Steady, incompressible flow.

(5) Internal energy and pressure is uniform at section (1) and (2)

As $\dot{m} = \rho VA =$ Mass Flow rate

Under these assumptions the energy equation reduces to

$$\dot{Q} = \int u\rho\vec{V}.d\vec{A} + \int gz\rho\vec{V}.d\vec{A} + \int (pv)\rho\vec{V}.d\vec{A} + \int \left(\frac{V^2}{2}\right)\rho\vec{V}.d\vec{A}$$

$$\dot{Q} = \dot{m}(u_2 - u_1) + \dot{m}\left(\frac{p_2}{\rho} - \frac{p_1}{\rho}\right) + \dot{m}g(z_2 - z_1) + \int_{A_2}\frac{V_2^2}{2}\rho V_2 dA_2 + \int_{A_1}\frac{V_1^2}{2}\rho V_1 dA_1$$

Note that we have not assumed only the velocity is uniform at section (1) and (2) because for viscous flow, the velocity is not uniform, however it is convenient to introduce the average velocity. For this we define **kinetic energy coefficient**.

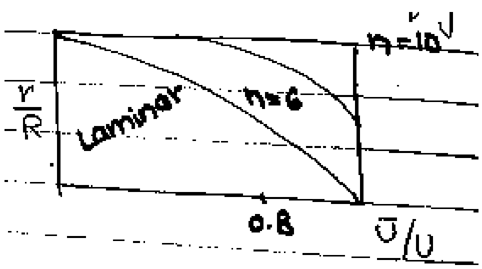
Kinetic Energy Coefficient

The kinetic energy coefficient, $\propto$, is defined such that;

$$\int_A \frac{V^2}{2}\rho V dA = \propto \int_A \frac{\bar{V}^2}{2}\rho V dA = \propto \dot{m}\frac{\bar{V}^2}{2}$$

Or $\quad \propto = \frac{\int_A \rho V^3 dA}{\dot{m}\bar{V}^2}$

Where for laminar flow $\propto = 2.0$ and for turbulent flow in a pipe $n = 6$ as shown in figure. In turbulent pipe flow, the velocity profile is quite flat



Where $\frac{\bar{V}}{u} = \left(\frac{y}{R}\right)^{\frac{1}{n}} = \left(1 - \frac{r}{R}\right)^{\frac{1}{n}} = \frac{2n^2}{(n+1)(2n+1)}$

And $\quad \propto = \left(\frac{u}{\bar{V}}\right)^3 \left(\frac{2n^2}{(n+3)(2n+3)}\right)$

Then for $n = 6, \propto = 1.08$ and for $n = 10, \propto = 1.03$

The overall result is that in the realistic range of n, from n = 6to n = 10 for high Reynolds numbers, $\propto$ varies from 1.08 to 1.03; for the one-seventh power profile (n = 7), $\propto$=1.06. Because $\propto$ is reasonably close to unity for high Reynolds numbers, and because the change in kinetic energy is usually small compared with the dominant terms in the energy equation, we shall almost always use the approximation α = 1 in our pipe flow calculations.

……………………………………..

$W_s = W_{shaft}$ = Shaft work

It is the kind of contact work because it occur through direct material contact with the surrounding at the boundary of the system.

Head Loss

Since $\dot{Q} = \int \left(u + \frac{V^2}{2} + gz + pv\right)\rho\vec{V}.d\vec{A}$

$$\dot{Q} = \int u\rho\vec{V}.d\vec{A} + \int gz\rho\vec{V}.d\vec{A} + \int (pv)\rho\vec{V}.d\vec{A} + \int \left(\frac{V^2}{2}\right)\rho\vec{V}.d\vec{A}$$

$$\dot{Q} = \dot{m}(u_2 - u_1) + \dot{m}\left(\frac{p_2}{\rho} - \frac{p_1}{\rho}\right) + \dot{m}g(z_2 - z_1) + \int \frac{V_2^2}{2}\rho V_2 dA_2 + \int \frac{V_1^2}{2}\rho V_1 dA_1$$

As $\propto = \frac{\int \rho V^3 dA}{\dot{m}\bar{V}^2}$ implies $\propto \dot{m}\bar{V}^2 = \int \rho V^3 dA$

Then $\int \frac{V_2^2}{2}\rho V_2 dA_2 = \frac{\propto \dot{m}\bar{V}^2}{2}$

$$\dot{Q} = \dot{m}(u_2 - u_1) + \dot{m}\left(\frac{p_2}{\rho} - \frac{p_1}{\rho}\right) + \dot{m}g(z_2 - z_1) + \dot{m}\left(\frac{\propto_2\bar{V}_2^2}{2} - \frac{\propto_1\bar{V}_1^2}{2}\right)$$

$$\frac{\dot{Q}}{\dot{m}} = (u_2 - u_1) + \left(\frac{p_2}{\rho} - \frac{p_1}{\rho}\right) + g(z_2 - z_1) + \left(\frac{\propto_2\bar{V}_2^2}{2} - \frac{\propto_1\bar{V}_1^2}{2}\right)$$

$$\Rightarrow \left(p_1 v_1 + \frac{\propto_1\bar{V}_1^2}{2} + gz_1\right) - \left(p_2 v_2 + \frac{\propto_2\bar{V}_2^2}{2} + gz_2\right) = u_2 - u_1 - \frac{dQ}{dm} \qquad \because \dot{Q} = \frac{dQ}{dt}, \dot{m} = \frac{dm}{dt}$$

The term $\frac{p}{\rho} + \frac{\propto\bar{V}^2}{2} + gz$ represents the mechanical energy per unit mass at a cross section. The term $(u_2 - u_1) - \frac{dQ}{dm}$ is equal to the difference in mechanical energy per unit mass between sections (1) and (2). It represents the (irreversible) conversion of mechanical energy at section (1) to unwanted thermal energy $(u_2 - u_1)$ and loss of energy via heat transfer $(-\frac{dQ}{dm})$.

We identify this group of terms as the total energy loss per unit mass and designate it by the symbol h_{l_T}, . Then

$$\left(p_1 v_1 + \frac{\propto_1\bar{V}_1^2}{2} + gz_1\right) - \left(p_2 v_2 + \frac{\propto_2\bar{V}_2^2}{2} + gz_2\right) = h_{l_T} \qquad \text{............(i)}$$

The dimensions of energy per unit mass FL/M are equivalent to dimensions of L^2/T^2 and equilateral for a viscous flow in pipe.
Above equation is one of the most important and useful equations in fluid mechanics. It enables us to compute the loss of mechanical energy caused by friction between two sections of a pipe.

One effect of friction may be to increase internal energy of flow. That is;

$$\left(\frac{p_1 v_1}{g} + \frac{\alpha_1 \bar{V}_1^2}{2g} + z_1\right) - \left(\frac{p_2 v_2}{g} + \frac{\alpha_2 \bar{V}_2^2}{2g} + z_2\right) = \frac{h_{l_T}}{g} \qquad \ldots\ldots\ldots\ldots(ii)$$

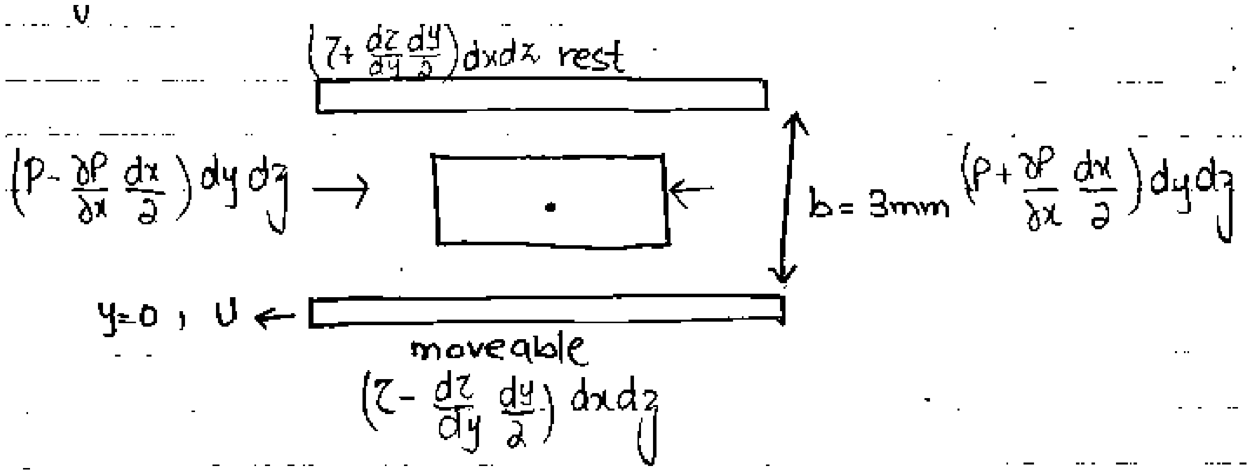
Here $H_{l_T} = \frac{h_{l_T}}{g}$ is energy per unit weight. And its dimension is $\frac{L^2}{T^2} \times \frac{T^2}{L} = L$

Equation (i) (or, ii) can be used to calculate the pressure difference between any two points in a piping system, provided the head loss, h_{l_T} (or H_{l_T}), can be determined.

Problem

Water at 60°C flows between two large flat plates. The lower plate moves to the left at a speed of 0.3 m/s; the upper plate is stationary. The plate spacing is 3 mm, and the flow is laminar. Determine the pressure gradient required to produce zero net flow at a cross section.

Solution



Assumptions: (1) Steady flow (2) Fully developed flow (3) $F_{B_x} = 0$

Using momentum equation

$$F_{S_x} + \cancelto{0}{F_{B_x}} = \cancelto{0}{\frac{\partial}{\partial t}\int_{cv} u\rho dv} + \int_{cs} u\rho \vec{V}.d\vec{A}$$

Also using $\tau_{yx} = \mu \frac{du}{dy}$ in $\frac{\partial p}{\partial x} = \frac{d\tau_{yx}}{dy}$ we get $\frac{\partial p}{\partial x} = \mu \frac{d^2u}{dy^2}$ or $\frac{d^2u}{dy^2} = \frac{1}{\mu}\frac{\partial p}{\partial x}$

Integrating twice, we obtain $u = \frac{1}{2\mu}\left(\frac{\partial p}{\partial x}\right)y^2 + c_1 y + c_2$

To evaluate the constants, c_1 and c_2, we must apply the boundary conditions.

At $y = 0, u = -u$. Consequently, $c_2 = -u$.

At $y = b, u = 0$. We get $0 = \frac{1}{2\mu}\left(\frac{\partial p}{\partial x}\right)b^2 + c_1 b - u$

This gives $c_1 = \frac{u}{b} - \frac{1}{2}\left(\frac{\partial p}{\partial x}\right)b$

And hence, $u = \frac{1}{2\mu}\left(\frac{\partial p}{\partial x}\right)y^2 + \frac{u}{b}y - \frac{1}{2}\left(\frac{\partial p}{\partial x}\right)by - u$

Or $u = \frac{1}{2\mu}\left(\frac{\partial p}{\partial x}\right)(y^2 - by) + u\left(\frac{y}{b} - 1\right)$

To find flow rate

$$\frac{Q}{l} = \int_a^b u dy = \int_a^b \left[\frac{1}{2\mu}\left(\frac{\partial p}{\partial x}\right)(y^2 - by) + u\left(\frac{y}{b} - 1\right)\right] = -\frac{1}{12\mu}\left(\frac{\partial p}{\partial x}\right)b^3 - \frac{ub}{2}$$

For pressure discharge $Q = 0$ with dynamic viscosity $\mu = 4.63 \times 10^4 Nsm^{-2}$

$$0 = \int_a^b u dy = -\frac{1}{12\mu}\left(\frac{\partial p}{\partial x}\right)b^3 - \frac{ub}{2}$$

$$\frac{\partial p}{\partial x} = -\frac{6u}{b^2}\mu$$

$$\frac{\partial p}{\partial x} = -6 \times 0.3ms^{-1} \times 4.63 \times 10^4 Nsm^{-2} \times \left(\frac{1}{0.003}\right)^2$$

$$\frac{\partial p}{\partial x} = -92.6Pam^{-1}$$

Thus pressure must decrease in x direction for zero flow rate.

Problem

For fully developed flow between parallel plates, upper plate is moving with velocity U = 2m/s and the distance between plats is 2.5mm, then find

a) $\frac{Q}{l}$ for $\frac{\partial p}{\partial x} = 0$
b) τ_{yx} at y = 0 for $\frac{\partial p}{\partial x} = 0$
c) $\frac{\partial p}{\partial x}$ at $y = 0.25a$ for $\tau_{yx} = 0$ where fluid is air.

Solution

The velocity profile if upper plate is moving with constant velocity

$$u = \frac{Uy}{a} + \frac{a^2}{2\mu}\left(\frac{\partial p}{\partial x}\right)\left[\left(\frac{y}{a}\right)^2 - \left(\frac{y}{a}\right)\right] \qquad \ldots\ldots\ldots\ldots\ldots\text{(i)}$$

a) Putting $\frac{\partial p}{\partial x} = 0$ in (i) $\Rightarrow u = \frac{Uy}{a}$
   Now $\frac{Q}{l} = \int_0^a u dy = \int_0^a \left(\frac{Uy}{a}\right) dy = \frac{Ua}{2} = \frac{2ms^{-1}\times 0.0025m}{2} = 0.00250m^2s^{-1}$
b) As $\tau_{yx} = \mu\frac{du}{dy}$ for $\frac{\partial p}{\partial x} = 0$ then for air as fluid
   $\Rightarrow \tau_{yx} = \mu\frac{U}{a} = 1.79\times 10^{-5}Nsm^{-2}\times\frac{2ms^{-1}}{0.0025m^2s^{-1}} = 0.0143Nsm^{-3}$
c) Given $\tau_{yx} = 0$ at $y = 0.25a$
   Since $u = \frac{Uy}{a} + \frac{a^2}{2\mu}\left(\frac{\partial p}{\partial x}\right)\left[\left(\frac{y}{a}\right)^2 - \left(\frac{y}{a}\right)\right]$
   $\Rightarrow \tau_{yx} = \mu\frac{du}{dy} = \mu\frac{U}{a} + a\left(\frac{\partial p}{\partial x}\right)\left[\frac{y}{a} - \frac{1}{2}\right]$
   $\Rightarrow 0 = \mu\frac{U}{a} + a\left(\frac{\partial p}{\partial x}\right)\left[\frac{0.25a}{a} - \frac{1}{2}\right] = \mu\frac{U}{a} + a\left(\frac{\partial p}{\partial x}\right)\left[\frac{1}{4} - \frac{1}{2}\right]$
   $\Rightarrow \frac{\partial p}{\partial x} = 4\mu\frac{U}{a^2}$
   $\Rightarrow \frac{\partial p}{\partial x} = 4\times 1.79\times\frac{2}{(0.0025)^2}$
   $\Rightarrow \frac{\partial p}{\partial x} = 22.9Pam^{-1}$

Problem

Consider incompressible flow in a circular channel. Derive general expressions for Reynolds number in terms of

a) Volume flow rate and tube diameter
b) Mass flow rate and tube diameter. The Reynolds number is 1800 in a section where the tube diameter is 10mm.
c) Find the Reynolds number for the same flow rate in a section where the tube diameter is 6 mm.

Solution

a) Since $\quad Re = \frac{\rho V D}{\mu}$

$\Rightarrow Re = \frac{\rho Q D}{\mu A}$ where we use $Q = AV$ then $V = \frac{Q}{A}$

$\Rightarrow Re = \frac{4\rho Q D}{\mu \pi D^2}$ where we use $A = \frac{\pi D^2}{4}$

$\Rightarrow Re = \frac{4\rho Q}{\mu \pi D} \Rightarrow Re = \frac{4Q}{v \pi D}$ where we use $v = \frac{\rho}{\mu}$

Thus Reynolds number is written in terms of Q and diameter of tube.

b) Since $\rho = \frac{m}{V} \Rightarrow m = \rho V \Rightarrow \dot{m} = \rho \dot{V}$

Now $V = Ax \Rightarrow \dot{V} = A\dot{x} \Rightarrow \dot{V} = AV$

Thus mass flow rate is $\Rightarrow \dot{m} = \rho A V \Rightarrow V = \frac{\dot{m}}{\rho A} = \frac{4\dot{m}}{\rho \pi D^2}$

Then $Re = \frac{\bar{V} D}{v} = \frac{4\dot{m} D}{\rho \pi D^2 v} = \frac{4\dot{m}}{\rho \pi D v}$

Thus Reynolds number is written in terms of mass flow rate and diameter of tube.

c) Since $\quad Re = \frac{4Q}{\pi D v} \Rightarrow Q = \frac{\pi D v Re}{4}$

$\Rightarrow \frac{\pi D_1 v Re_1}{4} = \frac{\pi D_2 v Re_2}{4} \Rightarrow D_1 Re_1 = D_2 Re_2 \Rightarrow Re_2 = \frac{D_1 Re_1}{D_2} = \frac{20 \times 1800}{6}$

$\Rightarrow Re_2 = 6000$

EXTERNAL INCOMPRESSIBLE VISCOUS FLOW

Our objective in this chapter is to quantify the behavior of viscous, incompressible fluids in external flow.

External flows: External flows are flows over bodies immersed in an unbounded fluid. The flow over a sphere and the flow over a streamlined body are examples of external flows. More interesting examples are the flow fields around such objects as airfoils, automobiles, and airplanes.

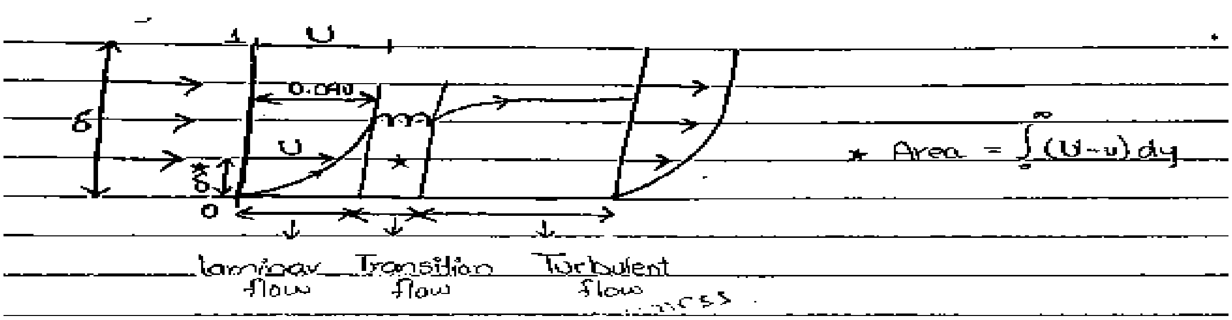
Transition flows: The process in which laminar flow become turbulent is called transition flow.

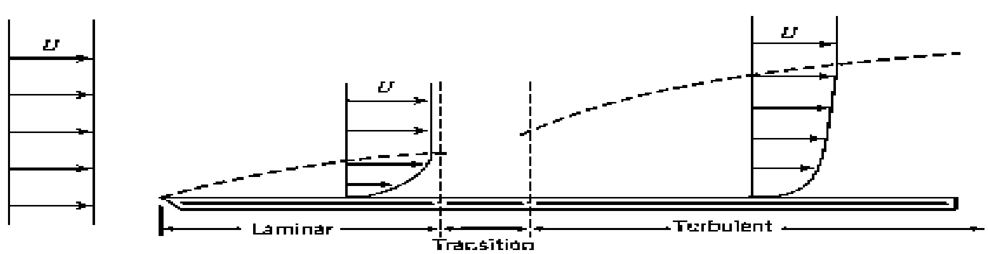
The Boundary-Layer Concept

The concept of a boundary layer was first introduced by Ludwig Prandtl's, a German aerodynamicist, in 1904. It is defined as follows;

The fluid layer in the neighborhood of a solid boundary where the effects of fluid friction (Viscous Effects) are predominant is known as **boundary**.

While the distance from the solid boundary at which the local value of the velocity reaches 99% of free stream velocity. i.e. Velocity = 0.09U is called **boundary layer**. It is the region adjacent to a solid surface in which viscous stresses are present.





Boundary-Layer Thicknesses

The boundary layer is the region adjacent to a solid surface in which viscous stresses are present. These stresses are present because we have shearing of the fluid layers, i.e., a velocity gradient, in the boundary layer. Both laminar and turbulent layers have such gradients, but the difficulty is that the gradients only asymptotically approach zero as we reach the edge of the boundary layer. Hence, the location of the edge, i.e., of the boundary-layer thickness, is not very obvious. We cannot simply define it as where the boundary-layer velocity u equals the freestream velocity U. Because of this, several boundary-layer definitions have been developed: the disturbance thickness $\boldsymbol{\delta}$, the displacement thickness $\boldsymbol{\delta}^*$, and the momentum thickness $\boldsymbol{\theta}$. The most straightforward definition is the disturbance thickness.

- **The disturbance thickness $\boldsymbol{\delta}$:** This is usually defined as the distance from the surface at which the velocity is within 1% of the free stream, $u \approx 0.99U$.
- **The displacement thickness $\boldsymbol{\delta}^*$:** it is the distance by which solid boundaries would have to displace in the frictionless flow to give the mass flow rate deficit that exists in the boundary layer.
  It is the distance the plate would be moved so that the loss of mass flux (due to reduction in uniform flow area) is equivalent to the loss the boundary layer causes.
  Its formula is $\delta^* = \int_0^\delta \left(1 - \frac{u}{U}\right) dy$
  Explanation
  Since mass flow rate is given as $\dot{m} = \rho VA = \rho U\delta^* w$ and decrease in mass flow rate caused by viscous force $\dot{m} = \int_0^\infty (U - u) w dy$ where w is the width of surface. Then combining both
  $\rho U\delta^* w = \int_0^\infty (U - u) w dy \Rightarrow \delta^* = \int_0^\infty \left(1 - \frac{u}{U}\right) dy$
  Here U is freestream velocity, u is velocity near boundary and for incompressible flow ρ will be constant.
  $\Rightarrow \delta^* = \int_0^\delta \left(1 - \frac{u}{U}\right) dy + \int_\delta^\infty \left(1 - \frac{u}{U}\right) dy$
  When $y \geq \delta$ then $u \approx U$ and we get
  $\delta^* = \int_0^\delta \left(1 - \frac{u}{U}\right) dy$

- **The momentum thickness $\boldsymbol{\theta}$:** it is the thickness of a layer of the fluid with velocity U for which momentum flow is equal to deficit of momentum flux through boundary layer. It is the distance the plate would be moved so that the loss of momentum flux is equivalent to the loss the boundary layer actually causes. Its formula is $\theta = \int_0^{\delta} \frac{u}{U}\left(1 - \frac{u}{U}\right) dy$

  Explanation

  Since mass flow rate is given as $\dot{m} = \rho VA = \rho U^2 \theta w$ it is the loss of momentum flux and actual mass flow rate through boundary layers is $\dot{m} = \int_0^{\infty} \rho u(U-u)w dy$ where w is the width of surface. Then combining both

  $\rho U^2 \theta w = \int_0^{\infty} \rho u(U-u)w dy \Rightarrow \theta = \int_0^{\infty} \frac{u}{U^2}(U-u)dy$

  $\Rightarrow \theta = \int_0^{\infty} \frac{u}{U}\left(1 - \frac{u}{U}\right) dy$

  $\Rightarrow \theta = \int_0^{\delta} \frac{u}{U}\left(1 - \frac{u}{U}\right) dy + \int_{\delta}^{\infty} \frac{u}{U}\left(1 - \frac{u}{U}\right) dy$

  When $y \geq \delta$ then $u \approx U$ and we get

  $\theta = \int_0^{\delta} \frac{u}{U}\left(1 - \frac{u}{U}\right) dy$

Keep in mind the displacement and momentum thicknesses, δ* and θ, are integral thicknesses, because their definitions, are in terms of integrals across the boundary layer. Because they are defined in terms of integrals for which the integrand vanishes in the freestream, they are appreciably easier to evaluate accurately from experimental data than the boundary-layer disturbance thickness, δ. This fact, coupled with their physical significance, accounts for their common use in specifying boundary-layer thickness.

Shape Factor

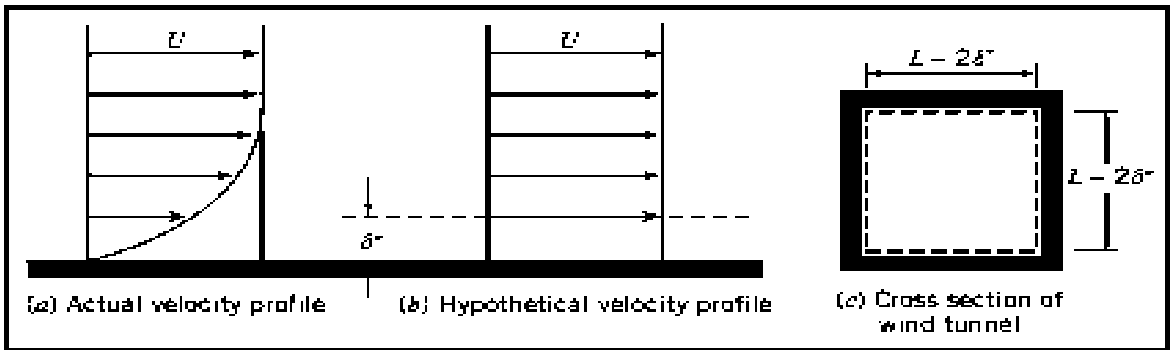
The ratio of displacement thickness to momentum thickness is called shape factor and it is denoted by H.

Its formula is given as $H = \frac{\delta^*}{\theta}$

Example **Boundary Layer In Channel Flow**

A laboratory wind tunnel has a test section that is 305 mm square. Boundary-layer velocity profiles are measured at two cross-sections and displacement thicknesses are evaluated from the measured profiles. At section CD, where the freestream speed is U = 26 m/s, the displacement thickness is δ^*= 1.5 mm. At section (2), located downstream from section (1), δ^* = 2.1 mm. Calculate the change in static pressure; between sections (1) and (2). Express the result as a fraction of the freestream dynamic pressure at section (1). Assume standard atmosphere conditions.

Solution:



Using the following formula

$\frac{\partial}{\partial t}\int_{cv}\cancelto{0}{\rho dv} + \int_{cs}\rho\vec{V}.d\vec{A} = 0$ or $\frac{p_1}{\rho} + \frac{V_1^2}{2} = \frac{p_2}{\rho} + \frac{V_2^2}{2}$ where $gz_1 = gz_2 = 0$

Assumptions: (1) Steady flow. (2) Incompressible flow.

(3) Flow uniform at each section outside δ^*.

(4) Flow along a streamline between sections (1) and (2) .

(5) No frictional effects in freestream. (6) Negligible elevation changes.

From the Bernoulli equation we obtain

$$p_1 - p_2 = \frac{1}{2}\rho(V_2^2 - V_1^2) = \frac{1}{2}\rho(U_2^2 - U_1^2) = \frac{1}{2}\rho U_1^2\left(\left(\frac{U_2}{U_1}\right)^2 - 1\right)$$

Or $\frac{p_1 - p_2}{\frac{1}{2}\rho U_1^2} = \left(\frac{U_2}{U_1}\right)^2 - 1$

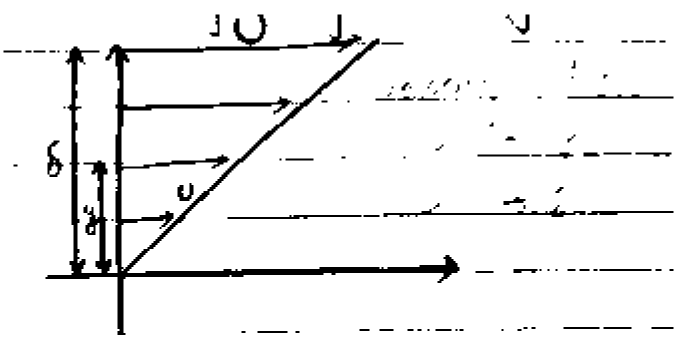
From continuity, $V_1A_1 = U_1A_1 = V_2A_2 = U_2A_2$ so $\frac{U_2}{U_1} = \frac{A_1}{A_2}$ where A = (L $-$ $2\delta^*$)^2 is the effective flow area. Substituting gives

$$\frac{p_1 - p_2}{\frac{1}{2}\rho U_1^2} = \left(\frac{A_1}{A_2}\right)^2 - 1 = \left(\frac{(L-2\delta_1^*)^2}{(L-2\delta_2^*)^2}\right)^2 - 1 = \left(\frac{305-2(1.5)}{305-2(2.1)}\right)^4 - 1 = 0.0161 = 1.61\%$$

This problem illustrates a basic application of the displacement-thickness concept. It is somewhat unusual in that, because the flow is confined, the reduction in flow area caused by the boundary layer leads to the result that the pressure in the inviscid flow region drops (if only slightly). In most applications the pressure distribution is determined from the inviscid flow and then applied to the boundary layer.

Example For a laminar distribution of velocity in a boundary layer on a flat plate, find the value of $\frac{\delta^*}{\theta}$, shape factor.

Solution:



Velocity distribution in boundary layer may be written as $\frac{u}{U} = \frac{y}{\delta}$

$$\delta^* = \int_0^\delta \left(1 - \frac{u}{U}\right) dy = \int_0^\delta \left(1 - \frac{y}{\delta}\right) dy = \left| y - \frac{y^2}{2\delta} \right|_0^\delta = \frac{\delta}{2}$$

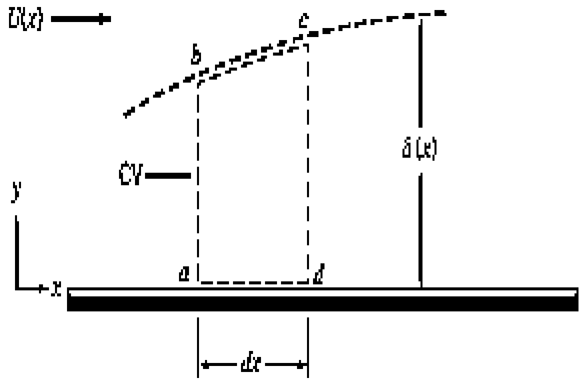
$$\theta = \int_0^\delta \frac{u}{U}\left(1 - \frac{u}{U}\right) dy = \int_0^\delta \frac{y}{\delta}\left(1 - \frac{y}{\delta}\right) dy = \left| \frac{y^2}{2\delta} - \frac{y^2}{3\delta^2} \right|_0^\delta = \frac{\delta}{6}$$

$$H = \frac{\delta^*}{\theta} = \frac{\frac{\delta}{2}}{\frac{\delta}{6}} = 3$$

This is required shape factor.

Momentum Integral Equation

Consider incompressible, steady, two-dimensional flow over a solid surface. The boundary-layer thickness, , grows in some manner with increasing distance, x. For our analysis we choose a differential control volume, of length dx, width w, and height δ (x), as shown in Figure. The freestream velocity is U(x).



We wish to determine the boundary-layer thickness, δ, as a function of x. There will be mass flow across surfaces ab and cd of differential control volume $abcd$. Since the edge of the boundary layer is not a streamline. Thus there will be mass flow across surface bc. Since control surface ad is adjacent to a solid boundary, there will not be flow across ad. Let us apply the continuity equation to determine the mass flux through each portion of the control surface. i.e.

$$\cancelto{0}{\frac{\partial}{\partial t}\int_{cv}\rho dv} + \int_{cs}\rho\vec{V}.d\vec{A} = 0$$

Assumptions:

(1) Steady flow. (2) Two-dimensional flow.

Then $\int_{cs}\rho\vec{V}.d\vec{A} = 0$

Hence $\dot{m}_{ab} + \dot{m}_{bc} + \dot{m}_{cd} = 0$

Or $\dot{m}_{bc} = -\dot{m}_{ab} - \dot{m}_{cd}$

Now let us evaluate these terms for the differential control volume of width w:

Surface	Mass Flux
ab	Surface ab is located at x. Since the flow is two-dimensional (no variation with z), the mass flux through ab is $$\dot{m}_{ab} = -\left\{\int_0^\delta \rho u\, dy\right\} w$$
cd	Surface cd is located at $x + dx$. Expanding $\dot{m}$ in a Taylor series about location x, we obtain $$\dot{m}_{x+dx} = \dot{m}_x + \frac{\partial \dot{m}}{\partial x}\bigg]_x dx$$ and hence $$\dot{m}_{cd} = \left\{\int_0^\delta \rho u\, dy + \frac{\partial}{\partial x}\left[\int_0^\delta \rho u\, dy\right] dx\right\} w$$
bc	Thus for surface bc we obtain, from the continuity equation and the above results, $$\dot{m}_{bc} = -\left\{\frac{\partial}{\partial x}\left[\int_0^\delta \rho u\, dy\right] dx\right\} w$$

Now let us consider the momentum fluxes and forces associated with control volume $abcd$. These are related by the momentum equation. i.e.

Apply the x component of the momentum equation to control volume $abcd$:

$$F_{S_x} + \cancelto{0}{F_{B_x}} = \cancelto{0}{\frac{\partial}{\partial t}\int_{cv}} u\rho dv + \int_{cs} u\rho \vec{V}.d\vec{A}$$

Assumption: $F_{B_x} = 0$.

Then $F_{S_x} = mf_{ab} + mf_{bc} + mf_{cd}$

Where mf represents the x component of momentum flux.

Let us consider the momentum flux first and again consider each segment of the control surface.

Surface	Momentum Flux (mf)
ab	Surface *ab* is located at x. Since the flow is two-dimensional, the x momentum flux through *ab* is $\mathrm{mf}_{ab} = -\left\{ \int_0^\delta u\,\rho u\,dy \right\} w$
cd	Surface *cd* is located at $x + dx$. Expanding the x momentum flux (mf) in a Taylor series about location x, we obtain $\mathrm{mf}_{x+dx} = \mathrm{mf}_x + \left.\frac{\partial \mathrm{mf}}{\partial x}\right]_x dx$ or $\mathrm{mf}_{cd} = \left\{ \int_0^\delta u\,\rho u\,dy + \frac{\partial}{\partial x}\left[\int_0^\delta u\,\rho u\,dy \right] dx \right\} w$
bc	Since the mass crossing surface *bc* has velocity component U in the x direction, the x momentum flux across *bc* is given by $\mathrm{mf}_{bc} = U\,\dot{m}_{bc}$ $\mathrm{mf}_{bc} = -U\left\{ \frac{\partial}{\partial x}\left[\int_0^\delta \rho u\,dy \right] dx \right\} w$

From the above we can evaluate the net x momentum flux through the control surface as

From the above we can evaluate the net x momentum flux through the control surface as

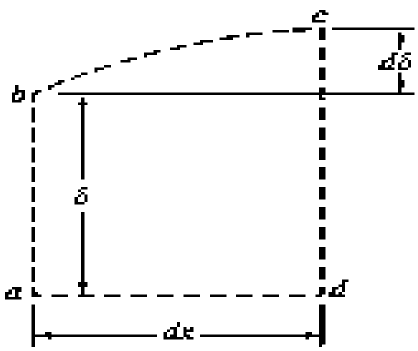
$$\int_{\mathrm{CS}} u\,\rho \vec{V} \cdot d\vec{A} = -\left\{ \int_0^\delta u\,\rho u\,dy \right\} w + \left\{ \int_0^\delta u\,\rho u\,dy \right\} w + \left\{ \frac{\partial}{\partial x}\left[\int_0^\delta u\,\rho u\,dy \right] dx \right\} w - U\left\{ \frac{\partial}{\partial x}\left[\int_0^\delta \rho u\,dy \right] dx \right\} w$$

Collecting terms, we find that

$$\int_{\mathrm{CS}} u\,\rho \vec{V} \cdot d\vec{A} = \left\{ \frac{\partial}{\partial x}\left[\int_0^\delta u\,\rho u\,dy \right] dx - U\frac{\partial}{\partial x}\left[\int_0^\delta \rho u\,dy \right] dx \right\} w$$

This is an expression for the x momentum flux through the control surface.

Let us consider the surface forces acting on the control volume in the x direction. We recognize that normal forces having nonzero components in the x direction act on three surfaces of the control surface. In addition, a shear force acts on surface *ad*



Since the velocity gradient goes to zero at the edge of the boundary layer, the shear force acting along surface *bc* is negligible.

Surface	Force
ab	If the pressure at x is p, then the force acting on surface *ab* is given by $$F_{ab} = pw\delta$$ [The boundary layer is very thin; its thickness has been greatly exaggerated in all the sketches we have made. Because it is thin, pressure variations in the y direction may be neglected, and we assume that within the boundary layer, $p = p(x)$ only.]
cd	Expanding in a Taylor series, the pressure at $x + dx$ is given by $$p_{x+dx} = p + \left.\frac{dp}{dx}\right]_x dx$$ The force on surface *cd* is then given by $$F_{cd} = -\left(p + \left.\frac{dp}{dx}\right]_x dx\right) w(\delta + d\delta)$$
bc	The average pressure acting over surface *bc* is $$p + \left.\frac{1}{2}\frac{dp}{dx}\right]_x dx$$

Then the x component of the normal force acting over bc is given by

$$F_{bc} = \left(p + \frac{1}{2}\left.\frac{dp}{dx}\right]_x dx\right) w\, d\delta$$

ad The average shear force acting on ad is given by

$$F_{ad} = -\left(\tau_w + \frac{1}{2}\, d\tau_w\right) w\, dx$$

Summing these x components, we obtain the total force acting in the x direction on the control volume,

$$F_{S_x} = \left\{-\frac{dp}{dx}\,\delta\, dx - \frac{1}{2}\frac{dp}{dx}\,\underbrace{\cancel{dx\, d\delta}}_{\simeq 0} - \tau_w\, dx - \frac{1}{2}\underbrace{\cancel{d\tau_w\, dx}}_{\simeq 0}\right\} w$$

where we note that $dx\, d\delta \ll \delta\, dx$ and $d\tau_w \ll \tau_w$, and so neglect the second and fourth terms.

Substituting the expressions, for $\int_{CS} u\,\rho \vec{V} \cdot d\vec{A}$ and F_{S_x} into the x momentum equation (Eq. 4.18a), we obtain

$$\left\{-\frac{dp}{dx}\,\delta\, dx - \tau_w\, dx\right\} w = \left\{\frac{\partial}{\partial x}\left[\int_0^\delta u\,\rho u\, dy\right] dx - U\frac{\partial}{\partial x}\left[\int_0^\delta \rho u\, dy\right] dx\right\} w$$

Dividing this equation by $w\, dx$ gives

$$-\delta\frac{dp}{dx} - \tau_w = \frac{\partial}{\partial x}\int_0^\delta u\rho u dy - U\frac{\partial}{\partial x}\int_0^\delta \rho u dy$$

This is a "momentum integral" equation that gives a relation between the x components of the forces acting in a boundary layer and the x momentum flux.

The pressure gradient, $\frac{dp}{dx}$, can be determined by applying the Bernoulli equation to the inviscid flow outside the boundary layer: $\frac{dp}{dx} = -\rho U\frac{dU}{dx}$.

If we use that $\delta = \int_0^\delta dy$ then above equation can be written as

$$\tau_w = -\frac{\partial}{\partial x}\int_0^{\delta} u\,\rho u\,dy + U\frac{\partial}{\partial x}\int_0^{\delta}\rho u\,dy + \frac{dU}{dx}\int_0^{\delta}\rho U\,dy$$

Since

$$U\frac{\partial}{\partial x}\int_0^{\delta}\rho u\,dy = \frac{\partial}{\partial x}\int_0^{\delta}\rho u U\,dy - \frac{dU}{dx}\int_0^{\delta}\rho u\,dy$$

we have

$$\tau_w = \frac{\partial}{\partial x}\int_0^{\delta}\rho u(U-u)\,dy + \frac{dU}{dx}\int_0^{\delta}\rho(U-u)\,dy$$

and

$$\tau_w = \frac{\partial}{\partial x}U^2\int_0^{\delta}\rho\frac{u}{U}\left(1-\frac{u}{U}\right)dy + U\frac{dU}{dx}\int_0^{\delta}\rho\left(1-\frac{u}{U}\right)dy$$

Using $\delta^* = \int_0^{\delta}\left(1-\frac{u}{U}\right)dy$ and $\theta = \int_0^{\delta}\frac{u}{U}\left(1-\frac{u}{U}\right)dy$ we obtain

$$\frac{\tau_w}{\rho} = \frac{d}{dx}(U^2\theta) + \delta^* U\frac{dU}{dx}$$

This is the momentum integral equation. This equation will yield an ordinary differential equation for boundary-layer thickness δ as a function of x.

Use Of The Momentum Integral Equation For Flow With Zero Pressure Gradient

Using momentum integral equation $\frac{\tau_w}{\rho} = \frac{d}{dx}(U^2\theta) + \delta^* U\frac{dU}{dx}$

$$\tau_w = \rho\frac{d}{dx}(U^2\theta) + \rho\delta^* U\frac{dU}{dx} \qquad \text{...........(i)}$$

For the special case of a flat plate (zero pressure gradient) the free-stream pressure p and velocity U are both constant, so we have U(x) = U = constant.

Then $\frac{dp}{dx} = 0 = \frac{dU}{dx}$

The momentum integral equation then reduces to

$$\tau_w = \rho U^2\frac{d\theta}{dx} = \rho U^2\frac{d}{dx}\int_0^{\delta}\frac{u}{U}\left(1-\frac{u}{U}\right)dy$$

The velocity distribution, $\frac{u}{U}$, in the boundary layer is assumed to be similar for all values of x and normally is specified as a function of $\frac{y}{\delta}$. (Note that $\frac{u}{U}$ is dimensionless and δ is a function of x only.) Consequently, it is convenient to change the variable of integration from y to $\frac{y}{\delta}$. Defining $\eta = \frac{y}{\delta}$. We get $dy = \delta d\eta$

And the momentum integral equation for zero pressure gradient is written

$$\tau_w = \rho U^2 \frac{d\theta}{dx} = \rho U^2 \frac{d\delta}{dx} \int_0^1 \frac{u}{U}\left(1 - \frac{u}{U}\right) d\eta \qquad \ldots\ldots\ldots\ldots(ii)$$

We wish to solve this equation for the boundary-layer thickness as a function of x. To do this, we must satisfy the remaining items:

Assume a velocity distribution in the boundary layer—a functional relationship of the form $\frac{u}{U} = f\left(\frac{y}{\delta}\right)$

(a) The assumed velocity distribution should satisfy the following approximate physical boundary conditions:

at y = o, u = 0

at y = δ, u = U

at y = $\delta, \frac{\partial u}{\partial y} = 0$

(b) Note that for any assumed velocity distribution, the numerical value of the integral is simply $\int_0^1 \frac{u}{U}\left(1 - \frac{u}{U}\right) d\eta = \beta =$ Constant

And the momentum integral equation becomes $\tau_w = \rho U^2 \frac{d\delta}{dx} \beta$

Obtain an expression for τ_w in terms of $\delta(x)$.

Laminar Flow

For laminar flow over a flat plate, a reasonable assumption for the velocity profile is polynomial in y:

$u = a + by + cy^2$ …………(1)

The physical boundary conditions are:

at y = 0, u = 0

at y = δ, u = U

at y = $\delta, \frac{\partial u}{\partial y} = 0$

$(1) \Rightarrow a = 0$ at y = 0, u = 0

$(1) \Rightarrow U = b\delta + c\delta^2$ ……….(2) at $a = 0, y = \delta, u = U$

$(1) \Rightarrow \frac{\partial u}{\partial y} = b + 2\,cy$

$\Rightarrow 0 = b + 2\,c\delta$ at y = $\delta, \frac{\partial u}{\partial y} = 0$

$\Rightarrow b = -2\,c\delta$ ……….(3)

$(2) \Rightarrow U = -2\,c\delta^2 + c\delta^2 \Rightarrow U = -c\delta^2 \Rightarrow c = -\frac{U}{\delta^2}$

$(3) \Rightarrow b = \frac{2U}{\delta}$ at $c = -\frac{U}{\delta^2}$

$(1) \Rightarrow u = \frac{2U}{\delta}y - \frac{U}{\delta^2}y^2$

$\Rightarrow \frac{u}{U} = 2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2$

$\Rightarrow \frac{u}{U} = 2\eta - \eta^2$ ……….(4)

Now $\tau_w = \mu \frac{\partial u}{\partial y})_{y=0}$

Substituting the assumed velocity profile, into this expression for τ_w gives

$$\tau_w = \mu \frac{\partial u}{\partial y}\bigg]_{y=0} = \mu \frac{U\,\partial(u/U)}{\delta\,\partial(y/\delta)}\bigg]_{y/\delta=0} = \frac{\mu U}{\delta}\frac{d(u/U)}{d\eta}\bigg]_{\eta=0}$$

or

$$\tau_w = \frac{\mu U}{\delta}\frac{d}{d\eta}(2\eta - \eta^2)\bigg]_{\eta=0} = \frac{\mu U}{\delta}(2 - 2\eta)\bigg]_{\eta=0} = \frac{2\mu U}{\delta}$$

Note that this shows that the wall stress τ_w is a function of x, since the boundary-layer thickness δ = δ(x). Now we can return to the momentum integral equation

$$\tau_w = \rho U^2 \frac{d\theta}{dx} = \rho U^2 \frac{d\delta}{dx}\int_0^1 \frac{u}{U}\left(1 - \frac{u}{U}\right) d\eta$$

Substituting for τ_w and u/U, we obtain

$$\frac{2\mu U}{\delta} = \rho U^2 \frac{d\delta}{dx}\int_0^1 (2\eta - \eta^2)(1 - 2\eta + \eta^2)\, d\eta$$

or

$$\frac{2\mu U}{\delta \rho U^2} = \frac{d\delta}{dx}\int_0^1 (2\eta - 5\eta^2 + 4\eta^3 - \eta^4)\, d\eta$$

Integrating and substituting limits yields

$$\frac{2\mu}{\delta \rho U} = \frac{2}{15}\frac{d\delta}{dx} \quad \text{or} \quad \delta\, d\delta = \frac{15\mu}{\rho U}\, dx$$

which is a differential equation for δ. Integrating again gives

$$\frac{\delta^2}{2} = \frac{15\mu}{\rho U}x + c$$

If we assume that $\delta = 0$ at $x = 0$, then $c = 0$, and thus

$$\delta = \sqrt{\frac{30\mu x}{\rho U}}$$

Note that this shows that the laminar boundary-layer thickness δ grows as $\sqrt{x}$; it has a parabolic shape. Traditionally this is expressed in dimensionless form:

$$\frac{\delta}{x} = \sqrt{\frac{30\mu}{\rho U x}} = \frac{5.48}{\sqrt{Re_x}}$$

This Equation shows that the ratio of laminar boundary-layer thickness to distance along a flat plate varies inversely with the square root of length Reynolds number.

Once we know the boundary-layer thickness, all details of the flow may be determined. The wall shear stress, or "skin friction," coefficient is defined as

$$C_f \equiv \frac{\tau_w}{\frac{1}{2}\rho U^2}$$

And substituting $\frac{\delta}{x} = \frac{5.48}{\sqrt{Re_x}}$ we get

$$C_f = \frac{\tau_w}{\frac{1}{2}\rho U^2} = \frac{2\mu(U/\delta)}{\frac{1}{2}\rho U^2} = \frac{4\mu}{\rho U \delta} = 4\frac{\mu}{\rho U x}\frac{x}{\delta} = 4\frac{1}{Re_x}\frac{\sqrt{Re_x}}{5.48}$$

Finally,

$$C_f = \frac{0.730}{\sqrt{Re_x}}$$

Once the variation of τ_w is known, the viscous drag on the surface can be evaluated by integrating over the area of the flat plate.

Example Laminar Boundary Layer On A Flat Plate

Approximate Solution Using Sinusoidal Velocity Profile

Consider two-dimensional laminar boundary-layer flow along a flat plate. Assume the velocity profile in the boundary layer is sinusoidal,

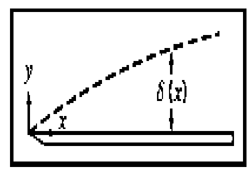
$$\frac{u}{U} = \sin\left(\frac{\pi}{2}\frac{y}{\delta}\right)$$

Find expressions for:

(a) The rate of growth of δ as a function of x.
(b) The displacement thickness, δ^*, as a function of x.
(c) The total friction force on a plate of length L and width b.

Given: Two-dimensional, laminar boundary-layer flow along a flat plate. The boundary-layer velocity profile is

$$\frac{u}{U} = \sin\left(\frac{\pi}{2}\frac{y}{\delta}\right) \qquad \text{for } 0 \leq y \leq \delta$$



and

$$\frac{u}{U} = 1 \qquad \text{for } y > \delta$$

Find: (a) $\delta(x)$. (b) $\delta^*(x)$.
(c) Total friction force on a plate of length L and width b.

Solution:
For flat plate flow, U = constant, $dp/dx = 0$, and

$$\tau_w = \rho U^2 \frac{d\theta}{dx} = \rho U^2 \frac{d\delta}{dx}\int_0^1 \frac{u}{U}\left(1 - \frac{u}{U}\right)d\eta \qquad (9.19)$$

Assumptions: (1) Steady flow.
(2) Incompressible flow.

Substituting $\frac{u}{U} = \sin\frac{\pi}{2}\eta$ into Eq. 9.19, we obtain

$$\tau_w = \rho U^2 \frac{d\delta}{dx}\int_0^1 \sin\frac{\pi}{2}\eta\left(1-\sin\frac{\pi}{2}\eta\right)d\eta = \rho U^2\frac{d\delta}{dx}\int_0^1\left(\sin\frac{\pi}{2}\eta - \sin^2\frac{\pi}{2}\eta\right)d\eta$$

$$= \rho U^2\frac{d\delta}{dx}\frac{2}{\pi}\left[-\cos\frac{\pi}{2}\eta - \frac{1}{2}\frac{\pi}{2}\eta + \frac{1}{4}\sin\pi\eta\right]_0^1 = \rho U^2\frac{d\delta}{dx}\frac{2}{\pi}\left[0+1-\frac{\pi}{4}+0+0-0\right]$$

$$\tau_w = 0.137\rho U^2\frac{d\delta}{dx} = \beta\rho U^2\frac{d\delta}{dx}; \quad \beta = 0.137$$

Now

$$\tau_w = \mu\frac{\partial u}{\partial y}\bigg]_{y=0} = \mu\frac{U}{\delta}\frac{\partial(u/U)}{\partial(y/\delta)}\bigg]_{y=0} = \mu\frac{U}{\delta}\frac{\pi}{2}\cos\frac{\pi}{2}\eta\bigg]_{\eta=0} = \frac{\pi\mu U}{2\delta}$$

Therefore,

$$\tau_w = \frac{\pi\mu U}{2\delta} = 0.137\rho U^2\frac{d\delta}{dx}$$

Separating variables gives

$$\delta\, d\delta = 11.5\frac{\mu}{\rho U}dx$$

Integrating, we obtain

$$\frac{\delta^2}{2} = 11.5\frac{\mu}{\rho U}x + c$$

But $c = 0$, since $\delta = 0$ at $x = 0$, so

$$\delta = \sqrt{23.0\frac{x\mu}{\rho U}}$$

or

$$\frac{\delta}{x} = 4.80\sqrt{\frac{\mu}{\rho U x}} = \frac{4.80}{\sqrt{Re_x}} \longleftarrow \underline{\delta(x)}$$

The displacement thickness, δ^*, is given by

$$\delta^* = \delta\int_0^1\left(1-\frac{u}{U}\right)d\eta$$

$$= \delta\int_0^1\left(1-\sin\frac{\pi}{2}\eta\right)d\eta = \delta\left[\eta + \frac{2}{\pi}\cos\frac{\pi}{2}\eta\right]_0^1$$

$$\delta^* = \delta\left[1-0+0-\frac{2}{\pi}\right] = \delta\left[1-\frac{2}{\pi}\right]$$

Since, from part (a),

$$\frac{\delta}{x} = \frac{4.80}{\sqrt{Re_x}}$$

then

$$\frac{\delta^*}{x} = \left(1 - \frac{2}{\pi}\right)\frac{4.80}{\sqrt{Re_x}} = \frac{1.74}{\sqrt{Re_x}} \qquad \delta^*(x)$$

The total friction force on one side of the plate is given by

$$F = \int_{A_p} \tau_w \, dA$$

Since $dA = b\,dx$ and $0 \leq x \leq L$, then

$$F = \int_0^L \tau_w b\,dx = \int_0^L \rho U^2 \frac{d\theta}{dx} b\,dx = \rho U^2 b \int_0^{\theta_L} d\theta = \rho U^2 b \theta_L$$

$$\theta_L = \int_0^{\delta_L} \frac{u}{U}\left(1 - \frac{u}{U}\right) dy = \delta_L \int_0^1 \frac{u}{U}\left(1 - \frac{u}{U}\right) d\eta = \beta \delta_L$$

From part (a), $\beta = 0.137$ and $\delta_L = \dfrac{4.80L}{\sqrt{Re_L}}$, so

$$F = \frac{0.658 \rho U^2 b L}{\sqrt{Re_L}} \qquad F$$

This problem illustrates application of the momentum integral equation to the laminar boundary layer on a flat plate.

The *Excel* workbook for this Example plots the growth of δ and δ^* in the boundary layer, and the exact solution (Eq. 9.13 on the Web). It also shows wall shear stress distributions for the sinusoidal velocity profile and the exact solution.

Pressure Gradients In Boundary-Layer Flow

The boundary layer (laminar or turbulent) with a uniform flow along an infinite flat plate is the easiest one to study because the pressure gradient is zero—the fluid particles in the boundary layer are slowed only by shear stresses, leading to boundary-layer growth. We now consider the effects caused by a pressure gradient, which will be present for all bodies except, as we have seen, a flat plate.

A favorable pressure gradient is one in which the pressure decreases in the flow direction (i.e., $\frac{\partial p}{\partial x} < 0$, arising when the free-stream velocity U is increasing with x, for example in a converging flow field) will tend to counteract the slowing of fluid particles in the boundary layer.

On the other hand an adverse pressure gradient in which pressure increases in the flow direction (i.e., $\frac{\partial p}{\partial x} > 0$, when U is decreasing with x, for example in a diverging flow field) will tend to contribute to the slowing of the fluid particles. If the adverse pressure gradient is severe enough, the fluid particles in the boundary layer will actually be brought to rest. When this occurs, the particles will be forced away from the body surface (a phenomenon called flow **separation**) as they make room for following particles, ultimately leading to a wake in which flow is turbulent.

This description, of the adverse pressure gradient and friction in the boundary layer together forcing flow separation, certainly makes intuitive sense; the question arises whether we can more formally see when this occurs. For example, can we have flow separation and a wake for uniform flow over a flat plate, where $\frac{\partial p}{\partial x} = 0$? We can gain insight into this question by considering when the velocity in the boundary layer will become zero. Consider the velocity u in the boundary layer at an infinitesimal distance Δy above the plate. This will be

$$u_{y=\Delta y} = u_0 + \left.\frac{\partial u}{\partial y}\right)_{y=0} \Delta y = \left.\frac{\partial u}{\partial y}\right)_{y=0} \Delta y$$

where u_0 = 0 is the velocity at the surface of the plate. It is clear that $u_{y=\Delta y}$ will ba zero (i.e., separation will occur) only when $\left.\frac{\partial u}{\partial y}\right)_{y=0}$ = 0. Hence, we can use this as our litmus test for flow separation. We recall that the velocity gradient near the surface in a laminar boundary layer, and in the viscous sublayer of a turbulent boundary layer, was related to the wall shear stress by

$$\tau_w = \mu \left.\frac{\partial u}{\partial y}\right)_{y=0}$$

Further, we learned in the previous section that the wall shear stress for the flat plate is given by $\frac{\tau_w(x)}{\rho U^2} = \frac{constant}{\sqrt{Re_x}}$

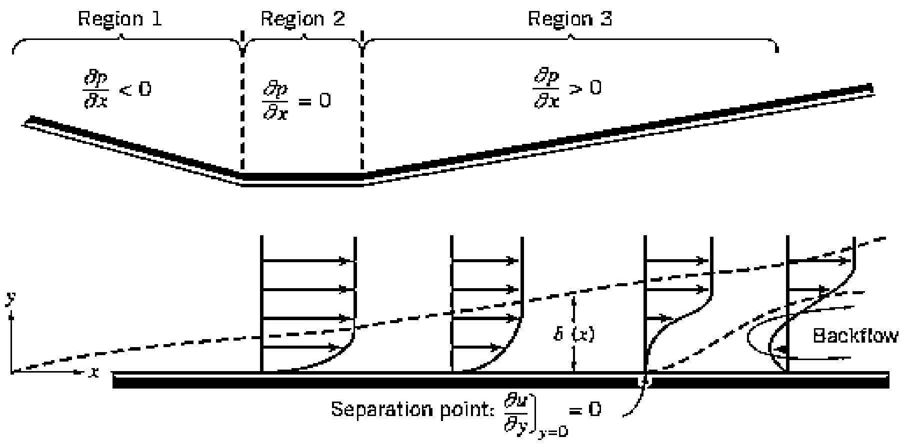
For a laminar boundary layer and $\frac{\tau_w(x)}{\rho U^2} = \frac{constant}{\sqrt[5]{Re_x}}$

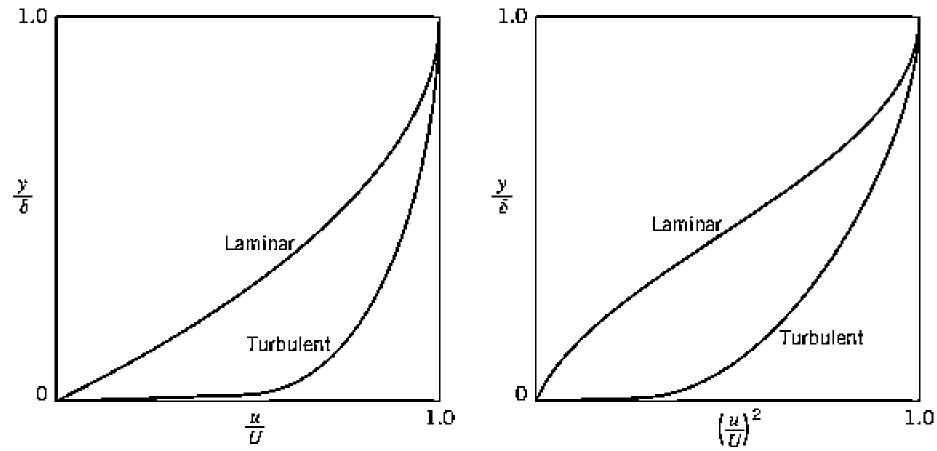
For a turbulent boundary layer. We see that for the flow over a flat plate, the wall stress is always τ_w >0. Hence, $\left.\frac{\partial u}{\partial y}\right)_{y=0}$ > 0 always; and therefore, finally, $u_{y=\Delta y}$ > 0 always. We conclude that for uniform flow over a flat plate the flow never separates, and we never develop a wake region, whether the boundary layer is laminar or turbulent, regardless of plate length.

We conclude that flow will not separate for flow over a flat plate, when $\frac{\partial p}{\partial x} = 0$. Clearly, for flows in which $\frac{\partial p}{\partial x} < 0$ (whenever the free-stream velocity is increasing), we can be sure that there will be no flow separation; for flows in which $\frac{\partial p}{\partial x} > 0$ (i.e., adverse pressure gradients) we could have flow separation. We should not conclude that an adverse pressure gradient always leads to flow separation and a wake; we have only concluded that it is a necessary condition for flow separation to occur.

To illustrate these results consider the variable cross-sectional flow shown in Figure. Outside the boundary layer the velocity field is one in which the flow accelerates (Region 1), has a constant velocity region (Region 2), and then a deceleration region (Region 3). Corresponding to these, the pressure gradient is favorable, zero, and adverse, respectively, as shown. (Note that the straight wall is not a simple flat plate— it has these various pressure gradients because the flow above the wall is not a uniform flow.) From our discussions above, we conclude that separation cannot occur in Region 1 or 2, but can (as shown) occur in Region 3. Could we avoid flow separation in a device like this? Intuitively, we can see that if we make the divergent section less severe, we may be able to eliminate flow separation. In other words, we may eliminate flow separation if we sufficiently reduce the magnitude of the adverse pressure gradient $\frac{\partial p}{\partial x}$. The final question remaining is how small the adverse pressure gradient needs to be to accomplish this. This, and a more rigorous proof that we must have $\frac{\partial p}{\partial x} > 0$ for a chance of flow separation, is beyond the scope of this text [3]. We conclude that flow separation is possible, but not guaranteed, when we have an adverse pressure gradient.

The nondimensional velocity profiles for laminar and turbulent boundary-layer flow over a flat plate are shown in Figure (a). The turbulent profile is much fuller (more blunt) than the laminar profile. At the same freestream speed, the momentum flux within the turbulent boundary layer is greater than within the laminar layer (Figure (b)).





(*a*) Velocity profiles (*b*) Momentum-flux profiles

Separation occurs when the momentum of fluid layers near the surface is reduced to zero by the combined action of pressure and viscous forces. As shown in Figure (b), the momentum of the fluid near the surface is significantly greater for the turbulent profile. Consequently, the turbulent layer is better able to resist separation in an adverse pressure gradient.

Adverse pressure gradients cause significant changes in velocity profiles for both laminar and turbulent boundary-layer flows. Approximate solutions for nonzero pressure gradient flow may be obtained from the momentum integral equation

$$\frac{\tau_w}{\rho} = \frac{d}{dx}(U^2\theta) + \delta^* U \frac{dU}{dx}$$

Expanding the first term, we can write

$$\frac{\tau_w}{\rho} = U^2 \frac{d\theta}{dx} + (\delta^* + 2\theta)\, U \frac{dU}{dx}$$

or

$$\frac{\tau_w}{\rho U^2} = \frac{C_f}{2} = \frac{d\theta}{dx} + (H + 2)\, \frac{\theta}{U} \frac{dU}{dx}$$

Where $H = \frac{\delta^*}{\theta}$ is a velocity-profile "shape factor." The shape factor increases in an adverse pressure gradient. For turbulent boundary-layer flow, H increases from 1.3 for a zero pressure gradient to approximately 2.5 at separation. For laminar flow with zero pressure gradient, H = 2.6; at separation H = 3.5.

The freestream velocity distribution, U(x), must be known before above Equation can be applied. Since $\frac{dp}{dx} = -\rho U \frac{dU}{dx}$, specifying U(x) is equivalent to specifying the pressure gradient. We can obtain a first approximation for U(x) from ideal flow theory for an inviscid flow under the same conditions.

Kelvin's Minimum Energy Theorem:

Statement:

The kinetic energy (K.E) of an irrotational flow for an incompressible fluid occupying the connected region is less than the K.E of any other flow of the fluid having the same normal velocity.

Proof:

Let S be the simply connected region enclosing a volume τ of an incompressible fluid, Let V be the velocity of fluid. Since the flow is irrotational. Therefore,

$$\overline{V} = -\nabla\phi$$

From equation of continuity $\frac{D\rho}{Dt} + \rho\left(\nabla.\overline{V}\right) = 0$ _____(i)

Since the fluid is incompressible $\Rightarrow$ $\frac{D\rho}{Dt} = 0$

Eq (i) becomes $\rho\left(\nabla.\overline{V}\right) = 0$

$$\nabla.\overline{V} = 0 \qquad \_\_\_\_\_(ii)$$

Let T be the kinetic energy for the flow then

$$T = \frac{1}{2}\iiint_{\tau} \rho V^2 d\tau$$

$$\because V^2 = \left|\overline{V}\right|^2 = \overline{V}^2$$

$$T = \frac{\rho}{2}\iiint_{\tau} \overline{V}^2 d\tau \qquad \_\_\_\_(iii)$$

Let T' and V' be the K.E and velocity of any other flow of the fluid respectively. So, that

$$V' = \overline{V} + \overline{V_0}$$

From equation of continuity

$$\frac{D\rho}{Dt}+\rho\left(\nabla.V'\right)=0$$

Since the fluid is incompressible i.e. $\frac{D\rho}{Dt}=0$

$$\Rightarrow \quad \rho\left(\nabla.V'\right)=0$$

$$\Rightarrow \quad \nabla.V'=0$$

$$\Rightarrow \quad \nabla.\left(\overline{V}+\overline{V_0}\right)=0$$

$$\Rightarrow \quad \nabla\overline{V}+\nabla\overline{V_0}=0 \qquad \_\_\_\_(iv)$$

It is also given that the flow has same normal velocity

$$\overline{V}.\hat{n}=V'.\hat{n}$$

$$\overline{V}.\hat{n}=\left(\overline{V}+\overline{V_0}\right).\hat{n}$$

$$\overline{V}.\hat{n}=\overline{V}.\hat{n}+\overline{V_0}.\hat{n}$$

$$\overline{V_0}.\hat{n}=0 \qquad \_\_\_(v)$$

The K.E T' of any other flow is

$$T'=\frac{1}{2}\iiint_\tau \rho V'^2 d\tau$$

$$T'=\frac{\rho}{2}\iiint_\tau\left(\overline{V}+\overline{V_0}\right)^2 d\tau$$

$$T'=\frac{\rho}{2}\iiint_\tau\left(\overline{V^2}+\overline{V_0^2}+2\overline{V}.\overline{V_0}\right)d\tau$$

$$T'=\frac{\rho}{2}\iiint_\tau\overline{V^2}d\tau+\frac{\rho}{2}\iiint_\tau\overline{V_0^2}d\tau+\rho\iiint_\tau\left(\overline{V}.\overline{V_0}\right)d\tau$$

$$T'=T+T_0+\rho\iiint_\tau\left(\overline{V}.\overline{V_0}\right)d\tau \quad \_\_\_\_\_(vi) \qquad \because by\ (iii)$$

Since the flow is irrotational $\overline{V} = -\nabla\phi$

$$T' = T + T_0 + \rho\iiint_\tau \left(-\nabla\phi.\overline{V_0}\right)d\tau$$

$$T' = T + T_0 - \rho\iiint_\tau \left(\nabla\phi.\overline{V_0}\right)d\tau \qquad \_\_\_\_(vii)$$

$$Since \quad \nabla\left(\phi\overline{V_0}\right) = \phi\nabla.\overline{V_0} + \nabla\phi.\overline{V_0}$$

$$\nabla\left(\phi\overline{V_0}\right) - \phi\nabla.\overline{V_0} = \nabla\phi.\overline{V_0}$$

$$eq\,(vii) \Rightarrow \quad T' = T + T_0 - \rho\iiint_\tau \left(\nabla\left(\phi\overline{V_0}\right) - \phi\nabla.\overline{V_0}\right)d\tau$$

From Eq (ii) $\nabla.V = 0 \Rightarrow \nabla.V_0 = 0$

$$T' = T + T_0 - \rho\iiint_\tau \nabla.\left(\phi\overline{V_0}\right)d\tau \qquad \_\_\_\_\_(viii)$$

By using the Gauss Divergence theorem

$$\iiint_V \nabla.\overline{A}\,dV = \iint_S \overline{A}.\hat{n}\,dS$$

$$T' = T + T_0 - \rho\iint_S \phi\overline{V_0}.\hat{n}\,dS$$

From eq (v) $\overline{V_0}.\hat{n} = 0$

$$T' = T + T_0$$

$$\Rightarrow T' > T$$

$$Or \quad T < T'$$

حرفِ آخر (03-12-2021)

خوش رہیں خوشیاں بانٹیں اور جہاں تک ہو سکے دوسروں کے لیے آسانیاں پیدا کریں۔

اللہ تعالیٰ آپ کو زندگی کے ہر موڑ پر کامیابیوں اور خوشیوں سے نوازے۔ (آمین)

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