MULTIVARIABLE CALCULUS

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Preface

Multivariable calculus is a fundamental subject that extends the concepts of single-variable calculus to higher-dimensional spaces. It provides a powerful framework for analyzing and modeling complex phenomena in fields such as physics, engineering, economics, and computer science.

This textbook is designed to provide a comprehensive introduction to multivariable calculus, covering topics such as Vectors, Functions, partial derivatives, multiple integrals, and differential equations, Laplace and Fourier Transformations, Sequence, Series and Complex Integration. Through a combination of theoretical foundations, practical applications, and numerous examples and exercises, we aim to equip students with a deep understanding of the subject matter and its relevance to real-world problems.

Throughout the book, we emphasize the development of problem-solving skills, critical thinking, and mathematical maturity. We also highlight the connections between multivariable calculus and other areas of mathematics, such as linear algebra and differential equations.

Our goal is to make this textbook a valuable resource for students, instructors, and researchers alike, providing a solid foundation for further study and exploration in mathematics, science, and engineering.

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Superior Group of Colleges

Sillanwali, Sargodha

Purpose of this course is to develop the skills to have ground knowledge of multivariate calculus and appreciation for their further computer science courses.

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MULTI VARIABLE FUNCTIONS AND PARTIAL DERIVATIVES

In this chapter we will learn about;

- Functions of Several Variables
- Limits and Continuity
- Partial Derivatives
- Differentiability
- Tangent Planes and Linear Approximations
- The Chain Rule
- Partial Derivatives with Constrained Variables
- Directional Derivatives and the Gradient Vector
- Maximum and Minimum Values
- Extreme Values, Saddle Points, Stationary Points, Critical Points
- Lagrange Multipliers
- Taylor's Formula

Functions of Several Variables

In this section we study functions of two or more variables from four points of view:

■ verbally (by a description in words)

■ numerically (by a table of values)

■ algebraically (by an explicit formula)

■ visually (by a graph or level curves)

Functions of Two Variables

A function of two variables is a rule that assigns to each ordered pair of real numbers (x,y) in a set D a unique real number denoted by *f(x,y)*. The set D is the domain of *f* and its range is the set of values that *f* takes on, that is, $\{f(x, y): (x, y) \in D\}$

We often write z = *f(x,y)* to make explicit the value taken on by *f* at the general point *(x,y)*. The variables x and y are independent variables and z is the dependent variable.

Example

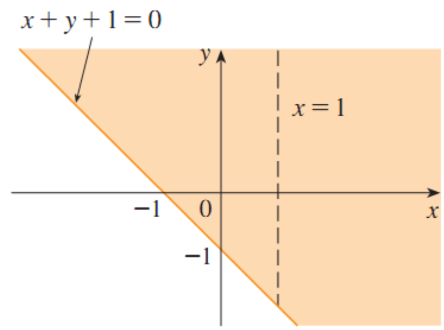
For the following function, evaluate and find and sketch the domain.

$$f(x, y) = \frac{\sqrt{x + y + 1}}{x - 1}$$

Solution

$$f(3, 2) = \frac{\sqrt{3 + 2 + 1}}{3 - 1} = \frac{\sqrt{6}}{2}$$

$$D = \{(x, y) \mid x + y + 1 \geqslant 0, \; x \neq 1\}$$



Example

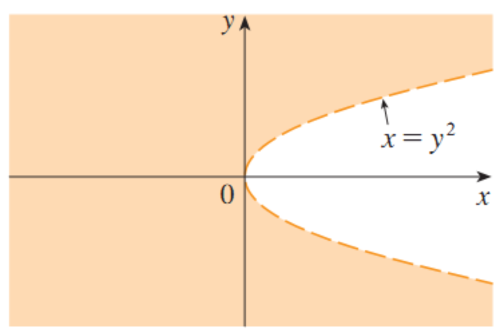
For the following function, evaluate and find and sketch the domain.

$$f(x, y) = x \ln(y^2 - x)$$

Solution

$$f(3, 2) = 3 \ln(2^2 - 3) = 3 \ln 1 = 0$$

$$D = \{(x, y) \mid x < y^2\}$$



Example

In regions with severe winter weather, the wind-chill index is often used to describe the apparent severity of the cold. This index W is a subjective temperature that depends on the actual temperature T and the wind speed v. So W is a function of T and v, and we can write W=*f* (T,v). Table records values of W compiled by the National Weather Service of the US and the Meteorological Service of Canada.

Wind speed (km/h)

Actual temperature (°C)

T \ v	5	10	15	20	25	30	40	50	60	70	80
5	4	3	2	1	1	0	−1	−1	−2	−2	−3
0	−2	−3	−4	−5	−6	−6	−7	−8	−9	−9	−10
−5	−7	−9	−11	−12	−12	−13	−14	−15	−16	−16	−17
−10	−13	−15	−17	−18	−19	−20	−21	−22	−23	−23	−24
−15	−19	−21	−23	−24	−25	−26	−27	−29	−30	−30	−31
−20	−24	−27	−29	−30	−32	−33	−34	−35	−36	−37	−38
−25	−30	−33	−35	−37	−38	−39	−41	−42	−43	−44	−45
−30	−36	−39	−41	−43	−44	−46	−48	−49	−50	−51	−52
−35	−41	−45	−48	−49	−51	−52	−54	−56	−57	−58	−60
−40	−47	−51	−54	−56	−57	−59	−61	−63	−64	−65	−67

For instance, the table shows that if the temperature is -5°C and the wind speed is 50 km/h, then subjectively it would feel as cold as a temperature of about -15°C with no wind. So

$$f(-5, 50) = -15$$

Example

In 1928 Charles Cobb and Paul Douglas published a study in which they modeled the growth of the American economy during the period 1899–1922. They considered a simplified view of the economy in which production output is determined by the amount of labor involved and the amount of capital invested. While there are many other factors affecting economic performance, their model proved to be remarkably accurate. The function they used to model production was of the form

$$P(L, K) = bL^{\alpha}K^{1-\alpha}$$

where P is the total production (the monetary value of all goods produced in a year), L is the amount of labor (the total number of person-hours worked in a year), and K is the amount of capital invested (the monetary worth of all machinery, equipment, and buildings).

Cobb and Douglas used economic data published by the government to obtain Table. They took the year 1899 as a baseline and P, L, and K for 1899 were each assigned the value 100. The values for other years were expressed as percentages of the 1899 figures.

Year	P	L	K
1899	100	100	100
1900	101	105	107
1901	112	110	114
1902	122	117	122
1903	124	122	131
1904	122	121	138
1905	143	125	149
1906	152	134	163
1907	151	140	176
1908	126	123	185
1909	155	143	198
1910	159	147	208
1911	153	148	216
1912	177	155	226
1913	184	156	236
1914	169	152	244
1915	189	156	266
1916	225	183	298
1917	227	198	335
1918	223	201	366
1919	218	196	387
1920	231	194	407
1921	179	146	417
1922	240	161	431

Cobb and Douglas used the method of least squares to fit the data of Table to the function

$$P(L, K) = 1.01L^{0.75}K^{0.25}$$

If we use the model given by the function in previous equation to compute the production in the years 1910 and 1920, we get the values

$$P(147, 208) = 1.01(147)^{0.75}(208)^{0.25} \approx 161.9$$

$$P(194, 407) = 1.01(194)^{0.75}(407)^{0.25} \approx 235.8$$

which are quite close to the actual values, 159 and 231.
The production function

$$P(L, K) = bL^{\alpha}K^{1-\alpha}$$

has subsequently been used in many settings, ranging from individual firms to global economics. It has become known as the **Cobb-Douglas production function**. Its domain is $\{(L, K): L \geq 0, K \geq 0\}$ because L and K represent labor and capital and are therefore never negative.

Example

Find the domain and range of $g(x, y) = \sqrt{9 - x^2 - y^2}$.

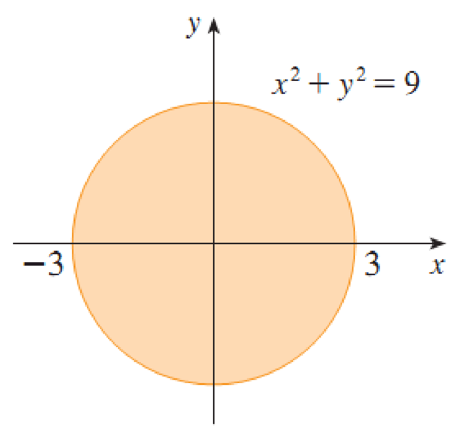
Solution

Domain is

$$D = \{(x, y) \mid 9 - x^2 - y^2 \geqslant 0\} = \{(x, y) \mid x^2 + y^2 \leqslant 9\}$$

Range is

$$\{z \mid 0 \leqslant z \leqslant 3\} = [0, 3]$$



Graph

If is a function of two variables with domain D, then the graph of $\boldsymbol{f}$ is the set of all points (x,y,z) in $\mathbf{R}^3$ such that z = f(x,y) and (x,y) is in D.

Linear Function

A function of the form $f(x, y) = ax + by + c$ is called a linear function. The graph of such a function has the equation

$z = ax + by + c$ or $ax + by - z + c = 0$

so it is a plane. In much the same way that linear functions of one variable are important in single-variable calculus, we will see that linear functions of two variables play a central role in multivariable calculus.

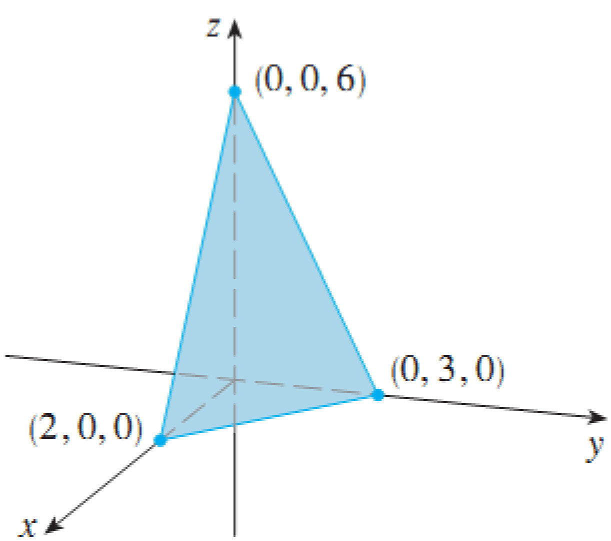
Example

Sketch the graph of the function $f(x, y) = 6 - 3x - 2y$.

Solution

The graph of f has the equation $z = 6 - 3x - 2y$, or $3x + 2y + z = 6$, which represents a plane. To graph the plane we first find the intercepts.

Putting $y = z = 0$ in the equation, we get x = 2 as the -intercept. Similarly, the y - intercept is 3 and the z-intercept is 6. This helps us sketch the portion of the graph that lies in the first octant in Figure.

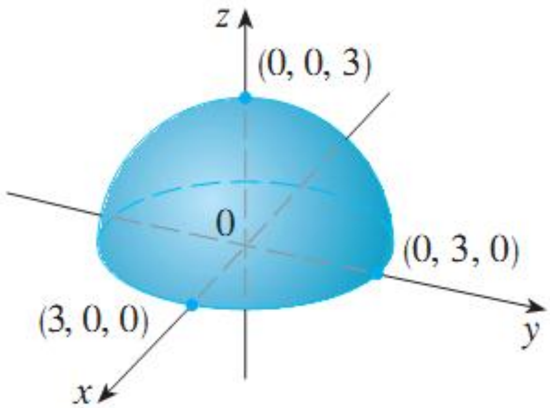


Example

Sketch the graph of $g(x, y) = \sqrt{9 - x^2 - y^2}$

Solution

The graph has equation $z = \sqrt{9 - x^2 - y^2}$.
We square both sides of this equation to obtain $z^2 = 9 - x^2 - y^2$,
or $x^2 + y^2 + z^2 = 9$, which we recognize as an equation of the sphere with center the origin and radius 3. But, since $z \geq 0$, the graph of g is just the top half of this sphere.



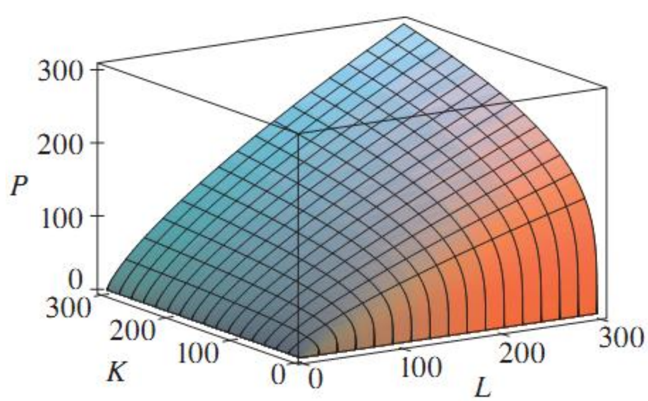
Example

Draw the graph of the Cobb-Douglas production function

$P(L, K) = 1.01L^{0.75}K^{0.25}$

Solution

Figure shows the graph of P for values of the labor L and capital K that lie between 0 and 300. The computer has drawn the surface by plotting vertical traces. We see from these traces that the value of the production P increases as either L or K increases, as is to be expected.



Example

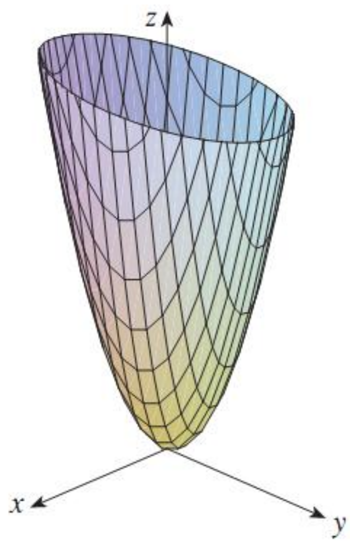
Find the domain and range and sketch the graph of $h(x, y) = 4x^2 + y^2$.

Solution

Domain is R^2

The range of h is $[0, \infty)$ the set of all non-negative real numbers.

The graph of h has the equation $z = 4x^2 + y^2$, which is the elliptic paraboloid. Horizontal traces are ellipses and vertical traces are parabolas.



Remark

So far we have two methods for visualizing functions: arrow diagrams and graphs. A third method, borrowed from mapmakers, is a contour map on which points of constant elevation are joined to form contour lines, or level curves.

Level Curves

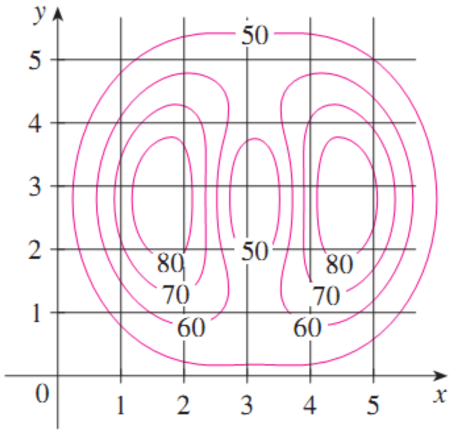
The level curves of a function f of two variables are the curves with equations $(x, y) = k$, where k is a constant (in the range of f).

Or

A level curve $f(x, y) = k$ is the set of all points in the domain of f at which f takes on a given value k. In other words, it shows where the graph of f has height k.

Example

A contour map for a function f is shown in Figure. Use it to estimate the values of $f(1,3)$ and $f(4,5)$.



Solution

The point (1, 3) lies partway between the level curves with z-values 70 and 80. We estimate that

$$f(1,3) \approx 73$$

Similarly, we estimate that

$$f(4,5) \approx 56$$

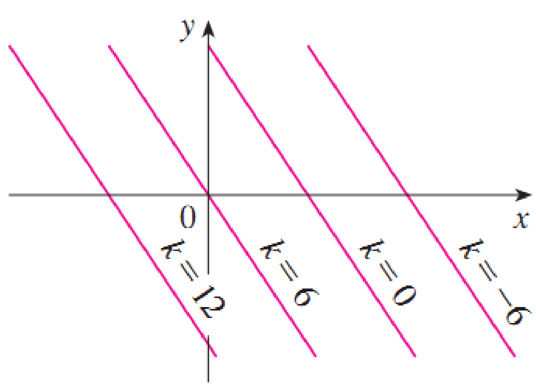
Example

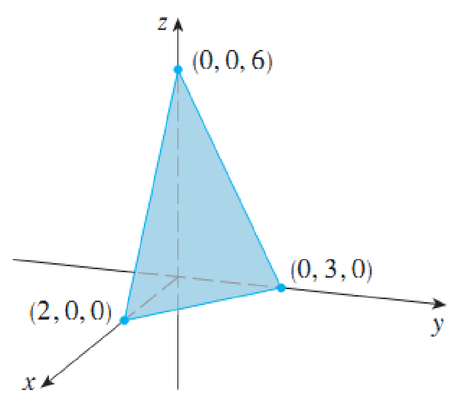
Sketch the level curves of the function $f(x,y) = 6 - 3x - 2y$ for the values $k = -6,0,6,12$

Solution

The level curves are $6 - 3x - 2y = k$ or $3x + 2y + (k - 6) = 0$

This is a family of lines with slope $-\frac{3}{2}$. The four particular level curves with $k = -6,0,6,12$ are $3x + 2y - 12 = 0, 3x + 2y - 6 = 0$ and $3x + 2y = 0$. They are sketched. The level curves are equally spaced parallel lines because the graph of f is a plane (see Figure on right).

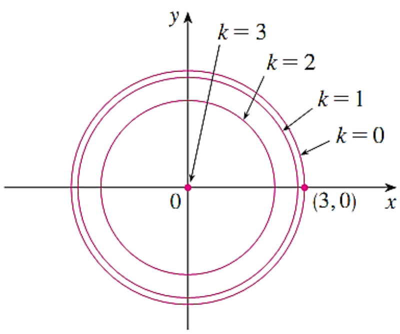


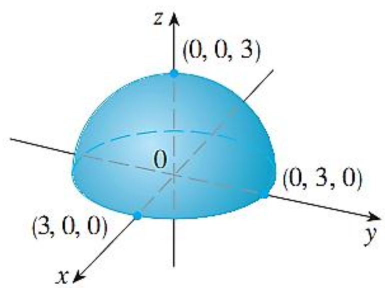


Example
Sketch the level curves of the function

$$g(x, y) = \sqrt{9 - x^2 - y^2} \qquad \text{for} \quad k = 0, 1, 2, 3$$

Solution
The level curves are $\sqrt{9 - x^2 - y^2} = k$ or $x^2 + y^2 = 9 - k^2$
This is a family of concentric circles with center (0,0) and radius $\sqrt{9 - k^2}$. The cases $k = 0,1,2,3$ are shown in Figure. Try to visualize these level curves lifted up to form a surface and compare with the graph of g (a hemisphere) in Figure on right.

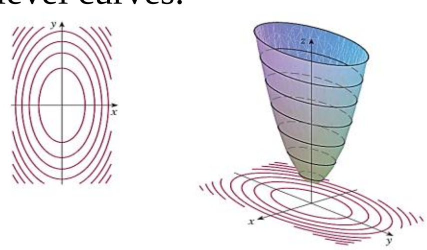




Example
Sketch the level curves of the function $f(x, y) = 4x^2 + y^2 + 1$
Solution
The level curves are

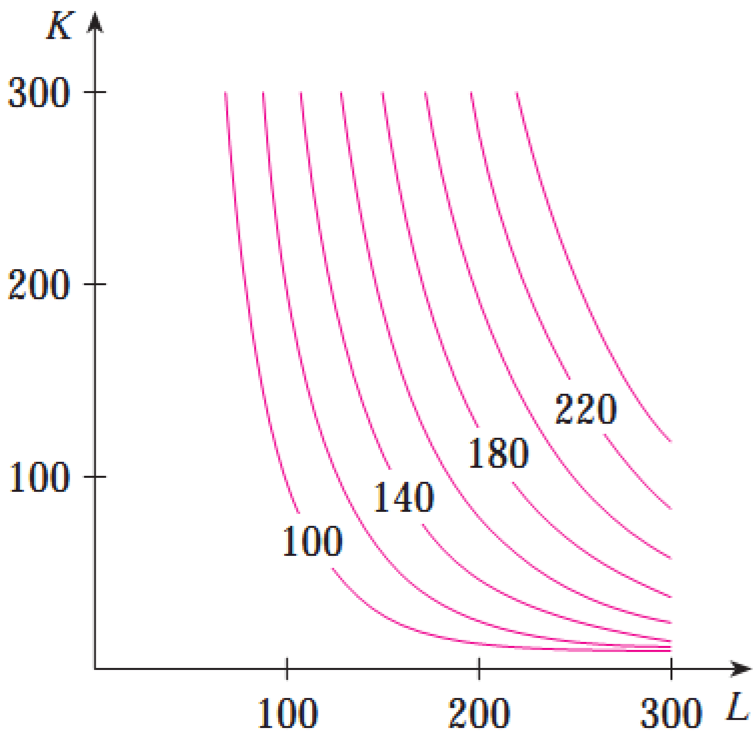
$$4x^2 + y^2 + 1 = k \qquad \text{or} \qquad \frac{x^2}{\frac{1}{4}(k-1)} + \frac{y^2}{k-1} = 1$$

which, for k > 1, describes a family of ellipses with semiaxes $\frac{1}{2}\sqrt{k-1}$ and $\sqrt{k-1}$.
Figure (a) shows a contour map of h drawn by a computer. Figure (b) shows these level curves lifted up to the graph of h (an elliptic paraboloid) where they become horizontal traces. We see from Figure how the graph of h is put together from the level curves.



(a) Contour map (b) Horizontal traces are raised level curves

Example
Plot level curves for the Cobb-Douglas production function.
Solution
In Figure



we use a computer to draw a contour plot for the Cobb-Douglas production function

$$P(L, K) = 1.01 L^{0.75} K^{0.25}$$

Level curves are labeled with the value of the production P. For instance, the level curve labeled 140 shows all values of the labor L and capital investment K that result in a production of P = 140. We see that, for a fixed value of P, as L increases K decreases, and vice versa.

Functions of Three or More Variables

A function of three variables, f, is a rule that assigns to each ordered triple (x, y, z) in a domain $D \subset R^3$ a unique real number denoted by $f(x, y, z)$. For instance, the temperature T at a point on the surface of the earth depends on the longitude x and latitude y of the point and on the time t, so we could write $T = f(x, y, t)$.

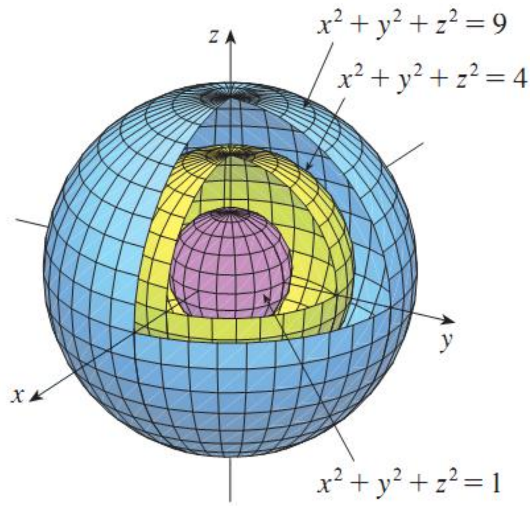
Example
Find the domain of f if

$$f(x, y, z) = \ln(z - y) + xy \sin z$$

Solution

$$D = \{(x, y, z) \in \mathbb{R}^3 \mid z > y\}$$

This is a half-space consisting of all points that lie above the plane z = y.
Example
Find the level surfaces of the function $f(x, y, z) = x^2 + y^2 + z^2$
Solution
The level surfaces are $x^2 + y^2 + z^2 = k$, where $k \geq 0$. These form a family of concentric spheres with radius $\sqrt{k}$. (See Figure)



Thus, as (x, y, z) varies over any sphere with center O, the value of $f(x, y, z)$ remains fixed.

Functions of n – Variables

A function of n variables, is a rule that assigns a number $z = f(x_1, x_2, x_3, \dots, x_n)$ to an –tuple $(x_1, x_2, x_3, \dots, x_n)$ of real numbers. We denote by $\mathbf{R^n}$ the set of all such n – tuples.

For example, if a company uses different ingredients in making a food product, c_i is the cost per unit of the i^{th} ingredient, and x_i units of the i^{th} ingredient are used, then the total cost C of the ingredients is a function of the n variables $x_1, x_2, x_3, \dots, x_n$:

$$C = f(x_1, x_2, \dots, x_n) = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n$$

Remember

In view of the one-to-one correspondence between points $(x_1, x_2, x_3, \dots, x_n)$ in R^n and their position vectors $\boldsymbol{x} = \langle x_1, x_2, x_3, \dots, x_n \rangle$ in V_n, we have three ways of looking at a function f defined on a subset of R^n:

1. As a function of real variables $x_1, x_2, x_3, \dots, x_n$

2. As a function of a single point variable $(x_1, x_2, x_3, \dots, x_n)$

3. As a function of a single vector variable $\boldsymbol{x} = \langle x_1, x_2, x_3, \dots, x_n \rangle$

Limits

Let f be a function of two variables whose domain D includes points arbitrarily close to (a,b). Then we say that the limit of f(x,y) as (x,y) approaches (a,b) is L and we write

$\text{lmt}_{(x,y)\to(a,b)}\ f(x,y) = L$

if for every number $\in > 0$ there is a corresponding number $\delta > 0$ such that

$$\text{if} \quad (x, y) \in D \quad \text{and} \quad 0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta \quad \text{then} \quad |f(x, y) - L| < \varepsilon$$

Remark

If $f(x,y)\to L_1$ as $(x,y)\to(a,b)$ along a path C_1 and $f(x,y)\to L_2$ as $(x,y)\to(a,b)$ along a path C_2, where $L_2 \neq L_2$, then limit does not exist.

Example

Show that $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2 - y^2}{x^2 + y^2}$ does not exist.

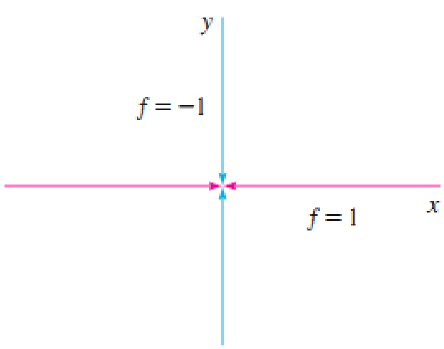
Solution

Given that $f(x,y) = \frac{x^2-y^2}{x^2+y^2}$

Along Horizontal Axis: put y = 0; $\lim_{(x,y)\to(x,0)} f(x,y) = \lim\limits_{(x,y)\to(x,0)} \frac{x^2-y^2}{x^2+y^2} = 1$

Along Vertical Axis: put x = 0; $\lim_{(x,y)\to(0,y)} f(x,y) = \lim\limits_{(x,y)\to(0,y)} \frac{x^2-y^2}{x^2+y^2} = -1$

Since we have obtained different limits along different paths, the given limit does not exist.



Example

If $f(x, y) = xy/(x^2 + y^2)$, does $\lim\limits_{(x, y)\to(0, 0)} f(x, y)$ exist?

Solution

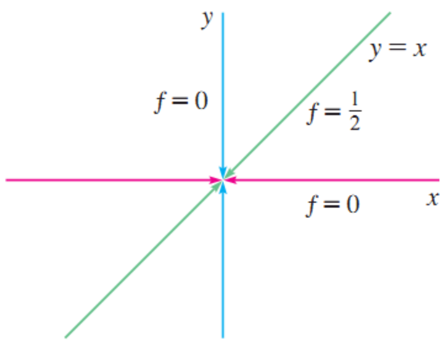
Given that $f(x,y) = \frac{xy}{x^2+y^2}$

Along Horizontal Axis: put y = 0; $\lim_{(x,y)\to(x,0)} f(x,y) = \lim\limits_{(x,y)\to(x,0)} \frac{xy}{x^2+y^2} = 0$

Along Vertical Axis: put x = 0; $\lim_{(x,y)\to(0,y)} f(x,y) = \lim\limits_{(x,y)\to(0,y)} \frac{xy}{x^2+y^2} = 0$

Along the line y = x; $\lim_{(x,y)\to(x,x)} f(x,y) = \lim\limits_{(x,y)\to(x,x)} \frac{xy}{x^2+y^2} = \frac{1}{2}$

Since we have obtained different limits along different paths, the given limit does not exist.



Example

If $f(x, y) = \dfrac{xy^2}{x^2 + y^4}$, does $\lim\limits_{(x, y)\to(0, 0)} f(x, y)$ exist?

Solution

Given that $f(x,y) = \frac{xy^2}{x^2+y^4}$

Along the line y = mx; $\lim_{(x,y)\to(x,mx)} f(x,y) = \lim\limits_{(x,y)\to(x,mx)} \frac{xy^2}{x^2+y^4} = \frac{m^2x}{1+m^4x^2}$

Thus f has the same limiting value along every nonvertical line through the origin. But that does not show that the given limit is 0, for if we now let $(x,y) \to (0,0)$ along the parabola $x = y^2$, we have

$\lim_{(x,y)\to(y^2,mx)} f(x,y) = \lim\limits_{(x,y)\to(y^2,mx)} \frac{m^2x}{1+m^4x^2} = \frac{1}{2}$

Since different paths lead to different limiting values, the given limit does not exist.

Example

Find $\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2}$ if it exists.

Solution

Let $\in > 0$. We want to find $\delta > 0$ such that

$$\text{if} \quad 0 < \sqrt{x^2+y^2} < \delta \quad \text{then} \quad \left| \frac{3x^2y}{x^2+y^2} - 0 \right| < \varepsilon$$

that is, $$\text{if} \quad 0 < \sqrt{x^2+y^2} < \delta \quad \text{then} \quad \frac{3x^2|y|}{x^2+y^2} < \varepsilon$$

But $x^2 \leqslant x^2 + y^2$ since $y^2 \geqslant 0$, so $x^2/(x^2+y^2) \leqslant 1$ and therefore

$$\frac{3x^2|y|}{x^2+y^2} \leqslant 3|y| = 3\sqrt{y^2} \leqslant 3\sqrt{x^2+y^2}$$

Thus if we choose $\delta = \varepsilon/3$ and let $0 < \sqrt{x^2+y^2} < \delta$, then

$$\left| \frac{3x^2y}{x^2+y^2} - 0 \right| \leqslant 3\sqrt{x^2+y^2} < 3\delta = 3\left(\frac{\varepsilon}{3}\right) = \varepsilon$$

Hence, by Definition,

$$\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} = 0$$

Continuity

A function f of two variable is said to be continuous at (a,b) if

$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$

We say f is continuous on D if f is continuous at every point (a,b) in D.

Example

Evaluate $\lim_{(x,y)\to(1,2)} (x^2y^3 - x^3y^2 + 3x + 2y)$

Solution

Since this is a polynomial, it is continuous everywhere, so we can find the limit by direct substitution:

$$\lim_{(x,y)\to(1,2)} (x^2y^3 - x^3y^2 + 3x + 2y) = 1^2 \cdot 2^3 - 1^3 \cdot 2^2 + 3 \cdot 1 + 2 \cdot 2 = 11$$

Example

Where is the function $f(x,y) = \dfrac{x^2 - y^2}{x^2 + y^2}$ continuous?

Solution

The function f is discontinuous at (0,0) because it is not defined there. Since f is a rational function, it is continuous on its domain, which is the set

$$D = \{(x,y) \mid (x,y) \neq (0,0)\}$$

Example (Previously Solved)

Discuss the continuity of

$$g(x,y) = \begin{cases} \dfrac{x^2 - y^2}{x^2 + y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

Solution

Here g is defined at (0,0) but is still discontinuous there because $\lim_{(x,y)\to(0,0)} g(x,y)$ does not exist.

Example (Previously Solved)
Discuss the continuity of

$$f(x, y) = \begin{cases} \dfrac{3x^2y}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

Solution

We know f is continuous for (x,y) ≠ (0,0) since it is equal to a rational function there. Also, we have

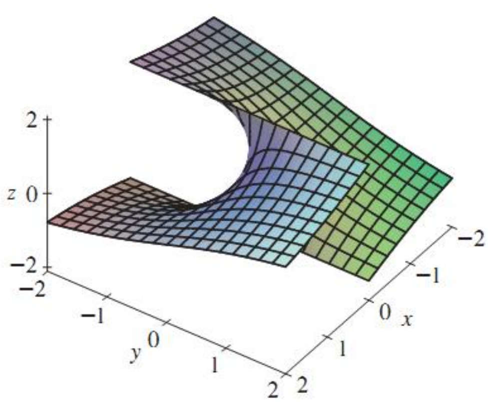
$$\lim_{(x, y) \to (0, 0)} f(x, y) = \lim_{(x, y) \to (0, 0)} \frac{3x^2y}{x^2 + y^2} = 0 = f(0, 0)$$

Therefore f is continuous at (0,0), and so it is continuous on R^2.

Example
Where is the function h(x,y) = arctan(y/x) continuous?
Solution
The function $f(x, y) = \frac{y}{x}$ is a rational function and therefore continuous except on the line x = 0. The function g(t) = arctan(t) is continuous everywhere. So the composite function

$$g(f(x, y)) = \arctan(y/x) = h(x, y)$$

is continuous except where . The graph in Figure shows the break in the graph of above the -axis.



Continuity of Function with three Variables

A function f of three variable is said to be continuous at (a,b,c) if

$\lim_{(x,y,z)\to(a,b,c)} f(x,y,z) = f(a,b)$

We say f is continuous on D if f is continuous at every point (a,b,c) in D.

Or

For every number $\in > 0$ there is a corresponding number $\delta > 0$ such that

$$\text{if} \quad (x, y, z) \text{ is in the domain of } f \quad \text{and} \quad 0 < \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2} < \delta$$

$$\text{then} \quad |f(x, y, z) - L| < \varepsilon$$

The function f is **continuous** at (a, b, c) if

$$\lim_{(x, y, z)\to(a, b, c)} f(x, y, z) = f(a, b, c)$$

For instance, the function

$$f(x, y, z) = \frac{1}{x^2 + y^2 + z^2 - 1}$$

is a rational function of three variables and so is continuous at every point in R^3 except where $x^2 + y^2 + z^2 = 1$. In other words, it is discontinuous on the sphere with center the origin and radius 1.

Remark

If we use the vector notation, then we can write the definitions of a limit for functions of two or three variables in a single compact form as follows.

If f is defined on a subset D of $\mathbb{R}^n$, then $\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = L$ means that for every number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that

$$\text{if} \quad \mathbf{x} \in D \quad \text{and} \quad 0 < |\mathbf{x} - \mathbf{a}| < \delta \quad \text{then} \quad |f(\mathbf{x}) - L| < \varepsilon$$

Partial Derivatives
If f is a function of two variables, its partial derivatives are the functions f_x and f_y defined by

$$f_x(x, y) = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

$$f_y(x, y) = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}$$

Notations for Partial Derivatives
If $z = f(x, y)$, we write

$$f_x(x, y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x, y) = \frac{\partial z}{\partial x} = f_1 = D_1 f = D_x f$$

$$f_y(x, y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x, y) = \frac{\partial z}{\partial y} = f_2 = D_2 f = D_y f$$

Rule for Finding Partial Derivatives of $z = f(x, y)$

1. To find f_x, regard y as a constant and differentiate $f(x, y)$ with respect to x.
2. To find f_y, regard x as a constant and differentiate $f(x, y)$ with respect to y.

Example

If $f(x, y) = x^3 + x^2y^3 - 2y^2$, find $f_x(2, 1)$ and $f_y(2, 1)$

Solution

$$f_x(x, y) = 3x^2 + 2xy^3$$

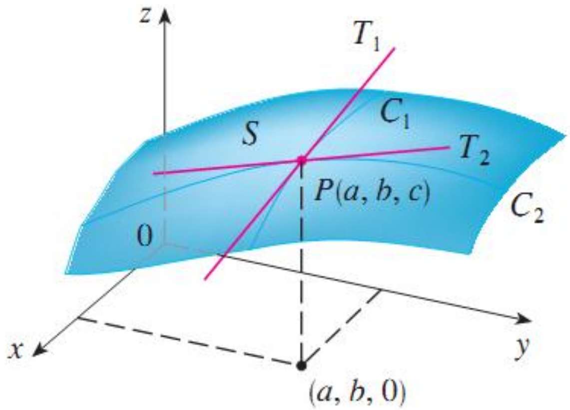
$$f_x(2, 1) = 3 \cdot 2^2 + 2 \cdot 2 \cdot 1^3 = 16$$

$$f_y(x, y) = 3x^2y^2 - 4y$$

$$f_y(2, 1) = 3 \cdot 2^2 \cdot 1^2 - 4 \cdot 1 = 8$$

Interpretations of Partial Derivatives

The partial derivatives $f_x(a, b)$ and $f_y(a, b)$ can be interpreted geometrically as the slopes of the tangent lines at $P(a, b, c)$ to the traces C_1 and C_2 of S in the planes y = b and x = a.



Remark

Partial derivatives can also be interpreted as rates of change. If z = f(x,y) , then $\frac{\partial z}{\partial x}$ represents the rate of change of z with respect to x when y is fixed. Similarly, $\frac{\partial z}{\partial y}$ represents the rate of change of z with respect to y when x is fixed.

Example

If $f(x, y) = 4 - x^2 - 2y^2$, find $f_x(1,1)$ and $f_y(1,1)$ and interpret these numbers as slopes.

Solution

We have

$$f_x(x, y) = -2x \qquad f_y(x, y) = -4y$$

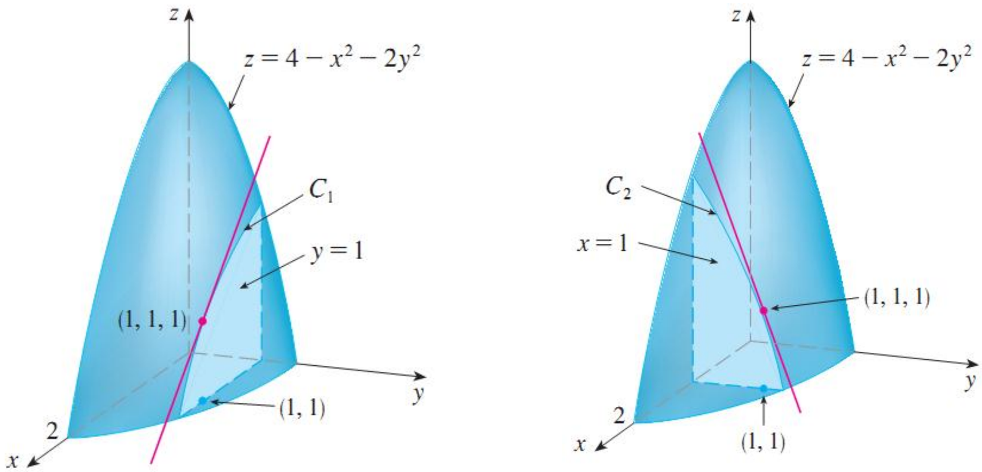
$$f_x(1, 1) = -2 \qquad f_y(1, 1) = -4$$

The graph of f is the paraboloid $z = 4 - x^2 - 2y^2$ and the vertical plane y = 1 intersects it in the parabola $z = 2 - x^2$, y = 1. (See Figure 1.)

The slope of the tangent line to this parabola at the point (1, 1, 1) is $f_x(1,1) = -2$.

Similarly, the plane x = 1 intersects the paraboloid is the parabola $z = 3 - 2y^2$, x = 1 , and the slope of the tangent line at (1, 1, 1) is $f_y(1,1) = -4$.

(See Figure 2.)



Example

If $f(x, y) = \sin\left(\dfrac{x}{1+y}\right)$, calculate $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$.

Solution

Using the Chain Rule for functions of one variable, we have

$$\frac{\partial f}{\partial x} = \cos\left(\frac{x}{1+y}\right) \cdot \frac{\partial}{\partial x}\left(\frac{x}{1+y}\right) = \cos\left(\frac{x}{1+y}\right) \cdot \frac{1}{1+y}$$

$$\frac{\partial f}{\partial y} = \cos\left(\frac{x}{1+y}\right) \cdot \frac{\partial}{\partial y}\left(\frac{x}{1+y}\right) = -\cos\left(\frac{x}{1+y}\right) \cdot \frac{x}{(1+y)^2}$$

Example

Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if z is defined implicitly as a function of x and y by the equation

$$x^3 + y^3 + z^3 + 6xyz = 1$$

Solution

$$\frac{\partial z}{\partial x} = -\frac{x^2 + 2yz}{z^2 + 2xy}$$

$$\frac{\partial z}{\partial y} = -\frac{y^2 + 2xz}{z^2 + 2xy}$$

Functions of More Than Two Variables
Partial derivatives can also be defined for functions of three or more variables. For example, if f is a function of three variables x, y, and z, then its partial derivative with respect to x is defined as

$$f_x(x, y, z) = \lim_{h \to 0} \frac{f(x + h, y, z) - f(x, y, z)}{h}$$

Generally we may write

$$\frac{\partial u}{\partial x_i} = \lim_{h \to 0} \frac{f(x_1, \ldots, x_{i-1}, x_i + h, x_{i+1}, \ldots, x_n) - f(x_1, \ldots, x_i, \ldots, x_n)}{h}$$

and we also write

$$\frac{\partial u}{\partial x_i} = \frac{\partial f}{\partial x_i} = f_{x_i} = f_i = D_i f$$

Example

Find f_x, f_y, and f_z if $f(x, y, z) = e^{xy} \ln z$.

Solution

$$f_x = ye^{xy} \ln z$$

$$f_y = xe^{xy} \ln z \qquad \text{and} \qquad f_z = \frac{e^{xy}}{z}$$

Higher Derivatives

If f is a function of two variables, then its partial derivatives f_x and f_y are also functions of two variables, so we can consider their partial derivatives $(f_x)_x$, $(f_x)_y$, $(f_y)_x$, and $(f_y)_y$, which are called the second partial derivatives of f.

If $z = f(x, y)$, we use the following notation:

$$(f_x)_x = f_{xx} = f_{11} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 z}{\partial x^2}$$

$$(f_x)_y = f_{xy} = f_{12} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y\,\partial x} = \frac{\partial^2 z}{\partial y\,\partial x}$$

$$(f_y)_x = f_{yx} = f_{21} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial^2 z}{\partial x\,\partial y}$$

$$(f_y)_y = f_{yy} = f_{22} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 z}{\partial y^2}$$

Thus the notation f_{xy} (or $\frac{\partial^2 f}{\partial y \partial x}$) means that we first differentiate with respect to x and then with respect to y, whereas in computing f_{yx} the order is reversed.

Example
Find the second partial derivatives of

$$f(x, y) = x^3 + x^2y^3 - 2y^2$$

Solution

$$f_x(x, y) = 3x^2 + 2xy^3 \qquad f_y(x, y) = 3x^2y^2 - 4y$$

Therefore

$$f_{xx} = \frac{\partial}{\partial x}(3x^2 + 2xy^3) = 6x + 2y^3 \qquad f_{xy} = \frac{\partial}{\partial y}(3x^2 + 2xy^3) = 6xy^2$$

$$f_{yx} = \frac{\partial}{\partial x}(3x^2y^2 - 4y) = 6xy^2 \qquad f_{yy} = \frac{\partial}{\partial y}(3x^2y^2 - 4y) = 6x^2y - 4$$

Clairaut′s Theorem
The following theorem, which was discovered by the French mathematician Alexis Clairaut (1713–1765), gives conditions under which we can assert that $f_{xy} = f_{yx}$

Clairaut's Theorem Suppose f is defined on a disk D that contains the point (a, b). If the functions f_{xy} and f_{yx} are both continuous on D, then

$$f_{xy}(a, b) = f_{yx}(a, b)$$

In need we may also use

$$f_{xyy} = f_{yxy} = f_{yyx}$$

Example

Calculate f_{xxyz} if $f(x, y, z) = \sin(3x + yz)$.

Solution

$$f_x = 3\cos(3x + yz)$$

$$f_{xx} = -9\sin(3x + yz)$$

$$f_{xxy} = -9z\cos(3x + yz)$$

$$f_{xxyz} = -9\cos(3x + yz) + 9yz\sin(3x + yz)$$

Partial Differential Equations

- Laplace's equation after Pierre Laplace (1749–1827). Solutions of this equation are called harmonic functions; they play a role in problems of heat conduction, fluid flow, and electric potential.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

- The wave equation describes the motion of a waveform, which could be an ocean wave, a sound wave, a light wave, or a wave traveling along a vibrating string.

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$$

- Heat conduction equation

$$u_t = \alpha^2 u_{xx}$$

Example

Show that the function $u(x, y) = e^x \sin y$ is a solution of Laplace's equation.

Solution

$$u_x = e^x \sin y \qquad\qquad u_y = e^x \cos y$$

$$u_{xx} = e^x \sin y \qquad\qquad u_{yy} = -e^x \sin y$$

$$u_{xx} + u_{yy} = e^x \sin y - e^x \sin y = 0$$

Therefore u satisfies Laplace's equation.

Example

Verify that the function $u(x, t) = \sin(x - at)$ satisfies the wave equation.

Solution

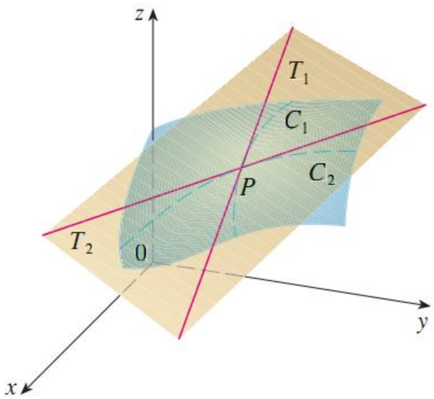
$$u_x = \cos(x - at) \qquad\qquad u_t = -a\cos(x - at)$$

$$u_{xx} = -\sin(x - at) \qquad\qquad u_{tt} = -a^2 \sin(x - at) = a^2 u_{xx}$$

Therefore u satisfies wave equation.

Tangent Planes

A tangent plane is a flat surface that touches a curved surface at a single point, called the point of tangency. It is a plane that contains all the tangent lines to a surface at that point.



Equation of Tangent Plane to a Surface

Suppose f has continuous partial derivatives. An equation of the tangent plane to the surface $z = f(x, y)$ at the point $P(x_0, y_0, z_0)$ is

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Example

Find the tangent plane to the elliptic paraboloid $z = 2x^2 + y^2$ at the point $(1,1,3)$.

Solution

Let $f(x, y) = 2x^2 + y^2$. Then

$$f_x(x, y) = 4x \qquad f_y(x, y) = 2y$$

$$f_x(1, 1) = 4 \qquad f_y(1, 1) = 2$$

Then the equation of the tangent plane at $(1,1,3)$ is

$$z - 3 = 4(x - 1) + 2(y - 1)$$

$$z = 4x + 2y - 3$$

Linearization
The linear function whose graph is this tangent plane, namely

$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

is called the linearization of f at (a, b).
Linear Approximations
The linear approximation of a function is approximating the value of the function at a point using a line.
Or the approximation

$$f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

is called the linear approximation or the tangent plane approximation of f at (a, b).

Example
Find the Linearization and Linear Approximation to the paraboloid $z = 2x^2 + y^2$ at the point $(1,1,3)$.
Solution

Let $f(x, y) = 2x^2 + y^2$. Then

$$f_x(x, y) = 4x \qquad f_y(x, y) = 2y$$

$$f_x(1, 1) = 4 \qquad f_y(1, 1) = 2$$

Then the equation of the tangent plane at $(1,1,3)$ is

$$z - 3 = 4(x - 1) + 2(y - 1)$$

$$z = 4x + 2y - 3$$

Then Linearization is

$$L(x, y) = 4x + 2y - 3$$

And Linear Approximation is

$$f(x, y) \approx 4x + 2y - 3$$

Increment
If $z = f(x, y)$, then the increment of z is

$$\Delta z = f(a + \Delta x, b + \Delta y) - f(a, b)$$

Differentiable Function/ Differentiability

If $z = f(x, y)$, then f is **differentiable** at (a, b) if Δz can be expressed in the form

$$\Delta z = f_x(a, b)\,\Delta x + f_y(a, b)\,\Delta y + \varepsilon_1\,\Delta x + \varepsilon_2\,\Delta y$$

where ε_1 and $\varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$.

Theorem
If the partial derivatives f_x and f_y exist near (a, b) and are continuous at (a, b), then f is differentiable at (a, b).

Example
Show that $f(x, y) = xe^{xy}$ is differentiable at (1, 0) and find its linearization there. Then use it to approximate $f(1.1, -0.1)$.

Solution

$$f_x(x, y) = e^{xy} + xye^{xy} \qquad f_y(x, y) = x^2e^{xy}$$

$$f_x(1, 0) = 1 \qquad f_y(1, 0) = 1$$

Both f_x and f_y are continuous functions, so f is differentiable. The linearization is

$$L(x, y) = f(1, 0) + f_x(1, 0)(x - 1) + f_y(1, 0)(y - 0)$$

$$= 1 + 1(x - 1) + 1 \cdot y = x + y$$

The corresponding linear approximation is

$$xe^{xy} \approx x + y$$

$$f(1.1, -0.1) \approx 1.1 - 0.1 = 1$$

Compare this with the actual value of

$$f(1.1, -0.1) = 1.1e^{-0.11} \approx 0.98542$$

Differentials
For a differentiable function of two variables, $z = f(x, y)$, we define the differentials dx and dy to be independent variables; that is, they can be given any values. Then the differential dz, also called the total differential, is defined by

$$dz = f_x(x, y)\,dx + f_y(x, y)\,dy = \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial y}\,dy$$

Example
(a) If $z = f(x, y) = x^2 + 3xy - y^2$, find the differential dz.
(b) If x changes from 2 to 2.05 and y changes from 3 to 2.96, compare the values of Δz and dz.
Solution
(a)

$$dz = \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial y}\,dy = (2x + 3y)\,dx + (3x - 2y)\,dy$$

(b) Putting $x = 2$, $dx = \Delta x = 0.05$, $y = 3$, and $dy = \Delta y = -0.04$, we get

$$dz = [2(2) + 3(3)]0.05 + [3(2) - 2(3)](-0.04) = 0.65$$

The increment of z is

$$\begin{aligned}\Delta z &= f(2.05, 2.96) - f(2, 3)\\ &= [(2.05)^2 + 3(2.05)(2.96) - (2.96)^2] - [2^2 + 3(2)(3) - 3^2]\\ &= 0.6449\end{aligned}$$

Notice that $\Delta z \approx dz$ but dz is easier to compute.

Example

The base radius and height of a right circular cone are measured as 10 cm and 25 cm, respectively, with a possible error in measurement of as much as 0.1cm in each. Use differentials to estimate the maximum error in the calculated cone.

Solution

The volume V of a cone with base radius r and height h is $V = \frac{1}{3}\pi r^2 h$. So the differential of V is

$$dV = \frac{\partial V}{\partial r}\, dr + \frac{\partial V}{\partial h}\, dh = \frac{2\pi r h}{3}\, dr + \frac{\pi r^2}{3}\, dh$$

Since each error is at most 0.1 cm, we have $|\Delta r| \leqslant 0.1$, $|\Delta h| \leqslant 0.1$. To estimate the largest error in the volume we take the largest error in the measurement of r and of h. Therefore we take $dr = 0.1$ and $dh = 0.1$ along with $r = 10$, $h = 25$. This gives

$$dV = \frac{500\pi}{3}(0.1) + \frac{100\pi}{3}(0.1) = 20\pi$$

Thus the maximum error in the calculated volume is about $20\pi\ \text{cm}^3 \approx 63\ \text{cm}^3$.

Functions of Three or More Variables

For such functions the linear approximation is

$$f(x, y, z) \approx f(a, b, c) + f_x(a, b, c)(x - a) + f_y(a, b, c)(y - b) + f_z(a, b, c)(z - c)$$

and the linearization L(x,y,z) is the right side of this expression.

If $w = f(x, y, z)$, then the increment of w is

$$\Delta w = f(x + \Delta x, y + \Delta y, z + \Delta z) - f(x, y, z)$$

The differential dw is defined in terms of the differentials dx, dy, and dz of the independent variables by

$$dw = \frac{\partial w}{\partial x}\, dx + \frac{\partial w}{\partial y}\, dy + \frac{\partial w}{\partial z}\, dz$$

Example

The dimensions of a rectangular box are measured to be 75 cm, 60 cm, and 40 cm, and each measurement is correct to within 0.2 cm. Use differentials to estimate the largest possible error when the volume of the box is calculated from these measurements.

Solution

If the dimensions of the box are x, y, and z, its volume is $V = xyz$ and so

$$dV = \frac{\partial V}{\partial x}\,dx + \frac{\partial V}{\partial y}\,dy + \frac{\partial V}{\partial z}\,dz = yz\,dx + xz\,dy + xy\,dz$$

We are given that $|\Delta x| \leqslant 0.2$, $|\Delta y| \leqslant 0.2$, and $|\Delta z| \leqslant 0.2$. To estimate the largest error in the volume, we therefore use $dx = 0.2$, $dy = 0.2$, and $dz = 0.2$ together with $x = 75$, $y = 60$, and $z = 40$:

$$\Delta V \approx dV = (60)(40)(0.2) + (75)(40)(0.2) + (75)(60)(0.2) = 1980$$

Thus an error of only 0.2 cm in measuring each dimension could lead to an error of approximately 1980 cm^3 in the calculated volume! This may seem like a large error, but it's only about 1% of the volume of the box.

The Chain Rule (Case – I)

Suppose that $z = f(x, y)$ is a differentiable function of x and y, where x = g(t) and y = h(t)are both differentiable functions of t. Then z is a differentiable function of t and

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

Since we often write $\frac{\partial z}{\partial x}$ in place of $\frac{\partial f}{\partial x}$, we can rewrite the Chain Rule in the form

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

Proof

A change of Δt in Δt produces changes of Δx in x and Δy in y. These, in turn, produce a change of Δz in z, and from we have

$$\Delta z = \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y + \varepsilon_1\,\Delta x + \varepsilon_2\,\Delta y$$

where $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$. [If the functions ε_1 and ε_2 are not defined at (0, 0), we can define them to be 0 there.] Dividing both sides of this equation by Δt, we have

$$\frac{\Delta z}{\Delta t} = \frac{\partial f}{\partial x}\frac{\Delta x}{\Delta t} + \frac{\partial f}{\partial y}\frac{\Delta y}{\Delta t} + \varepsilon_1\frac{\Delta x}{\Delta t} + \varepsilon_2\frac{\Delta y}{\Delta t}$$

If we now let $\Delta t \to 0$, then $\Delta x = g(t + \Delta t) - g(t) \to 0$ because g is differentiable and therefore continuous. Similarly, $\Delta y \to 0$. This, in turn, means that $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$, so

$$\begin{aligned}
\frac{dz}{dt} &= \lim_{\Delta t \to 0}\frac{\Delta z}{\Delta t} \\
&= \frac{\partial f}{\partial x}\lim_{\Delta t \to 0}\frac{\Delta x}{\Delta t} + \frac{\partial f}{\partial y}\lim_{\Delta t \to 0}\frac{\Delta y}{\Delta t} + \left(\lim_{\Delta t \to 0}\varepsilon_1\right)\lim_{\Delta t \to 0}\frac{\Delta x}{\Delta t} + \left(\lim_{\Delta t \to 0}\varepsilon_2\right)\lim_{\Delta t \to 0}\frac{\Delta y}{\Delta t} \\
&= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + 0 \cdot \frac{dx}{dt} + 0 \cdot \frac{dy}{dt} \\
&= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}
\end{aligned}$$

Example

If $z = x^2 y + 3xy^4$, where $x = \sin 2t$ and $y = \cos t$, find dz/dt when $t = 0$.

Solution

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$= (2xy + 3y^4)(2\cos 2t) + (x^2 + 12xy^3)(-\sin t)$$

It's not necessary to substitute the expressions for x and y in terms of t. We simply observe that when $t = 0$, we have $x = \sin 0 = 0$ and $y = \cos 0 = 1$. Therefore

$$\left.\frac{dz}{dt}\right|_{t=0} = (0 + 3)(2\cos 0) + (0 + 0)(-\sin 0) = 6$$

Example

The pressure P (in kilopascals), volume V (in liters), and temperature T (in kelvins) of a mole of an ideal gas are related by the equation $PV = 8.31T$. Find the rate at which the pressure is changing when the temperature is 300 K and increasing at a rate of 0.1 K/s and the volume is 100 L and increasing at a rate of 0.2 L/s.

Solution

If t represents the time elapsed in seconds, then at the given instant we have $T = 300$, $dT/dt = 0.1$, $V = 100$, $dV/dt = 0.2$. Since

$$P = 8.31\frac{T}{V}$$

$$\frac{dP}{dt} = \frac{\partial P}{\partial T}\frac{dT}{dt} + \frac{\partial P}{\partial V}\frac{dV}{dt} = \frac{8.31}{V}\frac{dT}{dt} - \frac{8.31T}{V^2}\frac{dV}{dt}$$

$$= \frac{8.31}{100}(0.1) - \frac{8.31(300)}{100^2}(0.2) = -0.04155$$

The pressure is decreasing at a rate of about 0.042 kPa/s.

The Chain Rule (Case – II)

Suppose that $z = f(x, y)$ is a differentiable function of x and y, where x = g(s,t) and y = h(s,t) are both differentiable functions of t. Then z is a differentiable function of s and t, then

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \qquad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

Example

If $z = e^x \sin y$, where $x = st^2$ and $y = s^2 t$, find $\partial z/\partial s$ and $\partial z/\partial t$.

Solution

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = (e^x \sin y)(t^2) + (e^x \cos y)(2st)$$

$$= t^2 e^{st^2} \sin(s^2 t) + 2ste^{st^2} \cos(s^2 t)$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = (e^x \sin y)(2st) + (e^x \cos y)(s^2)$$

$$= 2ste^{st^2} \sin(s^2 t) + s^2 e^{st^2} \cos(s^2 t)$$

The Chain Rule (General Version)

Suppose that u is a differentiable function of the n variables $x_1, x_2, \ldots, x_n$ and each x_j is a differentiable function of the m variables $t_1, t_2, \ldots, t_m$. Then u is a function of $t_1, t_2, \ldots, t_m$ and

$$\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_i} + \cdots + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_i}$$

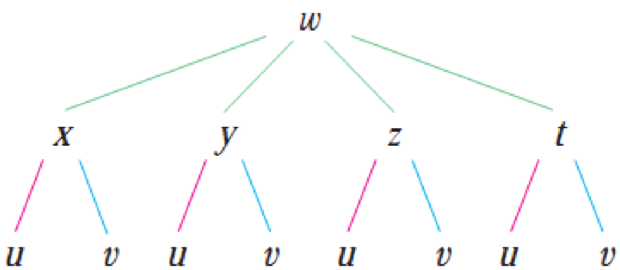
for each $i = 1, 2, \ldots, m$.

Example

Write out the Chain Rule for the case where $w = f(x, y, z, t)$ and $x = x(u, v)$, $y = y(u, v)$, $z = z(u, v)$, and $t = t(u, v)$.

Solution

We apply The Chain Rule (General Version) with n = 4 and m = 2. Figure shows the tree diagram.



Although we haven't written the derivatives on the branches, it's understood that if a branch leads from y to u, then the partial derivative for that branch is $\frac{\partial y}{\partial u}$. With the aid of the tree diagram, we can now write the required expressions:

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u} + \frac{\partial w}{\partial t}\frac{\partial t}{\partial u}$$

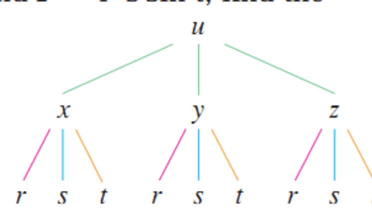
$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial v} + \frac{\partial w}{\partial t}\frac{\partial t}{\partial v}$$

Example

If $u = x^4y + y^2z^3$, where $x = rse^t$, $y = rs^2e^{-t}$, and $z = r^2s\sin t$, find the value of $\partial u/\partial s$ when $r = 2$, $s = 1$, $t = 0$.

Solution

With the help of the tree diagram in Figure, we have



$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial s}$$

$$= (4x^3y)(re^t) + (x^4 + 2yz^3)(2rse^{-t}) + (3y^2z^2)(r^2 \sin t)$$

When $r = 2$, $s = 1$, and $t = 0$, we have $x = 2$, $y = 2$, and $z = 0$, so

$$\frac{\partial u}{\partial s} = (64)(2) + (16)(4) + (0)(0) = 192$$

Example

If $g(s, t) = f(s^2 - t^2, t^2 - s^2)$ and f is differentiable, show that g satisfies the equation

$$t\frac{\partial g}{\partial s} + s\frac{\partial g}{\partial t} = 0$$

Solution

Let $x = s^2 - t^2$ and $y = t^2 - s^2$. Then $g(s, t) = f(x, y)$ and the Chain Rule gives

$$\frac{\partial g}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} = \frac{\partial f}{\partial x}(2s) + \frac{\partial f}{\partial y}(-2s)$$

$$\frac{\partial g}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} = \frac{\partial f}{\partial x}(-2t) + \frac{\partial f}{\partial y}(2t)$$

Therefore

$$t\frac{\partial g}{\partial s} + s\frac{\partial g}{\partial t} = \left(2st\frac{\partial f}{\partial x} - 2st\frac{\partial f}{\partial y}\right) + \left(-2st\frac{\partial f}{\partial x} + 2st\frac{\partial f}{\partial y}\right) = 0$$

Example

If $z = f(x, y)$ has continuous second-order partial derivatives and $x = r^2 + s^2$ and $y = 2rs$, find (a) $\partial z/\partial r$ and (b) $\partial^2 z/\partial r^2$.

Solution

(a) The Chain Rule gives

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial z}{\partial x}(2r) + \frac{\partial z}{\partial y}(2s)$$

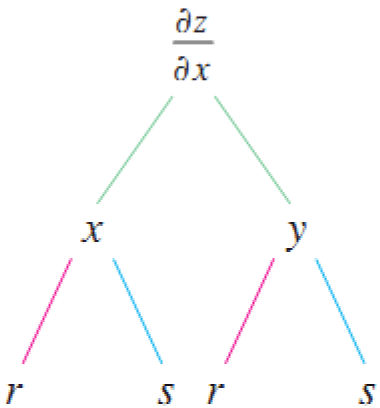
(b) Applying the Product Rule to the expression in part (a), we get

5

$$\frac{\partial^2 z}{\partial r^2} = \frac{\partial}{\partial r}\left(2r\frac{\partial z}{\partial x} + 2s\frac{\partial z}{\partial y}\right)$$

$$= 2\frac{\partial z}{\partial x} + 2r\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial x}\right) + 2s\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial y}\right)$$

But, using the Chain Rule again (see Figure),



we have

$$\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)\frac{\partial x}{\partial r} + \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)\frac{\partial y}{\partial r} = \frac{\partial^2 z}{\partial x^2}(2r) + \frac{\partial^2 z}{\partial y\,\partial x}(2s)$$

$$\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)\frac{\partial x}{\partial r} + \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)\frac{\partial y}{\partial r} = \frac{\partial^2 z}{\partial x\,\partial y}(2r) + \frac{\partial^2 z}{\partial y^2}(2s)$$

Putting these expressions into Equation 5 and using the equality of the mixed second-order derivatives, we obtain

$$\begin{aligned}\frac{\partial^2 z}{\partial r^2} &= 2\frac{\partial z}{\partial x} + 2r\left(2r\frac{\partial^2 z}{\partial x^2} + 2s\frac{\partial^2 z}{\partial y\,\partial x}\right) + 2s\left(2r\frac{\partial^2 z}{\partial x\,\partial y} + 2s\frac{\partial^2 z}{\partial y^2}\right)\\ &= 2\frac{\partial z}{\partial x} + 4r^2\frac{\partial^2 z}{\partial x^2} + 8rs\frac{\partial^2 z}{\partial x\,\partial y} + 4s^2\frac{\partial^2 z}{\partial y^2}\end{aligned}$$

Implicit Differentiation

Implicit differentiation makes use of the chain rule to differentiate a function which cannot be explicitly expressed in the form y= f(x).
It is defined as follows

$$\frac{dy}{dx} = -\frac{\dfrac{\partial F}{\partial x}}{\dfrac{\partial F}{\partial y}} = -\frac{F_x}{F_y}$$

The Implicit Function Theorem

The Implicit Function Theorem, proved in advanced calculus, gives conditions under which this assumption is valid:

It states that if F is defined on a disk containing (a,b) where, F(a,b) = 0, $F_y(a,b) \neq 0$ and F_x and F_y are continuous on the disk, then the equation F(x,y) = 0 defines y as a function of x near the point (a,b) and the derivative of this function is given by Equation

$$\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{F_x}{F_y}$$

We may also write as follows

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} \qquad \frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}$$

Example

Find y' if $x^3 + y^3 = 6xy$.

Solution

The given equation can be written as

$$F(x, y) = x^3 + y^3 - 6xy = 0$$

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{3x^2 - 6y}{3y^2 - 6x} = -\frac{x^2 - 2y}{y^2 - 2x}$$

Example

Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ if $x^3 + y^3 + z^3 + 6xyz = 1.$

Solution

Let $F(x, y, z) = x^3 + y^3 + z^3 + 6xyz - 1.$

Then, we have

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{3x^2 + 6yz}{3z^2 + 6xy} = -\frac{x^2 + 2yz}{z^2 + 2xy}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{3y^2 + 6xz}{3z^2 + 6xy} = -\frac{y^2 + 2xz}{z^2 + 2xy}$$

Directional Derivatives

The **directional derivative** of f at (x_0, y_0) in the direction of a unit vector $\mathbf{u} = \langle a, b \rangle$ is

$$D_{\mathbf{u}} f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

if this limit exists.

Theorem

If f is a differentiable function of x and y, then f has a directional derivative in the direction of any unit vector $\mathbf{u} = \langle a, b \rangle$ and

$$D_{\mathbf{u}} f(x, y) = f_x(x, y)\, a + f_y(x, y)\, b$$

Proof

If we define a function g of the single variable h by

$$g(h) = f(x_0 + ha, y_0 + hb)$$

then, by the definition of a derivative, we have

$$\boxed{4} \qquad g'(0) = \lim_{h \to 0} \frac{g(h) - g(0)}{h} = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$
$$= D_{\mathbf{u}} f(x_0, y_0)$$

On the other hand, we can write $g(h) = f(x, y)$, where $x = x_0 + ha$, $y = y_0 + hb$, so the Chain Rule (Theorem 14.5.2) gives

$$g'(h) = \frac{\partial f}{\partial x}\frac{dx}{dh} + \frac{\partial f}{\partial y}\frac{dy}{dh} = f_x(x, y)\, a + f_y(x, y)\, b$$

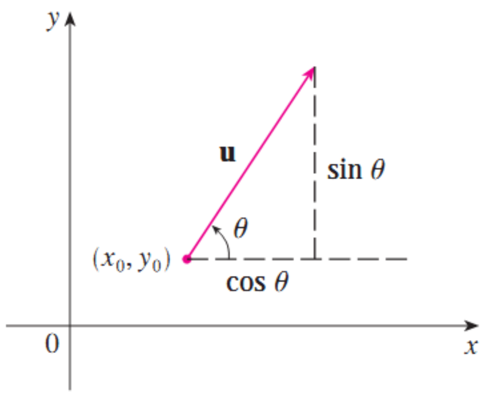
If we now put $h = 0$, then $x = x_0$, $y = y_0$, and

$$\boxed{5} \qquad g'(0) = f_x(x_0, y_0)\, a + f_y(x_0, y_0)\, b$$

Comparing Equations 4 and 5, we see that

$$D_{\mathbf{u}} f(x_0, y_0) = f_x(x_0, y_0)\, a + f_y(x_0, y_0)\, b$$

If the unit vector **u** makes an angle θ with the positive -axis (as in Figure),



then we can write **u** = $\langle cos\theta, sin\theta \rangle$ and the formula in Previous Theorem becomes

$$D_{\mathbf{u}} f(x, y) = f_x(x, y) \cos \theta + f_y(x, y) \sin \theta$$

Example

Find the directional derivative $D_u f(x, y)$ if

$$f(x, y) = x^3 - 3xy + 4y^2$$

and **u** is the unit vector given by angle $\theta = \pi/6$. What is $D_{\mathbf{u}} f(1, 2)$?

Solution

$$\begin{aligned} D_{\mathbf{u}} f(x, y) &= f_x(x, y) \cos \frac{\pi}{6} + f_y(x, y) \sin \frac{\pi}{6} \\ &= (3x^2 - 3y) \frac{\sqrt{3}}{2} + (-3x + 8y)\tfrac{1}{2} \\ &= \tfrac{1}{2}\left[3\sqrt{3}\,x^2 - 3x + \left(8 - 3\sqrt{3}\right)y\right] \end{aligned}$$

Therefore

$$D_{\mathbf{u}} f(1, 2) = \tfrac{1}{2}\left[3\sqrt{3}(1)^2 - 3(1) + \left(8 - 3\sqrt{3}\right)(2)\right] = \frac{13 - 3\sqrt{3}}{2}$$

The Gradient Vector

If f is a function of two variables x and y, then the gradient of f is the vector function ∇f defined by

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

Example

Find the gradient vector if

If $f(x, y) = \sin x + e^{xy}$,

Solution

$$\nabla f(x, y) = \langle f_x, f_y \rangle = \langle \cos x + ye^{xy}, xe^{xy} \rangle$$

$$\nabla f(0, 1) = \langle 2, 0 \rangle$$

Remark

With the notation for the gradient vector, we can rewrite the directional derivative of a differentiable function as

$$D_{\mathbf{u}} f(x, y) = \nabla f(x, y) \cdot \mathbf{u}$$

This expresses the directional derivative in the direction of a unit vector **u** as the scalar projection of the gradient vector onto **u**.

Example
Find the directional derivative of the function

$$f(x, y) = x^2 y^3 - 4y$$

at the point $(2,-1)$ in the direction of the vector.

$$\mathbf{v} = 2\,\mathbf{i} + 5\mathbf{j}$$

Solution

We first compute the gradient vector at $(2, -1)$:

$$\nabla f(x, y) = 2xy^3\,\mathbf{i} + (3x^2y^2 - 4)\mathbf{j}$$

$$\nabla f(2, -1) = -4\,\mathbf{i} + 8\mathbf{j}$$

Note that $\mathbf{v}$ is not a unit vector, but since $|\mathbf{v}| = \sqrt{29}$, the unit vector in the direction of $\mathbf{v}$ is

$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{2}{\sqrt{29}}\,\mathbf{i} + \frac{5}{\sqrt{29}}\,\mathbf{j}$$

Therefore, we have

$$D_{\mathbf{u}} f(2, -1) = \nabla f(2, -1) \cdot \mathbf{u} = (-4\,\mathbf{i} + 8\mathbf{j}) \cdot \left(\frac{2}{\sqrt{29}}\,\mathbf{i} + \frac{5}{\sqrt{29}}\,\mathbf{j}\right)$$

$$= \frac{-4 \cdot 2 + 8 \cdot 5}{\sqrt{29}} = \frac{32}{\sqrt{29}}$$

The directional derivative for the Functions of Three Variables

The **directional derivative** of f at (x_0, y_0, z_0) in the direction of a unit vector $\mathbf{u} = \langle a, b, c\rangle$ is

$$D_{\mathbf{u}} f(x_0, y_0, z_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb, z_0 + hc) - f(x_0, y_0, z_0)}{h}$$

if this limit exists.

Directional Derivative in the Compact Form
If we use vector notation, then we can write the directional derivative in the compact form

$$D_{\mathbf{u}} f(\mathbf{x}_0) = \lim_{h \to 0} \frac{f(\mathbf{x}_0 + h\mathbf{u}) - f(\mathbf{x}_0)}{h}$$

The Gradient Vector for the Functions of Three Variables
If f is a function of three variables x,y and z, then the gradient of f is the vector function ∇f defined by

$$\nabla f = \langle f_x, f_y, f_z \rangle = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

Remark
With the notation for the gradient vector, we can rewrite the directional derivative of a differentiable function as

$$D_{\mathbf{u}} f(x, y, z) = \nabla f(x, y, z) \cdot \mathbf{u}$$

Example
If $f(x, y, z) = x \sin yz$, (a) find the gradient of f and (b) find the directional derivative of f at $(1, 3, 0)$ in the direction of $\mathbf{v} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$.

Solution
(a) The gradient of f is

$$\begin{aligned} \nabla f(x, y, z) &= \langle f_x(x, y, z), f_y(x, y, z), f_z(x, y, z) \rangle \\ &= \langle \sin yz, xz \cos yz, xy \cos yz \rangle \end{aligned}$$

(b) At $(1, 3, 0)$ we have $\nabla f(1, 3, 0) = \langle 0, 0, 3 \rangle$. The unit vector in the direction of $\mathbf{v} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$ is

$$\mathbf{u} = \frac{1}{\sqrt{6}}\mathbf{i} + \frac{2}{\sqrt{6}}\mathbf{j} - \frac{1}{\sqrt{6}}\mathbf{k}$$

$$\begin{aligned} D_{\mathbf{u}} f(1, 3, 0) &= \nabla f(1, 3, 0) \cdot \mathbf{u} \\ &= 3\mathbf{k} \cdot \left(\frac{1}{\sqrt{6}}\mathbf{i} + \frac{2}{\sqrt{6}}\mathbf{j} - \frac{1}{\sqrt{6}}\mathbf{k} \right) \\ &= 3\left(-\frac{1}{\sqrt{6}} \right) = -\sqrt{\frac{3}{2}} \end{aligned}$$

Maximizing the Directional Derivative

Suppose we have a function f of two or three variables and we consider all possible directional derivatives of f at a given point. These give the rates of change of f in all possible directions. We can then ask the questions: In which of these directions does f change fastest and what is the maximum rate of change? The answers are provided by the following theorem.

Theorem

Suppose f is a differentiable function of two or three variables. The maximum value of the directional derivative $D_{\mathbf{u}} f(\mathbf{x})$ is $|\nabla f(\mathbf{x})|$ and it occurs when $\mathbf{u}$ has the same direction as the gradient vector $\nabla f(\mathbf{x})$.

Proof

Using equation

$$D_{\mathbf{u}} f(x, y) = \nabla f(x, y) \cdot \mathbf{u}$$

We have

$$D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u} = |\nabla f||\mathbf{u}| \cos\theta = |\nabla f| \cos\theta$$

where θ is the angle between ∇f and $\mathbf{u}$. The maximum value of $\cos\theta$ is 1 and this occurs when $\theta = 0$. Therefore the maximum value of $D_{\mathbf{u}} f$ is $|\nabla f|$ and it occurs when $\theta = 0$, that is, when $\mathbf{u}$ has the same direction as ∇f.

Example

(a) If $f(x, y) = xe^y$, find the rate of change of f at the point $P(2, 0)$ in the direction from P to $Q(\frac{1}{2}, 2)$.

(b) In what direction does f have the maximum rate of change? What is this maximum rate of change?

Solution

(a) We first compute the gradient vector:

$$\nabla f(x, y) = \langle f_x, f_y \rangle = \langle e^y, xe^y \rangle$$

$$\nabla f(2, 0) = \langle 1, 2 \rangle$$

The unit vector in the direction of $\overrightarrow{PQ} = \langle -1.5, 2 \rangle$ is $\mathbf{u} = \left\langle -\frac{3}{5}, \frac{4}{5} \right\rangle$, so the rate of change of f in the direction from P to Q is

$$D_{\mathbf{u}} f(2, 0) = \nabla f(2, 0) \cdot \mathbf{u} = \langle 1, 2 \rangle \cdot \left\langle -\tfrac{3}{5}, \tfrac{4}{5} \right\rangle$$

$$= 1\left(-\tfrac{3}{5}\right) + 2\left(\tfrac{4}{5}\right) = 1$$

(b) Here, f increases fastest in the direction of the gradient vector $\nabla f(2, 0) = \langle 1, 2 \rangle$. The maximum rate of change is

$$|\nabla f(2, 0)| = |\langle 1, 2 \rangle| = \sqrt{5}$$

Example

Suppose that the temperature at a point (x, y, z) in space is given by $T(x, y, z) = 80/(1 + x^2 + 2y^2 + 3z^2)$, where T is measured in degrees Celsius and x, y, z in meters. In which direction does the temperature increase fastest at the point $(1, 1, -2)$? What is the maximum rate of increase?

Solution

The gradient of T is

$$\nabla T = \frac{\partial T}{\partial x}\mathbf{i} + \frac{\partial T}{\partial y}\mathbf{j} + \frac{\partial T}{\partial z}\mathbf{k}$$

$$= -\frac{160x}{(1 + x^2 + 2y^2 + 3z^2)^2}\mathbf{i} - \frac{320y}{(1 + x^2 + 2y^2 + 3z^2)^2}\mathbf{j} - \frac{480z}{(1 + x^2 + 2y^2 + 3z^2)^2}\mathbf{k}$$

$$= \frac{160}{(1 + x^2 + 2y^2 + 3z^2)^2}(-x\mathbf{i} - 2y\mathbf{j} - 3z\mathbf{k})$$

At the point $(1, 1, -2)$ the gradient vector is

$$\nabla T(1, 1, -2) = \tfrac{160}{256}(-\mathbf{i} - 2\mathbf{j} + 6\mathbf{k}) = \tfrac{5}{8}(-\mathbf{i} - 2\mathbf{j} + 6\mathbf{k})$$

Here the temperature increases fastest in the direction of the gradient vector

$\nabla T(1, 1, -2) = \frac{5}{8}(-\mathbf{i} - 2\mathbf{j} + 6\mathbf{k})$ or, equivalently, in the direction of $-\mathbf{i} - 2\mathbf{j} + 6\mathbf{k}$ or the unit vector $(-\mathbf{i} - 2\mathbf{j} + 6\mathbf{k})/\sqrt{41}$. The maximum rate of increase is the length of the gradient vector:

$$|\nabla T(1, 1, -2)| = \tfrac{5}{8}|-\mathbf{i} - 2\mathbf{j} + 6\mathbf{k}| = \tfrac{5}{8}\sqrt{41}$$

Therefore the maximum rate of increase of temperature is $\frac{5}{8}\sqrt{41} \approx 4°\text{C/m}$.

Tangent Planes to Level Surfaces

Suppose S is a surface with equation $F(x, y, z) = k$, that is, it is a level surface of a function F of three variables, and let $P(x_0, y_0, z_0)$ be a point on S. Let C be any curve that lies on the surface S and passes through the point P. Then tangent plane to the level surface $F(x, y, z) = k$ at $P(x_0, y_0, z_0)$ as the plane that passes through P and has normal vector $\nabla F(x_0, y_0, z_0)$ is defined by the following equation

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0$$

A tangent plane to a level surface serves as a flat plane that touches the surface at a single point, effectively acting as a linear approximation of the surface's behavior in the immediate vicinity of that point, allowing us to analyze local properties like the gradient and normal vector at that specific location on the surface; essentially, it provides a way to understand how the surface changes near a particular point by representing it with a flat plane that best fits the curvature at that point.

Normal Line

A normal line to a point (x,y) on a curve is the line that goes through the point (x,y) and is perpendicular to the tangent line. Since the normal line and tangent line are perpendicular, they will have slopes that are opposite reciprocals of each other.

Equation of the Normal Line

The equation of a normal line to a curve at a given point is $y=mx+b$, where m is the slope and b is the y-intercept. The slope of the normal line is the negative reciprocal of the curve's derivative at the point.

Symmetric Equations of the Normal Line

$$\frac{x - x_0}{F_x(x_0, y_0, z_0)} = \frac{y - y_0}{F_y(x_0, y_0, z_0)} = \frac{z - z_0}{F_z(x_0, y_0, z_0)}$$

Example
Find the equations of the tangent plane and normal line at the point $(-2,1,-3)$ to the ellipsoid

$$\frac{x^2}{4} + y^2 + \frac{z^2}{9} = 3$$

Solution
The ellipsoid is the level surface (with k = 3) of the function

$$F(x, y, z) = \frac{x^2}{4} + y^2 + \frac{z^2}{9}$$

Therefore we have

$$F_x(x, y, z) = \frac{x}{2} \qquad F_y(x, y, z) = 2y \qquad F_z(x, y, z) = \frac{2z}{9}$$

$$F_x(-2, 1, -3) = -1 \qquad F_y(-2, 1, -3) = 2 \qquad F_z(-2, 1, -3) = -\tfrac{2}{3}$$

Then the equation of the tangent plane at $(-2,1,-3)$ is

$$-1(x + 2) + 2(y - 1) - \tfrac{2}{3}(z + 3) = 0$$

which simplifies to $3x - 6y + 2z + 18 = 0$.

Also, symmetric equations of the normal line are

$$\frac{x + 2}{-1} = \frac{y - 1}{2} = \frac{z + 3}{-\frac{2}{3}}$$

Significance of the Gradient Vector

- A gradient vector signifies the direction of the steepest ascent (or maximum rate of change) of a scalar field at a given point, essentially pointing in the direction where a function increases the fastest, with its magnitude representing the "steepness" of that increase.
- The gradient vectors always point to the direction where the function increases maximum. This property helps to find maxima/minima of the function using the steepest ascent/descent algorithm.

Extreme Value of a Function

An extreme value of a function is a maximum or minimum value of the function within a given interval. There are two types of extreme values: local and absolute.

Maximum Value of a Function

The "maximum value" of a function refers to the highest value that the function reaches across its entire domain, essentially the "peak" point on the graph of the function; it's the value where the function is greater than or equal to all other values it can produce.

Key points about maximum value:

Visual interpretation:

On a graph, the maximum value is the highest point on the curve representing the function.

Finding the maximum:

To find the maximum value, you typically need to calculate the derivative of the function, set it equal to zero to find critical points, and then evaluate the function at those points along with the endpoints of the domain to identify the highest value.

Local Maximum of a Function

A function of two variables has a local maximum at (a, b) if $f(x, y) \leq f(a, b)$ when (x, y) is near (a, b). [This means that $f(x, y) \leq f(a, b)$ for all points (x, y) in some disk with center (a, b).] The number $f(a, b)$ is called a local maximum value.

Absolute/ Global Maximum of a Function

A function of two variables has an absolute maximum at (a, b) if $f(x, y) \leq f(a, b)$ for all points in in the domain of f.

Local vs. Global maximum:
Local maximum: A point where the function is higher than its immediate neighbors but might not be the highest overall.

Global maximum: The absolute highest value of the function across its entire domain.

Minimum Value of a Function

The "minimum value" of a function refers to the lowest point on the graph of that function, essentially the smallest output value the function can produce across its entire domain; it's the point where the function reaches its lowest possible value.

Key points about minimum value:

Visualizing:

When looking at a graph, the minimum value is the "lowest point" on the curve.

Finding with calculus:

To mathematically find the minimum value, you typically take the derivative of the function, set it equal to zero to find critical points, then test those points to see which one gives the lowest output.

Local Minimum of a Function

A function of two variables has a local minimum at (a,b) if $f(x,y) \geq f(a,b)$ when (x,y) is near (a,b). [This means that $f(x,y) \geq f(a,b)$ for all points (x,y) in some disk with center (a,b).] The number $f(a,b)$ is called a local minimum value.

Absolute/ Global Minimum of a Function

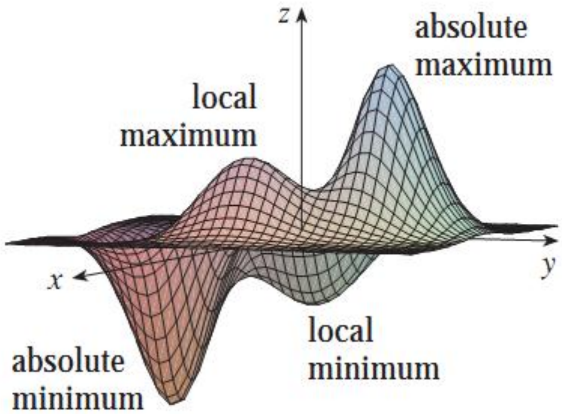
A function of two variables has an absolute minimum at (a,b) if $f(x,y) \geq f(a,b)$ for all points in in the domain of f.

Local vs. Global minimum:
Local minimum: A point where the function is lower than its immediate neighbors, but might not be the lowest value overall.

Global minimum: The absolute lowest value the function takes on across its entire domain.

Remark

Look at the hills and valleys in the graph of f shown in Figure. There are two points (a, b) where f has a local maximum, that is, where $f(a, b)$ is larger than nearby values of $f(x, y)$. The larger of these two values is the absolute maximum. Likewise, has two local minima, where $f(a, b)$ is smaller than nearby values. The smaller of these two values is the absolute minimum.



Theorem

If f has a local maximum or minimum at (a, b) and the first-order partial derivatives of f exist there, then $f_x(\mathrm{a}, \mathrm{b}) = 0$ and $f_y(\mathrm{a}, \mathrm{b}) = 0$.

Proof

Let $\mathrm{g}(x) = f(x, b)$. If f has a local maximum (or minimum) at (a, b), then g has a local maximum (or minimum) at a, so $\mathrm{g}(\mathrm{a}) = 0$ by Fermat's Theorem. But $\mathrm{g}'(\mathrm{a}) = f_x(x, b)$ and so $f_x(\mathrm{a}, \mathrm{b}) = 0$. Similarly, by applying Fermat's Theorem to the function $\mathrm{G}(y) = f(\mathrm{a}, y)$, we obtain $f_y(\mathrm{a}, \mathrm{b}) = 0$.

Critical / Stationary Point of a Function

A point (a, b) is called a critical point (or stationary point) of f if $f_x(\mathrm{a}, \mathrm{b}) = 0$ and $f_y(\mathrm{a}, \mathrm{b}) = 0$, or if one of these partial derivatives does not exist.

Remark

If has a local maximum or minimum at , then is a critical point of . However, as in single-variable calculus, not all critical points give rise to maxima or minima. At a critical point, a function could have a local maximum or a local minimum or neither.

Saddle Point of a Function

Saddle points in a multivariable function are those critical points where the function attains neither a local maximum value nor a local minimum value. Saddle points mostly occur in multivariable functions. For example $(0,0)$ is a saddle point of $z = \ y^2 - x^2$.

Example
Find the critical points of

$$f(x, y) = x^2 + y^2 - 2x - 6y + 14$$

Solution

Let $f(x, y) = x^2 + y^2 - 2x - 6y + 14$. Then

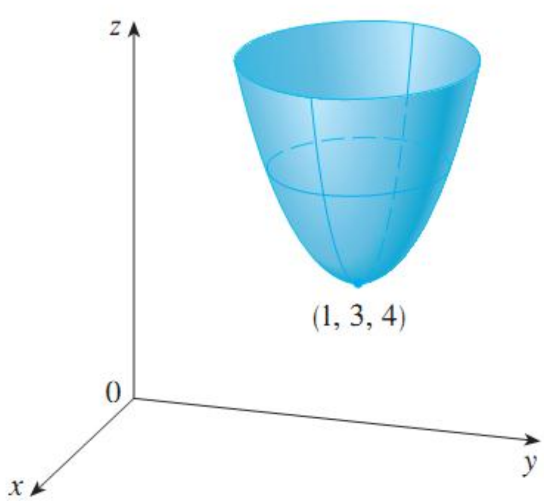
$$f_x(x, y) = 2x - 2 \qquad f_y(x, y) = 2y - 6$$

These partial derivatives are equal to 0 when $x = 1$ and $y = 3$, so the only critical point is $(1, 3)$. By completing the square, we find that

$$f(x, y) = 4 + (x - 1)^2 + (y - 3)^2$$

Since $(x - 1)^2 \geqslant 0$ and $(y - 3)^2 \geqslant 0$, we have $f(x, y) \geqslant 4$ for all values of x and y. Therefore $f(1, 3) = 4$ is a local minimum, and in fact it is the absolute minimum of f.

This can be confirmed geometrically from the graph of f which is the elliptic paraboloid with vertex (1,3,4) shown in Figure.



Example

Find the extreme values of $f(x, y) = y^2 - x^2$

Solution

Since $f_x = -2x$ and $f_y = 2y$, the only critical point is $(0, 0)$. Notice that for points on the x-axis we have $y = 0$, so $f(x, y) = -x^2 < 0$ (if $x \neq 0$). However, for points on the y-axis we have $x = 0$, so $f(x, y) = y^2 > 0$ (if $y \neq 0$). Thus every disk with center $(0, 0)$ contains points where f takes positive values as well as points where f takes negative values. Therefore $f(0, 0) = 0$ can't be an extreme value for f, so f has no extreme value.

Second Derivatives Test

Suppose the second partial derivatives of f are continuous on a disk with center (a,b), and suppose that $f_x(\mathrm{a,b}) = 0$ and $f_y(\mathrm{a,b}) = 0$[that is, $(\mathrm{a,b})$ is a critical point of f]. Let

$$D = D(a, b) = f_{xx}(a, b)\, f_{yy}(a, b) - [\, f_{xy}(a, b)]^2$$

(a) If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.

(b) If $D > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum.

(c) If $D < 0$, then $f(a, b)$ is not a local maximum or minimum.

Remark

- In case (c) the point $(\mathrm{a,b})$ is called a saddle point of f and the graph of f crosses its tangent plane at $(\mathrm{a,b})$.
- If D = 0, the test gives no information: f could have a local maximum or local minimum at $(\mathrm{a,b})$, or$(\mathrm{a,b})$ could be a saddle point of f.
- To remember the formula for D, it's helpful to write it as a determinant:

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}\, f_{yy} - (\, f_{xy})^2$$

Theorem

Suppose the second partial derivatives of f are continuous on a disk with center (a,b), and suppose that $f_x(\text{a}, \text{b}) = 0$ and $f_y(\text{a}, \text{b}) = 0$[that is, (a, b) is a critical point of f]. Let

$$D = D(a, b) = f_{xx}(a, b)\, f_{yy}(a, b) - [\, f_{xy}(a, b)]^2$$

If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.

Proof

We compute the second-order directional derivative of f in the direction of $\mathbf{u} = \langle h, k \rangle$. The first-order derivative is given by Theorem 14.6.3:

$$D_{\mathbf{u}} f = f_x h + f_y k$$

Applying this theorem a second time, we have

$$\begin{aligned} D_{\mathbf{u}}^2 f = D_{\mathbf{u}}(D_{\mathbf{u}} f) &= \frac{\partial}{\partial x}(D_{\mathbf{u}} f)h + \frac{\partial}{\partial y}(D_{\mathbf{u}} f)k \\ &= (f_{xx}h + f_{yx}k)h + (f_{xy}h + f_{yy}k)k \\ &= f_{xx}h^2 + 2 f_{xy}hk + f_{yy}k^2 \qquad \text{(by Clairaut's Theorem)} \end{aligned}$$

If we complete the square in this expression, we obtain

$$\boxed{10} \qquad D_{\mathbf{u}}^2 f = f_{xx}\left(h + \frac{f_{xy}}{f_{xx}}k\right)^2 + \frac{k^2}{f_{xx}}(f_{xx} f_{yy} - f_{xy}^2)$$

We are given that $f_{xx}(a, b) > 0$ and $D(a, b) > 0$. But f_{xx} and $D = f_{xx} f_{yy} - f_{xy}^2$ are continuous functions, so there is a disk B with center (a, b) and radius $\delta > 0$ such that $f_{xx}(x, y) > 0$ and $D(x, y) > 0$ whenever (x, y) is in B. Therefore, by looking at Equation 10, we see that $D_{\mathbf{u}}^2 f(x, y) > 0$ whenever (x, y) is in B. This means that if C is the curve obtained by intersecting the graph of f with the vertical plane through $P(a, b, f(a, b))$ in the direction of $\mathbf{u}$, then C is concave upward on an interval of length 2δ. This is true in the direction of every vector $\mathbf{u}$, so if we restrict (x, y) to lie in B, the graph of f lies above its horizontal tangent plane at P. Thus $f(x, y) \geqslant f(a, b)$ whenever (x, y) is in B. This shows that $f(a, b)$ is a local minimum.

Example
Find the local maximum and minimum values and saddle points of

$$f(x, y) = x^4 + y^4 - 4xy + 1$$

Solution
We first locate the critical points:

$$f_x = 4x^3 - 4y \qquad f_y = 4y^3 - 4x$$

Setting these partial derivatives equal to 0, we obtain the equations

$$x^3 - y = 0 \qquad \text{and} \qquad y^3 - x = 0$$

To solve these equations we substitute $y = x^3$ from the first equation into the second one. This gives

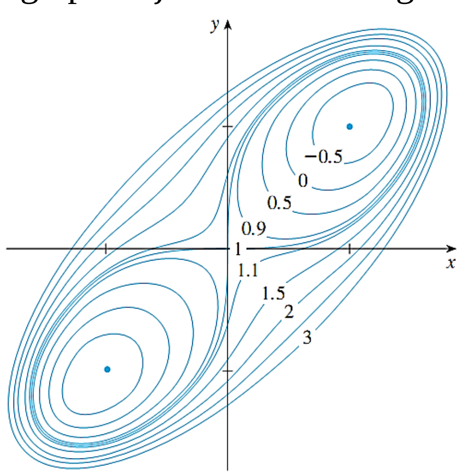
$$0 = x^9 - x = x(x^8 - 1) = x(x^4 - 1)(x^4 + 1) = x(x^2 - 1)(x^2 + 1)(x^4 + 1)$$

so there are three real roots: $x = 0, 1, -1$. The three critical points are $(0, 0)$, $(1, 1)$, and $(-1, -1)$.
Next we calculate the second partial derivatives and : D(x, y):

$$f_{xx} = 12x^2 \qquad f_{xy} = -4 \qquad f_{yy} = 12y^2$$

$$D(x, y) = f_{xx} f_{yy} - (f_{xy})^2 = 144x^2y^2 - 16$$

Since $D(0, 0) = -16 < 0$, it follows from case (c) of the Second Derivatives Test that the origin is a saddle point; that is, f has no local maximum or minimum at $(0, 0)$. Since $D(1, 1) = 128 > 0$ and $f_{xx}(1, 1) = 12 > 0$, we see from case (a) of the test that $f(1, 1) = -1$ is a local minimum. Similarly, we have $D(-1, -1) = 128 > 0$ and $f_{xx}(-1, -1) = 12 > 0$, so $f(-1, -1) = -1$ is also a local minimum.
The graph of f is shown in Figure



Example

Find and classify the critical points of the function

$$f(x, y) = 10x^2 y - 5x^2 - 4y^2 - x^4 - 2y^4$$

Also find the highest point on the graph of f.

Solution

The first-order partial derivatives are

$$f_x = 20xy - 10x - 4x^3 \qquad f_y = 10x^2 - 8y - 8y^3$$

So to find the critical points we need to solve the equations

$$\boxed{4} \qquad 2x(10y - 5 - 2x^2) = 0$$

$$\boxed{5} \qquad 5x^2 - 4y - 4y^3 = 0$$

From Equation 4 we see that either

$$x = 0 \qquad \text{or} \qquad 10y - 5 - 2x^2 = 0$$

In the first case ($x = 0$), Equation 5 becomes $-4y(1 + y^2) = 0$, so $y = 0$ and we have the critical point $(0, 0)$.

In the second case ($10y - 5 - 2x^2 = 0$), we get

$$\boxed{6} \qquad x^2 = 5y - 2.5$$

and, putting this in Equation 5, we have $25y - 12.5 - 4y - 4y^3 = 0$. So we have to solve the cubic equation

$$\boxed{7} \qquad 4y^3 - 21y + 12.5 = 0$$

Using a graphing calculator or computer to graph the function

$$g(y) = 4y^3 - 21y + 12.5$$

as in Figure 6, we see that Equation 7 has three real roots. By zooming in, we can find the roots to four decimal places:

$$y \approx -2.5452 \qquad y \approx 0.6468 \qquad y \approx 1.8984$$

(Alternatively, we could have used Newton's method or a rootfinder to locate these roots.) From Equation 6, the corresponding x-values are given by

$$x = \pm\sqrt{5y - 2.5}$$

If $y \approx -2.5452$, then x has no corresponding real values. If $y \approx 0.6468$, then $x \approx \pm 0.8567$. If $y \approx 1.8984$, then $x \approx \pm 2.6442$. So we have a total of five critical points, which are analyzed in the following chart. All quantities are rounded to two decimal places.

Critical point	Value of f	f_{xx}	D	Conclusion
$(0, 0)$	0.00	−10.00	80.00	local maximum
$(\pm 2.64, 1.90)$	8.50	−55.93	2488.72	local maximum
$(\pm 0.86, 0.65)$	−1.48	−5.87	−187.64	saddle point

Figures 7 and 8 give two views of the graph of f and we see that the surface opens downward. [This can also be seen from the expression for $f(x, y)$: The dominant terms are $-x^4 - 2y^4$ when $|x|$ and $|y|$ are large.] Comparing the values of f at its local maximum points, we see that the absolute maximum value of f is $f(\pm 2.64, 1.90) \approx 8.50$. In other words, the highest points on the graph of f are $(\pm 2.64, 1.90, 8.50)$.

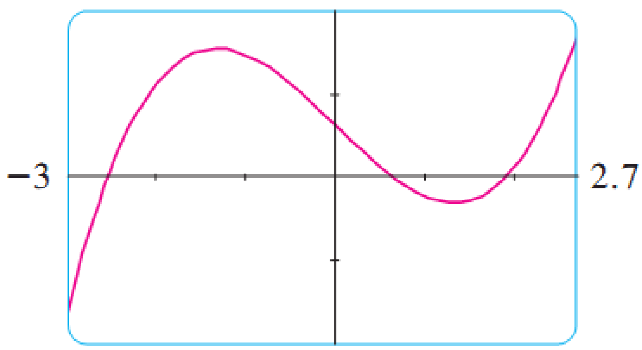


FIGURE 6

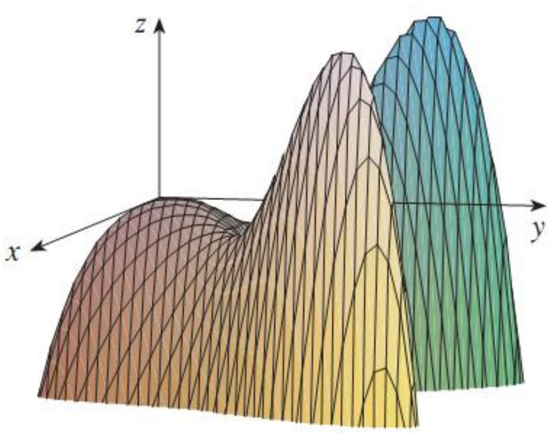


FIGURE 7

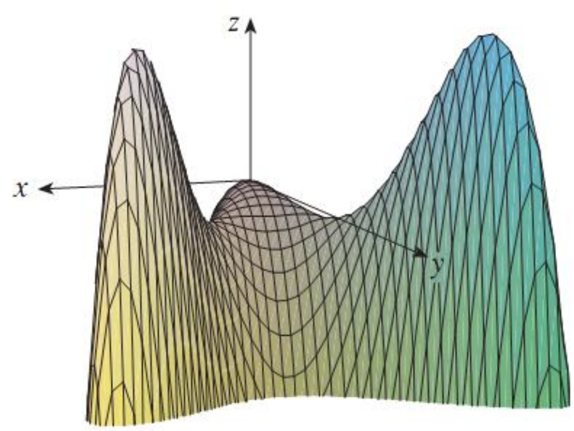


FIGURE 8

Example

Find the shortest distance from the point $(1,0,-2)$ to the plane

$$x + 2y + z = 4$$

Solution

The distance from any point (x, y, z) to the point $(1, 0, -2)$ is

$$d = \sqrt{(x-1)^2 + y^2 + (z+2)^2}$$

but if (x, y, z) lies on the plane $x + 2y + z = 4$, then $z = 4 - x - 2y$ and so we have $d = \sqrt{(x-1)^2 + y^2 + (6 - x - 2y)^2}$. We can minimize d by minimizing the simpler expression

$$d^2 = f(x, y) = (x-1)^2 + y^2 + (6 - x - 2y)^2$$

By solving the equations

$$f_x = 2(x-1) - 2(6 - x - 2y) = 4x + 4y - 14 = 0$$

$$f_y = 2y - 4(6 - x - 2y) = 4x + 10y - 24 = 0$$

we find that the only critical point is $\left(\frac{11}{6}, \frac{5}{3}\right)$. Since $f_{xx} = 4$, $f_{xy} = 4$, and $f_{yy} = 10$, we have $D(x, y) = f_{xx} f_{yy} - (f_{xy})^2 = 24 > 0$ and $f_{xx} > 0$, so by the Second Derivatives Test f has a local minimum at $\left(\frac{11}{6}, \frac{5}{3}\right)$. Intuitively, we can see that this local minimum is actually an absolute minimum because there must be a point on the given plane that is closest to $(1, 0, -2)$. If $x = \frac{11}{6}$ and $y = \frac{5}{3}$, then

$$d = \sqrt{(x-1)^2 + y^2 + (6 - x - 2y)^2} = \sqrt{\left(\tfrac{5}{6}\right)^2 + \left(\tfrac{5}{3}\right)^2 + \left(\tfrac{5}{6}\right)^2} = \tfrac{5}{6}\sqrt{6}$$

The shortest distance from $(1, 0, -2)$ to the plane $x + 2y + z = 4$ is $\frac{5}{6}\sqrt{6}$.

Example

A rectangular box without a lid is to be made from 12 m^2 of cardboard. Find the maximum volume of such a box.

Solution

Let the length, width, and height of the box (in meters) be x, y, and z, as shown in Figure 10. Then the volume of the box is

$$V = xyz$$

We can express V as a function of just two variables x and y by using the fact that the area of the four sides and the bottom of the box is

$$2xz + 2yz + xy = 12$$

Solving this equation for z, we get $z = (12 - xy)/[2(x + y)]$, so the expression for V becomes

$$V = xy\frac{12 - xy}{2(x + y)} = \frac{12xy - x^2y^2}{2(x + y)}$$

We compute the partial derivatives:

$$\frac{\partial V}{\partial x} = \frac{y^2(12 - 2xy - x^2)}{2(x + y)^2} \qquad \frac{\partial V}{\partial y} = \frac{x^2(12 - 2xy - y^2)}{2(x + y)^2}$$

If V is a maximum, then $\partial V/\partial x = \partial V/\partial y = 0$, but $x = 0$ or $y = 0$ gives $V = 0$, so we must solve the equations

$$12 - 2xy - x^2 = 0 \qquad 12 - 2xy - y^2 = 0$$

These imply that $x^2 = y^2$ and so $x = y$. (Note that x and y must both be positive in this problem.) If we put $x = y$ in either equation we get $12 - 3x^2 = 0$, which gives $x = 2$, $y = 2$, and $z = (12 - 2 \cdot 2)/[2(2 + 2)] = 1$.

We could use the Second Derivatives Test to show that this gives a local maximum of V, or we could simply argue from the physical nature of this problem that there must be an absolute maximum volume, which has to occur at a critical point of V, so it must occur when $x = 2$, $y = 2$, $z = 1$. Then $V = 2 \cdot 2 \cdot 1 = 4$, so the maximum volume of the box is 4 m^3.

Closed Set

A closed set in R^2 is one that contains all its boundary points.

Bounded Set

A bounded set in R^2 is one that is contained within some disk. In other words, it is finite in extent.

Extreme Value Theorem for Functions of Two Variables

If f is continuous on a closed, bounded set D in $\mathbb{R}^2$, then f attains an absolute maximum value $f(x_1, y_1)$ and an absolute minimum value $f(x_2, y_2)$ at some points (x_1, y_1) and (x_2, y_2) in D.

Procedure

To find the absolute maximum and minimum values of a continuous function f on a closed, bounded set D:

1. Find the values of f at the critical points of f in D.
2. Find the extreme values of f on the boundary of D.
3. The largest of the values from steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

Example

Find the absolute maximum and minimum values of the function $f(x, y) = x^2 - 2xy + 2y$ on the rectangle $D = \{(x, y) \mid 0 \leqslant x \leqslant 3, 0 \leqslant y \leqslant 2\}$.

Solution

Since f is a polynomial, it is continuous on the closed, bounded rectangle D, so Theorem 8 tells us there is both an absolute maximum and an absolute minimum. According to step 1 in $\boxed{9}$, we first find the critical points. These occur when

$$f_x = 2x - 2y = 0 \qquad f_y = -2x + 2 = 0$$

so the only critical point is $(1, 1)$, and the value of f there is $f(1, 1) = 1$.

In step 2 we look at the values of f on the boundary of D, which consists of the four line segments L_1, L_2, L_3, L_4 shown in Figure 12. On L_1 we have $y = 0$ and

$$f(x, 0) = x^2 \qquad 0 \leqslant x \leqslant 3$$

This is an increasing function of x, so its minimum value is $f(0, 0) = 0$ and its maximum value is $f(3, 0) = 9$. On L_2 we have $x = 3$ and

$$f(3, y) = 9 - 4y \qquad 0 \leqslant y \leqslant 2$$

This is a decreasing function of y, so its maximum value is $f(3, 0) = 9$ and its minimum value is $f(3, 2) = 1$. On L_3 we have $y = 2$ and

$$f(x, 2) = x^2 - 4x + 4 \qquad 0 \leqslant x \leqslant 3$$

By the methods of Chapter 3, or simply by observing that $f(x, 2) = (x - 2)^2$, we see that the minimum value of this function is $f(2, 2) = 0$ and the maximum value is $f(0, 2) = 4$. Finally, on L_4 we have $x = 0$ and

$$f(0, y) = 2y \qquad 0 \leqslant y \leqslant 2$$

with maximum value $f(0, 2) = 4$ and minimum value $f(0, 0) = 0$. Thus, on the boundary, the minimum value of f is 0 and the maximum is 9.

In step 3 we compare these values with the value $f(1, 1) = 1$ at the critical point and conclude that the absolute maximum value of f on D is $f(3, 0) = 9$ and the absolute minimum value is $f(0, 0) = f(2, 2) = 0$. Figure 13 shows the graph of f.

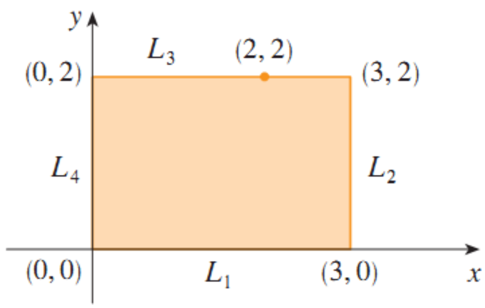


FIGURE 12

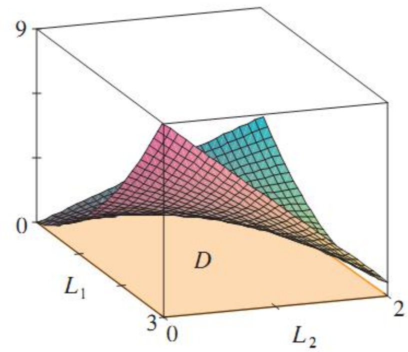


FIGURE 13
$f(x, y) = x^2 - 2xy + 2y$

Lagrange Multipliers

In mathematical optimization, the **method of Lagrange multipliers** is a strategy for finding the local maxima and minima of a function subject to equation constraints (i.e., subject to the condition that one or more equations have to be satisfied exactly by the chosen values of the variables). It is named after the mathematician Joseph-Louis Lagrange.

The basic idea is to convert a constrained problem into a form such that the derivative test of an unconstrained problem can still be applied.

If we have the following equation

$$\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0)$$

Then the number λ in Equation is called a Lagrange multiplier.

Method of Lagrange Multipliers

To find the maximum and minimum values of $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$ [assuming that these extreme values exist and $\nabla g \neq \mathbf{0}$ on the surface $g(x, y, z) = k$]:

(a) Find all values of x, y, z, and λ such that

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$$

and

$$g(x, y, z) = k$$

(b) Evaluate f at all the points (x, y, z) that result from step (a). The largest of these values is the maximum value of f; the smallest is the minimum value of f.

Example
A rectangular box without a lid is to be made from 12 m^2 of cardboard. Find the maximum volume of such a box.
Solution

Let x, y, and z be the length, width, and height, respectively, of the box in meters. Then we wish to maximize

V = xyz

subject to the constraint

$$g(x, y, z) = 2xz + 2yz + xy = 12$$

Using the method of Lagrange multipliers, we look for values of x, y, z, and λ such that $\nabla V = \lambda \nabla g$ and $g(x, y, z) = 12$. This gives the equations

$$V_x = \lambda g_x$$

$$V_y = \lambda g_y$$

$$V_z = \lambda g_z$$

$$2xz + 2yz + xy = 12$$

which become

$$\boxed{2} \qquad yz = \lambda(2z + y)$$

$$\boxed{3} \qquad xz = \lambda(2z + x)$$

$$\boxed{4} \qquad xy = \lambda(2x + 2y)$$

$$\boxed{5} \qquad 2xz + 2yz + xy = 12$$

There are no general rules for solving systems of equations. Sometimes some ingenuity is required. In the present example you might notice that if we multiply (2) by x, (3) by y, and (4) by z, then the left sides of these equations will be identical. Doing this, we have

$\boxed{6}$ $$xyz = \lambda(2xz + xy)$$

$\boxed{7}$ $$xyz = \lambda(2yz + xy)$$

$\boxed{8}$ $$xyz = \lambda(2xz + 2yz)$$

We observe that $\lambda \neq 0$ because $\lambda = 0$ would imply $yz = xz = xy = 0$ from $\boxed{2}$, $\boxed{3}$, and $\boxed{4}$ and this would contradict $\boxed{5}$. Therefore, from $\boxed{6}$ and $\boxed{7}$, we have

$$2xz + xy = 2yz + xy$$

which gives $xz = yz$. But $z \neq 0$ (since $z = 0$ would give $V = 0$), so $x = y$. From $\boxed{7}$ and $\boxed{8}$ we have

$$2yz + xy = 2xz + 2yz$$

which gives $2xz = xy$ and so (since $x \neq 0$) $y = 2z$. If we now put $x = y = 2z$ in $\boxed{5}$, we get

$$4z^2 + 4z^2 + 4z^2 = 12$$

Since x, y, and z are all positive, we therefore have $z = 1$ and so $x = 2$ and $y = 2$.

Example

Find the extreme values of the function $f(x, y) = x^2 + 2y^2$ on the circle $x^2 + y^2 = 1$.

Solution

We are asked for the extreme values of f subject to the constraint $g(x, y) = x^2 + y^2 = 1$. Using Lagrange multipliers, we solve the equations $\nabla f = \lambda \nabla g$ and $g(x, y) = 1$, which can be written as

$$f_x = \lambda g_x \qquad f_y = \lambda g_y \qquad g(x, y) = 1$$

or as

$\boxed{9}$ $$2x = 2x\lambda$$

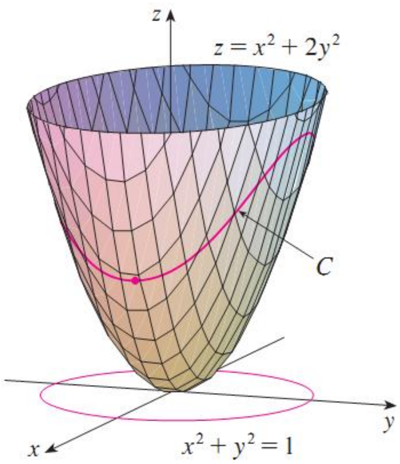
$\boxed{10}$ $$4y = 2y\lambda$$

$\boxed{11}$ $$x^2 + y^2 = 1$$

From $\boxed{9}$ we have $x = 0$ or $\lambda = 1$. If $x = 0$, then $\boxed{11}$ gives $y = \pm 1$. If $\lambda = 1$, then $y = 0$ from $\boxed{10}$, so then $\boxed{11}$ gives $x = \pm 1$. Therefore f has possible extreme values at the points $(0, 1)$, $(0, -1)$, $(1, 0)$, and $(-1, 0)$. Evaluating f at these four points, we find that

$$f(0, 1) = 2 \qquad f(0, -1) = 2 \qquad f(1, 0) = 1 \qquad f(-1, 0) = 1$$

Therefore the maximum value of f on the circle $x^2 + y^2 = 1$ is $f(0, \pm 1) = 2$ and the minimum value is $f(\pm 1, 0) = 1$. Checking with Figure 2, we see that these values look reasonable.



Example

Find the extreme values of the function $f(x, y) = x^2 + 2y^2$ on the circle $x^2 + y^2 = 1$.

Solution

According to the procedure, we compare the values of f at the critical points with values at the points on the boundary. Since $f_x = 2x$ and $f_y = 4y$, te only critical point is (0,0). We compare the value of f at that point with the extreme values on the boundary;

$$f(0, 0) = 0 \qquad f(\pm 1, 0) = 1 \qquad f(0, \pm 1) = 2$$

Therefore the maximum value of f on the disk $x^2 + y^2 \leqslant 1$ is $f(0, \pm 1) = 2$ and the minimum value is $f(0, 0) = 0$.

Example
Find the points on the sphere $x^2+y^2+z^2=4$ that are closest to and farthest from the point $(3,1,-1)$.
Solution

The distance from a point (x, y, z) to the point $(3, 1, -1)$ is

$$d=\sqrt{(x-3)^2+(y-1)^2+(z+1)^2}$$

but the algebra is simpler if we instead maximize and minimize the square of the distance:

$$d^2=f(x, y, z)=(x-3)^2+(y-1)^2+(z+1)^2$$

The constraint is that the point (x, y, z) lies on the sphere, that is,

$$g(x, y, z)=x^2+y^2+z^2=4$$

According to the method of Lagrange multipliers, we solve $\nabla f=\lambda \nabla g$, $g=4$. This gives

$$\boxed{12} \qquad 2(x-3)=2x\lambda$$

$$\boxed{13} \qquad 2(y-1)=2y\lambda$$

$$\boxed{14} \qquad 2(z+1)=2z\lambda$$

$$\boxed{15} \qquad x^2+y^2+z^2=4$$

The simplest way to solve these equations is to solve for x, y, and z in terms of λ from $\boxed{12}$, $\boxed{13}$, and $\boxed{14}$, and then substitute these values into $\boxed{15}$. From $\boxed{12}$ we have

$$x-3=x\lambda \qquad \text{or} \qquad x(1-\lambda)=3 \qquad \text{or} \qquad x=\frac{3}{1-\lambda}$$

[Note that $1-\lambda \neq 0$ because $\lambda=1$ is impossible from $\boxed{12}$.] Similarly, $\boxed{13}$ and $\boxed{14}$ give

$$y=\frac{1}{1-\lambda} \qquad z=-\frac{1}{1-\lambda}$$

Therefore, from $\boxed{15}$, we have

$$\frac{3^2}{(1-\lambda)^2}+\frac{1^2}{(1-\lambda)^2}+\frac{(-1)^2}{(1-\lambda)^2}=4$$

which gives $(1 - \lambda)^2 = \frac{11}{4}$, $1 - \lambda = \pm\sqrt{11}/2$, so

$$\lambda = 1 \pm \frac{\sqrt{11}}{2}$$

These values of λ then give the corresponding points (x, y, z):

$$\left(\frac{6}{\sqrt{11}}, \frac{2}{\sqrt{11}}, -\frac{2}{\sqrt{11}}\right) \quad \text{and} \quad \left(-\frac{6}{\sqrt{11}}, -\frac{2}{\sqrt{11}}, \frac{2}{\sqrt{11}}\right)$$

It's easy to see that f has a smaller value at the first of these points, so the closest point is $\left(6/\sqrt{11}, 2/\sqrt{11}, -2/\sqrt{11}\right)$ and the farthest is $\left(-6/\sqrt{11}, -2/\sqrt{11}, 2/\sqrt{11}\right)$.

Lagrange Multipliers (Two Constraints Form)

$$\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0) + \mu \nabla h(x_0, y_0, z_0)$$

In this case Lagrange's method is to look for extreme values by solving five equations in the five unknowns x, y, z, λ, and μ. These equations are obtained by writing above Equation in terms of its components and using the constraint equations:

$$f_x = \lambda g_x + \mu h_x$$

$$f_y = \lambda g_y + \mu h_y$$

$$f_z = \lambda g_z + \mu h_z$$

$$g(x, y, z) = k$$

$$h(x, y, z) = c$$

Example

Find the maximum value of the function $f(x, y, z) = x + 2y + 3z$ on the curve of intersection of the plane $x - y + z = 1$ and the cylinder $x^2 + y^2 = 1$.

Solution

We maximize the function $f(x, y, z) = x + 2y + 3z$ subject to the constraints $g(x, y, z) = x - y + z = 1$ and $h(x, y, z) = x^2 + y^2 = 1$. The Lagrange condition is $\nabla f = \lambda \nabla g + \mu \nabla h$, so we solve the equations

$$1 = \lambda + 2x\mu \tag{17}$$

$$2 = -\lambda + 2y\mu \tag{18}$$

$$3 = \lambda \tag{19}$$

$$x - y + z = 1 \tag{20}$$

$$x^2 + y^2 = 1 \tag{21}$$

Putting $\lambda = 3$ [from $\boxed{19}$] in $\boxed{17}$, we get $2x\mu = -2$, so $x = -1/\mu$. Similarly, $\boxed{18}$ gives $y = 5/(2\mu)$. Substitution in $\boxed{21}$ then gives

$$\frac{1}{\mu^2} + \frac{25}{4\mu^2} = 1$$

and so $\mu^2 = \frac{29}{4}$, $\mu = \pm\sqrt{29}/2$. Then $x = \mp 2/\sqrt{29}$, $y = \pm 5/\sqrt{29}$, and, from $\boxed{20}$, $z = 1 - x + y = 1 \pm 7/\sqrt{29}$. The corresponding values of f are

$$\mp\frac{2}{\sqrt{29}} + 2\left(\pm\frac{5}{\sqrt{29}}\right) + 3\left(1 \pm \frac{7}{\sqrt{29}}\right) = 3 \pm \sqrt{29}$$

Therefore the maximum value of f on the given curve is $3 + \sqrt{29}$.