Ordinary Differential Equations

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Partial Contents

Differential Equations

Definition:- An equation containing the derivatives of one or more dependent variables with respect to one or more Independent variables is said to be a Differential Equation OR

A differential equation is an equation which contains one or more terms and derivatives of one or more dependent variables with respect to other variables (Independent variables) OR

Equations that contain derivatives of dependent variables with respect to Independent variables.

* **Examples:-**

(i) $\frac{d^2y}{dx^2}+5\left(\frac{dy}{dx}\right)^3-4y=e^x$

(ii) $\frac{dy}{dx}-5y$

(iii) $(y-x)dx=4xdy$

(iv) $y''-y'+6y=0$, $\because y'=\frac{dy}{dx}$

(v) $\frac{d^2\theta}{dt^2}+\frac{g}{l}\sin\theta=F(t)$, (the pendulum equation)

(vi) $\frac{d^2y}{dt^2}+\varepsilon(y^2+1)\frac{dy}{dt}+y=0$ (the van der Pol equation)

* **Note:-** A differential equation (DE) contains more than one dependent variable. For example

$$\frac{dx}{dt}+\frac{dy}{dt}=2x+y.$$

* Classification by Types :-

Differential equations are classified into two types:

(i) Ordinary Differential Equation (ODE)
(ii) Partial Differential Equations (PDE)

(i) are already defined on previous page.

* A partial differential equation abbreviated as (PDE) is an equation involving one or more partial derivatives of an unknown functions of several variables. For example

(i) $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0$ (ii) $\frac{\partial^2 u}{\partial x^2} = \frac{1}{k}\frac{\partial u}{\partial t}$

(iii) $u_{xx} + u_{yy} = 0$, (iv) $u_{xy} + u_{yy} = 0$

* **Note** :- Ordinary differential equation is mainly abbreviated as ODE.

ORDER OF DE

Order of a DE (either PDE or ODE) is the order of the highest derivative appearing in equation. For example

$$\frac{d^2y}{dx^2} + 5\left(\frac{dy}{dx}\right)^3 - 4y = e^x$$

is a second order differential equation.

Similarly (i) $\frac{dy}{dx} + y\cos x = \sin x$,

(ii) $\frac{d^2y}{dx^2} + xy\left(\frac{dy}{dx}\right)^2 = 0$ are ODEs having order 1 and 2 respectievely.

* $\frac{d^ny}{dx^n}$ = is nth order ODE.

"Degree of ODE.

The degree of a DE (either ODE or PDE) is the power of highest order derivative present in a D.E. OR
The power of the order in a D.E is known as degree of D.E. For example

(i) $\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2+3=0$; order=2, degree=1

(ii) $(y''')^3+3y''+6y-12=0$, order=degree=3

(iii) $\frac{dy}{dx}+y\cos x=\sin x$; order= degree=1

Note :- We can use both the Leibniz notation or prime notation to express the derivative term in a DE. For example

$$\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2+3=0 \quad \text{and} \quad y''+(y')^2+3=0$$

have the same meaning.

Definition :- A differential equation is said to be linear if it is of the degree first OR it is linear in the Independent variable. OR
A differential equation is said to be linear if

(i) The dependent variable (say y) and its derivatives $y, y', y'', \ldots, y^{(n)}$ are of the first degree, that is the power of each term involving y is 1.

(ii) The coefficients of the terms depends at most on the Independent variable.

For example

$2y'' + 5y' + 3y = 0$, $\frac{dy}{dx} - x^2 y = \cos x$ are linear ordinary differential equations.

Definition:- A DE is said to be non-linear if it is not linear i.e any one of the condition or both of the linear DE fails. For example:

* $(1-y)y' + 2y = e^x$, non-linear because of yy'
* $\frac{d^4y}{dx^4} + y^2 = 0$, non-linear because of y^2.

Solution of the DE

A function f is said to be the solution of DE or Integral of the DE if it free of derivatives and satisfy the given DE.

Examples: Verify that the Indicated functions are solution of Given ODE.s.

(i) $\frac{dy}{dx} = x\sqrt{y}$, $y = \frac{1}{16}x^4$

Sol:- Since we have

$$y = \frac{1}{16}x^4$$

Differentiating b/s w.r.t "x",

$$\frac{dy}{dx} = \frac{4}{16}x^3 = \frac{1}{4}x^3$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{4}x^3$$

Substitute value of "y" and y' in Given DE, we get

$$\frac{1}{4}x^3 = x\sqrt{\frac{1}{16}x^4}$$

$$= x \cdot \frac{x^2}{4} = \frac{x^3}{4}$$

i.e $\frac{1}{4}x^3 = \frac{1}{4}x^3 = \frac{x^3}{4}$.

We see that each side of the equation is the same for every real number x. Note that $y^{1/2} = \frac{1}{4}x^2$ is, by definition the non-negative square root of $\frac{1}{16}x^4$. (This type of solution is known as explicit solution of ODE.

(ii) $y'' - 2y' + y = 0, \quad y = xe^x$

Sol:- Since $y = xe^x$

Differentiating w.r.t "x" b/s we get

$$y' = xe^x + e^x \quad \text{and}$$

$$y'' = xe^x + e^x + e^x = xe^x + 2e^x$$

Putting values of y, y', y'' into given DE, we get.

$$xe^x + 2e^x - 2(xe^x + e^x) + xe^x = 0$$

$$xe^x + 2e^x - 2xe^x - 2e^x + xe^x = 0$$

$$\cancel{2xe^x} - \cancel{2xe^x} + \cancel{2e^x} - \cancel{2e^x} = 0$$

$$0 = 0$$

So for every real number x in $y = xe^x$ is a solution of $y'' - 2y' + y = 0$

(iii) $y' = 9 + y^2, \quad y = 3\tan 3x$

Sol:- Since $y = 3\tan 3x$

Differentiating w.r.t "x" b/s, we get

$$y' = 3\sec^2 3x\,(3)$$

$$y' = 9\sec^2 3x$$

Putting values of y and y' in Given DE

$$9\sec^2 3x = 9 + (3\tan 3x)^2$$

$$= 9 + 9\tan^2 3x$$

$$= 9(1 + \tan^2 3x)$$

Note:- We can also verify the solution of DE by putting given value in D.E.

$9\sec^2 3x = 9(\sec^2 3x)$ $\quad \because 1+\tan^2\theta = \sec^2\theta$

$9\sec^2 3x = 9\sec^2 3x$

Hence given value of y in (iii) is a solution of DE in example III.

(iv) $y^3 - 3x + 3y = 5$; $y'' + 2y(y')^3 = 0$

Sol: Since $y^3 - 3x + 3y = 5$

$$\Rightarrow 3y^2y' - 3 + 3y' = 0 \quad \text{(differentiating)}$$

$$\Rightarrow 3y^2y' + 3y' = 3$$

$$\Rightarrow y'(3y^2+3) = 3$$

$$\Rightarrow \cancel{3}y'(y^2+1) = \cancel{3}$$

$$\Rightarrow y' = \frac{1}{y^2+1}$$

differentiating wrt "x" again

$$y'' = \frac{-1}{(y^2+1)^2}(2yy')$$

$$= \frac{-2}{(y^2+1)^2}\left(\frac{1}{(y^2+1)^2}\right) = \frac{-2y}{(y^2+1)^3}$$

Using values of y, y', y'' in given DE, we get

$$\frac{-2y}{(y^2+1)^3} + 2y\left(\frac{1}{y^2+1}\right)^3 = 0$$

$$\frac{-\cancel{2y}}{(y^2+1)^3} + \frac{\cancel{2y}}{(y^2+1)^3} = 0$$

$$0 = 0$$

Hence $y^3 - 3x + 3y = 5$ is a solution of $y'' + 2y(y')^3 = 0$, and this type of solution is called implicit solution of a DE.

(v) $(y-x)y' + y = 0$; $\ln y + \frac{x}{y} = c$

Sol:- Since $\ln y + \frac{x}{y} = c$

Differentiating w.r.t "x"

$$\frac{1}{y}y' + \left(\frac{-x}{y^2}y'\right) + \frac{1}{y} = 0$$

$$\frac{dy}{dx}\left(\frac{1}{y} - \frac{x}{y^2}\right) + \frac{1}{y} = 0 \; ; \quad y' = \frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx}\left(\frac{y-x}{y^2}\right) = -\frac{1}{y}$$

$$\Rightarrow \frac{dy}{dx} = \frac{y^2}{(y-x)} \cdot \frac{-1}{y}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-y}{y-x}$$

$$\Rightarrow y' = \frac{y}{x-y}$$

Putting y' in given DE.

$$(y-x)\frac{y}{x-y} + y = 0$$

$$-\cancel{(x-y)}\frac{y}{\cancel{(x-y)}} + y = 0$$

$$-y + y = 0$$

Hence $\ln y + x/y = c$ is an Implicit solution of $(y-x)y' + y = 0$.

H.W:- (i) $x\left(\frac{dy}{dx}\right) = x^2 + y$; $\quad y = x^2 + cx$

(ii) $xy' + y = 1$; $\quad y = \frac{c}{x} + 1$

(iii) $y'' + y = 2e^{-x}$; $\quad y = e^{-x} + \sin x$

(iv) $\cos x \frac{dy}{dx} + y \sin x = 1$; $\quad y = A\cos x + B\sin y$

(v) $x^2y'' - 3xy' + 4y = 0$; $\quad y = x^2 + x^2\ln x, \; x > 0$

"Formation of ODE"

Suppose we have a family of curves containing n arbitrary constants, then we can find/obtain nth order differential equation whose solution is the given family.

For example: Let us consider equation of circle centered at origin with radius "a", i.e

$$x^2+y^2=a^2 \longrightarrow (1)$$

Taking derivative of (1) w.r.t "x"

$$2x+2y\frac{dy}{dx}=0$$

$$\cancel{2}y\frac{dy}{dx}=-\cancel{2}x$$

$$\frac{dy}{dx}=-\frac{x}{y}$$

<u>OR</u>

$$\frac{dy}{dx}+\frac{x}{y}=0 \longrightarrow (A)$$

So equation (A) is the D.E of a given circle equation. Now is this solution of this circle equation is true? Let us solve (A) by variable separable method i.e

$$\frac{dy}{dx}+\frac{x}{y}=0$$

$$\frac{dy}{dx}=-\frac{x}{y}$$

$$y\,dy=-x\,dx$$

$$y\,dy+x\,dx=0$$

Integrating b/s

$$\int y\,dy+\int x\,dx=\int 0$$

$$y^2/2+\frac{x^2}{2}=c$$

$$x^2+y^2=2c$$

$$x^2+y^2=a^2 \qquad \text{where } 2c=a^2$$

Working Rules:-

(i) Write the equation of family of curves.
(ii) Differintiate eq in step (i).
(iii) Eliminate arbitrary constants from step(ii) equation. we get the required ODE.

Example: Let $y = e^{mx} \longrightarrow (i)$
Differentiating w.r.t "x" b/s

$$\frac{dy}{dx} = me^{mx} \longrightarrow (ii)$$

Using (i) in (ii)

$$\frac{dy}{dx} = my \longrightarrow (ii)^*$$

$$\Rightarrow m = \frac{1}{y}\frac{dy}{dx} \longrightarrow (iii)$$

Again from (i), we have

$$y = e^{mx}$$

$$\Rightarrow \ln y = \ln e^{mx} = mx$$

$$\Rightarrow \ln y = mx$$

$$\Rightarrow m = \frac{1}{x}\ln y \longrightarrow (iv)$$

Putting eq(iv) in eq(ii)* we get

$$\frac{dy}{dx} = \frac{1}{x}\ln y \; y$$

$$\frac{dy}{dx} - \frac{y\ln y}{x} = 0 \longrightarrow (A)$$

Hence "A" is the DE of (i), and (i) is the solution of (A). It is the task for you to check that (i) is a solution of (A).

Question:- Form a DE whose general solution is given in each part:

(i) $y = mx + 3$, "m" arbitrary constant

Sol:-

step (i) Since $y = mx + 3$

Differentiating b/s w.r.t "x"

step (ii) $\frac{dy}{dx} = m$

step (iii) For eliminating "m" in (ii), put (ii) in (i), we get

$$y = \frac{dy}{dx}x + 3$$

or $\boxed{x\frac{dy}{dx} - y + 3 = 0}$ Required DE.

(ii) $y = (x+c)^2$; family of parabolas

Sol:- Since $y = (x+c)^2 \longrightarrow (i)$

$$\Rightarrow \frac{dy}{dx} = 2(x+c)$$

$$\Rightarrow \frac{1}{2}\frac{dy}{dx} = x + c \longrightarrow (ii)$$

Put (ii) in (i), we obtain

$$y = \left(\frac{1}{2}\frac{dy}{dx}\right)^2$$

$$y = \frac{1}{4}\left(\frac{dy}{dx}\right)^2 \quad \text{or}$$

$$y = \frac{1}{4}(y')^2$$

$$\Rightarrow \boxed{(y')^2 - 4y = 0}$$

(iii) $x = A\cos(nt + \beta)$; A, β are constants

Sol:- Since $x = A\cos(nt + \beta) \longrightarrow (i)$

$$\Rightarrow \frac{dx}{dt} = -A\sin(nt+\beta)n \quad (\text{Differentiating})$$

$$\Rightarrow \frac{d^2x}{dt^2} = -n^2 A\cos(nt+\beta) \longrightarrow (ii)$$

Using (i) in (ii)

$$\frac{d^2x}{dt^2} = -n^2(x)$$

$$\Rightarrow \boxed{\frac{d^2x}{dt^2} + n^2x = 0}$$

Note:- An equation which contains "n" different arbitrary constants will be differentiated "n" times to form a DE of this equation for which the general solution will be that equation.

(iii) $y = ax + bx^3$

Sol:- Since $y = ax + bx^3 \longrightarrow (i)$

$$\Rightarrow \frac{dy}{dx} = a + 3bx^2 \longrightarrow (ii)$$

& $$\frac{d^2y}{dx^2} = 6bx \longrightarrow (iii)$$

From eq (iii) $\frac{d^2y}{dx^2}/6x = b$ put in (ii)

$$\frac{dy}{dx} = a + \cancel{3}x^2\left(\frac{d^2y}{dx^2}/\cancel{6}x\right)$$

$$y' = a + x\frac{y''}{2}$$

$$\Rightarrow a = y' - xy''/2$$

Now put value of a and b in (i)

$$y = \left(y' - \frac{xy''}{2}\right)x + \left(\frac{y''}{6x}\right)x^3$$

$$y = xy' - \frac{1}{2}x^2y'' + \frac{1}{6}x^2y''$$

$$y = xy' - \left(\frac{1}{2} - \frac{1}{6}\right)x^2y''$$

$$y = xy' - \frac{1}{3}x^2y''$$

$$3y = 3xy' - x^2y''$$

$$\Rightarrow \boxed{x^2y'' - 3xy' + 3y = 0}$$

(iv) $y = c_1e^{2x} + c_2e^{-x} + x$

Sol: Since $y = c_1e^{2x} + c_2e^{-x} + x \longrightarrow (i)$

$$\Rightarrow y' = 2c_1e^{2x} - c_2e^{-x} + 1 \longrightarrow (ii)$$

$$\Rightarrow y'' = 4c_1e^{2x} + c_2e^{-x} \longrightarrow (iii)$$

Eq (ii) $\Rightarrow$ $y' - 1 = 2c_1e^{2x} - c_2e^{-x} \longrightarrow (B)$

$$y' - 1 - 2c_1e^{2x} + c_2e^{-x} = 0$$

Eq (iii) $\Rightarrow$ $y'' - 4c_1e^{2x} - c_2e^{-x} = 0 \longrightarrow (C)$

Eq (i) $\Rightarrow$ $x - y + c_1e^{2x} + c_2e^{-x} = 0 \longrightarrow (A)$

To eliminate c_1 & c_2 from (A), (B) & (C) we take its determinant

$$\begin{vmatrix} x-y & e^{2x} & e^{-x} \\ y'-1 & 2e^{2x} & -e^{-x} \\ y'' & 4e^{2x} & e^{-x} \end{vmatrix} = 0$$

$$\Rightarrow e^{2x}e^{-x}\begin{vmatrix} x-y & 1 & 1 \\ y'-1 & 2 & -1 \\ y'' & 4 & 1 \end{vmatrix} = 0$$

$$\Rightarrow e^{x}\begin{vmatrix} x-y & 1 & 1 \\ y'-1+x-y & 3 & 0 \\ y''+y'-1 & 6 & 0 \end{vmatrix} = 0 \quad \text{by } \begin{matrix} R_2 + R_1 \\ R_3 + R_2 \end{matrix}$$

Since $e^x \neq 0 \Rightarrow \begin{vmatrix} y'+x-y-1 & 3 \\ y''+y'-1 & 6 \end{vmatrix} = 0$

$\Rightarrow 6(y'+x-y-1)-3(y''+y'-1)=0$

$\Rightarrow 2(y'+x-y-1)-(y''+y'-1)=0$

$\Rightarrow \boxed{y''-y'+2y-2x+1=0}$

(v) $\frac{x^2}{4}+\frac{y^2}{9}=k \longrightarrow (A)$

Sol:- Since $\frac{x^2}{4}+\frac{y^2}{9}=k \longrightarrow (i)$

Differentiating w.r.t "x"

$\frac{2x}{4}+\frac{2y}{9}\cdot\frac{dy}{dx}=0$

$\frac{x}{2}+\frac{2}{9}y\frac{dy}{dx}=0$

$\frac{x}{2}+\frac{2}{9}yy'=0$

$\frac{9x+4yy'}{18}=0$, L.C.M

$9x+4yy'=0$

$4yy'+9x=0$

or $\boxed{4y\frac{dy}{dx}+9x=0}$ Req. D.E of (A)

(vi) $y=e^x(A\cos x+B\sin x)$

Sol:- Since $y=e^x(A\cos x+B\sin x)$

Differentiating twice w.r.t "x"

$\frac{dy}{dx}=e^x(-A\sin x+B\cos x)+e^x(A\cos x+B\sin x)$

$$\frac{dy}{dx} = e^x(-A\sin x + B\cos x) + y$$

Now $$\frac{d^2y}{dx^2} = e^x(-A\cos x - B\sin x) + e^x(-A\sin x + B\cos x) + \frac{dy}{dx}$$

$$= -e^x(A\cos x + B\sin x) + e^x(-A\sin x + B\cos x) + y$$

$$= -y + \frac{dy}{dx} - y + \frac{dy}{dx}$$

$$\Rightarrow \frac{d^2y}{dx^2} = -2y + 2\frac{dy}{dx}$$

$$\Rightarrow \frac{d^2y}{dx^2} + 2y - 2\frac{dy}{dx} = 0$$

$$\Rightarrow \boxed{\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0}$$

Home Work :-

(i) $x^2 + y^2 - 2kx = 0$

(ii) $(x^3 + c)e^{-3x} = y$

(iii) $y = A\cos kx + B\sin kx + C\cosh kx$

First Order & First degree DE

Definition :- Differential Equation of first order and first degree can be written in the form of

(i) $\frac{dy}{dx} = F(x, y)$ or

(ii) $M(x,y)\,dx + N(x,y)\,dy = 0$

A DE of first order and first degree contains Independent variable x, dependent variable y and its derivative i.e

$$\frac{dy}{dx} = f(x,y) \quad \text{or} \quad f\left(x, y, \frac{dy}{dx}\right) = 0$$

For example let us consider the DE's

(i) $xy(y+1)\,dy = (x^2+1)\,dx$

(ii) $\dfrac{dy}{dx} = \dfrac{x+y}{x-y}$

(iii) $\dfrac{dy}{dx} + y = \sin x$ etc

Now to solve such types of ODE,s we shall discuss several types of ODE,s i.e

(i) The Separable Equation
(ii) Homogeneous Equations
(iii) Exact Equations
(iv) Linear Equations

⇒ Let us try to discuss it in detail.

(i) Separable Equations

Overview:- If the solution of DE is not possible by direct Integration method then the Integral technique called separable variable method will be used for solving DE. Separation of variable is a technique commonly used to solve first order DE. It is so called because we try to arrange the equation to be solved in such a way that all terms involving the dependent variable (say y) appear on one side of the equation and all terms involving the independent variable (x) appear on the other side. It is not possible always to rearrange all first order DE in this way, so this method is not alway appropriate. Furthermore, it is not always possible to perform Integration even if the variable are separable.

Mathematical Form:- A DE of the form $M(x)\,dx + N(y)\,dy = 0$, where M is a pure function of x and N is a pure function of y is called a separable equation.

$$\underline{M(x)}\,dx + \underline{N(y)}\,dy = 0 \text{ (or constant)}$$

can be constant / can be constant

function of "x" only

function of "y" only

In general, a DE of the form

$$\frac{dy}{dx} = \frac{f(x)}{g(y)}, \quad g(y) \neq 0$$

then by shifting, we get

$$g(y)\,dy = f(x)\,dx \longrightarrow \text{(SDE)}$$

The solution to "SDE" can be found by integrating L.H.S w.r.t "y" and R.H.S w.r.t "x".

* **Steps:-** To solve a separable equation, we perform the following steps

<u>Step (i):-</u> Let $\frac{dy}{dx} = f(x)\,g(y)$ be the given equation, where $f(x)$ is a function of x-alone and $g(y)$ is a function of y alone.

<u>Step (ii):-</u> For the constant solution of equation we solve the equation $g(y)=0$.

Step III :- For a non-constant solution, we separate variable in the form

$$\frac{dy}{g(y)} = f(x)\,dx$$

from step (i),

Step (iv) Integrate b/s w.r.t the required variable to get the required non-constant solution.

Examples :- Solve the following SDEs.

(1) $\frac{dy}{dx} = \frac{y}{(1+x)}$

Sol:- Since

$$\frac{dy}{dx} = \frac{y}{(1+x)} \longrightarrow (i)$$

The only constant solution here is $y=0$. $\rightarrow$ (ii)

For a non-constant solution, we separate the variables

$$\frac{dy}{y} = \frac{dx}{1+x} \longrightarrow (iii)$$

Integrating b/s w.r.t required variables

$$\int \frac{dy}{y} = \int \frac{dx}{1+x}$$

$$\ln y = \ln|1+x| + \ln c$$

$$e^{\ln y} = e^{\ln|1+x| + \ln c}$$

$$\cancel{e}^{\cancel{\ln} y} = \cancel{e}^{\cancel{\ln}|c(1+x)|}$$

$$y = c(1+x), \quad \because c = \pm e^{c}$$

$\therefore$ Solution is & $\begin{cases} y = c(1+x) \\ y = 0 \end{cases}$

(ii) $\frac{dy}{dx} = (y-1)^2$

Sol:- Since $\frac{dy}{dx} = (y-1)^2$

As $(y-1)^2 = 0 \Rightarrow y = 1$

Therefore, the only constant solution is $y=1$.

For a non-constant solution, separate variables

$$\frac{dy}{(y-1)^2} = dx$$

$$(y-1)^{-2}\,dy = dx$$

Integrating b/s

$$\int (y-1)^{-2}\,dy = \int dx$$

$$\frac{(y-1)^{-2+1}}{-2+1} = x+c$$

$$-\frac{1}{y-1} = x+c$$

So the solutions of equation are

$$\begin{cases} -\frac{1}{y-1} = x+c \\ y = 1 \end{cases}$$

Question:- Find the non-constant solutions of the following DE.

(3) $(1-x)\,dy = (1+y)\,dx$

Sol:- Separating variables

$$\frac{(1-x)}{dx} = \frac{(1+y)}{dy}$$

Integrating b/s

$$\int \frac{dy}{1+y} = \int \frac{dx}{1-x}$$

$$\ln(1+y) = -\ln(1-x) + \ln c$$

$$\ln(1+y) + \ln(1-x) = \ln c$$

$$\ln((1+y)(1-x)) = \ln c$$

$$\cancel{e}^{\ln((1+y)(1-x))} = \cancel{e}^{\ln c}$$

$$\boxed{(1+y)(1-x) = c}$$

(4) $\frac{dy}{dx} = 2x(1+y^2)$

Sol:- Separating variables

$$\frac{dy}{(1+y^2)} = 2x\,dx$$

Integrating b/s

$$\int \frac{dy}{(1+y^2)} = 2\int x\,dx$$

$$\tan^{-1}y = \cancel{2}\cdot\frac{x^2}{\cancel{2}} + c$$

$$\tan^{-1}y = x^2 + c$$

$$\Rightarrow \boxed{y = \tan(x^2+c)}$$

(5) $x\ln x\,dy = y\,dx$

Sol:- Separating variables

$$\frac{dy}{y} = \frac{dx}{x\ln x}$$

Integrating b/s

$$\int \frac{dy}{y} = \int \frac{dx}{x\ln x}$$

Let $u = \ln x$
$du = \frac{1}{x}dx$

$$\ln y = \int \frac{1}{u}\,du$$

$$\ln y = \ln u + \ln c$$

$$\ln y = \ln(\ln x) + \ln c$$

$$\ln y = \ln(c\ln x)$$

$$e^{\ln y} = e^{\ln(c\ln x)}$$

$$\boxed{y = c\ln x}$$ ANS

(6) $\sqrt{1-x^2}\,dy = \sqrt{1-y^2}\,dx$

Sol:- Separating variables

$$dx/\sqrt{1-x^2} = dy/\sqrt{1-y^2}$$

Integrating b/s

$$\int \frac{dx}{\sqrt{1-x^2}} = \int \frac{dy}{\sqrt{1-y^2}}$$

$$\sin^{-1}(x) = \sin^{-1}(y) + c$$

$$\Rightarrow \sin^{-1}(x) - \sin^{-1}(y) = c$$

$$\sin^{-1}\left(x\sqrt{1-x^2} - y\sqrt{1-y^2}\right) = c$$

$$\left(\because \sin^{-1}A - \sin^{-1}B = \sin^{-1}\left(A\sqrt{1-A^2} - B\sqrt{1-B^2}\right)\right)$$

So $x\sqrt{1-x^2} - y\sqrt{1-y^2} = \sin c$

$$\boxed{x\sqrt{1-x^2} - y\sqrt{1-y^2} = C} \quad \because \sin c = C$$

(7) $(\sin x + \cos x)\, dy + (\cos x - \sin x)\, dx = 0$

Sol:- Separating variables

$$dy + \frac{(\cos x - \sin x)}{\sin x + \cos x}\, dx = 0$$

Integrating b/s

$$\int dy + \int \frac{(\cos x - \sin x)}{\sin x + \cos x}\, dx$$

$\because \int \frac{f'}{f}\, dx = \ln|f| + c$

$$y + \ln(\sin x + \cos x) + c$$

$$\boxed{y = -\ln(\sin x + \cos x) + c}$$

(8) $\sin(2x)\, dy = y \cos(2x)\, dx$

Sol:- Separating variables

$$\frac{dy}{y} = \frac{\cos(2x)}{\sin(2x)}\, dx$$

Integrating b/s

$$\int \frac{dy}{y} = \int \frac{\cos(2x)}{\sin(2x)}\, dx$$

$$\ln y = \frac{1}{2} \ln|\sin 2x| + \ln c$$

$$\ln y = \ln|\sin(2x)|^{\frac{1}{2}} + \ln c$$

$$\ln y = \ln[c(\sin 2x)^{\frac{1}{2}}]$$

$$e^{\ln y} = e^{\ln|c(\sin(2x))^{\frac{1}{2}}|}$$

$$\boxed{y = c\sqrt{\sin(2x)}} \quad \text{ANS}$$

"Initial value Problems"

Sometimes when we solve any DE we will obtain infinitly many solutions. For example consider the DE $\frac{dy}{dx} = y$. All solutions to this DE are given as $y = ce^x$ where c is a constant. We can verify this because $\frac{d}{dx}(ce^x) = ce^x$. However, suppose that we want to find a specific solution to our DE. Consider we look at $\frac{dy}{dx} = y$ that $y(0) = 3$. Since our solution set is $y = ce^x$, we see that $3 = y(0)$ i.e $3 = y(0) = ce^0 = c$ and $c = 3$. Therefore the solution $y = 3e^x$ both satisfies $\frac{dy}{dx} = y$ and $y(0) = 3$. This is what we essentially call an Initial value problem where $y(0) = 3$ is an Initial value.

* **Definition:-** An Initial value problem (often abbreviated as I.V.P) is a problem where we want to find a solution to some DE,s that satisfies a given Initial condition $y(x_0) = y_0$.

While solving an Initial value problem in DE, the following steps are necessary.

(1) Find the general solution of given DE involving arbitrary constants.

(2) Plug an Initial value to get an equation involving c, and then solve for the value of c.

(iii) Put the value of c obtained in step II in the result of step (i), to get the required particular solution of Given DE.

Note:- Usually, an Initial value problem has only one solution.

Question:- Solve the following Initial value problems (IVPs).

(1) $\frac{dy}{dx} = y\tan(2x)$; $y(0)=2$

Sol:- Since $\frac{dy}{dx} = y\tan 2x$

Separating variables

$$\frac{dy}{y} = \tan(2x)\,dx$$

Integrating b/s

$$\int \frac{dy}{y} = \int \tan(2x)\,dx$$

$$\ln y = -\frac{1}{2}\ln|\cos 2x| + \ln c$$

$$2\ln y = \ln(c) - \ln|\cos 2x|$$

$$\ln y^2 = \ln\left(\frac{c}{\cos 2x}\right)$$

$$e^{\ln y^2} = e^{\ln\left(\frac{c}{\cos 2x}\right)}$$

$$y^2 = \frac{c}{\cos 2x} \longrightarrow (A)$$

Now we have $y(0) = 2 \Rightarrow x=0, y=2$

putting values we obtain

$$(2)^2 = \frac{c}{\cos(2\cdot 0)} = \frac{c}{1}$$

$\Rightarrow 2^2 = c \Rightarrow \boxed{c = 4}$

putting values of $c = 4$ in eq(A)

$$y^2 = \frac{4}{\cos 2x}$$

$\Rightarrow \boxed{y = \frac{2}{\sqrt{\cos 2x}}}$ req. solution

(2) $2x(y+1)dx - y\,dy = 0$; $y(0) = -2$

Sol:-

Separating variables

$$2x\,dx - \frac{y}{y+1}dy = 0$$

Integrating

$$\int 2x\,dx - \int \frac{y}{y+1}dy = \int 0$$

$$2\frac{x^2}{2} - \left(\int \frac{y}{y}dy - \int \frac{dy}{y+1}\right) = 0$$

$$x^2 - \int dy + \int \frac{dy}{y+1} = 0 \qquad \because \; y+1 \overline{\big)\, y \,}\; 1, \; \frac{\pm y \pm 1}{-1}$$

$$x^2 - y + \ln(y+1) = c \longrightarrow (A) \qquad \boxed{\frac{y}{y+1} = 1 - \frac{1}{y+1}}$$

Now $y(0) = -2 \Rightarrow x = 0$ & $y = -2$

putting this condition

$$0 - (-2) + \ln(-2+1) = c$$

$$2 + \ln|-1| = c$$

$$2 + \ln|1| = c \qquad \because \ln|1| = 0$$

$$\boxed{2 = c}$$

Thus (A) $\Rightarrow$ $x^2 - y + \ln(y+1) = 2$

or

$$\boxed{x^2 = y - \ln(y+1) + 2}$$

(3) $\frac{dy}{dx} = 2e^x y^3$; $y(0) = \frac{1}{2}$

Sol:- Since $\frac{dy}{dx} = 2e^x y^3$

Separating variables

$$\frac{dy}{y^3} = 2e^x dx$$

Integrating b/s

$$\int \frac{dy}{y^3} = 2\int e^x dx$$

$$\frac{-1}{2y^2} = 2e^x + c \longrightarrow (A)$$

Now we have $y(0) = \frac{1}{2} \Rightarrow x=0$ & $y = \frac{1}{2}$

putting values we obtain

$$\frac{-1}{2(\frac{1}{2})^2} = 2e^{(0)} + c$$

$$\frac{-1}{2\cdot\frac{1}{4}} = 2(1) + c$$

$$\frac{-1}{1/2} = 2 + c$$

$$-2 = 2 + c$$

$$\Rightarrow \boxed{c = -4}$$

Thus (A) $\Rightarrow$ $\frac{-1}{2y^2} = 2e^x - 4$

$$\frac{1}{y^2} = -4e^x + 8$$

$$y^2 = \frac{1}{8 - 4e^x}$$

$$y^2 = \frac{1}{4(2 - e^x)}$$

$$y = \frac{1}{2\sqrt{2 - e^x}} \quad \underline{\text{ANS}}$$

(7) $(1+2y^2)\,dy = y\cos x\,dx \quad ; \quad y(0)=1$

Sol:- Since $(1+2y^2)\,dy = y\cos x\,dx$

Separating variables

$$\frac{(1+2y^2)}{y}\,dy = \cos x\,dx$$

Integrating b/s

$$\int \frac{1+2y^2}{y}\,dy = \int \cos x\,dx$$

$$\int \frac{dy}{y} + 2\int \frac{y^2}{y}\,dy = \int \cos x\,dx$$

$$\int \frac{dy}{y} + 2\int y\,dy = \int \cos x\,dx$$

$$\ln y + \cancel{2}\frac{y^2}{\cancel{2}} = \sin x + c$$

$$y^2 + \ln y = \sin x + c \longrightarrow (A)$$

Now as $y(0)=1 \Rightarrow x=0,\ y=1$

putting values, we get

$$(1)^2 + \ln(1) = \sin(0) + c$$

$$1+0 = 0+c$$

$$c = 1$$

Using value of c in eq.(A), we get

$$\boxed{y^2 + \ln y = \sin x + 1} \quad \text{ANS}$$

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(4) $e^x \sec y\, dx + (1+e^x)\sec y \tan y\, dy = 0$; $x=3$, $y=60^\circ$

Sol: Since

$$e^x \cancel{\sec y}\, dx + (1+e^x)\cancel{\sec y} \tan y\, dy = 0$$

$$\Rightarrow e^x dx = -(1+e^x)\tan y\, dy = 0$$

Separating variables,

$$\frac{e^x dx}{1+e^x} = -\tan y\, dy$$

Integrating b/s

$$\int \frac{e^x dx}{1+e^x} = \int -\tan y\, dy$$

$$\ln|1+e^x| = -(-\ln|\cos y|) + \ln c$$

$$\Rightarrow 1+e^x = c\cos y \longrightarrow (A)$$

$$\Rightarrow c = \frac{1+e^x}{\cos y}$$

Now, we have $x=3$, $y=60^\circ$, so

$$c = \frac{1+e^3}{\cos(60^\circ)}$$

$$c = \frac{1+e^3}{1/2} \qquad \because \cos(60^\circ) = \frac{1}{2}$$

$$c = 2(1+e^3)$$

Using c value in (A), we get

$$2(1+e^3)\cos y = 1+e^x$$

$$\boxed{\cos y = \frac{1+e^x}{2(1+e^3)}}$$

(5) $8\cos^2 y\,dx + \operatorname{cosec}^2 x\,dy = 0, \quad y\left(\frac{\pi}{12}\right) = \frac{\pi}{4}$.

Sol:- Since $8\cos^2 y\,dx + \operatorname{cosec}^2 x\,dy = 0$

Separating variables

$$\int \frac{8\,dx}{\operatorname{cosec}^2 x} + \int \frac{dy}{\cos^2 y} = 0$$

or $8\sin^2 x\,dx + \sec^2 y\,dy$

Integrating b/s

$$8\int \sin^2 x\,dx + \int \sec^2 y\,dy = \int 0\,dy$$

$$8\int \left(\frac{1-\cos 2x}{2}\right)dx + \int \sec^2 y\,dy = 0$$

$$4\int (1-\cos 2x)\,dx + \int \sec^2 y\,dy = 0$$

$$4\left[x - \frac{\sin 2x}{2}\right] + \tan y = c$$

$$4x - 2\sin 2x + \tan y = c \longrightarrow (A)$$

Now as $y\left(\frac{\pi}{12}\right) = \frac{\pi}{4}$

$\Rightarrow x = \pi/12$ & $y = \pi/4$

Hence

$$4(\pi/12) - 2\sin(2\cdot\pi/12) + \tan(\pi/4) = c$$

$$\pi/3 - 2\sin(2\pi/12) + \tan\frac{\pi}{4} = c$$

$$\frac{\pi}{3} - 2\sin(\pi/6) + \tan\frac{\pi}{4} = c$$

$$\frac{\pi}{3} - 2\left(\frac{1}{2}\right) + 1 = c$$

$$c = \pi/3 \cancel{-1} \cancel{+1} \Rightarrow \boxed{c = \frac{\pi}{3}}$$

Now putting c value in (A), we get

$$\boxed{4x - 2\sin 2x + \tan y = \frac{\pi}{3}}$$

(ii) Differential Equations reducible to variable separable method

DE of the first order cannot be solved directly by variable separable method. But by some substitution, we can reduce it to a DE with separable variable. Let the DE of the form

$$\frac{dy}{dx} = f(ax+by+c)$$

This DE can be reduced to separable form by the substitution

$$ax+by+c = z$$

$$\therefore \quad a+b\frac{dy}{dx} = \frac{dz}{dx}$$

$$\therefore \left(\frac{dz}{dx}-a\right)\frac{1}{b} = f(z)$$

$$\frac{dz}{dx} = a+b\,f(z)$$

Now, apply variable separable method

Examples:- Solve the following ODE.s

(1) $\frac{dy}{dx} = \sin(x+y)+\cos(x+y)$ (Exase

Sol:- Since we have

$$\frac{dy}{dx} = \sin(x+y)+\cos(x+y) \longrightarrow (1)$$

Let $z = x+y$

$$\Rightarrow \frac{dz}{dx} = 1+\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{dz}{dx}-1$$

So $(1) \Rightarrow \frac{dz}{dx} - 1 = \sin z+\cos z$

Separating variables

$$\Rightarrow \frac{dz}{\sin z+\cos z+1} = dx$$

Integrating b/s

$$\int \frac{dz}{\sin z+\cos z+1} = \int dx \longrightarrow (2)$$

Now since

$$\sin(z) = \frac{2\tan(z/2)}{1+\tan(z/2)^2}$$ (weierstrass substitution)

$$\cos(z) = \frac{1-(\tan(z/2))^2}{1+\tan(\frac{z}{2})^2}$$

So (2) $\Rightarrow$

$$\int \frac{dz}{\frac{2\tan(z/2)}{1+\tan(\frac{z}{2})^2} + \frac{1-(\tan\frac{z}{2})^2}{1+\tan(\frac{z}{2})^2} + 1} = \int dx$$

(2) $\frac{dy}{dx} = (x+y)^2 \longrightarrow (1)$

Sol:- Let $z = x+y$

$$\Rightarrow \frac{dz}{dx} = 1 + \frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{dz}{dx} - 1$$

So (1) $\Rightarrow$ $\frac{dz}{dx} - 1 = (z)^2$

$$\frac{dz}{dx} = z^2 + 1$$

Separating variables

$$\frac{dz}{z^2+1} = dx$$

Integrating b/s

$$\int \frac{dz}{z^2+1} = \int dx$$

$$\tan^{-1}(z) = x + c$$

put $z = x+y$ again

$$\tan^{-1}(x+y) = x + c$$

$$\Rightarrow x+y = \tan(x+c)$$

$$\boxed{y = \tan(x+c) - x}$$ ANS

(3) $(3x-4y-3)\,dy = (3x-4y-2)\,dx$

Sol:- Since

$$(3x-4y-2)\,dy = (3x-4y-2)\,dx$$

$$\Rightarrow \frac{dy}{dx} = \frac{3x-4y-2}{3x-4y-3} \longrightarrow (2)$$

So Let $3x-4y=v$

$$\Rightarrow \frac{dv}{dx}=3-4\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx}=\frac{-1}{4}\left(\frac{dv}{dx}-3\right)$$

Hence (2) becomes

$$\frac{-1}{4}\left(\frac{dv}{dx}-3\right)=\frac{v-2}{v-3}$$

$$\left(\frac{dv}{dx}-3\right)=-4\left(\frac{v-2}{v-3}\right)$$

$$\frac{dv}{dx}-3=\frac{-4v+8}{v-3}$$

$$\frac{dv}{dx}=\frac{-4v+8}{v-3}+3$$

$$\frac{dv}{dx}=\frac{-4v+8+3v-9}{v-3}$$

$$\frac{dv}{dx}=\frac{-(v+1)}{v-3}$$

Separating variables

$$\frac{(v-3)}{v+1}dv=-dx$$

Integrating b/s

$$\int\frac{v-3}{v+1}dv=-\int dx$$

From partial fraction, we get

$$\int\left(1-\frac{4}{v+1}\right)dv=-\int dx$$

$$\int dv-4\int\frac{dv}{v+1}=-\int dx$$

$$v-4\ln(v+1)=-x+c_1$$

$$3x-4y-4\ln(3x-4y+1)=-x+c_1 \quad \text{(putting value of } v \text{ again)}$$

$$x-y-\ln(3x-4y+1)=\frac{c_1}{4}=C$$

$$\boxed{x-y-\ln(3x-4y+1)=C}$$ ANS

(4) $(2x+y+1)\,dx + (4x+2y-1)\,dy = 0$

Sol: Since $(2x+y+1)dx + (4x+2y-1)dy = 0$

$$\Rightarrow \frac{dy}{dx} + \frac{2x+y+1}{4x+2y-1} = 0 \quad \rightarrow (1)$$

Let $2x+y = u$

$$\frac{du}{dx} = 2 + \frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{du}{dx} - 2$$

So eq (1) becomes

$$\frac{du}{dx} - 2 + \frac{u+1}{2u-1} = 0$$

$$\frac{du}{dx} \frac{-2(2u-1)+u+1}{2u-1} = 0$$

$$\frac{du}{dx} - \frac{4u+2+u+1}{2u-1} = 0$$

$$\frac{du}{dx} + \frac{3-3u}{2u-1} = 0$$

$$\frac{du}{dx} + \frac{3(1-u)}{2u-1} = 0$$

Separating variables

$$\frac{2u-1}{1-u}\,du = -3\,dx$$

Integrating b/s

$$\int \frac{2u-1}{1-u}\,du = -3\int dx$$

By using partial fraction, we get

$$\int\left(-2+\frac{1}{1-u}\right)du = -3\int dx$$

$$-2\int du + \int \frac{du}{1-u} = -3\int dx$$

$$-2u - \ln(1-u) = -3x + c_1$$

put $u = 2x+y$ again

$$-2(2x+y) - \ln(1-(2x+y)) = -3x+c_1$$

$$-4x-2y-\ln(1-2x-y) = -3x+c_1$$

$$-2(2x+y) - \ln(1-2x-y) = -3x+c_1$$

$$-2(2x+y) - \ln(1-2x-y) + 3x = c_1$$

$$-4x+3x-2y-\ln(1-2x-y) = c_1$$

$$-x-2y-\ln(1-2x-y) = c_1$$

$$x+2y+\ln(1-2x-y) = -c_1$$

$$x+2y+\ln(1-2x-y) = c \quad , \quad c = -c_1$$

(5) $(x+2y)(dx-dy) = dx+dy$

Sol:- Since $(x+2y)(dx-dy) = dx+dy$

$$(x+2y)(dx-dy) - dx - dy = 0$$

$$(x+2y)dx - (x+2y)dy - dx - dy = 0$$

$$(x+2y-1)dx - (x+2y+1)dy = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{x+2y-1}{x+2y+1} \longrightarrow (A)$$

Let $x+2y = u$

$$\frac{du}{dx} = 1 + 2\frac{dy}{dx}$$

$$2\frac{dy}{dx} = \frac{du}{dx} - 1$$

$$\frac{dy}{dx} = \frac{1}{2}\left(\frac{du}{dx} - 1\right)$$

Using values in (A), we get

$$\frac{1}{2}\left(\frac{du}{dx}-1\right) = \frac{u-1}{u+1}$$

$$\Rightarrow \frac{du}{dx}-1 = 2\left(\frac{u-1}{u+1}\right)$$

$$\frac{du}{dx} = 2\left(\frac{u-1}{u+1}\right)+1$$

$$\frac{du}{dx} = \frac{2u-2}{u+1}+1$$

$$\frac{du}{dx} = \frac{2u-2+u+1}{u+1}$$

$$\frac{du}{dx} = \frac{3u-1}{u+1}$$

Separating variables

$$\left(\frac{u+1}{3u-1}\right) du = dx$$

xing b/s by "3"

$$3\left(\frac{u+1}{3u-1}\right) = 3dx$$

$$\frac{3u+3}{3u-1} du = 3dx$$

Integrating b/s

$$\int \frac{3u+3}{3u-1} du = 3\int dx$$

Now $\frac{3u+3}{3u-1}$ is an Improper fraction

So $(3u+3) \div (3u-1)$

By $(3u+3) \div (3u-1)$, we get

$$\frac{3u+3}{3u-1} = 1 + \frac{4}{3u-1}$$

$$\therefore \int\left(1+\frac{4}{3u-1}\right)du = 3\int dx$$

$$\int du + \frac{4}{3}\int\frac{3du}{3u-1} = 3\int dx$$

$$u + \frac{4}{3}\ln|3u-1| = 3x + c_1$$

x ing by 3 b/s

$$3u + 4\ln|3u-1| = 9x + 3c_1$$

reput $u = x+2y$ again

$$3(x+2y) + 4\ln(3(x+2y)-1) = 9x + 3c_1$$

$$3x+6y + 4\ln(3x+6y-1) = 9x+3c_1$$

$$4\ln(3x+6y-1) + 3x+6y-9x = 3c_1$$

$$4\ln(3x+6y-1) - 6x+6y = 3c_1$$

÷ ing b/s by 2

$$2\ln(3x+6y-1) - 3x+3y = \frac{3}{2}c_1$$

$$2\ln(3x+6y-1) - 3x+3y = \frac{3}{2}c_1$$

$$2\ln(3x+6y-1) = 3x-3y+\frac{3}{2}c_1 \; ; \text{ let } c = \frac{3c_1}{2}$$

$$2\ln(3x+6y-1) = 3x-3y+c$$ ANS.

Home Work :-

(1) $(2x+3y-1)dx + (2x+3y+2)dy = 0$

(2) $\dfrac{dy}{dx} = \dfrac{2x-6y+7}{x-3y+4}$

Home Work:-

(1) $\frac{dy}{dx}+y^2\sin x=0$

(2) $\frac{dy}{dx}=\frac{e^{x-y}}{1+e^x}$

(3) $\frac{dy}{dx}=2x^{-1}\sqrt{y-1}$

(4) $(e^{2y}-y)\cos x\frac{dy}{dx}=e^y\sin 2x$; $y(0)=0$

(5) $x^2\frac{dy}{dx}=y-xy$; $y(-1)=-1$

(6) $(1+x^4)dy+x(1+4y^2)dx$; $y(1)=0$

(iii) Homogeneous differential Equations

Before we discuss something about homogeneous DE,s, let us try to understand homogeneous function.

Homogeneous function:-

A function $f(x,y)$ is said to be a homogeneous function of degree n, if each of its term is of degree "n"

Mathematically

$$f(tx,ty)=\underbrace{t^n}_{\text{degree}}(f(x,y))$$

It is a function of severale variables such that, if all the arguments are multiplied by a scaler, then its value is multiplied by some power of this scaler, called the degree of homogenity OR simple the degree

For example, Consider
$f(x,y) = x^2+y^2$, then this function is a homogeneous function of degree 2 because put $x = tx$, $y = ty$ we get

$$f(tx,ty) = (tx)^2+(ty)^2$$
$$= t^2x^2+t^2y^2$$
$$= t^2(x^2+y^2)$$
$$f(tx,ty) = \underbrace{t^2}_{\text{degree}} f(x,y)$$

Similarly
$f(x,y) = x^3+7xy^2+8x^2y+y^3$ is a homogeneous function of degree 3.

A first order DE $\frac{dy}{dx} = f(x,y)$ is said to be homogeneous if f is a homogeneous of any degree. If this equation is written in the form

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

then it is called homogeneous if $M(x,y)$ and $N(x,y)$ are homogeneous functions of the same degree

$$\frac{dy}{dx} = \frac{x^3+y^3}{x^2y+xy^2}, \quad \frac{dy}{dx} = \frac{x^2+3y^2}{2xy}$$

$$(x-y)\,dx + (x+y)\,dy = 0$$

are examples of homogeneous D.Eqn's of degree's 3, 2 and 1 respectivelly.

For the solution of such DE, we use the method discussed in the next theorem.

Theorem: A homogeneous equation $\frac{dy}{dx} = g\left(\frac{y}{x}\right)$ can be transformed into a separable equations (in the variables v and x) by substitution $y = vx$.

proof: Since we have

$$\frac{dy}{dx} = g\left(\frac{y}{x}\right) \longrightarrow (1)$$

Put $y = vx$ into (1). Then

$\frac{dy}{dx} = v + x\frac{dv}{dx}$ and (1) becomes

$$v + x\frac{dv}{dx} = g(v)$$

or $$v - g(v) + x\frac{dv}{dx} = 0$$

or $$[v - g(v)]\,dx + x\,dv = 0$$

So this equation is separable and can be solved as in the previous section.

More briefly, we use the following method to solve a homogeneous D.E by reducing it to variable separable by substituting $y = vx$ in given DE

$$\Rightarrow \frac{dy}{dx} = v\frac{d}{dx}(x) + x\frac{d}{dx}(v)$$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

We use $y = vx$ when we have $\frac{dy}{dx}$. For $\frac{dx}{dy}$ we assume $x = vy$

Thus the equation given will reduce to variable separable equation with variables v and x. At last put the value of v as $\frac{y}{x}$ $\left(\because y = vx \Rightarrow v = \frac{y}{x}\right)$

Question:- Solve the following D·Eqns:

(1) $\frac{dy}{dx} = \frac{x^3+y^3}{x^2y+xy^2} \longrightarrow (1)$

Sol:- Since we see that (1) is homogeneous of degree (3)

So using $y = vx$ into (1) &

$$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

Hence (1) $\Rightarrow$

$$v + x\frac{dv}{dx} = \frac{x^3+(vx)^3}{x^2vx+x(vx)^2}$$

$$= \frac{\cancel{x^3}(1+v^3)}{\cancel{x^3}(v+v^2)}$$

$$+x\frac{dv}{dx} = \frac{1+v^3}{v+v^2} = \frac{1+v^3}{v(1+v)}$$

$$x\frac{dv}{dx} = \frac{1+v^3}{v(1+v)} - v$$

$$x\frac{dv}{dx} = \frac{1+v^3-v^2(1+v)}{v(1+v)}$$

$$= \frac{1+\cancel{v^3}-v^2-\cancel{v^3}}{v(1+v)}$$

$$= \frac{1-v^2}{v(1+v)}$$

$$x\frac{dv}{dx} = \frac{(1-v)\cancel{(1+v)}}{v\cancel{(1+v)}}$$

$$x\frac{dv}{dx} = \frac{1-v}{v}$$

$$\frac{v}{1-v}\frac{dv}{dx} = \frac{1}{x}$$

$$\frac{v}{1-v}dv = \frac{dx}{x}$$

Which is separable D.E
Integrating b/s

$$\int \frac{v}{1-v}\,dv = \int \frac{dx}{x}$$

$$\int\left(-1+\frac{1}{1-v}\right)dv = \int \frac{dx}{x} \qquad \because \boxed{\frac{v}{1-v} = -1+\frac{1}{1-v}}$$

$$\int -dv + \int \frac{dv}{1-v} = \int \frac{dx}{x}$$

$$-v+(-\ln|1-v|) = \ln|x| + \ln|c|$$

$$-(v+\ln|1-v|) + \ln|cx|$$

$$\ln|cx| + v + \ln|1-v| = 0$$

$$\Rightarrow \ln|cx(1-v)| + v = 0 \longrightarrow (2)$$

Again put $v = \frac{y}{x}$ ~~in~~ (2)

$$\ln\left|cx\left(1-\frac{y}{x}\right)\right| + \frac{y}{x} = 0$$

$$\ln\left|cx\left(\frac{x-y}{x}\right)\right| + \frac{y}{x} = 0$$

$$\ln|c(x-y)| + \frac{y}{x} = 0$$

OR

$$x\ln|c(x-y)| + y = 0$$

ANS

(2) $(x^2+y^2)\,dx + 2xy\,dy = 0 \rightarrow (1)$

Sol: The given DE is a homogeneous DE of order 2. So put

$$y = vx$$

$$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$(1) \Rightarrow \quad \frac{dy}{dx} = -\frac{(x^2+y^2)}{2xy}$$

So putting these values in given DE, we get

$$v+x\frac{dv}{dx} = -\left[\frac{x^2+(vx)^2}{2xvx}\right]$$

$$v+x\frac{dv}{dx} = -\left[\frac{x^2+v^2x^2}{2x^2v}\right]$$

$$v+x\frac{dv}{dx} = -\cancel{x^2}\left(\frac{1+v^2}{2\cancel{x^2}v}\right)$$

$$x\frac{dv}{dx} = -\left(\frac{1+v^2}{2v}\right)-v$$

$$x\frac{dv}{dx} = \frac{-1-v^2-2v^2}{2v} = \frac{-(1+3v^2)}{2v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{-(1+3v^2)}{2v}$$

Separating variables

$$\frac{2v}{(1+3v^2)}dv = -\frac{dx}{x}$$

Integrating b/s

$$\int\frac{2v}{1+3v^2}dv = -\int\frac{dx}{x}$$

$$2\int\frac{6v}{6(1+3v^2)}dv = -\int\frac{dx}{x}$$

$$\frac{2}{6}\int\frac{6v\,dv}{(1+3v^2)} = -\int\frac{dx}{x}$$

$$\frac{1}{3}\ln|1+3v^2| = -\ln|x|+\ln|c|$$

$$\ln|1+3v^2| = -3\ln|x|+\ln|c|$$

$$\ln|1+3v^2| = 3\ln\left|\frac{c}{x}\right|$$

$$\ln|1+3v^2| = \ln\left|\frac{c}{x}\right|^3 \qquad \left[\because a\ln b = \ln b^a\right.$$

$$\Rightarrow \quad 1+3v^2 = \left(\frac{c}{x}\right)^3$$

Now $\quad 1+3v^2 = \frac{c^3}{x^3}$

$$x^3(1+3v^2) = c^3$$

put $v = \frac{y}{x}$

$$x^3\left(1+3\left(\frac{y}{x}\right)^2\right) = c^3$$

$$x^3\left(1+\frac{3y^2}{x^2}\right) = c^3$$

$$x^3\left(\frac{x^2+3y^2}{x^2}\right) = c^3$$

$$x(x^2+3y^2) = c^3$$

$$x^3+3xy^2 = c^3 \quad \underline{\text{Ans}}$$

(3) $(3x-y)y' - y - x = 0$

Sol:- Since we know that

$$(3x-y)y' = (x+y)$$

$$y' = \frac{x+y}{3x-y}$$

OR

$$\frac{dy}{dx} = \frac{x+y}{3x-y} \longrightarrow (A)$$

So DE(A) is a homogeneous of degree 1.

Now put $y = vx$ &

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

So $(A) \Rightarrow \quad v + x\frac{dv}{dx} = \frac{x+vx}{3x-vx}$

$$v + x\frac{dv}{dx} = \frac{x(1+v)}{x(3-v)} = \frac{(1+v)}{(3-v)}$$

$$v+x\frac{dv}{dx} = \frac{1+v}{3-v}$$

$$x\frac{dv}{dx} = \frac{1+v}{3-v} - v$$

$$x\frac{dv}{dx} = \frac{1+v-3v+v^2}{3-v}$$

$$x\frac{dv}{dx} = \frac{v^2-2v+1}{3-v}$$

Separating variables

$$\frac{3-v}{v^2-2v+1}dv = \frac{dx}{x}$$

Integrating b/s

$$+\int \frac{3-v}{v^2-2v+1}dv = \int \frac{dx}{x}$$

$$-\int \frac{v-3}{v^2-2v+1}dv = \int \frac{dx}{x}$$

$$\Rightarrow 3\int \frac{dv}{v^2-2v+1} - \int \frac{v}{v^2-2v+1}dv = \int \frac{dx}{x}$$

$$3\int \frac{dv}{v^2-2v+1} - \frac{1}{2}\int \frac{2v-2+2}{v^2-2v+1}dv = \int \frac{dx}{x}$$

$$3\int \frac{dv}{v^2-2v+1} - \frac{1}{2}\int \frac{2v-2}{v^2-2v+1}dv - \frac{1}{2}\int \frac{2dv}{v^2-2v+1} = \int \frac{dx}{x}$$

$$3\int \frac{dv}{(v-1)^2} - \frac{1}{2}\int \frac{2v-2}{(v-1)^2} - \frac{1}{2}\int \frac{2\,dv}{(v-1)^2} = \int \frac{dx}{x}$$

$$3\int \frac{dv}{(v-1)^2} - \int \frac{dv}{(v-1)^2} - \frac{1}{2}\int \frac{2v-2}{(v-1)} = \int \frac{dx}{x}$$

$$2\int \frac{dv}{(v-1)^2} - \frac{1}{2}\int \frac{2v-2}{(v-1)^2} = \int \frac{dx}{x}$$

$$\frac{2(v-1)^{-2+1}}{-2+1} - \frac{1}{2}\ln|(v-1)^2| = \ln x + \ln c$$

$$\frac{2(v-1)^{-1}}{-1} - \frac{1}{2}\cdot 2\ln|v-1| = \ln|cx|$$

$$\frac{-2}{v-1} = \ln|v-1| + \ln|cx|$$

$$\frac{2}{1-v} = \ln|cx(v-1)|$$

use $v = \frac{y}{x}$, we get

$$\frac{2}{1-y/x} = \ln\left|cx\left(\frac{y}{x}-1\right)\right|$$

$$\frac{2}{\frac{x-y}{x}} = \ln\left|cx\left(\frac{y-x}{x}\right)\right|$$

$$\frac{2x}{x-y} = \ln|c(y-x)| \text{ ANS}$$

(4) $\frac{dy}{dx} = \frac{y-x}{y+x} \longrightarrow (B)$

Sol:- Since it is clearly, that this is a homogeneous ODE of degree 1.
So use $y = vx$ &

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

Eq(B) $\Rightarrow$ $v + x\frac{dv}{dx} = \frac{vx-x}{vx+x}$

$$v + x\frac{dv}{dx} = \frac{x(v-1)}{x(v+1)}$$

$$x\frac{dv}{dx} = \frac{(v-1)}{(v+1)} - v$$

$$x\frac{dv}{dx} = \frac{(v-1) - v(v+1)}{v+1}$$

$$x\frac{dv}{dx} = \cancel{v} - 1 - v^2 - \cancel{v}$$

$$x\frac{dv}{dx} = \frac{-(1+v^2)}{v+1}$$

Separating variables, we have

$$\int \frac{(v+1)}{v^2+1}\,dv = -\int \frac{dx}{x}$$

$$\int \frac{v}{v^2+1}\,dv + \int \frac{dv}{1+v^2} = -\int \frac{dx}{x}$$

$$\frac{1}{2}\int \frac{2v}{v^2+1}\,dv + \int \frac{dv}{1+v^2} = -\int \frac{dx}{x}$$

$$\frac{1}{2}\ln|v^2+1| + \tan^{-1}(v) = -\ln x + \ln c$$

$$\tan^{-1}(v) = \ln\left|\frac{c}{x}\right| - \frac{1}{2}\ln|v^2+1|$$

$$\tan^{-1}(v) = \ln\left|\frac{c}{x}\right| - \ln|v^2+1|^{\frac{1}{2}}$$

$$= \ln\left|\frac{c}{x}\Big/\sqrt{v^2+1}\right|$$

$$\tan^{-1}(v) = \ln\left|\frac{c}{x\sqrt{v^2+1}}\right|$$

put $v = \frac{y}{x}$ again

$$\tan^{-1}\left(\frac{y}{x}\right) = \ln\left|\frac{c}{x\sqrt{(y/x)^2+1}}\right|$$

$$= \ln\left|\frac{c}{x\sqrt{\frac{y^2+x^2}{x^2}}}\right|$$

$$= \ln\left|\frac{c}{\cancel{x}\frac{\sqrt{y^2+x^2}}{\cancel{x}}}\right|$$

$$\left.\begin{aligned}\tan^{-1}\left(\frac{y}{x}\right) &= \ln\left|\frac{c}{\sqrt{x^2+y^2}}\right| \\ \frac{y}{x} &= \tan\left[\ln\left|\frac{c}{\sqrt{x^2+y^2}}\right|\right]\end{aligned}\right\}$$

OR

ANS

(5) $(x^2+3y^2)\,dx - 2xy\,dy = 0$; $y(1)=6$

Sol: Since $(x^2+3y^2)dx - 2xy\,dy = 0$

$$\Rightarrow (x^2+3y^2)dx = 2xy\,dy$$

$$\Rightarrow \frac{dy}{dx} = \frac{x^2+3y^2}{2xy}$$

Which shows that the equation is homogeneous of degree 2. So

Let us use $y = vx$ so

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$\therefore \quad v + x\frac{dv}{dx} = \frac{x^2+3(vx)^2}{2xvx}$$

$$v + x\frac{dv}{dx} = \frac{x^2+3v^2x^2}{2x^2v}$$

$$v + x\frac{dv}{dx} = \frac{x^2(1+3v^2)}{2x^2v}$$

$$v + x\frac{dv}{dx} = \frac{(1+3v^2)}{2v}$$

$$x\frac{dv}{dx} = \frac{1+3v^2}{2v} - v$$

$$= \frac{1+3v^2-2v^2}{2v}$$

$$x\frac{dv}{dx} = \frac{1+v^2}{2v}$$

Separating variables

$$\int \frac{2v}{1+v^2}\,dv = \int \frac{dx}{x}$$

Integrating

$$\ln|1+v^2| = \ln|x| + \ln|c|$$

$$\Rightarrow \quad 1+v^2 = cx$$

Replacing v by $\frac{y}{x}$, we obtain

$$1+\left(\frac{y}{x}\right)^2=|cx|$$

$$1+\frac{y^2}{x^2}=|cx|$$

$$\frac{x^2+y^2}{x^2}=|cx|$$

$$x^2+y^2=|cx|\cdot x^2$$

$$x^2+y^2=cx^3$$

or $y^2=cx^3-x^2$

$$y=\pm\sqrt{cx^3-x^2} \longrightarrow *$$

Now $y(2)=6 \Rightarrow x=2,\ y=6$

$$6=\pm\sqrt{c\cdot 2^3-2^2}$$

$$6=\pm\sqrt{8c-4}$$

$$8c-4=36$$

$$8c=36+4$$

$$\Rightarrow c=5$$

Using in $(*)$, we obtain

$$y=\sqrt{5x^3-x^2} \quad \underline{\text{ANS}}$$

<u>Note</u>:- As from the given Initial condition, we have $y(2)=6$ i.e $y(2)=+ive$. that's why we have taken the plus sign in the radical.

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BS Maths (AWKUM)

(6) $x\sin\left(\frac{y}{x}\right)dy = \left(y\sin\left(\frac{y}{x}\right) - x\right)dx \longrightarrow (A)$

Sol:- Since

$$x\sin\left(\frac{y}{x}\right)dy = \left(y\sin\left(\frac{y}{x}\right) - x\right)dx$$

$$\Rightarrow \frac{dy}{dx} = \frac{y\sin\left(\frac{y}{x}\right) - x}{x\sin\left(\frac{y}{x}\right)}$$

$\div$ing Numerator & Denominator by "x",

$$\frac{dy}{dx} = \frac{\frac{y}{x}\sin\left(\frac{y}{x}\right) - 1}{\sin\left(\frac{y}{x}\right)}$$

This is a homogeneous DE of degree 1.

So Let $\frac{y}{x} = v$

$$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

Using in eq(A)

$$v + x\frac{dv}{dx} = \frac{v\sin v - 1}{\sin v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{v\sin v - 1}{\sin v} - v$$

$$= \frac{\cancel{v\sin v} - 1 - \cancel{v\sin v}}{\sin v}$$

$$x\frac{dv}{dx} = \frac{-1}{\sin v}$$

Separating variables

$$\frac{dx}{x} = -\sin v\, dv$$

Integrating b/s

$$\int \frac{dx}{x} = -\int \sin v\, dv$$

$$\ln x = -(-\cos v) + c$$

$$\ln x = \cos v + c$$

Replace the value of $v = \frac{y}{x}$ again

$\ln x = \cos\left(\frac{y}{x}\right) + c$ ANS

Solve Initial value problems

(7) $x\,dy = (x+y)\,dx$; $y(1) = -1$

Sol:- Since $x\,dy = (x+y)\,dx$

$$= \frac{dy}{dx} = \frac{(x+y)}{x} \longrightarrow (A)$$

Which is homogeneous DE of degree 1.

So let $y = vx$

$$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

using values in eq (A)

$$v + x\frac{dv}{dx} = \frac{x+vx}{vx}$$

$$= \frac{x(1+v)}{xv}$$

$$x\frac{dv}{dx} = \left(\frac{1+v}{v}\right) - v$$

$$x\frac{dv}{dx} = \frac{1+v-v^2}{v}$$

$$x\frac{dv}{dx} = \frac{-v^2+v+1}{v}$$

Separating variables

$$\frac{v}{-v^2+v+1}\,dv = \frac{dx}{x}$$

Integrating b/s w.r.t required variables

$$\int \frac{v}{-v^2+v+1}\,dv = \int \frac{dx}{x}$$

further solve it (Class Work)

2nd method

As $xdy = (x+y)\,dx$

$\Rightarrow xdy = (x+y)\,dx$ (Homogeneous)

$\Rightarrow x\frac{dy}{dx} = (x+y) \rightarrow (1)$

Let $y = vx$

$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

Hence (1) $\Rightarrow x\left[v + x\frac{dv}{dx}\right] = x + vx$

$\Rightarrow \cancel{xv} + x^2\frac{dv}{dx} = x + \cancel{vx}$

$x^2\frac{dv}{dx} = x$

Separating variables

$dv = \frac{x}{x^2}\,dx$

$dv = \frac{dx}{x}$

Integrating b/s

$\int dv = \int \frac{dx}{x}$

$v = \ln x + c$

Replace v by $\frac{y}{x}$

$\frac{y}{x} = \ln x + c$

$y = x(\ln x + c) \rightarrow (2)$

Now applying $y(1) = -1$

$-1 = 1(\ln(1) + c)$ $\quad$ $y(1) = -1 \Rightarrow x = 1, y = -1$

$-1 = 1(0 + c)$

$\Rightarrow c = -1$

Hence (2) $\Rightarrow y = x(\ln x - 1)$ Ans

(8) $xy\,dy = (x^2+y^2)dx$; $y(1) = -2$

Sol:- Since $xy\,dy = (x^2+y^2)\,dx$

$$\Rightarrow \frac{dy}{dx} = \frac{(x^2+y^2)}{xy} \longrightarrow (A)$$

(Since this DE is just like the DE in question (2). So solve it by yourself)

After solving, we obtain general solution as

$$y^2 = x^2 \ln(x^2) + cx^2 \longrightarrow (B)$$

Now we have $y(1) = -2 \Rightarrow x=1$, $y=-2$

$\therefore \quad (-2)^2 = (1)^2 \times \ln(1^2) + c(1)^2$

$4 = 0 \times 1 + c \qquad \because \ln(1) = 0$

$\Rightarrow \boxed{c=4}$

So eq.(B) becomes

$y^2 = x^2\ln(x^2) + 4x^2$

$\Rightarrow \quad y = -\sqrt{x^2\ln(x^2)+4x^2}$ **ANS**

Where we have taken -ive square root, because $y(1) = -2$ required it.

(9) $x\,dy - y\,dx = \sqrt{x^2+y^2}\,dx$; $y(1)=0$

Sol:- Since $x\,dy - y\,dx = \sqrt{x^2+y^2}\,dx$

$\Rightarrow x\frac{dy}{dx} - y = \sqrt{x^2+y^2} \qquad \div$ing by dx

$\Rightarrow x\frac{dy}{dx} = \sqrt{x^2+y^2} + y$

$$\frac{dy}{dx} = \frac{y+\sqrt{x^2+y^2}}{x}$$

$$\frac{dy}{dx} = \frac{y}{x} + \frac{\sqrt{x^2+y^2}}{x} \longrightarrow (1)$$

Which is homogeneous of degree 1

Let $y = vx$

$$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

So (1) becomes

$$v + x\frac{dv}{dx} = \frac{vx + \sqrt{x^2+x^2v^2}}{x}$$

$$v + x\frac{dv}{dx} = \frac{vx + \sqrt{x^2(1+v^2)}}{x}$$

$$= \frac{vx + x\sqrt{1+v^2}}{x}$$

$$= \frac{\cancel{x}(v+\sqrt{1+v^2})}{\cancel{x}}$$

$$\Rightarrow \cancel{v} + x\frac{dv}{dx} = \cancel{v} + \sqrt{1+v^2}$$

$$x\frac{dv}{dx} = \sqrt{1+v^2}$$

Separating variables

$$\frac{dv}{\sqrt{v^2+1}} = \frac{dx}{x}$$

Integrating b/s

$$\int \frac{dv}{\sqrt{1^2+v^2}} = \int \frac{dx}{x}$$

$$\ln\left|v+\sqrt{v^2+1^2}\right| = \ln x + \ln c \qquad \because \int \frac{dx}{\sqrt{x^2+a^2}} = \ln\left|x+\sqrt{x^2+a^2}\right|$$

$$\Rightarrow v + \sqrt{v^2+1} = cx$$

put $v = \frac{y}{x}$

$$\frac{y}{x} + \sqrt{\left(\frac{y}{x}\right)^2+1} = cx$$

$$\frac{y}{x} + \sqrt{\frac{y^2}{x^2} + 1} = cx$$

$$\frac{y}{x} + \frac{\sqrt{y^2 + x^2}}{x} = cx$$

$$\frac{y + \sqrt{y^2 + x^2}}{x} = cx$$

$$\frac{y + \sqrt{y^2 + x^2}}{1} = cx^2 \longrightarrow (C)$$

Now $y(1) = 0 \Rightarrow x = 1,\ y = 0$

$$\therefore \quad \frac{0 + \sqrt{0^2 + 1^2}}{1} = c(1)^2$$

$$\sqrt{1^2} = 1 = c$$

$$\Rightarrow \boxed{c = 1}$$

So putting c value in eq(C), we get

$$y + \sqrt{y^2 + x^2} = x^2 \quad \underline{\underline{\text{ANS}}}$$

(10) $(x^2 + 3y^2)\,dx = 2xy\,dy$; $y(2) = 6$

Sol:- Since $(x^2 + 3y^2)\,dx = 2xy\,dy$

$$\Rightarrow \frac{dy}{dx} = \frac{x^2 + 3y^2}{2xy} \longrightarrow (1)$$

Which is homogeneous DE of degree (2)

Let $y = vx$

$$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

So eq(1) becomes

$$v + x\frac{dv}{dx} = \frac{x^2 + (vx)^2 \cdot 3}{2x\,vx}$$

$$v + x\frac{dv}{dx} = \frac{x^2 + 3v^2x^2}{2vx^2}$$

$$v + x\frac{dv}{dx} = \frac{\cancel{x^2}(1+3v^2)}{2\cancel{x^2}v}$$

$$v + x\frac{dv}{dx} = \frac{1+v^2}{2v}$$

$$x\frac{dv}{dx} = \frac{1+3v^2}{2v} - v$$

$$x\frac{dv}{dx} = \frac{1+3v^2-2v^2}{2v}$$

$$x\frac{dv}{dx} = \frac{1+v^2}{2v}$$

Separating variables

$$\frac{2v}{1+v^2}\,dv = \frac{dx}{x}$$

Integrating b/s

$$\int \frac{2v}{1+v^2}\,dv = \int \frac{dx}{x}$$

$$\ln|1+v^2| = \ln x + \ln c$$

$$\ln|1+v^2| = \ln(cx)$$

$$\Rightarrow 1+v^2 = cx \longrightarrow (*)$$

put $v = \frac{y}{x}$ in (*),

$$1 + \left(\frac{y}{x}\right)^2 = cx$$

$$\frac{x^2+y^2}{x^2} = cx$$

$$x^2+y^2 = cx^3$$

$$y^2 = cx^3 - x^2$$

$$y = \pm\sqrt{cx^3 - x^2} \longrightarrow (**)$$

Now $y(2) = 6$

$\Rightarrow x = 2,\ y = 6$

So from $(**)$ we get

$$6 = \pm\sqrt{c(2)^3 - 2^2}$$

$$6 = \pm\sqrt{8c - 4}$$

$$6 = \sqrt{8c - 4}$$ (+ive sign of radical because $y(2) = 6$ (+ive)

$$36 = 8c - 4$$

$$8c = 40$$

$$\Rightarrow \boxed{c = 5}$$

So eq. $(**)$ becomes

$$\boxed{y = \sqrt{5x^3 - x^2}} \text{ ANS}$$

Home Work:-

Solve the following ODES

(1) $\dfrac{dy}{dx} = -\left(\dfrac{x^{2} - 3y^2}{2xy}\right)$ ANS: $|y^2 - x^4| = |cx| x^2$

(2) $y\,dy + x\,dx = \sqrt{x^2 + y^2}\,dx$

ANS:- $y^2 - 2cx + c^2 = 0$

(3) $(x^2 + xy + y^2)\,dx - x^2\,dy$

ANS:- $\tan^{-1}\left(\frac{y}{x}\right) - \ln x = c$

(4) $\dfrac{dy}{dx} = \dfrac{2x}{y + x^2 y}$, $y(0) = 2$

ANS: $y^2 = \ln(e^4(1 + x^2)^2)$

(5) $\dfrac{dy}{dx} = -\left(\dfrac{2x - 5}{4x - y}\right)$; $y(1) = 4$

ANS:- $y + \sqrt{x^2 + y^2} = x^2$

(iv) "Differential Equations reducible to Homogeneous differential Equations"

A non-homogeneous DE is the same as homogeneous DE, except they have terms involving only constant and x. It is of the form

$$\frac{dy}{dx} = \frac{a_1x+b_1y+C_1}{a_2x+b_2y+C_2} \longrightarrow (1)$$

where $a_1, b_1, c_1, a_2, b_2, c_2$ are real constants. DE of the form (1) can be reduced to homogeneous form by taking new variable x and y such that $x = X+h$ and $y = Y+K$, where h & k are constants to be choosen as to make the given homogeneous.

Examples:- $y' = \frac{3x+2y-1}{x+y+3}$

$$y' = \frac{x-y-1}{2x-2y-1}$$

Now for solving such type of ODEs we use the following steps. Keep in mind these steps will be followed in case when $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$.

(i) Substitute $x = X+h \Rightarrow dx = dX$
and $y = Y+K \Rightarrow dy = dY$

(ii) Put these values in equation (1)

$$\frac{dY}{dX} = \frac{a_1(X+h)+b_1(Y+K)+C_1}{a_2(X+h)+b_2(Y+K)+C_2}$$

$$\frac{dY}{dX} = \frac{a_1X + a_1h + b_1Y + b_1K + c_1}{a_2X + a_2h + b_2Y + b_2K + c_2}$$

$$\frac{dY}{dX} = \frac{(a_1X + b_1Y) + (a_1h + b_1k + c_1)}{(a_2X + b_2Y) + (a_2h + b_2K + c_2)}$$

(iii) Put $a_1h + b_1k + c_1 = 0$ and
$a_2h + b_2k + c_2 = 0$

(iv) Then $\frac{dY}{dX} = \frac{a_1X + b_1Y}{a_2X + b_2Y}$

which is homogeneous DE, and we know that how to solve homogeneous DE (see solving steps in (iii)).

(v) Find the value of h and k in step(iii) and then put all the values back in equation (1).

* If however $\frac{a_1}{a_2} = \frac{b_1}{b_2} = m$ (say), then the DE becomes of the form

$$\frac{dy}{dx} = \frac{m(a_1x + b_1y) + c_1}{a_1x + b_1y + c_2}$$

and we know that how to solve such type of DE (see techinque ii on page 29) For this we substitute $v = a_1x + b_1y$. and then further solve it.

Examples :- Solve the following DEqs.

(1) $\frac{dy}{dx} = \frac{x + y - 1}{x - y + 3} \longrightarrow (1)$

Sol:- Since this is a non-homogeneous DE of first order and first degree, and it can be reduced to homogeneous DE by transformation of variables on substituting

$$x = X + h \quad \& \Rightarrow dx = dX$$
$$y = Y + k \quad \Rightarrow \quad dy = dY$$

So putting values in eq(1) gives us

$$\frac{dY}{dX} = \frac{X+h+Y+k-1}{X+h-Y-k+3}$$

$$\frac{dY}{dX} = \frac{(X+Y)+(h+k-1)}{(X-Y)+(h-k+3)}$$

Now $h+k-1 = 0 \rightarrow (i)$

& $h-k+3 = 0 \rightarrow (ii)$

adding (i) & (ii)

$$\begin{array}{r} h+k-1=0 \\ h-k+3=0 \\ \hline 2h+2=0 \Rightarrow \boxed{h=-1} \end{array}$$

and by substituting $h=-1$ in (i), we get

$$\boxed{k=2}$$

So we have

$$\frac{dY}{dX} = \frac{X+Y}{X-Y} \longrightarrow (2)$$

Equation (2) is now homogeneous DE of degree 1. So using the procedure of homogeneous DE, we assume that

$$Y = vX$$
$$\Rightarrow \frac{dY}{dX} = v + X\frac{dv}{dX}$$

so eq(2) becomes

$$v + X\frac{dv}{dX} = \frac{X+vX}{X-vX} = \frac{X(1+v)}{X(1-v)}$$

$$v + x\frac{dv}{dx} = \frac{1+v}{1-v}$$

$$x\frac{dv}{dx} = \frac{1+v}{1-v} - v$$

$$x\frac{dv}{dx} = \frac{1+v-v+v^2}{1-v}$$

$$x\frac{dv}{dx} = \frac{v^2+1}{1-v}$$

Separating variables

$$\int\frac{1-v}{1+v^2}\,dv = \frac{dx}{x}$$

Integrating

$$\int\frac{(1-v)}{(1+v^2)}\,dv = \int\frac{dx}{x}$$

$$\int\frac{dv}{1+v^2} - \int\frac{v}{1+v^2}\,dv = \int\frac{dx}{x}$$

$$\tan^{-1}(v) - \frac{1}{2}\int\frac{2v}{1+v^2}\,dv = \ln|x| + \ln c$$

$$\tan^{-1}(v) - \frac{1}{2}\ln|1+v^2| = \ln x + \ln c_1$$

$$\tan^{-1}(v) = \frac{1}{2}\ln|1+v^2| + \ln x + \ln c_1$$

$$2\tan^{-1}(v) = \ln|1+v^2| + 2\ln x + 2\ln c_1$$

$$2\tan^{-1}(v) = \ln|1+v^2| + \ln x^2 + \ln c_1^2$$

$$2\tan^{-1}(v) = \ln|1+v^2| + \ln x^2 + \ln c \qquad c_1^2 = c$$

$$2\tan^{-1}(v) = \ln|cx^2\cdot(1+v^2)| \longrightarrow (3)$$

Now we have

$$y = vx \Rightarrow v = \frac{y}{x}$$

So eq(3) becomes

$$2\tan^{-1}\left(\frac{y}{x}\right) = \ln\left|cx^2\left(1+\left(\frac{y}{x}\right)^2\right)\right|$$

$$2\tan^{-1}\left(\frac{y}{x}\right) = \ln\left|c\cancel{x^2}\left(\frac{x^2+y^2}{\cancel{x^2}}\right)\right|$$

$$2\tan^{-1}\left(\frac{Y}{X}\right) = \ln\left|c(X^2+Y^2)\right| \longrightarrow (4)$$

Now $x = X + h \Rightarrow X = x - h$

& $y = Y + k \Rightarrow Y = y - k$

Since $h = -1,\ k = 2$

So $X = x - (-1) = x + 1$

$Y = y - 2$

So eq.(4) becomes

$$2\tan^{-1}\left(\frac{y-2}{x+1}\right) = \ln\left|c\left((x+1)^2 + (y-2)^2\right)\right|$$

$$\therefore\ 2\tan^{-1}\left(\frac{y-2}{x+1}\right) = \ln\left|c(x+1)^2 + c(y-2)^2\right| \ \underline{\text{Ans}}$$

(2) $$\frac{dy}{dx} = \frac{x+3y-5}{x-y-1} \longrightarrow (A)$$

<u>Sol:-</u> Here $\frac{a_1}{a_2} = \frac{1}{1},\ \frac{b_1}{b_2} = \frac{3}{-1}$

$$\Rightarrow \frac{a_1}{a_2} \neq \frac{b_1}{b_2}$$

So Let $x = X + h \Rightarrow dx = dX$

& $y = Y + k \Rightarrow dy = dY$

So eq(A) becomes

$$\frac{dY}{dX} = \frac{X+h+3(Y+k)-5}{X+h-Y-k-1}$$

$$\frac{dY}{dX} = \frac{X+h+3Y+3k-5}{X+h-Y-k-1}$$

$$\frac{dY}{dX} = \frac{(X+3Y)+(h+3k-5)}{(X-Y)+(h-k-1)} \longrightarrow (B)$$

Now
$$h+3k-5=0 \rightarrow (i)$$
$$-\quad h+k-1=0 \rightarrow (ii)$$
$$4k-4=0$$
$$\Rightarrow \boxed{k=1}$$
subtracting (ii) from (i)

and by putting $k=1$ in (i), we get
$h=2$
putting values of h and k in eq (B) yields

$$\frac{dy}{dx} = \frac{x+3y}{x-y} \longrightarrow (C)$$

Equation (C) is now homogeneous of degree 1. So
Let $y = vx$
$$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$
Using these values in eq (C) gives

$$v + \frac{dv}{dx} = \frac{x+3vx}{x-vx}$$
$$v + \frac{dv}{dx} = \frac{x(1+3v)}{x(1-v)}$$
$$v + \frac{dv}{dx} = \frac{1+3v}{1-v}$$
$$\frac{dv}{dx} = \frac{1+3v}{1-v} - v$$
$$= \frac{1+3v-v+v^2}{1-v}$$
$$\frac{dv}{dx} = \frac{v^2+2v+1}{1-v}$$
$$\frac{dv}{dx} = \frac{(1+v)^2}{1-v}$$
Separating variables

$$\frac{1-v}{(v+1)^2}\,dv = \frac{dx}{x}$$

Integrating b/s

$$\int \frac{1-v}{(v+1)^2}\,dv = \int \frac{dx}{x}$$

$$\int\left(\frac{1}{(v+1)^2} - \frac{v}{(v+1)^2}\right)dv = \int \frac{dx}{x}$$

$$\int \frac{dv}{(v+1)^2} - \int \frac{v\,dv}{(v+1)^2} = \int \frac{dx}{x}$$

$$\int (v+1)^{-2}dv - \int \frac{1+v-1}{(v+1)^2}\,dv = \int \frac{dx}{x}$$

$$\int (v+1)^{-2}dv - \int \frac{1+v}{(v+1)^2}\,dv + \int \frac{dv}{(v+1)^2} = \int \frac{dx}{x}$$

$$\int (v+1)^{-2}dv - \int \frac{1+v}{(v+1)^2}\,dv + \int (v+1)^{-2}dv = \int \frac{dx}{x}$$

$$2\int (v+1)^{-2}dv - \int \frac{1+v}{(v+1)^2}\,dv = \int \frac{dx}{x}$$

$$2\int (v+1)^{-2}dv - \int \frac{dv}{v+1} = \int \frac{dx}{x}$$

$$2\left(\frac{(v+1)^{-2+1}}{-2+1}\right) - \ln(v+1) = \ln(x) + \ln c$$

$$\frac{-2}{1+v} - \ln(v+1) = \ln x + \ln c$$

Now put $v = \frac{y}{x}$ again

$$\frac{-2}{1+\frac{y}{x}} - \ln\left(\frac{y}{x}+1\right) = \ln x + \ln c$$

$$\frac{-2}{\frac{x+y}{x}} - \ln\left(\frac{y+x}{x}\right) = \ln x + \ln c$$

SafiMaths(Alvicum) (64)

$$\frac{-2X}{X+Y}-\left(\ln(X+Y)-\ln X\right)=\ln X+\ln c$$

$$\frac{-2X}{X+Y}-\ln(X+Y)+\cancel{\ln X}=\cancel{\ln X}+\ln c$$

$$\frac{-2X}{X+Y}=\ln(X+Y)+\ln c$$

$$\frac{-2X}{X+Y}=\ln c(X+Y) \longrightarrow (D)$$

Now as $x=X+h \Rightarrow X=x-h$

& $y=Y+k \Rightarrow Y=y-k$

and by putting values of h & k, we get

$X=x-2$ and $\quad \because h=2$

$Y=y-1 \quad k=1$

putting these values in eq (D)

$$\frac{-2(x-2)}{x-2+y-1}=\ln(c(x-2+y-1)$$

$$\frac{-2(x-2)}{x+y-3}=\ln(c(x+y-3))$$

$$\frac{4-2x}{x+y-3}=\ln(c(x+y-3))$$

$$4-2x=(x+y-3)\ln(c(x+y-3))$$ Ans

(3) $\dfrac{dy}{dx} = \dfrac{3x-4y-2}{3x-4y-3} \longrightarrow (1)$

Sol:- Here we have, $a_1 = 3$, $a_2 = 3$

and $b_1 = -4$, $b_2 = -4$

$\therefore \dfrac{a_1}{a_2} = \dfrac{+3}{3}$, $\dfrac{b_1}{b_2} = \dfrac{-4}{-4}$

$\Rightarrow \dfrac{a_1}{a_2} = \dfrac{b_1}{b_2}$

So let $3x-4y = v$

$\Rightarrow 3-4\dfrac{dy}{dx} = \dfrac{dv}{dx} \Rightarrow \dfrac{dy}{dx} = \dfrac{-1}{4}\left(\dfrac{dv}{dx}-3\right)$

so by putting values, eq (1) becomes

$$\frac{-1}{4}\left(\frac{dv}{dx}-3\right) = \frac{v-2}{v-3}$$

xing by -4 b/s

$$\frac{dv}{dx}-3 = -4\left(\frac{v-2}{v-3}\right)$$

$$\frac{dv}{dx}-3 = \frac{-4v+8}{v-3}$$

$$\frac{dv}{dx} = \frac{-4v+8}{v-3}+3$$

$$\frac{dv}{dx} = \frac{-4v+8+3v-9}{v-3}$$

$$\frac{dv}{dx} = \frac{-(v+1)}{v-3}$$

separating variables

$$\frac{v-3}{v+1}\,dv = -dx$$

Integrating b/s

$$\int \frac{v-3}{v+1}\, dv = -\int dx$$

Improper fraction

$$\begin{array}{r|l} & 1 \\ v+1 & v-3 \\ & \underline{\pm v \pm 1} \\ & -4 \end{array}$$

$$= 1 - \frac{4}{v+1}$$

$$\int \left(1 - \frac{4}{v+1}\right) dv = -\int dx$$

$$\int dv - 4\int \frac{dv}{v+1} = -\int dx$$

$$v - 4\ln(v+1) = -x + c_1$$

repute $v = 3x - y$ again

$$3x - 4y - 4\ln(3x - 4y + 1) = -x + c_1$$

$$4x - 4y - 4\ln(3x - 4y + 1) = c_1$$

$\div$ ing b/s by 4, we get

$$x - y - \ln(3x - 4y + 1) = \frac{c_1}{4}$$

Let $C = \frac{c_1}{4}$

$$\therefore \quad x - y - \ln(3x - 4y + 1) = C \quad \text{Ans}$$

(4) $(2x + 3y - 1)\, dx + (2x + 3y + 2)\, dy = 0$

where $y(1) = 3$.

Sol:- $\frac{dy}{dx} + \frac{2x+3y-1}{2x+3y+2} \longrightarrow (A)$

Since $a_1 = 2, \; a_2 = 2$

$b_1 = 3, \; b_2 = 3$

$$\Rightarrow \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{2}{2} = \frac{3}{3} = 1$$

So Let $2x + 3y = v$

$$\Rightarrow \frac{dv}{dx} = 2 + 3\frac{dy}{dx}$$

$$\Rightarrow 3\frac{dy}{dx} = \frac{dv}{dx} - 2$$

$$\frac{dy}{dx} = \frac{1}{3}\left(\frac{dv}{dx} - 2\right)$$

so eq (A) becomes

$$\frac{1}{3}\left(\frac{dv}{dx} - 2\right) + \frac{v-1}{v+2} = 0$$

$$\frac{dv}{dx} - 2 + \frac{3(v-1)}{v+2} = 0$$

$$\Rightarrow \frac{dv}{dx} - 2 + \frac{3(v-1)}{v+2} = 0$$

$$\frac{dv}{dx} + \frac{3v-3}{v+2} - 2 = 0$$

$$\frac{dv}{dx} + \frac{3v-3-2v-4}{v+2} = 0$$

$$\frac{dv}{dx} + \frac{v-7}{v+2} = 0$$

Separating variables

$$\frac{v+2}{v-7}\,dv + dx = 0$$

Integrating b/s

$$\int \frac{v+2}{v-7}\,dv + \int dx = 0$$

$$\int \left(1 + \frac{9}{v-7}\right) dv + \int dx = 0$$

$$\begin{array}{r|l} & 1 \\ \hline v-7 & v+2 \\ & \underline{\pm v \mp 7} \\ & 9 \end{array}$$

$$\int dv + 9\int \frac{dv}{v-7} + \int dx = 0$$

$$= 1 + \frac{9}{v-7}$$

$$\Rightarrow v + 9\ln(v-7) + x = c_1$$

replacing v by $2x+3y$

$$2x+3y + 9\ln(2x+3y-7) + x = c_1$$

$$3x+3y+9\ln(2x+3y-7)=C_1$$

$$\Rightarrow x+y+3\ln(2x+3y-7)=\frac{C_1}{3}=C$$

$$x+y+3\ln(2x+3y-7)=C \longrightarrow (B)$$

Now $y(1)=3$

$\Rightarrow x=1,\ y=3$

So $1+3+3\ln(2(1)+3(3)-7)=C$

$$4+3\ln(11-7)=C$$

$$4+3\ln(4)=C$$

putting values of c in eq (B)

$$x+y+3\ln(2x+3y-7)=4+3\ln(4) \quad \underline{\underline{\text{ANS}}}$$

Home Works: Solve the non-homogeneous ODEs.

(1) $\dfrac{dy}{dx}=\dfrac{2y-x+5}{2x-y-4}$

ANS:- $(y-x+3)=c(y+x+1)$

(2) $(2x-3y+4)\,dx+(3x-2y+1)\,dy=0$

ANS:- $(x+y-3)^5=c(y-x-1)$

(3) $\dfrac{dy}{dx}+\dfrac{2x+y+1}{4x+2y-1}=0$

ANS: $x+2y+\ln(2x+y-1)=C$

(v) Exact Differential Equations

The expression

$$M(x,y)dx\,N(x,y)dy = 0 \longrightarrow (1)$$

is called an exact DE if there exists a continuously differentiable function $f(x,y)$ of two variables x and y such that the expression equals the total differential df. We know from calculus

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

Thus, if (1) is exact then

$$M(x,y) = \frac{\partial f}{\partial x} = f_x \quad \text{and}$$

$$N(x,y) = \frac{\partial f}{\partial y} = f_y$$

If (1) is an exact differential then the differential equation

$$M(x,y)dx + N(x,y)dy = 0$$

is called an exact equation.

Theorem:- The DE

$$M(x,y)dx + N(x,y)dy = 0 \longrightarrow (1^*)$$

is said to be an exact DE if and only if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

where the functions $M(x,y)$ and $N(x,y)$ have continuous first order partial derivatives.

Proof:- Suppose that the eq (1^*) is exact so that $Mdx + Ndy = 0$ is an exact DE. By definition, $\exists$ a function $f(x,y)$ such that

$$M(x,y) = \frac{\partial f}{\partial x} = f_x \text{ and}$$

$$N(x,y) = \frac{\partial f}{\partial y} = f_y$$

Then $M_y = \frac{\partial M}{\partial y} = f_{xy} = \frac{\partial^2 f}{\partial y \partial x}$

and $N_x = \frac{\partial N}{\partial x} = f_{yx} = \frac{\partial^2 f}{\partial x \partial y}$

Since M and N possess continuous first order partial derivatives. We have $f_{xy} = f_{yx}$ and therefore

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad \text{as desired}$$

In case this condition holds $\left(\text{i.e } \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}\right)$ then we use the following steps.

(1) Integrate M w.r.t "x" keeping y as a constant i.e ($\int M dx$, y is constant)

(2) Integrate N w.r.t y those terms of N which are independent of x.

(3) The sum of the above 2 steps is equal to a constant is a solution of the given exact DE OR Equating results of step (1) and step 2

(4) If you are given an IVP, plug in the initial condition to find the constant c.

Examples:- Solve the following exact DEqs, by checking whether it is exact or not.

(1) $(3x^2y+2)\,dx + (x^3+y)\,dy = 0$

Sol:- Since we see that the given DE is of the form $M\,dx + N\,dy = 0$ where $M = 3x^2y+2$, $N = x^3+y$

To show that the given DE is exact we need to show that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Now $\frac{\partial M}{\partial y} = M_y = 3x^2$

$\frac{\partial N}{\partial x} = N_x = 3x^2$

So we see that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = 3x^2.$$

This shows that this is an exact DE

Now using the steps

(i) Integrating M w.r.t x keeping y as a constant i.e

$$\int (3x^2y+2)\,dx$$

$$= \int 3x^2y\,dx + 2\int dx$$

$$= 3y\int x^2\,dx + 2\int dx$$

$$\cancel{3}y\frac{x^3}{\cancel{3}} + 2x + c_1$$

$$x^3y + 2x + c_1$$

(ii) Integrating N w.r.t y those terms of N which are Independent of x-So

$\int N\,dy$ (Independent of x terms)

$\int y\,dy$ $\qquad N = x^3+y$

$= \frac{y^2}{2} + c_2$

(iii) Equating steps (1) & (2) results or adding step(1) and step(2), we get

$2x + x^3y + c_1 + y^2/_2 + c_2 = 0$

$x^3y + 2x + \frac{y^2}{2} + c_1 + c_2 = 0$

$x^3y + 2x + \frac{y^2}{2} + C \quad ; \quad c_1 + c_2 = C$

or $x^3y + 2x + \frac{y^2}{2} = C^* \quad , \quad -C = C^*$

(2) $(y + 2xy^2)\,dx + (x + 2x^2y)\,dy = 0 \; ; \; y(1) = 1$

Sol:- This is a DE of the form

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

where $M = y + 2xy^2$, &

$N = x + 2x^2y$

Now $\frac{\partial M}{\partial y} = M_y = 1 + 4xy$

& $\frac{\partial N}{\partial x} = N_x = 1 + 4xy$

$\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = 1 + 4xy$

$\Rightarrow$ DE which is given is exact.

Now

(i) $\int M\,dx$; y is constant

$$I_1 = \int (y + 2xy^2)\,dx$$

$$= y\int dx + 2y^2\int x\,dx$$

$$= yx + 2y^2\frac{x^2}{2} + c_1$$

$$I_1 = xy + x^2y^2 + c_1$$

(ii) $\int N\,dy$ (free of x terms)

i.e $\int N(\text{free of } x \text{ terms})\,dy$

$$I_2 = \int (x + 2x^2y)\,dy = 0$$

because there are no free of "x" terms

$\therefore$ $I_2 = 0$

(iii) $I_1 + I_2 = xy + x^2y^2 + c_1$

$$= xy + x^2y^2 = -c_1$$

$$xy + x^2y^2 = c \longrightarrow (A)$$

Now $y(1) = 1 \Rightarrow x = y = 1$

so $1\cdot 1 + 1^2\cdot 1^2 = c$

$\Rightarrow c = 2$

Hence eq (A) becomes

$\boxed{xy + x^2y^2 = 2}$ ANS

(3) $x(2y^2+3x)\,dx + 2x^2y\,dy = 0$

Sol:- Since this is a DE of the form
$$M(x,y)\,dx + N(x,y)\,dy = 0$$
where $M(x,y) = x(2y^2+3x) = 2xy^2+3x^2$
& $N(x,y) = 2x^2y$

For exact DE, we know that
$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$
Now $\frac{\partial M}{\partial y} = 4xy$ &
$$\frac{\partial N}{\partial x} = 4xy$$
$$\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial y} = 4xy$$
Which shows that this is an exact DE.

Now

(i) $I_1 = \int M\,dx$; y is a constant
$$= \int (2xy^2+3x^2)\,dx$$
$$= y^2\int 2x\,dx + 3\int x^2\,dx$$
$$= y^2\cdot 2\frac{x^2}{2} + 3\frac{x^3}{3}$$
$$\Rightarrow I_1 = x^2y^2 + x^3$$
& $I_2 = \int N(\text{free of } x \text{ terms})\,dy$
$= 0$ ($\because$ no terms free of x)

Hence the required general solution is
$$I_1 + I_2 = C$$
$$x^2y^2 + x^3 + 0 = C$$
$$\Rightarrow \boxed{x^2y^2 + x^3 = C}$$ ANS

(4) $(ax+hy+g)\,dx+(hx+by+f)\,dy=0$

Sol:- Given DE is of the form

$$M\,dx+N\,dy=0$$

where $M=ax+hy+g$ &

$N=hx+by+f$

To show that this is an exact DE, we need to show that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

so $\frac{\partial M}{\partial y}=h$ and

$\frac{\partial N}{\partial x}=h$

$$\Rightarrow \frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}=h$$

Hence this is an exact DE.

Now

(i) $I_1=\int M\,dx$; keeping y is a constant

$$I_1=\int(ax+hy+g)\,dx$$

$$I=\frac{ax^2}{2}+hxy+gx$$

and

(ii) $I_2=\int N(\text{free of } x \text{ terms})\,dy$

$$=\int(by+f)\,dy$$

$$I_2=\frac{by^2}{2}+fy$$

(iii) Hence the required general solution is

$$I_1+I_2=C$$

$$\Rightarrow \boxed{\frac{ax^2}{2}+hxy+gx+\frac{by^2}{2}+fy=0}$$ ANS

(5) $x\cos y\, dy = (2x - \sin y)\, dx$; $y(2) = 0$

Sol:- Since $x\cos y\, dy = (2x - \sin y)\, dx$

$$x\cos y\, dy + (\sin y - 2x)\, dx = 0$$
$$\text{or } (2x - \sin y)\, dx - x\cos y\, dy = 0$$

For exact DE, we have

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Here $M = 2x - \sin y$, $N = -x\cos y$

so $\dfrac{\partial M}{\partial y} = -\cos y$

& $\dfrac{\partial N}{\partial x} = -\cos y$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = -\cos y$$

So this is an exact DE.

Now

(i) $I_1 = \int M\, dx$ (keeping y constant)

$$= \int (2x - \sin y)\, dx$$
$$I_1 = \frac{2x^2}{2} - x\sin y$$
$$I_1 = x^2 - \sin y$$

(ii) $I_2 = \int N$ (free of x terms) dy

$$= 0$$

(iii) $\therefore$ $I_1 + I_2 = c_1$

$$x^2 - x\sin y + 0 = c_1 \Rightarrow x^2 - x\sin y = c$$
$$x\sin y - x^2 = -c_1 \quad , \quad \text{let } -c_1 = c$$
$$x\sin y - x^2 = c \longrightarrow (A)$$

putting $y(2) = 0$, i.e $x = 2$, $y = 0$

$$2\sin(0) - (2)^2 = c$$
$$\Rightarrow \boxed{c = -4}$$

so by putting value of c in eq(1), we get

$$x\sin y - x^2 = -4$$

$$\Rightarrow \boxed{x\sin y - x^2 + 4 = 0} \quad \underline{\text{Ans}}$$

(6) $\left(\frac{3-y}{x^2}\right)dx - \left(\frac{2x-y^2}{xy^2}\right)dy = 0 \; ; \; y(-1)=2$

Sol:- Given DE is of the form

$$Mdx + Ndy = 0$$

where $M = \frac{3-y}{x^2}$, $N = -\left(\frac{2x-y^2}{xy^2}\right)$

For exact DE, we need to show that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Now $\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(\frac{3}{x^2} - \frac{y}{x^2}\right) = \frac{-1}{x^2}$

$$\Rightarrow \frac{\partial M}{\partial y} = \frac{-1}{x^2}$$

&

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}-\left(\frac{2x}{xy^2} - \frac{y^2}{xy^2}\right)$$

$$= -\left(0 - \left(\frac{-1}{x^2}\right)\right)$$

$$\Rightarrow \frac{\partial N}{\partial x} = \frac{-1}{x^2}$$

$$\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = \frac{-1}{x^2}$$

This shows that this is an exact DE

Now,

(i) $\int M dx$; (Keeping y as a constant)

$$\int \frac{3-y}{x^2}dx = \int\left(\frac{3}{x^2} - \frac{y}{x^2}\right)dx$$

$= 3-y\int x^{-2}\,dx$

$I_1 = 3-y\left(\frac{-1}{x}\right)$

(ii) $I_2 = \int N(\text{free of } x \text{ terms})\,dy$

$= \int -\frac{2}{y^2}\,dy$

$= -2\int y^{-2}\,dy$

$I_2 = \frac{2}{y}$

III $I_1 + I_2 = c$

$3-y\left(\frac{-1}{x}\right) + \frac{2}{y} = c$

$\Rightarrow \frac{y-3}{x} + \frac{2}{y} = c \longrightarrow (A)$

Now $y(-1) = 2 \Rightarrow x = -1, y = 2$

$\frac{2-3}{-1} + \frac{2}{2} = c$

$1 + 1 = c \Rightarrow \boxed{c = 2}$

putting value of c in eq(A)

$\frac{y-3}{x} + \frac{2}{y} = 2$

$\Rightarrow \frac{y(y-3)+2x}{xy} = 2$

$y^2 - 3y + 2x = 2xy$

$\boxed{y^2 - 2xy - 3y + 2x = 0}$ ANS

(7) $x(2\cos y+3xy)dx-(y+x^2\sin y-x^3)dy=0$
$y(0)=2$.

Sol:- Here we have
$x(2\cos y+3xy)dx-(y+x^2\sin y-x^3)dy=0$

$\Rightarrow(2x\cos y+3x^2y)dx-(y+x^2\sin y-x^3)dy=0$
This equation is of the form
$M(x,y)dx+N(x,y)dy=0$
Where
$M(x,y)=2x\cos y+3x^2y$, and
$N(x,y)=-y-x^2\sin y+x^3$

To show that given DE is exact, we need to show that
$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$
$$\therefore \quad \frac{\partial M}{\partial y}=-2x\sin y+3x^2$$
and $$\frac{\partial N}{\partial x}=-2x\sin y+3x^2$$
$$\Rightarrow \frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}=-2x\sin y+3x^2$$
Which means that the given DE is exact DE. So
(i) $I_1=\int Mdx$; (keeping y as a constant)
$$=\int(2x\cos y+3x^2y)dx$$
$$=\cos y\int 2xdx+y\int 3x^2dx$$
$$=\cos y\left(\frac{2x^2}{2}\right)+y\cdot\frac{3x^3}{3}$$
$$I_1=x^2\cos y+x^3y$$

(II) $I_2 = \int N(\text{free of } x \text{ terms})\, dy$

$$= \int -y\, dy$$

$$I_2 = \frac{-y^2}{2}$$

(iii) Now $I_1 + I_2 = c$

$$\Rightarrow x^2\cos y + x^3 y - \frac{y^2}{2} = c \longrightarrow (A)$$

Using initial condition, $y(0) = 2$ in (A)

$$\Rightarrow x = 0,\ y = 2$$

$$0\cos(2) + 0(2) - \frac{(2)^2}{2} = c$$

$$-\frac{4}{2} = c \Rightarrow \boxed{c = -2}$$

putting value of c in (A), we get

$$x^2\cos y + x^3 y - \frac{y^2}{2} = -2$$

$$x^2\cos y + x^3 y - \frac{y^2}{2} + 2 = 0$$

$$\Rightarrow \boxed{2x^2\cos y + 2x^3 y - y^2 + 4 = 0} \text{ ANS}$$

Home Work :- Check whether the given equations are exact, if it is exact then solve it.

(1) $6xy^3\,dx + y^2(4y + 9x^2)\,dy = 0$

ANS:- $3x^2y^3 + y^4 = c$

(2) $(x+y)(x-y)\,dx + x(x-2y)\,dy = 0$

ANS:-

(3) $(ye^{-x} - \sin x)\,dx - (e^{-x} + 2y)\,dy = 0$

ANS:- $y = e^{-x} - \cos x + y^2 = c$

(4) $(xy^2 - 1)\,dx - (1 - x^2y)\,dy = 0$; $y(0) = 1$

ANS:- $x^2y^2 - 2(x+y) + 2 = 0$

(5) $(\cos x \sin x - xy^2)\,dx + y(1 - x^2)\,dy = 0$; $y(0) = 2$

ANS:- $y^2(1 - x^2) - \cos^2 x = 3$

Some Important Formulae of DEs.

(1) $d(xy) = x\,dy + y\,dx$, $d(yx) = y\,dx + x\,dy$

(2) $d(x/y) = (y\,dx - x\,dy)\frac{1}{y^2}$

& $d(y/x) = (x\,dy - y\,dx)\frac{1}{x^2}$

(3) $d(x^2 \pm y^2) = 2(x\,dx \pm y\,dy)$

(4) $d(x^m y^n) = x^{m-1} y^{n-1}(my\,dx + nx\,dx)$

(5) $d(\tan^{-1} y/x) = (x\,dy - y\,dx)\frac{1}{x^2+y^2}$

(6) $d(\frac{1}{2}\ln(x^2+y^2)) = (x\,dx + y\,dy)\frac{1}{x^2+y^2}$

(7) $d(x^2/y) = (2xy\,dx - x^2\,dy)\frac{1}{y^2}$

& $d(y^2/x) = (2xy\,dy - y^2\,dx)\frac{1}{x^2}$

(8) $d(\frac{x^2}{y^2}) = \frac{2xy^2\,dx - 2x^2y\,dy}{y^4}$

& $d(\frac{y^2}{x^2}) = (2x^2y\,dy - 2y^2x\,dx)\frac{1}{x^4}$

(9) $d(\log xy) = \frac{x\,dy + y\,dx}{xy}$

(10) $d(\log \frac{x}{y}) = \frac{x\,dy - y\,dx}{xy}$

(11) $d(\frac{-1}{xy}) = \frac{x\,dy + y\,dx}{x^2y^2}$

(12) $d(\sin(\frac{1}{x})) = \frac{-1}{x^2}\cos(\frac{1}{x})$ & $d(\cos(\frac{1}{x})) = \frac{1}{x^2}\sin(\frac{1}{x})$

<u>**Note**</u> :- The above rules are used to determine the general solution of the given Non-Exact differential equations.

× These are basically the Integrating factors found by inspection method

Q:- Solve the following D-Eqns by using the previous formulae (Inspection method).

(1) $y\,dx - (x - 2y^3)\,dy = 0$

Sol:- By checking we see that this is a non-exact DE (i.e $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$).

$$\therefore \quad y\,dx - (x - 2y^3)\,dy = 0$$

$$y\,dx - x\,dy + 2y^3\,dy = 0$$

$\div$ ing by y^2, we get

$$\frac{y\,dx - x\,dy + 2y^3\,dy}{y^2} = \frac{0}{y^2}$$

$$\frac{y\,dx - x\,dy}{y^2} + \frac{2y^3}{y^2}\,dy = 0$$

$$\frac{y\,dx - x\,dy}{y^2} + 2y\,dy = 0 \longrightarrow (A)$$

Since by formula (2), we know that

$$d\left(\frac{x}{y}\right) = \frac{y\,dx - x\,dy}{y^2}, \text{ using in (A)}$$

$$\Rightarrow \quad d\left(\frac{x}{y}\right) + 2y\,dy = 0$$

Integrating b/s, we get

$$\int d\left(\frac{x}{y}\right) + \int 2y\,dy = \int 0$$

$$\frac{x}{y} + \frac{2y^2}{2} = c$$

$$\Rightarrow x/y + y^2 = c$$

$$\Rightarrow \boxed{x + y^3 = cy} \text{ ANS}$$

(2) $x\,dy - y\,dx - \cos\left(\frac{1}{x}\right)dx = 0 \longrightarrow (1)$

Sol:-

Given $x\,dy - y\,dx - \cos\left(\frac{1}{x}\right)dx = 0$

$\div$ing b/s by "x^2", we get

$$\frac{x\,dy - y\,dx}{x^2} - \cos\left(\frac{1}{x}\right)\cdot\frac{1}{x^2}dx = 0$$

$$\frac{x\,dy - y\,dx}{x^2} - \frac{1}{x^2}\cos\left(\frac{1}{x}\right)dx = 0$$

$$d\left(\frac{y}{x}\right) + d\left(\sin\left(\frac{1}{x}\right)\right) = 0 \text{ ; by using (2) \& (12)}$$

Integrating b/s

$$\int d(y/x) + \int d(\sin(1/x)) = \int 0$$

$$\frac{y}{x} + \sin\left(\frac{1}{x}\right) = c$$

$$\Rightarrow y + x\sin\left(\frac{1}{x}\right) = cx \quad \text{ANS}$$

OR

$$\boxed{y + x\sin\left(\frac{1}{x}\right) - cx = 0}$$

(3) $(y^2e^x + 2xy)dx - x^2dy = 0$

Sol:- Given $(y^2e^x + 2xy)\,dx - x^2dy = 0$

$$y^2e^x dx + 2xy\,dx - x^2dy = 0$$

$\div$ing b/s by y^2, we obtain

$$e^x dx + \frac{2xy\,dx - x^2dy}{y^2} = 0$$

$$e^x dx + d\left(\frac{x^2}{y}\right) = 0 \text{ ; by using (7)}$$

$$e^x dx + d(x^2/y) = 0 \text{ ; Integrating}$$

$$\int e^x dx + \int d(x^2/y) = 0$$

$$e^x + x^2/y = c$$

$$\Rightarrow \boxed{ye^x + x^2 = yc} \quad \underline{\text{ANS}}$$

(4) $x\,dy - y\,dx = (x^2+y^2)\,dx$

Sol:- $x\,dy - y\,dx = (x^2+y^2)\,dx$

$$\Rightarrow \frac{x\,dy - y\,dx}{x^2+y^2} = dx$$

$$\Rightarrow \frac{\frac{x\,dy - y\,dx}{x^2}}{\frac{x^2+y^2}{x^2}} = dx$$

$$\Rightarrow d\left(\tan^{-1}\frac{y}{x}\right) = dx \quad ; \text{ by using 5}$$

Integrating b/s

$$\int d\left(\tan^{-1}\left(\frac{y}{x}\right)\right) = \int dx$$

$$\tan^{-1}\left(\frac{y}{x}\right) = x + c$$

$$\Rightarrow \frac{y}{x} = \tan(x+c)$$

$$\Rightarrow \boxed{y = x\tan(x+c)} \quad \underline{\text{ANS}}$$

(5) $y(1+xy)\,dx + (1-xy)\,x\,dy = 0$

Sol:- Since $(1+xy)\,dx + (1-xy)\,x\,dy = 0$

$$y\,dx + xy^2\,dx + x\,dy - x^2y\,dy = 0$$

$$y\,dx + x\,dy + xy(y\,dx - x\,dy) = 0$$

÷ing by "x^2y^2" b/s to obtain

$$\frac{y\,dx + x\,dy}{(xy)^2} + \frac{y\,dx - x\,dy}{xy} = 0$$

$$\Rightarrow \underbrace{d\left(\frac{-1}{xy}\right)}_{\text{(by (11))}} + \left(\frac{dx}{x} - \frac{dy}{y}\right) = 0$$

Integrating b/s

$$\int d\left(-\frac{1}{xy}\right) + \int \frac{dx}{x} - \int \frac{dy}{y} = \int 0$$

$$-\frac{1}{xy} + \ln x - \ln y = c$$

$$\Rightarrow \frac{1}{xy} = \ln y - \ln x + c$$

$$\Rightarrow \boxed{\frac{1}{xy} = \ln\left(\frac{y}{x}\right) + c}$$

(6) $(2x^2y + e^x)\,y\,dx = (e^x + y^3)\,dy$

Sol:- Given $(2x^2y + e^x)y\,dx = (e^x + y^3)\,dy$

$$2x^2y^2dx + ye^xdx - e^xdy + y^3dy = 0$$

$\div$ ing by "y^2" we get

$$\Rightarrow \frac{ye^xdx - e^xdy}{y^2} + 2x^2dx - y\,dy = 0$$

$$\Rightarrow d\left(\frac{e^x}{y}\right) + 2x^2dx - y\,dy = 0$$

Integrating b/s

$$\int d(e^x/y) + 2\int x^2dx - \int y\,dy = \int 0$$

$$e^x/y + 2\frac{x^3}{3} - \frac{y^2}{2} = c$$

$$e^x/y + \frac{2}{3}x^3 - \frac{1}{2}y^2 = c \quad \text{OR}$$

$$\boxed{6e^x + 4x^3y - 3y^3 = 6cy}$$

(7) $(y+\ln x)\,dx = x\,dy$

Sol:- Since $(y+\ln x)\,dx = x\,dy$

$\Rightarrow y\,dx + \ln x\,dx - x\,dy = 0$

÷ing by "x^2"

$$\frac{y\,dx - x\,dy}{x^2} + \frac{\ln x}{x^2}\,dx = 0$$

$$-d\left(\frac{y}{x}\right) + \ln x \cdot \frac{1}{x^2}\,dx = 0$$

Integrating b/s

$$-\int d(y/x) + \int \ln x \cdot \frac{1}{x^2}\,dx = \int 0$$

$$-\frac{y}{x} + \ln x\left(-\frac{1}{x}\right) - \int \frac{1}{x}\left(-\frac{1}{x}\right)dx = c$$

$$\Rightarrow -\frac{y}{x} - \frac{1}{x}\ln x - \frac{1}{x} = c$$

or $\boxed{y + \ln x + 1 + cx = 0}$ ANS

Home Work:- Using the previous formulae to solve the following D Eqns:

(1) $x\,dy - y\,dx = x^2 y^3\,dx$

ANS: $2\left(\frac{x}{y}\right)^2 + x^4 + 4c = 0$

(2) $y\,dx - x\,dy + (1+x^2)\,dx + x^2 \sin y\,dy = 0$

ANS:- $x^2 - y - 1 - x\cos y = c$

(3) $(x^2+y^2)(x\,dx + y\,dy) = a^2(x\,dy - y\,dx)$

ANS: $x^2+y^2 = a^2\tan^{-1}(y/x) + c$

(4) $y(x-2y)\,dx = x(x-3)\,dy$

ANS: $\frac{x}{y} + \ln(y^3/x^2) = c$

Note: $d\left(\sqrt{x^2+y^2}\right) = \frac{x\,dx + y\,dy}{\sqrt{x^2+y^2}}$

DE reducible to Exact Form

* If a DE of the form $Mdx+Ndy=0$, where M = N are both functions of x and y ie
$$M(x,y)\,dx+N(x,y)\,dy=0 \longrightarrow (1)$$
is not an exact DE (ie $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$) but when it is multiplied by a function $u(x,y)$ such that the equivalent equation obtained by multiplying both sides of (1) by $u(x,y)$.

$$u(x,y)\,M(x,y)\,dx+u(x,y)\,N(x,y)=0 \rightarrow (2)$$
is exact. Such a function (x,y) is called an Integrating factor of the original equation. Integrating factors turned nonexact equations into exact ones. The number of Integrating factors of an equation may be infinite. The question is, how do you find an Integrating factor? Following are some special cases.

* **Case 1:-** If the DE $M(x,y)dx+N(x,y)dy=0$ is not exact, then $M_y \neq N_x$; i.e $M_y-N_x \neq 0$

However, if
$$\frac{M_y-N_x}{N}=f(x)$$
is a function of x only, then $e^{\int f(x)\,dx}$ is an Integrating factor of Given DE (1).

Proof:- Let u is an Integrating factor (I.F) of DE (1). then by hypothesis
$$u\,M(x,y)\,dx+u\,N(x,y)\,dy=0$$
is an exact DE. Thus
$$\frac{\partial}{\partial y}(uM)=\frac{\partial}{\partial x}(uN)$$

or

$$u\frac{\partial M}{\partial y} + M\frac{\partial u}{\partial y} = u\frac{\partial N}{\partial x} + N\frac{\partial u}{\partial x}$$

or $$u\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = N\frac{\partial u}{\partial x} - M\frac{\partial u}{\partial y}$$

Since u is a function of x only, $\frac{\partial u}{\partial y} = 0$, therefore

$$u\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = N\frac{\partial u}{\partial x}$$

or $$\frac{u\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)}{N} = \frac{\partial u}{\partial x}$$

$$= \frac{u(M_y - N_x)}{N} = \frac{\partial u}{\partial x} \longrightarrow (B)$$

As $\frac{M_y - N_x}{N} = f(x)$

$\Rightarrow u f(x) = \frac{du}{dx}$, separating variables

$$\Rightarrow \frac{du}{u} = f(x)\,dx$$

Integrating b/s

$$\int \frac{du}{u} = \int f(x)\,dx$$

$$\ln u = \int f(x)\,dx$$

$$\Rightarrow u = e^{\int f(x)dx}$$

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Example:- Solve the DEqn $(3xy^2+2y)dx+(2x^2y+x)dy=0$

Sol:- here $M = 3xy^2+2y$ and
$N = 2x^2y+x$

$$\Rightarrow \frac{\partial M}{\partial y} = 6xy+2 \quad \&$$

$$\frac{\partial N}{\partial x} = 4xy+1$$

$$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

therefore, the given DE is not exact

Now

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{6xy+2-4xy-1}{2x^2y+x}$$

$$= \frac{\cancel{2xy+1}}{x(\cancel{2xy+1})} = \frac{1}{x} = f(x)$$

$$\Rightarrow \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} \text{ is a function of } x \text{ alone}$$

So, $e^{\int f(x)dx} = e^{\int \frac{1}{x}dx} = e^{\ln x} = x$

$\therefore$ x is an Integrating factor.
xing the given DE by Integrating factor x b/s, we get

$$x\left[(3xy^2+2y)dx+(2x^2y+x)dy\right] = 0 \cdot x$$

$$(3x^2y^2+2xy)dx+(2x^3y+x^2)dy = 0$$

Now $M = 3x^2y^2+2xy$, $N = 2x^3y+x^2$

$\frac{\partial M}{\partial y} = 6x^2y + 2x$ &

$\frac{\partial N}{\partial x} = 6x^2y + 2x$

$\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Hence by xing x^4, the given DE becomes exact. Hence the general is given by

$\int M(x,y)dx + \int N(\text{free of } x \text{ terms})dy = c$

$\int (3x^2y^2 + 2xy)dx + \int 0\,dy = c$

↳ No free of x terms

$3y^2 \int x^2 dx + 2y \int x\,dx = c$

$3y^2 \cdot \frac{x^3}{3} + 2y \cdot \frac{x^2}{2} = c$

$\boxed{x^3y^2 + x^2y = c}$

(2) $(x^2 + y^2 + x)dx + xy\,dy = 0 \longrightarrow (1)$

Sol:- Here we have

$M = x^2 + y^2 + x$ & $N = xy$

$\frac{\partial M}{\partial y} = 2y$, $\frac{\partial N}{\partial x} = y$

$\Rightarrow \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

$\therefore$ Eq.(1) is not exact.

Now to find the Integrating factor, we use the rule No 1. As

$\frac{M_y - N_x}{N} = \frac{2y - y}{xy} = \frac{y(2-1)}{xy} = \frac{1}{x}$

i.e $\frac{1}{x} = f(x)$ (function of x alone)

So I.F $e^{\int f(x)dx} = e^{\int \frac{1}{x} dx}$

$= e^{\ln x} = x$

$I \cdot F = x$

xing eq.(1) by Integrating factor

$x\left[(x^2+y^2+x)dx + xy\,dy\right] = 0 \cdot x$

$(x^3+xy^2+x^2)dx + x^2y\,dy = 0$

Now here $M = x^3+xy^2+x^2$

$\Rightarrow \frac{\partial M}{\partial y} = 2xy$

& $N = x^2y$

$\Rightarrow \frac{\partial N}{\partial x} = 2xy$

$\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = 2xy$

so by xing I.F. this eq is now exact and its general solution is

$\int M(x,y)\,dx + \int N(\text{free of } x \text{ term})\,dy = C_1$

$\int (x^3+xy^2+x^2)\,dx + \int (0)\,dy = C_1$

$\frac{x^4}{4} + \frac{x^2y^2}{2} + \frac{x^3}{3} = C_1$

xing b/s by 12, we get

$3x^4 + 4x^3 + 6x^2y^2 = 12C_1$

Let $C = 12C_1$

$\therefore \boxed{3x^4+4x^3+6x^2y^2 = C}$

(3) $y(x+y)\,dx + (x+2y-1)\,dy = 0 \longrightarrow (A)$

Sol:- Here $M = y(x+y) = xy+y^2$

$$\frac{\partial M}{\partial y} = x+2y$$

& $N = x+2y-1 \Rightarrow \frac{\partial N}{\partial x} = 1$

$\because \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

$\therefore$ Given DE is not exact.

For Integrating factor I.F, we have

$$I.F = \frac{M_y - N_x}{N} = \frac{x+2y-1}{x+2y-1} = 1 = f(x)$$

$$\therefore I.F = e^{\int 1dx} = e^x$$

xing eq(A) by e^x, we obtain

$$(xy\,e^x + y^2e^x)\,dx + (xe^x + 2ye^x - e^x)\,dy = 0 \rightarrow (B)$$

Now Here $M = xye^x + y^2e^x$

$$\Rightarrow \frac{\partial M}{\partial y} = xe^x + 2ye^x$$

& $N = xe^x + 2ye^x - e^x$

$$\frac{\partial N}{\partial x} = xe^x + e^x + 2ye^x - e^x = xe^x + 2ye^x$$

$$\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$\therefore$ Eq.(B) is now exact DE. So its general solution will be

$$\int M dx + \int N(\text{free of } x \text{ term})\,dy = c$$

$$\int (xye^x + y^2e^x)\,dx + \int (0)\,dy = c$$

$$y\int xe^x dx + y^2\int e^x dx = c$$

Now using Integration by parts, we see that

$$\int xe^x dx = xe^x - e^x$$

$$\therefore \; [xe^x - e^x] + y^2 e^x = c$$

$$(x-1+y)\, e^x = c$$

$$\boxed{(x-1+y) = ce^{-x}} \; \underline{\text{ANS}}$$

(4) $(4xy+3y^2-x)\,dx + x(x+2y)\,dy = 0$

Sol:- By checking we can clearly see that this is a non exact DE $\left(\text{ie } \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}\right)$.

So for I.F we have

$$\frac{M_y - N_x}{N} = f(x) \quad (\text{valid only when RHS is function of } x \text{ only})$$

$$M_y = 4x + 6y \qquad (\because M = 4xy + 3y^2 - x)$$

$$N_y = 2x + 2y \qquad (\because N = x(x+2y))$$

$$\Rightarrow \frac{M_y - N_x}{N} = \frac{4x+6y-2x-2y}{x(x+2y)}$$

$$= \frac{2x+4y}{x^2+2xy} = \frac{2(x+2y)}{x(x+2y)} = \frac{2}{x} = f(x)$$

$$\text{So } \; I.F = e^{\int f(x)dx} = e^{\int \frac{2}{x}dx} = e^{2\ln x}$$

$$e^{\ln x^2} = x^2 = I.F$$

xing given DE by I.F (ie x^2), we get

$$(4x^3y+3x^2y^2-x^3)\,dx+x^3(x+2y)\,dy=0 \rightarrow (B)$$

Here $M = 4x^3y+3x^2y^2-x^3$

$$\frac{\partial M}{\partial y} = 4x^3+6x^2y$$

& $N = x^4+2x^3y$

$$\frac{\partial N}{\partial x} = 4x^3+6x^2y$$

$$\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = 4x^3+6x^2y$$

$\therefore$ Eq (B) is now exact. And its general solution is

$$\int M\,dx+\int N(\text{free of } x \text{ terms})\,dy = c$$

$$\int(4x^3y+3x^2y^2-x^3)\,dx+\int 0\,dy = c$$

$$\frac{4x^4y}{4}+\frac{3x^3y^2}{3}-\frac{x^4}{4} = c_1$$

$$x^4y+x^3y^2-\frac{x^4}{4} = c_1$$

xing b/s by 4

$$4x^4y+4x^3y^2-x^4 = 4c_1$$

Let $4c_1 = c$

$$\therefore \boxed{4x^4y+4x^3y^2-x^4=c} \text{ ANS}$$

(5) $(3xy+y^2)\,dx+(x^2+xy)\,dy=0 \rightarrow (A)$

Sol:- Here $M = 3xy+y^2$

$$M_y = \frac{\partial M}{\partial y} = 3x+2y$$

& $N = x^2+xy$

$$N_x = \frac{\partial N}{\partial x} = 2x+y$$

$$\Rightarrow \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$\therefore$ Given DE is not exact.

Now to find Integrating factor, we evaluate

$$\frac{M_y - N_x}{N}$$

$$I.F = \frac{(3x+2y)-(2x+y)}{x^2+xy}$$

$$= \frac{x+\cancel{y}}{x(\cancel{x+y})} = \frac{1}{x} = f(x)$$

So Integrating factor $= e^{\int \frac{1}{x}dx}$

$$= e^{\ln x} = x$$

x-ing D.E (A) with Integrating factor

$$x\left[(3xy+y^2)\,dx+(x^2+xy)\,dy\right]=0\cdot x$$

$$(3x^2y+xy^2)\,dx+(x^3+x^2y)\,dy=0 \rightarrow (B)$$

Here in D.E (B), we have

$M = 3x^2y+xy^2 \Rightarrow M_y = \frac{\partial M}{\partial y} = 3x^2+2xy$

& $N = x^3+x^2y \Rightarrow N_x = \frac{\partial N}{\partial x} = 3x^2+2xy$

$$\because \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad (\text{ie } M_y = N_x)$$

This means that DE (B) is now exact and its general solution will be

$$\int M dx + \int N(\text{free of } x \text{ terms}) dy = c$$

$$\int (3x^2y + xy^2)\, dx + \int 0\, dy = c$$

$$\int x^2 3y\, dx + \int xy^2 dx + 0 = c_1$$

$$3y \frac{x^3}{3} + y^2 \cdot \frac{x^2}{2} = c_1$$

$$x^3y + \frac{x^2y^2}{2} = c_1$$

or $\quad 2x^3y + x^2y^2 = 2c_1$

Let $c = 2c_1$

$$\boxed{2x^3y + x^2y^2 = c}$$

Rule # 02 :- If $M(x,y)dx + N(x,y)dy = 0$ is not exact and if $\frac{-M_y + N_x}{M} = g(y)$, a function of y alone, then $e^{\int g(y) dy}$ is an integrating factor of DE.

The proof of this rule is quite similar to the proof in case (1).

Q:- Solve the following DEqns:

(1) $(3x^2y^4 + 2xy) + (2x^3y^3 - x^2)\, dy = 0 \longrightarrow (1)$

Here $M = 3x^2y^4 + 2xy \Rightarrow M_y = 12x^2y^3 + 2x$

& $\quad N = 2x^3y^3 - x^2 \Rightarrow N_x = 6x^2y^3 - 2x$

$$\Rightarrow \quad M_y \neq N_x$$

This means that the given DE is not exact. So for Integrating factor

$$I\cdot F = \frac{N_x - M_y}{M}$$

$$= \frac{6x^2y^3 - 2x - 12x^2y^3 - 2x}{3x^2y^4 + 2xy}$$

$$= \frac{-6x^2y^3 - 4x}{xy(3xy^3+2)} = \frac{-2x(3xy^3+2)}{xy(3xy^3+2)}$$

$$= \frac{-2}{y} = g(y)$$

i.e function of y only

So Integrating factor is

$$I\cdot F = e^{\int g(y)dy} = e^{\int -2/y\, dy}$$

$$= e^{-2\int 1/y\, dy} = e^{-2\ln y} = e^{\ln y^{-2}}$$

$$I\cdot F = y^{-2} = 1/y^2$$

xing DE(1) by Integrating factor

$$(3x^2y^4 + 2xy)\frac{1}{y^2}dx + \frac{1}{y^2}(2x^3y^3 - x^2)dy = 0$$

$$\left(3x^2y^2 + \frac{2x}{y}\right)dx + \left(2x^3y - \frac{x^2}{y^2}\right)dy = 0 \rightarrow (B)$$

Here $M = 3x^2y^2 + \frac{2x}{y}$

$\Rightarrow M_y = 6x^2y - 2x/y^2$

& $N = 2x^3y - x^2/y^2$

$\Rightarrow N_x \; 6x^2y - 2x/y^2$

$\Rightarrow M_y = N_x$

So eq(B) is now exact DE and its general solution is

$$\int M\,dx + \int N(\text{free of } x \text{ terms})\,dy = c$$

$$\int \left(3x^2y^2 + \frac{2x}{y}\right)dx + \int 0\,dy = c$$

$$3y^2\int x^2dx + \frac{2}{y}\int x\,dx + 0 = c$$

$$\cancel{3}y^2\cdot\frac{x^3}{\cancel{3}} + \frac{\cancel{2}}{y}\cdot\frac{x^2}{\cancel{2}} + 0 = c$$

$$x^3y^2 + \frac{x^2}{y} = c$$

or

$$x^3y^3 + x^2 = cy$$

$$\boxed{x^2 + x^3y^3 - cy = 0} \quad \underline{\text{ANS}}$$

(2) $(y^4 + 2y)\,dx + (xy^3 + 2y^4 - 4x)\,dy = 0 \rightarrow (B)$

Sol:- Here $M = y^4 + 2y$

$$\Rightarrow M_y = 4y^3 + 2$$

& $N = xy^3 + 2y^4 - 4x$

$$N_x = y^3 - 4$$

$$\Rightarrow M_y \neq N_x$$

$\therefore$ Given DE is not exact.

So for integrating factor, we assume

$$\text{I.F} = \frac{N_x - M_y}{M}$$

$$= \frac{y^3 - 4 - 4y^3 - 2}{y^4 + 2y} = \frac{-3y^3 - 6}{y(y^3 + 2)}$$

$$= \frac{-3(y^3+2)}{y(y^3+2)} = \frac{-3}{y} = g(y)$$

means this is a function of y alone

So

$$I.F = e^{\int g(y)dy}$$

$$= e^{\int \frac{-3}{y} dy} = e^{-3\int \frac{dy}{y}}$$

$$= e^{-3\ln y} = e^{\ln y^{-3}} = y^{-3}$$

$$I.F = \frac{1}{y^3}$$

×ing Eq.(B) by I.F

$$\frac{1}{y^3}(y^4+2y)\,dx + \frac{1}{y^3}(xy^3+2y^4-4x)\,dy = 0$$

$$\left(y+\frac{2}{y^2}\right)dx + \left(x+2y-\frac{4x}{y^3}\right)dy = 0$$

Here $M = y + \frac{2}{y^2}$

$$\frac{\partial M}{\partial y} = 1 - \frac{4}{y^3}$$

& $N = x+2y - \frac{4x}{y^3}$

$$\frac{\partial N}{\partial x} = 1 - \frac{4}{y^3}$$

$$\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Which is now exact DE. and its general solution is

$$\int M dx + \int N(\text{free of } x \text{ terms})\, dy =$$

$$\int \left(y+\frac{2}{y^2}\right)dx + \int 2y\, dy = c$$

$$xy + \frac{2}{y^2}x + \frac{2y^2}{2} = c$$

$$y^2 + xy + \frac{2x}{y^2} = c$$

$$\boxed{y^4 + xy^3 + 2x = cy^2} \quad \underline{\underline{\text{ANS}}}$$

(3) $(2xy^4e^y + 2xy^3 + y)dx + (x^2y^4e^y - x^2y^2 - 3x)dy = 0$

Sol:-

Here $M = 2xy^4e^y + 2xy^3 + y$

$$\Rightarrow \frac{\partial M}{\partial y} = 8xy^3e^y + 2xy^4e^y + 6xy^2 + 1$$

& $N = x^2y^4e^y - x^2y^2 - 3x$

$$\frac{\partial N}{\partial x} = 2xy^4e^y - 2xy^2 - 3$$

$$\Rightarrow \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$\therefore$ Given DE is not exact

So for I.F, we have

$$\frac{N_x - M_y}{M}$$

$$= \frac{2xy^4e^y - 2xy^2 - 3 - 8xy^3e^y - 2xy^4e^y - 6xy^2 - 1}{2xy^4e^y + 2xy^3 + y}$$

$$= \frac{-8xy^3e^y - 8xy^2 - 4}{y(2xy^3e^y + 2xy^2 + 1)}$$

$$= \frac{-4\cancel{(2xy^3e^y + 2xy^2 + 1)}}{y\cancel{(2xy^3e^y + 2xy^2 + 1)}}$$

$$= -4/y = g(y)$$

So $e^{\int g(y)dy} = e^{\int -4/y\, dy} = e^{-4\int \frac{dy}{y}}$

$$= e^{-4\ln y} = e^{\ln y^{-4}} = \frac{1}{y^4}$$

xing given DE by $1/y^4$ and solve it.

Rule #3 :- If $M(x,y)\,dx + N(x,y)\,dy = 0$ —(1) is homogeneous and $Mx + Ny \neq 0$ then $\frac{1}{Mx+Ny}$ is an Integrating

Proof :- If $\frac{1}{Mx+Ny}$ is an Integrating factor of eq(1), then we are to show that

$$\frac{M}{Mx+Ny}\,dx + \frac{N}{Mx+Ny}\,dy = 0$$

is an exact DE. i.e

$$\frac{\partial}{\partial y}\left(\frac{M}{Mx+Ny}\right) = \frac{\partial}{\partial x}\left(\frac{N}{Mx+Ny}\right)$$

Now

$$\frac{\partial}{\partial y}\left(\frac{M}{Mx+Ny}\right) = \frac{(Mx+Ny)\frac{\partial M}{\partial y} - M\left(x\frac{\partial M}{\partial y} + N + y\frac{\partial N}{\partial y}\right)}{(Mx+Ny)^2}$$

$$= \frac{Ny\frac{\partial M}{\partial y} - MN - My\frac{\partial N}{\partial y}}{(Mx+Ny)^2} \longrightarrow (2)$$

and

$$\frac{\partial}{\partial x}\left(\frac{N}{Mx+Ny}\right) = \frac{(Mx+Ny)\frac{\partial N}{\partial x} - N\left(x\frac{\partial M}{\partial x} + M + y\frac{\partial N}{\partial x}\right)}{(Mx+Ny)^2}$$

$$\frac{Mx\frac{\partial N}{\partial x} - MN - Nx\frac{\partial M}{\partial x}}{(Mx+Ny)^2} \longrightarrow (3)$$

Subtracting eq(3) from eq(2)

$$\frac{\partial}{\partial y}\left(\frac{M}{Mx+Ny}\right) - \frac{\partial}{\partial x}\left(\frac{N}{Mx+Ny}\right) = \frac{N\left(x\frac{\partial M}{\partial x} + y\frac{\partial M}{\partial y}\right) - M\left(x\frac{\partial N}{\partial x} + y\frac{\partial N}{\partial y}\right)}{(Mx+Ny)^2}$$

$$= \frac{N(nM) - M(nN)}{(Mx+Ny)^2} = 0$$

Using Euler's theorem on homogeneous function $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nu$

Note:- If $Mx+Ny=0$ identically, then $\frac{M}{N}=-\frac{y}{x}$ and DE (1) becomes

$$\frac{dy}{dx}=-\frac{M}{N}=\frac{y}{x} \text{ or } y\,dx-x\,dy=0$$

for which $\frac{1}{xy}$ is an Integrating factor

Q:- Solve the following D.Eqns.

(1) $(y^2+xy)\,dx=x^2dy$

Sol:- Since we see that given D.E is homogeneous of degree 2 and it is also not exact DE because, $\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$

So for Integrating factor we have, $(y^2+xy)\,dx-x^2dy=0 \longrightarrow (A)$

$$\frac{1}{Mx+Ny}=\frac{1}{x(y^2+xy)-x^2(y)}$$

$$=\frac{1}{xy^2+x^2y-x^2y}=\frac{1}{xy^2}$$

So I.F is $\frac{1}{xy^2}$

xing DE(A) by I.F

$$\frac{1}{xy^2}\left((y^2+xy)\,dx-x^2dy\right)=0$$

$$\left(\frac{1}{x}+\frac{1}{y}\right)dx-\frac{x}{y^2}dy=0 \longrightarrow (B)$$

Here $M=\frac{1}{x}+\frac{1}{y} \Rightarrow \frac{\partial M}{\partial y}=\frac{-1}{y^2}$

& $N = \frac{-x}{y^2} \Rightarrow \frac{\partial N}{\partial x} = \frac{-1}{y^2}$

$\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = \frac{-1}{y^2}$

So DE(2) is now exact and its general solution is

$\int M dx + \int N(\text{free of } x\text{ term}) dy = c$

$\int \left(\frac{1}{x} + \frac{1}{y}\right) dx + \int 0 \, dy = c$

$\int \frac{1}{x} dx + \frac{1}{y} \int dx + 0 = c$

$\ln x + \frac{1}{y} \cdot x = c$

$\ln x + \frac{x}{y} = c$

$\boxed{y \ln x + x = cy}$ ANS

(2) $(x^2y - 2xy^2)dx - (x^3 - 3x^2y)dy = 0$

Sol:- Here $M = x^2y - 2xy^2$

so $\frac{\partial M}{\partial y} = x^2 - 4xy$

& $N = -x^3 + 3x^2y$

so $\frac{\partial N}{\partial x} = -3x^2 + 6xy$

$\Rightarrow \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

so given DE is not exact.

But we see clearly that given

DE is homogeneous of degree 3.
so

$$Mx + Ny = (x^2y - 2xy^2)x + (-x^3 + 3x^2y)y$$

$$= \cancel{x^3y} - 2x^2y^2 - \cancel{x^3y} + 3x^2y^2$$

$$= x^2y^2 \neq 0$$

So

$$\frac{1}{Mx+Ny} = \frac{1}{x^2y^2} \text{ is I.F}$$

xing Given D.E by I.F

$$\frac{1}{x^2y^2}\left[(x^2y - 2xy^2)dx - (x^3 - 3x^2y)dy\right] = 0$$

$$\left(\frac{1}{y} - \frac{2}{x}\right)dx - \left(\frac{x}{y^2} - \frac{3}{y}\right)dy = 0$$

Here $M = \frac{1}{y} - \frac{2}{x}$

& $\frac{\partial M}{\partial y} = \frac{-1}{y^2}$

and $N = -\frac{x}{y^2} + \frac{3}{y}$

$\frac{\partial N}{\partial x} = -\frac{1}{y^2}$

$$\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = -\frac{1}{y^2}$$

so this DE is now exact and its general solution is

$$\int M dx + \int N(\text{free of } x\text{-terms})dy = c$$

$$\int\left(\frac{1}{y} - \frac{2}{x}\right)dx + \int \frac{3}{y} dy = c$$

$$\int \frac{1}{y}\,dx - 2\int \frac{dx}{x} + 3\int \frac{dy}{y} = c$$

$$\frac{x}{y} - 2\ln x + 3\ln y = c$$

or

$$\boxed{x - 2y\ln x + 3y\ln y = cy} \quad \underline{\underline{\text{ANS}}}$$

(3) $(3xy + y^2)\,dx + (x^2 + xy)\,dy = 0 \longrightarrow (1)$

Sol:

Here $M = 3xy^2 + y^2$

$$\frac{\partial M}{\partial y} = 3x + 2y$$

& $N = x^2 + xy$

$$\frac{\partial N}{\partial x} = 2x + y$$

$$\Rightarrow \quad \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

So this is not an exact D.E

We see that this is a homogeneous D.E. So for I.F, we have

$$Mx + Ny = (3xy + y^2)x + (x^2 + xy)y$$

$$= 3x^2y + xy^2 + x^2y + xy^2$$

$$= 2xy(2x + y) \neq 0$$

So $\quad I.F = \dfrac{1}{2xy(2x+y)}$

xing D.E (1) by Integrating factor

$$\frac{1}{2xy(2x+y)}\left[(3xy + y^2)\,dx + (x^2 + xy)\,dy\right] = 0$$

$$\frac{3xy+y^2}{2xy(2x+y)}dx + \frac{x(x+y)}{2xy(2x+y)}dy = 0$$

$$\frac{y(3x+y)}{2xy(2x+y)}dx + \frac{x(x+y)}{2xy(2x+y)}dy = 0$$

$$\frac{(3x+y)}{2x(2x+y)}dx + \frac{(x+y)}{2y(2x+y)}dy = 0 \rightarrow B_1$$

It is your home work to check the exactness of DE (B_1).

Now its general solution will be

$$\int M\,dx + \int N(\text{free of } x \text{ terms})\,dy = C_1$$

$$\int \frac{(3x+y)}{2x(2x+y)}dx + \int 0\,dy = C_1$$

$$\frac{1}{2}\int \frac{3x+y}{x(2x+y)}dx + 0 = C_1$$

$$\frac{1}{2}\int \frac{3x+y}{x(2x+y)}dx = C_1$$

$$\frac{1}{2}\int\left(\frac{1}{x} + \frac{1}{2x+y}\right)dx = C_1$$

$$\frac{1}{2}\int \frac{dx}{x} + \frac{1}{2\cdot 2}\int \frac{2dx}{2x+y} = C_1$$

$$\frac{1}{2}\ln x + \frac{1}{4}\ln(2x+y) = C_1$$

$$\frac{1}{4}\left(2\ln x + \ln(2x+y)\right) = C_1$$

$$2\ln x + \ln(2x+y) = 4C_1 \quad , \text{ Let } C = 4C_1$$

$$\boxed{2\ln x + \ln(2x+y) = C}$$

$\because \frac{3x+y}{x(2x+y)} = \frac{A}{x} + \frac{B}{2x+y}$

$3x+y = A(2x+y) + Bx$

$\boxed{A=1}$, $\boxed{B=1}$

By partial fraction

Rule # 4 :- When the non-exact D.E is of the form $y\,M(x,y)dx + x\,N(x,y)dy = 0$ and $Mx - Ny \neq 0$ then $\frac{1}{Mx-Ny}$ is an integrating factor.

Question :- Solve the following D.Eqns

(1) $(y - xy^2)dx + x\,dy = 0$

Sol :- By checking, we get that this is not an exact D.E i.e

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

Now from given DE, we see that

$$y(1-xy)dx + x \cdot 1\,dy = 0$$

Which is of the form

$$y\,M(x,y)dx + x\,N(x,y)dy = 0$$

So for I.F, we have

$$\frac{1}{Mx - Ny} = \frac{1}{xy(1-xy) - xy}$$

$$I.F = \frac{1}{xy - x^2y^2 - xy} = \frac{-1}{x^2y^2}$$

xing given D.E by I.F

$$\frac{-1}{x^2y^2}\left[y(1-xy)dx + x\,dy\right] = 0$$

$$\frac{y(xy-1)}{x^2y^2}dx + \frac{x}{-x^2y^2}dy = 0$$

$$\left(\frac{xy-1}{x^2y}\right)dx - \frac{1}{xy^2}dy = 0$$

or $\left(\frac{1}{x} - \frac{1}{x^2y}\right)dx - \frac{1}{xy^2}dy = 0$

Here $M = \frac{1}{x} - \frac{1}{xy}$

$\frac{\partial M}{\partial y} = \frac{1}{x^2y^2}$

& $N = \frac{-1}{xy'}$

$\frac{\partial N}{\partial x} = \frac{1}{x^2y^2}$

$\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = \frac{1}{x^2y^2}$

means this D'E is now exact and its general solution will be

$\int M dx + \int N(\text{free of } x \text{ term}) dy = c$

$\int\left(\frac{1}{x} - \frac{1}{x^2y}\right)dx + \int 0\, dy = c$

$= \int \frac{dx}{x} - \frac{1}{y}\int \frac{dx}{x^2} + 0 = c$

$\ln x - \frac{1}{y} \cdot \frac{-1}{x} = c$

$\ln x + \frac{1}{xy} = c$

$\boxed{xy \ln x + 1 = cxy}$ ANS

(2) $(3y + 4xy^2)dx + (2x + 3x^2y)dy = 0 \rightarrow (A)$

Sol: Here $M = 3y + 4xy^2$

$\frac{\partial M}{\partial y} = 3 + 8xy$

& $N = 2x + 3x^2y \Rightarrow \frac{\partial N}{\partial x} = 2 + 6xy$

$$\Rightarrow \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

So this is a non-exact DE.
But we see that

$$(3y + 4xy^2)\,dx + (2x + 3x^2y)\,dy = 0$$

$$y(3 + 4xy)\,dx + x(2 + 3xy)\,dy = 0 \to (B)$$

which is of the form

$$yM(x,y)\,dx + xN(x,y)\,dy = 0$$

$$\therefore \; I \cdot F = \frac{1}{Mx - Ny} = \frac{1}{xy(3+4xy) - xy(2+3xy)}$$

$$= \frac{1}{3xy + 4x^2y^2 - 2xy - 3x^2y^2} = \frac{1}{xy + x^2y^2}$$

xing given D·E (B) by $\frac{1}{xy + x^2y^2}$

$$\frac{y(3+4xy)}{xy + x^2y^2}\,dx + \frac{x(2+3xy)}{xy + x^2y^2}\,dy = 0$$

This is now exact D·E (check it) and its general solution is

$$\int M\,dx + \int N(\text{free of } x \text{ terms})\,dy = \ln c$$

$$\int \frac{y(3+4xy)}{xy + x^2y^2}\,dx + \int 0\,dy = \ln c$$

$$\int \frac{y(3+4xy)}{y(x + x^2y)}\,dx + 0 = \ln c$$

$$\int \frac{3+4xy}{x(1+xy)}\,dx = \ln c$$

$$3\int \frac{dx}{x(1+xy)} + 4y\int \frac{dx}{(1+xy)} =$$

$$3\int \left(\frac{1}{x} - \frac{y}{1+xy}\right)dx + 4y\int \frac{dx}{1+xy} =$$

$$3\ln x - 3\ln(1+xy) + 4\ln(1+xy) = \ln c$$

$$\ln x^3 + \ln(1+xy) = \ln c$$

$$\ln\left(x^3(1+xy)\right) = \ln c$$

$$\boxed{x^3(1+xy) = c} \text{ ANS}$$

Home Work :- Solve the following D.Eqns by the use of integrating factors discussed in Rule (1) — to Rule (4)

(1) $(3xy + y^2)\,dx + (x^2 + xy)\,dy = 0$

(2) $(x^2 - 2x + 2y^2)\,dx + 2xy\,dy = 0$

(3) $(x^2y - 2xy^2)\,dx - (x^3 - 3x^2y)\,dy = 0$

(4) $(xy^2 + 2x^2y^3)\,dx + (x^2y - x^3y^2)\,dy = 0$

(5) $(x^2y^2 + y)\,dx - x\,dy = 0$

(6) $(x^2 + x - y)\,dx + x\,dy = 0$

(7) $(3y + 4xy^2)\,dx + (2x + 3x^2y)\,dy = 0$

(8) $(y^2 + xy)\,dx - x^2\,dy = 0$

(9) $\frac{dy}{dx} = e^{2x} + y - 1$

(10) $dy + \frac{y - \sin x}{x}\,dx = 0$

(V) Linear Equation of order 1.

Overview :-

A first order ODE is said to be linear when the dependent variable and its derivative appears only in the first degree. For Instance, consider the DE $\frac{dy}{dx} + 3xy = 0$ is a DE of 1st degree.

Definition :- A first order ordinary DE is linear in the dependent variable y and independent variable x if it is or can be written in the form

$$\frac{dy}{dx} + P(x)y = Q(x) \longrightarrow (1)$$

where $P(x)$ and $Q(x)$ are functions of x only. DE (1) is also called "linear nonhomogeneous ODE of 1st order. If $P(x)$ and $Q(x)$ both are zeros then DE (1) is reduced to variable separable equation.

Let us find a formula for the general solution of equation (1). DE (1) can be written as

$$(P(x)y - Q(x))dx + dy = 0 \longrightarrow (2)$$

Which is of the form

$$Mdx + Ndy = 0$$

where

$$M = P(x)y - Q(x) \quad \& \quad N = 1$$

Now

$$\frac{\partial M}{\partial y} = P(x) \quad \text{and} \quad \frac{\partial N}{\partial x} = 0$$

Thus (2) is non-exact DE unless $P(x) = 0$.

Now since

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{P(x) - 0}{1} = P(x)$$

i.e

$$\frac{M_y - N_x}{N} = P(x) = f(x)$$

depends only on x so by case (1) we know that $e^{\int P(x)dx}$ is an Integrating factor. Multiplying DE (1) by $e^{\int P(x)dx}$

$$e^{\int P(x)dx}\left(\frac{dy}{dx} + P(x)y\right) = Q\, e^{\int P(x)dx}$$

which can be written as

$$\frac{d}{dx}\left(e^{\int P(x)dx}(y)\right) = Q\, e^{\int P(x)dx}$$

then we have

$$d\left(e^{\int P(x)dx} \cdot y\right) = \left(Q(x)\, e^{\int P(x)dx}\right) dx$$

On Integrating b/s

$$\int d\left(e^{\int P(x)dx}\, y\right) = \int \left(Q(x)\, e^{\int P(x)dx}\right) dx$$

$$y \times I \cdot F = \int Q(I \cdot F)\, dx + c$$

Note :- The choice of the value of constant of Integration in $\int P(x)dx$ does not matter so that we may chose it to be zero.

Question :- Solve the following first order ODEs.

(1) $\frac{dy}{dx} + 2xy = 2e^{-x^2}$

Sol:- Here $P(x) = 2x$, $Q(x) = 2e^{-x^2}$

So

$$I \cdot F = e^{\int P(x)dx} = e^{\int 2x\,dx} = e^{2\int x\,dx}$$

$$e^{\cancel{2}\frac{x^2}{\cancel{2}}} = e^{x^2} = I \cdot F$$

Hence the general solution of the given DE is

$$y \cdot (I \cdot F) = \int (Q(x) \cdot I \cdot F)\,dx + C$$

$$y \cdot e^{x^2} = \int (2e^{-x^2} \cdot e^{x^2})\,dx + C$$

$$y e^{x^2} = \int (2e^{x^2 - x^2})\,dx + C$$

$$y e^{x^2} = \int 2(1)\,dx + C$$

$$y e^{x^2} = 2x + C$$

$$\boxed{y = (2x + C)\,e^{-x^2} \quad \text{ANS}}$$

Which is an explicit solution.

(2) $\frac{dy}{dx} + y\tan x = \sin(2x)$; $y(0) = 1$

Sol:- Here $P = \tan x$, $Q = \sin 2x$

So Integrating factor is

$$e^{\int P(x)dx} = e^{\int \tan x\,dx} = e^{\ln|\sec x|}$$

$$I \cdot F = \sec x$$

Since $e^{\int \tan x\,dx}$
$= e^{-\ln\cos x}$
$= e^{\ln \cos x^{-1}}$
$= \cos x^{-1}$
$= \frac{1}{\cos x}$
$= \sec x$

So the general solution of the given DE is

$$y\cdot(I\cdot F)=\int(Q(x)\cdot I\cdot F)\,dx+C$$

$$y\cdot\sec x=\int(\sin 2x\cdot\sec x)\,dx+C$$

$$=\int(2\sin x\cos x\cdot\sec x)\,dx+C$$

$$y\cdot\sec x=\int\left(2\sin x\cancel{\cos x}\cdot\frac{1}{\cancel{\cos x}}\right)dx+C$$

$$=2\int\sin x\,dx+C$$

$$=2(-\cos x)+C$$

$$y\cdot\sec x=-2\cos x+C$$

$$y=\frac{1}{\sec x}\left(-2\cos x+C\right)$$

$$=\cos x(C-2\cos x)$$

$$y=C\cos x-2\cos^2 x\longrightarrow(1)$$

Using the initial condition $y(0)=1$

$\Rightarrow x=0,\ y=1$

$$1=C\cos 0-2\cos^2(0)$$

$$1=C(1)-2(1)$$

$$\boxed{C=3}$$

putting value of C in (1)

$$\boxed{y=3\cos x-2\cos^2 x}$$

(3) $x\dfrac{dy}{dx}+2y=\sin x\longrightarrow(A)$

Sol:- Now we use 2nd method in the solution of this question as

compare to the previous 2 questions which is quite similar to the previous one.
Here, we have

$$x\frac{dy}{dx}+2y=\sin x$$

$$\Rightarrow \frac{dy}{dx}+\frac{2y}{x}=\frac{\sin x}{x} \longrightarrow (2)$$

Now we see that

$$P(x)=2/x, \quad Q(x)=\frac{\sin x}{x}$$

$$\therefore \quad I\cdot F=e^{\int P(x)dx}=e^{\int 2/x\,dx}$$

$$=e^{2\int\frac{dx}{x}}=e^{2\ln x}=e^{\ln x^2}=x^2$$

So $I\cdot F=x^2$

xing eq(2) by "x^2",

$$x^2\left(\frac{dy}{dx}+\frac{2y}{x}\right)=x^2\,\frac{\sin x}{x}$$

$$\frac{d}{dx}(x^2\cdot y)=x\sin x$$

$$d(x^2y)=x\sin x\,dx$$

Integrating b/s

$$\int d(x^2y)=\int x\sin x\,dx$$

$$x^2y=x\int\sin x\,dx-\int\left(\frac{d}{dx}(1)\int\sin x\,dx\right)dx$$

$$=x(-\cos x)-\int 1\cdot(-\cos x)dx$$

$$=-x\cos x+\int\cos x\,dx$$

$$x^2y=-x\cos x+\sin x+C$$

$$\Rightarrow y=\frac{1}{x^2}\left(C-x\cos x+\sin x\right) \text{ ANS}$$

(4) $\frac{dr}{d\theta} + r\tan\theta = \cos^2\theta, \quad r(\pi/4) = 1$

Sol:- Here $P(\theta) = \tan\theta$
& $Q(\theta) = \cos^2\theta$

So $I \cdot F = e^{\int P(\theta) d\theta} = e^{\int \tan\theta\, d\theta}$
$= e^{\int \tan\theta d\theta} = e^{-\ln|\cos\theta|} = e^{\ln|\cos\theta|^{-1}}$

$I \cdot F = \sec\theta$

Hence for general solution, we have

$r \cdot I \cdot F = \int (Q(\theta)(I \cdot F)) d\theta + C$

$r \cdot \sec\theta = \int (\cos^2\theta \cdot \sec\theta) d\theta + C$

$r \sec\theta = \int \left(\cos^2\theta \cdot \frac{1}{\cos\theta}\right) d\theta + C$

$r \sec\theta = \int \cos\theta\, d\theta + C$

$r \sec\theta = \sin\theta + C$

Now

$r = \frac{1}{\sec\theta}\left[\sin\theta + C\right]$

$r = \sin\theta\cos\theta + C\cos\theta \rightarrow (3)$

Again, we have

$r(\pi/4) = 1$

$\Rightarrow \theta = \pi/4 \Rightarrow 1 = r$

$\Rightarrow 1 = \sin(\pi/4)\cos(\pi/4) + C\cos(\pi/4)$

$1 = \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} + C \cdot \frac{1}{\sqrt{2}}$

$1 = \frac{1}{2} + \frac{1}{\sqrt{2}} C$

$C \cdot \frac{1}{\sqrt{2}} = \frac{1}{2} \Rightarrow C = \frac{\sqrt{2}}{2}$

$$c = \frac{1}{\sqrt{2}}$$

putting in eq(3), we get

$$r = \sin\theta\cos\theta + \frac{1}{\sqrt{2}}\cos\theta$$

$$\Rightarrow 2r = 2\sin\theta\cos\theta + \frac{2}{\sqrt{2}}\cos\theta$$

$$2r = \sin 2\theta + \frac{\sqrt{2}\cdot\cancel{\sqrt{2}}}{\cancel{\sqrt{2}}}\cos\theta$$

$$\boxed{2r = \sin 2\theta + \sqrt{2}\cos\theta} \text{ ANS}$$

(5) $y' + \tan(x)\, y = \cos^2(x)$; $y(0) = 2$

Sol:- Given equation is of the form

$$y' + P(x)y = Q(x)$$

here

$$P(x) = \tan x, \quad Q(x) = \cos^2 x$$

So for Integrating factor, we have

$$e^{\int P(x)dx} = e^{\int \tan x\, dx} = e^{-\ln|\cos|}$$

$$e^{\ln|\cos|^{-1}} = (\cos)^{-1} = \frac{1}{\cos} = \sec$$

Now the general solution becomes

$$(I\cdot F)\cdot y = \int (I\cdot F)\, Q(x)\, dx + C$$

$$\therefore \quad \sec x \cdot y = \int (\sec x \cdot \cos^2 x)\, dx + C$$

$$\sec x \cdot y = \int \frac{1}{\cos x}\cdot \cos^2 x\, dx + C$$

$$y = \frac{1}{\sec x}\int \cos x\, dx + C$$

$$y = \cos x((\sin x) + c) \longrightarrow (1)$$

Now given that $y(0) = 2$

$$\therefore \quad y(0) = \cos(0)(\sin(0) + c)$$

$$2 = 1(0 + c)$$

$$\Rightarrow c = 2$$

Using in the above equation (1)

$$y = \cos x(\sin x + 2) \cdot \text{ ANS.}$$

(6) $x \dfrac{dy}{dx} + (2x+1)y = xe^{-2x} \longrightarrow (1)$

Sol:- Given that

$$x\frac{dy}{dx} + (2x+1)y = xe^{-2x}$$

$$\frac{dy}{dx} + \left(\frac{2x+1}{x}\right)y = \frac{xe^{-2x}}{x}$$

$$\frac{dy}{dx} + \left(2 + \frac{1}{x}\right)y = e^{-2x} \longrightarrow (*)$$

This is frist order linear ODE of the form $\dfrac{dy}{dx} + P(x)y = Q(x) \longrightarrow (2)$

By comparing given ODE (1) with (2) we have

$$P(x) = \left(2 + \frac{1}{x}\right) \;\&\; Q(x) = e^{-2x}$$

Now for integrating factor, we have

$$e^{\int p(x)dx} = e^{\int(2+1/x)dx} = e^{2x+\ln x}$$

$$I \cdot F = e^{2x} \cdot e^{\ln x}$$

i.e $I \cdot F = xe^{2x}$

Now multiplying eq (x) by $I \cdot F$ we get

$$xe^{x}\left(\frac{dy}{dx}+\left(2+\frac{1}{x}\right)y\right)=xe^{2x}\cdot e^{-2x}$$

$$xe^{x}\frac{dy}{dx}+xe^{x}\left(\frac{2x+1}{x}\right)y=xe^{2x-2x}$$

$$\frac{d}{dx}\left(xe^{2x}\cdot y\right)=xe^{0}=x(1)=x$$

$$\Rightarrow \frac{d}{dx}(xe^{2x}\cdot y)=x$$

$$d(xe^{2x}\cdot y)=x\,dx$$

Integrating b/s

$$\int d(xe^{2x}\cdot y)=\int x\,dx$$

$$xe^{2x}\cdot y=\frac{x^2}{2}+c$$

$$y=\frac{1}{xe^{2x}}\left(\frac{x^2}{2}+c\right)$$

$$y=\frac{xe^{-2x}}{2}+\frac{c}{x}e^{-2x} \quad \underline{\text{ANS}}$$

(7) $dr+(2r\cot\theta+\sin 2\theta)\,d\theta=0$

Sol: Given that

$$dr+(2r\cot\theta+\sin 2\theta)\,d\theta=0$$

$$\Rightarrow \frac{dr}{d\theta}+2r\cot\theta+\sin 2\theta=0$$

$$\Rightarrow \frac{dr}{d\theta}+2r\cot\theta=-\sin 2\theta \longrightarrow (A)$$

Equation (A) is linear ODE of first order in r of the form

$$\frac{dr}{d\theta} + P(\theta)r = Q(\theta)$$

Here $P(\theta) = 2\cot\theta$ &

$Q(\theta) = -\sin 2\theta$

Now for integrating factor, we have

$$e^{\int P(\theta)d\theta} = e^{\int 2\cot\theta d\theta} = e^{2\int \frac{\cos\theta}{\sin\theta} d\theta}$$

$$e^{2\ln\sin\theta} = e^{\ln\sin^2\theta} = \sin^2\theta$$

So general solution of given ODE becomes

$$r\cdot(I\cdot F) = \int I\cdot F \times Q(\theta)d\theta + C$$

$$r\cdot \sin^2\theta = \int \sin^2\theta\cdot(-\sin 2\theta)d\theta + C$$

$$r\sin^2\theta = \int -\sin 2\theta \sin^2\theta \, d\theta + C$$

$$= -2\int (\sin\theta\cos\theta\sin^2\theta)\, d\theta + C$$

$$= -2\int \sin^3\theta\cos\theta\, d\theta + C$$

Let $u = \sin\theta$

$\Rightarrow du = \cos\theta \, d\theta$

$$\therefore -2\int u^3 du$$

$$= -2\frac{u^4}{4} + C$$

putting values of u again, we have

$$\int \sin^3\theta \cos\theta \, d\theta = -\frac{2}{4}\sin^4\theta$$

Hence

$$r\sin^2\theta = -2\int \sin^3\theta \cos\theta \, d\theta + c$$

becomes

$$r\sin^2\theta = -\frac{1}{2}\sin^4\theta + c$$

$$2r\sin^2\theta = -\sin^4\theta + 2c_1$$

$$\Rightarrow 2r\sin^2\theta + \sin^4\theta = 2c_1 = c$$

or

$$\boxed{2r\sin^2\theta + \sin^4\theta = c} \quad \underline{\text{ANS}}$$

(8) $x\dfrac{dy}{dx} - 3y = x - 1 \quad ; \quad y(1) = 0$

Sol:- Given that

$$x\frac{dy}{dx} - 3y = x - 1$$

$$\Rightarrow \frac{dy}{dx} - \frac{3}{x}y = \frac{x-1}{x}$$

$$\text{or} \quad \frac{dy}{dx} - \frac{3}{x}y = 1 - \frac{1}{x}$$

which is linear first order ODE of the form $\dfrac{dy}{dx} + P(x)y = Q(x)$

Here $P(x) = \dfrac{-3}{x}$, $Q(x) = 1 - \dfrac{1}{x}$

Now for Integrating factor, we have

$$e^{\int P(x)dx} = e^{\int \frac{-3}{x}dx} = e^{-3\int \frac{dx}{x}} = e^{-3\ln x}$$

$$= e^{\ln x^{-3}} = x^{-3} = \frac{1}{x^3}$$

So general solution becomes

$$(I \cdot F)\, y = \int (I \cdot F)\, Q(x)\, dx + C$$

$$\therefore \frac{1}{x^3} y = \int \frac{1}{x^3}\left(1 - \frac{1}{x}\right) dx + C$$

$$\frac{1}{x^3} y = \int \left(\frac{1}{x^3} - \frac{1}{x^4}\right) dx + C$$

$$\frac{1}{x^3} y = \int (x^{-3} - x^{-4})\, dx + C$$

$$\frac{1}{x^3} y = \frac{x^{-3+1}}{-3+1} - \frac{x^{-4+1}}{-4+1} + C$$

$$\frac{1}{x^3} y = -\frac{1}{2x^2} + \frac{1}{3x^3} + C$$

$$y = +x^3\left[-\frac{1}{2x^2} + \frac{1}{3x^3}\right] + Cx^3$$

$$y = \frac{-x^3}{2x^2} + \frac{x^3}{3x^3} + Cx^3$$

$$y = -\frac{x}{2} + \frac{1}{3} + Cx^2 \longrightarrow (A)$$

Now from initial condition we have $y(1) = 0$

$$\therefore y(1) = -\frac{1}{2} + \frac{1}{3}(1) + C(1)^2$$

$$0 = -\frac{1}{2} + \frac{1}{3} + C$$

$$-\frac{1}{6} + C = 0$$

$$\Rightarrow \boxed{C = 1/6}$$

$c = \frac{1}{6}$

Using value of c in eq(A), we get

$$y = \frac{-x}{2} + \frac{1}{3} + \frac{1}{6}x^3$$

$$\Rightarrow \boxed{6y = 2 - 3x + x^3} \quad \underline{\text{ANS}}$$

Home Work:-

Solve the following ODEs.

(1) $\frac{dy}{dx} + x^2 y = (x^2+1)e^x$

ANS:- $y = e^x + ce^{-x^3/3}$

(2) $\frac{dy}{dx} + 2xy = 0$

ANS:- $y = e^{-x^2+c_1}$

(3) $\frac{dy}{dx} + y = \sin x \quad ; \quad y(\pi) = 1$

(4) $y' \sin x - (\cos x)y = \cot x$

(5) $(x^2+1)\frac{dy}{dx} + 2xy = 4x^2 \quad ; \quad y(0) = 0$

ANS:- $3y(x^2+1) = 4x^3$

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(9) $\frac{d\bar{u}}{dx} + xs\bar{u} = 0 \longrightarrow (1)$

Sol:- Given ODE is linear of first order in $\bar{u}$

Here $P(x) = sx$ & $Q(x) = 0$

So by Integrating factor, we have

$$e^{\int P(x)dx} = e^{\int sx\,dx} = e^{s\int x\,dx}$$

$$I \cdot F = e^{\frac{sx^2}{2}}$$

Hence general solution of (1) becomes

$$\bar{u}\cdot(I\cdot F) = \int Q(x)(I\cdot F)\,dx + C$$

but $Q(x) = 0$ (here)

$$\Rightarrow e^{\frac{sx^2}{2}}\bar{u} = 0 + C$$

$$\Rightarrow \bar{u} = e^{-sx^2/2}(0 + C)$$

$$\Rightarrow \boxed{\bar{u} = Ce^{-sx^2/2}}$$ ANS

Home Work:-

* $\frac{d\bar{u}}{dx} - (2s+1)\bar{u} = -12e^{-3x}$

ANS:- $\bar{u} = \frac{6}{s+2}e^{-3x} + Ce^{(2s+1)x}$.

The Bernoulli Equation

Overview:- Sometimes non-linear ODEs can be reduced to linear form by using appropriate substitution. Let us consider the differential equation

$$y' + P(x)y = Q(x)y^n \longrightarrow (A)$$

Equation (A) is known as Bernoulli's differential equation. This is the example of non-linear ordinary differential equation which can be reduced to a linear one by a clever substitution. The new equation is a first order linear differential equation, and can be solved explicitly. The Bernoulli equation was one of the first differential equation, and still one of very few non-linear differential equations that can be solved explicitly. Most other such equations have no solutions, or substitution that cannot be written in a closed form, but the Bernoulli equation is an exception.

Note:- In equation (A) n is any real number but not 0 and 1 because, when $n=0$, then it becomes

$$y' + P(x)y = Q(x)$$

Which is first order linear differential equation, and we know that how to solve first order linear differential equation. And if $n=1$ then from (A)

$$y' + P(x)y = Q(x)y$$

Which can be solved by using the method of separation of verriable i.e

$$y' + P(x)y = Q(x)y$$

$$\Rightarrow y' + P(x)y - Q(x)y = 0$$

$$\Rightarrow y' + [P(x) - Q(x)]y = 0$$

However if n is 0 and 1, then Bernoulli equation can be reduced to linear equation.

Working Rule:- Now to reduce eq(A) into first order linear DE, we write eq(A) here again

$$y' + P(x)y = Q(x)y^n$$

* ÷ing b/s by y^n

$$\frac{y'}{y^n} + \frac{P(x)y}{y^n} = Q(x)\frac{y^n}{y^n}$$

$$\frac{y'}{y^n} + P(x)y^{1-n} = Q(x)$$

$$y'y^{-n} + P(x)y^{1-n} = Q(x) \longrightarrow (B)$$

* Let $v = y^{1-n}$, then

$$\frac{dv}{dx} = (1-n)y^{-n}y'$$

$$y^{-n}y' = \frac{1}{1-n}\frac{dv}{dx}$$

So by using the value of $y^{-n}y'$ in Eq(B), we get

$$\frac{1}{1-n}\frac{dv}{dx} + vP(x) = Q(x)$$

$$\Rightarrow \frac{dv}{dx} + (1-n)P(x)v = Q(x) \longrightarrow (C)$$

Now clearly eq(C) is a linear first order ordinary differential equation.

* Now for Integrating factor I·F we have

$$e^{\int P(x)dx}$$

Here $P(x) = (1-n)P(x)$

$$\therefore e^{\int (1-n)p(x)dx} = e^{\int (1-n)p(x)dx} = I\cdot F$$

Let $I\cdot F = f(x)$

then $f(x) = e^{\int(1-n)p(x)dx}$

Now for the general solution, we have from eq(3).

$$v\cdot I\cdot F = \int (I\cdot F \times Q(x))dx + C$$

$$v\cdot f(x) = \int f(x)\,Q(x)\,dx + C$$

$$v = \frac{1}{f(x)}\int (f(x)\cdot Q(x))dx + C \longrightarrow (D)$$

Now again substitute value of v

i.e $v = y^{1-n}$

By using this value of v in eq(D)

We will get the required solution of DE (A).

* If there is any Initial condition, then use it in the general solution. Now we try to learn this concept with the help of few examples.

Question:- Solve the following D.Eqns.

(1) $y' + \frac{1}{x}y = x^2 y^6 \longrightarrow (A)$

Sol:- Given DE is of the form

$$y' + P(x)y = Q(x)y^n$$

$\Rightarrow$ Given DE is a Bernoulli DE.

To solve eq(A), $\div$ing b/s of (A) by y^6, we have

$$\frac{y'}{y^6} + \frac{1}{xy^6}\cdot y = \frac{x^2\cancel{y^6}}{\cancel{y^6}}$$

$$y'y^{-6} + \frac{1}{x}y^{-5} = x^2 \longrightarrow (B)$$

Let $y^{-5} = v$, then

$$\frac{dv}{dx} = -5y^{-6}y'$$

$$\Rightarrow y'y^{-6} = \frac{-1}{5}\frac{dv}{dx} \longrightarrow (C)$$

putting eq(C) in eq(B)

$$\frac{-1}{5}\frac{dv}{dx} + \frac{1}{x}v = x^2$$

xing b/s by -5

$$\frac{dv}{dx} - \frac{5}{x} v = -5x^2$$

$$v' + \left(\frac{-5}{x}\right) v = -5x^2 \longrightarrow (D)$$

Eq (D) is now linear in v of the form $v' + P(x)v = Q(x)$
where $P(x) = \frac{-5}{x}$, $Q(x) = -5x^2$
Now Integrating factor is

$$e^{\int P(x)dx} = e^{\int \frac{-5}{x} dx} = e^{-5\ln x}$$

$$= e^{\ln x^{-5}} = x^{-5} = \frac{1}{x^5} = I \cdot F$$

Hence general solution of (D) is

$$v \cdot (I \cdot F) = \int (Q(x) \cdot I \cdot F)\, dx + C$$

$$v \cdot x^{-5} = \int (-5x^2 \cdot x^{-5})\, dx + C$$

$$v \cdot x^{-5} = -5 \int x^{-3} dx + C$$

$$v x^{-5} = -5 \frac{x^{-2}}{-2} + C$$

$$v x^{-5} = \frac{5}{2} x^{-2} + C$$

$$v = \left(\frac{5}{2} x^{-2} + C\right) x^5$$

$$v = 5/2\, x^{-2} \cdot x^5 + C x^5$$

$$v = \frac{5}{2} x^3 + C x^5$$

$\because$ $v = y^{-5}$

$\therefore$ $\boxed{y^{-5} = \frac{5}{2} x^3 + C x^5}$ ANS

(2) $\frac{dy}{dx} + \frac{x}{1-x^2}y = xy^{\frac{1}{2}} \longrightarrow (1)$

Sol:- Given DE is Bernoulli DE of the form

$$\frac{dy}{dx} + P(x)y = Q(x)y^n$$

To solve this ODE, ÷ing both sides of eq (1) by $y^{\frac{1}{2}}$, we obtain

$$y^{-\frac{1}{2}}\left(\frac{dy}{dx} + \frac{x}{1-x^2}y\right) = xy^{\frac{1}{2}}y^{-\frac{1}{2}}$$

$$y^{-\frac{1}{2}}y' + \frac{x}{1-x^2}y^{-\frac{1}{2}}y = x \longrightarrow (2)$$

$$y^{-\frac{1}{2}}y' + \frac{x}{1-x^2}y^{\frac{1}{2}} = x$$

Let $v = y^{+\frac{1}{2}}$

$$\Rightarrow \frac{dv}{dx} = \frac{+1}{2}y^{-\frac{1}{2}}\frac{dy}{dx}$$

or $\frac{dv}{dx} = \frac{1}{2}y^{-\frac{1}{2}}y'$

$$2\frac{dv}{dx} = \frac{\cancel{2}}{2}y'y^{-\frac{1}{2}}$$

$$2\frac{dv}{dx} = y'y^{-\frac{1}{2}}$$

Using values in eq(2), we get

$$2\frac{dv}{dx} + \frac{x}{(1-x^2)}v = x$$

$$\frac{dv}{dx} + \frac{x}{2(1-x^2)}v = \frac{x}{2} \longrightarrow (3)$$

Equation (3) is now linear in v.

For integrating factor (I·F), we have

$$e^{\int P(x)dx} = e^{\int \frac{x}{(1-x^2)2} dx}$$

$$= e^{\frac{-1}{4}\int \frac{-4x}{2(1-x^2)} dx}$$

$$e^{\frac{-1}{4}\ln(1-x^2)} = e^{\ln(1-x^2)^{\frac{-1}{4}}}$$

So $I \cdot F = (1-x^2)^{\frac{-1}{4}}$

So general solution of eq(3) becomes

$$V \cdot (I \cdot F) = \int Q(x) \cdot (I \cdot F)\, dx + C$$

$$V \cdot (1-x^2)^{\frac{-1}{4}} = \int \frac{x}{2} \cdot (1-x^2)^{\frac{-1}{4}} dx + C$$

$$V \cdot (1-x^2)^{\frac{-1}{4}} = \int \frac{x}{2}(1-x^2)^{\frac{-1}{4}} dx + C \rightarrow (4)$$

Now we need to solve $\int \frac{x}{2}(1-x^2)^{\frac{-1}{4}} dx$

Let $1-x^2 = t$

$$dt = -2x\,dx$$

$$\frac{dt}{-2} = x\,dx$$

$$\therefore \frac{-1}{2}\int \frac{1}{2}(t)^{\frac{-1}{4}} dt = \frac{-1}{4}\int t^{\frac{-1}{4}} dt$$

$$= \frac{-1}{4}\left(\frac{t^{\frac{-1}{4}+1}}{\frac{-1}{4}+1}\right)$$

$$= \frac{-1}{4}\, t^{\frac{3}{4}} / {}^{3}/_{4}$$

$$= \frac{-t^{3/4}}{3}$$

put $t = 1-x^2$ again, we get

$$-\frac{(1-x^2)}{3} = \int \frac{x}{2}(1-x^2)^{-\frac{1}{4}}dx$$

putting this value in eq(4)

$$v\cdot(1-x^2)^{-\frac{1}{4}} = -\frac{(1-x^2)}{3} + c$$

$$v = \left(-\frac{(1-x^2)}{3} + c\right)(1-x^2)^{\frac{1}{4}}$$

$$v = -\frac{(1-x^2)}{3}\cdot(1-x^2)^{\frac{1}{4}} + c(1-x^2)^{\frac{1}{4}}$$

$$= \frac{(x^2-1)}{3}(1-x^2)^{\frac{1}{4}} + c(1-x^2)^{\frac{1}{4}}$$

$$= \frac{(x^2-1)}{3}(1-x^2)^{\frac{1}{4}} + c(1-x^2)^{\frac{1}{4}}$$

$$= -\frac{(1-x^2)}{3}(1-x^2)^{\frac{1}{4}} + c(1-x^2)^{\frac{1}{4}}$$

$$= -\frac{1}{3}(1-x^2)^{1+\frac{1}{4}} + c(1-x^2)^{\frac{1}{4}}$$

$$v = -\frac{1}{3}(1-x^2)^{5/4} + c(1-x^2)^{\frac{1}{4}}$$

putting value of v again

$$y^{\frac{1}{2}} = -\frac{1}{3}(1-x^2)^{5/4} + c(1-x^2)^{\frac{1}{4}} \quad \underline{\underline{\text{Ans}}}$$

(3) $y' - \frac{1}{x}y = y^9 \longrightarrow (A)$

<u>Sol:-</u> We see that (A) is Bernoulli <u>DE</u>. So $\div$ing b/s of (A) by y^9 we get

$$\frac{1}{y^9}\left(y' - \frac{1}{x}y\right) = \frac{y^9}{y^9}$$

$$y'y^{-9} - \frac{1}{x}yy^{-9} = 1$$

$$y^{-9}y' - \frac{1}{x}y^{-8} = 1 \longrightarrow (1)$$

Let $v = y^{-8}$, then

$$\frac{dv}{dx} = -8y^{-9}y'$$

$$\frac{-1}{8}\frac{dv}{dx} = y^{-9}y'$$

Using values in (1), we get

$$\frac{-1}{8}\frac{dv}{dx} - \frac{1}{x}v = 1$$

$$\frac{dv}{dx} + \frac{8}{x}v = -8 \longrightarrow (2)$$

Equation (2) is now linear DE in v of the form

$$v' + P(x)v = Q(x)$$

Now for Integrating factor, we have

$e^{\int P(x)dx}$ here $P(x) = 8/x$

$$\therefore e^{\int P(x)dx} = e^{\int 8/x\, dx} = e^{8\int \frac{dx}{x}}$$

$$e^{8\ln x} = e^{\ln x^8} = x^8$$

So $I \cdot F = x^8$

Hence general solution of (A) becomes

$$I \cdot F \times v = \int Q(x)(I \cdot F)\,dx + A$$

$$v \times x^8 = \int (-8 \cdot x^8)\, dx + A$$

$$v \cdot x^8 = \frac{-8x^9}{9} + A$$

$$v = \left(\frac{-8}{9}x^9 + A\right)\frac{1}{x^8}$$

$$v = \frac{-8}{9}x^9 x^{-8} + A x^{-8}$$

$$v = \frac{-8}{9}x + A x^{-8}$$

put $v = y^{-6}$ again, we get

$$y^{-6} = \frac{-8}{9}x + Ax^{-8}$$

where A is constant.

(4) $2xy\,dy - (x^2 + y^2 + 1)\,dx = 0$; $y(1) = 1$

Sol:- Frist we need to write given DE in appropriate form. So

$$2xy\frac{dy}{dx} - x^2 - y^2 - 1 = 0$$

$$2xy\frac{dy}{dx} - x^2 - y^2 = 1$$

$$2xy\frac{dy}{dx} - y^2 = 1 + x^2$$

$$y\frac{dy}{dx} - \frac{y^2}{2x} = \frac{(1+x^2)}{2x}$$

$$yy' - \frac{1}{2x}y^2 = \frac{1+x^2}{2x} \longrightarrow (A)$$

Eq(A) is now Bernoulli DE.

So Let $v = y^2$

$$\Rightarrow \frac{dv}{dx} = 2yy'$$

$$\Rightarrow \frac{1}{2}\frac{dv}{dx} = yy'$$

Using values in eq (A)

$$\frac{1}{2}\frac{dv}{dx} - \frac{1}{2x}v = \frac{x^2+1}{2x}$$

$$\frac{dv}{dx} - \frac{\cancel{2}}{2x}v = \frac{\cancel{2}(x^2+1)}{2x}$$

$$\frac{dv}{dx} - \frac{1}{x}v = \frac{x^2+1}{x}$$

$$\frac{dv}{dx} - \frac{1}{x}v = x + \frac{1}{x} \longrightarrow (2)$$

Eq (2) is now linear in v of the form

$$v' + P(x)v = Q(x)$$

where $P(x) = \frac{-1}{x}$, $Q(x) = x + \frac{1}{x}$

Now for Integrating factor, we have

$$I \cdot F = e^{\int P(x)dx}$$

$$e^{\int -\frac{1}{x}dx} = e^{-\ln x} = e^{\ln x^{-1}} = x^{-1}$$

$$\therefore I \cdot F = x^{-1} = \frac{1}{x}$$

So general solution of given ODE becomes

$$v(I \cdot F) = \int Q(x)(I \cdot F)\,dx + c$$

$$v\cdot\frac{1}{x} = \int\left(x+\frac{1}{x}\right)\cdot\frac{1}{x} + c$$

$$v\cdot\frac{1}{x} = \int\left(1+\frac{1}{x^2}\right)dx$$

$$v\cdot\frac{1}{x} = \int dx + \int x^{-2}dx \quad + c$$

$$v\cdot\frac{1}{x} = x + \frac{x^{-1}}{-1} + c$$

$$v\cdot\frac{1}{x} = x - \frac{1}{x} + c$$

$$v = x\left(x-\frac{1}{x}\right) + cx$$

$$v = x^2 - 1 + cx$$

put $v = y^2$ again

$$y^2 = x^2 - 1 + cx \longrightarrow (i)$$

Now from initial condition we have $y(1)=1 \Rightarrow x=y=1$

$$\therefore \quad (1)^2 = (1)^2 - 1 + c(1)$$

$$\Rightarrow \quad c = 1$$

putting value of c in (i), we get

$$y^2 = x^2 - 1 + x \quad \underline{\underline{\text{ANS}}}$$

(5) $y' + \frac{2y}{x} = x^2y^2\sin x$

<u>Sol:</u> $y' + \frac{2y}{x} = x^2y^2\sin x$

$$y' + 2y/x = x^2\sin x(y^2) \longrightarrow (1)$$

$\div$ing b/s by y^{-2}

$$y^{-2}y' + \frac{2y}{x}y^{-2} = x^2 \sin x$$

$$y^{-2}y' + \frac{2}{x}y^{-1} = x^2 \sin x \longrightarrow (A)$$

Equation (A) is now a Bernoulli DE, So to solve it, we let

$$v = y^{-1}$$

$$\Rightarrow \frac{dv}{dx} = -y^{-2}\frac{dy}{dx}$$

$$\Rightarrow \frac{dv}{dx} = -y^{-2}y'$$

$$-\frac{dv}{dx} = y^{-2}y'$$

Using values in eq (A), we get

$$-\frac{dv}{dx} + \frac{2}{x}v = x^2 \sin x$$

$$\Rightarrow \frac{dv}{dx} - \frac{2}{x}v = -x^2 \sin x \longrightarrow (B)$$

Which is linear DE of the form

$$\frac{dv}{dx} + P(x)v = Q(x)$$

For Integrating Factor, we have

$$e^{\int P(x)dx} = e^{-\int \frac{2}{x}dx} = e^{-2\int \frac{dx}{x}} = e^{-2\ln x}$$

$$e^{\ln x^{-2}} = x^{-2} = \frac{1}{x^2}$$

So $I \cdot F = \frac{1}{x^2}$

Hence the general solution of eq (B) becomes

$$(V)\,I\cdot F = \int Q(x)\, I\cdot F \, dx + C$$

$$V\cdot \frac{1}{x^2} = \int -\cancel{x^2}\sin x \cdot \frac{1}{\cancel{x^2}}\, dx + C$$

$$V\cdot \frac{1}{x^2} = \int -\sin x \, dx + C$$

$$V = \frac{-1}{x^2}\int \sin x \, dx + C$$

$$V = -\frac{1}{x^2}\left((-\cos x) + C\right)$$

$$V = \frac{+1}{x^2}\left(\cos x + C\right)$$

putting value of v again

$$y^{-1} = \frac{\cos x}{x^2} + \frac{c}{x^2} \quad \text{ANS}$$

Home Work:

Solve the following ODEs.

(1) $y + x + y + 1 = (x+y)^2 e^{3x}$
Hint: Let $v = x+y$

(2) $x\frac{dy}{dx} + 3y = x^3 y^2$; $y(1) = 2$
ANS: $y = \frac{1}{x^3(\frac{1}{2} - \ln x)}$, $0 < x < \sqrt{e}$

(3) $x\frac{dy}{dx} + y = y^2 \ln x$
ANS:- $\frac{1}{y} = 1 + \ln x + cx$

(4) $\frac{dy}{dx} + y = xy^3$
ANS:- $\frac{1}{y^2} = x + \frac{1}{2} + ce^{2x}$

1st Order Non-Linear ODEs

A non-linear DE is one that is not linear w.r.t the unknown function and its derivatives. In this section we shall consider first order ODEs with degree more than one. We have already studied various methods of finding the solution of some special type of first order non-linear first order and first degree ODEs such equations were separable, exact, homogeneous and so on. We shall briefly discuss techniques to find solution of special types of first order non-linear ordinary DEs of higher degree. This differential equation will be involve $\frac{dy}{dx}$ in higher degree. And its general form is,

$$f(x, y, dy/dx).$$

Examples:-

* $y\left(\frac{dy}{dx}\right)^2 + (x-y)\frac{dy}{dx} - x = 0$
* $xy(dy/dx)^2 - (x^2+y^2)\frac{dy}{dx} + xy = 0$ etc

In solving such type of ODEs we shall denote $\frac{dy}{dx}$ by p.

Such equation can be solved by the following methods

(1) Equation solvable for p.
(2) Equation solvable for x.
(3) Equation solvable for y.

Now we try to understand these methods one-by-one.

(i) Equation solvable for P

If the DE is factorizable for p, then we can solve it for p.

Examples (1):- $x^2p^2 + xp - y^2 - y = 0$

Sol: We factorize the left hand side, i.e

$$(x^2p^2 - y^2) + (xp - y) = 0$$

$$(xp-y)(xp+y+1) = 0$$

Therefore either

$$xp - y = 0 \longrightarrow (1)$$

or $$xp + y + 1 = 0 \longrightarrow (2)$$

(1) gives

$$x\frac{dy}{dx} - y = 0 \quad , \quad \because p = \frac{dy}{dx}$$

or $$x\frac{dy}{dx} = y$$

$$\frac{dy}{y} = \frac{dx}{x}$$

$$\ln y = \ln x + \ln c \quad , \ln c = \text{constant}$$

$$\ln y = \ln cx$$

$$\Rightarrow y = cx \longrightarrow (3)$$

and (2) gives

$$xp = -(y+1)$$

or $$x\frac{dy}{dx} = -(y+1)$$

or $$\frac{dy}{y+1} = -\frac{dx}{x}$$

$$\int \frac{dy}{y+1} = \int -\frac{dx}{x}$$

$$\ln(y+1) = -\ln x + \ln c$$

$$\ln(y+1) = \ln c - \ln x$$

$$\ln(y+1) + \ln x = \ln c$$

$$\ln(x(y+1)) = \ln c$$

$$x(y+1) = c \longrightarrow (4)$$

Combining eq (3) & eq (4), the required solution of given ODE becomes

$$(y-cx)(xy+x-c) = 0 \quad \underline{\underline{\text{ANS}}}$$

(2) $P^2 - 7P + 12 = 0$

Sol:- Given that $P^2 - 7P + 12 = 0$

which is quadratic in P, so it can be solvable for P

$$P^2 - 3P - 4P + 12 = 0$$

$$(P-3)(P-4) = 0$$

$$\Rightarrow P = 3, 4$$

$\therefore$ For $P = 3 \Rightarrow \frac{dy}{dx} = 3$

$$\Rightarrow dy = 3dx$$

$$y = 3x + c$$

& For $P = 4 \Rightarrow \frac{dy}{dx} = 4$

$\Rightarrow \quad dy = 4dx$

$\Rightarrow \quad y = 4x + c$

Hence the general solution of given ODE is

$$(y-4x-c)(y-3x-c) = 0 \quad \underline{\underline{\text{ANS}}}$$

(3) $y\left(\frac{dy}{dx}\right)^2 + (x-y)\frac{dy}{dx} - x = 0$

Sol:- Given that

$$y\left(\frac{dy}{dx}\right)^2 + (x-y)\frac{dy}{dx} - x = 0$$

$\because \; P = \frac{dy}{dx}$

$\Rightarrow \quad yp^2 + (x-y)p - x = 0$

$\Rightarrow \quad yp^2 + xp - yp - x = 0$

$\Rightarrow \quad yp^2 - yp + xp - x = 0$

$yp(p-1) + x(p-1) = 0$

$(yp + x)(p-1) = 0$

This is possible only when

$yp + x = 0$ or $p - 1 = 0$

So for $p - 1 = 0$

$\Rightarrow \frac{dy}{dx} - 1 = 0$

$\Rightarrow dy = dx$

$\Rightarrow y = x + c$

∧ for $yp + x = 0$

$$\Rightarrow y\frac{dy}{dx} = -x$$

$$y\,dy = -x\,dx$$

$$\Rightarrow \frac{y^2}{2} = -\frac{x^2}{2} + c$$

$$\Rightarrow y^2 = -x^2 + c$$

$$\Rightarrow y^2 + x^2 = c$$

Hence the general solution of given is

$$(y - x - c)(y^2 + x^2 + c) = 0 \quad \underline{\underline{\text{ANS}}}$$

(4) $p^2 - 2p\sinh x - 1 = 0 \rightarrow (A)$

Sol: Given that $p^2 - 2p\sinh x - 1 = 0 \rightarrow (A)$

Eq(A) is quadratic in P, so from quadratic formula, we have

$$P = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-(-2\sinh x) \pm \sqrt{(2\sinh x)^2 - 4(1)(-1)}}{2(1)}$$

$$= \frac{2\sinh x \pm \sqrt{4\sinh^2 x + 4}}{2}$$

$$= \frac{2\sinh x \pm 2\sqrt{\sinh^2 x + 1}}{2}$$

$$= \frac{2(\sinh x \pm \sqrt{\cosh^2 x}}{2} \qquad \because \sinh^2 x - \cosh^2 x = 1$$

$$p = \sinh x \pm \cosh x$$

$$\because \cosh x = \frac{e^x + e^{-x}}{2}, \quad \sinh x = \frac{e^x - e^{-x}}{2}$$

$$\therefore p = \frac{e^x - e^{-x}}{2} \pm \frac{e^x + e^{-x}}{2}$$

$$p = \frac{e^x - e^{-x}}{2} + \frac{e^x + e^{-x}}{2} \longrightarrow (1)$$

& $$p = \frac{e^x - e^{-x}}{2} - \frac{e^x + e^{-x}}{2} \longrightarrow (2)$$

From (1), we have

$$p = \frac{e^x - \cancel{e^{-x}} + e^x + \cancel{e^{-x}}}{2} = \frac{2e^x}{2}$$

$$\frac{dy}{dx} = e^x \qquad \because p = \frac{dy}{dx}$$

$$dy = e^x dx$$

$$\Rightarrow y = e^x + c$$

& from eq (2), we have

$$p = \frac{e^x - e^{-x}}{2} - \frac{e^x + e^{-x}}{2}$$

$$\frac{dy}{dx} = \frac{\cancel{e^x} - e^{-x} - \cancel{e^x} - e^{-x}}{2} = \frac{-\cancel{2}e^{-x}}{\cancel{2}}$$

$$\frac{dy}{dx} = -e^{-x}$$

$$dy = -e^{-x} dx$$

$$\Rightarrow y = e^{-x} + c$$

Thus the general solution is

$$(y - e^x - c)(y - e^{-x} - c) = 0 \quad \underline{\text{ANS}}$$

(5) $4xp^2-(3x-a)^2=0$

Sol:- Given that

$$4xp^2-(3x-a)^2=0$$

$$4xp^2=(3x-a)^2$$

$$p^2=(3x-a)^2/4x$$

$$p=\pm\frac{3x-a}{2\sqrt{x}}$$

$$p=\pm\frac{3x}{2\sqrt{x}}-\frac{a}{2\sqrt{x}}$$

$$=\pm\frac{3}{2}\sqrt{x}-\frac{a}{2}x^{-\frac{1}{2}}$$

$$p=\pm\frac{3}{2}x^{\frac{1}{2}}-\frac{a}{2}x^{-\frac{1}{2}}$$

$$\Rightarrow\quad\frac{dy}{dx}=\pm\frac{3}{2}x^{\frac{1}{2}}-\frac{a}{2}x^{-\frac{1}{2}}$$

$$\frac{dy}{dx}=\pm\frac{1}{2}\left(3x^{\frac{1}{2}}-ax^{-\frac{1}{2}}\right)$$

$$dy=\pm\frac{1}{2}\left(3x^{\frac{1}{2}}-ax^{-\frac{1}{2}}\right)dx$$

$$\int dy=\pm\frac{1}{2}\int\left(3x^{\frac{1}{2}}-ax^{-\frac{1}{2}}\right)dx$$

$$=\pm\frac{1}{2}\left(x^{\frac{3}{2}}-ax^{\frac{1}{2}}\right)\cdot\frac{1}{1/2}$$

$$y+c=\pm\sqrt{x}\cdot x-a\sqrt{x}$$

$$y+c=\sqrt{x}\,(x-a)\ \underline{\underline{\text{ANS}}}$$

(6) $xp^3-(x^2+x+y)p^2+(x^2+xy+y)p-xy=0$

Sol:- By inspection we find that $p-1$ is a factor of left hand side of above ODE. Thus given ODE is

$$(p-1)[xp^2-(x^2+y)p+xy]=0$$

or $(p-1)[(xp-y)(p-x)]=0$

Therefore either

$p-1=0$

or $xp-y=0$

or $p-x=0$

If $p-1=0 \Rightarrow \frac{dy}{dx}-1=0$

$\frac{dy}{dx}=1$

$\Rightarrow dy=dx$

$\Rightarrow y=x+c \longrightarrow (1)$

If $xp-y=0 \Rightarrow y=px$

$x\frac{dy}{dx}=y$

$\Rightarrow \frac{dy}{y}=\frac{dx}{x}$

$\Rightarrow \ln y=\ln cx$

$\Rightarrow y=cx \longrightarrow (2)$

& If $p-x=0 \Rightarrow p=x$

$\Rightarrow \frac{dy}{dx}=x$

$\Rightarrow dy=x\,dx$

$\Rightarrow y=x^2/2+c$

or $y-x^2/2-c=0$

So the general solution of given ODE is obtained by combining (1), (2), (3). Thus

$$(y-x-c)(y-cx)(x^2-2y+2c)=0 \quad \underline{\text{ANS}}$$

(7) $\frac{dy}{dx}-\frac{dx}{dy}=\frac{x}{y}-\frac{y}{x}$

Sol: Given that $\frac{dy}{dx}-\frac{dx}{dy}=x/y-y/x$

$$\because \frac{dy}{dx}=P \Rightarrow \frac{dx}{dy}=1/P$$

$$\therefore \quad P-1/P=x/y-y/x$$

$$P^2-1=P(x/y-y/x)$$

or

$$P^2+P(y/x-x/y)-1=0$$

$$\Rightarrow P^2+P(y/x)-P(x/y)-1=0$$

or

$$P(P+y/x)-x/y(P+y/x)=0$$

$$(P+y/x)(P-x/y)=0$$

If $P+y/x=0 \Rightarrow \frac{dy}{dx}+\frac{y}{x}=0$

$$\Rightarrow \frac{dy}{y}+\frac{dx}{x}=0$$

$$\ln y+\ln x=\ln c$$

$$\ln xy=\ln c$$

$$\Rightarrow xy-c=0$$

& if $P-x/y=0 \Rightarrow P=x/y$

$$\frac{dy}{dx}=x/y$$

$$\frac{dy}{dx} - x/y = 0$$

$$y\,dy - x\,dx = 0$$

$$y^2/2 - x^2/2 = c_1$$

$$y^2 - x^2 = 2c_1 \Rightarrow x^2 - y^2 = -2c_1$$

$$x^2 - y^2 = c \quad \text{where } c = -2c_1$$

$$x^2 - y^2 - c = 0$$

Hence the general solution is

$$(xy - c)(x^2 - y^2 - c) = 0 \quad \underline{\text{ANS}}$$

(5) $xyp^2 + (3x^2 - 2y^2)p - 6xy = 0$

Sol:- Given that $xyp^2 + (3x^2 - 2y^2)p - 6xy = 0$

$$xyp^2 + 3x^2p - 2y^2p - 6xy = 0$$

$$(yp + 3x)(xp - 2y) = 0$$

If $yp + 3x = 0$

$$\Rightarrow y\frac{dy}{dx} + 3x = 0$$

$$\Rightarrow y\frac{dy}{dx} = -3x$$

$$y\,dy = -3x\,dx$$

$$y\,dy + 3x\,dx = 0$$

$$\int y\,dy + 3\int x\,dx = 0$$

$$\frac{y^2}{2} + \frac{3x^2}{2} + c_1$$

$$y^2 + 3x^2 = 2c_1$$

$$y^2 + 3x^2 = c \rightarrow (1) \quad \text{where } 2c_1 = c$$

And when $xp - 2y = 0$

$$\Rightarrow x\frac{dy}{dx} - 2y = 0$$

$$\Rightarrow \frac{dy}{y} - 2\frac{dx}{x} = 0$$

$$\Rightarrow \int\frac{dy}{y} - 2\int\frac{dx}{x} = \int 0$$

$$\ln y - 2\ln x = \ln c$$

$$\ln y = 2\ln x + \ln c$$

$$\ln y = \ln x^2 + \ln c$$

$$\ln y = \ln cx^2$$

$$y = cx^2$$

$$\text{or} \quad y - cx^2 = 0 \longrightarrow (2)$$

Hence the general solution of given ODE is,

$$(3x^2 + y^2 - c)(y - x^2c) = 0 \quad \underline{\text{ANS}}$$

Home Work :- Solve the following ODEs

(1) $y + px = p^2x^4$

ANS:- $xy + c = c^2x$

(2) $y = 2px + p^4x^2$

ANS:- $y = 2\sqrt{xc} + c^2$

(3) $p^2 + p - 6 = 0$

ANS:- $(y - 2x - c)(y + 3x - c) = 0$

(4) $p^2y + (x - y)p - x = 0$

ANS:- $(x^2 + y^2 - c)(y - x - c) = 0$

(ii) Equation solvable for x

Steps for solution:-

* If given DE of the form $f(x, y, p) = 0$ is solvable for x, takes the form

$$x = f(y, p) \longrightarrow (1)$$

then differentiate w.r.t y and we get

$$\frac{1}{p} = \frac{dx}{dy} = \phi\left(y, p, \frac{dp}{dy}\right) \longrightarrow (2)$$

* Solving eq (2) we get

$$\psi(y, p, c) = 0 \longrightarrow (3)$$

* Eliminate p from eq (1) & eq (3), we get the required solution.

Question:- Solve the following ODEs.

(1) $y = 2px + y^2p^3$

Sol: Given that $y = 2px + y^2p^3$

$$2px = y - y^2p^3$$

$$x = \frac{y}{2p} - \frac{y^2p^3}{2p}$$

$$x = \frac{y}{2p} - \frac{y^2p^2}{2} \longrightarrow (A)$$

Differentiating eq (A) b/s w.r.t y.

$$\frac{dx}{dy} = \frac{1}{2p}\frac{d}{dy}(y) + \frac{y}{2}\frac{dp}{dy} - \left(\frac{p^2}{2}\frac{d}{dy}(y^2) + \frac{y^2}{2}\frac{d}{dy}(p^2)\right)$$

$$= \frac{1}{2p} - \frac{y}{2p^2}\frac{dp}{dy} - \frac{2yp^2}{2} - \frac{y^2 2p}{2}\frac{dp}{dy}$$

$$\frac{dx}{dy} = \frac{1}{2p} - \frac{y}{2p^2}\frac{dp}{dy} - yp^2 - y^2p\frac{dp}{dy}$$

$$\frac{1}{p} - \frac{1}{2p} + yp^2 + \frac{y}{2p^2}\frac{dp}{dy} + y^2p\frac{dp}{dy} = 0$$

$$\frac{1}{2p} + yp^2 + \frac{y}{2p^2}\frac{dp}{dy} + y^2p\frac{dp}{dy} = 0$$

$$\left(\frac{1}{2p}+yp^2\right)+\frac{y}{p}\left(\frac{1}{2p}+yp^2\right)\frac{dp}{dy}=0$$

$$\left(\frac{1}{2p}+yp^2\right)\left(1+\frac{y}{p}\frac{dp}{dy}\right)=0$$

$$\Rightarrow \quad 1+\frac{y}{p}\frac{dp}{dy}=0$$

$$\frac{y}{p}\frac{dp}{dy}=-1$$

$$\frac{dp}{p}=-\frac{dy}{y}$$

$$\int\frac{dp}{p}=-\int\frac{dy}{y}$$

$$\ln p=-\ln y+\ln c$$

$$\ln p=\ln\frac{c}{y}$$

$$\Rightarrow p=c/y$$

Using this value of P in Given ODE, we have

$$2(c/y)x=y-y^2(c/y)^3$$

$$y^2=2cx+c^3 \quad \underline{\underline{\text{ANS}}}.$$

(2) $xp=1+p^2$

Sol:- We have $xp=1+p^2 \rightarrow (1)$

$$x=\frac{1}{p}+p \longrightarrow (2)$$

Differentiating (2) wrt y b/s

$$\frac{dx}{dy}=-\frac{1}{p^2}\frac{dp}{dy}+\frac{dp}{dy}$$

$$\because \frac{dy}{dx} = P \Rightarrow \frac{dx}{dy} = \frac{1}{P}$$

$$\therefore \quad \frac{dx}{dy} = \frac{1}{P} = \frac{-1}{P^2}\frac{dP}{dy} + \frac{dP}{dy}$$

$$\frac{1}{P} = \left(1 - \frac{1}{P^2}\right)\frac{dP}{dy}$$

$$1 = P\left(1 - \frac{1}{P^2}\right)\frac{dP}{dy}$$

$$1 = \left(P - \frac{1}{P}\right)\frac{dP}{dy}$$

Which is a separable ODE.

$$\therefore \quad dy = \left(P - \frac{1}{P}\right)dP$$

$$\int dy = \int\left(P - \frac{1}{P}\right)dP$$

$$c + y = P^2/_2 - \ln p$$

or

$$2y = P^2 - 2\ln p - 2c \longrightarrow (3)$$

Hence (2) and (3) constitute the solution of (1).

(3) $x - y = p^2 \longrightarrow (1)$

Sol:- Given $x - y = p^2$

$$x = y + p^2 \longrightarrow (2)$$

Which is of the form $x = f(y, p)$

Differentiate w.r.t y we get

$$\frac{dx}{dy} = 1 + 2p\frac{dP}{dy}$$

$$\frac{1}{P} = 1 + 2p\frac{dP}{dy}$$

$$\frac{1}{p} - 1 = 2p\frac{dp}{dy}$$

Separating the variables

$$\frac{1-p}{p} = 2p\frac{dp}{dy}$$

$$\therefore \quad \frac{2p^2}{1-p}dp = dy$$

or $$-\frac{2p^2}{p-1}dp = dy$$

or $$-2\left[\frac{(p^2-1)+1}{p-1}\right]dp = dy$$

or $$-2\left[\frac{(p^2-1)}{p-1} + \frac{1}{p-1}\right]dp = dy$$

or $$-2\left[\int(p+1)dp + \int\frac{dp}{p-1}\right] = \int dy$$

or $$-2\left[\frac{p^2}{2} + p + \ln(p-1)\right] = y + c_1$$

$$y = -c_1 - 2\left[p^2/_2 + p + \ln(p-1)\right]$$

$$y = c - 2\left[p + \ln(p-1) + p^2/_2\right] \rightarrow (3) \text{ where } c = -c_1$$

Hence eq (2) and eq (3) constitute, the required general solution.

(4) $y = 2px + 4p^2y$

Sol: Given $y = 2px + 4p^2y \longrightarrow (1)$

$$y/_p = 2x + 4py$$

or $$2x = y/_p - 4py \longrightarrow (2)$$

which is now in the form of $x = f(y, p)$

Differentiating b/s of x w.r.t y

$$2\frac{dx}{dy} = \frac{1}{p} + y\left(\frac{-1}{p^2}\right)\frac{dp}{dy} - 4p - 4y\frac{dp}{dy}$$

or

$$\frac{2}{p} - \frac{1}{p} + 4p + \left(\frac{y}{p^2} + 4y\right)\frac{dp}{dy} = 0$$

or $$\frac{1}{p} + 4p + y\left(\frac{1}{p^2} + 4\right)\frac{dp}{dy} = 0$$

or $$p\left(\frac{1}{p^2} + 4\right) + y\left(\frac{1}{p^2} + 4\right)\frac{dp}{dy} = 0$$

or $$\left(\frac{1}{p} + 4\right)\left(p + y\frac{dp}{dy}\right) = 0$$

Now we shall take only $p + y\frac{dp}{dy} = 0$ because it involves $\frac{dp}{dy}$ i.e

$$p + y\frac{dp}{dy} = 0$$

$$\Rightarrow \frac{dp}{p} + \frac{dy}{y} = 0$$

$$\Rightarrow \int \frac{dp}{p} + \int \frac{dy}{y} = \int 0$$

$$\Rightarrow \ln p + \ln y = \ln c$$

$$\Rightarrow \ln py = \ln c$$

$$\Rightarrow py = c$$

$$\Rightarrow c/y = p$$

putting value of p in eq (1) to eliminate p

$$y = 2(c/y)x + 4(c/y)^2 y$$

$$y = \frac{2cx}{y} + \frac{4c^2}{y}$$

$$\therefore\ y^2 = 2cx + 4c^2 \quad \text{Ans}$$

(5) $y = 2px + y^2p^2$

Sol:- Given $y = 2px + y^2p^2 \longrightarrow (1)$

$$2px = y - y^2p^2$$

$$2x = y/p - y^2p \longrightarrow (2)$$

which is in the form of $x = f(y,p)$

Differentiating (2) b/s wrt y

$$2\frac{dx}{dy} = \frac{1}{p} + y\left(\frac{-1}{p^2}\frac{dp}{dy}\right) - 2yp^2 - 2y^2p\frac{dp}{dy}$$

$$2\frac{1}{p} - \frac{1}{p} + \frac{y}{p^2}\frac{dp}{dy} + 2yp^2 + 2y^2p\frac{dp}{dy} = 0$$

$$\frac{1}{p} + \frac{y}{p}\left(\frac{1}{p}\frac{dp}{dy}\right) + 2yp^2 + 2y^2p\frac{dp}{dy} = 0$$

$$\left(\frac{1}{p} + 2yp^2\right) + \frac{y}{p}\left(\frac{1}{p} + 2yp^2\right)\frac{dp}{dy} = 0$$

$$\left(\frac{1}{p} + 2yp^2\right)\left(1 + y/p\,\frac{dp}{dy}\right) = 0$$

Again we will consider only

$$\left(1 + y/p\,\frac{dp}{dy}\right) = 0$$

$$\therefore \quad 1 + y/p\,\frac{dp}{dy} = 0$$

$$\Rightarrow \frac{dy}{y} + \frac{dp}{p} = 0$$

$$\int \frac{dx}{y} + \int \frac{dp}{p} = \int 0$$

$$\ln y + \ln p = \ln c$$

$$\Rightarrow \ln yp = \ln c$$

$$\Rightarrow yp = c$$

$$yp = c$$
$$\Rightarrow p = c/y$$

Using this value of p in eq (1) to eliminate p

$$y = 2(c/y)x + y^2(c/y)^3$$
$$= \frac{2cx}{y} + \frac{y^2 c^3}{y^3}$$
$$\Rightarrow y = \frac{2cx}{y} + \frac{c^2}{y}$$
$$\boxed{y^2 = 2cx + c^2}$$

Which is the required general solution.

Home Work :- Solve the following ODEs.

(1) $xp^3 = a + bp$.

ANS:- $y = \frac{3a}{2p^2} + \frac{2b}{p} + c$.

(2) $4yp = y^4 - 4xp^2$.

ANS:- $4c^2xy = y - 4c$.

(3) $\ln y - xyp = p^2$.

ANS:- $\ln y = cx + c^2$.

(iii) Equation solvable for y

* If the DE $f(x,y,p)=0$ is solvable for then it can be written in the form of

$$y = f(x,p)$$

* Differentiate w.r.t x we get

$$p = \frac{dy}{dx} = \phi\left(x, p, \frac{dp}{dx}\right).$$

Solve it and we get $\psi(x,p,c)=0$.

* Eliminate p and we get the required general solution.

Question:- Solve the following ODEs

(1) $p^2 + x^3p - 2x^2y = 0$

Sol:- Given that $p^2 + x^3p - 2x^2y = 0$

$$2x^2y = p^2 + x^3p$$

$$2y = p^2/x^2 + \frac{x^3p}{x^2}$$

$$2y = (p/x)^2 + xp \longrightarrow (1)$$

Which is now in the form $y = f(x,p)$

Differentiating (1) b/s w.r.t x

$$2\frac{dy}{dx} = p^2\left(\frac{-2}{x^3}\right) + \frac{1}{x^2}\cdot 2p\frac{dp}{dx} + x\frac{dp}{dx} + p(1)$$

$$2p = x\frac{dp}{dx} + p + p^2(-2/x^3) + \frac{2p}{x^2}\frac{dp}{dx}$$

$$2p - p - x\frac{dp}{dx} + p^2\left(\frac{2}{x^3}\right) - \frac{2p}{x^2}\frac{dp}{dx} = 0$$

$$p - x\frac{dp}{dx} + p^2\left(\frac{2}{x^3}\right) - \frac{2p}{x^2}\frac{dp}{dx} = 0$$

$$p + p^2\left(\frac{2}{x^3}\right) - x\frac{dp}{dx} - \frac{2p}{x^2}\frac{dp}{dx} = 0$$

$$p\left(1 + \frac{2p}{x^3}\right) - x\left(1 + \frac{2p}{x^3}\right)\frac{dp}{dx} = 0$$

$$\left(1 + \frac{2p}{x^3}\right)\left(p - x\frac{dp}{dx}\right) = 0$$

Now we will consider $p - x\frac{dp}{dx} = 0$ because it Involves $\frac{dp}{dx}$.

$$\therefore \quad p - x\frac{dp}{dx} = 0$$

$$p = x\frac{dp}{dx}$$

$$\Rightarrow \frac{dx}{x} = \frac{dp}{p}$$

$$\Rightarrow \int\frac{dx}{x} = \int\frac{dp}{p}$$

$$\ln p = \ln x + \ln c$$

$$\ln p = \ln cx$$

$$\Rightarrow p = cx$$

putting value of p in eq (1)

$$2y = x(cx) + \frac{(cx)^2}{x^2}$$

$$2y = cx^2 + \frac{c^2\cancel{x^2}}{\cancel{x^2}}$$

$$\boxed{2y = cx^2 + c^2}$$

Which is the required general solution.

(2) $y = (x-a)p - p^2 \longrightarrow (1)$

Sol: Given that $y = (x-a)p - p^2$

So this equation is already in the form of $y = f(x, p)$, so by differentiating b/s w.r.t x, we get

$$\frac{dy}{dx} = (x-a)\frac{dp}{dx} + p(1) - 2p\frac{dp}{dx}$$

$$p = (x-a)\frac{dp}{dx} + p - 2p\frac{dp}{dx}$$

$$\cancel{p} = (x-a-2p)\frac{dp}{dx} + \cancel{p}$$

$$(x-a-2p)\frac{dp}{dx} = 0$$

$$\Rightarrow \frac{dp}{dx} = 0 \Rightarrow dp = 0$$

$$\Rightarrow \int dp = 0 \Rightarrow p = c$$

putting value of p in eq (1)

$$y = (x-a)(c) - c^2$$

$$\boxed{y = (x-a)c - c^2} \text{ ANS}$$

(3) $y = 2px + \tan^{-1}(xp^2)$

Sol: Given that $y = 2px + \tan^{-1}(xp^2)$

which is already in the form of $y = f(x, p)$. So differentiating b/s of eq (1) w.r.t to "x", we get

$$\frac{dy}{dx} = 2\left(p + x\frac{dp}{dx}\right) + \frac{1}{1+x^2p^4}\left(p^2 + 2xp\frac{dp}{dx}\right)$$

$$P = 2P + 2x\frac{dP}{dx} + \frac{P}{1+x^2P^4}\left(P + 2x\frac{dP}{dx}\right)$$

$$P + 2x\frac{dP}{dx} + \frac{P}{1+x^2P^4}\left(P + 2x\frac{dP}{dx}\right) = 0$$

$$\Rightarrow \left(P + 2x\frac{dP}{dx}\right)\left(1 + \frac{P}{1+x^2P^4}\right) = 0$$

Here we will consider that only

$P + 2x\frac{dP}{dx} = 0$ because it

involves $\frac{dP}{dx}$ term.

$$\therefore \quad P + 2x\frac{dP}{dx} = 0$$

$$\frac{2dP}{P} + \frac{dx}{x} = 0$$

$$2\int \frac{dP}{P} + \int \frac{dx}{x} = \int 0$$

$$2\ln P + \ln x = \ln c$$

$$\ln P^2 + \ln x = \ln c$$

$$\Rightarrow \ln xP^2 = \ln c$$

$$\Rightarrow xP^2 = c$$

$$\Rightarrow P^2 = c/x$$

$$\text{or} \quad P = \sqrt{c/x}$$

putting value of P in eq (1)

$$y = 2x\sqrt{c/x} + \tan^{-1}\left(x\left(\sqrt{c/x}\right)^2\right)$$

$$= 2x\frac{\sqrt{c}}{\sqrt{x}} + \tan^{-1}(c)$$

$$\boxed{y = 2\sqrt{xc} + \tan^{-1}(c)} \quad \text{ANS}$$

(4) $y - 2px = f(xp^2)$

Sol:- Given that $y - 2px = f(xp^2)$

$$y = 2px + f(xp^2) \longrightarrow (1)$$

Which is now in the form of

$$y = f(x, p)$$

So differentiating b/s w.r.t x of (1)

$$\frac{dy}{dx} = 2p(1) + 2x\frac{dp}{dx} + f'(xp^2)\left(x2p\frac{dp}{dx} + p^2\right)$$

$$p = 2p + 2x\frac{dp}{dx} + f'(xp^2)\left(2px\frac{dp}{dx} + p^2\right)$$

$$p + 2x\frac{dp}{dx} + f'(xp^2)\left(2px\frac{dp}{dx} + p^2\right) = 0$$

$$\left(p + 2x\frac{dp}{dx}\right)\left(p + p\, f'(xp^2)\right) = 0$$

We shall consider $p + 2x\frac{dp}{dx} = 0$ only.

$$\therefore \quad p + 2x\frac{dp}{dx} = 0$$

$$\Rightarrow 2\frac{dp}{p} + \frac{dx}{x} = 0$$

$$\Rightarrow 2\int\frac{dp}{p} + \int\frac{dx}{x} = 0$$

$$2\ln p + \ln x = \ln x$$

$$\ln p^2 + \ln x = \ln c$$

$$\ln p^2 x = \ln c \Rightarrow p^2 x = c$$

$$\Rightarrow p^2 = c/x \Rightarrow p = \sqrt{c/x}$$

putting value of p in eq (1)

$$y = 2\sqrt{c/x}\cdot x + f\left(x(\sqrt{c/x})^2\right)$$

$$\boxed{y = 2\sqrt{cx} + f(c)}$$ ANs

Home Work :-

Solve the following ODEs:

(1) $xp^2 - 2py + ax = 0$

ANS:- $c^2x^2 - 2cxy + ax = 0$

(2) $y + px = p^2x^4$

ANS:- $xy + c = c^2x$

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Clairaut's Equation

In mathematics, Clairaut's equation or Clairaut equation is a differential equation. It is named after the French mathematician Alexis Clairaut (1713-1765). It is an equation of the form

$$y(x) = x\frac{dy}{dx} + f\left(\frac{dy}{dx}\right) \longrightarrow (A)$$

where f is a continuously differentiable.
Since we know that

$$\frac{dy}{dx} = P \quad \text{and} \quad y(x) = y$$

So equation (A) can also be written as:

$$y = xp + f(P) \longrightarrow (B)$$

Now to solve Clairaut's equation.
differentiating (B) w.r.t x b/s

$$\frac{dy}{dx} = x\frac{dP}{dx} + P(1) + f'(P)\frac{dP}{dx}$$

$$\because \frac{dy}{dx} = P$$

$$\therefore \frac{dy}{dx} = P = P + x\frac{dP}{dx} + f'(P)\frac{dP}{dx}$$

$$\cancel{P} = \cancel{P} + x\frac{dP}{dx} + f'(P)\frac{dP}{dx}$$

$$x\frac{dP}{dx} + f'(P)\frac{dP}{dx} = 0$$

$$\left(x + f'(P)\right)\frac{dP}{dx} = 0$$

Since one of the factor must be 0, two ODEs arises.

(i) If $\frac{dP}{dx} = 0$, then

$$dP = 0$$

$$\int dP = \int 0$$

$$\Rightarrow P = C$$

By substituting this value of P in eq (B) yields the general solution

$$y = cx + f(c).$$

(ii) If $x + f'(P) = 0$ then

$$x = -f'(P)$$

and eq (B) becomes

$$y = -Pf'(P) + f(P)$$

Thus x and y are both expressed as functions of P and we obtain the parametric equations

$$\left.\begin{aligned} x &= -f'(P) \\ y &= f(P) - Pf'(P) \end{aligned}\right\} \rightarrow (C)$$

of a curve representing a solution of eq (A) or eq (B). This solution (C) is called the singular solution. This solution is not deductiable from the general solution. p may be eliminated between the two equations in (C) to get a relation in x and y involving no constant.

Question: Solve the following ODEs:

(1) $xp^2 - yp + a = 0$

Sol: Given that $xp^2 - yp + a = 0$

$$\text{or } yp = xp^2 + a$$

$$\text{or } y = xp^2/p + a/p$$

$$y = xp + a/p \longrightarrow (1)$$

Equation (1) is now in a Clairaut's form, so differentiating b/s w.r.t 'x'.

$$\frac{dy}{dx} = (1)p + x\frac{dp}{dx} - \frac{a}{p^2}\frac{dp}{dx}$$

$$p = p + x\frac{dp}{dx} - \frac{a}{p^2}\frac{dp}{dx}$$

$$x\frac{dp}{dx} - \frac{a}{p^2}\frac{dp}{dx} = 0$$

$$\left(x - \frac{a}{p^2}\right)\frac{dp}{dx} = 0$$

We will consider $\frac{dp}{dx} = 0$ only.

So $\frac{dp}{dx} = 0$

$$dp = 0$$

$$\Rightarrow \int dp = \int 0$$

$$\Rightarrow p = c$$

Using value of p in eq (1), we get

$$\boxed{y = xc + a/c}$$

Which is the required general solution.

Note:- If the given DE is in a Clairaut's form, then we just replace p by some constant c to obtain the general solution as in the above example, we see that P is just replaced by c in the general solution of given DE.

(2) $x^2(y-px) = yp^2 \longrightarrow (A)$

Sol:- Given that $x^2(y-px) = yp^2$

We see that eq(A) is not in a Clairaut's equation, so first we need to bring it in the form of Clairauts.

Let us consider $x^2 = u,\ y^2 = v$

$\because x^2 = u$
$\therefore x = \sqrt{u}$
$\because y^2 = v$
$\therefore y = \sqrt{v}$

$$\Rightarrow 2x\,dx = du \quad 2y\,dy = dv$$

$$\Rightarrow \frac{y}{x}\frac{dy}{dx} = \frac{dv}{du}$$

$$\Rightarrow \frac{dv}{du} = \frac{y}{x}P = \frac{\sqrt{v}}{\sqrt{u}}P$$

putting these values in eq (A)

$$u\left(\sqrt{v} - \frac{\sqrt{u}}{\sqrt{v}}\frac{dv}{du}\sqrt{u}\right) = \sqrt{v}\left(\frac{\sqrt{u}}{\sqrt{v}}\frac{dv}{du}\right)^2$$

$$u\left(\frac{\sqrt{v}\cdot\sqrt{v} - \sqrt{u}\cdot\sqrt{u}\frac{dv}{du}}{\sqrt{v}}\right) = \sqrt{v}\frac{(\sqrt{u})^2}{(\sqrt{v})^2}\left(\frac{dv}{du}\right)^2$$

$$\frac{u}{\sqrt{v}}\left(v - u\frac{dv}{du}\right) = \frac{\sqrt{v}}{v}u\left(\frac{dv}{du}\right)^2$$

$$\frac{1}{\sqrt{v}}\left(v - u\frac{dv}{du}\right) = \frac{\sqrt{v}}{\cancel{\sqrt{v}}\sqrt{v}}\left(\frac{dv}{du}\right)^2$$

$$\left(v - u\frac{dv}{du}\right) = \left(\frac{dv}{du}\right)^2$$

$$v = u\frac{dv}{du} + \left(\frac{dv}{du}\right)^2$$

Let $\frac{dv}{du} = p'$, then

$$v = up' + (p')^2 \longrightarrow (B)$$

Equation B is now in a Clairuits form, so its general can be obtained by replacing p' by a constant c.

$$\therefore \quad v = uc + c^2 \longrightarrow (C)$$

Now put the values of u & v again in eq (C), we get

$$y^2 = x^2 c + c^2$$

Which is the general solution of given eq (A)

(3) $p = \ln(px - y)$

<u>Sol:-</u> Given that $p = \ln(px - y)$

$$\Rightarrow e^p = e^{\ln(px-y)}$$

$$\Rightarrow e^p = px - y$$

$$\Rightarrow y = px - e^p \longrightarrow (1)$$

Equation (1) is now in Clairaut form

$$y = px + f(p)$$

So its solution is obtained by replacing p by c, we get

$$\boxed{y = cx - e^c} \quad \text{ANS}$$

(4) $(y - px)(p-1) = p$

Sol: Given that $(y - px)(p-1) = p$

First we need to bring it to Clairaut's equation.

$$(y - px) = \frac{p}{p-1}$$

$$y = px + \frac{p}{p-1} \longrightarrow (1)$$

Eq(1) is now a Clairaut's equation of the form $y = px + f(p)$ and replacing p by c is the general solution of given DE.

$$\therefore \boxed{y = cx + \frac{c}{c-1}} \quad \text{ANS} \quad \text{OR}$$

$$(y - cx)(c-1) = c \quad \text{ANS}$$

(5) $\sin px \cos y = \cos px \sin y + p \longrightarrow (A)$

Sol: Given that $\sin px \cos y = \cos px \sin y + p$

We need to write eq(A) in the form of $y = px + f(p)$

$$\sin px \cos y - \cos px \sin y = p$$

$$\Rightarrow \sin(px - y) = p$$

$$\Rightarrow px - y = \sin^{-1}(p)$$

$$\Rightarrow y = px - \sin^{-1}(p) \longrightarrow (B)$$

Equation B is now a Clairaut equation. So replacing p by c, we have

$$\boxed{y = cx - \sin^{-1}(c)} \quad \text{ANS}$$

Required general solution.

(5) $x^2(y-px)=yp^2 \longrightarrow (1)$

Sol: Given that $x^2(y-px)=yp^2$

We need suitable substitutions to reduce eq (1) Clairaut's form.

For this Let $x^2=u \Rightarrow x=\sqrt{u}$

$$\Rightarrow \frac{d}{du}(u)=2x\frac{dx}{du}$$

$$\Rightarrow 1=2x\frac{dx}{du}$$

$$\Rightarrow du=2x\,dx$$

and Let $y^2=v \Rightarrow y=\sqrt{v}$

$$\Rightarrow \frac{d}{dv}(v)=2y\frac{dy}{dv}$$

$$\Rightarrow 1=2y\frac{dy}{dv}$$

$$\Rightarrow dv=2y\,dy$$

Also $p=\frac{dy}{dx}=\frac{dv/2y}{du/2x}=\frac{x}{y}\frac{dv}{du}$

Consider $p'=\frac{dv}{du}$

$\therefore\ p=\frac{x}{y}p'$

Using these substitutions in eq (1) after simplifying

$$x^2\left(y-\frac{x}{y}p'\cdot x\right)=y\left(\frac{x}{y}p'\right)^2$$

$$x^2\frac{(y^2-x^2p')}{\cancel{y}}=\cancel{y}\cdot\frac{x^2}{\cancel{y^2}}p'^2$$

$$x^2(y^2-x^2p')=x^2p'^2 \longrightarrow (*)$$

Now use the substitutions in eq (x)

$$u(v-up') = up'^2$$

$$\Rightarrow (v-up') = \frac{up'^2}{u}$$

$$\Rightarrow v-up' = p'^2$$

$$\Rightarrow v = up'+p'^2 \longrightarrow (2)$$

Now eq(2) is a Clairaut's equation. So its general solution can be obtained by replacing p' by c.

$$\therefore \quad v = uc+c^2$$

But $v=y^2$ & $u=x^2$

$$\therefore \quad \boxed{y^2 = x^2c+c^2} \text{ ANS}$$

(6) $e^{4x}(p-1)+e^{2y}p^2=0 \longrightarrow (1)$

Sol: We need to bring eq(1) to Clairauts form by using some useful substitutions.

For this Let $u=e^{2x}$ & $v=e^{2y}$

$$\Rightarrow 2e^{2x}dx = du, \quad 2e^{2y}dy = dv$$

Now $$\frac{dv}{du} = \frac{2e^{2y}dy}{2e^{2x}dx}$$

$$\Rightarrow p = \frac{dy}{dx} = \frac{e^{2x}dv}{e^{2y}du} = \frac{u}{v}\frac{dv}{du}$$

Using these values in eq(1)

$$u^2\left[\left(\frac{u}{v}\frac{dv}{du}\right)-1\right]+v\left(\frac{u}{v}\frac{dv}{du}\right)^2 = 0$$

$$\frac{u^3}{v}\frac{dv}{du}-u^2+\frac{u^2}{v}\left(\frac{dv}{du}\right)^2 = 0$$

$$v^2\left(u\frac{dv}{du} - v + \left(\frac{dv}{du}\right)^2\right) = 0$$

$$\Rightarrow u\frac{dv}{du} - v + \left(\frac{dv}{du}\right)^2 = 0$$

Let $p' = \frac{dv}{du}$

$\therefore \quad up' - v + p'^2 = 0$

$$v = up' + p'^2 \longrightarrow (2)$$

Equation (2) is now a clairaut's equation of the form. $v = up' + p'^2$
So its general solution can be obtained by replacing p' by c,

$\therefore \quad v = uc + c^2$

but we know that
$v = e^{2y}$, & $u = e^{2x}$

$\therefore \quad \boxed{e^{2y} = ce^{2x} + c^2}$ ANS

(7) $(y-px)(p-1)=p$

Sol: Given that $(y-px)(p-1)=p$

$$\Rightarrow (y-px)=\frac{p}{p-1}$$

$$\Rightarrow y=px+\frac{p}{p-1}$$

Which is now a Clairauts equation of the form $y=px+f(p)$. so replacing p by c, we get

$$y=cx+\frac{c}{c-1}$$

$$y(c-1)=cx(c-1)+c$$

$$yc-y=c^2x-cx+c$$

$$(y-cx)(c-1)=c \quad \text{ANS}$$

Home Works: Solve the following CDEs:

(1) $y=xp+\frac{1}{4}p^4$.

(2) $y=x^2(y-px)=yp^2$.

(3) $(x-py)(x-\frac{y}{p})=a^2$

ANS: $y^2=cx^2-\frac{a^2c}{1-c}$.

(4) $y=yp^2+2px$

ANS: $4y^2=4cx+c^2$.

(5) $2yp=xp^2+ax$

ANS: $y=cx^2+\frac{a}{4c}$.

Singular Solutions

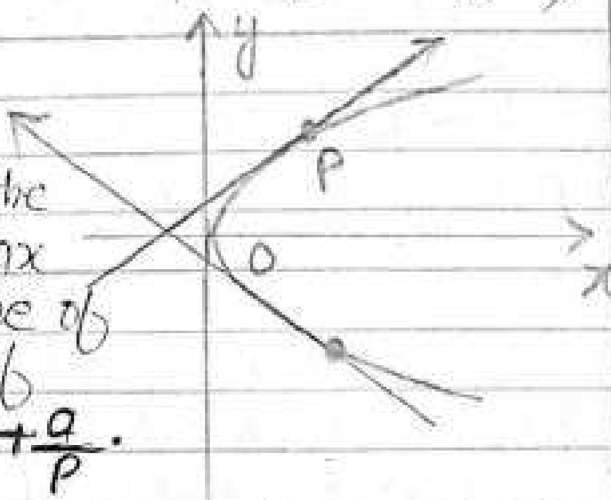
Let $f(x,y,p)=0$ be a non-linear first order differential equation in which the left hand member is a polynomial in p. The general solution of this differential equation will be a one-parameter family

$$f(x,y,c)=0$$

Now those solutions which do not contain the arbitrary constant and which cannot be obtained from the general solution are called singular solutions.

Note :- (1) A curve which at each points is tangent to some one of the curves is called an envelope of that family.

For example; the parabola $y^2=4ax$ is the envelope of the family of lines $y=px+\frac{a}{p}$.



(2) The envelope represents a singular solution of the differential equation

Working Rules:

(1) Let $f(x, y, p) = 0$ be a given DE then differentiating w.r.t "p" partially b/s. $\frac{\partial f}{\partial p} = 0$

( ) Eliminate p from the above two equations we get singular solution

Question:- Solve and find singular solution of each of the following.

(1) $9p^2(y-2)^2 = 4(3-y) \longrightarrow (A)$

Sol:- Given that

$$9p^2(y-2)^2 = 4(3-y)$$

$$\Rightarrow F = 9p^2(y-2)^2 - 4(3-y) = 0 \rightarrow (1)$$

Differentiating (1) partially w.r.t "p".

$$\frac{\partial F}{\partial p} = 18p(y-2)^2 = 0$$

$$\Rightarrow p = 0$$

put value of p in eq (1), so

$$y = 3$$

$\therefore$ $y = 3$ is a singular solution.

Now we check whether $y = 3$ is a solution of given DE (A). For this

As $y = 3$

$$\Rightarrow \frac{dy}{dx} = p = 0$$

$$\therefore \quad y = 3 \ \& \ p = 0$$

put value of y & p in eq (A),

$$90^2(3-2)^2 = 4(3-0)$$

$$0 = 0$$

$\therefore$ $y=3$ is a solution of eq(A).

(2) $p^2 - xp + y = 0 \longrightarrow (A)$

Sol:- Given that

$$p^2 - xp + y = 0$$

$$y = xp + p^2$$

Which is a Clairauts equation of the form $y = xp + f(p)$ and its general solution is obtained by replacing p by c. i.e

$$y = xc + c^2 \longrightarrow (1)$$

$$f(x,y,c) = y - cx + c^2 = 0 \longrightarrow 1'$$

Now differentiating (1') b/s partially w.r.t c

$$\frac{\partial f}{\partial c} = 0 - x + 2c$$

$$\frac{\partial f}{\partial c} = -x + 2c$$

$$2c = x$$

$$\Rightarrow c = x/2$$

putting value of c in (1'), we have

$$f(x,y,c) = y - x(x/2) + (x/2)^2 = 0$$

$$y - \frac{x^2}{2} + \frac{x^2}{4} = 0$$

$$y - \frac{2x^2+x^2}{4} = 0$$

$$y - \frac{x^2}{4} = 0$$

$$y = x^2/4$$

$$4y = x^2 \longrightarrow (2)$$

Which is the required singular solution of eq (A).
Now we check whether (2) is a solution of given DE. For this
As

$$4y = x^2 \Rightarrow y = x^2/4$$

$$\Rightarrow 4\frac{dy}{dx} = 2x$$

$$4p = 2x$$

$$\Rightarrow 2p = x \Rightarrow p = x/2$$

So $y = x^2/4$ & $p = x/2$

putting these values in eq (A).

$$(x/2)^2 - x(x/2) + x^2/4 = 0$$

$$x^2/4 - \frac{x^2}{2} + \frac{x^2}{4} = 0$$

$$\frac{x^2 - 2x^2 + x^2}{4} = 0$$

$$\frac{2x^2 - 2x^2}{4} = 0$$

$$0 = 0$$

Thus eq (2) is a solution of given DE (A) & is a singular solution.

Note:- In example (2), we had discussed 2nd method of finding singular solution of DE, whose solution steps are the following.

Steps:- (1) Let $f(x,y,p)=0$ be a given DE and $\phi(x,y,c)=0$ be the general solution of $f(x,y,p)=0$.

(2) Find $\frac{\partial \phi}{\partial c}=0$.

(3) Eliminating c from $\phi(x,y,c)=0$ and $\frac{\partial \phi}{\partial c}=0$ we get singular solution.

Example # 3:- $y=px+\frac{a}{p}$

Sol:-

Given that $y=px+\frac{a}{p}$

$$\Rightarrow f=px-y+a/p=0 \longrightarrow (1)$$

Now using method (1), differentiating partially w.r.t p b/s of (1)

$$\frac{\partial f}{\partial p}=x-\frac{a}{p^2}$$

$$\Rightarrow x=a/p^2$$

$$\Rightarrow p^2=a/x$$

$$\Rightarrow p=\pm\sqrt{a/x}$$

Putting value of p in

$$y=\pm\sqrt{a/x}\cdot x+a/\sqrt{x/a}$$

$$y=\frac{\sqrt{a}}{\sqrt{x}}\cdot\sqrt{x}\sqrt{x}+a\sqrt{a/x}$$

$$y=\sqrt{ax}+\sqrt{ax}$$

$$y = 2\sqrt{ax} \quad \text{OR}$$

$$y^2 = 4ax \quad \underline{\text{ANS}}$$

(4) $y = 2px + y^2p^3 \longrightarrow (A)$

Sol: Given that $y = 2px + y^2p^3$
Frist we find the general solution and then the singular solution, so

$$2px = y - y^2p^3$$

$$2x = \frac{y}{p} - y^2p^2 \longrightarrow (1)$$

Differentiating w.r.t y

$$2\frac{dx}{dy} = \frac{1}{p} - \frac{y}{p^2}\frac{dp}{dy} - 2yp^2 - 2py^2\frac{dp}{dy}$$

$$\because \frac{dy}{dx} = p$$

$$\Rightarrow \frac{dx}{dy} = \frac{1}{p}$$

$$\therefore \quad 2\frac{1}{p} = \frac{1}{p} - \frac{y}{p^2}\frac{dp}{dy} - 2yp^2 - 2py^2\frac{dp}{dy}$$

$$\frac{2}{p} - \frac{1}{p} + \frac{y}{p^2}\frac{dp}{dy} + 2yp^2 + 2py^2\frac{dp}{dy} = 0$$

$$\frac{1}{p} + 2yp^2 + \left(\frac{y}{p^2} + 2py^2\right)\frac{dp}{dy} = 0$$

$$\frac{1+2yp^3}{p} + \frac{y(1+2p^3y)}{p^2}\frac{dp}{dy} = 0$$

$$1+2yp^3\left[\frac{1}{p} + \frac{y}{p^2}\right]\frac{dp}{dy} = 0$$

Now for its general solution

$$1 + \frac{y}{p}\frac{dp}{dy} = 0$$

$$\Rightarrow 1 = -\frac{y}{p}\frac{dp}{dy}$$

$$\frac{dy}{y} = \frac{-dp}{p}$$

$$\frac{dy}{y} + \frac{dp}{p} = 0$$

$$\int\frac{d}{y} + \int\frac{dp}{p} = \int 0$$

$$\ln y + \ln p = \ln c$$

$$\ln yp = \ln c$$

$$yp = c$$

$$\Rightarrow y = c/p$$

put in eq (2)

$$2x = \frac{c/p}{p} - (c/p)^2 p^2$$

$$2x = c/p^2 - \frac{c^2}{\cancel{p^2}}\cancel{p^2}$$

$$x = \frac{c}{2p^2} - c^2$$

If we put $p = c/y$ in eq(2), then

$$2x = y/c/y - (c/y)^2 y^2$$

$$2x = y^2/c - c^2$$

$$y^2 = 2(x + c^3$$

For singular solution we have

$$1+2yp^3 = 0$$

$$\Rightarrow p^3 = \frac{-1}{2y}$$

put in eq (A), we have

$$y = 2px + y^2\left(\frac{-1}{2y}\right)$$

$$y = 2px - \frac{y}{2}$$

$$y + \frac{y}{2} = 2px$$

$$2px = \tfrac{3}{2}y$$

$$\Rightarrow 8p^3x^3 = \tfrac{27}{8}y^3$$

$$\text{or } \frac{27}{8}y^3 = 8\left(\frac{-1}{2y}\right)x^3$$

$$\Rightarrow \boxed{27y^4 + 32x^3 = 0} \text{ ANS}$$

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RICATTI EQUATION

We have already studied first order linear differential equation of the form

$$y' + P(x)y = R(x) \longrightarrow (1)$$

If we add the term $Q(x)y^2$ to the left hand side of eq (1), we obtain a non-linear ODE

$$y' + P(x)y + Q(x)y^2 = R(x) \longrightarrow (2)$$

This equation (2) is called the Riccati equation. Here P, Q, R are functions of x or constants. Riccati equation is exactly linear when P is identically 0. If $R = 0$ then the Reccati equation becomes the Bernoulli's equation.

In many cases the solution of (2) cannot be expressed in terms of elementary functions. However the Ricatti equation

$$y' + Py + Qy^2 = R$$

can be reduced to a linear equation by the substitution $y = y_1 + 1/u$ where y_1 is a particular solution of (1) and u is a unknown function of x

Proof:- Let $y = y_1 + \frac{1}{u}$ be as given

differentiating wrt x, we have

$$y' = \frac{dy}{dx} = \frac{dy_1}{dx} - \frac{1}{u^2}\frac{du}{dx}$$

Substituting for y and y' in (1) we get,

$$\frac{dy_1}{dx} - \frac{1}{u^2}\frac{du}{dx} + P\left(y_1 + \frac{1}{u}\right) + Q\left(y_1 + \frac{1}{u}\right)^2 = R$$

$$\frac{dy_1}{dx} - \frac{1}{u^2}\frac{du}{dx} + P\left(y_1 + \frac{1}{u}\right) + Q\left(y_1^2 + \frac{1}{u^2} + \frac{2y_1}{u}\right) = R$$

<u>OR</u>

$$\frac{dy_1}{dx} + Py_1 + Qy_1^2 - R - \frac{1}{u^2}\left(\frac{du}{dx} - Pu - 2Qy_1u - Q\right) = 0 \rightarrow (2)$$

Since y_1 is a solution of (1), we have

$$\frac{dy_1}{dx} + Py_1 + Qy_1^2 - R = 0$$

and so (2) reduces to,

$$\frac{du}{dx} - (P + 2Qy_1)u = Q$$

which is a linear equation.
Thus if a particular solution of (1) is known then its general solution can be found.

Question:- Solve $\frac{dy}{dx} - y^2 = -1$; $y(0) = 3$. given that $y_1 = 1$ is a particular solution of given DE.

Sol:- Here $P = 0$, $Q = -1$, $R = -1$

Writing $y = 1 + \frac{1}{u}$, the given equation reduces to

$$\frac{d}{dx}\left(1 + \frac{1}{u}\right) - \left(1 + \frac{1}{u}\right)^2 = -1$$

$$\frac{du}{dx}-(0+2(-1)(1))u=-1 \quad \left(\because \frac{du}{dx}-(P+2Qy_1)u=Q\right)$$

step

$$\frac{du}{dx}+2u=-1 \longrightarrow (1)$$

Which is now a linear DE in u

For Integrating factor, we have

$$e^{\int P(x)dx} \quad e^{\int 2dx} \qquad \text{Here } p(x)=2$$

$$I.F = e^{2x}$$

So general solution of 1 becomes

$$I.F(u) = \int I.F(Q(x))\,dx + C$$

$$e^{2x}\cdot u = \int e^{2x}(-1)\,dx + C$$

$$e^{2x}\cdot u = -\frac{e^{2x}}{2} + C$$

$$u = \frac{-1}{2}\frac{\cancel{e^{2x}}}{e^{2x}} + \frac{C}{e^{2x}}$$

$$u = \frac{-1}{2} + \frac{C}{e^{2x}}$$

Hence $y = 1 + \frac{1}{u}$ becomes

$$y = 1 - \frac{1}{u} \longrightarrow (2)$$

Now by Initial condition $y(0)=3$

we have $x=0$, $y=3$, So (2) becomes

$$y(0) = 3 = 1 + \frac{1}{u(0)} \text{ or } u(0) = \frac{1}{2}$$

Hence from (2)

$$u(0) = \frac{1}{2} = \frac{-1}{2} + C \Rightarrow \boxed{C=1}$$

Thus

$$u = \frac{-1}{2} + \frac{1}{e^{2x}}$$

$$= \frac{2-e^{2x}}{2e^{2x}}$$

Hence the required solution is

$$y = 1 + \frac{1}{u}$$

$$= 1 + \frac{1}{\frac{2-e^{2x}}{2e^{2x}}}$$

$$= 1 + \frac{2e^{2x}}{2-e^{2x}}$$

$$= \frac{2-e^{2x}+2e^{2x}}{2-e^{2x}}$$

$$y = \frac{2+e^{2x}}{2-e^{2x}} \quad \underline{\text{ANS}}$$

(2) Solve $y' - \frac{y}{x} - x^3y^2 = -x^5$ by finding a particular solution.

<u>Sol:</u>

Given DE is a Riccati equation with $P = \frac{-1}{x}$, $Q = -x^3$, $R = -x^5$

An obvious solution of given DE is $y_1 = x$.

Substituting $y = x + \frac{1}{u}$ into given DE, we have

$$\frac{du}{dx} - \left(\frac{-1}{x} + 2(-x^3)x\right)u = -x^3$$

$$\frac{du}{dx} + \left(\frac{1}{x} + 2x^4\right)u = -x^3 \longrightarrow (A)$$

Which is now a linear DE in u. For Integrating factor we have

$$I \cdot F = e^{\int(\frac{1}{x}+2x^4)dx}$$

$$= e^{\int\frac{dx}{x}+2\int x^4 dx}$$

$$= e^{\ln x+\frac{2x^5}{5}}$$

$$= e^{\ln x} \cdot e^{\frac{2}{5}x^5}$$

$$I \cdot F = x e^{\frac{2}{5}x^5}$$

So the general solution of (A)

$$u(I \cdot F) = \int Q\,(I \cdot F)\,dx + C$$

$$u \cdot x e^{\frac{2}{5}x^5} = \int (-x^3)\left(x e^{\frac{2}{5}x^5}\right)dx + C$$

$$u x e^{\frac{2}{5}x^5} = -\int x^4 e^{\frac{2}{5}x^5}dx + C$$

$-\int x^4 e^{\frac{2}{5}x^5}dx$ can be solved by using substitution

So Let $x^5 = t$

$$\Rightarrow \frac{dt}{dx} = 5x^4 \Rightarrow \frac{1}{5}dt = x^4 dx$$

$$\therefore -\int x^4 e^{\frac{2}{5}x^5}dx = -\int \frac{1}{5} e^{\frac{2}{5}t}dt$$

$$= -\frac{1}{5}\left(\frac{e^{\frac{2}{5}t}}{\frac{2}{5}}\right) \text{, put value of } t$$

$$= -\frac{1}{5}\left(\frac{e^{\frac{2}{5}x^5}}{\frac{2}{5}}\right) = -\frac{1}{2}e^{\frac{2}{5}x^5}$$

$$\therefore \quad u x e^{\frac{2}{5}x^5} = -\frac{1}{2} e^{\frac{2}{5}x^5} + c$$

$$u = \frac{-1}{2} \frac{x e^{\frac{2}{5}x^5}}{x e^{\frac{2}{5}x^5}} + \frac{c}{x e^{\frac{2}{5}x^5}}$$

$$u = \frac{c - \frac{1}{2} x e^{\frac{2}{5}x^5}}{x e^{\frac{2}{5}x^5}}$$

Hence the general solution of given differential equation (DE) is

$$y = x + \frac{1}{u}$$

$$= x + \frac{1}{\frac{c - \frac{1}{2} x e^{\frac{2}{5}x^5}}{x e^{\frac{2}{5}x^5}}}$$

$$= x + \frac{x e^{\frac{2}{5}x^5}}{c - \frac{1}{2} x e^{\frac{2}{5}x^5}}$$

$$= x\left(1 + \frac{x e^{\frac{2}{5}x^5}}{c - \frac{1}{2} x e^{\frac{2}{5}x^5}} \right)$$

$$= x\left[\frac{c - \frac{1}{2} x e^{\frac{2}{5}x^5} + x e^{\frac{2}{5}x^5}}{c - \frac{1}{2} x e^{\frac{2}{5}x^5}} \right]$$

$$= x\left[\frac{c + \frac{1}{2} x e^{\frac{2}{5}x^5}}{c - \frac{1}{2} x e^{\frac{2}{5}x^5}} \right]$$

$$\frac{y}{x} = \frac{c + \frac{1}{2} x e^{\frac{2}{5}x^5}}{c - \frac{1}{2} x e^{\frac{2}{5}x^5}} \quad \underline{\underline{\text{ANS}}}$$

<u>Note</u>: More simplification is possible.

(3) Solve the Ricatt equation

$$y' - y - \frac{2}{x^3} y^2 = -x^3 \longrightarrow (1)$$

given that $y_1 = x^2$ is a particular solution.

Sol:- Given that,

$$y' - y - \frac{2}{x^3} y^2 = -x^3$$

which is of the form

$$y' + P(x) y + Q(x) y^2 = R(x) \longrightarrow (2)$$

By comparing (1) & (2), we get

$$P(x) = -1, \quad Q(x) = \frac{-2}{x^3}, \quad R(x) = -x^3$$

As we know that the solution of Ricatti's equation is

$$y = y_1 + \frac{1}{u}$$

where y_1 is the particular solution

$\because$ $y_1 = x^2$ is the particular solution of eq (1) in given

$\therefore$ $y = x^2 + \frac{1}{u}$ (A) is the solution of (1)

Now we need to find u, so

$$\frac{du}{dx} - (P + 2Qy_1) u = Q.$$

$$\therefore \frac{du}{dx} - \left(-1 + 2\left(\frac{-2}{x^3}\right)x^2\right) u = {}^{-2}/_{x^3}$$

$$\frac{du}{dx} - \left(-1 - \frac{4}{x}\right) u = \frac{-2}{x^3}$$

$$\frac{du}{dx} + \left(1 + \frac{4}{x}\right) u = \frac{-2}{x^3} \longrightarrow (2)$$

Equation (2) is now a linear D·E of the form

$$\frac{du}{dx} + P(x)\, u = Q(x) \longrightarrow (3)$$

Note: The values of $P(x)$ and $Q(x)$ in eq(3) are not the values of Riccati equation but these are the values which by comparing we get from eq(2).

In eq(2) & eq(3) comparison, we see that

$$P(x) = 1 + \frac{4}{x} \quad Q(x) = \frac{-2}{x^3}$$

For Integrating factor, we have

$$I \cdot F = e^{\int P(x) dx} = e^{\int (1 + \frac{4}{x}) dx}$$

$$= e^{x + 4\ln x} = e^{x \cdot \ln x^4}$$

$$= e^{x} \cdot e^{\ln x^4}$$

$$I \cdot F = x^4 e^x$$

So the general solution eq(2) will be,

$$u(I \cdot F) = \int Q(x) \cdot (I \cdot F)\, dx + C$$

$$u \cdot x^4 e^x = \int \frac{-2}{x^3} x^4 e^x \, dx + C$$

$$u \cdot x^4 e^x = -2 \int x e^x \, dx + C$$

$$u \cdot x^4 e^x = -2\left(x\int e^x dx - \int\left(\frac{d}{dx}(x)\int e^x dx\right)dx + c\right)$$

$$ux^4 e^x = -2xe^x + 2\int 1 \cdot e^x\, dx + c$$

$$= -2xe^x + 2e^x + c$$

$$u = \frac{-2xe^x + 2e^x + c}{x^4 e^x}$$

$$\Rightarrow \frac{1}{u} = \frac{x^4 e^x}{-2xe^x + 2e^x + c}$$

Now from eq(A) on page (187) we know that

$$y = x^2 + \frac{1}{u}$$

putting value of $1/u$ here, we get

$$y = x^2 + \frac{x^4 e^x}{-2xe^x + 2e^x + c}$$

OR $$y = x^2\left(1 + \frac{x^2 e^x}{-2xe^x + 2e^x + c}\right)$$

OR $$y = x^2\left(1 + \frac{x^2 e^x}{c + 2(1-x)e^x}\right)$$ Ans

(4): Solve the Reccati Equation

$$y' - 2y^2 + 3y = 1 \quad . \quad y_1(x) = 1$$

Sol:- Given that

$$y' - 2y^2 + 3y = 1 \longrightarrow (1)$$

which is of the form
$$y' + P(x)y + Q(x)y^2 = R(x) \longrightarrow (2)$$

Comparing eq(1) and eq(2), we get

$P(x) = 3$, $Q(x) = -2$ and $R(x) = 1$

As we know that the solution of Riccati's equation is
$$y = y_1 + \frac{1}{u}$$
where y_1 is the particular solution which is given i.e
$$y_1 = 1$$
$$\therefore \quad y = 1 + \frac{1}{u} \longrightarrow (A)$$

Now we need to find the value of u, for this we know that
$$\frac{du}{dx} - (P + 2Qy_1)u = Q$$
$$\therefore \quad \frac{du}{dx} - (3 + 2(-2)(1))u = -2$$
$$\frac{du}{dx} - (3-4)u = -2$$
$$\frac{du}{dx} + u = -2 \longrightarrow (B)$$
Which is now a linear DE of the form,
$$y' + P(x)u = Q(x)$$
where $p(x) = 1$, $Q(x) = -2$

For Integrating factor, we have.

$$I \cdot F = e^{\int P(x)dx}$$

$$= e^{\int 1 dx} \qquad \because P(x) = 1$$

$$I \cdot F = e^{x}$$

Hence the general solution of eq(B) becomes

$$u(I \cdot F) = \int Q(x)(I \cdot F)\, dx + c$$

$$u \cdot e^{x} = \int -2e^{x} dx + c$$

$$u \cdot e^{x} = -2\int e^{x} dx + c$$

$$u \cdot e^{x} = -2e^{x} + c$$

$$u = -2e^{x} \cdot e^{-x} + ce^{-x}$$

$$u = -2(1) + ce^{-x}$$

$$\Rightarrow \frac{1}{u} = \frac{1}{-2 + ce^{-x}}$$

Now from eq(A), we know that

$$y = 1 + \frac{1}{u}$$

putting values we get

$$y = 1 + \frac{1}{-2 + ce^{-x}}$$

Which is the required solution of (1)

(5) $y' + \frac{3}{x}y - y^2 = \frac{1}{x^2} \qquad y_1(x) = \frac{1}{x}$

Sol:- Given that

$$y' + \frac{3}{x}y - y^2 = \frac{1}{x^2} \longrightarrow (1)$$

and $y_1(x) = \frac{1}{x}$ is a particular solution

Eqn(1) is of the form

$$y' + P(x)y + Q(x)y^2 = R(x) \longrightarrow (2)$$

By comparing (1) & (2), we see that

$$P(x) = \frac{3}{x}, \quad Q(x) = -1, \quad R(x) = \frac{1}{x^2}$$

Since we know that the solution of Riccati equation is of the form

$$y = y_1 + \frac{1}{u} \longrightarrow (ii)$$

where y_1 is the particular solution which is given as $y_1 = \frac{1}{x}$

So (ii) becomes

$$y = \frac{1}{x} + \frac{1}{u} \longrightarrow (A)$$

We need to find the value of u For which we know that

$$\frac{du}{dx} - (P + 2Qy_1)u = Q \longrightarrow (B)$$

Using value in (B), we get

$$\frac{du}{dx} - \left(\frac{3}{x} + 2(-1)\cdot\frac{1}{x}\right)u = -1$$

$$\frac{du}{dx} - \left(\frac{3}{x} - \frac{2}{x}\right) u = -1$$

$$\frac{du}{dx} - \frac{1}{x} u = -1 \longrightarrow (B^*)$$

Eq (B^*) is now a linear first order ODE of the form,

$$\frac{du}{dx} + P(x) u = Q(x)$$

where $P(x) = \frac{-1}{x}$, $Q(x) = -1$

For Integrating factor, we have,

$$I \cdot F = e^{\int P(x) dx}$$

$$= e^{\int \frac{-1}{x} dx} = e^{-\ln x} = e^{(\ln x)^{-1}}$$

$$I \cdot F = x^{-1} = \frac{1}{x}$$

Hence general solution of (B^*) is

$$u = \frac{1}{I \cdot F}\left(\int Q(x) \cdot I \cdot F \, dx + \ln c\right)$$

$$= \frac{1}{1/x}\left(\int (-1) \cdot \frac{1}{x} dx + \ln c\right)$$

$$= -x\left(\int \frac{1}{x} dx + \ln c\right)$$

$$u = -x \ln x + x \ln c = x \ln \frac{c}{x}$$

so $$\frac{1}{u} = \frac{-1}{x \ln \frac{c}{x}}$$

Now from eq (A), we have

$$y = \frac{1}{x} + \frac{1}{x \ln \frac{c}{x}}$$

which is the required general solution of DE (1).

Home Work :-

Solve the following ODEs.

(1) $y' - y - \frac{2}{x^3} y^2 = -x^2$ $y_1 = x^2$ is particular solution. ANS: (solved already)

(2) $\frac{dy}{dx} = 7 - 6y - y^2$ ANS:- $y = \frac{e^{16(x+c)} + 15}{e^{15(x+c)} - 1}$

(3) $\frac{dy}{dx} - 4y - y^2 = 4$ ANS:- $y = \frac{2x - 2c + 1}{c - x}$

(4) $\frac{dy}{dx} + (\cot x) y - y^2 = -\csc^2 x$ ANS:- $y = \frac{1 + c\cos x}{(c + \cos x)\sin x}$

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Orthogonal Trajectories

Overview:

Since it has been observed that the general solution of a first order DE contains one arbitrary constant. When this constant is assigned different values, one obtains a one-parameter family of curves. Each of these curves represents a particular solution of the given DE.

On the other hand, given a one-parameter family of curves

$$f(x, y, c) = 0 \longrightarrow (1)$$

c being parameter, then each member of the family is a particular solution of some DE. In fact, this DE is obtained by elimination of the parameter c between (1) and the relation obtained by differentiating (1).

Trajectories:- A curve which cuts every member of a given family of curves is called trajectories.

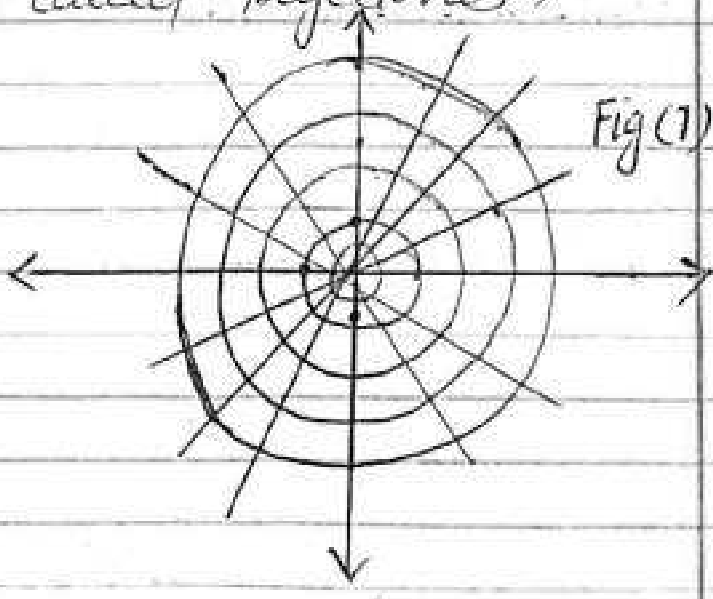

Fig (1)

Orthogonal Trajectories:- Let $f(x,y,c)=0$ and $F(x,y,K)=0$ be two family of curves with parameters c and K. If each curves in either family cuts/intersects every member of the other family at right angle/orthogonally, then each family is said to be orthogonal trajectory of the other. Recall that two curves are said to be orthogonal (intersect orthogonally) if their tangents at the point of intersection are perpendicular to each other.

For example, the families of curves given by

$$x^2+y^2=c^2$$

$$\Rightarrow f(x,y,c)=x^2+y^2-c^2=0$$

and $y=kx$

$$\Rightarrow F(x,y,k)=y-kx=0$$

are orthogonal as illustrated in Fig 1.

Working rule of orthogonal trajectories

(1) Let $f(x,y,c)=0$ be the given family of curves.

(2) Differentiating given family of curves and eliminate parameter c, i.e

$$\frac{dy}{dx}=F(x,y)$$

(3) Replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$, i.e

$$-\frac{dx}{dy} = F(x,y)$$

or $$\frac{dy}{dx} = \frac{-1}{F(x,y)} \longrightarrow (A)$$

(4) The solution of (A) is the orthogonal trajectory of the given family of curves in step (1).

Example: Find the orthogonal trajectories (OT) of family of circles $x^2+y^2=c^2$. $\longrightarrow$ (i)

Sol: Given that

$$x^2+y^2=c^2 \longrightarrow (i)$$

Differentiating (i) w.r.t "x", we have

$$2x+2y\frac{dy}{dx} = 0$$

$$2\left(x+y\frac{dy}{dx}\right) = 0$$

$$\Rightarrow x+y\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{x}{y}$$

Replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$.

$$-\frac{dx}{dy} = -\frac{x}{y}$$

or $$\frac{dy}{dx} = \frac{1}{x/y}$$

$$\frac{dy}{dx} = \frac{y}{x} \longrightarrow (ii)$$

Eq (11) is now separable. So

$$\frac{dy}{y} = \frac{dx}{x}$$

Integrating we get

$$\int \frac{dy}{y} = \int \frac{dx}{x}$$

$$\Rightarrow \ln y = \ln x + \ln k$$

$$\Rightarrow \ln y = \ln kx$$

$$\Rightarrow y = kx$$

which is the required equation of the orthogonal trajectories of (1). This equation represents a family of straight lines through the origin, which is also shown in Fig (19).

(2) $x^2 - y^2 = c$

Sol: Given that

$$x^2 - y^2 = c \longrightarrow (1)$$

Differentiating (1) b/s w.r.t "x"

$$2x - 2y\frac{dy}{dx} = 0$$

$$2\left(x - y\frac{dy}{dx}\right) = 0$$

$$\Rightarrow x - y\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{x}{y}$$

Replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$

$$-\frac{dx}{dy} = \frac{x}{y}$$

$$\frac{dx}{dy} = -\frac{x}{y}$$

or $$\frac{dy}{dx} = \frac{-y}{x} \rightarrow \text{(ii)}$$

For solving (ii), separating variables

$$\frac{dy}{y} = -\frac{dx}{x}$$

Integrating b/s

$$\int \frac{dy}{y} = -\int \frac{dx}{x}$$

$$\ln y = -\ln x + \ln k$$

$$\ln y + \ln x = \ln k$$

$$\ln xy = \ln k$$

$$\Rightarrow \quad xy = k$$

$$\Rightarrow \quad x = \frac{k}{y}$$ **ANS**

(3) $xy = c$

Sol: Given that

$$xy = c$$

Differentiating b/s w.r.t x

$$x\frac{dy}{dx} + y(1) = c$$

$$x\frac{dy}{dx}+y=0$$

$$\frac{dy}{dx}=\frac{-y}{x}$$

Replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$,

$$-\frac{dx}{dy}=\frac{-y}{x}$$

$$\frac{dx}{dy}=\frac{y}{x}$$

or $\frac{dy}{dx}=\frac{x}{y} \rightarrow (*)$

Now to solve $(*)$, separating variables

$$y\,dy=x\,dx$$

Integrating b/s

$$\int y\,dy=\int x\,dx$$

$$\frac{y^2}{2}=\frac{x^2}{2}+K$$

$$\frac{y^2}{2}-\frac{x^2}{2}=K$$

$$\frac{1}{2}(y^2-x^2)=K$$

$y^2-x^2=2K$ ANS

Orthogonal Trajectories in Polar Form-

Now we consider the case where the curve is in polar coordinate form. Let $f(r,\theta,c)=0$ be the given family of curves in polar form where c is the parameter.

* Working Rule.

(1) Let the family of curves be

$$f(r,\theta,c)=0 \longrightarrow (1)$$

(2) Suppose the differential equation of family (1) is

$$P\,dr + Q\,d\theta = 0 \longrightarrow (A)$$

where P & Q are functions of r and θ.

then

$$\frac{d\theta}{dr} + \frac{P}{Q} = 0$$

$$\frac{d\theta}{dr} = -\frac{P}{Q}$$

$$r\frac{d\theta}{dr} = -\frac{Pr}{Q}$$

Hence the family of orthogonal trajectories of the solutions of eq(A) must be the solution of

$$r\frac{dr}{d\theta} + \frac{Q}{Pr} = 0$$

or $\quad r^2P\,d\theta = Q\,dr$

$$Q\,dr - r^2P\,d\theta = 0.$$

OR

(i) Differentiating given family of curves w.r.t "θ" and eliminate parameter.
(ii) Replace $\frac{dr}{d\theta}$ by $-r^2\left(\frac{d\theta}{dr}\right)$ to obtain DE of orthogonal trajectories.
(iii) Solution of DE in step(ii) be the required family of orthogonal trajectories.

Example (4):- Find orthogonal trajectories of $r\theta = a$.

Sol:- Method (1)

Given that

$$r\theta = a \longrightarrow (i)$$

Differentiating (i) w.r.t "θ".

$$r(1)+\theta\frac{dr}{d\theta} = 0$$

$$r+\theta\frac{dr}{d\theta} = 0$$

Replacing $\frac{dr}{d\theta}$ by $-r^2\frac{d\theta}{dr}$

$$r+\theta\left(-r^2\frac{d\theta}{dr}\right) = 0$$

$$r\left(1-r\theta\frac{d\theta}{dr}\right) = 0$$

$$\Rightarrow 1-r\theta\frac{d\theta}{dr} = 0 \qquad \because r \neq 0$$

$$\Rightarrow r\theta\frac{d\theta}{dr} = 1$$

$$\Rightarrow \theta\, d\theta = \frac{dr}{r}$$

$$\int \theta\, d\theta = \int \frac{dr}{r}$$

or $$\int \frac{dr}{r} = \int \theta\, d\theta$$

$$\ln r = \theta^2/2 + c_1$$

$$e^{\ln r} = e^{\theta^2/2 + c_1}$$

$$r = e^{\theta^2/2} \cdot e^{c_1}$$

Let $e^{c_1} = c$

$$r = c e^{\theta^2/2} \quad \underline{\text{ANS}}$$

Method (II) :-

Given that

$$r\theta = a \longrightarrow (i)$$

Differentiating (i) w.r.t r b/s

$$r\frac{d\theta}{dr} + \theta(1) = 0$$

$$r\frac{d\theta}{dr} = -\theta$$

The DE of orthogonal trajectories is

$$r\frac{d\theta}{dr} = -\left(\frac{-1}{\theta}\right)$$

$$r\frac{d\theta}{dr} = \frac{1}{\theta}$$

Separating variables

$$\theta\, d\theta = \frac{dr}{r}$$

$$\frac{dr}{r} = \theta\, d\theta$$

Integrating

$$\int \frac{dr}{r} = \int \theta\, d\theta$$

$$\ln r = \theta^2/2 + c_1$$

$$e^{\ln r} = e^{\theta/2 + c_1}$$

$$r = e^{\theta/2} \cdot e^{c_1}$$

Let $e^{c_1} = c$ then

$$r = c\,e^{\theta/2} \quad \underline{\text{ANS}}.$$

(5) Find the orthogonal trajectories of the family of cardiods $r = a(1-\cos\theta)$.

Sol: Given that

$$r = a(1-\cos\theta) \longrightarrow (i)$$

Differentiating (i) b/s w.r.t θ

$$\frac{dr}{d\theta} = a(0 - (-\sin\theta))$$

$$\frac{dr}{d\theta} = a\sin\theta$$

$$\Rightarrow \frac{d\theta}{dr} = \frac{1}{a\sin\theta}$$

$$\Rightarrow r\frac{d\theta}{dr} = \frac{r}{a\sin\theta} \longrightarrow (ii)$$

putting value of r in R.H.S of (ii)

$$r\frac{d\theta}{dr} = \frac{\cancel{a}(1-\cos\theta)}{\cancel{a}\sin\theta}$$

$$r\frac{d\theta}{dr} = \frac{\cancel{2}\sin^{\cancel{2}}\theta/2}{\cancel{2}\cancel{\sin\theta/2}\cos\theta/2}$$

$\because 1-\cos\theta$
$= 1-(\cos^2\theta/2 - \sin^2\theta/2)$
$= 1-(1-\sin^2\theta/2 - \sin^2\theta/2)$
$= 2\sin^2\theta/2$

$$r\frac{d\theta}{dr} = \frac{\sin\theta/2}{\cos\theta/2}$$

$$r\frac{d\theta}{dr} = \tan\theta/2$$

Now D.E of orthogonal trajectories is,

$$r\frac{d\theta}{dr} = \frac{-1}{\tan\theta/2}$$

Separating variables

$$\frac{dr}{r} = -\frac{d\theta}{\tan\theta/2}$$

Integrating b/s

$$\int\frac{dr}{r} = -\int\frac{d\theta}{\tan\theta/2}$$

$$\ln r = -\int\frac{\cos\theta/2}{\sin\theta/2}\,d\theta$$

$$\ln r = -2\int\frac{1}{2}\frac{\cos\theta/2}{\sin\theta/2}\,d\theta$$

$$\ln r = 2\ln\sin\theta/2 + \ln c_1$$

$$\ln r = \ln(\sin\theta/2)^2 + \ln c_1$$

$$\ln r = \ln c_1 \sin^2\theta/2$$

$$r = c_1\sin^2\theta/2$$

$$r = c_1\left(\frac{1-\cos\theta}{2}\right)$$

$$r = \frac{c_1}{2}(1-\cos\theta)$$

Let $\frac{c_1}{2} = C$

$$r = C(1-\cos\theta) \quad \text{ANS}$$

This is the same family of curve as the one we started with. Thus we see that the cardiod $r = a(1-\cos\theta)$ is self orthogonal.

(6) Find orthogonal trajectory of $r = a(1+\cos\theta)$.

Sol:- Given that

$$r = a(1+\cos\theta) \longrightarrow (1)$$

Differentiating (1) w.r.t "θ"

$$\frac{dr}{d\theta} = a(0-\sin\theta)$$

$$\frac{dr}{d\theta} = -a\sin\theta$$

$$\Rightarrow \frac{d\theta}{dr} = \frac{-1}{a\sin\theta}$$

$$\Rightarrow r\frac{d\theta}{dr} = \frac{-r}{a\sin\theta}$$

put value of r on R.H.S

$$r\frac{d\theta}{dr} = -\frac{\cancel{a}(1+\cos\theta)}{\cancel{a}\sin\theta}$$

$$r\frac{d\theta}{dr} = -\left(\frac{1+\cos\theta}{\sin\theta}\right)$$

$$r\frac{d\theta}{dr} = -\frac{\cancel{2}\cos^{\cancel{2}}\theta/2}{\cancel{2}\sin\theta/2\,\cancel{\cos\theta/2}}$$

$$r\frac{d\theta}{dr} = -\frac{\cos\theta/2}{\sin\theta/2} = -\cot\theta/2$$

$$r\frac{d\theta}{dr} = -\cot(\theta/2)$$

Now D.E of orthogonal trajectories is,

$$r\frac{d\theta}{dr} = -\left(\frac{1}{-\cot\theta/2}\right)$$

$$r\frac{d\theta}{dr} = \frac{1}{\cot\theta/2}$$

$$r\frac{d\theta}{dr} = \tan\theta/2$$

Separating variables

$$\frac{dr}{r} = \frac{d\theta}{\tan\theta/2}$$

Integrating b/s

$$\int\frac{dr}{r} = \int\frac{d\theta}{\tan\theta/2}$$

$$\int\frac{dr}{r} = \int\frac{\cos\theta/2}{\sin\theta/2}\,d\theta$$

$$\int\frac{dr}{r} = 2\int\frac{1}{2}\,\frac{\cos\theta/2}{\sin\theta/2}\,d\theta$$

$$\int\frac{dr}{r} = 2\ln|\sin\theta/2| + \ln c_1$$

$$\ln r = \ln\sin^2\theta/2 + \ln c_1$$

$$\ln r = \ln c_1 \sin^2\theta/2$$

$$\Rightarrow r = c_1\sin^2\theta/2$$

$$r = c_1\left(\frac{1-\cos\theta}{2}\right)$$

$$r = \frac{c_1}{2}(1-\cos\theta)$$

Let $\frac{c_1}{2} = C$

$r = C(1-\cos\theta)$ <u>ANS</u>

(7) Find the orthogonal trajectories of the family of curves $y = ce^{-x/4}$.

Sol: Given that

$$y = ce^{-x/4} \longrightarrow (1)$$

Differentiating w.r.t x b/s of (1)

$$\frac{dy}{dx} = -\frac{c}{4}e^{-x/4}$$

$$\frac{dy}{dx} = -\frac{1}{4}y \longrightarrow (2)$$

Equation (2) is the differential equation of (1)

Now replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$. i.e

$$-\frac{dx}{dy} = -\frac{1}{4}y$$

$$\frac{dx}{dy} = \frac{1}{4}y$$

or $$\frac{dy}{dx} = \frac{4}{y}$$

Separating variables

$$y\,dy = 4\,dx$$

Integrating b/s

$$\int y\,dy = 4\int dx$$

$$\frac{y^2}{2} = 4x + K \qquad K = \text{constant}$$

$$y^2 = 8x + 2K \quad \text{ANS.}$$

(8) Find the orthogonal trajectories of the family of $r^2 = a\sin 2\theta$.

Sol:- Given that

$$r^2 = a\sin 2\theta \longrightarrow (i)$$

Differentiating b/s w.r.t θ of (i)

$$\cancel{2}r\frac{dr}{d\theta} = \cancel{2}a\cos 2\theta$$

$$r\frac{dr}{d\theta} = a\cos 2\theta$$

$$\Rightarrow \frac{1}{r}\frac{d\theta}{dr} = \frac{1}{a\cos 2\theta}$$

$$\frac{d\theta}{dr} = \frac{r}{a\cos 2\theta}$$

$$\Rightarrow r\frac{d\theta}{dr} = \frac{r^2}{a\cos 2\theta}$$

put value of r^2 on R.H.S

$$r\frac{d\theta}{dr} = \frac{\cancel{a}\sin 2\theta}{\cancel{a}\cos 2\theta}$$

$$r\frac{d\theta}{dr} = \tan 2\theta$$

Now D.E of orthogonal trajectories is,

$$r\frac{d\theta}{dr} = -\frac{1}{\tan 2\theta}$$

$$r\frac{d\theta}{dr} = -\frac{\cos 2\theta}{\sin 2\theta}$$

Separating variables

$$\frac{dr}{r} = -\frac{\sin 2\theta}{\cos 2\theta}\,d\theta$$

Integrating b/s

$$\int \frac{dr}{r} = -\int \frac{\sin 2\theta}{\cos 2\theta}\, d\theta$$

$$\int \frac{dr}{r} = -\frac{1}{2}\int 2\,\frac{\sin 2\theta}{\cos 2\theta}\, d\theta$$

$$\ln r = -\left(-\frac{1}{2}\ln(\cos 2\theta) + \ln k\right)$$

$$\ln r = \frac{1}{2}\ln(\cos 2\theta) - \ln k$$

$$2\ln r = \ln(\cos 2\theta) - \ln k$$

$$\ln r^2 = \ln\left(\frac{\cos 2\theta}{k}\right)$$

$$\Rightarrow \quad r^2 = \frac{1}{k}\cos 2\theta$$

$$r^2 = \frac{1}{k}\cos 2\theta$$

Let $1/k = b$ (another constant)

$r^2 = b\cos 2\theta$ ANS

Home Work :- Find orthogonal trajectories of the given family of curves

(1) $y = x - 1 + ce^{-x}$

ANS: $x = y - 1 + ke^{-y}$

(2) $y = (x - c)^2$

ANS :- $16y^3 = 9(k - x)^2$

(3) $r = a(1 + \sin\theta)$

ANS: $r = b(1 - \sin\theta)$

(4) $r = a\sin n\theta$

ANS:- $r^{n^2} = b\cos n\theta$