QUANTITATIVE REASONING-I

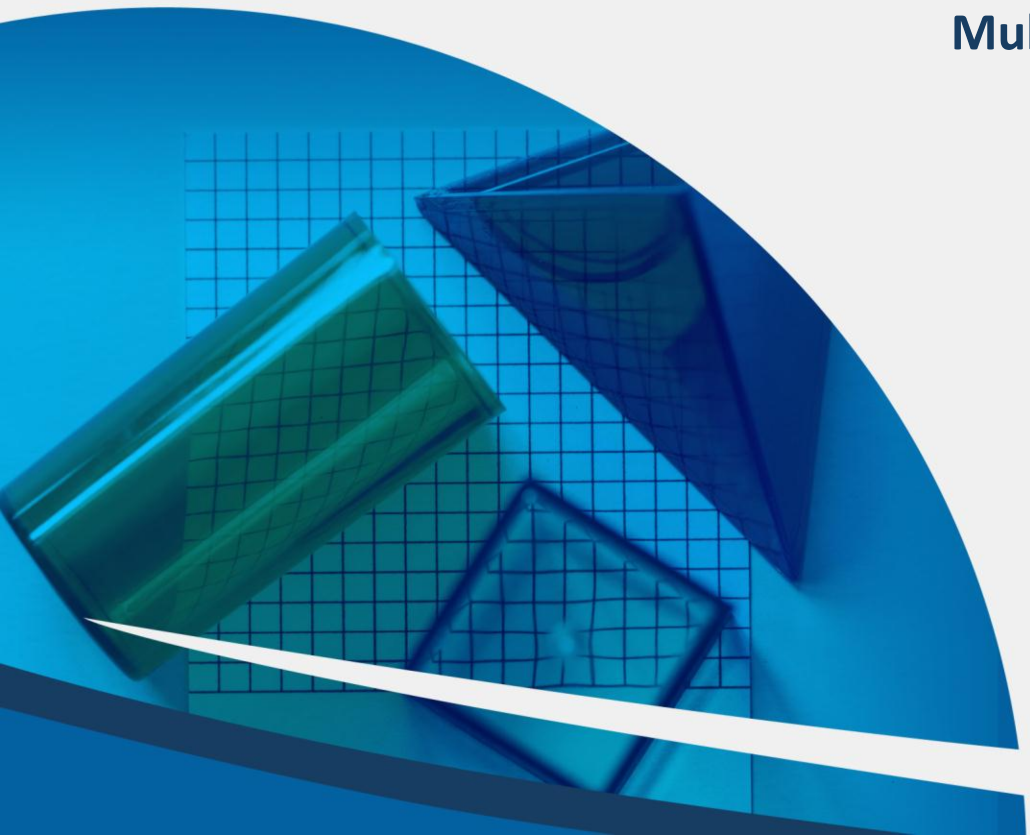
Exploring Quantitative Skills

Muhammad Usman Hamid
University of Sargodha

Ayesha Ghazanfar

Ammara Batool

Raniya Hayat
Superior College, Sillanwali



Preface

Since ancient times, numbers, quantification, statistics and mathematics has played a central role in scientific and technological development. In the 21st century, Quantitative Reasoning (QR) skills are essential for life as they help to better understand socio-economic, political, health, education, and many other issues, an individual now faces in daily life. The skills acquired by taking this course will help the students to apply QR methods in their daily life and professional activities. This course will also change student’s attitude about statistics and mathematics. It will not only polish their QR skills, but also enhance their abilities to apply these skills.

We would like to extend our sincerest gratitude to **Ayesha Ghazanfar, Ammara Batool,** and **Rania Hayat** for their invaluable contributions to this project. Their tireless efforts in research, content development, and editing have been instrumental in shaping this book into its current form. Their dedication, expertise, and passion for quantitative skills have enriched this publication, and we are deeply grateful for their involvement.

Through this book, we aim to equip readers with a solid understanding of quantitative concepts, techniques, and applications. Our goal is to foster a community of individuals who can navigate complex data, think critically, and make informed decisions in an increasingly data-driven world.

We hope that Exploring Quantitative Skills will serve as a valuable resource for students, professionals, and anyone seeking to enhance their numerical literacy. Thank you for joining us on this journey, and we wish you success in your pursuit of quantitative excellence.

Muhammad Usman Hamid
University of Sargodha

Mathematical Thinking and Quantitative Reasoning presents an analytical investigation of topics and concepts that are relevant to modern society. Our goal is to demonstrate the power of mathematics and quantitative reasoning in solving contemporary problems.

The purpose of this text is to strengthen students’ quantitative reasoning skills by having them solve a variety of real-world problems. Although we assume that the reader has an intermediate algebra background, each topic is carefully developed, and appropriate material is reviewed whenever necessary. When deciding on the depth of coverage, our singular criterion was to make mathematics accessible.

Dr. Aamir Shahzad
Baba Guru Nanak University

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CHAPTER

1

INTRODUCTION TO NUMERACY AND BASIC DEFINATIONS

Welcome to Quantitative Skills and Reasoning! Just what are quantitative skills and reasoning? The simple answer is working with numbers, making sense of data, and using your brain to figure things out. This will cover fundamental concepts from problem solving, statistics, probability, graphs, logic, sets, measurements, and finance.

In this chapter we will learn about;

- Introduction to quantitative reasoning
- Types of quantitative reasoning
- Overview of contributions of Mathematicians and Statisticians especially Muslim scholars.
- Types of standard numbers system & basic arithmetic operations
- Logical fallacies, Inductive, deductive and abductive Reasoning

What is Reasoning?

Reasoning is the ability to assess things rationally by applying logic based on new or existing information when making a decision or solving a problem. It allows you to weigh the benefits and disadvantages of two or more courses of action before choosing the one with the most benefit or the one that suits your needs.

Types of Reasoning

- **Deductive Reasoning:** Reasoning that uses formal logic and observations to prove a theory or hypothesis. It can be used to apply a general law to a specific case or test an induction. Its results typically have a logical certainty.
- **Inductive Reasoning:** Inductive reasoning is the process of reaching a general conclusion by examining specific examples. It uses theories and assumptions to validate observations. It can be used to apply a specific law to a general. Its results are not always certain because it uses conclusions from observations to make generalizations. It is helpful for extrapolation, prediction and part – to – whole arguments.
- **Analogical Reasoning:** Form of thinking that finds similarities between two or more things and then use those characteristics to find other qualities common to them. It is based on brain tendency. It can help you expand your understanding by looking for similarities between different things.
- **Abductive Reasoning:** Type of reasoning that uses an observation or set of observations to reach a logical conclusion. It is similar to inductive reasoning; however it permits making best guesses to arrive at the simplest conclusions.
- **Cause and Effect Reasoning:** Type of thinking in which you show the linkage between two events. It explains what may happen if an action takes place or why things happen when some conditions are present.
- **Critical Thinking:** It involves extensive rational thought about a specific object in order to come to a definitive conclusion. It is helpful in logic, computing and social sciences.
- **Decompositional Reasoning:** It is the process of breaking things into constituent parts to understand the function of each component and how it contributes to the operation of the item as a whole. It is helpful in logic, computing, game theory, product development, marketing and social sciences.

Quantitative Skills

Any skills that use or manipulate numbers are called quantitative skills. They help to make sense of numerical, categorical or ordinal data and scientific concepts. It is helpful in statistics, economics, algebra, finance, business, logic and social sciences.

Quantitative Reasoning / Quantitative Literacy / Numeracy

Quantitative Reasoning is the ability to assess mathematical ideas or things rationally by applying logic based on new or existing information when making a decision or solving a problem. It is application of mathematical concepts or skills to solve real world problems.

Importance of Quantitative Skills / Numeracy

Numeracy is simply the application of critical thinking skills like analysis and interpretation along with mathematical basics like algebra to quantitative information. It refers to the ability to solve quantitative reasoning problems, or to making judgment derived from quantitative reasoning in a variety of context. It helps to make sense of numerical, categorical or ordinal data and scientific concepts. It is helpful in statistics, economics, algebra, finance, business, logic and social sciences.

Quantitative Reasoning Examples

- **Statical Analysis:** Analysts apply quantitative reasoning when they assess large dataset to derive meaningful conclusions. They use statical methods like regression analysis and hypothesis testing to interpret data and distinguish patterns.
- **Financial Planning:** A financial planner utilizes quantitative reasoning for a client's investment strategy. This involves analyzing expected returns, tax implications and risk factors.

What is Mathematics?

The branch of science that deals with the numbers is called Mathematics. The world "Mathematics" is derived from the Greek word "Mathematikos" which means "inclined to learn"

Mathematics is based on deductive reasoning though man's first experience with mathematics was of an inductive nature. This means that the foundation of mathematics is the study of some logical and philosophical notions. We elaborate in simple terms that the deductive system involves four things:

Known Branches of Mathematics

- **Logi:** The Study of Principles of Reasoning.
- **Arithmetic:** Method for operating on numbers.
- **Algebra:** Method for working with unknown quantities.
- **Geometry:** The study of size and shape.
- **Trigonometry:** The study of triangles and their uses.
- **Probability:** The study of chance.
- **Statistics:** Method for analyzing data.
- **Calculus:** The study of quantities that change.

Remember

- $N = \{1,2,3,4, \dots\}$
- $W = \{0,1,2,3,4, \dots\}$
- $Z = \{0, \pm 1, \pm 2, \pm 3, \dots\}$
- $Q = \left\{x : x = \frac{p}{q}, q \neq 0, p, q \in Z\right\}$
- $Q' = \left\{x : x \neq \frac{p}{q}, q \neq 0, p, q \in Z\right\}$
- $R = Q \cup Q'$
- $C = \{a + ib : a, b \in R\}$
- $N \subseteq W \subseteq Z \subseteq R \subseteq C$

Number System

A number is a mathematical object used to count, measure, & label. It is the mathematical notation for representing numbers of a given set by using digits or other symbols in a consistent manner. It provides a unique representation of every number and represents the arithmetic and algebraic structure of the figures.

Number System: A system of writing to express numbers. It presents a unique representation of numbers

Types

- Binary Number system has only two digits that 0 & 1
- Octal Number System has only eight digits from 0 to 7
- Decimal Number System has only ten digits from 0 to 9
- Hexadecimal Number System has sixteen alphanumeric values from 0 to 9 & A to F

Types of Standard Numbers

1. Natural Numbers

The set of positive integers that start at 1 and continue infinitely. The set of natural numbers is represented by the letter “N” or the symbol ℕ.

N = {1,2,3,4,5….}

2. Whole Numbers

In math, whole numbers are positive integers, including zero, that do not have any decimal or fractional parts. The symbol for whole numbers is “W”

W = {0, 1, 2, 3, 4, 5...}

3. Integers

Integers, also known as whole numbers or round numbers, are positive or negative numbers that don't have fractional or decimal parts. The symbol for integers is Z

Z = {−1,−2,0,1,2,...}

4. Rational Numbers

The set of rational numbers includes all the integers, each of which can be written as a quotient with the integer as the numerator and 1 as the denominator. rational number, in arithmetic, a number that can be represented as the quotient p/q of two integers such that q ≠ 0. i.e. $Q = \left\{x: x = \frac{p}{q}, q \neq 0, p, q \in Z\right\}$

Q = {3/2,⅕, ¾…}

5. Irrational Numbers

An irrational number is a real number that cannot be written as a fraction of two integers, or in the form of p/q, where p and q are integers and q ≠ 0

F = $\{\sqrt{2}, \sqrt{3}, \sqrt{5}, \pi\}$

6. Real Numbers

Real numbers can be positive or negative & include fractions, integers& irrational numbers. They can be used in arithmetic operations and represented on a number line.

R = {23, −12, 6.99, 5/2, π, $\sqrt{2}$, 12.6}

7. Prime or Composite Numbers

A prime number is a whole number that is greater than 1 and can only be divided by itself and 1 without a remainder. For example, 19 is a prime number because it can only be divided by 1 and 19 these are P ={2,3,5,7,11…}

8. Complex Numbers

A complex number is a number that has both real and imaginary parts, and is written in the form $C = \{a + ib: a, b \in R\}$. For example $C = \{2 + 0i = 2, 1 + 3i\}$

9. Even & Odd Numbers

Even numbers are numbers that can be divided into two equal parts, while odd numbers are numbers that cannot.

Even numbers: End in 0, 2, 4, 6, or 8

Odd numbers: End in 1, 3, 5, 7, or 9

Arithmetic

Arithmetic is a field of mathematic that studies the characteristics of classical operations on numbers, such as addition, subtraction, multiplication, division, exponentiation and root extraction.

Arithmetic Operations

Arithmetic is the fundamental of mathematics that includes the operations of numbers. These operations are addition, subtraction, multiplication and division. Arithmetic is one of the important branches of mathematics that lays the foundation of the subject 'Maths', for students.

Addition

Combines objects into a larger collection, or increases a value. It is represented by the plus sign (+) and the answer is called the sum. For example, 4 + 7 = 11.

Subtraction

Finds the difference between numbers or quantities, or decreases a value. It is represented by the minus sign (-) and the answer is called the difference. For example, 9 - 7 = 2.

Multiplication

Multiplication is represented by the multiplication signs (×) or (*), and division is represented by the division signs (÷) or (/4, 5, 6). For example, 8 multiplied by 4 is equal to 32, which can be written as 8 × 4 = 32.

Division

Division is a method of dividing or distributing a number into equal parts. For example, 16 divided by 4 is equal to 4, which can be written as 16 ÷ 4 = 4.

Contributions of Mathematicians and Statisticians
Especially Muslim Scholars
Here is the list with era, date of birth, and date of death:

Mathematicians:

Isaac Newton (1643-1727)
Developed calculus, laws of motion, and universal gravitation
Published "Philosophiæ Naturalis Principia Mathematica" (1687)
Laid the foundation for classical mechanics and modern physics

Archimedes (c. 287 BC - c. 212 BC)
Discovered the principle of buoyancy and developed fluid mechanics
Made significant contributions to geometry and the study of spheres, cylinders, and cones
Invented various machines, including the Claw of Archimedes and the Archimedes' screw

Euclid(fl. 300 BC)
Authored the famous book "Elements," systematizing geometry and establishing axioms
Developed the concept of theorems and proofs
Introduced the concept of irrational numbers

Pierre-Simon Laplace (1749-1827)
Developed probability theory and the concept of expected value
Made significant contributions to celestial mechanics and the study of planetary motion
Published "A Philosophical Essay on Probabilities" (1812)

Albert Einstein (1879-1955)
Developed the theory of special relativity and the famous equation $E = mc^2$
Introduced the concept of spacetime and the speed of light as a universal constant
Made significant contributions to the development of quantum mechanics

Statisticians:

Ronald Fisher (1890-1962)
Developed modern statistical inference and experimental design
Introduced the concept of null hypothesis testing
Made significant contributions to the development of maximum likelihood estimation

Karl Pearson (1857-1936)
Developed the correlation coefficient and principal component analysis
Introduced the concept of the p-value
Published "The Grammar of Science" (1892)

William Gosset (1876-1937)
Developed the t-distribution and statistical hypothesis testing
Introduced the concept of the t-test
Made significant contributions to quality control and statistical process control

John Tukey (1915-2000)
Developed exploratory data analysis and the Fast Fourier Transform
Introduced the concept of the box plot
Made significant contributions to statistical graphics and visualization

Florence Nightingale (1820-1910)
Developed statistical graphics and applied statistics to medicine
Introduced the concept of the polar area chart
Made significant contributions to hospital sanitation and public health

Note: "fl." stands for "floruit," meaning "he/she flourished" and indicates the period of activity or prominence.

Muslim Scholars

Muhammad ibn Musa al-Khwarizmi

Muhammad ibn Musa al-Khwarizmi (780AD–850AD) was a Persian mathematician, astronomer, astrologer geographer and a scholar in the House of Wisdom in Baghdad. He was born in Persia of that time around 780. Al-Khwarizmi was one of the learned men who worked in the House of Wisdom. The House of Wisdom was a scientific research and teaching center. Al-Khwarizmi developed the concept of the algorithm in mathematics Al-Khwarizmi’s algebra is regarded as the foundation and cornerstone of the sciences .He is known as the “father of algebra”, a word derived from the title of his book, Kitab al-Jabr. Muhammad ibn Musa al-Khwarizmi died in c. 850 being remembered as one of the most seminal scientific minds of early Islamic culture.

Ibn al-Haytham

Ibn al-Haytham Latinised as Alhazen (965AD–1040AD) was a medieval mathematician, astronomer, and physicist of the Islamic Golden Age from present-day Iraq. Referred to as "the father of modern optics".he made significant contributions to the principles of optics and the use of scientific experiments. His most influential work is titled Kitāb al-Manāẓir "Book of Optics" in Latin Edition. Ibn al-Haytham, who lived a thousand years ago, is finally being recognized as the world's first true scientist.

Omar Khayyam

Omar Khayyam (1048AD–1131AD) was a Persian mathematician, astronomer, and poet. He made great contributions to these areas. He lived during the period of the Seljuk dynasty, around the time of the First Crusade. As a mathematician, he is most notable for his work on the classification and solution of cubic equations. He is best known for his work in geometric algebra, the Jalil calendar, and his poetry collected as, The Rubaiyat.

Ibrahim ibn Sinan

Ibrahim ibn Sinan (908AD-946AD) was born in Baghdad. He was a mathematician and astronomer who belonged to a family of scholars originally from Harran in northern Mesopotamia. He belonged to a religious sect of star worshippers known as the Sabians of Harran. Ibrahim ibn Sinan studied geometry, in particular tangents to circles. He made advances in the quadrature of the parabola and the theory of integration, generalizing the work of Archimedes, which was unavailable at the time. Ibrahim ibn Sinan is often considered to be one of the most important mathematicians of his time.

Sharaf al-Din al-Tusi

Sharaf al-Din al-Tusi (1135AD-1213AD) was an Iranian mathematician and astronomer of the Islamic Golden Age (during the Middle Ages). Sharaf al-Tusi was an Islamic mathematician who wrote a treatise on cubic equations. Al-Tusi is best known for his mathematically impressive study of the conditions under which cubic equations have a positive real root and of numerical methods for finding a solution of such equations. He made significant contributions to development of Algebraic geometry & cubic equation.

Abu Rayhan Muhammad ibn Ahmad al-Biruni

Abu Rayhan Muhammad ibn Ahmad al-Biruni (973AD–1050AD) known as al-Biruni, was a Khwarazmian Iranian scholar and polymath during the Islamic Golden Age. He has been called variously "Father of Comparative Religion", "Father of modern geodesy", Founder of Indology and the first anthropologist. Al-Biruni was well versed in physics, mathematics, astronomy, and natural sciences, and also distinguished himself as a historian, chronologist, and linguist. In 1017, he travelled to the Indian subcontinent and wrote a treatise on Indian culture entitled Tārīkh al-Hind ("The History of India"). Al-Biruni developed many instruments for astronomy and geography measurements. He was also a very good encyclopedia writer. His famous achievements were, studying geography of India, accurately measuring Earth's radius, comparing different calendars & He enabled direction of Qibla.

Inductive and Deductive Reasoning

Inductive Reasoning

Inductive reasoning is the process of reaching a general conclusion by examining specific examples. The conclusion formed by using inductive reasoning is often called a **conjecture**, since it may or may not be correct.

Deductive Reasoning

Deductive reasoning is the process of reaching a conclusion by applying general assumptions, procedures, or principles.

1. **Use Inductive Reasoning to Predict a Number**
   Use inductive reasoning to predict the most probable next number in each of the following lists.
   a. 3, 6, 9, 12, 15, ?
   b. 1, 3, 6, 10, 15, ?
   Solution
   a. Each successive number is 3 larger than the preceding number. Thus we predict that the most probable next number in the list is 3 larger than 15, which is **18**.
   b. The first two numbers differ by 2. The second and the third numbers differ by 3. It appears that the difference between any two numbers is always 1 more than the preceding difference. Since 10 and 15 differ by 5, we predict that the next number in the list will be 6 larger than 15, which is **21**.

2. Use inductive reasoning to predict the most probable next number in each of the following lists.
   a. 5, 10, 15, 20, 25, ?
   b. 2, 5, 10, 17, 26, ?
   Solution
   a. Each successive number is 5 larger than the preceding number. Thus we predict that the next number in the list is 5 larger than 25, which is 30.
   b. The first two numbers differ by 3. The second and third numbers differ by 5. It appears that the difference between any two numbers is always 2 more than the preceding difference. Since 17 and 26 differ by 9, we predict that the next number will be 11 more than 26, which is 37

3. Use Inductive Reasoning to Make a Conjecture

Consider the following procedure: Pick a number. Multiply the number by 8, add 6 to the product, divide the sum by 2, and subtract 3.

Complete the above procedure for several different numbers. Use inductive reasoning to make a conjecture about the relationship between the size of the resulting number and the size of the original number.

Solution

Suppose we pick 5 as our original number. Then the procedure would produce the following results:

Original number:	5
Multiply by 8:	$8 \times 5 = 40$
Add 6:	$40 + 6 = 46$
Divide by 2:	$46 \div 2 = 23$
Subtract 3:	$23 - 3 = 20$

We started with 5 and followed the procedure to produce 20. Starting with 6 as our original number produces a final result of 24. Starting with 10 produces a final result of 40. Starting with 100 produces a final result of 400. In each of these cases the resulting number is four times the original number. We conjecture that following the given procedure will produce a resulting number that is four times the original number.

4. Consider the following procedure: Pick a number. Multiply the number by 9, add 15 to the product, divide the sum by 3, and subtract 5.

Complete the above procedure for several different numbers. Use inductive reasoning to make a conjecture about the relationship between the size of the resulting number and the size of the original number.

Solution

If the original number is 2, then $\frac{2\times9+15}{3} - 5 = 6$ which is three times the original number. If the original number is 7, then $\frac{7\times9+15}{3} - 5 = 21$ which is three times the original number.

If the original number is -12 then $\frac{-12\times9+15}{3} - 5 = -36$ which is three times the original number. It appears, by inductive reasoning, that the procedure produces a number that is three times the original number.

5. Examine the table below. Identify a pattern and then use that pattern to find the missing terms in the sequence.

Term	1	2	3	4	5	6	7	8
Value	1	3	9	27	81			

Solution

With this problem we see that the pattern to get the next number in the sequence is to multiply the previous term in the sequence by 3. So to find the 6th, 7th, and 8th terms in the sequence we will use this pattern. The 5th term is 81. The 6th term is $3 \times 81 = 243$, the 7th term is $3 \times 243 = 729$, and the 8th term is $3 \times 729 = 2187$.

6. Examine the table below. Identify a pattern and then use that pattern to find the missing terms in the sequence.

Term	1	2	3	4	5	6	7	8
Value	58	46	34	22	10			

Solution

With this problem we see that the pattern to get the next number in the sequence is to subtract 12 from the previous term in the sequence. To find the 6th, 7th, and 8th terms in the sequence we will use this pattern. The 5th term is 10. The 6th term is $10 - 12 = -2$, 7th term is $-2 - 12 = -14$, and 8th term is $-14 - 12 = -26$.

7. Examine the table below. Identify a pattern and then use that pattern to find the missing terms in the sequence.

Term	1	2	3	4	5	6	7	8	9	10
Value	5	10	30	120	240	720	2880			

Solution

With this sequence we see to go from 5 to 10 we multiply by 2. To go from 10 to 30 we multiply by 3. To go from 30 to 120 we multiply by 4. Then we see that this pattern repeats to get the next three terms in the sequence. $2 \times 120 = 240$, $3 \times 240 = 720$, and $4 \times 720 = 2880$. So we will use this same pattern to get the 8th , 9th , and 10th terms. The 8th term is $2 \times 2880 = 5760$, the 9th term is $3 \times 5760 = 17280$, and the 10th term is $4 \times 17280 = 69120$.

8. Use Inductive Reasoning to Solve an Application

Use the data in the table on the preceding page and inductive reasoning to answer each of the following.

a. If a pendulum has a length of 25 units, what is its period?

b. If the length of a pendulum is quadrupled, what happens to its period?

Solution

a. In the table, each pendulum has a period that is the square root of its length. Thus we conjecture that a pendulum with a length of 25 units will have a period of 5 heartbeats.

b. In the table, a pendulum with a length of 4 units has a period that is twice that of a pendulum with a length of 1 unit. A pendulum with a length of 16 units has a period that is twice that of a pendulum with a length of 4 units. It appears that quadrupling the length of a pendulum doubles its period.

9. A tsunami is a sea wave produced by an under-water earthquake. The velocity of a tsunami as it approaches land depends on the height of the tsunami. Use the table at the left and inductive reasoning to answer each of the following questions.

a. What happens to the height of a tsunami when its velocity is doubled?

b. What should be the height of a tsunami if its velocity is 30 feet per second?

Solution

a. It appears that when the velocity of a tsunami is doubled, it height is quadrupled.
b. A tsunami with a velocity of 30 feet per second will have a height that is four times that of a tsunami with a speed of 15 feet per second. Thus, we predict a height of $4 \times 25 = 100$ feet for a tsunami with a velocity of 30 feet per second.

10. The last four times I have driven downtown at 6pm there has been traffic. use inductive reasoning to draw your conclusion.

Solution

My conclusion is that there is always traffic downtown around 6pm.

11. Consider the statement and determine if it is inductive or deductive:

" Every month has 30 days in it. July is a month. Therefore it has 30 days in it. "

Solution:

This statement starts with a generalization and it’s then applied to a specific case. This follows the pattern of **deductive reasoning**. The statements are not necessarily true, but if every month had 30 days in it, then it would be true.

12. Use Deductive Reasoning to Establish a Conjecture

Use deductive reasoning to show that the following procedure produces a number that is four times the original number.

Procedure: Pick a number. Multiply the number by 8, add 6 to the product, divide the sum by 2, and subtract 3.

Solution

Let n represent the original number.

Multiply the number by 8: $8n$

Add 6 to the product: $8n + 6$

Divide the sum by 2: $\frac{8n + 6}{2} = 4n + 3$

Subtract 3: $4n + 3 - 3 = 4n$

We started with n and ended with 4n. The procedure given in this example produces a number that is four times the original number.

13. Use deductive reasoning to show that the following procedure produces a number that is three times the original number.

Procedure: Pick a number. Multiply the number by 6, add 10 to the product, divide the sum by 2, and subtract 5.

Solution

Let n represent the original number.

$6n$

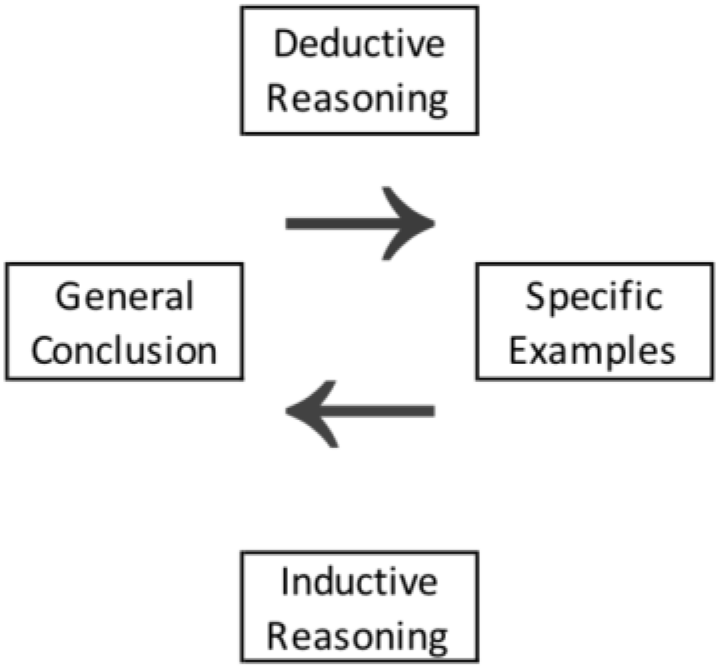
$6n + 10$

$\frac{6n + 10}{2} = 3n + 5$

$3n + 5 - 5 = 3n$

The procedure always produces a number that is three times the original number.

Inductive Reasoning vs. Deductive Reasoning



- For Inductive Reasoning we start with examples or cases, and then draw general conclusions.
- For Deductive Reasoning we start with a general statement and apply it to examples or cases.

In next Example we analyze arguments to determine whether they use inductive or deductive reasoning.

14. Determine Types of Reasoning

Determine whether each of the following arguments is an example of inductive reasoning or deductive reasoning.

a. During the past 10 years, a tree has produced plums every other year. Last year the tree did not produce plums, so this year the tree will produce plums.

b. All home improvements cost more than the estimate. The contractor estimated my home improvement will cost $35,000. Thus my home improvement will cost more than $35,000.

Solution

a. This argument reaches a conclusion based on specific examples, so it is an example of inductive reasoning.

b. Because the conclusion is a specific case of a general assumption, this argument is an example of deductive reasoning.

15. Determine whether each of the following arguments is an example of inductive reasoning or deductive reasoning.

a. All Janet Evanovich novels are worth reading. The novel *To the Nines* is a Janet Evanovich novel. Thus *To the Nines* is worth reading.

b. I know I will win a jackpot on this slot machine in the next 10 tries, because it has not paid out any money during the last 45 tries.

Solution

a. The conclusion is a specific case of a general assumption, so the argument is an example of deductive reasoning.
b. The argument reaches a conclusion based on specific examples, so the argument is an example of inductive reasoning.

16. Consider the statement and determine if it is inductive or deductive:

" Pizza Hut has a lunch buffet. Stevi B's has a lunch buffet. Therefore all pizza restaurants have a lunch buffet."

Solution

This statement starts with two examples about pizza restaurants having lunch buffets. Based on these examples a generalization is made. This follows the pattern of **inductive reasoning**.

17. Consider the statement and determine if it is inductive or deductive:

" All pro wrestlers have a catch phrase. Macho Man Randy Savage was a pro wrestler. Therefore he had a catch phrase. "

Solution

This statement starts with a generalization about pro wrestlers having catch phrases. It's then applied to the specific case of Macho Man Randy Savage. This follows the pattern of **deductive reasoning**.

UNIT ANALYSIS AS PROBLEM SOLVING TOOL

(UNITS & THEIR CONVERSION, DIMENSIONS, AREA, PERIMETER, VOLUME)

Units

The words that describe what we are measuring or counting such kg, m, cm are called the units associated with the numbers.

Unit Analysis

The technique of working with units to help solve problems is called unit analysis or dimensional analysis.

Keywords Per (divide) and Of (multiplication)

The word per (which means "for every") is a keyword in mathematical problems, because it tells us to divide. For example 50 miles per hour. The word of is a keyword in mathematical problems, because it tells us to multiply. For example 10 apples at a price of 2 dollars.

1. What is the total distance travelled when you run 7 laps around a 400 meter track?
   Solution
   7 laps of a 400 meter track = $7 \times 400\text{m} = 2800\text{m}$

2. How many crates do you need to hold 2000 apples if each crate holds 40 apples?
   Solution
   Per crate apples = 40
   Total needed crates = $\frac{2000}{40} = 50$ crates

3. How much will you pay for 3.5 pounds of bananas at a price of \$ 0.90 per pound?
   Solution
   Total weight = 3.5 pounds
   Per pound rate = \$ 0.90
   Total price = $3.5\text{ lb} \times \frac{\$\ 0.90}{1\text{ lb}} = \$\ 3.15$

4. What is the total weight of 23 baseballs that weigh 5.25 ounces each?
Solution
Weight of each ball = 5.25 ounces
Total balls = 23
Total weight = 23 baseballs $\times \frac{5.25\text{ oz}}{1\text{ baseballs}} = 120.75\text{ oz}$

5. How much will you earn working for 6 months at a salary of $3200 per month?
Solution
Per month salary = $3200 ; Total months = 6
Total earning = 6 months $\times \frac{\$\,3200}{1\text{month}} = \$\,19200$

6. How many apartment buildings are needed to house 3000 people if each building can house 150 people?
Solution
Given ratio = $\frac{1\text{ apartment}}{150\text{ people}}$
3000 people need apartment = $\frac{3000}{150} = 20\text{ apartment}$

7. Identifying units
Identify the unit of the answer in each of the following cases
a) The price you paid for gasoline, found by dividing it's total cost in Dollars by the number of gallons of gas that you bought.
b) The area of a circle, found with the formula πr^2, where the radius "r" is measured in centimetres (note that π is a number and has no unit)
c) A volume found by multiplying an area measured in acres by a depth measured in feet

Solution

a) The price of the gasoline has units of dollars divided by gallons, which we write as $/gal and read as" dollars per gallon."
b) The area of the circle has units of centimeters to the second power, which we write as cm^2 and read as" centimetre square".
c) In this case ,the volume has the units of *acres* × *feet*, which we read as " acre- feet" .The unit of this volume is commonly used by hydrologists (water engineer) in the United states

8. Identifying units

a) Your average speed is on a long walk, found by dividing distance traveled in Miles by time elapsed in hours?
miles/hour

b) The unit price of oranges, found by dividing the price in dollars by the weight in pounds?
Dollars/pounds

c) The cost of a piece of a carpet, found by dividing its price in dollars by its area in square yards?
Dollars/ square yards

d) The floor rate of a river in which 500 cubic feet of water flow past a particular location every second?
Cubic feet/second

e) The unit price of rice in Japan, found by dividing the price in yen by the weight in kilogram?
Yen/kilogram

f) The production rate of a bagel bakery, found by dividing the number of bagels produced by the time required in hours?
bagels/hours

g) The per capita daily oil consumption by the residents of a town, found by dividing the amount of oil used per day in gallons by the population of the town?
Galon/person

h) The density of a rock, found by dividing its weight in grams by its volume in cubic centimeters?
Gram/cubic centimeter

Area

Area is a fundamental concept in mathematics, particularly in geometry. It refers to the amount of space inside a two-dimensional shape or region. In other words, it's the size of the surface enclosed by a shape.

Here are some key aspects of area:

- **Unit:** Area is typically measured in square units, such as square meters (m^2), square feet (ft^2), or square centimeters (cm^2).
- **Dimension:** Area is a two-dimensional concept, meaning it's used to describe the size of flat shapes like triangles, quadrilaterals, polygons, circles, and more.
- **Calculation:** Area can be calculated using various formulas, depending on the shape. For example:
  Rectangle: length × width ($A = lw$)
  Triangle: ½ × base × height ($A = \frac{1}{2}bh$)
  Circle: π × radius² ($A = \pi r^2$)
  Trapezoid: ½ × (base1 + base2) × height ($A = \frac{1}{2}(b1 + b2)h$)
  Area of a Square: $A = s^2$
  Area of a Parallelogram: $A = bh$

Properties: Area has several important properties, including:

- Positive: Area is always positive (or zero).
- Additive: The area of a composite shape is the sum of the areas of its individual parts.
- Scaling: When a shape is scaled (enlarged or reduced), its area changes by the square of the scaling factor.

Applications: Area is used in various real-world contexts, such as:

- Architecture: building design, room layout
- Engineering: structural analysis, material usage
- Physics: surface area, volume calculations
- Computer graphics: texture mapping, rendering
- Agriculture: land area, crop yields

In summary, area is a fundamental concept in mathematics that describes the size of two – dimensional shapes and regions. Its properties and applications make it a crucial tool in various fields.

Volume

Volume is the amount of three-dimensional space occupied by a substance, object, or container. It's a fundamental concept in mathematics, particularly in geometry. Volume measures the amount of space inside a 3D shape or region.

Here are some key aspects of volume:

- **Unit:** Volume is typically measured in cubic units, such as cubic meters (m^3), cubic feet (ft^3), or cubic centimeters (cm^3).
- **Dimension:** Volume is a three-dimensional concept, describing the size of 3D shapes like cubes, spheres, cylinders, cones, and more.
- **Calculation:** Volume can be calculated using various formulas, depending on the shape:
  Cube: side^3 ($V = s^3$)
  Sphere: $(4/3) \times \pi \times$ radius^3 ($V = (4/3)\pi r^3$)
  Cylinder: $\pi \times$ radius^2 $\times$ height ($V = \pi r^2 h$)
  Cone: $(1/3) \times \pi \times$ radius^2 $\times$ height ($V = (1/3)\pi r^2 h$)
  Regular Square Pyramid: ($V = (1/3)s^2 h$)

Properties: Volume has several important properties:

- Positive: Volume is always positive (or zero).
- Additive: The volume of a composite object is the sum of the volumes of its individual parts.
- Scaling: When an object is scaled, its volume changes by the cube of the scaling factor.

Applications: Volume is used in various real-world contexts:

- Physics: mass, density, buoyancy
- Engineering: fluid dynamics, structural analysis
- Chemistry: molar volume, concentration
- Biology: cell volume, tissue structure
- Everyday life: measuring liquids, packaging, and storage

In summary, volume is a fundamental concept in mathematics that describes the amount of three-dimensional space occupied by an object or substance. Its properties and applications make it a crucial tool in various fields.

Hyphen

A hyphen (-) is a punctuation mark used to join two or more words, phrases, or parts of a word together. It is a short horizontal line that connects the words or parts, indicating that they are linked or related. **Hyphen implies multiplication**.

Here are some ways hyphens are used:

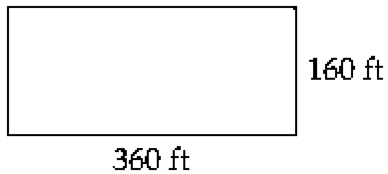
- **Compound words:** Hyphens connect two or more words to form a new word, like self-portrait, merry-go-round, or well-being.
- **Prefixes and suffixes:** Hyphens attach prefixes and suffixes to words, like co-pilot, re-write, or un-happy.
- **Phrases and expressions:** Hyphens join phrases and expressions, like two-thirds, five-year-old, or high-speed.
- **Numbers:** Hyphens separate numbers, like phone numbers (123-456-7890) or dates (2023-02-15).
- **Connection:** Hyphens indicate a connection or relationship between words, like father-in-law or passers-by.
  Remember, hyphens are different from dashes (-), which are longer and used for different purposes.

Keywords Operations with Units

Key word or Symbol	Operation	Example
Per	Division	Miles per hour
Of or Hyphen	Multiplication	Kilowatt hours
Square	Raising to second power	Square feet
Cube or Cubic	Raising to third power	Cubic feet

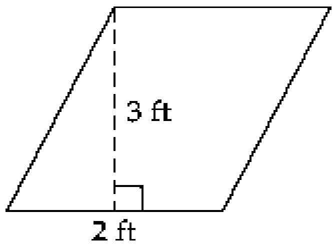
9. How many squares, each 1 inch on a side, are needed to cover a rectangle that has an area of 18 in^2?
Eighteen squares, each 1 inch on a side, are needed to cover the rectangle.

10. How many square feet of sod are needed to cover a football field? A football field measures 360 ft by 160 ft.



$\text{Area} = L \times W = 360\text{ft} \times 160\text{ft} = 57600\text{ft}^2$

11.A solar panel is in the shape of a parallelogram that has a base of 2 ft and a height of 3 ft. Find the area of the solar panel.

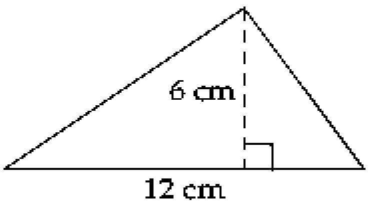


Area $= bh = 2 \times 3 = 6\text{ft}^2$

12. A homeowner wants to carpet the family room. The floor is square and measures 6 m on each side. How much carpet should be purchased?

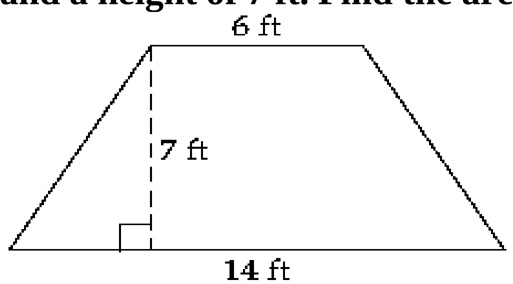
Area $= s^2 = 6^2 = 36\text{m}^2$

13. A riveter uses metal plates that are in the shape of a triangle with a base of 12 cm and a height of 6 cm. Find the area of one metal plate.



Area $= \frac{1}{2}\text{bh} = \frac{1}{2} \times 12 \times 6 = 36\text{cm}^2$

14. A boat dock is built in the shape of a trapezoid with bases measuring 14 ft and 6 ft and a height of 7 ft. Find the area of the dock.



Area $= \frac{1}{2}h(\text{b}_1 + \text{b}_2) = 12\text{ft} \times 10\text{ft} = 120\text{ft}^2$

15. How large a cover is needed for a circular hot tub that is 8 ft in diameter? Round to the nearest tenth of a square foot.

Area $= \pi\text{r}^2 = \pi(\text{d}/2)^2 = 3.14(4)^2 = 3.14(25) \approx 50.3$

16.Find the area of a circle with a diameter of 10 m. Round to the nearest hundredth of a square meter.

Area $= \pi\text{r}^2 = \pi(\text{d}/2)^2 = 3.14(5)^2 = 3.14(25) \approx 78.54$

17. Find the area of a room if the room is 12*ft* long and 10*ft* wide.
Area $= L \times W = 12ft \times 10ft = 120ft^2$

18. Find the volume of a box if the box is 6*in.* wide, 4*in.* deep and 10*in.* high.
Volume $= L \times W \times D = 6in. \times 4in. \times 10in. = 240\ in^3$.

19. A large box shaped Arena has a rectangular floor that measures 200 feet by 150 feet and a flat ceiling that is 35 feet above the floor. Find the area of the floor and the volume of the arena?
Area $= L \times W = 200ft \times 150ft = 30000ft^2$
Volume $= L \times W \times H = 200ft \times 150ft \times 35ft = 1050000ft^3$

20. A flat bottom reflecting pool has length 30 yards, width 10 yards and depth 0.3 yard. Find the surface area of the pool and the volume of water it holds?
Area $= 30\ yards \times 10\ yards = 300\ yd^2$
Volume $= 30\ yards \times 10\ yards \times 0.3 yards = 90\ yd^3$

21. A raised flower bed is 25 feet long, 8 feet wide and 1.5 feet deep. Find the area of the bed and the volume of soil it holds?
Area $= 25ft \times 8ft = 200ft^2$
Volume $= 25ft \times 8ft \times 1.5ft = 300ft^3$

22. A warehouse is 40 yards long and 25 yards wide and it is piled with curtains to a height of 3 yards. What is the area of the warehouse floor? What is the total volume of the cartons?
Area $= 25\ yards \times 40\ yards = 1000\ yd^2$
Volume $= 25\ yards \times 40\ yards \times 3\ yards = 3000\ yd^3$

23. The bed of a truck is 3.5 feet deep, 12 feet long and 5 feet wide. What is the area of the bed's floor? What is the volume of the bed?
Area $= 5ft \times 12ft = 60ft^2$
Volume $= 5ft \times 12ft \times 3.5ft = 210ft^3$

24. A can has a circular base with an area of 6 square inches and is 4 inches tall. What is its total volume?
Area $= 6\ in^2$.
Volume $= A \times H = 6in^2. \times 4in. = 24\ in^3$.

25. Which of the following are rectangular solids: a juice box, a milk carton, a can of soup, a compact disc, and a jewel case (plastic container in which a compact disc is packaged)?

A juice box and a jewel case are rectangular solids.

26. Find the volume of a rubber ball that has a diameter of 6 in.

$$\text{Volume} = \frac{4}{3}\pi r^3 = \frac{4}{3} \times 3.14 \times \left(\frac{6}{2}\right)^3 = \frac{4}{3} \times 3.14 \times (3)^3 \approx 113.10$$

27. Which of the following units could not be used to measure the volume of a regular square pyramid?

a. ft^3 b. m^3 c. yd^2 d. cm^3 e. mi

Volume is measured in cubic units. Therefore, the volume of a regular square pyramid could be measured in ft^3, m^3 or cm^3 but not in yd^2 or mi.

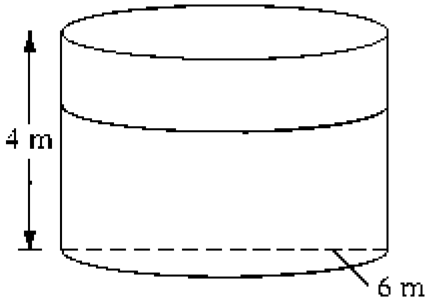
28. Find the volume of a cube that measures 1.5 m on a side.

$$\text{Volume} = s^3 = (1.5)^3 = 3.375 m^3$$

29. The radius of the base of a cone is 8 cm. The height of the cone is 12 cm. Find the volume of the cone. Round to the nearest hundredth of a cubic centimeter.

$$\text{Volume} = \frac{1}{3}\pi r^2 h = \frac{1}{3} \times 3.14 \times (2)^2 \times (12) \approx 804.25 \text{ cm}^3$$

30. An oil storage tank in the shape of a cylinder is 4 m high and has a diameter of 6 m. The oil tank is two-thirds full. Find the number of cubic meters of oil in the tank. Round to the nearest hundredth of a cubic meter.



$$\text{Volume} = \pi r^2 h = 3.14 \times \left(\frac{6}{2}\right)^2 \times (4) = 3.14 \times (3)^2 \times (4) \approx 75.40 \text{ m}^3$$

Unit Conversions: The U.S. Customary System of Measurements

In the first State of the Union Address, George Washington advocated “uniformity in the currency, weights, and measures of the United States.” His recommendation led to the adoption of the U.S.C.S. The standard U.S. C.S. units of length are inch, foot, yard, and mile.

Lengths	1 inch = 2.54cm	1 furlong = 40 rods = 1.8 miles 1 mile = 1760 yards = 5280 ft 1 nautical mile = 1.852 km = 6076.1 ft 1 league = 3 nautical miles	
	1 ft = 12 inc		
	1 yard = 3 ft		
	1 rod = 5.5 yards		
	1 fathom = 6 ft		
Weights	**Avoirdupois**	**Troy**	**Apothecary**
	1 grain = 0.0648 gram	1 gran = 0.0648 gram	1 grain = 0.0648 gram
	1 ounce = 437.5 grains	1 carat = 0.2 gram = 3.086 grans	1 scruple = 20 grains
	1 pound = 16 ounce	1 pennyweight = 24 grains	1 dram = 3 scruples
	1 ton = 2000 ponds	1 troy ounce = 480 grains	1 apoth.oz = 8 dram
	1 long ton = 2240 pound	1 troy pound = 12 troy ounce	1 apoth.lb = 10 oz
Volumes	**Liquid Measures**		**Dry Measures**
	1 tablespoon = 3 teaspoon 1 fluid ounce = 2 tablespoons = 1.805 in^3 1 cup = 8 fluid ounce 1 pint = 16 fluid ounces = 28.8 in^3 1 quart = 2 pints = 57.75 in^3 1 gallon = 4 quarts 1 barrel of petroleum = 42 gallons 1 barrel of liquid = 31 gallons		1 in^3 = 16.387 cm^3
			1 ft^3 = 1728 in^3 =7.48 gallon
			1 yd^3 = 27 ft^3
			1 dry pint = 33.60 in^3
			1 dry qrt = 2 dry pints = 67.2 in^3
			1 peck = 8 dry quarts (qt)
			1 bushel = 4 pecks
			1 cord = 128 t^3

31. What conversion rate would you use to convert each of the following?

a. Feet to inches b. Inches to feet

c. Pounds to ounces d. Ounces to pounds

Answer

a. $\frac{12\text{ in.}}{1\text{ ft}}$ ***b.*** $\frac{1\text{ ft}}{12\text{ in.}}$ ***c.*** $\frac{16\text{ oz}}{1\text{ lb}}$ ***d.*** $\frac{1\text{ lb}}{16\text{ oz}}$

32. Convert.

a. 36 fl oz to cups b. $4\frac{1}{2}$ tons to pounds

Solution

a. $1\text{ c} = 8\text{ fl oz} \Rightarrow 1\text{fl oz} = \frac{1}{8}\text{ c} \Rightarrow 36\text{fl oz} = \frac{36}{8}\text{ c} = 4\frac{1}{2}\text{c}$

b. $1\text{ ton} = 2000\text{ lb} \Rightarrow 4\frac{1}{2}\text{ tons} = \frac{9}{2}\text{ tons} \times 1 = \frac{9}{2}\text{ tons} \times \frac{2000\text{ lb}}{1\text{ ton}} = 900\text{ lb}$

33. In 2005, a horse named Shaniko ran a 1.125-mile race in 1.829 min. Find Shaniko's average speed for that race in miles per hour. Round to the nearest tenth.

Solution

$\text{Shaniko's rate} = \frac{1.125\text{ mi}}{1.829\text{ min}} = \frac{1.125\text{ mi}}{1.829\text{ min}} \times \frac{60\text{ min}}{1\text{ h}} = \frac{1.125\text{ mi}}{1.829\text{ h}} \approx 36.9\text{ mph}$

34. A carpet is to be installed in a room that is 20 ft wide and 30 ft long. At $28.50 per square yard, how much will it cost to carpet the room?

Solution

$A = LW = 30\text{ft} \times 20\text{ft} = 600\text{ f}t^2$

$600\text{ ft}^2 = 600\text{ ft}^2 \times \frac{1}{9}\text{ yd}^2 = \frac{200}{3}\text{ yd}^2$

$\text{Cost} = \frac{200}{3}\text{ yd}^2 \times \frac{\$28.50}{\text{yd}^2} = \$1900$

The cost to carpet the room is $1900.

35. Convert 2.5 h to seconds.

Solution

1 h = 60 min and 1 min = 60 s

$2.5\text{ h} = \frac{25}{10} \times 3600\text{ sec} = 9000\text{ sec}$

36. Inches to feet

Convert a length of 102 inches to feet.

Solution

$102\text{ in.} = 102 \times \frac{1}{12}\text{ft} = 8.5\text{ft}$

A length of 102 inches is equal to 8.5 feet.

37. Seconds to Minutes

Convert a time of 3000 second into minutes.

Solution

$1\text{min} = 60\text{sec} \Rightarrow 1\text{sec} = \frac{1}{60}\text{min}$

$\Rightarrow 3000\text{sec} = \frac{3000}{60}\text{min} = 50\text{min}$

A time of 3000 seconds is equal to 50 minutes.

38. Using a chain of conversion

How many seconds are there in one day?

Solution

We know that 1 day = 24 hrs = 60 minutes, and 1 min = 60 s. We can answer a question by setting up a chain of unit conversions in which we start with day and end up with seconds

1 day × 24 hrs/1day × 60 mint/1hr × 60 s/1 min = 86400s

By using the conversation factor needed to cancel the appropriate units, we are left with the answer in seconds .There 86400 s in one day.

39.Conversions

i. **Convert 24 feet to inches?**

$24\text{ ft} = 24 \times 12\text{ in.} = 288\text{ in.}$

ii. **Convert 24 feet to yards?**

$24\text{ ft} = 24 \times \frac{1}{3}\text{ yards} = 8\text{ yd}$

iii. **Convert 25 minutes to seconds?**

$25\text{ min} = 25 \times 60\text{ sec} = 1500\text{ sec}$

iv. **Convert 32 years to days (neglecting leap years)?**

$32\text{ years} = 32 \times 365\text{ days} = 11680\text{ days}$

v. **Convert 2.5 hours to seconds?**

$2.5\text{h} = 2.5 \times 3600\text{ sec} = 9000\text{ sec}$

vi. **Convert the space station's orbital speed of 17,200 miles per hour to units of miles per second?**

$17{,}200\text{ miles per hour} = 17{,}200\frac{\text{min}}{\text{h}} \times \frac{1\text{ h}}{60\text{ min}} \times \frac{1\text{ min}}{60\text{ sec}} = 4.78\frac{\text{min}}{\text{sec}}$

vii. **Convert 3 years to hours (neglecting leap years)?**

$3\text{ years} = 3 \times 365\text{ days} = 1095\text{ days} \Rightarrow 1095\text{ days} = 1095 \times 24\text{ h} = 26280\text{ h}$

viii. **Convert 26,500 inches to miles, using the facts 1mile = 1760yd, 1yd = 3ft, 1ft = 12in.**

$26500\text{ in.} = 26500\text{ in.} \times \frac{1\text{ ft}}{12\text{ in.}} \times \frac{1\text{ yd}}{3\text{ ft}} \times \frac{1\text{ mi}}{1760\text{ yd}} = 0.42\text{ miles}$

Conversion with Units Raised to Powers

40. Carpeting a room

You want to carpet a room that measures 10 *ft* by 12 *ft*, making an area of 120 square feet. But carpet is usually sold by the square yard .How many square yards of carpet do you need?

Solution

$120\ \text{ft}^2 = 120 \times \left(\frac{1}{3}\text{yd}\right)^2 = 120 \times \frac{1}{9}\ \text{yd}^2 = 13.3\text{yd}^2$

41. Cubic units: Purchasing Garden Soil

You are preparing a vegetable garden that is 40 feet long and 16 feet wide ,and you need enough soil to fill it to a depth of 1 foot .The landscape supply stores sells soil by the cubic yard .How much soil should you order?

Solution

Volume of soil = 40 feet ×16feet × 1 feet = 640 ft^3

$640\ \text{ft}^3 = 640\ \times \left(\frac{1}{3}\text{yd}\right)^3 = 640 \times \frac{1}{27}\ \text{yd}^3 = 23.7\text{yd}^3$

You will need to order about 24 cubic yards of soil for your garden

42. The Kentuchy Derby

The length of Kentucky derby horse race is 10 furlongs. How long is race in mile?

Solution

$1\ \text{furlong} = \frac{1}{8}\ \text{mi} = 0.125\text{mi}$ then $10\ \text{furlong} = 10 \times 0.125\text{mi} = 1.25$ mile

43. 20,000 League Under the Sea

In Jules Verne's Novel 20,000 League Under the Sea, does the title refer to an ocean depth? How do you know?

Solution

1 league $=$ 3 nautical miles

20,000 league $= 3 \times 20000$ nautical miles = 60000 nautical miles

This distance is several time the diameter of Earth, so 20000 League cannot possibly refer to an ocean depth. The book's title refers to the distance travelled by Captain Nemo's submarine, Nautilus.

44. Find a conversion factor between square feet and Square inches. Write it in three forms?

45.Solution

$1\ \text{ft} = 12\ \text{in.} \Rightarrow 1\ \text{ft}^2 = 144\ \text{in}^2.$

So $1\ \text{ft}^2 = 144\ \text{in}^2.$ Or $\frac{1\text{ft}^2}{144\text{in}^2.}$ Or $\frac{144\text{in}^2.}{1\text{ft}^2}$

46. Find a conversion factor between cubic meters and cubic centimeters. Write it in three forms?

Solution

$1\text{ m} = 100\text{ cm} \Rightarrow\ 1\text{ m}^3 = 1000000\text{ cm}^3$

So $1\text{ m}^3 = 1000000\text{ cm}^3$ or $\frac{1\text{ m}^3}{1000000\text{ cm}^3}$ or $\frac{1000000\text{ cm}^3}{1\text{ m}^3}$

47. A new sidewalk will be 4 feet wide, 200 feet long, and filled to a depth of 6 inches (0.5 feet) with concrete. How many cubic yards of concretes are needed?

Solution

$\text{cubic} = 4 \times 200 \times 0.5 = 400\text{ft}^3 \Rightarrow\ 27\text{ ft}^3 = 1\text{ yd}^3 \Rightarrow \frac{400}{27} = 14.8\text{ yd}^3$

48. Find the area in square feet of a rectangular yard that measures 20 yards by 12 yards?

Solution

$1\text{yd} = 3\text{ft} \Rightarrow 1\text{ yd}^2 = 9\text{ ft}^2$

$\text{Area} = 20\text{yd} \times 12\text{yd} = 240\text{ yd}^2$

$\Rightarrow\ 240\text{ yd}^2 = 240\ \times 9\text{ ft}^2 = 2160\text{ft}^2$

49. An air conditioning system can circulate 320 cubic feet of air per minute. How many cubic yards of air can it circulate per minute?

Solution

$1\text{ yd}^3 = 27\text{ ft}^3 \Rightarrow \frac{320\text{ ft}^3}{27\text{ ft}^3\text{yd}^{-3}} = 11.85\text{ yd}^3$. It can circulate about 11.9 yd^3 per minute.

50. A hot tub pump circulates 4 cubic feet of water per minute. How many cubic inches of water does it circulate each minute?

Solution

$12\text{ in.} = 1\text{ ft} \Rightarrow 4\text{ft} = 12 \times 4 = 48\text{ in.} \Rightarrow 4\text{ft}^3 = 6912\text{ in}^3$.

The International Metric System

SI, which is the official abbreviation in all languages for the System International d'Unites, is an extension and refinement of the traditional metric system. More than 30 countries have declared it to be the only legally accepted system. It was invented in France late 18th century for two primary reasons;

To replace many customary units with just a few units.

To simplify conversions through use of a decimal (base 10) system.

Common Metric Prefixes

Small Values			Small Values		
prefix	Abbrev.	Value	prefix	Abbrev.	Value
deci	d	10^{-1}	deca	da	10^{1}
centi	c	10^{-2}	hecto	h	10^{2}
milli	m	10^{-3}	kilo	k	10^{3}
micro	μ	10^{-6}	mega	M	10^{6}
nano	n	10^{-9}	giga	G	10^{9}
pico	p	10^{-12}	tera	T	10^{12}

Conversion in MLtT with Dimensions

Quantity	Formula	Unit (SI System)	Dimension
Power	$P = \frac{W}{t} = \frac{Fd}{t} = \frac{ma.d}{t}$	$P = \frac{kg \times ms^{-2} \times m}{s} = \frac{kgm^2}{s^3}$	$\left[\frac{ML^2}{t^3}\right]$
Pressure	$P = \frac{F}{A} = \frac{ma}{A}$	$P = \frac{kg \times ms^{-2}}{m^2} = \frac{kg}{ms^2}$	$\left[\frac{M}{Lt^2}\right]$
Modulus of Elasticity	$\frac{\text{Stress}}{\text{Strain}} = \frac{F/A}{\frac{\text{Change inlayer}}{\text{origional layer}}} = \frac{F}{A}$	$\frac{kgms^{-2}}{m^2} = \frac{kg}{ms^2}$	$\left[\frac{M}{Lt^2}\right]$
Momentum	$P = mv$	$P = \frac{kgm}{s}$	$\left[\frac{ML}{t}\right]$
K.E	$K.E = \frac{1}{2}mv^2$	$K.E = kg(ms^{-1})^2$ $K.E = \frac{kgm^2}{s^2}$	$\left[\frac{ML^2}{t^2}\right]$

51. Which unit in the metric system is one-thousandth of a meter?

Answer

The millimeter is one-thousandth of a meter.

52. Which unit in the metric system is equal to one thousand grams?

Answer

The kilogram is equal to one thousand grams.

53. Convert 2759 centimeters to meters.

Solution

$$1\text{m} = 100\text{cm} \Rightarrow 1\text{cm} = \frac{1}{100}\text{m} \Rightarrow 2759\text{cm} = 2759 \times \frac{1}{100}\text{m} = 27.59\text{m}$$

54. How many nanoseconds are in a microsecond?

Solution

We compare the quantities by dividing the longer time (microsecond) by the shorter time (nanosecond).

$$\frac{1\mu\text{s}}{1\text{ns}} = \frac{10^{-6}\text{s}}{10^{-9}\text{s}} = 10^{-6+9}\text{s} = 10^{3}\text{s}$$

There is 1000 nanosecond in a micro second.

55. Solve an Application Involving Metric Units

The thickness of a single sheet of paper is 0.07 mm. Find the height in centimeters of a ream of paper. A ream is 500 sheets of paper.

Solution

0.07(500) = 35

35 mm = 3.5 cm

The height of a ream of this paper is 3.5 cm.

37. Conversion

a. 4.08 m to centimeters **b.** 5.93 g to milligrams

c. 82 ml to liters **d.** 9 kl to liters

Solution

a. Write the units of length from largest to smallest.

Km hm dam m dm cm mm

Converting from meter to centimeter requires moving 2 places to the right

4.08 m = 408 cm

b. Write the units of mass from largest to smallest.

Kg hg dag g dg cg mg

Converting from gram to milligram requires moving 3 places to the right.

5.93 g = 5930 mg

c. Write the units of capacity from largest to smallest.

Kl hl dal L dl cl ml

Converting from mililiter to liter requires moving 3 places to the left.

82 ml = 0.082 L

d. Write the units of capacity from largest to smallest.

Kl hl dal L dl cl ml

Converting from kiloliter to liter requires moving 3 places to the right.

9kl = 9 000 L

38. Question

Complete the following sentence with the number. All chances should be greater than 1

- **A meter is 1000 times as large as a millimeter.**

$\frac{1\text{m}}{1\text{mm}} = \frac{10^0\text{m}}{10^{-3}\text{m}} = 10^{0-(-3)} = 10^{0+3} = 10^3 = 1000$.

Hence the statement.

- **A kilogram is 1,000,000 times as large as a milligram.**

$\frac{1\text{kg}}{1\text{mg}} = \frac{10^3\text{g}}{10^{-3}\text{g}} = 10^{3-(-3)} = 10^{3+3} = 10^6 = 1000{,}000.$

Hence the statement.

- **A liter is 1000 times as large as a milliliter.**

$\frac{1\text{L}}{1\text{mL}} = \frac{10^0\text{L}}{10^{-3}\text{L}} = 10^{0-(-3)} = 10^{0+3} = 10^3 = 1000.$

Hence the statement.

- **A kilometer is 1,000,000,000 times as large as a micrometer.**

$\frac{1\text{km}}{1\mu\text{m}} = \frac{10^3\text{m}}{10^{-6}\text{m}} = 10^{3-(-6)} = 10^{3+6} = 10^9 = 1000{,}000{,}000.$

Hence the statement.

- **A square meter is 10,000 times as large as a Square centimeter.**

$\frac{1\text{m}^2}{1\text{cm}^2} = \frac{1\text{m}^2}{(10^{-2}\text{m})^2} = \frac{10^0\text{m}^2}{10^{-4}\text{m}^2} = 10^{0-(-4)} = 10^4 = 10{,}000.$

Hence the statement.

- **A cubic meter is 1,000,000,000 times as large as a cubic millimeter.**

$\frac{1\text{m}^3}{1\text{mm}^3} = \frac{1\text{m}^3}{(10^{-3}\text{m})^3} = \frac{10^0\text{m}^3}{10^{-9}\text{m}^3} = 10^{0-(-9)} = 10^9 = 1000{,}000{,}000$

Hence the statement.

Metric U.S Customary System of Measurements (USCS) Conversions

We carry out conversions between metric and USCS units like any other unit conversions. Table lists a few handy Conversions factors. It's useful to memorize approximate conversions, particularly if you plan to travel internationally or if you work with metric units in sports and business. For example, if you remember that a kilometer is about 0.6 mile, you will know that a 10 kilometer road race is about 6 miles. Similarly, if you remember that a meter is about 10% longer than a yard, you'll know that a 100 meter race is about the same as a 110 yard race.

USCS – Metric Conversions

USCS to Metric	Metric to USCS
1 in = 2.540 cm	1 cm = 0.3937 in
1 ft = 0.3048 m	1 m = 3.28 ft
1 yd = 0.9144 m	1 m = 1.094 yd
1 mi = 1.6093 km	1 km = 0.6214 mi
1 lb = 0.4536 kg	1 kg = 2.205 lb
1 fl oz = 29.574 mL	1 mL = 0.03381 fl oz
1 qt = 0.9464 L	1 L = 1.057 qt
1 gal = 3.785 L	1 L = 0.2642 gal

39. Marathon Distance

The Marathon running race is about 26.2 miles. About how far is it in kilometers?

Solution

We know that 1 mi = 1.6093km.

$26.2\text{mi} \times 1.6093\text{km} = 42.2\text{km}$

Rounded to one decimal place, a marathon is 42.2 kilometers.

40. Square kilometers are in one square mile?

Solution

We know that 1 mi =1.6093 km.

Squaring both sides $(1\text{ mi})^2 = (1.6093\text{ km})^2$

$1\text{ mi}^2 = 2.5898\text{km}^2$

One square mile is approximately 2.6 square kilometers.

41. The Kentucky Derby distance is 10 furlongs. How far is the Kentucky Derby in (a)Rods (b)Fathoms?

Solution

$$10\text{ furlongs} = 10\text{ furlongs} \times \frac{1\text{ mi}}{8\text{ furlongs}} \times \frac{5280\text{ ft}}{1\text{ mi}} \times \frac{1\text{ yd}}{3\text{ ft}} \times \frac{1\text{ rod}}{5.5\text{ yd}} = 400\text{ rods}$$

$$10\text{ furlongs} = 10\text{ furlongs} \times \frac{1\text{ mi}}{8\text{ furlongs}} \times \frac{5280\text{ ft}}{1\text{ mi}} \times \frac{1\text{ fathom}}{6\text{ ft}} = 1100\text{ fathoms}$$

42. The depth of the Challenger deep is 36,198 feet. How deep is it in (a)Fathoms (b)Leagues (marine)?

Solution

$$36{,}198\text{ feet} = 36{,}198\text{ ft} \times \frac{1\text{ fathom}}{6\text{ ft}} = 6033\text{ fathoms}$$

$$36{,}198\text{ feet} = 36{,}198\text{ ft} \times \frac{1\text{ naut.mile}}{6076.1\text{ ft}} \times \frac{1\text{ marine league}}{3\text{ naut.mile}} = 1.99\text{ marine leagues}$$

43. One cubic foot holds 7.48 gallons of water, and one gallon of water weighs 8.33 pounds. How much does a cubic foot of water weigh in pounds?
In ounces (avoirdupois)?

Solution

$1\text{ gallon} = 8.33\text{ lb} \Rightarrow 7.48\text{ gallon} = 7.48 \times 8.33\text{ lb} = 62.30\text{ lb}$

$1\text{ lb} = 16\text{ oz} \Rightarrow 62.30\text{ lb} = 62.30 \times 16\text{ oz} = 996.8\text{ oz}$

44. Suppose you bought 10 six packs of soda, each six pack containing six 12-ounce cans. How many gallons of soda did you buy?

Solution

$\text{amount of soda} = 10 \times 6 = 60\text{ packs} \Rightarrow \text{amount of soda} = 60 \times 12\text{ oz} = 720\text{ oz}$

$1\ \text{ gallon} = 128\text{ oz} \Rightarrow \text{ten 6 pack to gallon} = \frac{720}{128} = 5.625$

45. The price of gasoline is \$2.89 per gallon. Find the price per liter. Round to the nearest cent.

Solution

$$\$2.89\text{ per gallon} = \frac{\$2.89\text{ per}}{\text{gallon}} = \frac{\$2.89\text{ per}}{1\text{ gallon}} \times \frac{1\text{ gallon}}{3.79\text{ L}} \approx \$0.76\text{ per liter}$$

46. A speed boat has a top speed of 46 knots (nautical miles per hour). What is this speed in miles per hour?

Solution

$$46\text{ knots} = \frac{46\text{ naut.mile}}{\text{hour}} \times \frac{6076.1\text{ ft}}{1\text{ naut.mile}} \times \frac{1\text{ mile}}{5280\text{ ft}} = 52.94\text{ miles per hour}$$

47. The price of gasoline is $.512 per liter. Find the price per gallon. Round to the nearest cent.

Solution

$$\$.512 \text{ per liter} = \frac{\$.512}{\text{L}} \times \frac{3.79 \text{ L}}{\text{gal}} \approx \$1.94 \text{ per gallon}$$

Energy

Energy is defined as the ability to do work. Energy is stored in coal, in gasoline, in water behind a dam, and in one’ s own body.

One foot-pound (1 ft-lb) of energy is the amount of energy necessary to lift 1 pound a distance of 1 foot. One ft-lb is read as 1 foot pound.

To lift 500 lb a distance of 3 ft requires $(3\text{ ft})(500\text{ lb}) = 150\text{ ftlb}$ of energy

48. Find the energy required for a 150-pound person to climb a mile-high mountain.

Solution

In climbing the mountain, the person is lifting 150 lb a distance of 5280 ft.

$\text{Energy} = (5280\text{ ft})(150\text{ lb}) = 792{,}000\text{ ft} - \text{lb}$

The energy required is 792,000 ft-lb.

Note

Consumer items that use energy, such as furnaces, stoves, and air conditioners, are rated in terms of the British thermal unit (Btu). For example, a furnace might have a rating of 35,000 Btu per hour, which means that it releases 35,000 Btu of energy in 1 hour.

Because 1 Btu is approximately 778 ft-lb, the following conversion rate, equivalent to 1, is used: $\frac{778\text{ ft}-\text{lb}}{1\text{ Btu}}$

49. A furnace is rated at 80,000 Btu per hour. How many foot-pounds of energy are released in 1 h?

Solution

$$80{,}000\text{ Btu} = \frac{80{,}000\text{ Btu}}{1} \times \frac{778\text{ ft}-\text{lb}}{1\text{ Btu}} = 62{,}240{,}000\text{ ft} - \text{lb}$$

The furnace releases 62,240,000 ft-lb of energy in 1 h.

50. Convert.

a. 200 m to feet b. 45 mph to kilometers per hour

Solution

$200\text{ m} = 200\text{ m} \times 3.28\text{ ft} = 656\text{ ft}$

$45\text{ mph} = 45\text{mph} \times 1.61\text{ km} = 72.45\text{ km/h}$

51. How many cords a wood could you fit in a room that is 4 yards long, 4 yards wide and 2 yards high?

Solution

$\text{volume} = 4yd \times 4yd \times 2yd = 32\text{ }yd^3$

$32\text{ yd}^3 = 32\text{ yd}^3 \times \left(\frac{3\text{ ft}}{1\text{ yd}}\right)^3 \times \frac{1\text{ cord}}{5128\text{ ft}^3} = 6.75\text{ cords}$

52. Question

- **22 kilograms to pounds?**

  $22kg = 22 \times 2.205\text{lb} = 48.51\text{ lb}$

- **160 cm to inches?**

  $160\ cm = 160 \times \frac{1}{2.54}\text{in.} = 62.99\ in.$

- **16 quarts to liters?**

  $16\ qt = 16 \times \frac{1}{1.057}\text{L} = 15.14\text{ L}$

- **2 square kilometers to square miles?**

  $2\ km^2 = 2 \times 0.38614\text{ mi}^2 = 0.77\ \text{mi}^2$

- **55 miles per hour to kilometers per hour?**

  $55\frac{\text{mi}}{\text{h}} = 55\frac{\text{mi}}{\text{h}} \times 1.6093\frac{\text{km}}{\text{mi}} = 88.51\frac{\text{km}}{\text{h}}$

- **23 meters per second to miles per hour?**

  $23\frac{\text{m}}{\text{s}} = 23\frac{\text{m}}{\text{s}} \times \frac{1\text{ km}}{1000\text{ m}} \times \frac{1\text{ mi}}{1.6093\text{ km}} \times \frac{60\text{ s}}{1\text{min}} \times \frac{60\text{ min}}{\text{h}} = 51.45\frac{\text{mi}}{\text{h}}$

- **300 cubic inches to cubic centimeters?**

  $1\text{ in.} = 2.54\text{ cm} \Rightarrow 1\text{ in}^3. = 16.387\text{ cm}^3$

  $300\text{ in}^3. = 300 \times 16.387\text{ cm}^3 = 4916.12\text{ cm}^3$

- **18 grams per cubic centimeter to pounds per cubic inch?**

  $18\frac{\text{g}}{\text{cm}^3} = 18\frac{\text{g}}{\text{cm}^3} \times \frac{1\text{ kg}}{1000\text{ g}} \times \frac{2.205\text{ lb}}{1\text{ kg}} \times \left(\frac{2.54\text{ cm}}{1\text{ in.}}\right)^3 = 0.650\frac{\text{lb}}{\text{in}^3.}$

Standardized Temperature Units

Another important set of standardized units are those we use to measure temperature. There temperature scales are commonly used.

- **The Fahrenheit Scale**, commonly used in the United States, is defined so water freezes at 32°F and boils at 212°F.
- The rest of the word uses **the Celsius Scale**, which places the freezing point of water at 0°C and the boiling point at 100°C.
- In science, we use **the Kelvin Scale** which is the same as Celsius scale except for its zero point, which corresponds to−273.15°C. A temperature of 0 K is known as **absolute zero**, because it is the coldest temperature. (The degree symbol [°] is not used on the Kelvin scale).

To convert From	**Conversion Formula**
Celsius to Fahrenheit	$F = 1.8C + 32$
Fahrenheit to Celsius	$C = \frac{F - 32}{1.8}$
Celsius to Kelvin	$K = C + 273.15$
Kelvin to Celsius	$C = K - 273.15$

53. Human Body Temperature

Average human body temperature is 98.6°F.

What is it in the Celsius and Kelvin?

Solution

$C = \frac{F-32}{1.8} = \frac{98.6-32}{1.8} = \frac{66.6}{1.8} = 37.0°\text{C}$

$K = C + 273.15 = 37 + 273.15 = 310.15°K$

Human body temperature is 37°C or 310.15K

54. Convert the temperature 125.5 K in Degree Fahrenheit and Degree Celsius.

Solution

Temperature = 125.5 K

$K = C + 273.15 \Rightarrow 125.5 = C + 273.5 \Rightarrow C = 125.5 - 273.5 = -148°C$

$F = 1.8C + 32 \Rightarrow F = 1.8 \times -148 + 32 \Rightarrow F = -234.4°F$

55. Convert from Celsius-Fahrenheit and Celsius-Kelvin.

(a) 45°F (b) 20°C (c) 50K (d) -10°C

a) $C = \frac{F-32}{1.8} = \frac{45-32}{1.8} = 7.22°C$

b) $F = 1.8C + 32 = 1.8 \times 20 + 32 = 68°F$

c) $C = K - 273.15 = 50 - 273.5 = -223.15°C$

d) $K = C + 273.15 = 10 + 273.5 = -283.15$ K

Currency Conversion

Sample Currency Exchange Rate

Currency	Dollars Per Foreign	Foreign Per Dollar
British Pound	1.624	0.6158
Canadian Dollar	1.005	0.9950
European Euro	1.320	0.7576
Japanese Yen	0.0120	83.33
Mexican Peso	0.07855	12.73

56. At a French department store, the price for a pair of jean is 45 euros. What is the price in US dollars? If 1 Euro = 1.320 Dollars.

Solution

1 Euro = 1.320 Dollars

€45 = 45 × \$1.320 = \$59.40

57. You are on holiday in Mexico and need cash. How many pesos can you buy with \$100? If \$1 = 12.73 Peso.

Solution

\$1 = 12.73 Peso

\$100 = 100 × 12.73 Peso = 1273 Peso

58. (CAD: Canadian Dollars)

A gas station in Canada sells gasoline for CAD 1.34 per liter. What is the price in dollars per gallon?

Solution

1 CAD = $1.005; 1 gallon = 3.785 liter

Gasoline per gallon = 1.34 × 1.005 × 3.785 = $5.10 per gallon

59. Conversions

- Your dinner in London costs 60 British pounds. How much was it in U.S dollars?
  $1\text{ Pound} = \$1.624 \Rightarrow 60\text{ Pound} = 60 \times \$1.624 = \$97.44$

- Your hotel rate in Tokyo is 31,000 yen per night. What is the nightly rate in U.S dollars?
  $1\text{ Yen} = \$0.0120 \Rightarrow 31{,}000\text{ Yen} = 31{,}000 \times \$0.0120 = \$372.00$

- As you leave Paris, you convert 450 Euros to dollars. How much dollars do you receive?
  $1\text{ Euro} = \$1.320 \Rightarrow 450\text{ Euro} = 450 \times \$1.320 = \$594.00$

- You return from Mexico with 3000 pesos. How much are they worth in U.S dollars?
  $1\text{ Peso} = \$0.07855 \Rightarrow 3000\text{ Peso} = 3000 \times \$0.07855 = \$235.65$

- Gasoline sells for 1.5 euros per liter in Bonn. What is the price in US dollars per gallon?
  $1\text{ Euro} = \$1.320 \Rightarrow 1.5\text{ Euro} = \frac{1.5\text{ Euro}}{\text{L}} \times \frac{3.785\text{ L}}{\text{gal}} \times \$1.320 = \frac{\$7.49}{\text{gal}}$

- You purchase fresh strawberries in Mexico for 28 pesos per kilogram. What is the price in US dollars per pound?
  $1\$ = 12.73\text{ Pesos} \Rightarrow 28\text{ Peso} = \frac{28\text{ Peso}}{\text{kg}} \times \frac{1\ \$}{12.73\text{ Pesos}} \times \frac{1\text{ kg}}{2.205\text{ lb}} = \frac{\$1.00}{\text{lb}}$

60. In July the federal minimum wage was $5.15 per hour, and the minimum wage in California was $6.75 per hour. How much grater is an employ's pay for working 35 hours and earning the California minimum wage rather than the federal minimum wage?

Solution

Hourly difference in wage = $\$6.75 - \$5.15 = \$1.60$

Employ's pay for working 35 hours = $\$1.60 \times 35 = \$56$

An employ's pay for working 35 hours and earning the California minimum wage is $56 grater.

61. Solve an application using exchange rates

a. How many Japanese Yen are needed to pay for an order costing $10,000?

b. Find the number of British pounds that would be exchanged for $5000?

Solution

Required Yen $= \$10{,}000 \times 140 = 1400000$ JPY

Required British pounds $= \$5000 \times 0.83 = 4150$ BP

62. Solve an application using exchange rates

a. How many Canadian Dollars are needed to pay for an order costing $20,000?

b. Find the number of Euros that would be exchanged for $25,000?

Solution

Required Canadian Dollars $= \$20{,}000 \times 1.1616 = \$23232$

Required Euros $= \$25{,}000 \times 0.92 = 2300$ EUR

63. The table below shows the exchange rates per U.S. dollar for four foreign countries on December 2, 2005. Use this table for Exercises 25 to 28.

a. How many Danish kroner are equivalent to $10,000?
b. Find the number of Indian rupees that would be exchanged for $45,000.
c. Find the cost, in Mexican pesos, of an order of American computer hardware costing $38,000.
d. Calculate the cost, in Australian dollars, of an American car costing $29,000.

Solution

a. Danish kroner equivalent to $10,000 $= 6.3652 \times 10{,}000 = 63652$ kroner
b. Indian rupees exchanged for $45,000 $= 46.151 \times 45{,}000 = 2076795$
c. Required cost $= 10472 \times 38{,}000 = 397936$ Pesos
d. Cost of American Car $= 1.3359 \times 29{,}000 = 38741.1$

Exchange Rates Per US Dollar	
Australian Dollar	1.3359
Danish Krone	6.3652
Indian Rupee	46.151
Mexican Peso	10.472

64. The median salary for the New York Yankees in 2008 was $1,875,000. Assuming a 160 game season express this salary in dollars per game?

Solution

Salary = $1,875,000; Game seasons = 160

Salary in dollars per game $= \frac{\$1{,}875{,}000}{160 \text{ games}} = \frac{\$11.719}{\text{games}}$

Unit Analysis as Problem Solving Tool

Identify the units involved in the problem and the units that you expect for the answer.

Use the given units and the expected answer units to help you find a strategy for solving the problem.

Remember, you cannot add or subtract numbers with different units, but you can combine different units through multiplication, division or rising to power.

65. A car travels 25 miles every hour. How fast is it going?

Solution

$$S = \frac{v}{t} = \frac{25\text{mi}}{\frac{1}{2}\text{h}} = 25 \times 2\text{mi. h}^{-1} = 50\text{mi. h}^{-1}$$

66. An Airliner travels 45 miles in 5 minutes. What is its speed in miles per hour?

Solution

$$S = \frac{v}{t} = \frac{45\text{ mi}}{5\text{ minute}} = \frac{45\text{ mi}}{5\text{ minute}} \times \frac{60\text{ miute}}{\text{hour}} = 540\text{ mi. h}^{-1}$$

67. Competition speed Skydivers have reached record speeds of 614 miles per hour. At this speed how many feet would you fall every second?

Solution

Given speed = 614 miles per hour

$$\text{Converted speed} = \frac{614\text{ miles}}{\text{hour}} \times \frac{1\text{ hour}}{60\text{ minute}} \times \frac{1\text{ minute}}{60\text{ seconds}} \times \frac{5280\text{ feet}}{\text{minute}} = 901\frac{\text{ft}}{\text{sec}}$$

68. You are buying 30 acres of farm land at $12000 per acre. What is the total cast?

Solution

Per acre cost = $12000 per acre

Total cost for 30 acres = 30 × 12000 = $360, 000

69. What is the total cost of 1.2 cubic yards of soil if it sells for $24 per cubic yard?

Solution

Cubic yards = 1.2

Sell per cubic yard = $24

Total cost = 1. 2 × 24 = $28.80

70. You are a grader for math course an exam questions reads Eli purchased 5 pounds of apples at a price of 50 cents per pound how much did he pay for the apples ? On the paper you are grading a student has written 50 ÷ 5=10 he paid 10 cents .write a note to the student explaining what went wrong?

Solution

5 pounds of apples at a price of 50 cents per pound = $50\frac{c}{lb} \div 5lb = 50\frac{c}{lb} \times \frac{1}{5lb} = 10\frac{c}{lb^2}$

It cannot correct for a question that asks for a price. The correct calculation is as follows;

5 pounds of apples at a price of 50 cents per pound = $50\frac{c}{lb} \times 5lb = 250c = \$2.50$

71. You are planning to make pesto and need to buy basil. At the grocery store you can buy small containers of basil price at $ 2.99 for each 2/3 ounce container. At the farmer's market you can buy basil in bunches for $12 per pound which is a better deal?

Solution

1 pound = 16 ounce

Container price = $\frac{\$\,2.99}{\frac{2}{3}oz} \times \frac{16oz}{1lb} = \frac{\$71.16}{1lb}$

The small containers are priced at most $72 per pound. Which is six times as much as the farmer's market price.

72. How much would you pay for 2.5 ounces of gold at a price of $920 per ounce?

Solution

Price of gold per ounce = $920

Total pay = 2.5 × $920 = $2300

73. Your destination is 90 miles away and your fuel gauge shows that your gas tank is one quarter full. Your tank holds 12 gallons of gas and your car average is about 25 miles per gallon do you need to stop for gas?

Solution

Present fuel in car = one quarter of 12 gallons = 3 gallons

Total distance = 90 miles

Car average = 25 miles per gallon

Required fuel for 90 miles = $\frac{90 \text{ mi}}{25 \text{ mi per gallon}} = 3.6\text{gal}$

Since we have only 3 gallons fuel. So we should stop for gas.

74. During a long road trip, you drive for 420 miles on a 12 gallon tank of gas. What is your gas mileage (in miles per gallon)?

Solution

Distance = 420 miles; Tank of gas = 12 gallon

Gas mileage $= \frac{420 \text{ miles}}{12 \text{ galon}} = 35 \frac{\text{miles}}{\text{galon}}$

75. If your car gets 28 miles per gallon, how much does it cost to drive 250 miles when gasoline costs \$2.90 per gallon?

Solution

Distance = 28 miles per gallon; Gasoline cost = \$2.90 per gallon

Cost to drive 250 miles $= 250$ miles $\times \frac{1 \text{ gal}}{28 \text{ mi}} \times \frac{\$2.90}{1 \text{ gal}} = \$25.89$

76. A hose fills a hot tub at a rate of 3.2 gallons per minute. How many hours will it take to fill a 300 gallons hot tub?

Solution

Required time $= 300 \text{ gal} \times \frac{1 \text{ minute}}{3.2 \text{ gal}} \times \frac{1 \text{ hour}}{60 \text{ minute}} = 1.56 \text{ hr}$

77. Suppose you earn \$8.5 per hour and work 24 eight hour days in a month. How much do you earn in that month?

Solution

Earning per hour = \$8.5 per hour; Work = 24 eight – hour days

Total earning in moth $= 24\text{days} \times \frac{8 \text{ hour}}{\text{day}} \times \frac{\$ 8.5}{\text{hour}} = \$1632$

78. You found that melting the Antarctica ice would release about 25 million cubic kilometers of water over Earth's 340 million square kilometers of ocean surface. Use unit analysis to find the amount that sea level will rise?

Solution

Rise in sea level $= \frac{25 \text{ million km}^3}{340 \text{ million km}^2} \approx 0.074 \text{ km} = 74 \text{ m}$

Sea level would rise by 74 meter, or about 240 feet.

79. As you ride an exercise bicycle that display states that you are using 500 calories per hour are you generating in a power to light a 100 watt bulb? (1 Calorie = 4184Joule)

Solution

We use a chain of conversions to go from calories per hour to joule per second.

$500\text{ cal.}h^{-1} = 500 \times 4184\text{ j } \times \frac{1}{3600}\text{sec} \approx 581\text{ js}^{-1}$

Electric Utility Bills

The watt-hour is used for measuring electrical energy. One watt-hour is the amount of energy required to lift 1 kg a distance of 370 m. A light bulb rated at 100 watts (W) will emit 100 watt-hours (Wh) of energy each hour.

Recall that the prefix kilo- means 1000.

1 kilowatt-hour (kWh) = 1000 watt-hours (Wh)

Utility bills electrical energy is usually measured in units of kilowatt hours.

$1000\text{ watt} = \frac{1000\text{ j}}{\text{s}} \times 1\text{ hr} \times \frac{60\text{ m}}{\text{h}} \times \frac{60\text{ s}}{\text{min}} = \frac{1000}{3600}\text{js}^{-1} = 3600{,}000\text{ joule}$

Kilowatt hour: a kilowatt hour is a unit of energy (1KWH = 3.6MJ)

80. Your utility company charges 15 ¢ per kilowatt hour of electricity how much does it cost to keep a 100 watt light bulb on per week? How much will you save in a year if you replace the bulb with an LED bulb that provide the same amount of light for only 25 watts of power?

Solution

Energy used by bulb in a week $= 100\ \frac{\text{watt}}{\text{week}}$

Energy used by bulb in a week $= 100\text{ watt} \times \frac{1\text{kw}}{1000\text{ watt}} \times 1\text{ week} \times \frac{7\text{days}}{\text{week}} \times \frac{24\text{hours}}{\text{day}}$

Energy used by bulb in a week $= 16.8\text{ kilowatt} - \text{hour}$

Energy charges 15 ¢ per kwh $= 16.8\text{ kilowatt} - \text{hour} \times \frac{15¢}{\text{kilowatt}-\text{hour}} = 252¢ = \$2.52$

If we replace bulb with 25 watt per week then we get $63¢ = \$0.63$ and our saving will be $\$2.52 - \$0.63 = \$1.89$. Thus

Per annum saving $= \frac{\$1.89}{\text{week}} = \frac{\$1.89}{\text{week}} \times \frac{52\text{ week}}{\text{year}} \approx \frac{\$99}{\text{yr}}$ (more efficient bulb)

81. A 150-watt bulb is on for 8 h. At 8¢ per kilowatt-hour, find the cost of the energy used.

Solution

Cost of Energy $= 150(8) = 1200$	Find the number of watt-hours used
$1\,200\text{ Wh} = 1.2\text{ kWh}$	Convert watt-hours to kilowatt-hours.
$1.2(0.08) = 0.096$	Multiply the number of kilowatt-hours used by the cost, in dollars, per kilowatt-hour.

The cost of the energy used is $.096.

82. Manhattan Island has a population of about 1.6 million people living in an area of about 57 Square kilometers. What is its population density if there were no high rise apartments? How much space would be available person?

Solution

Population density $= \frac{1.6\text{ million}}{57\text{ km}^2} = \frac{1600000\text{ people}}{57\text{ km}^2} \approx \frac{2800\text{ people}}{\text{km}^2}$

If there we no high rises, each resident would have $\frac{1\text{km}^2}{2800\text{ people}}$ of land. Then

$\frac{1\text{km}^2}{2800\text{ people}} = \frac{1\text{km}^2}{2800\text{ people}} \times \left(\frac{1000m}{\text{km}}\right)^2 \approx \frac{36\text{m}^2}{\text{person}}$

Without high rise, each person would have only 36 square meters. And no enough space for roads, schools and other common properties. Clearly Manhattan Island could not fit so many residents without high rises.

83. In 2008, 565,650 Americans died of (all forms of) Cancer. Assuming a population of 305 million what was the mortality rate in units of death per 1 lakh people?

Solution

Total deaths = 565,650

Total population = 305 million = 305000000

Mortality rate in units of death per 1 lakh people $= \frac{\text{death}}{\text{population}} \times 100000$

Mortality rate $= \frac{565650\text{ deaths}}{305000000\text{ people}} \times \frac{305000000\text{ people}}{3050\text{ groups of }100000} = \frac{185\text{ deaths}}{100{,}000\text{ people}}$

84. In 2008 about 310,000 Americans died of sudden cardiac attack (about half of all deaths from coronary heart disease). Assuming a population of 305 million, what was the mortality rate in units of death per 1 lakh people?

Solution

Total deaths = 310,000

Total population = 305 million = 305000000

Mortality rate in units of death per 1 lakh people $= \frac{\text{death}}{\text{population}} \times 100000$

Mortality rate $= \frac{310{,}000\text{ deaths}}{305000000\text{ people}} \times \frac{305000000\text{ people}}{3050\text{ groups of }100000} = \frac{102\text{ deaths}}{100{,}000\text{ people}}$

85. There were approximately 3 million births in the United States each year. Find the birth rate in units of births per minute?

Solution

Total birth = 3 million per year

Birth rate $= \frac{3000{,}000\ \text{birth}}{\text{year}} \times \frac{1\ \text{yr}}{365\ \text{days}} \times \frac{1\ \text{day}}{24\ \text{hours}} \times \frac{1\ \text{hour}}{60\ \text{minutes}} = 5.7\,\frac{\text{births}}{\text{minute}}$

86. A child weighing 15 kilograms has a bacterial ear infection. A Physician order treatment with amoxicillin at a dosage based on 30 milli grams per kilogram of body weight per day divided into doses every 12 hours.

a) How much amoxicillin should the child be prescribed every 12 hours?
b) If the medicine is to be taken in a liquid suspension with concentration 25 mg per *ml* how much should that child take every 12 hours?

Solution

Child weight = 15 kilogram
Daily 24 hour dosage = 30 milli grams per kilogram of body weight per day
Daily 12 hour dosage = 15 milli grams per kilogram of body weight per day

Dose every 12 hours $= \frac{15\ \text{mg}}{\text{kg}} \times 15\ \text{kg} = 225\ \text{mg}$

Liquid suspension with concentration = 25 mg per *ml*

Liquid Dose $= 225\ \text{mg} \div \frac{15\ \text{mg}}{\text{ml}} = 225\ \text{mg} \times \frac{1\ \text{ml}}{25\ \text{mg}} =$ 9ml

87. And average size man has about 5 liters (5000 milliliters) of blood and average 12 ounce can of Beer contain about 15 grams of alcohol (assuming the beer is about 6% alcohol by volume). If all the alcohol was immediately absorb into the blood stream what blood alcohol contain content would we find in an average size men who quickly drank a single can of beer? How much Beer would make him legally intoxicated (BAC of 0.08)?

Solution

Concentration in grams per milliliters $= \frac{15\ \text{g alchohol}}{5000\ \text{ml blood}} = 0.003\,\frac{\text{g}}{\text{ml}}$

Concentration in grams per 100 milliliters $= 0.003\,\frac{\text{g}}{\text{ml}} \times \frac{100}{100} = \frac{0.3\ \text{g}}{100\ \text{ml}}$

The men’s blood alcohol concentration would be $\frac{0.3\ g}{100\ ml}$ of blood. Almost four time the legal limit of $\frac{0.08\ g}{100\ ml}$. Therefore it would take only about one quarter of the can, or 3 ounce of beer, to reach the legal limit. This example points out how quickly and easily a person can become dangerously intoxicated.

88. If you sleep an average of 8 hours each night, how many hours do you sleep in a year?

Solution

Sleep = 8 hours; 1 year = 365 days

Sleep years $= 1\text{ yr} \times \frac{365\text{ days}}{\text{yr}} \times \frac{8\text{ hours}}{\text{day}} = 2920\text{ hours}$

89. A human heart beats about 70 times per minute. If an average human being lives to the age of 80 how many times does the average heart beat in a lifetime?

Solution

Lifetime to heart beat $= 1\text{ lifetime} \times \frac{80\text{ yr}}{\text{lifetime}} \times \frac{365\text{ days}}{\text{yr}} \times \frac{24\text{ hours}}{\text{day}} \times \frac{60\text{ minute}}{\text{hr}} \times \frac{70\text{ beats}}{\text{minute}}$

Lifetime to heart beat $= 2{,}943{,}360{,}000\text{ beats}$

90. A Candy store cells chocolate for $7.70 per Pound. The piece you want to buy weighs 0.11 pound. How much will it cost to the nearest cent?

Solution

Student solution = 0.11 ÷ 7.70 = $0.014 = 1.4¢ **wrong method**

Correct solution = 0.11 × 7.70 = $0.85 **right method**

91. You ride your bike up a steep Mountain Road at 5 miles per hour. How far do you go in 3 hours?

Solution

Student solution = 5 miles per hour ÷ 3 hour = 1.7 $\frac{\text{miles}}{\text{hr}^2}$ **wrong method**

Correct solution = 5 miles per hour × 3 hour = 15 miles **right method**

92. You can buy a 50 pound bag of floor for $11 or you can buy a 1 pound bag for $0.39. Compare the per pound cost for the large and small bags?

Solution

Student solution = 50 pound ÷ $11 = $4.55 per pound **wrong method no sense**

Correct solution = $11 ÷ 50 pound = $0.22 per pound **right method**

93. The average person needs 1500 Calories per day. A can of Coke contains 140 calories. How many cokes would you need to drink to fill your daily caloric needs?

Solution

Student solution $= \frac{1500\text{ Calories}}{\text{day}} \times \frac{140\text{ Calories}}{\text{coke}} = 210{,}000\frac{\text{Cal}^2}{\text{d.coke}}$ **wrong method**

Correct solution $= \frac{1500\text{ Calories}}{\text{day}} \times \frac{\text{coke}}{140\text{ Calories}} = 10.7\frac{\text{coke}}{\text{day}}$ **right method**

94. You can buy shampoo in a 6 ounce bottle for \$3.99 or in a 14 ounce bottle for \$9.49. Which one is best deal?

Solution

6 ounce bottle; $\frac{\$3.99}{6\text{ oz}} = \frac{\$0.665}{\text{oz}}$ **better deal**

14 ounce bottle; $\frac{\$9.49}{4\text{ oz}} = \frac{\$0.678}{\text{oz}}$

95. You can buy one dozen eggs for \$2.30 or 30 eggs for \$5.50. Which one is best deal?

Solution

12 eggs; $\frac{\$2.30}{12\text{ eggs}} = \frac{\$0.19}{\text{egg}}$

30 eggs; $\frac{\$5.50}{30\text{ eggs}} = \frac{\$0.18}{\text{egg}}$ **better deal**

96. You can fill a 15 gallon tank of gas for \$55.20 or buy gas for \$3.60/gallon. Which one is best deal?

Solution

15 gallon cost; $\frac{\$55.20}{15\text{ gal}} = \frac{\$3.68}{\text{gal}}$. So \$3.60/gallon is best deal.

97. You can rent a storage locker for \$32 per yield square per month or for \$2 per feet square per week. Which one is best deal?

98. Solution

$\$32\text{ per yield square per month} = \frac{\frac{\$32}{\text{yd}^2}\times\left(\frac{\text{yd}}{3\text{ft}}\right)^2}{30\text{ days}} = \$0.12/\text{ft}^2/\text{day}$ **Better deal**

$\$2\text{per feet square per week} = \frac{\$2/\text{ft}^2}{7\text{ days}} = \$0.29/\text{ft}^2/\text{day}$

99. You plan to take a 2000 Mile trip in your car which averages 32 miles per gallon. How many gallons of gasoline should you expect to use? Would a car that has only half the gas mileage (16 miles per gallon) require twice as much gasoline for the same trip? Explain.

Solution

Gas needed = $2000\text{ Miles} \div \frac{32\text{ Miles}}{\text{gal}} = 2000\text{ Miles} \times \frac{\text{gal}}{32\text{ Miles}} = 62.5\text{ gal}$

Yes, a car that has half the gas mileage would need twice as much gas. Halving the value of the denominator has the same effect as doubling the value of the fraction.

100. Two friends take a 3000 Mile cross country trip together, but they drive their own cars. Car A has a 12 gallon gas tank and averages 40 miles per gallon, while Car B has a 20 gallon gas tank and average is 30 miles per gallon. Assume both drivers pay an average of $3.90 per gallon of gas

a) What is the cost of one full tank of gas for Car A? for Car B?

b) How many tanks of gas do cars A and B each use for the trip?

c) How much tanks of gas do cars A and B each pay for gas for the trip?

Solution

a) Cost for one full tank of gas for car A $= 12\text{ gal} \times \frac{\$3.90}{\text{gal}} = \$46.80$

Cost for one full tank of gas for car B $= 20\text{ gal} \times \frac{\$3.90}{\text{gal}} = \$78.00$

b) Tank will used by car A $= 3000\text{ miles} \times \frac{\text{gal}}{40\text{ miles}} \times \frac{\text{tank}}{12\text{ gal}} = 6.25\text{ tank}$

Tank will used by car B $= 3000\text{ miles} \times \frac{\text{gal}}{30\text{ miles}} \times \frac{\text{tank}}{20\text{ gal}} = 5\text{ tank}$

c) Payment spent for car A $= 6.25\text{ tank} \times \frac{\$46.80}{\text{tank}} = \$292.50$

Payment spent for car B $= 5\text{ tank} \times \frac{\$78.00}{\text{tank}} = \$390.00$

101. Gas Mileage actually varies slightly with the driving speed of a car (as well as with highway versus city driving). Suppose your car averages 38 miles per gallons on the highway if your average speed is 55 miles per hour and it averages 32 miles per gallon on the highway if your average speed is 70 miles per hour.

a) What is the driving time for a 2000 mile trip if you drive at an average speed of 55 miles per hour what is the driving time at 70 miles per hour?

b) Assume a gasoline price of $3.90 per gallon. What is the gasoline cost for the 2000 mile trip if you drive at an average speed of 55 miles per the hour? What is the gasoline cost at 70 miles per hour?

Solution

a) Driving time travelling at 55 miles per hour $= 2000\text{ Miles} \div \frac{55\text{ Miles}}{\text{hr}}$

Driving time travelling at 55 miles per hour $= 2000\text{ Miles} \times \frac{\text{hr}}{55\text{ Miles}} = 36.36\text{h}^{-1}$

Driving time travelling at 70 miles per hour $= 2000\text{ Miles} \div \frac{70\text{ Miles}}{\text{hr}}$

Driving time travelling at 70 miles per hour $= 2000\text{ Miles} \times \frac{\text{hr}}{70\text{ Miles}} = 28.57\text{h}^{-1}$

b) Cost for 38 miles trip at 55 mph $= \frac{2000\text{ miles}}{38\text{ miles}/\text{gal}} \times \frac{\$3.90}{\text{gal}} = \$205.26$

Cost for 32 miles trip at 70 mph $= \frac{2000\text{ miles}}{32\text{ miles}/\text{gal}} \times \frac{\$3.90}{\text{gal}} = \$243.75$

102. Suppose your car averages 32 miles per gallon on the highway if your average speed is 60 miles per hour and averages 25 miles per gallon on the highway .If your average speed is 75 miles per hour?

a) What is the driving time for a 1500 Mile trip if you drive at an average speed of 60 miles per hour? What is the driving time at 75 miles per hour?

b) Assuming a gasoline price of $3.90 per gallon. What is the gasoline cost for a 1500 Mile trip if you drive at an every speed of 60 miles per hour? What is the gasoline cast at 75 miles per hour?

Solution

a) Driving time travelling at 60 miles per hour $= 1500\text{ Miles} \times \frac{\text{hr}}{60\text{ minutes}} = 25\text{ hr}$

Driving time travelling at 75 miles per hour $= 1500\text{ Miles} \times \frac{\text{hr}}{75\text{ minutes}} = 20\text{ hr}$

b) Cost for 32 miles trip at 60 mph $= 1500\text{ Miles} \times \frac{\text{gal}}{32\text{ miles}} \times \frac{\$3.90}{\text{gal}} = \$182.81$

Cost for 25 miles trip at 75 mph $= 1500\text{ Miles} \times \frac{\text{gal}}{25\text{ miles}} \times \frac{\$3.90}{\text{gal}} = \$234.00$

103. The Greenland Ice Sheet contains about 3 million cubic kilometers of ice. If completely melted, this ice would release about 2.5 million cubic kilometers of water, which would spread out over Earth's 340 million square kilometers of ocean surface. How much would sea level rise?

Solution

Sea level rise $= \frac{\text{volume of water}}{\text{earth surface area}} = \frac{2.5\times10^6\ \text{km}^3}{340\times10^6\ \text{km}^2} = 0.007\ \text{km}$ about 7 meters.

104. The greatest volcanic eruption in recorded history took place in 1815 on the Indonesian island of Sumbawa, when the volcano Tambora expelled an estimated 100 cubic kilometers of molten rock. Suppose all of the ejected material fell on a region with an area of 600 square kilometers. Find the average depth of the resulting layer of ash and rock?

Solution

Average depth $= \frac{\text{volume}}{\text{area}} = \frac{100\ \text{km}^3}{600\ \text{km}^2} = 0.167\ \text{km}$

105. Assume running consumes 100 Calories per mile. If you run 10 minutes miles. What is your average power output in watts during a 1 hour run?

Solution

Power consumption $= \frac{100\ \text{Cal}}{\text{mile}} \times \frac{\text{mile}}{10\ \text{min}} \times \frac{1\ \text{min}}{60\ \text{s}} \times \frac{8184\ \text{J}}{\text{Cal}} = \frac{8184{,}000\ \text{J}}{600\ \text{s}} = 697\text{W}$

106. Assume that riding a bike burns 50 Calories per mile. If you ride at a speed of 15 miles per hour, what is your average power output in watts?

Solution

Power consumption $= \frac{50\ \text{Cal}}{\text{mile}} \times \frac{15\ \text{mile}}{\text{hr}} \times \frac{1\ \text{hr}}{60\ \text{min}} \times \frac{1\ \text{min}}{60\ \text{s}} \times \frac{8184\ \text{J}}{\text{Cal}} = \frac{8184{,}000\ \text{J}}{600\ \text{s}} = 871.7\text{W}$

107. Assuming 365 day in a year. Your utility company charges 13¢ per kilowatt hour of electricity. What is the daily cost of keeping it 75 watt light bulb for 12 hours each day? How much will you save in a year if you replace the bulb with an LED bulb that provide the same amount of light using only 15 watts of power?

Solution

75 watt bulb's one day cost $= 75\text{W} \times \frac{12\ \text{hr}}{\text{day}} \times \frac{1\ \text{kW}}{1000\text{W}} \times \frac{\$0.13}{1\text{kW}-\text{hr}} = \$0.117$

15 watt bulb's one day cost $= 15\text{W} \times \frac{12\ \text{hr}}{\text{day}} \times \frac{1\ \text{kW}}{1000\text{W}} \times \frac{\$0.13}{1\text{kW}-\text{hr}} = \$0.0234$

Saving $= (\$0.117 - \$0.0234) \times 365\text{days} = \$34.16$ in one year

108. Suppose you have a cloth dryer that uses 4000 watts of power and you run it for an average of 1 hour each day. If you pay the utility company 14 cents per kilowatt hours of electricity, what is the daily cost to run your dryer? How much would you save in a year if you replace it with the more efficient model that uses only 2000 watts?

Solution

4000 watt daily average cost $= 4000\text{W} \times 1\text{ hr} \times \frac{1\text{ kW}}{1000\text{W}} \times \frac{\$0.14}{1\text{kW}-\text{hr}} = \$0.56$

2000 watt daily average cost $= 2000\text{W} \times 1\text{ hr} \times \frac{1\text{ kW}}{1000\text{W}} \times \frac{\$0.14}{1\text{kW}-\text{hr}} = \$0.28$

Saving $= (\$0.56 - \$0.28) \times 365\text{days} = \$3102.20$ in one year

109. A cube of wood measures 3 centimeters on a side and it weighs 20 grams? What is its density? Will it float in water?

Solution

Density $= \frac{\text{mass}}{\text{volume}} = \frac{20\text{g}}{27\text{cm}^3} = 0.74\frac{\text{g}}{\text{cm}^3}$

It will float in water because the density of the water is $1\frac{\text{g}}{\text{cm}^3}$

110. At room temperature is 0.1 cubic centimetre sample of plutonium weigh 1.98 grams. What is its density? Will it float in water?

Solution

Density $= \frac{\text{mass}}{\text{volume}} = \frac{1.98\text{g}}{0.1\text{cm}^3} = 19.8\frac{\text{g}}{\text{cm}^3}$

It will sink in water because the density of water is $1\frac{\text{g}}{\text{cm}^3}$

111. The land area of the United States is about 3.5 million square miles and the population is about 306 million people? What is the average population density?

Solution

Density $= \frac{\text{People}}{\text{Area}} = \frac{306{,}000{,}000\text{ people}}{3{,}500{,}000\text{ mi}^2} = 87\frac{\text{people}}{\text{mi}^2}$

112. The country with the greatest population density in Monaco, where approximately 32,500 people live in an area of 1.95 square kilometers. What is a population density of Monaco in people per square kilometers? Compare this density to that of the United States which is approximately 31 people per square kilometer?

Solution

Monaco′s Density $= \frac{\text{People}}{\text{Area}} = \frac{32{,}500\text{ people}}{1.95\text{ km}^2} = 6{,}700\frac{\text{people}}{\text{k m}^2}$

Which is about $\frac{6{,}700\frac{\text{people}}{\text{k m}^2}}{31\frac{\text{people}}{\text{k m}^2}} = 538$ or more than 500 times greater than that of United States.

113. New Jersey and Alaska have population of 8.7 million and 680,000 respectively. There areas are 7417 and 571,951 square miles respectively. Compute the population density of both States.

Solution

New Jersey's Density $= \frac{\text{People}}{\text{Area}} = \frac{8700{,}000\text{ people}}{7417\text{ mi}^2} = 1173\,\frac{\text{people}}{\text{mi}^2}$

Alaska's Density $= \frac{\text{People}}{\text{Area}} = \frac{680{,}000\text{ people}}{571{,}951\ \text{mi}^2} = 1.2\,\frac{\text{people}}{\text{mi}^2}$

Which is smaller than New Jersey's population density.

114. A standard DVD has a surface area of 134 square centimetres. Depending on formatting, it holds either 4.7 or 8.5 gigabytes. Find the data density in both cases?

Solution

4.7GB Density $= \frac{\text{byte}}{\text{Area}} = \frac{4.7\text{ GB}}{134\text{ cm}^2} \times \frac{10^9\text{ byte}}{1\text{ GB}} = 35{,}000{,}000\,\frac{\text{Byte}}{\text{cm}^2}$

8.5GB Density $= \frac{\text{byte}}{\text{Area}} = \frac{8.5\text{ GB}}{134\text{ cm}^2} \times \frac{10^9\text{ byte}}{1\text{ GB}} = 63{,}000{,}000\,\frac{\text{Byte}}{\text{cm}^2}$

115. The antihistamine Benadryl is often prescribed for allergies. A typical dose for a hundred pound person is 25 mg every 6 hours (a) Following this dosage how many 12.5 mg chewable tablets would be taken in a week? (b) Benadryl also comes in liquid form with a concentration of 12.5mg / 5mL. Following the prescribe dosage, how much liquid Benadryl should a 100-pound person take in a week?

Solution

a) 100 pound person dosage in a week $= 1\text{ week} \times \frac{7\text{ days}}{\text{week}} \times \frac{25\text{mg}}{6\text{hr}} \times \frac{24\text{hr}}{\text{day}} = 700\text{ mg}$

Chewable tablets taken in a week $= 700\text{ mg} \times \frac{1\text{tablet}}{12.5\text{mg}} = 56\text{ tablets}$

b) A 100 pound person should take $700\text{ mg} \times \frac{5\text{mL}}{12.5\text{mg}} = 280\text{mL}$ of liquid Benadryl.

116. Suppose a dose of 9000 units per kg of penicillin is prescribed every 6 hours for treatment of a bacterial infection. For penicillin, 400,000 units is equal to 250 mg

a) Express the dose in mg per kg of body weight.

b) How milligrams of penicillin would a 20 kg child take in one?

Solution

a) The dose is $\frac{9000\text{units}}{1\text{kg}} \times \frac{250\text{mg}}{400{,}000\text{units}} = 5.625\,\frac{\text{mg}}{\text{kg}}$

b) A child would take 4 doses in a day,

So a 20-kg child would receive $4 \times 20\text{ kg} \times \frac{5.625\text{mg}}{1\text{kg}} = 450\text{ mg}$

117. Blood Alcohol Content: A typical glass of wine contains about 20 grams of alcohol. Consider a 110 – pound women with approximately 4 litters (4000mL) of blood, who drink two glasses of wine.

(a) If all alcohol were immediately absorbed into her bloodstream, what would her blood alcohol content be? Explain why it is fortunate that, in reality, the alcohol is not absorbed immediately?

(b) Again assume all the alcohol is absorbed immediately, but now assume her body eliminates the alcohol (through metabolism) at a rate of 10 grams per hour. What is her blood alcohol content 3 hours after drinking the wine? Is it safe for her to drive at this time? Explain.

Solution

a) BAC is usually measured in units of grams of alcohol per 100 mL of blood. A woman who drinks two glass of wine, each with 20 grams of alcohol, has consumed 40 grams of alcohol. If she has 4000 mL of blood, her BAC is as follows;

$$\text{BAC} = \frac{40\text{g}}{4000\text{mL}} = \frac{0.01\text{g}}{\text{mL}} \times \frac{100}{100} = \frac{1\text{g}}{100\text{mL}}$$

It is fortunate that alcohol is not absorbed immediately, because if it were, the woman would most likely die. A **BAC** above $0.4\frac{\text{g}}{\text{mL}}$ is typically enough to induce comma or death.

b) If alcohol is eliminated from the body at a rate of 10g per hour, then after 3 hours, 30g would have been eliminated. This leaves 10g in the woman’s system, which means her BAC is

$$\text{BAC} = \frac{10\text{g}}{4000\text{mL}} = \frac{0.0025\text{g}}{\text{mL}} \times \frac{100}{100} = \frac{0.25\text{g}}{100\text{mL}}$$

This is well about the legal limit for driving, so it is not safe to drive. Of course this solution assumes the woman survives 3 hours of lethal levels of alcohol in her body, because we have assumed all the alcohol is absorbed immediately. In reality the situation is somewhat more complicated.

118. Blood Alcohol Content: Hard Liquor; 8 ounces of a hard liquor (such as whisky) typically contain about 70 grams of alcohol. Consider a 200 pound man with approximately 6L (6000mL) of blood, who quickly drinks 8 ounces of hard liquor.

(a) If all the alcohol were immediately absorbed into his blood stream, what would his blood alcohol content be? Explain why it is fortunate that, in reality, the alcohol is not absorbed immediately?

(b) Again assume all the alcohol is absorbed immediately, but now assume his body eliminates the alcohol (through metabolism) at a rate of 15 grams per hour. What is his blood alcohol content 4 hours after drinking the liquor? Is it safe for him to drive at this time? Explain.

Solution

a) BAC is usually measured in units of grams of alcohol per 100 mL of blood. A man who drinks 8 ounces of a hard liquor has consumed 70 grams of alcohol and with 6000 mL of blood, his BAC is as follows;

$$\text{BAC} = \frac{70\text{g}}{6000\text{mL}} = \frac{0.0117\text{g}}{\text{mL}} \times \frac{100}{100} = \frac{1.17\text{g}}{100\text{mL}}$$

It is fortunate that alcohol is not absorbed immediately, because if it were, the man would most likely die. A **BAC** above $0.4\frac{\text{g}}{\text{mL}}$ is typically enough to induce comma or death.

b) If alcohol is eliminated from the body at a rate of 15g per hour, then after 4 hours, 60g would have been eliminated. This leaves 10g in the man’s system, which means his BAC is

$$\text{BAC} = \frac{10\text{g}}{6000\text{mL}} = \frac{0.0017\text{g}}{\text{mL}} \times \frac{100}{100} = \frac{1.17\text{g}}{100\text{mL}}$$

This is well about the legal limit for driving, so it is not safe to drive. Of course this solution assumes the man survives 4 hours of lethal levels of alcohol in his body, because we have assumed all the alcohol is absorbed immediately. In reality the situation is somewhat more complicated.

PROPORTIONS

(RATES, RATIO AND PERCENTAGE)

Ratios

A comparison of two numbers or quantities in the same units.

Ratios can be written as a fraction $\frac{2}{3}$, with a colon $2:3$, or as two numbers separated by a the word to **2 to 3**."

For example, the ratio of children to students in a music class could be written as 20:30, which simplifies to 2:3. Also the fractional form of the ratio 12 to 5 is $\frac{12}{5}$.

Rates

A comparison of two number or quantities in different units and a special type of ratio where the two quantities are measured in different units.

For example, miles per hour are a rate that compares a number of miles in one hour. Rates can be expressed as a number or a percentage, and can be equal to any value, including negative numbers.

Unit Rate

A unit rate is a rate in which the number in the denominator is 1. To find a unit rate, divide the number in the numerator of the rate by the number in the denominator of the rate. For example, for the rate $\frac{135 \text{ miles}}{6 \text{ gal}} = 22.5$ the unit rate is $\frac{22.5 \text{ miles}}{\text{gal}}$.

1. **Calculate a Unit Rate**
   A dental hygienist earns $780 for working a 40-hour week. What is the hygienist's hourly rate of pay?
   Solution
   The hygienist's rate of pay is $\frac{\$780}{40 \text{ h}}$
   To find the hourly rate of pay, divide 780 by 40. i.e. 780 ÷ 40 = 19.5
   $\frac{\$780}{40 \text{ h}} = \frac{\$19.5}{\text{h}}$
   The hygienist's hourly rate of pay is $19.50 per hour.

2. Solve an Application of Unit Rates

A teacher earns a salary of $34,200 per year. Currently the school year is 180 days. If the school year were extended to 220 days, as is proposed in some states, what annual salary should the teacher be paid if the salary is based on the number of days worked per year?

Solution

Find the current salary per day.

$\frac{\$34{,}200}{180\text{ days}} = \frac{\$190}{\text{day}}$

Multiply the salary per day by the number of days in the proposed school year.

$\frac{\$190}{\text{day}} \times 220\text{days} = \$41{,}800$

The teacher's annual salary should be $41,800.

3. Determine the more economical purchase

Which is the more economical purchase? An 18 ounce jar of peanut butter priced at $3.49 or a 12 ounce jar of peanut butter priced $2.59?

Solution

Finding the unit price we have $\frac{\$3.49}{18\text{ oz}} = \frac{\$0.194}{\text{oz}}$ and $\frac{\$2.59}{12\text{ oz}} = \frac{\$0.1216}{\text{oz}}$ that is $\$0.194 < \$0.1216$. The item with the lower unit price is the more economical purchase. The more economical purchase is the 18 ounce jar of peanut butter priced at $3.49.

4. Determine a ratio in simplest form

A survey revealed that, on average, eighth-graders watch approximately 21 hours of television each week. Find the ratio, as a fraction in simplest form, of the number of hours spent watching television to the total number of hours in a week.

Solution

1 week to hours $= \frac{24\text{ h}}{\text{day}} \times 7\text{ days} = 168\text{ hours}$

Required ratio $= \frac{21\text{ h}}{\text{week}} = \frac{21\text{ h}}{168\text{ hours}} = \frac{1}{8}$

5. What does the rate given for Oklahoma mean?

Oklahoma's rate of 1.6 means that 1.6 out of every million people living in Oklahoma die in bicycle accidents.

6. Calculate the student–faculty ratio at Oregon State University. Round to the nearest whole number. Write the ratio using the word to.

University	Men	Women	Faculty
Oregon State University	7509	6478	1352
University of Oregon	6740	7710	798

Solution

Total number of students = 7509 + 6478 = 13,987

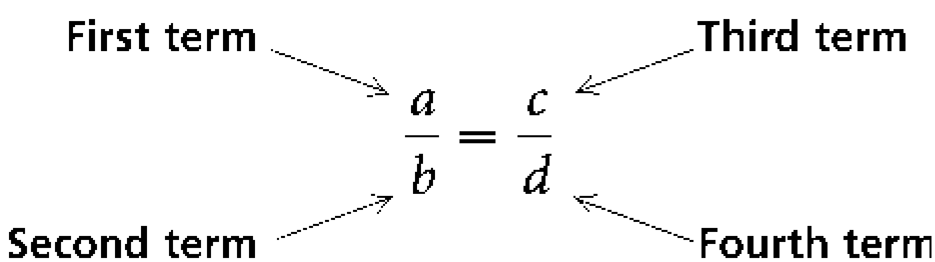
Required ratio $= \frac{13,987}{1352} \approx \frac{10.3454}{1} \approx \frac{10}{1}$

The ratio is approximately 10 to 1.

Proportions

A proportion is an equation that states that two ratios or rates are equal to each other or Equality of two ratios or rates.

Proportions are often denoted using the symbol "::" or "=". The definition of a proportion can be stated as follows: If $\frac{a}{b}$ and $\frac{c}{d}$ are equal ratios or rates, then $\frac{a}{b} = \frac{c}{d}$ is a proportion. Each of the four members in a proportion is called a **term**. Each term is numbered as shown below.



The second and third terms of the proportion are called the **means** and the first and fourth terms are called the **extremes**.

For example, if a train travels 100 kilometers per hour, it would take 5 hours to travel 500 kilometers. The proportion would be written as 100km/hr = 500km/5hrs. Also the proportion of 3 is to 5 as 12 is to 20 will be 3:5::12:20 or $\frac{3}{5} = \frac{12}{20}$

7. For the proportion $\frac{5}{8} = \frac{10}{16}$

a. Name the first and third terms
b. Write the product of the means
c. Write the product of the extremes.

Solution

a. The first term is 5. The third term is 10. b. The product of the means is 8(10) = 8
c. The product of the extremes is 5(16) = 80.

Cross-Products Method of Solving a Proportion

$$\text{If } \frac{a}{b} = \frac{c}{d} \text{ then } ad = bc.$$

8. Solve $\frac{8}{5} = \frac{n}{6}$

Solution

$\frac{8}{5} = \frac{\mathrm{n}}{6} \Rightarrow 5 \times \mathrm{n} = 8 \times 6 \Rightarrow 5\mathrm{n} = 48 \Rightarrow \mathrm{n} = \frac{48}{5} \Rightarrow \mathrm{n} = 9.6$

9. Solve $\frac{42}{x} = \frac{5}{8}$

Solution

$\frac{42}{\mathrm{x}} = \frac{5}{8} \Rightarrow 5 \times \mathrm{x} = 8 \times 42 \Rightarrow 5\mathrm{x} = 336 \Rightarrow \mathrm{x} = \frac{336}{5} \Rightarrow \mathrm{x} = 67.5$

10. If you travel 290 miles in your car on 15 gallons of gasoline, how far can you travel in your car on 12 gallons of gasoline under similar driving conditions?

Solution

$\frac{290 \text{ miles}}{15 \text{ gallons}} = \frac{\text{x miles}}{12 \text{ gallons}}$

$\Rightarrow \frac{290}{15} = \frac{\mathrm{x}}{12} \Rightarrow 15 \times \mathrm{x} = 290 \times 12 \Rightarrow 15\mathrm{x} = 3480 \Rightarrow \mathrm{x} = \frac{3480}{15} \Rightarrow \mathrm{x} = 232$

You can travel 232 miles on 12 gallons of gasoline.

11. On a map, a distance of 2 centimeters represents 15 kilometers. What is the distance between two cities that are 7 centimeters apart on the map?

Solution

$\frac{15 \text{ kilometers}}{2 \text{ centimeters}} = \frac{\text{x kilometers}}{7 \text{ centimeters}}$

$\frac{15}{2} = \frac{\mathrm{x}}{7} \Rightarrow 2\mathrm{x} = 105 \Rightarrow \mathrm{x} = \frac{105}{2} \Rightarrow \mathrm{x} = 52.5$

The distance between two cities 52.5 kilometers.

12. The table below shows three of the universities in the Big Ten Conference and their student–faculty ratios. (Source: Barron's Profile of American Colleges, 26th edition, c. 2005) There are approximately 31,100 full-time undergraduate students at Michigan State University. Approximate the number of faculty at Michigan State University.

University	Student – Faculty Ratio
Michigan State University	13 to 1
University of Illinois	15 to 1
University of Iowa	11 to 1

Solution

Let F = the number of faculty members.

Write a proportion and then solve the proportion for F.

$\frac{13\text{ students}}{1\text{ Faculty}} = \frac{31{,}100\text{ students}}{F\text{ Faculty}}$

$\Rightarrow \frac{13}{1} = \frac{31{,}100}{F} \Rightarrow 13F = 31{,}100 \Rightarrow F = \frac{31{,}100}{13} \Rightarrow F \approx 2392$

There are approximately 2392 faculty members at Michigan State University.

13. In the United States, the average annual number of deaths per million people aged 5 to 34 from asthma is 3.5. Approximately how many people aged 5 to 34 die from asthma each year in this country? Use a figure of 150,000,000 for the number of U.S. residents who are 5 to 34 years old.

Solution

Let D = the number of people aged 5 to 34 who die each year from asthma in the United States. Write and solve a proportion. One rate is 3.5 deaths per million people.

$\frac{3.5\text{ deaths}}{1000{,}000\text{ people}} = \frac{D\text{ deaths}}{150{,}000{,}000\text{ people}} \Rightarrow \frac{3.5}{1000{,}000} = \frac{D}{150{,}000{,}000}$

$\Rightarrow 1000{,}000D = 3.5 \times 150{,}000{,}000 \Rightarrow 1000{,}000D = 525{,}000{,}000$

$\Rightarrow D = \frac{525{,}000{,}000}{1000{,}000} \Rightarrow D = 525$

In the United States, approximately 525 people aged 5 to 34 die each year from asthma.

Earned Run Average

One measure of a pitcher's success is earned run average. Earned run average (ERA) is the number of earned runs a pitcher gives up for every nine innings pitched.

14. During the 2005 baseball season, Pedro Martinez gave up 68 earned runs and pitched 217 innings for the New York Mets. Calculate Pedro Martinez's ERA.

Solution

To calculate Pedro Martinez's ERA, let x equal the number of earned runs for every nine innings pitched. Write a proportion and then solve it for x.

$\frac{68 \text{ earned runs}}{217 \text{ innings}} = \frac{\text{x earned runs}}{9 \text{ innings}} \Rightarrow \frac{68}{217} = \frac{x}{9} \Rightarrow 217x = 68 \times 9 \Rightarrow 217x = 612$

$\Rightarrow x = \frac{612}{217} \Rightarrow x = 2.82$

Pedro Martinez's ERA for the 2005 season was 2.82

15. A Machinist earns \$490 for working a 35 hour week. What is the Machinist's hourly rate of pay?

Solution

Machinist's hourly rate of pay $= \frac{490}{35} = 14$ hours

16. Each of the Space Shuttle's solid rocket motors burns 680,400 kilograms of propellant in 2.5 minutes. How much propellant does each motor burn in 1 minute?

Solution

Motor burn in 1 minute $= \frac{680{,}400}{2.5} = 272160$ kg

17. During filming, an IMAX camera uses 65 mm film at a rate of 5.6 feet per second. At what rate per minute does the camera go through film? How quickly does the camera use a 500-foot roll of 65-mm film? Round to the nearest second.

Solution

IMAX camera rate $= \frac{65}{5.6} = 11.61$ sec

Rate per minute $= 11.61 \times 60 = 696.43$ sec

18. Which is the more economical purchase, a 32 ounce jar of mayonnaise for \$2.79, or a 24 ounce jar of mayonnaise for \$2.09?

Solution

First jar $= \frac{\$2.79}{32 \text{ oz}} = \frac{\$0.0872}{\text{oz}}$

Second jar $= \frac{\$2.09}{24 \text{ oz}} = \frac{\$0.0871}{\text{oz}}$ **more economical**

That is $\$0.0872 > \$0.0871$ approximately equal prices.

19. Which is the more economical purchase, and 18 ounce box of corn flakes for \$2.79, or a 24 ounce box of corn flakes for \$3.89?

Solution

First box $= \frac{\$2.79}{18\text{ oz}} = \frac{\$0.161}{\text{oz}}$

Second box $= \frac{\$3.89}{24\text{ oz}} = \frac{\$0.1621}{\text{oz}}$ **more economical**

That is $\$0.161 < \$0.1621$

20. You have a choice of receiving a wage of \$34,000 per year, \$2840 per month, \$650 per week or \$16.50 per hour. Which pay choice would you take?

Assume a 40 hour work week and 52 weeks of work per year?

Solution

Let's calculate the total annual salary for each option:

\$34,000 per year

No calculation needed, it's already an annual salary.

\$2,840 per month

\$2,840 × 12 months = \$34,080 per year

\$650 per week

\$650 x 52 weeks = \$33,800 per year

4. \$16.50 per hour

\$16.50 × 40 hours/week = \$660/week

\$660/week x 52 weeks = \$34,320 per year

Based on these calculations, the best option would be the hourly wage of \$16.50, which translates to an annual salary of \$34,320. This is slightly higher than the other options.

21. Baseball statisticians calculate a hitter's at-bats per home run by dividing the number of times the player has been at bat by the number of home runs the player has hit.

a) Calculate the at-bats per home run ratio for each player in the table on the following page. Round to the nearest tenth.

b) Which player has the lowest rate of at-bats per home run? Which player has the second lowest rate?

c) Why is this rate used for comparison rather than the number of home runs a player has hit?

Players with 50 or more Runs Per Season

Year	Baseball Player	Number of Times at Bat	Number of Home Runs Hit	Number of at –Bats Per Home run
1921	Babe Ruth	540	59	
1927	Babe Ruth	540	60	
1930	Hack Wilson	585	56	
1932	Jimmie Fox	585	58	
1938	Hank Greenberg	556	58	
1961	Roger Maris	590	61	
1961	Mickey Mantle	514	54	
1964	Willie Mays	558	52	
1977	George Foster	615	52	
1998	Mark McGwire	509	70	
1998	Sammy Sosa	643	66	
2001	Barry Bonds	476	73	
2002	Alex Rodriguez	624	57	

Solution

Year	Baseball Player	Number of Times at Bat	Number of Home Runs Hit	Number of At-Bats per Home Run
1921	Babe Ruth	540	59	540/59 =9.15
1927	Babe Ruth	540	60	9
1930	Hack Wilson	585	56	10.45
1932	Jimmie Foxx	585	58	10.08
1938	Hank Greenberg	556	58	9.59
1961	Roger Maris	590	61	9.67
1964	Willie Mays	558	52	10.73
1977	George Foster	615	52	11.83
1998	Mark McGwire	509	70	7.27
2001	Barry Bonds	476	73	6.52
2002	Alex Rodriguez	624	57	10.95

Which player has the lowest rate of at-bats per home run? Which player has the second lowest rate?

Barry Bonds has the lowest rate of at-bats that is 6.52 per home run.

Mark McGwire has the second lowest rate of at-bats that is 7.27 per home run.

Why is this rate used for comparison rather than the number of home runs a player has hit?

Average is more compatible than the number of home runs a player has hit.

22. The table below shows the populations and areas of three countries. The population density of a country is the number of people per square mile.

a) Which country has the lowest population density?

b) How many more people per square mile are there in India than in the United States? Round to the nearest whole number.

Country	Population	Area (In square miles)
Australia	20,090,000	2,938,000
India	1,080,264,000	1,146,000
United States	295,734,000	3,535,000

Solution

Population density $= \frac{\text{number of people}}{\text{land area}}$

Australia's Population density $= \frac{20090000}{2938300} = 6.84$

India's Population density $= \frac{1080264000}{1146000} = 942.64$

Australia has the lowest population density.

859 more people per square mile are there in India than in the United States

23. Forrester Research, Inc., compiled the following estimates on consumer use of e-mail in the United States.

a. Complete the last column of the table on the following page by calculating the estimated number of messages per day that each user receives. Round to the nearest tenth.

b. The predicted number of messages per person per day in 2005 is how many times the estimated number in 1993?

Solution

Year	Number of users (in millions)	Messages per day(in millions)	Message per person per day
1993	8	17	17/8 = 2.125%
1997	55	150	150/55 = 2.763%
2001	135	500	500/135 = 3.703%
2005	170	5000	5000/170 = 29.411%

The predicted number of messages per person per day in 2005 is 27.286% many times the estimated number in 1993.

24. Researchers at the Centers for Disease Control and Prevention estimate that 5 million young people living today will die of tobacco-related diseases. Almost one-third of children who become regular smokers will die of a smoking-related illness such as heart disease or lung cancer. The table below shows, for eight states in our nation, the numbers of children under 18 who are expected to become smokers and the numbers who are projected to die of smoking-related illnesses.

State	Projected Number of Smokers	Projected Number of Deaths
Alabama	260,639	83,404
Alaska	56,246	17,999
Arizona	307,864	98,516
Arkansas	155,690	49,821
California	1,446,550	462,896
Colorado	271,694	86,942
Connecticut	175,501	56,160
Delaware	51,806	16,578

Find the ratio of the projected number of smokers in each state listed to the projected number of deaths, Round to the nearest thousandth. Write the ratio using the word do?

Solution

State	Projected Number of Smokers	Projected Number of Deaths	Ratio
Alabama	260,639	83,404	3.125%
Alaska	56,246	17,999	3.12%
Arizona	307,864	98,821	3.12%
Arkansas	155,690	49,821	3.124
California	1,446,550	462,896	3.388
Colorado	271,694	86,942	3.125
Connecticut	175,501	56,160	3.125
Delaware	51,806	16,578	3.124

25. Researchers at the Centers for Disease Control and Prevention estimate that 5 million young people living today will die of tobacco-related diseases. Almost one-third of children who become regular smokers will die of a smoking-related illness such as heart disease or lung cancer. The table below shows, for eight states in our nation, the numbers of children under 18 who are expected to become smokers and the numbers who are projected to die of smoking-related illnesses.

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Connecticut	175,501	56,160
Delaware	51,806	16,578

a. Did the researchers calculate different possibilities of death from smoking related illnesses for each state?

b. If the projected number of smokers in Florida is 928,464. What would you expect the researcher's to project as a number of deaths from smoking related illnesses in Florida?

Solution

State	Projected Number of Smokers	Projected Number of Deaths	Ratio
Alabama	260,639	83,404	3.125%
Alaska	56,246	17,999	3.12%
Arizona	307,864	98,821	3.12%
Arkansas	155,690	49,821	3.124
California	1,446,550	462,896	3.388
Colorado	271,694	86,942	3.125
Connecticut	175,501	56,160	3.125
Delaware	51,806	16,578	3.124

26. The table below shows the numbers of full-time men and women undergraduates and the numbers of full-time faculty at universities in the Big East. Use this table for Exercises 31 to 34. Round ratios to the nearest whole number.

University	Men	Women	Faculty
Boston College	6292	9756	1283
Georgetown University	2940	3386	655
Syracuse University	4722	6024	815
University of Connecticut	6762	7489	842
West Virginia University	8878	7665	1289

a. Calculate the student faculty ratio at Syracuse University. Write the ratio using a colon and using the word to. What does this ratio mean?
b. Which school listed has the lowest student faculty ratio?
c. Which school listed has the highest student faculty ratio?
d. Which schools listed have the same student faculty ratio?

Solution

a. Boston College = 6292 + 9756 = 16048 **Ratio:** 16048/1283 = 6299.6

Georgetown University = 2940 + 3386 = 6326 **Ratio:** 6326/655 = 9.6580

Syracuse University = 4722 + 6024 = 10746 **Ratio:** 10746/815 = 4729.3

University of Connecticut = 6762 + 7489 = 14251 **Ratio:** 14251/842 = 6770.9

West Virginia University = 8878 + 7665 = 16543 **Ratio:** 16543/1289 = 8883.9

b. Georgetown University listed the lowest student faculty ratio.
c. West Virginia University listed the highest student faculty ratio.
d. Boston College and University of Connecticut almost have the same student faculty ratio.

27. A bank uses the ratio of a borrower's total monthly debt to total monthly income to determine eligibility for a loan. This ratio is called the debt–equity ratio. First National Bank requires that a borrower have a debt–equity ratio that is less than $\frac{2}{5}$. Would the homeowner whose monthly income and debt are given below qualify for a loan using these standards?

Monthly Income (in dollars)		Monthly Debt (in dollars)	
Salary	3400	Mortgage	1800
Interest	83	Property tax	104
Rent	640	Insurance	27
Dividends	34	Credit cards	354
		Car loan	199

Solution

To determine if the borrower qualifies for a loan based on the debt-to-income ratio, we need to calculate the total monthly debt and total monthly income first.

Total Monthly Income:

$3400 + $83 + $640 + $34 = $4157

Total Monthly Debt:

$1800 + $104 + $27 + $354 + $199 = $2484

Now, we can calculate the debt-to-income ratio:

Debt-to-Income Ratio = Total Monthly Debt / Total Monthly Income

Debt-to-Income Ratio = $2484 / $4157 ≈ 0.597

Since the debt-to-income ratio is approximately 0.597, which is less than 2/5 (0.4), the borrower qualifies for a loan based on the bank's requirement of a debt-to-income ratio less than 2/5.

Percentage/Percent

The term percent means "per hundred, or hundredths". Percents are ratios that are often used to represent parts of a whole, where the whole is considered as having 100 parts. Percents can be converted to fractions or decimal equivalents.

Percent means "for every 100." Therefore, unemployment of 5% means that 5 out of every 100 people are unemployed. An increase in tuition of 10% means that tuition has gone up $10 for every $100 it cost previously.

Examples

- 1 percent means one part out of 100 parts. The fraction equivalent of 1 percent is 1/100, and the decimal equivalent is 0.01.
- 32 percent means 32 parts out of 100 parts. The fraction equivalent of 32 percent is 32/100, and the decimal equivalent is 0.32.
- 50 percent means 50 parts out of 100 parts. The fraction equivalent of 50 percent is 50/100, and the decimal equivalent is 0.50.

Note that in the fraction equivalent, the part is the numerator of the fraction and the whole is the denominator. Percents are often written using the percent symbol, %, instead of the word "percent ". Here are five example of percents written using the% symbol, along with their fraction and decimal equivalents.

Examples Percent as a Fraction

- $100\% = \frac{100}{100} = 1$
- $25\% = \frac{25}{100} = \frac{1}{4}$
- $120\% = \frac{120}{100} = 1\frac{20}{100} = 1\frac{1}{5}$
- $12\frac{1}{2}\% = \frac{25}{2}\% = \left(\frac{25}{2}\right)\left(\frac{1}{100}\right) = \frac{1}{8}$

Examples Fraction as a Percent

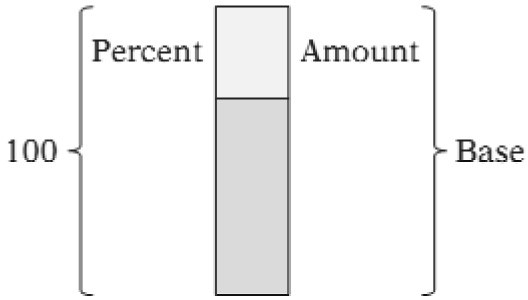
- $\frac{3}{4} = \frac{3}{4} \times 100\% = 75\%$
- $\frac{5}{8} = \frac{5}{8} \times 100\% = 62.5\%$
- $1\frac{1}{2} = \frac{3}{2} = \frac{3}{2} \times 100\% = 150\%$

Percent Problems: The Proportion Method

The proportion method of solving a percent problem is based on writing two ratios. One ratio is the percent ratio, written $\frac{\text{Percent}}{100}$ The second ratio is the amount-to-base ratio, written $\frac{\text{amount}}{\text{base}}$. These two ratios form the proportion used to solve percent problems. That is

$$\frac{\textbf{Percent}}{\textbf{100}} = \frac{\textbf{amount}}{\textbf{base}}$$

The proportion method can be illustrated by a diagram. The rectangle at the left is divided into two parts. On the left, the whole rectangle is represented by 100 and the yellow part by percent. On the right, the whole rectangle is represented by the base and the grey part by the amount. The ratio of percent to 100 is equal to the ratio of the amount to the base.



When solving a percent problem, first identify the percent, the base, and the amount. It is helpful to know that the base usually follows the phrase "percent of."

28. In the statement "15% of 40 is 6," which number is the percent? Which number is the base? Which number is the amount?
Solution
The percent is 15. The base is 40. (It follows the phrase "percent of.") The amount is 6.

29. The average size of a house in 2003 was 2137 square feet. This is approximate 130% of the average size of a house in 1979. What was the average size of house in 1979? Round to the nearest whole number.
Solution
$\frac{\text{Percent}}{100} = \frac{\text{amount}}{\text{base}}$
$\frac{130}{100} = \frac{2137}{\text{B}} \Rightarrow 130B = 213700 \Rightarrow B = \frac{213700}{130} \Rightarrow B \approx 1644$
The average size of a house in 1979 was 1644 square feet.

30. During 1996, Texas suffered through one of its longest droughts in history. Of the $5 billion in losses caused by the drought, $1.1 billion was direct losses to ranchers. What percent of the total losses was direct losses to ranchers?

Solution

$\frac{\text{Percent}}{100} = \frac{\text{amount}}{\text{base}}$

$\frac{P}{100} = \frac{1.1}{5} \Rightarrow 5P = 110 \Rightarrow P = \frac{110}{5} \Rightarrow P = 22$

Direct losses to ranchers represent 22% of the total losses.

31. In a recent year, Blockbuster Video customers rented 24% of the approximately 3.7 billion videos rented that year. How many million videos did Blockbuster Video rent that year?

Solution

$\frac{\text{Percent}}{100} = \frac{\text{amount}}{\text{base}}$

$\frac{24}{100} = \frac{A}{3.7} \Rightarrow 100A = 88.8 \Rightarrow A = \frac{88.8}{100} \Rightarrow A = 0.888$

The number 0.888 is in billions. We need to convert it to millions.

0.888 billion = 888 million

Blockbuster Video rented approximately 888 million videos that year.

The Basic Percent Equation

$PB = A$ where P is the percent, B is the base, and A is the amount.

Remark

When solving a percent problem using the proportion method, we first have to identify the percent, the base, and the amount. The same is true when solving percent problems using the basic percent equation. Remember that the base usually follows the phrase “percent of.” When using the basic percent equation, the percent must be written as a decimal or a fraction.

32. A real estate broker receives a commission of 3% of the selling price of a house. Find the amount the broker receives on the sale of a $275,000 home.

Solution

We want to answer the question “3% of $275,000 is what number?” Use the basic percent equation. The percent is 3% = 0.03. The base is 275,000. The amount is the amount the broker receives on the sale of the home.

$PB = A \Rightarrow 0.03(275{,}000) = A \Rightarrow A = 8250$

The real estate broker receives a commission of $8250 on the sale.

33. An investor received a payment of $480, which was 12% of the value of the investment. Find the value of the investment.

Solution

We want to answer the question "12% of what number is 480?" Use the basic percent equation. The percent is 12% = 0.12. The amount is 480. The base is the value of the investment.

$PB = A \Rightarrow 0.12B = 480 \Rightarrow B = \frac{480}{12} \Rightarrow B = 4000$

The value of the investment is $4000.

34. If you answer 96 questions correctly on a 120-question exam, what percent of the questions did you answer correctly?

Solution

We want to answer the question "What percent of 120 questions is 96 questions?" Use the basic percent equation. The base is 120. The amount is 96. The percent is unknown.

$PB = A \Rightarrow P \times 120 = 96 \Rightarrow P = \frac{96}{120} \Rightarrow P = 0.8 = 80\%$

You answered 80% of the questions correctly.

35. Find the difference between the cost of adding an attic bedroom to your home and the amount by which the addition increases the sale price of your home.

Home Remodeling Project	Average Cost	Percent Recouped
Addition to the master suite	$36,472	84%
Attic bedroom	$22,840	84%
Major kitchen remodeling	$21,262	90%
Bathroom addition	$11,645	91%
Minor kitchen remodeling	$8,507	94%

Solution

The cost of building the attic bedroom is $22,840, and the sale price increases by 84% of that amount. We need to find the difference between $22,840 and 84% of $22,840. Use the basic percent equation to find 84% of $22,840. The percent is 84% = 0.84. The base is 22,840. The amount is unknown.

$PB = A \Rightarrow 0.48(22480) = A \Rightarrow A = 19{,}185.60$

Subtract 19,185.60 (the amount of the cost that is recouped when the home is sold) from 22,840 (the cost of building the attic bedroom).
i.e. $22{,}840 - 19{,}185.60 = 3654.40$
The difference between the cost of the addition and the increase in value of your home is $3654.40.

Remember

- The percent used to determine the increase in the cost of living is a **percent increase**. Percent increase is used to show how much a quantity has increased over its original value.
- The percent used to measure the decrease in the federal deficit is a **percent decrease**. Percent decrease is used to show how much a quantity has decreased from its original value.
- The **federal debt** is the amount the government owes after borrowing the money it needs to pay for its expenses. It is considered a good measure of how much of the government's spending is financed by debt as opposed to taxation.
- The **federal deficit** is the amount by which government spending exceeds the federal budget.

36. How much would a family in Denver, Colorado living on $55,000 per year need in New York City to maintain a comparable lifestyle?

If you live in	and are moving to	you will need to make this percent of your current salary
Cincinnati, Ohio	San Francisco, California	236%
St. Louis, Missouri	Boston, Massachusetts	213%
Denver, Colorado	New York, New York	239%

Solution
In New York City, the family would need $55,000(2.39) = $131,450 per year to maintain a comparable lifestyle.

37. Find the percent increase in the federal debt $4.97 -$ to $-\ 0.91$ respectively from 1980 to 1995. Round to the nearest tenth of a percent.
Solution
Calculate the amount of increase in the federal debt from 1980 to 1995.
$4.97 - 0.91 = 4.06$
$PB = A \Rightarrow P(0.91) = 4.06 \Rightarrow P \approx 4.462$
The percent increase in the federal debt from 1980 to 1995 was 446.2%.

38. Find the federal deficit projected to decrease by from 2005 to 2006.

Year	Federal Deficit
2005	\$363.570 billion
2006	\$267.632 billion
2007	\$241.272 billion
2008	\$238.969 billion
2009	\$237.076 billion

Solution

First find the amount of decrease in the deficit from 2005 to 2006.

$363.570 - 267.632 = 95.938$

We will use the basic percent equation to find the percent. The base is the deficit in 2005. The amount is the amount of decrease.

$PB = A \Rightarrow P(363.570) = 95.938 \Rightarrow P \approx 0.264$

The federal deficit is projected to decrease by 26.4% from 2005 to 2006.

39. In 1998, GM plants took an average of 32 hours to produce one vehicle. From 1998 to 2005, that time decreased 28.125%. Find the average time for GM to produce one vehicle in 2005.

Solution

We will write and solve a proportion. (The basic percent equation could also be used.) The percent is 28.125%. The base is 32. The amount is unknown.

$\frac{\text{Percent}}{100} = \frac{\text{amount}}{\text{base}} \Rightarrow \frac{28.125}{100} = \frac{A}{32} \Rightarrow 100A = 900 \Rightarrow A = 9$

Subtract the decrease in time from the 1998 time. i.e. $32 - 9 = 23$

In 2005, GM plants took an average of 23 hours to produce one vehicle.

40. When adults were asked to name their favorite cookie, 52% said chocolate chip. What does this statistic mean?

Solution

This statistic means that 52 out of every 100 people surveyed responded that their favorite cookie was chocolate chip. (In the same survey, the following responses were also given: oatmeal raisin, 10%; peanut butter, 9%; oatmeal, 7%; sugar, 4%; molasses, 4%; chocolate chip oatmeal, 3%.)

Finding a percent of a number

$\text{Percentage} = \text{Base} \times \text{Rate\%}$

41. Find 34% of 450.
Solution
Here 34% is the rate and 450 is the base. Thus
Percentage = Base ×Rate% = 450×34% = 450×34/100 = 450×0.34 = 153

42. Find 25% of 1200.
Solution
Here 25% is the rate and RS.1200 is the base .Thus
Percentage = Base × Rate% = 1200 × 25% = 1200 × 25/100 = 1200 × 0.025 = 300

43. A shopkeeper sold am article at 20% more than its cost. What was the selling price if the cost of the article was Rs.320?
Solution
Here 20% is the rate and 320 is the base.
Profit = Base × Rate% = 320 × 20% = 320 × 20/100 = 320 × 0.2 = Rs.64
Selling price = cost + profit = Rs.320 + Rs.64 = Rs.384.

Finding what percent one number is of another number

$\text{Rate \%} = \frac{\text{Number}}{\text{Base}}$

44. Find what percent Rs.45 is of Rs.250.
Solution
Rs.45 is the number for which percent (rate) is required on the base Rs.250
Rate % = Number /Base = 45/250 = 0.18 =18%

45. A business man on an incitement of Rs.10,000 made a profit of Rs.300. What percent did he make on his investment?
Solution
Rs.300 is the number for which percent (rate) is required on the base Rs.10,000.
Rate = Number/base =300/10000= 3/100 = 0.03 = 3%

46. Find rate % on the cost of a chair, which is sold for Rs.560 and cost Rs.415.
Solution
Selling price= Rs.560, Cost price = Rs.415
Profit = selling - cost price = Rs.560-Rs.414 = Rs.145
Here, Rs.145 is the number for which percent (rate) is required on the basis of Rs.15
Rate = number/ base = 145/415 = 0.35 = 35%

Finding a number when a certain percent of it is given

$$\text{Base} = \frac{\text{percentage of a number}}{\text{rate \%}}$$

47. If 4% of the amount is Rs 80, what is the amount or base?

Solution

Amount = base, Rate% = 4%=0.04, percentage of number = Rs.80

$$\text{Amount} = \text{Base} = \frac{\text{percentage of a number}}{\text{rate \%}} = \frac{80}{0.04} = \text{Rs. } 2000$$

48. The selling price of T.V. is Rs.8800 and the gross profit is 10% of the cost. What is the cost?

Solution

The cost is the base from which gross profit is 10% has been calculated.

The given selling price= cost + gross profit = 100 + 10% =110% of cost

Since Rs.8800 is 110% of the cost, the cost is:

$$\text{Cost} = \frac{8800}{110\%} = \frac{8800}{1.10} = \text{Rs. } 8000$$

49. An increase of 12% in the price of a new car makes the price Rs.96320.

(i) What was the price of car before the increase?

(ii) What is the amount of increase?

Solution

(i) If 100% represents the price of car before increase, then 112% represents the price of car after increase.

Price of car before increase = $\frac{10}{112} \times 96320 = \text{Rs. } 86000$

(ii) The amount of increase = 12% of 86000 = $\frac{12}{100} \times 86000 = \text{Rs. } 10320$

50. Mr. Ali gets 20% profit from his investment. His profit is Rs.6580. Find his amount of incitement.

Solution

Rate of profit = 20 %

The investment is the base 100% from which profit of 20% has been calculated .His total profit is Rs.6580.

$$\text{Amount of investment} = \frac{\text{profit}}{\text{rate\%}} = \frac{6580}{20\%} = \frac{6580}{20} \times 100 = \text{Rs. } 32900$$

51. If the whole is 20 and the part is 13, find the percent.

Solution

$$\text{Percentage} = \frac{\text{Part}}{\text{Whole}} = \frac{13}{20} = 0.65 = 650\%$$

52. What percent of 150 is 12.9?

Solution

Here the whole is 159 and the part is 12.9

Percentage = $\frac{\text{Part}}{\text{Whole}} = \frac{12.9}{150} = 0.86 = 8.6\%$

53. Find 30% of 350.

Solution

30% of 350 = $\frac{30}{100} \times 350 = (0.3)(350) = 105$

54. 15 is 60 % of what number?

Solution

60% of x $= 15 \Rightarrow \frac{60}{100}x = 15 \Rightarrow 0.6x = 15 \Rightarrow x = \frac{15}{0.6} = 25$

55. A man spends \$880 in a month. 26% out of this goes to rent .How much is his rent?

Solution

Rent = $880 \times 26\% = 880 \times \frac{26}{100} = \$\,228.8$

56. If a person salary increases from \$ 200 per week to \$ 234 per week, what is the percent increase in the person's salary?

Solution

Percent increase = $\frac{\$\,234-\$200}{200} = 17\%$

57. If an athlete weight decreases from 160 to 152 pound, what is the percent decrease in the athlete weight?

Solution

Percent decrease = $\frac{152-160}{152} = 5\ \%$

58. A Particular book is valued at \$40 per share .If the value increases by 20 percent and then decreases by 25 percent, what will be the value of the stock per share after the decrease?

Solution

Price after increase by 20% = 20% × 40 = \$ 8

Price after increase = \$40 + \$8 = \$ 48

Now as per the second statement the 25% decrease can be calculated by multiplying the NEW BASE (48\$) with the decimal equivalent of 100% - 25% = 75% which is 0.75. Therefore

$0.75(\$48) = 36\$$

USES AND ABUSES OF PERCENTAGE

News reports frequently express quantitative information with percentages .Unfortunately, while percentages themselves are rather basic- they are just an alternative form of fractions - they are often used in very subtle ways. For example, consider the following quotes that appeared in the front -page news article:

The rate of smoking eighth grader was up 44 percent, to 10.4 percent. The percentage in this statement is used correctly, but the phrase “up 44% to 10.4%” is not easy to interpret correctly.

Three Ways of Using Percentage

Consider the following statements from new reports:

1. A total of 13000 newspaper employees, 2.6% of the newspaper work force, lost their jobs.
2. Citigroup stock fell 15% last week, to $ 44.25.
3. The advanced battery lasts 125% longer than the standard one, but costs 200 more.

Using Percentages as Fraction

Percent is just a fancy way of saying "divided by 100," so P% simply means P/100. For example, 10.4% means 10.4/100, or 0.104. Therefore if 10.4% of eighth -grader smoke and there are 100,000 eighth grader, then the number who smoke is 10.4% of 100,000, or 10.4% × 100,000 = 0.104 × 100,000 = 10,400. Notice that the word of told us to multiply. We've found that if 10.4% of 109,000 eighth graders smoke, there are 10,499 smokers.

Example

An opinion poll finds that 64% of 1069 people surveyed said that the president is doing a good job .How many said that president is doing a good job?

Solution

Because of indicates multiplication, 64% of the 1069 respondents is

$$64\% \times 1069 = 0.64 \times 1069 = 684.16 \approx 684$$

About 684 people said the president is doing a good job.

We rounded the answer to 684 to obtain a whole number of people.

DATA HANDLILNG & ERRORS

(SMALL AND LARGE DATA, ABSOLUTE AND RELATIVE ERRORS)

In statistics and data analysis, "large data" and "small data" refer to the size and scope of the dataset being worked with. Here's what I mean by large and small data handling:

Large Data Handling

Deals with vast amounts of data (thousands, millions, or billions of rows and columns)

Often requires specialized techniques and tools to process and analyze

Examples: social media data, sensor data, customer transaction data, genomic data

Challenges: data storage, processing power, data quality, and noise

Techniques: data aggregation, sampling, data compression, distributed computing, machine learning

Small Data Handling

Deals with limited or restricted data (dozens, hundreds, or thousands of rows and columns)

Often requires careful attention to detail and statistical precision

Examples: survey data, experimental data, clinical trial data, pilot study data

Challenges: data scarcity, sampling bias, measurement error

Techniques: data augmentation, non-parametric methods, Bayesian analysis, meta-analysis

In general, large data handling focuses on managing and extracting insights from vast amounts of data, while small data handling focuses on making the most of limited data.

Some key differences between large and small data handling include

Data size: Large data involves massive datasets, while small data involves smaller, more manageable datasets.

Data complexity: Large data often requires specialized tools and techniques to handle complexity, while small data can be analyzed using traditional statistical methods.

Data quality: Large data may suffer from noise and data quality issues, while small data requires careful attention to detail to ensure accuracy.

Analysis goals: Large data often aims to identify patterns and trends, while small data focuses on making precise estimates and inferences.

By understanding the differences between large and small data handling, analysts and researchers can choose the right techniques and tools to extract valuable insights from their data.

Here are some common methods for handling large and small data in statistics:

Large Data Handling

Sampling: Select a representative subset of data for analysis.

Example: Conducting a national survey with a random sample of 10,000 respondents.

Data Aggregation: Group data into categories to reduce its size.

Example: Analyzing website traffic by month instead of individual page views.

Data Compression: Reduce data size using algorithms or formats.

Example: Using lossless compression for image data.

Distributed Computing: Process data across multiple machines or nodes.

Example: Using Hadoop for big data analysis.

Resampling Methods: Use techniques like bootstrapping or jackknifing to analyze large data.

Example: Estimating population parameters using bootstrapping.

Small Data Handling

Pilot Studies: Conduct small-scale studies to test hypotheses or refine methods.

Example: Conducting a pilot survey with 100 respondents before a larger study.

Data Augmentation: Increase data size using techniques like interpolation or extrapolation.

Example: Using regression to fill missing values.

Non-Parametric Methods: Use methods that don't require large sample sizes.

Example: Using Wilcoxon rank-sum test for comparing two groups.

Bayesian Analysis: Use prior knowledge to inform analysis of small data.

Example: Using Bayesian inference for estimating population parameters.

Meta-Analysis: Combine results from multiple small studies.

Example: Conducting a meta-analysis of clinical trials.

Example

Suppose we have a dataset of exam scores for 1000 students:

Student ID	Score
1	80
2	90
...	...

Large Data Handling

Sampling: Select a random sample of 100 students to analyze.

Data Aggregation: Calculate mean scores by school instead of individual students.

Small Data Handling

Pilot Study: Analyze data from a small pilot study of 20 students to refine methods.

Data Augmentation: Use interpolation to fill missing scores for 10 students.

Example

Suppose we have a dataset of customer purchases:

Customer ID	Purchase Amount	Date
1	10.99	2022-01-01
2	5.99	2022-01-05
...	...	...

Large Data Handling:

Data Aggregation: Group by region, calculate total sales

Data Sampling: Analyze 1% of transactions (e.g., 1000 rows)

Small Data Handling:

Data Enrichment: Add customer demographics (e.g., age, location)

Data Normalization: Scale purchase amounts to a common range (e.g., 0-100)

By applying these methods, we can effectively handle large and small data sets to extract valuable insights and inform business decisions.

Some statistical techniques for large and small data include:

Regression Analysis: For modeling relationships between variables.

Hypothesis Testing: For testing statistical significance.

Confidence Intervals: For estimating population parameters.

Time Series Analysis: For analyzing data with temporal dependencies.

By applying these methods, we can effectively handle large and small data sets to extract valuable insights and inform decision-making.

Putting Numbers in Perspective

We hear numbers in the millions, billions, or trillions nearly every day, sometimes in the context of government spending and sometimes in other contexts, such as memory storage on phones and computers. Yet relatively few people understand what these large numbers really mean. In this unit, we will study several techniques for putting large (or small) numbers into a perspective that gives them real meaning.

Writing Large & Small Numbers

Working with large and small numbers is much easier when we write them in a special format known as scientific notation. We express numbers in this format by writing a number between} and 10 multiplied by a power of 10. (See the Brief Review of powers of 10.) For example, a billion is ten to the ninth power, or 10^9, so we write 6 billion in scientific notation as 6×10^9. Similarly, we write 420 in scientific notation as 4.2×10^2, and 0.67 as 6.7×10^{-1}.

Scientific Notation

Scientific notation is a method for handling large and small numbers, making them easier to work with and express. It's a way to represent very large or very small numbers in a compact form, using powers of 10.

Scientific notation is a format in which a number is expressed as a number between 1 and 10 multiplied by a power of 10. For example 20 in scientific notation as 4.2×10^2, and 0.67 as 6.7×10^{-1}.

Scientific notation makes it easy to write numbers no matter how large or small. We must be careful, however, not to let this ease of writing deceive us. For example, it's so easy to write the number 10^{80} that we might think it's not all that big but it is larger than the total number of atoms in the known universe.

Scientific notation is useful for

Large data: Representing extremely large numbers, such as astronomical distances or population sizes.

Small data: Representing extremely small numbers, such as molecular sizes or probability values.

Examples

Large data: 1,234,567,890 $= 1.23456789 \times 10^9$

Small data: 0.000000123 $= 1.23 \times 10^{-7}$

Scientific notation is a fundamental concept in mathematics, physics, and engineering, making it easier to work with extreme values. So, it's both a large and small data handling method!

Example

Rewrite each of the following statements using scientific notation. (a) Total spending in the new federal budget is $3,900,000,000,000. (b) The diameter of a hydrogen nucleus is about 0.000000000000001 meter.

Solution

(a) Total spending in the new federal budget is $\$3.9 \times 10^{12}$ or $3.9 trillion.

(b) The diameter of a hydrogen nucleus is about 1×10^{-15} meter.

Example Convert 0.0006 in to scientific notation.

Solution $0.0006 = 6 \times 10^{-4}$

Example: Checking Answers with Approximations

You and a friend are doing a rough calculation of how much garbage New City residents produce every day. You estimate that, on average, each of the 83 million residents produces 1.8 pounds, or 0.0009 ton, of garbage each day. The total amount of garbage is

$$8{,}300{,}000 \text{ persons} \times 0.0009 \frac{\text{ton}}{\text{person}}$$

Your friend quickly presses calculator buttons and tells you that the answer is 225 tons. Without using your calculator, determine whether this answer is reasonable.

Solution

$$8.3 \text{ million} = 8.3 \times 10^6 \approx 10^7$$

$$0.0009 = 9 \times 10^{-4} \approx 10^{-3}$$

Therefore, the product should be approximately:

$$10^7 \times 10^{-3} = 10^{7-3} = 10^4 = 10{,}000$$

Clearly, your friend’s answer-of 225 tons is too small. This simple approximation technique provided a useful check, even though it did not tell us the exact answer which you confirm to be 7470 tons.

Examples: Convert into Standard Form

- **My new music player has a capacity of 340 gigabytes.**
  $340\text{GB} = 340 \times 10^9 B = 3.40 \times 10^2 \times 10^9 B = 3.40 \times 10^{11} B$
- **The diameter of a typical bacterium is about 0.000001 meter.**
  $0.000001\text{m} = 1.0 \times 10^{-6} m$
- **A beam of light can travel the length of a football field in about 30 nanosecond. Express your answer in seconds.**
  $30 \text{ nanosec} = 30 \times 10^{-9}\text{sec} = 3.0 \times 10^1 \times 10^{-9}\text{sec} = 3 \times 10^{-8}\ sec$
- **The number of different eight character password that can be made with 26 letters and ten numerals is approximately 2.8 trillion.**
  $2.8 \text{ trillion} = 2.8 \times 1\text{trillion} = 2.8 \times 10^{12}$

Error

Error is a term used to denote the amount by which an approximation fails to equal the exact solution. $\text{Error} = \text{Exact solution} - \text{Approximation}$

For example: $\pi = 3.14159265 \ldots$ is irrational and $\frac{22}{7} = 3.14285 \ldots$ and a rational. And $\pi \approx \frac{22}{7}$ then $\text{Error} = 3.14159265 \ldots - 3.142857 \ldots = -0.00126449 \ldots$ is accurate.

Note that the word **error** in statistics does not mean **mistake** it is a chance of inaccuracy or change.

Absolute Error/Change

If the observed/reference value and true value are denoted by x and $x{+}\in$ respectively then then absolute error is given as;

$$\text{Absolute Error} = \text{True Value} - \text{Reference Value} = (x{+}\in) - x = \in$$

Relative Error/Change

An absolute error divided by the true value is called the relative error.

$$\text{Relative Error} = \frac{\text{True Value} - \text{Reference Value}}{\text{True Value}} = \frac{(x{+}\in) - x = \in}{x{+}\in}$$

$$\text{Percentage Error} = \frac{\text{True Value} - \text{Reference Value}}{\text{True Value}} \times 100\% = \frac{\in}{x{+}\in} \times 100\%$$

Biased Error

An error is said to be biased when the observed value is consistently and constantly higher or lower than the true value. Also called **cumulative** or **systematic errors**.

Unbiased Error

An error is said to be unbiased when the deviation. i.e. the excesses and defects, from the true value tend to occur equally often. Also called **compensating** or **random/accidental errors**.

Example

During a sixth month period, Xerox stock double in price from \$7 to \$14. What were the absolute and relative changes in the stock price?

Solution

Reference value = \$7; True value = \$14

The absolute change is the difference between true and starting stock prices:
$$\text{Absolute change} = \text{True value} - \text{Reference value} = \$14 - \$7 = \$7$$

$$\text{Relative change} = \frac{\text{True value} - \text{Reference value}}{\text{Ture value}} = \frac{\$14 - \$7}{\$14} = \$0.5$$

$$\text{Percentage change} = \frac{\text{True value} - \text{Reference value}}{\text{Ture value}} \times 100\% = \frac{\$14 - \$7}{\$14} \times 100\% = 50\%$$

That is, the doubling of the stock price means a relative increase in value of 50%

Example

Estimated world population was 2.7 billion in 1953 and 7.1 billion in 2013. Describe the absolute and relative change in world population over this 60 years period.

Solution

The reference value is the 1954 population of 2.7 billion, and the new/true value is the 2013 population of 7.1 billion.

$$\text{Absolute change} = \text{New value} - \text{Reference value}$$

$$\text{Absolute change} = 7.1 \text{ billion} - 2.7 \text{ billion} = 4.4 \text{ billion}$$

$$\text{Relative change} = \frac{\text{New value} - \text{Reference value}}{\text{New value}} \times 100\%$$

$$\text{Relative change} = \frac{7.1 \text{ billion} - 2.7 \text{ billion}}{7.1 \text{ billion}} \approx 0.62 \text{ billion}$$

$$\text{Percentage change} = \frac{\text{New value} - \text{Reference value}}{\text{New value}} \times 100\%$$

$$\text{Percentage change} = \frac{7.1 \text{ billion} - 2.7 \text{ billion}}{7.1 \text{ billion}} \times 100\% \approx 61.97\%$$

World population increased by 4.4 billion people, or by 61.97%, from 1953 to 2013.

Example

You bought a computer three years ago for \$ 1000 .Today it is worth only \$300. Describe the absolute and relative change in computer's value.

Solution

Reference value = \$1000

New value = \$300

$$\text{Absolute change } = \text{New value} - \text{Reference value} = \$300 \; - \; \$1000 \; = \; -700\$$$

The negative sign tell us that the computer's current worth is \$700 less than the price you paid three years ago.

$$\text{Relative change} = \frac{\text{New value} - \text{Reference value}}{\text{New value}}$$

$$\text{Relative change} = \frac{\$300 - \$1000}{\$300} \; = -2.33\ \%$$

$$\text{Percentage change} = \frac{\text{New value} - \text{Reference value}}{\text{Reference value}} \times 100\%$$

$$\text{Percentage change} = \frac{\$300 - \$1000}{\$300} \times 100\% = -233.33\ \%$$

Again, the negative sign tells us that the computer is now worth 233% less than it was three years ago.

FINANCIAL INDICATOR ANALYSIS AND MONEY MANAGEMENT

(PROFIT, LOSS, TAX, ZAKAT AND USHR, SIMPLE AND COMPOUND INTEREST)

Profit

If selling price of an article is greater than the cost price. Then the difference between them is called the profit. That is

$$\text{Profit} = \text{Sales Price} - \text{Cost Price}$$

$$\text{Profit Percentage} = \frac{\text{Profit}}{\text{Cost Price}} \times 100\%$$

Loss

If selling price of an article is less than the cost price. Then the difference between them is called the loss. That is

$$\text{Loss} = \text{Cost Price} - \text{Sales Price}$$

$$\text{Loss Percentage} = \frac{\text{Loss}}{\text{Cost Price}} \times 100\%$$

Marked Price

The price printed on the wrapper of article is called marked price.

List Price

The price of article given in the list provided by the factory is called list price.

Discount

A deduction of price offered by the seller on the marked price or list price is called discount.

Mathematical Relations regarding Discount

- $\text{Discount} = (\text{Marked Price or List Price}) \times \text{Discount}\ \%$
- $\text{Sale Price} = (\text{Marked Price or List Price}) - \text{Discount}$
- $\text{Discount Percentage} = \frac{\text{Discount}}{\text{Marked Price}} \times 100\%$

Partnership

An association of two or more persons who run a business to get profit is called partnership. There are two types of partnership.

- **Simple Partnership**
  When the partners invest capital for the same period of time, the partnership is called simple partnership. In this case, profit or loss is distributed among partners in the ratio of capital invested by each of them.
- **Compound Partnership**
  When the different partners invest capital for the different period of time, the partnership is called compound partnership. In this case, profit or loss is distributed among partners in the ratio of products of capital and period of investment of each partner.

1. A woman purchased a dress for Rs.4300. she sold it at a profit of 5%. Find the selling price of the dress.

Solution

The cost price of the dress = Rs. 4300
Profit = 5% of Rs. 4300 = 5/100×4300 = Rs. 215
Selling Price = C.P. + Profit = 4300+215 = Rs. 4515
So, the selling price of the dress is Rs. 4515.

2. The marked price of an electric heater is Rs. 4800. A discount of 15% is announced on sale. What is the amount of discount and its selling price?

Solution

Marked price of an electric heater = Rs.4800
Discount = 15% of Rs.4800 = $\frac{15}{100} \times 4800$ = Rs.720
Selling price of the electric heater = marked price - discount amount
S.P. = 4800 – 720 = Rs. 4080
Hence, the Amount of discount is Rs. 720 and its selling price is Rs. 4080.

3. Tom buys 10 units of a product at $20 each and sells them at $25 each. What is his total profit?
Solution
Profit per unit = Selling Price - Cost Price = $25 - $20 = $5.
Total Profit = Profit per unit × Number of units = $5 × 10 = $50

4. A bakery sells 200 loaves of bread at $2 each. The cost of ingredients and labor is $300. What is the profit?
Solution
Total Revenue = Number of loaves × Selling Price = 200 x×$2 = $400.
Profit = Total Revenue - Total Cost = $400 - $300 = $100

5. A store buys a shipment of 50 units of a product at $30 each. They sell 30 units at $40 each and the remaining units at $35 each. What is the total profit?
Solution
Profit from first 30 units = (Selling Price - Cost Price) × Number of units
Profit from first 30 units = ($40 - $30) × 30 = $10 × 30 = $300.
Profit from remaining units = (Selling Price - Cost Price) × Number of units
Profit from remaining units = ($35 - $30) × 20 = $5 × 20 = $100.
Total Profit = $300 + $100 = $400

6. Two companies, A and B, sell the same product at the same price. Company A has a cost price of $25 and Company B has a cost price of $30. If they both sell 100 units, who makes more profit and by how much?
Solution
Company A's Profit = Selling Price - Cost Price = $40 - $25 = $15.
Company B's Profit = Selling Price - Cost Price = $40 - $30 = $10.
Company A makes more profit by $5 per unit,
so total difference in profit is $5 × 100 = $500

7. A company has two products, X and Y. Product X has a profit of $10 per unit and Product Y has a profit of $20 per unit. If the company sells 50 units of Product X and 25 units of Product Y, what is the total profit?
Solution
Total Profit from Product X = Profit per unit × Number of units = $10 × 50 = $500.
Total Profit from Product Y = Profit per unit × Number of units = $20 × 25 = $500.
Total Profit = $500 + $500 = $1000

8. If the cost price of an item is $50 and the selling price is $60, what is the profit percentage?
Solution
Profit Percentage = (Profit / Cost Price) × 100 = ($10 / $50) × 100 = 20%

9. A company sells 100 units of a product at $25 each. The cost price is $20 each. What is the profit percentage?
Solution
Total Revenue = Number of units × Selling Price = 100 × $25 = $2500.
Total Cost = Number of units × Cost Price = 100 × $20 = $2000.
Profit = Total Revenue - Total Cost = $2500 - $2000 = $500.
Profit Percentage = (Profit / Total Cost) × 100 = ($500 / $2000) × 100 = 25%

10. A store buys a product at $30 and sells it at $39. What is the profit percentage?
Solution
Profit Percentage = (Profit / Cost Price) × 100 = ($9 / $30) × 100 = 30%

11. Two companies, A and B, sell the same product at the same price. Company A has a cost price of $25 and Company B has a cost price of $30. If they both sell 100 units, who has a higher profit percentage and by how much?
Solution
Company A's Profit Percentage = (Profit / Cost Price) × 100
Company A's Profit Percentage = ($15 / $25) × 100 = 60%.
Company B's Profit Percentage = (Profit / Cost Price) × 100
Company B's Profit Percentage = ($10 / $30) × 100 = 33.3%.
Company A has a higher profit percentage by 26.7%

12. A company sells a product at a profit percentage of 30%. If the cost price is $50, what is the selling price?
Solution
Selling Price = Cost Price + (Profit Percentage x Cost Price)
Selling Price = $50 + (30% × $50) = $50 + $15 = $65

13. Tom buys a book for $20 and sells it for $15. What is his loss?
Solution
Loss = Cost Price - Selling Price = $20 - $15 = $5

14. A store purchases 100 units of a product at $10 each and sells them at $8 each. What is the total loss?

Solution

Loss per unit = Cost Price - Selling Price = $10 - $8 = $2.

Total Loss = Loss per unit × Number of units = $2 × 100 = $200

15. A company produces 500 units of a product at a cost of $5 each. Due to market conditions, they have to sell them at $4 each. What is the total loss?

Solution

Loss per unit = Cost Price - Selling Price = $5 - $4 = $1.

Total Loss = Loss per unit × Number of units = $1 × 500 = $500

16. Sarah buys a dress for $30 and alters it for $10. She then sells it for $25. What is her loss?

Solution

Total Cost = Cost Price + Alteration Cost = $30 + $10 = $40.

Loss = Total Cost - Selling Price = $40 - $25 = $15

17. A farmer grows wheat at a cost of $100 per acre and sells it at $80 per acre. If he grows 50 acres, what is his total loss?

Solution

Loss per acre = Cost Price - Selling Price = $100 - $80 = $20.

Total Loss = Loss per acre × Number of acres = $20 × 50 = $1000

18. If the cost price of an item is $50 and the selling price is $40, what is the loss percentage?

Solution

Loss Percentage = (Loss / Cost Price) × 100 = ($10 / $50) × 100 = 20%

19. A store sells a product at a loss of 15%. If the cost price is $60, what is the selling price?

Solution

Selling Price = Cost Price - (Loss Percentage × Cost Price)

Selling Price = $60 - (15% × $60) = $60 - $9 = $51

20. A company produces a product at a cost of $20 each and sells them at a loss of 25%. What is the selling price?

Solution

Selling Price = Cost Price - (Loss Percentage × Cost Price)

Selling Price = $20 - (25% × $20) = $20 - $5

Selling Price = $15

21. Tom buys a bike for $100 and sells it at a loss of 10%.
If he also pays a commission of 5% on the selling price.
What is the total loss percentage?

Solution

Selling Price = Cost Price - (Loss Percentage × Cost Price)

Selling Price = $100 - (10% × $100) = $100 - $10 = $90.

Commission = 5% × $90 = $4.5.

Total Loss = $10 + $4.5 = $14.5.

Total Loss Percentage = (Total Loss / Cost Price) × 100

Total Loss Percentage = ($14.5 / $100) × 100 = 14.5%

22. A retailer buys a product at $50 and sells it at a loss of 12%. If he sells 200 units, what is the total loss amount?

Solution

Loss per unit = Cost Price × Loss Percentage = $50 × 12% = $6.

Total Loss = Loss per unit x× Number of units = $6 × 200 = $1200

Tax

A tax is the amount that government imposes on public to give them facilities, like education, health, security, justice, roads , electricity, etc. Tax is the most important source of government income. Some taxes are paid directly and some indirectly by the public.

Direct tax

A tax in which the tax payer pays directly to the government. For example, income tax, property tax etc. Direct tax different for everyone.

Indirect tax

In such taxes, taxes are charged on goods and services or on commodities. Indirect tax is same for everyone.

23. Ahmad earns Rs. 80000 per month. Calculate the income tax on Ahmad's annual income.

Solution

Ahmad's monthly income = Rs. 80000 = 8000 × 12 = Rs. 960000

Exempted amount = Rs. 600000. (According to the given income slab)

Taxable income = Gross income - exempted amount = 960000 – 600000 = Rs. 360000

Rate of income tax = 2.5%. (According to the given taxable income slab)

Income tax = $\frac{25}{1000} \times 360000$ = Rs. 9000

Property Tax

Property tax is a tax imposed by government on the properties such as house, land and shops. Government imposes property tax on the annual value of a property. The amount of this tax depends on the location of the property and varies from location to location. The value of the property is assessed by the government departments.

24. Ayyan owns a house of worth Rs. 4500000. Calculate the amount of property tax at the rate of 4.5%.

Solution

Total value of the house = Rs. 4500000

Rate of tax = 4.5%

Amount of tax = $\frac{4.5}{100} \times 4500000$ = Rs. 202500

25. Abdullah owns a property. If he has to pay property tax of Rs. 22000 at the rate of 2%. Find the total worth of the property.

Solution

Let the rate of property = x

Amount of tax = Rs. 22000

Rate of tax = 2 %

2% of x = Rs. 22000

$\frac{2}{100} \times$ x=Rs. 22000

x = Rs. 22000 × $\frac{2}{100}$ = Rs. 1100000

Hence, the total worth of the property is Rs. 1100000.

General sales Tax

When a customer purchases an item, he pays an extra amount in addition to the original price of the item. This extra amount is called general sales tax. Usually, government imposes this tax in expensive items. In Pakistan, rate of GST varies from 0% to 25% depending on the type of item.

26. The price of motorcycle is Rs. 110000. Find the GST on it at the rate of 17%.

Solution

Price of motorcycle = Rs. 110000,   Rate of GST= 17%

Amount of GST = $\frac{17}{100} \times 110000$ = Rs. 18700

Hence the GST on this motorcycle is Rs. 18700.

27. The price of fan is Rs. 6000. Find the price of 10 fans including GST.

Solution

The price of one fan = Rs. 6000

Price of ten fan = 6000×10 = 60000

Rate of GST = 17%

Amount of GST = $\frac{17}{100} \times 60000$ = Rs. 10200

Total price payable = 6000 + 10200 = Rs. 70200

28. What does each of these statements imply about the precipitation during this year? Do the two statements have the same meaning? Explain.

Solution

100% of normal would mean 100% × 90 = 90 inches

Normal 200% of normal would be 200% × 90 = 180 inches

Double normal 100% above normal would be 90 + 180 = 270 inches

29. You purchase a bicycle with a retail price of $760. The local sales tax rate is 7.6%. What is the final cost?

Solution

Final cost $= \$760 + 0.076 \times \$760 = \$760 + \$57.76 = \$817.76$

30. The final cost of your new shoes is $107.69. The local sales tax rate is 6.2% . What was the retail price?

Solution

Price = (100 + 6.2)% = 106.2%

Label Cost $= \frac{\$107.69}{106.2\%} = \$101.4$

31. Between 2000 and 2010, the percentage of U.S. households with cordless phones increased by 13.7% to 91%. What percentage of households had cordless phones in 2000?

Solution

It tells us that the new rate is 13.7% more than the previous rate which means it is (100+13.7)% =113.7% of the previous rate.

New rate=113.7%× previous rate

Previous rate $= \frac{\text{new rate}}{113.7\%} = \frac{\text{new rate}}{1.137} = \frac{91\%}{1.137} = 80\%$

80% of households had cordless phones in 2000.

32. Between 2000 and 2010, the percentage of fatal automobile accidents due to speeding decreased by 34% to 16%. What percentage of fatal automobile accidents were due to speeding in 2000?

Solution

It tells us that the new rate is 34% decrease the previous rate.

Which means it is $(100 - 34)\% = 66\%$ of the previous rate.

New rate $= 66\% \times$ previous rate

Previous rate $= \frac{\text{new rate}}{66\%} = \frac{16\%}{0.66} = 24.24\%$

Commission

Commission is an amount of money which is paid by the seller or purchaser to the agent for his services. In other words, commission is an amount of money paid to an employee or agent by the seller or purchaser for selling or purchasing something.

Rate of commission: Commission = rate of commission × selling price

33. If a property dealer gets Rs.100000 as commission on the sale of a shop for Rs. 4000000. Find the rate of commission.

Solution

Selling price of a shop = Rs. 4000000

Amount of commission= Rs. 100000

Commission = rate of commission × selling price

100000 = rate of commission × 4000000

Rate of commission = $\frac{100000}{4000000}$ = 0.025 = 2.5%

34. What is the commission if the rate of commission is 10% and the sale value is $100?

Solution

Commission = Rate of Commission × Sale Value = 10% × $100 = $10

35. If a salesperson earns a commission of $50 on a sale of $500, what is the rate of commission?

Solution

Rate of Commission = Commission / Sale Value = $50 / $500 = 10%

36. A real estate agent sells a property for $200,000 and earns a commission of 5%. How much is the commission?

Solution

Commission = Rate of Commission × Sale Value = 5% × $200,000 = $10,000

37. If a company pays a commission of 15% on all sales, and an employee makes a sale of $800, how much is the commission?

Solution

Commission = Rate of Commission × Sale Value = 15% × $800 = $120

38. A salesperson earns a commission of $25 on a sale of $250. What is the rate of commission?

Solution

Rate of Commission = Commission / Sale Value = $25 / $250 = 10%

Zakat

There are 5 pillars of Islam. Zakat is one of the pillars of Islam. It imposes on those Muslims who have certain amount of wealth the whole year. There are types of recipients of Zakat. The purpose of Zakat is to help the poor and needy among the Muslims to create a welfare Muslim state. The Muslims pay Zakat if their annual savings reaches a certain level.

Nisaab of Zakat

Zakat is the minimum amount of annual savings on which Zakat has to be paid. Nisaab for Zakat is 7.5 tola (8748 grams) gold or 52.5 tola (612.36 gram) silver or equivalent amount.

Rate of Zakat

The rate of Zakat is 2.5% of total wealth. The amount of Zakat is calculated by the following formula.

Amount of Zakat = rate of Zakat × total amount

39. Khalid saved Rs. 2000000 for one year in his account. Calculate the amount of Zakat Khalid has to pay.
Solution
Total amount = Rs. 2000000
Rate of Zakat = 2.5%
Amount of Zakat = rate of Zakat × total amount = $\frac{2.5}{100} \times 2000000$ = Rs.50000
Hence, Khalid has to pay Rs.50000 as a Zakat.

40. Saif paid Zakat of Rs.15000 on gold. Find the total price of the gold.
Solution
Amount of Zakat paired = Rs. 15000
Amount of Zakat = rate of Zakat × total amount
Rs. 150000 = 2.5%× total amount = $\frac{2.5}{100}$ × total amount
Total amount = $15000 \times \frac{100}{2.5}$ = Rs. 600000
Hence, the total price of the gold is Rs. 600000

41. Calculate the amount of Zakat on an amount of Rs. 5,00,000.
Solution
Total amount = Rs. 5,00,000
Rate of Zakat = 2.5%
Amount of Zakat = rate of Zakat × total amount = $\frac{2.5}{100} \times 5{,}00{,}000$ = Rs.12500

42. Amira has $1,000 in her savings account. What is the Zakat she needs to pay if the rate is 2.5%?

Solution

Zakat = Total Amount × Rate = $1,000 × 2.5% = $25

43. A business owner has a annual profit of $100,000. If he has already paid $20,000 in taxes, what is the Zakat he needs to pay if the rate is 2.5%?

Solution

Zakat = (Total Profit - Taxes) × Rate = ($100,000 - $20,000) × 2.5%

Zakat = $80,000 × 2.5% = $2,000

44. A person has investments worth $500,000, but they are currently valued at $400,000 due to market fluctuations. What is the Zakat they need to pay if the rate is 2.5%?

Solution

Zakat = Current Value x× Rate = $400,000 × 2.5% = $10,000

45. A company has a total asset value of $1,500,000, but $500,000 is exempt from Zakat. If the rate is 2.5%, what is the Zakat they need to pay?

Solution

Zakat = (Total Assets - Exempt Amount) × Rate

Zakat = ($1,500,000 - $500,000) × 2.5%

Zakat = $1,000,000 × 2.5% = $25,000

46. A farmer has an annual harvest worth $200,000, but he has already paid $50,000 in taxes and has $30,000 in expenses. What is the Zakat he needs to pay if the rate is 2.5%?

Solution

Zakat = (Total Harvest - Taxes - Expenses) × Rate

Zakat = ($200,000 - $50,000 - $30,000) × 2.5%

Zakat = $120,000 × 2.5% = $3,000

Ushr

Ushr is paid at the rate of 10% from the agricultural products of the land which is irrigated by natural resources. However the rate of Ushr is 5% on the agricultural products of the land which is irrigated by artificial resources, that is canals, tube wells etc.

47. If the wheat crop is produced 40000kg by natural resources, calculate the amount of ushr, if the price of wheat is Rs.950 per kg.

Solution

Weight of wheat = 40000kg

Price of 40kg wheat = Rs.950

Price of 1kg wheat = Rs.$\frac{950}{40}$

Price of 40000kg wheat = Rs.$\frac{950}{40} \times 40000 = \text{Rs.}\,950000$

Amount of Ushr = $\frac{10}{100} \times 950000$ = Rs.95000

48. Calculate amount of ushr on a rice crop of weight 3000kg produced by artificial sources, if the price of 40 kg rice is Rs.2000.

Solution

Weight of rice crop = 3000kg

Price of 3000kg wheat = Rs.$\frac{2000}{40} \times 3000 = \text{Rs.}\,150000$

Amount of Ushr = $\frac{5}{100} \times 150000$ = Rs.7500

49. A landowner has 100 acres of agricultural land, and the annual produce is worth $50,000. What is the Ushr they need to pay if the rate is 5%?

Solution

Ushr = Total Produce × Rate = $50,000 × 5% = $2,500

50. A farmer has 50 acres of land, and the annual produce is worth $20,000. If they have already paid $5,000 in taxes, what is the Ushr they need to pay if the rate is 5%?

Solution

Ushr = (Total Produce - Taxes) × Rate

Ushr = ($20,000 - $5,000) × 5%

Ushr = $15,000 × 5%

Ushr = $750

51. A person owns 200 acres of land, but 50 acres are not cultivable. If the annual produce from the remaining land is worth $100,000, what is the Ushr they need to pay if the rate is 5%?

Solution

Ushr = Total Produce × Rate
Ushr = $100,000 × 5%
Ushr = $5,000

52. A company owns 500 acres of agricultural land, and the annual produce is worth $250,000. If they have already paid $50,000 in taxes and have $20,000 in expenses, what is the Ushr they need to pay if the rate is 5%?

Solution

Ushr = (Total Produce - Taxes - Expenses) × Rate
Ushr = ($250,000 - $50,000 - $20,000) × 5%
Ushr = $180,000 × 5%
Ushr = $9,000

53. A farmer has 150 acres of land, and the annual produce is worth $75,000. If they have already paid $15,000 in taxes and have $10,000 in expenses, what is the Ushr they need to pay if the rate is 5%?

Solution

Ushr = (Total Produce - Taxes - Expenses) × Rate
Ushr = ($75,000 - $15,000 - $10,000) × 5%
Ushr = $50,000 × 5%
Ushr = $2,500

Interest

When you deposit money in a bank for example, in a savings account you are permitting the bank to use your money. The bank may lend the deposited money to customers to buy cars or make renovations on their homes. The bank pays you for the privilege of using your money. The amount paid to you is called interest. If you are the one borrowing money from a bank, the amount you pay for the privilege of using that money is also called inter

Principal

The amount deposited in a bank or borrowed from a bank.

Or the principal in financial formulas is the balance upon which interest is paid.

Interest Rate

The percent used to determine the amount of interest is called the interest rate. If you deposit $1000 in a savings account paying 5% interest, $1000 is the principal and the interest rate is 5%.

Simple Interest

Simple Interest is interest paid only on the original investment and not on any interest added at later dates.

Simple Interest Formula

The simple interest formula is

I = Prt

Where I is the interest, P is the principal, r is the interest rate, and t is the time period.

54. Calculate the simple interest earned in 1 year on a deposit of $1000 if the interest rate is 5%.

Solution

$I = Prt = 1000(0.05)(1) = \$50$

55. Calculate the simple interest due on a three-month loan of $2000 if the interest rate is 6.5%.

Solution

$\text{Time} = t = \frac{3\text{ months}}{1\text{ year}} = \frac{3}{12}$

$I = Prt = 2000(0.065)\left(\frac{3}{12}\right) = \$32.5$

56. Calculate the simple interest due on a two-month loan of $500 if the interest rate is 1.5% per month.

Solution

Because the interest rate is per month, the time period of the loan is expressed as the number of months: $t = 2$

$I = Prt = 500(0.015)(2) = \$15$

Simple Interest: Calculate the amount of money you will have in the following accounts after 5 years, assuming that you earn simple interest.

57. You deposit $700 in an account with an annual interest rate of 4%

Solution

Simple interest for one year = 700 $ × 4% = 28 $

Simple interest for five year = 28 × 5 $ = 140$

Amount of money after five year = 700 + 140 = 840 $

58. You deposit $1200 in an account with an annual interest rate of 3%

Solution

Simple interest for one year = 1200 $ × 3% = 36 $

Simple interest for five year = 36 × 5$ = 180$

Amount of money after five year = 1200 + 180 = 1380 $

59. You deposit $3200 in an account with an annual interest rate of 3.5%

Solution

Simple interest for one year = 3200 $ × 0.035 = 112 $

Simple interest for five year = 112 × 5 = 560$

Amount of money after five year = 3200 + 560 = 3760 $

60. You deposit \$1800 in an account with an annual interest rate of 3.8%

Solution

Simple interest for one year = 1800 \$ × 0.038 = 68.4 \$

Simple interest for five year 68.4\$×5=342\$

Amount of money after five year=1800+342\$=2142\$

Remember

Remember that in the simple interest formula, time t is measured in the same period as the interest rate. Therefore, if the time period of a loan with an annual interest rate is given in days, it is necessary to convert the time period of the loan a fractional part of a year. There are two methods for converting time from days to years: the exact method and the ordinary method. Using the exact method, t number of days of the loan is divided by 365, the number of days in a year.

Exact Method: $t = \frac{\text{No.of days}}{365}$

The ordinary method is based on there being an average of 30 days in a month and 12 months in a year $(30.12 = 360)$. Using this method, the number of days of the loan is divided by 360.

Ordinary Method: $t = \frac{\text{No.of days}}{360}$

The ordinary method is used by most businesses. Therefore, unless otherwise stated, the ordinary method will be used in this text.

61. Calculate the simple interest due on a 45-day loan of \$3500 if the annual interest rate is 8%.

Solution

Because the interest rate is per month, the time period of the loan is expressed as the number of months: $t = \frac{\text{No.of days}}{360} = \frac{45}{360}$

$I = Prt = 3500(0.08)\left(\frac{45}{360}\right) = \$35$

62. The simple interest charged on a six-month loan of \$3000 is \$150. Find the simple interest rate.

Solution

$I = Prt$

$150 = 3000(r)\left(\frac{6}{12}\right)$

$r = 0.10 = 10\%$

63. Savings Bond: While banks almost always pay compound i interest, bonds usually pay simple interest. Suppose you invest \$1000 in a savings bond that pays simple interest of 10% per year. How much total interest will you receive in 5 year? If the bond paid compound interest, would you receive more or less total interest? Explain.

Solution

With simple interest, every year you receive the same interest payment of $10\% \times \$1000 = \$100$, Therefore, you receive a total of \$500 in interest over 5 years. With compound interest, you receive more than \$500 in interest because the interest each year is calculated on your growing balance rather than on your original investment. For example, because your first interest payment of \$100 raises your balance to \$1100, your next compound interest payment is $10\% \times \$1100 = \$110$, which is more than the simple interest payment of \$100. For the same interest rate, compound interest always raises your balance faster than simple interest.

Future Value or Maturity Value Formula for Simple Interest

The future value or maturity value formula for simple interest is

$A = P + I = P(1 + rt)$

Where A is the amount after the interest I has been added to the principal P.

64. Is the stated sum a maturity value or a future value?

a. The sum of the principal and the interest on an investment

b. The sum of the principal and the interest on a loan

Answer

a. Future value   b. Maturity value

65. Calculate the maturity value of a simple interest, eight-month loan of \$8000 if the interest rate is 9.75%.

Solution

$I = Prt = 8000(0.0975)\left(\frac{8}{12}\right) = \$520$

$A = P + I = 8000 + 520$

The maturity value of the loan is \$8520.

66. Calculate the maturity value of a simple interest, three-month loan of \$3800. The interest rate is 6%.

Solution

$A = P(1 + rt) = 3800\left(1 + 0.06\left(\frac{3}{12}\right)\right) = \$3857$

The maturity value of the loan is \$3857.

67. Find the future value after 1 year of \$850 in an account earning 8.2% simple interest.

Solution

$A = P(1 + rt) = 850\big(1 + 0.082(1)\big) = \$919.70$

The future value of the account after 1 year is \$919.70.

68. The maturity value of a three-month loan of \$4000 is \$4085.What is the simple interest rate?

Solution

$A = P + I \Rightarrow I = A - P \Rightarrow I = 4085 - 4000 = 85$

Find the simple interest rate by solving the simple interest formula for r.

$I = Prt \Rightarrow 85 = 4000(r)\left(\frac{3}{12}\right) \Rightarrow r = 0.085 = 8.5\%$

The simple interest rate on the loan is 8.5%.

Compound Interest

Compound Interest is interest paid both on the original investment and on all interest that has been added to the original investment.

Simple interest is generally used for loans of 1 year or less. For loans of more than 1 year, the interest paid on the money borrowed is called **compound interest**. Compound interest is interest calculated not only on the original principal, but also on any interest that has already been earned.

Compounding Period

The frequency with which the interest is compounded is called the compounding period. For example annually, semiannually (twice a year), quarterly (four times a year), monthly, or daily.

The Compound Interest Formula (For Interest Paid Once a Year)

$A = P \times (1 + APR)^{Y}$

A = Accumulated balance after Y years

P = Starting principal

APR = Annual percentage rate (as a decimal)

Y= Number of years

Notes

- The starting principal, P, is often called **the present value (PV)**, because we usually begin a calculation with the amount of money in an account at present.
- The accumulated balance, A, is often called **the future value (FV)**, because it is the amount that will be accumulated at some time in the future.
- When using this formula, you must express the APR as a decimal rather than as a percentage.

Compound Amount Formula

For the more general case in which the interest rate may not be set on an annual (APR) basis, the compound interest formula is written

$A = P(1 + i)^N$

Where i is the interest rate and N is the total number of compounding periods.

$$i = \frac{\text{annual interest rate}}{\text{number of compounding periods per year}} = \frac{r}{n}$$

$$N = (\text{number of compounding periods per year})(\text{number of years}) = nt$$

69. What is the value of i when the interest rate is 6%, compounded monthly?

Answer

$$i = \frac{r}{n} = \frac{6\%}{12} = 0.5\% = 0.005$$

70. You deposit $500 in an account earning 6% interest, compounded semiannually. How much is in the account at the end of 1 year?

Solution

To calculate the amount in the account at the end of 1 year, we can use the formula for compound interest:

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

Where: A = final amount; P = principal (initial deposit) = $500

r = annual interest rate = 6% = 0.06

n = number of times interest is compounded per year = 2 (semiannually)

t = time in years = 1

Plugging in the values, we get:

$$A = 500\left(1 + \frac{0.06}{2}\right)^{2\times 1} = 500\left(1 + \frac{0.06}{2}\right)^{2} = 530.45$$

So, at the end of 1 year, the account will have $530.45.

Remember

In calculations that involve compound interest, the sum of the principal and the interest that has been added to it is called the **compound amount**. In Example, the compound amount is $530.45.

71. Calculate the compound amount when $10,000 is deposited in an account earning 8% interest, compounded semiannually, for 4 years.

Solution

$$i = \frac{r}{n} = \frac{0.08}{2} = 0.04 \text{ , } N = nt = 2(4) = 8$$

$$A = P(1 + i)^N = 10{,}000(1 + 0.04)^8 \approx 13{,}685.69$$

The compound amount after 4 years is approximately $13,685.69.

72. Calculate the future value of $5000 earning 9% interest, compounded daily, for 3 years.

Solution

$i = \frac{r}{n} = \frac{0.09}{360} = 0.00025$, $N = nt = 360(3) = 1080$

$A = P(1+i)^N = 5000(1+0.00025)^{1080} \approx 6549.60$

The future value after 3 years is approximately $6549.60.

73. How much interest is earned in 2 years on $4000 deposited in an account paying 6% interest, compounded quarterly?

Solution

$i = \frac{r}{n} = \frac{0.06}{4} = 0.0015$, $N = nt = 4(2) = 8$

$A = P(1+i)^N = 4000(1+0.0015)^8 \approx 4505.97$

To Calculate the interest earned.

$I = A - P = 4505.97 - 4000 = 505.97$

The amount of interest earned is approximately $505.97.

74. Mattresses Investment: Your grandfather put $100 under his mattress 50 years ago. If he had instead invested it in a bank account paying 3.5% interest compounded yearly (roughly the average U.S. rate of inflation during that period), how much would it be worth now?

Solution

The starting principal is p= $100. The annual percentage rate is APR = 3.5% = 0.035. The number of years is Y= 50. So the accumulated balance is

$A = P \times (1 + APR)^Y = 100 \times (1 + 0.035)^{50} = \$558.49$

Invested at a rate of 3.5%, the $100 would be worth over $550 today. Unfortunately, the $100 was put under 4 mattresses, so it still has a face value of only $100.

75. You invest \$100 in two accounts that each pays an interest rate of 10% per year, but one pays simple interest and the other pays compound interest. Make a table to show the growth of each account over a 5-year period. Use the compound interest formula to verify the result in the table for the compound interest case.

Solution

The simple interest is the same absolute amount each year: $10\% \times \$100 = \$10$. The compound interest grows from year to year, because it is paid on the accumulated interest as well as on the starting principal. Table summarizes the calculations.

To verify the final entry in the table with the compound interest formula, we use a starting principal P = \$100 and an annual interest rate APR = 10% = 0.10 with interest paid for Y= 5 years. The accumulated balance A is

$A = P \times (1 + APR)^Y = 100 \times (1 + 0.1)^5 = \$161.05$

This result agrees with the one in the table. Overall, the account paying compound interest builds to \$161.05 while the simple interest account reaches only \$150, even though both pay at the same 10% rate. Although the 10% interest rate that we assumed here is quite high compared to what most banks pay, the basic point should be clear: For the same interest rate, compound interest is always better for the investor than simple interest.

Simple Interest Account			Compound Interest Account	
End of Year	**Interest Paid**	**O.B+Int = N.B**	**Int.Paid**	**O.B+Int = N.B**
1	$10\% \times \$100 = \$10$	$\$100 + \$10 = \$110$	$10\% \times \$100 = \$10$	$\$100 + \$10 = \$110$
2	$10\% \times \$100 = \$10$	$\$110 + \$10 = \$120$	$10\% \times \$110 = \$11$	$\$110 + \$11 = \$121$
3	$10\% \times \$100 = \$10$	$\$120 + \$10 = \$130$	$10\% \times \$121 = \$12.10$	$\$121 + \$12.10 = \$133.10$
4	$10\% \times \$100 = \$10$	$\$130 + \$10 = \$140$	$10\% \times \$133.10 = \$13.31$	$\$133.10 + \$13.31 = \$146.41$
5	$10\% \times \$100 = \$10$	$\$140 + \$10 = \$150$	$10\% \times \$146.41 = \$14.64$	$\$146.41 + \$14.64 = \$161.05$

76. New College Debt at 2%: If the interest rate is 2%, calculate the amount due to New College using a. Simple interest b. Compound interest

Solution

a) The following steps show the simple interest rate calculation for a starting principal P = $224 and an annual interest rate of 2%

1) The simple interest due each year is 2% of the starting principal:
$2\% \times \$224 = 0.02 \times \$224 = \$4.48$
2) Over 535 years, the total interest due is
$535 \times \$4.48 = \$2396.80$
3) The total due after 535 years is the starting principal plus the interest
$\$224 + \$2396.80 = \$2620.80$
With simple interest, the payoff amount after 535 years is $2620.80.

b) To find the amount due with compound interest, we set the annual interest rate to APR = 2% = 0.02 and the number of years to Y= 535. Then we use the formula for compound interest paid once a year:
$A = P \times (1 + APR)^{Y} = 224 \times (1 + 0.02)^{535} = \$8.94 \times 10^{6}$

The amount due with compound interest is about $8.94 million—far higher than the amount due with simple interest.

Simple versus Compound Interest: Complete the following tables, which show the performance of two investments over a 5 year period. Round all figures to the nearest dollar.

77. Suzanne deposits 53000 in an account that earns simple interest at an annual rate of 2.5%. Derek deposits \$3000 in an account that earns compound interest at an annual rate of 2.5%.

Solution

Year	Simple Interest		Compound Interest	
	Suzanne's		Derek's	
	Annual Interest	Balance	Annual Interest	Balance
1	3000 × 2.5% = \$75	3000 + 75 = \$3075	3000 × 2.5% = \$75	3000 + 75 = \$3075
2	3000 × 2.5% = \$75	3075 + 75 = \$3150	3075 × 2.5% = \$77	3075 + 77 = \$3152
3	3000 × 2.5% = \$75	3150 + 75 = \$3225	3152 × 2.5% = \$79	3152 + 79 = \$3231
4	3000 × 2.5% = \$75	3225 + 75 = \$3300	3231 × 2.5% = \$81	3231+ 81 = \$3312
5	3000 × 2.5% = \$75	3300 + 75 = \$3375	3312 × 2.5% = \$83	3312 + 83 = \$3395

78. Ariel deposits 55000 in an account that earns simple interest at an annual rate of 3%. Travis deposits 55000 in an account that earns compound interest at an annual rate of 3%.

Solution

Year	Simple Interest		Compound Interest	
	Ariel		Travis	
	Annual Interest	Balance	Annual Interest	Balance
1	5000 × 3% = \$150	5000 + 150 = 5150	5000 × 3% = \$150	5000 + 150 = 5150
2	5000 × 3% = \$150	5150 + 150 = 5300	5150 × 3% = \$154.5	5150 + 154.5 = 5304.5
3	5000 × 3% = \$150	5300 + 150= 5450	5304.5 × 3% = \$159	5304 + 159= 5463
4	5000 × 3% = \$150	5450 + 150= 5600	5463 × 3% = \$163	5463 + 163= 5626
5	5000 × 3% = \$150	5600 + 150 = 5750	5626 × 3% = \$168	5626 + 168= 5794

Compound Interest: Use the compound interest formula to compute the balance in the following accounts after the stated period of time, assuming interest is compounded annually.

79. \$10,000 is invested at an APR of 4% for 10 years.

Solution

$A = P\,(1 + APR)^{Y} = \$10{,}000\,(1 + 0.04)^{10} = \$10.000\,(1.04)^{10}$

$A = \$10{,}000 \times 1.4802 = \$14802.44$

80. \$10,000 is invested at an APR of 2.5% for 20 years.

Solution

$A = P\,(1 + APR)^{Y} = \$10{,}000\,(1 + 0.025)^{20} = \$10{,}000(1.025)^{20}$

$A = \$10{,}000 \times 1.6386 = \$16386$

81. \$15,000 is invested at an APR of 3.2% for 25 years.

Solution

$A = P(1 + APR)^{Y} = \$15{,}000\,(1 + 0.032)^{25} = \$15.000\,(1.032)^{25}$

$A = \$15{,}00 \times 2.1978 = \$32967.32$

82. \$3000 is invested at an APR of 1.8% for 12 years

Solution

$A = P(1 + APR)^{Y} = \$3000(1 + 0.018)^{12} = \$3000 \times (1.018)^{12}$

$A = \$3000 \times 1.2387 = \$3716.16$

83. 5000\$ is invested at an APR of 3.1% for 12 years

Solution

$A = P(1 + APR)^{Y} = \$\,5000(1 + 0.031)^{12} = \$5000(1.031)^{12}$

$A = \$5000 \times 1.4424 = \$7212.30$

84. \$40,000 is invested at an APR of 2.8% for 30 years.

Solution

$A = P(1 + APR)^{Y} = \$\,40{,}000(1 + 0.028)^{30} = \$\,40{,}000(1.028)^{30}$

$A = \$\,40{,}000 \times 2.2897 = \$91588$

The Compound Interest Formula (For Interest Paid more than Once a Year)

$A = P \times \left(1 + \frac{APR}{n}\right)^{(nY)}$

A = Accumulated balance after Y years

P.= Starting principal

APR = Annual percentage rate (as a decimal)

n= Number of compounding periods per year

Y= Number of years.

Note that Y is not necessarily an integer; for example

4 calculations for six months would have Y= 0.5.

85. Monthly Compounding at 3%: You deposit $5000 in a bank account that pays an APR of 3% and compounds interest monthly. How much money will you have after 5 years? Compare this amount to the amount you’d have if interest were paid only once each year.

Solution

Monthly compounding means that interest is paid n = 12 times a year and we are considering a period of Y = 5 years.

$$\text{Accumulated Balance} = A = P\left(1 + \frac{APR}{n}\right)^{(nY)}$$

$$\text{Accumulated Balance} = A = 5000\left(1 + \frac{0.03}{12}\right)^{(12\times 5)} = \$5808.08$$

$$\text{Accumulated Balance once a year} = A = P(1 + APR)^{Y}$$

$$\text{Accumulated Balance once a year} = A = 5000(1 + 0.03)^{5} = \$5796.37$$

After 5 years, monthly compounding gives you a balance of $5808.08 while annual compounding gives you a balance of $\$5796.37$. That is monthly compounding earns $\$5808.08 - \$5796.37 = \$11.71$ more even though the APR is the same in both cases.

Use the appropriate compound interest formula to compute the balance in the following accounts after the stated period of time.

86. $10,000 is invested for 10 years with an APR of 2% and quarterly compounding.

Solution

$$A = P\left(1+\frac{APR}{n}\right)^{nY} = \$\,10{,}000 \times \left(1+\frac{0.02}{4}\right)^{4\times 10}$$

$$A = \$\,10{,}000\,(1.005)^{40} = \$\,10{,}000 \times 1.2207 = \$12207.94$$

87. $2000 is invested for 5 years with an APR of 3% and daily compounding.

Solution

$$A = P\left(1+\frac{APR}{n}\right)^{nY}$$

$$A = \$2000\left(1+\frac{0.03}{365}\right)^{5\times 365} = \$200 \times 1.1618 = \$232.6$$

88. $25,000 is invested for 5 years with an APR of 3% and daily compounding.

Solution

$$A = P\left(1+\frac{APR}{n}\right)^{nY}$$

$$A = \$25000\left(1+\frac{0.03}{365}\right)^{5\times 365} = \$\,200 \times (1.1618) = \$29045.68$$

89. $10,000 is invested for 5 years with an APR of 2.75% and monthly compounding.

Solution

$$A = P\left(1+\frac{APR}{n}\right)^{nY}$$

$$A = \$10{,}000\left(1+\frac{0.0275}{12}\right)^{12\times 5} = \$\,10{,}000 \times (1.6022)^{60}$$

$$A = \$\,10{,}000 \times 1.14722 = 11472.21\$$$

90. $2000 is invested for 15 years with an APR of 5% and monthly compounding.

Solution

$$A = P\left(1+\frac{APR}{n}\right)^{nY}$$

$$A = \$\,2000\left(1+\frac{0.05}{12}\right)^{12\times 15} = \$\,2000 \times (1.004)^{180}$$

$$A = \$\,20002.1137 = \$4227.40$$

91. $30,000 is invested for 15 years with an APR of 4.5% and daily compounding.

Solution

$$A = P\left(1 + \frac{APR}{n}\right)^{nY}$$

$$A = \$30{,}000 \times \left(1 + \frac{0.045}{365}\right)^{15\times 365}$$

$$A = \ \$30000 \times (1.9640) \ = \ \$58918.54$$

92. $25,000 is invested for 30 years with an APR of 3.7% and quarterly compounding.

Solution

$$A = P\left(1 + \frac{APR}{n}\right)^{nY}$$

$$A = \$\ 25000 \times \left(1 + \frac{0.037}{4}\right)^{30\times 4}$$

$$A = \$\ 25000 \times (3.01892) \ = \ \$75472.89$$

93. $15,000 is invested for 15 years with an APR of 4.2% and monthly compounding.

Solution

$$A = P\left(1 + \frac{APR}{n}\right)^{nY}$$

$$A = \$\ 15000 \times \left(1 + \frac{0.042}{12}\right)^{15\times 12}$$

$$A \ = \ \$\ 15000 \times (1.0035)^{180}$$

$$A \ = \ 15000 \times (1.8755) \ = \$\ 28133.19$$

Annual Percentage Yield (APY)

The annual percentage yield (APY) also called the effective yield or simply the yield is the actual percentage by which a balance increases in one year. It is equal to the APR if interest is compounded annually. It is greater than the APR if interest is compounded more than once a year.

Banks usually list both the annual percentage rate (APR) and the annual percentage yield (APY). However, the APY is what your money really earns and is the more important number when you are comparing interest rates. Banks are required by law to state the APY on interest-bearing accounts.

94. More Compounding means a higher Yield: You deposit \$1000 into an investment with APR = 8%. Find the annual percentage yield with monthly compounding and with daily compounding.

Solution

$$\text{Monthly Compounding} = A = P\left(1+\frac{APR}{n}\right)^{(nY)} = 1000\left(1+\frac{0.08}{12}\right)^{12\times 1} = \$1083.00$$

Your balance increases by \$83 so the annual percent yield is

$$APY = \text{relative increase in 1 year} = \frac{\$83}{\$1000} = 0.083 = 8.3\% \qquad \text{(more that APR)}$$

$$\text{Daily Compounding} = A = P\left(1+\frac{APR}{n}\right)^{(nY)} = 1000\left(1+\frac{0.08}{365}\right)^{365\times 1} = \$1083.28$$

Your balance increases by \$83.28 so the annual percent yield is

$$APY = \text{relative increase in 1 year} = \frac{\text{absolute increase}}{\text{starting principal}} = \frac{\$83.28}{\$1000}$$

$$APY = \text{relative increase in 1 year} = 0.08328 = 8.328\% \quad \text{(slightly higher than APR)}$$

Find the annual percentage yield (to the nearest 0.01%) in the following situations.

95. A bank offers an APR of 3.1% compounded daily.

Solution

$$APY = \left[\left(1+\frac{APR}{365^{365}}\right)-1\right]\times 100\% = \left[\left(1+\frac{3.1\%}{365^{365}}\right)-1\right]\times 100\% = 3.15\%$$

96. A bank offers an APR of 3.2% compounded monthly

Solution

$$APY = \left[\left(1+\frac{APR}{12^{12}}\right)-1\right]\times 100\% = \left[\left(1+\frac{3.2\%}{12^{12}}\right)-1\right]\times 100\% = 3.25\%$$

97. A bank offers an APR of 1.23% compounded monthly.

Solution

$$APY = \left[\left(1+\frac{APR}{12^{12}}\right)-1\right]\times 100\% = \left[\left(1+\frac{1.23\%}{12^{12}}\right)-1\right]\times 100\% = 1.24\%$$

98. A bank offers an APR of 2.25% compounded quarterly.

Solution

$$APY = \left[\left(1+\frac{APR}{4^{4}}\right)-1\right]\times 100\% = \left[\left(1+\frac{2.25\%}{4^{4}}\right)-1\right]\times 100\% = 2.27\%$$

Continuous Compounding

Compounding infinitely many times per year is called Continuous Compounding. It represents the best possible compounding for a particular APR. With continuous compounding, the compound interest formula takes the following special form.

$$A = P\times e^{APR\times Y}$$

A = accumulated balance after Y years

P = Starting principal

APR = annual percentage rate (as a decimal)

Y = number of years

The number e is a special irrational number with a value of $e \approx 2.71828$.

99. Continuous Compounding: You deposit $100 in an account with an APR 8% and continuous compounding. How much will you have after 10 years?

Solution

$A = P\,e^{APR\times Y} = 100\,e^{0.08\times 10} = \$222.55$

100. Suppose you put money in an investment with an interest rate of APR = 3% compounded monthly and leave it there for the next 18 years. How much would you have to deposit now to realize $100,000 after 18 years?

Solution

$A = P\left(1+\frac{APR}{n}\right)^{(nY)}$

$P = \frac{A}{\left(1+\frac{APR}{n}\right)^{(nY)}} = \frac{100{,}000}{\left(1+\frac{0.03}{12}\right)^{(12\times 18)}} = \$58314.11$

Depositing $58314.11 now will yield the desire $100,000 in 18 years, assuming that the 3% APR does not change and that you make no withdrawals or additional deposits.

Present Value

The present value of an investment is the original principal invested, or the value of the investment before it earns any interest. Therefore, it is the principal (P) in the compound amount formula. Present value is used to determine how much money must be invested today in order for an investment to have a specific value at a future date.

The present value formula is $\mathbf{P} = \frac{\mathbf{A}}{(\mathbf{1}+\boldsymbol{i})^{\mathbf{N}}}$

Where P is the original principal invested at an interest rate of *i* per compounding period for N compounding periods, and A is the compound amount.

101. How much money should be invested in an account that earns 8% interest, compounded quarterly, in order to have $30,000 in 5 years?

Solution

$i = \frac{r}{n} = \frac{0.08}{4} = 0.02\ , N = nt = 4(5) = 20$

$P = \frac{A}{(1+i)^N} = \frac{30{,}000}{(1+0.02)^{20}} \approx 20{,}189.14$

To Calculate the interest earned.

$I = A - P = 4505.97 - 4000 = 505.97$

$20,189.14 should be invested in the account in order to have $30,000 in 5 years.

Effective Interest Rate

When interest is compounded, the annual rate of interest is called the **nominal rate**. The effective rate is the simple interest rate that would yield the same amount of interest after 1 year. When a bank advertises a "7% annual interest rate compounded daily and yielding 7.25%," the nominal interest rate is 7% and the effective rate is 7.25%.

102. A bank offers a savings account that pays 2.75% annual interest, compounded daily and yielding 2.79%.What is the effective rate on this account? What is the nominal rate?

Answer

The effective rate is 2.79%. The nominal rate is 2.75%.

103. A credit union offers a certificate of deposit at an annual interest rate of 3%, compounded monthly. Find the effective rate. Round to the nearest hundredth of a percent.

Solution

$i = \frac{r}{n} = \frac{0.03}{12} = 0.0025 \ , N = \ nt = 12(1) = 12$

$A = P(1 + i)^N = 100(1 + 0.0025)^{12} \approx 103.04$

To Find the interest earned on the $100.

$I = A - P = 103.04 - \ 100 = 3.04$

The effective interest rate is 3.04%.

104. One bank advertises an interest rate of 5.5%, compounded quarterly, on a certificate of deposit. Another bank advertises an interest rate of 5.25%, compounded monthly. Which investment has the higher annual yield?

Solution

Calculate $(1 + i)^N$ for each investment

$i = \frac{r}{n} = \frac{0.055}{4} = 0.0025 \ , N = \ nt = 4(1) = 4$

$(1 + i)^N = \left(1 + \frac{0.055}{4}\right)^4 \approx 1.0561448$

Also

$i = \frac{r}{n} = \frac{0.0525}{12} = 0.0025 \ , N = \ nt = 12(1) = 12$

$(1 + i)^N = \left(1 + \frac{0.0525}{12}\right)^{12} \approx 1.0537819$

Compare the two compound amounts.

$1.0561448 > 1.0537819$

An investment of 5.5% compounded quarterly has a higher annual yield than an investment that earns 5.25% compounded monthly.

NUMBERS

(SEQUENCE AND SERIES)

Sequences also called Progressions, are used to represent ordered lists of numbers. As the members of a sequence are in a definite order, so a correspondence can be established by matching them one by one with the numbers 1, 2, 3, 4,..... For example, if the sequence is 1, 4, 7, 10, ..., nth member, then such a correspondence can be set up as shown in the diagram below:

Position		the member of the sequence
1	→	1
2	→	4
3	→	7
4	→	10
⋮		⋮
n	→	nth member

Sequence

An arrangement of numbers in any specific order is called sequence or progression.

Or A sequence is a function whose domain is a subset of the set of natural numbers.

A sequence is a special type of a function from a subset of N to R or C. Sometimes, the domain of a sequence is taken to be a subset of the set {0, 1, 2, 3,...}, i.e., the set of non-negative integers. If all members of a sequence are real numbers, then it is called a real sequence.

Sequences are usually named with letters a, b, c etc., and n is used instead of x as a variable. If a natural number n belongs to the domain of a sequence a, the corresponding element in its range is denoted by a_n. For convenience, a special notation a_n is adopted for a(n)and the symbol $\{a_n\}$ or $a_1, a_2, a_3,....,a_n$,...is used to represent the sequence a. The elements in the range of the sequence $\{a_n\}$ are called its **terms**; that is, a_1 is the first term, a_2 the second term and a_n the nth term or the general term.

Types of Sequence

- Arithmetic Sequence (Progression)
- Geometric Sequence (Progression)
- Harmonic Sequence (Progression)

If the domain of a sequence is a finite set, then the sequence is called a finite sequence otherwise, an infinite sequence.
An infinite sequence has no last term.
Some examples of sequences are;
i) 1, 4, 9,...,121
ii) 1, 3, 5, 7, 9,...,21
iii) 1, 2, 4,...
iv) 1, 3, 7, 15, 31,...
v) 1, 6, 20, 56,...
The sequences (i) and (ii) are finite whereas the sequences (iii) to (v) are infinite.

1. Write first two, 21st and 26th terms of the sequence whose general term $(-1)^{n+1}$ is.
   Solution
   Given that $a_n = (-1)^{n+1}$
   For getting required terms, we put n = 1, 2, 21 and 26.
   $a_1 = (-1)^{1+1} = 1$
   $a_2 = (-1)^{2+1} = -1$
   $a_{21} = (-1)^{21+1} = 1$
   $a_{26} = (-1)^{26+1} = -1$

2. Write first five terms of the sequence if $a_n = 2n - 3$.
   Solution
   Given that $a_n = 2n - 3$
   For getting required terms, we put n = 1, 2, 3, 4, 5
   $a_1 = 2(1) - 3 = 2 - 3 = -1$
   $a_2 = 2(2) - 3 = 4 - 3 = 1$
   $a_3 = 2(3) - 3 = 6 - 3 = 3$
   $a_4 = 2(4) - 3 = 8 - 3 = 5$
   $a_5 = 2(5) - 3 = 10 - 3 = 7$

Arithmetic Progression (A.P)
A sequence $\{a_n\}$ is an Arithmetic Sequence or Arithmetic progression (A.P), if $a_n - a_{n-1}$ is the same number for all $n \in N$ and $n > 1$. The diference $a_n - a_{n-1}(n > 1)$ i.e., the difference of two consecutive terms of an A.P., is called the **common difference** and is usually denoted by **d**.
Its general formula is $\mathbf{a_n = a_1 + (n - 1)d}$

3. Find 7^{th} term of the sequence 7,11,15, ...
Solution
Given that $a_1 = 7, d = 11 - 7 = 4, n = 7$
Using formula: $a_n = a_1 + (n-1)d$
$a_7 = 7 + (7-1)(4)$
$a_7 = 31$

4. Find the general term and the eleventh term of the A.P. whose first term and the common difference are 2 and -3 respectively. Also write its first four terms.
Solution
Given that $a_1 = 2, d = -3$
Using formula: $a_n = a_1 + (n-1)d$
$a_n = 2 + (n-1)(3)$
$a_n = 2 - 3n + 3$
$a_n = 5 - 3n$ **General Term**
For getting 11^{th} term, we put n = 11
$a_{11} = 5 - 3(11) = 5 - 33 = -28$
For getting next terms, we put n = 2, 3, 4
$a_2 = 5 - 3(2) = 5 - 6 = -1$
$a_3 = 5 - 3(3) = 5 - 9 = -4$
$a_4 = 5 - 3(4) = 5 - 12 = -7$
Hence the first four terms of the sequence are: $2, -1, -4, -7$

5. If the 5th term of an A.P. is 13 and 17th term is 49, find a_n and a_{13}.
Solution
Given that $a_5 = 13, a_{17} = 49$
Using formula: $a_n = a_1 + (n-1)d$
$a_5 = a_1 + (5-1)d \Rightarrow a + 4d = 13$ ……….. (i)
$a_{17} = a_1 + (17-1)d \Rightarrow a_1 + 16d = 49$ ……….. (ii)
Solving (i) and (ii) we have
$a_1 = 1, d = 3$
Thus
$a_{13} = 1 + (13-1)3 = 37$
$a_n = 1 + (n-1)3 = 3n - 2$

6. Find the number of terms in the A.P. if; $a_1 = 3$, $d = 7$ and $a_n = 59$.
Solution
Given that $a_1 = 3, d = 7, a_n = 59$
Using formula: $a_n = a_1 + (n-1)d$
$59 = 3 + (n-1)7$
$56 = (n-1)7$
$n - 1 = 8$
$n = 9$
Thus the terms in the A.P. are 9.

7. If $a_{n-2} = 3n - 11$, find the nth term of the sequence.
Solution
Putting n = 3, 4, 5 in $a_{n-2} = 3n - 11$ we have
$a_1 = 3 \times 3 - 11 = -2$
$a_2 = 3 \times 4 - 11 = 1$
$a_3 = 3 \times 5 - 11 = 4$
Thus using
$a_n = a_1 + (n-1)d$
$a_n = -2 + (n-1)3$
$a_n = 3n - 5$

8. Find the missing value $4, 1, -2, \dots, -77$
Solution
Given that $a_1 = 4, d = 1 - 4 = -3, a_n = -77$
Using formula: $a_n = a_1 + (n-1)d$
$-77 = 4 + (n-1)(-3)$
$n = 28$

9. If the nth term of A.P. is $3n - 1$, find A.P
Solution
Given that $a_n = 3n - 1$
For getting required terms, we put n = 1, 2, 3, 4, 5
$a_1 = 3(1) - 1 = 3 - 1 = 2$
$a_2 = 3(2) - 1 = 6 - 1 = 5$
$a_3 = 3(3) - 1 = 9 - 1 = 8$
$a_4 = 3(4) - 1 = 12 - 1 = 11$
$a_5 = 3(5) - 1 = 15 - 1 = 14$
Required A.P. is 2,5,8,11,14

10. Which term of A.P. $5, 2, -1, \ldots$ is -85?

Solution

Given that $a_1 = 5, d = 2 - 5 = -3, a_n = -85$

Using formula: $a_n = a_1 + (n-1)d$

$-85 = 5 + (n-1)(-3)$

$n = 31$

so -85 is 31^{st} term.

11. Find the nth term of the sequence $\left(\frac{4}{3}\right)^2, \left(\frac{7}{3}\right)^2, \left(\frac{10}{3}\right)^2, \ldots$

Solution

$\left(\frac{4}{3}\right)^2, \left(\frac{7}{3}\right)^2, \left(\frac{10}{3}\right)^2, \ldots$

$4, 7, 10, \ldots$

Given that $a_1 = 4, d = 7 - 4 = 3$

Using formula: $a_n = a_1 + (n-1)d$

$a_n = 4 + (n-1)3$

$a_n = 3n + 1$

So $a_n = \left(\frac{3n+1}{3}\right)^2$

12. If $\frac{1}{a}, \frac{1}{b}$ and $\frac{1}{c}$ are in A.P. then show that $b = \frac{2ac}{a+c}$

Solution

Given that $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in A.P. so $\frac{1}{c} - \frac{1}{b} = \frac{1}{b} - \frac{1}{a}$

Given that $a_1 = 4, d = 7 - 4 = 3$

$\frac{1}{c} + \frac{1}{a} = \frac{1}{b} - \frac{1}{b} \Rightarrow \frac{a+c}{ac} = \frac{2}{b} \Rightarrow \mathbf{b} = \frac{\mathbf{2ac}}{\mathbf{a+c}}$

13. If $\frac{1}{a}, \frac{1}{b}$ and $\frac{1}{c}$ are in A.P. then show that common difference is $\frac{a-c}{2ac}$

Solution

Given that $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in A.P. so

$d = \frac{1}{c} - \frac{1}{b}$ ………. (i)

$d = \frac{1}{b} - \frac{1}{a}$ ………. (ii)

$2d = \frac{1}{c} - \frac{1}{a}$ adding both

$2d = \frac{a-c}{ac}$

$d = \frac{a-c}{2ac}$

Series
The sum of an indicated number of terms in a sequence is called a series.
For example, the sum of the first seven terms of the sequence $\{n^2\}$ is the series,
1 + 4 + 9 + 16 + 25 + 36 + 49.
The above series is also named as the 7th partial sum of the sequence $\{n^2\}$. If the number of terms in a series is finite, then the series is called a finite series, while a series consisting of an unlimited number of terms is termed as an infinite series.

Sum of first n terms of an arithmetic series

- $S_n = \frac{n}{2}(a_1 + a_n)$
- $S_n = \frac{n}{2}[2a_1 + (n-1)d]$

1. Find the 19th term and the partial sum of 19 terms of the arithmetic series:
$2 + \frac{7}{2} + 5 + \frac{13}{2} + \cdots$
Solution
Given that $a_1 = 2$, $d = \frac{7}{2} - 2 = \frac{3}{2}$
Using formula: $a_n = a_1 + (n-1)d$
$a_n = 2 + (19-1)\frac{3}{2} = 29$
Using formula: $S_n = \frac{n}{2}(a_1 + a_n)$
$S_{19} = \frac{19}{2}(2 + 29) = \frac{589}{2}$

2. Find the sum of following series
$5 + 8 + 11 + 14 + \cdots$ upto n terms
Solution
Given that $a_1 = 5$, $d = 3$
Using formula: $S_n = \frac{n}{2}[2(5) + (n-1)3]$
$S_n = \frac{n}{2}[10 + 3n - 3]$
$S_n = \frac{n}{2}(3n + 7)$

3. How many terms of the series $-9 - 6 - 3 + 0 + \cdots$ amount to 66?

Solution

Given that $a_1 = -9, d = 3, S_n = 66, n = ?$

Using formula: $S_n = \frac{n}{2}[2a_1 + (n-1)d]$

$66 = \frac{n}{2}[2(-9) + (n-1)3] = \frac{n}{2}[-18 + 3n - 3]$

$132 = n[3n - 21] \Rightarrow n(n-7) = 44 \Rightarrow n^2 - 7n - 44 = 0 \Rightarrow n = 11, -4$

But n cannot be negative in this case, so n = 11, that is, the sum of eleven terms amount to 66.

4. Find the arithmetic series if its fifth term is 19 and $S_4 = a_9 + 1$.

Solution

Given that $a_5 = 19$

$a_1 + 4d = 19$ ………. (i)

Also $S_4 = a_9 + 1$

$\frac{4}{2}[2a_1 + (4-1)d] = a_1 + 8d + 1$

$4a_1 + 6d = a_1 + 8d + 1$

$2d = 3a_1 - 1$ ………. (ii)

Using (ii) in (i) we have $a_1 + 2(3a_1 - 1) = 19 \Rightarrow a_1 + 6a_1 - 2 = 19 \Rightarrow a_1 = 3$

From (i) using $a_1 = 3$ we have $d = 4$

Thus the series is $3 + 7 + 11 + 15 + 19 + \cdots$

5. Find the sum of all the integral multiple of 3 between 4 and 97.

Solution

$6 + 9 + 12 + 15 + \cdots + 96$

Given that $a_1 = 6, d = 9 - 6 = 3, a_n = 96$

Using formula: $a_n = a_1 + (n-1)d$

$96 = 6 + (n-1)3 \Rightarrow n = 31$

Using formula: $S_n = \frac{n}{2}(a_1 + a_n)$

$S_{31} = \frac{31}{2}(6 + 96) = 1581$

6. If $S_n = n(2n-1)$ then find the series.

Solution

Given that $S_n = n(2n-1)$

For getting required terms, we put n = 1, 2, 3, 4, 5

$S_1 = a_1 = 1(2(1) - 1) = 1 \Rightarrow a_1 = 1$

$S_2 = a_1 + a_2 = 2(2(2) - 1) = 6 \Rightarrow 1 + a_2 = 6 \Rightarrow a_2 = 5$

$S_3 = a_1 + a_2 + a_3 = 3(2(3) - 1) = 15 \Rightarrow 1 + 5 + a_3 = 15 \Rightarrow a_3 = 9$

Required series $1 + 5 + 9 + \cdots$

Geometric Progression (G.P)

A sequence $\{a_n\}$ is a geometric sequence or geometric progression if $\frac{a_n}{a_{n-1}}$ is the same non-zero number for all $n \in N$ and $n > 1$. It is the sequence of numbers in which every term after the first is obtain from the preceding term by multiplying it with a constant number. The quotient $\frac{a_n}{a_{n-1}}$ is usually denoted by **r** and is called **common ratio** of the G.P .It is Clear that r is the ratio of any term of the G.P., to its predecessor. The common ratio $r = \frac{a_n}{a_{n-1}}$ is defined only if $a_{n-1} \neq 0$, i.e., no term of the geometric sequence is zero.

Remember for nth term of a G.P

- $\mathbf{a + ar + ar^2 + \cdots + ar^{n-1}}$ is the general form of a G.P.
- Each term after the first term is an r multiple of its preceding term.
- $\mathbf{a_n = ar^{n-1}}$ is the general term of a G.P.

1. Find the 5th term of the G.P., 3,6,12, ...

Solution

Given that $a = 3$, $r = \frac{6}{3} = 2$

Using $a_n = ar^{n-1}$ for n = 5

$a_5 = ar^{5-1} = ar^4 = 3 \times 2^4 = 48$

2. Find a_n if $a_4 = \frac{8}{27}$ and $a_7 = -\frac{64}{729}$ of a G.P.

Solution

To find a_n we have to find a_1 and r.

Using $a_n = ar^{n-1}$

$a_4 = ar^{4-1} = ar^3 = \frac{8}{27} \Rightarrow ar^3 = \frac{8}{27}$ ……….. (i)

$a_7 = ar^{7-1} = ar^6 = -\frac{64}{729} \Rightarrow ar^6 = -\frac{64}{729}$ ……….. (ii)

$\frac{a_7}{a_4} = \frac{ar^6}{ar^3} = \frac{-\frac{64}{729}}{\frac{8}{27}} \Rightarrow r^3 = -\frac{8}{27} \Rightarrow \mathbf{r} = -\frac{\mathbf{2}}{\mathbf{3}}$

(i) $\Rightarrow a\left(-\frac{2}{3}\right)^3 = \frac{8}{27} \Rightarrow \mathbf{a = -1}$

Thus $a_n = ar^{n-1} = (-1)\left(-\frac{2}{3}\right)^{n-1} = (-1)(-1)^{n-1}\left(\frac{2}{3}\right)^{n-1}$

$a_n = (-1)^n\left(\frac{2}{3}\right)^{n-1}$; $n \geq 1$

3. Find nth term of G.P., if $a = 8, r = \frac{3}{2}, n = 5$

Solution

Using $a_n = ar^{n-1} \Rightarrow a_5 = 8\left(\frac{3}{2}\right)^{5-1} = 8\left(\frac{3}{2}\right)^4 = \frac{81}{2}$

4. Find the geometric progression for which
First term = 2, Second term = −6, n = 5

Solution

Given that $a = a_1 = 2, a_2 = -6, n = 5, r = \frac{a_2}{a_1} = -\frac{6}{2} = -3$

Using $a_n = ar^{n-1}$

$a_1 = ar^{1-1} = ar^0 = 2(-3)^0 \Rightarrow a_1 = 2$

$a_2 = ar^{2-1} = ar^1 = 2(-3)^1 \Rightarrow a_2 = -6$

$a_3 = ar^{3-1} = ar^2 = 2(-3)^2 \Rightarrow a_3 = 18$

$a_4 = ar^{4-1} = ar^3 = 2(-3)^3 \Rightarrow a_4 = -54$

$a_5 = ar^{5-1} = ar^4 = 2(-3)^4 \Rightarrow a_5 = 162$

So the progression is $2, -6, 18, -54, 162$

5. Find the missing term if $a_n = 192, n = 7, r = 2$

Solution

Using $a_n = ar^{n-1}$

$192 = a(2)^{7-1} = a(2)^6 = a(64)$

$192 = a(64) \Rightarrow a = 3$

6. Find the missing term if $a = 4, a_n = \frac{1}{8}, r = \frac{1}{2}$

Solution

Using $a_n = ar^{n-1}$

$\frac{1}{8} = 4\left(\frac{1}{2}\right)^{n-1} \Rightarrow \frac{1}{32} = \left(\frac{1}{2}\right)^{n-1} \Rightarrow \left(\frac{1}{2}\right)^5 = \left(\frac{1}{2}\right)^{n-1}$

$\Rightarrow n - 1 = 5 \Rightarrow n = 6$

7. If $\frac{1}{a}, \frac{1}{b}$ and $\frac{1}{c}$ are in G.P. then show that common ratio is $\pm\sqrt{\frac{a}{c}}$

Solution

Given that $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in G.P. so

$r = \frac{1}{b}/\frac{1}{a} = \frac{a}{b}$ ………. (i) and $r = \frac{1}{c}/\frac{1}{b} = \frac{b}{c}$ ………. (ii)

$r^2 = \frac{a}{c}$ multiplying both

$r = \pm\sqrt{\frac{a}{c}}$

Sum of first n term of a G.P

- $\mathbf{S_n = \frac{a(1-r^n)}{1-r}}$ if $r < 1$
- $\mathbf{S_n = \frac{a(r^n-1)}{1-r}}$ if $r > 1$

Sum of infinite terms of a G.P

- $\mathbf{S_\infty = \frac{a}{1-r}}$ if $r \neq 1$

1. Find the sum of the series $9 + 3 + 1 + \cdots$ to 20th term.

Solution

Given that $a = 9, r = \frac{3}{9} = \frac{1}{3} < 1, n = 20$

Using $S_n = \frac{a(1-r^n)}{1-r}$

$S_{20} = \frac{9\left[1-\left(\frac{1}{3}\right)^{20}\right]}{1-\frac{1}{3}}$

$S_{20} = 13.5$

2. Find the sum of the series $3 + 3^2 + 3^3 + \cdots$

Solution

Given that $a = 3, r = \frac{3^2}{3} = 3 > 1$

Using $S_n = \frac{a(r^n-1)}{1-r}$

$S_n = \frac{3[(3)^n-1]}{3-1}$

$S_n = \frac{3}{2}(3^n - 1)$

3. Find the sum of the given information $a = 2, n = 5, l = a_n = 32$

Solution

Given that $a = 2, n = 5, l = a_n = 32$

Using $a_n = ar^{n-1}$

$32 = 2(r)^{5-1} \Rightarrow r^4 = 16 \Rightarrow r = 2 > 1$

Using $S_n = \frac{a(r^n-1)}{1-r}$

$S_5 = \frac{2[(2)^5-1]}{2-1}$

$S_5 = 62$

4. Find the sum of infinite G.P. 2, $\sqrt{2}$,1,...

Solution

Given that $a = 2, r = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$

Using $S_\infty = \frac{a}{1-r}$

$$S_\infty = \frac{2}{1-\frac{1}{\sqrt{2}}} = \frac{2\sqrt{2}}{\sqrt{2}-1} = \frac{2\sqrt{2}(\sqrt{2}+1)}{(\sqrt{2}-1)(\sqrt{2}+1)} = \frac{4+2\sqrt{2}}{2-1}$$

$$S_\infty = 4 + 2\sqrt{2}$$

5. Find the sum of the given information $a = 3, r = \frac{2}{3}$

Solution

Given that $a = 3, r = \frac{2}{3}$

Using $S_\infty = \frac{a}{1-r}$

$$S_\infty = \frac{3}{1-\frac{2}{3}}$$

$$S_\infty = 9$$

6. Convert the recurring decimal $2.\dot{2}\dot{3}$ into an equivalent common fraction (vulgur fraction).

Solution

Given that $2.\dot{2}\dot{3} = 2.232323\ ...$

$$2.\dot{2}\dot{3} = 2 + (.23 + .0023 + .000023 + \cdots)$$

Here $a = .23, r = \frac{.0023}{.23} = \frac{1}{100}$

Using $S_\infty = \frac{a}{1-r}$

$$2.\dot{2}\dot{3} = 2 + \left[\frac{.23}{1-\frac{1}{100}}\right] = 2 + \left[\frac{100\times.23}{99}\right]$$

$$2.\dot{2}\dot{3} = \frac{221}{99}$$

Harmonic Progression (H.P)
Reciprocal of Arithmetic Sequence/Progression is called Harmonic Sequence/Progression.

1. Find 9th term of the sequence $\frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \ldots$

Solution

Given that $\frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \ldots$ in H.P.

Then $3, 5, 7, \ldots$ in A.P.

$a_1 = 3, d = 5 - 3 = 2, n = 9$

Using formula: $a_n = a_1 + (n - 1)d$

$a_9 = 3 + (9 - 1)(2)$

$a_9 = 19$ in A.P.

$a_9 = \frac{1}{19}$ in H.P.

2. If the first term of an H.P is $-\frac{1}{3}$ and the fifth term is $\frac{1}{5}$. Find its 9th term.

Solution

Given that $a_1 = -\frac{1}{3}, a_5 = \frac{1}{5}$ in H.P.

Then Given that $a_1 = -3, a_5 = 5$ in A.P.

Using formula: $a_n = a_1 + (n - 1)d$

$a_5 = a_1 + 4d = 5$

$-3 + 4d = 5$

$d = 2$

Again using formula: $a_n = a_1 + (n - 1)d$

$a_9 = a_1 + (9 - 1)d$

$a_9 = a_1 + 8d$

$a_9 = -3 + (8)(2)$

$a_9 = 13$ in A.P.

$a_9 = \frac{1}{13}$ in H.P.

STATISTICS

(DESCRIPTIVE STATISTICS, SCALES OF MEASUREMENTS)

The word "Statistics" is derived from the Latin word **Status** or the Italian word **Statista** or the German word **Statistik** or the French word **Statistique** meaning "a political state" or "the state – man's art". It is the discipline that includes procedures or techniques used to collect process, analyzes numerical data to inferences and to reach decision in the face of uncertainty. It is the science of collection, presentation, analysis and interpretation of numerical data.

According to this definition, there are four stages.

- Collection of data
- Presentation of data
- Analysis of data
- Interpretation

Use/Importance of Statistical Information

- To inform general public
- To explain things that have happened
- To justify a claim
- To provide general comparison
- To estimate the unknown quantities

Limitations of Statistical Information

- Deals with aggregates and not with individuals.
- Deals with numerically specified characteristics.

Types of Statistics

- **Descriptive/Deductive Statistics:** Branch which deals with concepts and methods concerned with summarization and description of the important aspects of numerical data. In it no conclusion is drawn about the population.
- **Inferential/Inductive Statistics:** Branch which deals with drawing inferences about the population on the basis of sample information.

Below are some important terminologies we will be using in statistical data analysis.

- **Data** is the collection of all observations for a particular variable or variables, from one more people or things.
- The branch of mathematics that covers the methods and procedures in analyzing data is called **Statistics**. Statistics includes methods for planning studies and experiments, obtaining data, and then organizing, summarizing, presenting, analyzing, interpreting, and drawing conclusions.
- A **Population** is the collection of all individuals or items under consideration in a study. **e.g.** Total number of absent students, number of colour TV sets, Monthly salaries of all employies, number of computers sold out. Total number of objects in a population is called **population size**.
- A **Sample** is the part of a population from which information is actually collected. **e.g.** wheat yield per acre for 5 pieces of land. Total number of objects in a sample is called **sample size**.
- **Sample Survey** is the collection of information from a representative part of the population. It is carried out by an experimental design.
- A **Census** is information (data) obtained from the entire population.
- A **Parameter** is a numerical measurement describing some characteristic of a population.
  Examples: The average starting salary of elementary school teachers in Georgia is $33,673.
  The average for the whole United States is $35,763.
- A **Statistic** is a numerical measurement describing some characteristic of a sample. Example: A survey of ten job postings for elementary school teachers in the Atlanta area, had an average starting salary of $38,541.
- **Variable:** A **Variable** is a general type characteristic (or type of status), which can be different for each person or thing. It is a characteristic that varies with an individual or an object.
- **Constant:** Quantities which do not vary from individual to individual, place to place or time to time. e.g. π, 10 etc.

Some Facts about Data

- **Observation:** It is a fact or figure; we collect about a given variable. It can be expressed as a number or as a quality.
- **Data:** The collection of raw fact and figures is called data.
- **Data Set:** The collection of observations on one or more variables.
- **Cross Section Data:** Data collected on different elements at the same point in time or for the same period of time are called cross section data.
- **Time Series Data:** Data collected on the same element for the same variable at different points in time or for different periods of time are called time series data.
- **Discrete Data:** A data which is generated by a discrete variable is called discrete data.
- **Continuous Data:** A data which is generated by a continuous variable is called continuous data.
- **Datum:** A single numerical fact is datum.
- There are two types of data: Primary data and Secondary data.
- **Primary Data:** The data that have been initially collected and have not undergone any statistical treatment are called primary data.
- **Source of Primary Data:**
  Direct personal investigation.
  Indirect investigation or interviews.
  Collection through questionnaires.
  Collection through local sources.
  Through internet.
  Experimental research.
- **Secondary Data:** The data which has undergone any statistical treatment at least once is called primary data.
- **Source of Secondary Data:**
  - **Official:** Using the publications of statistical divisions, ministry of finance, the federal and provincial bureau of statistics, ministries of food, agriculture and industry etc.
  - **Semi Official:** State Bank of Pakistan, Railway Board, Central Cotton Committee, Board of Economic Inquiry, District Councils, Municipalities etc.
  - Publications of Trade Association, Chamber of Commerce etc.
  - Technical and Trade Journals and newspapers.
  - Research organizations such as universities and other institutions.

Presentation of Data

The device of gathering data often results in a massive volume of statical data which are in the form of individual measurement of counts. These are as follows;

- **Classification:** The process of dividing a set of observations or objects into classes or groups. It is the sorting of data into homogeneous classes or groups according to their being alike or not.
- **Tabulation:** A systematic presentation of data classified under suitable heads and subheads and placed in columns and rows. It is an orderly arrangement of data in columns and rows.
- **Graphical Display:** The visual display of statical data in the form of point lines, areas and other geometrical forms and symbols is in the most general term called graphical display. Such graphical representation divided into **graphs** and **diagrams**.

Data handling

Data handling is the process of securing the research data is gathered, archived or disposed of in a protected and safe way during and after the completion of the analysis process. Data handling means collecting the set of data and presenting in a different form.

Data Handling Steps

The steps involved in the data handling process are as follows:

- Problem Identification
- Data Collection
- Data Presentation
- Graphical Representation
- Data Analysis
- Conclusion
  From the analysis of the data, we can derive the solution to our problem statement. The data can be usually represented in any one of the following ways. They are:
  Bar Graph, Line Graphs, Histograms, Stem and Leaf Plot, Dot Plots, Frequency Distribution, Cumulative Tables and Graphs

Sampling

A **Variable** is a general type characteristic (or type of status), which can be different for each person or thing. It is a characteristic that varies with an individual or an object. Some variables are: name, height, color, texture, mood, wingspan, density, anxiety level, etc**.** The given set of all possible values from which the variable takes on a values is called its domain. There are two main types of variables.

- **Qualitative Variables** are variables which have values that are words, symbols, or categories. They can also be numbers that have no absolute measure, order or units. Example: gender, job title, letter grade, eye – colours, poverty etc. These are also called **attributes**. These also referred to as **Categorical Variables**.
- **Quantitative Variables** are variables which have values that are numerical values with a specific order and units. They represent counts or measures. Examples: age, weight, income, height, weight temperature, number of siblings, hours of sleep, etc. The **discrete data** can take only certain values such as **whole numbers**. The **continuous data** can take a value within the provided range.
- **Qualitative Data:** Data which is generated by qualitative variables is called qualitative data. It cannot be measured numerically.
- **Quantitative Data:** Data which is generated by quantitative variables is called quantitative data. It can be measured numerically.

Scale of Variables/ Measurement:

In statistics, there are few scales of measurements which are used in order to measure the statistical variables. Each of these measurements scales does measure a certain type of variable. Each measurement scales have some fundamental property by which they can be classified.

Properties

Every scale of measurement has few or all of the following properties explained below:

- Each and every value on a measurement scale does have a unique identity or meaning
- These values usually have some magnitude or an ordered relationship with one another. We may say that some of the values are smaller, while some are bigger.
- The intervals of the values (data points) on scales are equal to one another.
- A minimum value of zero. A minimum value of zero means that scale has a true zero point

Types of Measuring Variable Scales:

- **Nominal Scale:** Variables with categories or labels without any order or hierarchy.
  Example: Gender (male/female), Color (red/green/blue), Brands of vehicles (BMW, Honda, Toyota, Hyundai), Types of seasons (Summer, Winter, Autumn, Spring), Triangle, Right triangle, Parallelogram, Square, etc.
- **Ordinal Scale:** Variables with categories or ranks in a specific order, but without equal intervals between them.
  Example: Education level (dropout, high school, college), Socioeconomic status (low, middle, high), Grades (A, B, C, D), Positions (1st, 2nd, 3rd), Ranking of players
- **Interval Scale:** Variables with equal intervals between categories, but without a true zero point.
  Example: Temperature (°C or °F), IQ scores, Digital watches display 00:00 as the time at 12 AM, Kids' clothing sizes, where a zero size does not imply that a size does not exist., Pass and fail, where failing does not imply that the student received no credit.
- **Ratio Scale:** Variables with equal intervals, a true zero point, and a meaningful ratio between categories.
  Example: Height, Weight, Blood pressure, Age, Time, Salary, Distance

1. Examples of Measurement Scales

Nominal	Ordinal	Interval	Ratio
Gender	Grades	Temperature	Age
Eye Colour	Positions	IQ score	Weight
Religion	Rankings	SAT score	Height
Specialization	Ratings		Time
Nationality	Socio – Economic Status		Salary ……………………… Distance

2. Identify Types of Variables and at which type of Scale following variables fall

Variable Name	Type of Variable	Scale of Variable
Person’s CNIC number	Qualitative as well as Quantitative	Nominal Scale
Review of a product by giving number of Stars	Quantitative	Ordinal Scale
Number of shirts of players	Qualitative as well as Quantitative	Nominal Scale
Mobile phone sim slot sizes	Quantitative	Ordinal Scale
Temperature	Quantitative	Interval Scale (for Celsius and Fahrenheit scales) Ratio Scale (for Kelvin scale)
Number of product sale on different days of week	Quantitative	Ratio Scale
Growth rate of countries	Quantitative	Ratio Scale
Colors of flowers	Qualitative	Nominal Scale
Speed of vehicles	Quantitative	Ratio Scale
Ratio of different metals in an alloy	Quantitative	Ratio Scale

Write the Procedure to get summary statistics of a variable in MS Excel:

1. Select the entire range of data for the variable you want to analyze.
2. Click on the "Data" tab in the ribbon.
3. In the "Data Tools" group, click on "Data Analysis".
4. In the "Data Analysis" dialog box, select "Summary Statistics" and click OK.
5. Select the output options you want, such as mean, median, mode, standard deviation, variance, and range.
6. Click OK to generate the summary statistics.
7. The summary statistics will be displayed in a new worksheet or in a dialog box, depending on your output options.

Alternatively, you can also use functions to calculate summary statistics:

- Mean: = AVERAGE(range)
- Median: = MEDIAN(range)
- Mode: = MODE(range)
- Standard Deviation: = STDEV(range)
- Variance: = VAR(range)
- Range: = MAX(range)-MIN(range)

Replace "range" with the actual range of data you want to analyze.

Distributions

- A listing of all classes of the data and their frequencies is called a **Frequency distribution**. It is a tabular arrangement for classifying data into different groups and the number of observations falling in each group corresponds to the respective group. On the basis of type of variables, it has two types;
  Discrete frequency distribution
  Continuous frequency distribution
- The data presented in the form of frequency distribution is called **Grouped Data**.
- A listing of all classes and their relative frequencies is called a **Relative Frequency distribution**. Most distributions show frequencies as well as relative frequencies.

3. Discrete Frequency Table by using a Tally Column:

20 coins are tossed 5 times and the number of heads recorded at each toss are given below;
3,4,2,3,3,5,2,2,2,1,1,2,1,4,2,2,3,3,4,2.
Make frequency distribution of number of heads observed.
Solution: Let X = number of heads. The frequency distribution is given below;

X	Tally Marks	frequency (f)
1	|||	3
2	~~||||~~ |||	8
3	~~||||~~	5
4	|||	3
5	|	1

4. Continuous Frequency Table by listing Actual Values:

For data given below;
51,55,32,41,22,30,35,53,30,60,59,15,7,18,40,49,40,25,14,18,19,2,43,22,39,26,34,19,10,17, 47,38,13,30,34,54,10,21,51,52.
Make frequency distribution with a class interval of size 10.
Solution:

Class/Groups	Observations	frequency (f)
0 – 9	2,7	2
10 – 19	10,10,13,14,15,17,18,18,19,19	10
20 – 29	21,22,22,25,26	5
30 – 39	30,30,30,32,34,34,35,38,39	9
40 – 49	40,40,41,43,47,49	6
50 – 59	51,51,52,53,54,55,59	7
60 – 69	60	1

5. Continuous Frequency Table:

Bradley worked a summer job to earn money for college. His weekly hours over a 12 week period were 25, 32, 36, 32, 18, 28, 30, 36, 12, 16, 35, 36.

The Distribution would be as follows:

Hours	Frequency	Relative Frequency
10-19	3	$\frac{3}{12} = 0.25 = 25\%$
20-29	2	$\frac{2}{12} = 0.167 = 17\%$
30-39	7	$\frac{7}{12} = 0.583 = 58\%$
Total	12	100%

Remember;

- **Class Limits:** The minimum and the maximum values defined for a class or group are called Class Limits. The minimum value is called the **lower class limit** and maximum value is called the **upper class limit** of the class.
- **Class Boundaries:** The real class limits of a class are called **class boundaries**. A class boundary is obtained by adding two successive class limits and dividing the sum by 2. The value so obtained is taken as **upper class boundary** for the previous class and **lower class boundary** for the next class.
- **Mid – Point/ Class Mark:** For a given class the average of that class obtained by dividing the sum of upper and lower class by 2, is called the mid – point of class mark of that class.
- **Interval/ Class Width:** Difference between the class boundaries.
- **Cumulative Frequency:** The total of frequency up to an upper class limit or boundary is called the cumulative frequency.

Classes	Frequency (f)	Class Boundaries	Mid Point	Cumulative Frequency
10 – 14	5	9.5 – 14.5	12	5
15 – 19	12	14.5 – 19.5	17	5 + 12 = 17
20 – 24	30	19.5 – 24.5	22	17 + 30 = 47
25 – 29	25	24.5 – 29.5	27	47 + 25 = 72
30 – 34	6	29.5 – 34.5	32	72 + 6 = 78

Measures of Center/Averages

The Round-Off Rule: Round all calculations to one more decimal place than is present in the data. Round only the final answer, not the steps along the way. For example, if the data set has values that are go out to 2 decimal places, then all calculated statistics should be reported showing 3 decimal places.

Measures of Center (or measures of central tendency) are descriptive measures that indicate where the center or most typical value of a data set lies. Some of the specific measures of center are shown below.

Types of Averages

- **Mathematical Averages**
  - Mean / Arithmetic Mean
  - Geometric Mean
  - Harmonic Mean
- **Positional Averages**
  - Median
  - Mode

Arithmetic Mean / Mean

The Mean is sum of all the values of the observations, divided by the number of observations. The mean is also more commonly just called **average**.

The **Sample mean for ungrouped data** is represented by the symbol $\bar{X}$, which is called ’X-bar’: $\bar{X} = \frac{\sum X}{n}$, where n is the number of values in the sample.

The **Mean of Frequency Table/ Sample mean for grouped data** is represented by the symbol $\bar{X}$, which is called ’X-bar’: $\bar{X} = \frac{\sum fX}{\sum f}$, where *f* is the frequency in the sample.

The **Population mean** is represented by the symbol μ, which is called ’mu’: $\mu = \frac{\sum X}{N}$, Where N is the number of values in the population.

Example

For Bradley’s weekly hours at a summer job: 25, 32, 36, 32, 18, 28, 30, 36, 12, 16, 35, 36. Find the mean (average) hours he worked in a week.

Solution

$$\mu = \frac{\sum X}{N} = \frac{336}{12} = 28$$

Which we will report, according to the round off rule, as 28.0 hours worked in an average week.

6. (**Sample**)

Calculate mean for ungrouped data; 2,3,5,7,4,1.

$\bar{X} = \frac{\sum X}{n} = \frac{22}{6} = 3.6$

7. (**Population**)

Calculate mean for ungrouped data; 1,4,8,4,3,5,1,2,6,3.

$\mu = \frac{\sum X}{N} = \frac{37}{10} = 3.7$

8. Calculate mean for grouped data;

Classes	Frequency (f)	Mid Point (X)	fX
20 – 24	5	22	110
25 – 29	8	27	216
30 – 34	13	32	416
35 – 39	22	37	814
40 – 44	15	42	630
45 – 49	10	47	470
50 – 54	8	52	416
	$\sum f = 81$		$\sum fX = 3072$

$$\bar{X} = \frac{\sum fX}{\sum f} = \frac{3072}{81} = 37.9$$

9. Calculate mean for grouped data;

Classes	Frequency (f)	Mid Point (X)	fX
1 – 3	2	2	4
4 – 6	5	5	25
7 – 9	7	8	56
10 – 12	5	11	55
13 – 15	6	14	84
16 – 18	5	17	85
	$\sum f = 30$		$\sum fX = 309$

$$\bar{X} = \frac{\sum fX}{\sum f} = \frac{309}{30} = 10.3$$

10. Six friends in a biology class received test grades of 92, 84, 65, 76, 88, and 90. Find the mean of these test scores.

$\bar{X} = \frac{\sum X}{n} = \frac{495}{6} = 82.5$

11. Estimate the mean of the following frequency distribution.

Class	1-8	9-16	17-24	25-32	33-40
Frequency	3	5	7	2	1

Solution: The midpoints of the classes (X) are: 4.5; 12.5; 20.5; 28.5; 36.5.
Then the formula would be

$$\frac{\sum fX}{\sum f} = \frac{3(4.5)+5(12.5)+7(20.5)+2(28.5)+1(36.5)}{3+5+7+2+1} = \frac{313}{18} = 17.388$$

12. Compute the overall semester GPA for a college student who earned the following grades.

Course	Grade	Credits
Physics	B=3.0	4
English	C=2.0	3
Math	C=2.0	3
Study Skills	B=3.0	1
History	A=4.0	3

Solution: here we are looking for the average grade points, so the data values are grade (quality) points. The credit amounts are the weights.

$$GPA = \frac{\sum fX}{\sum f} = \frac{4(3.0) + 3(2.0) + 3(3.0) + 1(3.0) + 3(4.0)}{4 + 3 + 3 + 1 + 3} = \frac{39}{14} = 2.79$$

13. Find the mean of the data in Table.

Observed Event Number of Cable Television Connections, *x*	*Frequency* Number of Households, *f*, with *x* Cable Television Connections
0	5
1	12
2	14
3	3
4	2
5	3
6	0
7	1
	40 total

This row indicates that there are 14 households with two cable television connections.

Solution

The numbers in the right-hand column of Table are the frequencies f of the numbers in the first column. The sum of all the frequencies is 40.

$$\text{Mean} = \frac{\sum fX}{\sum f} = \frac{4(3.0) + 3(2.0) + 3(3.0) + 1(3.0) + 3(4.0)}{4 + 3 + 3 + 1 + 3} = \frac{39}{14} = 2.79$$

$$\text{Mean} = \frac{(0.5)+(1.12)+(2.14)+(3.3)+(4.2)+(5.3)+(6.0)+(7.1)}{40}$$

$$\text{Mean} = \frac{79}{40} = 1.975$$

The mean number of cable connections per household for the homes in the sub-division is 1.975.

Outlier

An outlier in a data set is a data value that is much higher or much lower than almost all other values. An outlier can change the mean of a data set but does not affect the median or mode.

14. Calculate effect of outlier on mean for ungrouped data;

245670, 176200, 360280, 272440, 450394, 310160, 393610, **3874480**

$$\text{Mean without Outlier} = \frac{245670+176200+360280+272440+450394,310160,+393610}{7}$$

$$\text{Mean without Outlier} = \frac{2208754}{7} = 315536.29$$

$$\text{Mean with Outlier} = \frac{245670+176200+360280+272440+450394,310160,+393610+3874480}{7}$$

$$\text{Mean with Outlier} = \frac{6083234}{7} = 760404.25$$

15. Calculate effect of outlier on mean for ungrouped data;

0.8161, 0.8194, 0.8165, 0.8176, 0.7901, 0.8143, 0.8126

$$\text{Mean without Outlier} = \frac{0.8161+0.8194+0.8165+0.8176+0.8143+0.8126}{7}$$

$$\text{Mean without Outlier} = 0.816$$

$$\text{Mean with Outlier} = \frac{0.8161+0.8194+0.8165+0.8176+0.7901+0.8143+0.8126}{7}$$

$$\text{Mean with Outlier} = 0.812$$

Geometric Mean

When we have the data in grades, ratio and percentage form, then we apply geometric mean. Geometric Mean of a variable X is the n^th positive root of the product of the $x_1, x_2, x_3, \dots, x_n$ observations. In symbol we write; $G.M = (x_1 x_2 x_3 \dots x_n)^{1/n}$

For ungrouped data

Geometric Mean $= antilog\left(\frac{\sum logX}{n}\right)$, where n is the number of values in the sample.

For grouped data

Geometric Mean $= antilog\left(\frac{\sum f logX}{\sum f}\right)$

16. Calculate geometric mean for ungrouped data; 8,40,175,1209,2000.

Solution

X	logX
8	0.91
40	1.61
175	2.24
1209	3.08
2000	3.30

Geometric Mean $= antilog\left(\frac{\sum logX}{n}\right) = 169.04$

17. Calculate geometric mean for ungrouped data; 3,13,11,15,5,4,2.

Solution

X	logX
3	0.477
13	1.113
11	1.041
15	1.176
5	0.698
4	0.602
2	0.301

Geometric Mean $= antilog\left(\frac{\sum logX}{n}\right) = 5.923$

18. Calculate geometric mean for grouped data;

Classes	Frequency (f)
15 – 19	5
20 – 24	3
25 – 29	8
30 – 34	2
35 – 39	4

Solution

Classes	Frequency (f)	Mid Point (X)	log$X$	$f$log$X$
15 – 19	5	17	1.23	6.15
20 – 24	3	22	1.34	4.02
25 – 29	8	27	1.43	11.44
30 – 34	2	32	1.50	3
35 – 39	4	37	1.56	6.24
	$\sum f = 22$			$\sum f\log X = 30.85$

Geometric Mean $= antilog\left(\frac{\sum flogX}{\sum f}\right) = antilog\left(\frac{30.85}{22}\right) = 25.24$

19. Calculate geometric mean for grouped data;

Solution

Classes	Frequency (f)	Mid Point (X)	log$X$	$f$log$X$
60 – 80	5	70	1.84	9.2
80 - 100	14	90	1.95	27.3
100 - 120	17	110	2.04	34.68
120 - 140	10	130	2.11	21.1
140 - 160	1	150	2.17	2.17
160 - 180	0	170	2.23	0
180 - 200	2	190	2.27	4.54
	$\sum f = 49$			$\sum f\log X = 166.31$

Geometric Mean $= antilog\left(\frac{\sum flogX}{\sum f}\right) = antilog\left(\frac{166.31}{49}\right) = 2477.88$

Harmonic Mean

Harmonic mean is defined as the reciprocal of mean and the reciprocal of the values. This type of average is also used as grades, ratio and percentage form of data.

- **For ungrouped data**

  Harmonic Mean $= \frac{n}{\Sigma\left(\frac{1}{x}\right)}$, where n is sample size.

- **For grouped data**

  Harmonic Mean $= \frac{\Sigma f}{\Sigma\left(\frac{f}{x}\right)}$

20. Calculate Harmonic mean for ungrouped data; 13.2,14.2,14.8,15.2,16.1.

Solution

X	logX
13.2	0.07
14.2	0.07
14.8	0.06
15.2	0.06
16.1	0.06

Harmonic Mean $= \frac{n}{\Sigma\left(\frac{1}{x}\right)} = \frac{5}{0.32} = 15.62$

21. Calculate Harmonic mean for grouped data;

Solution

Classes	Frequency (f)	Mid Point (X)	$\frac{f}{x}$
15 – 19	5	17	0.29
20 – 24	3	22	0.13
25 – 29	2	27	0.07
30 – 34	4	32	0.125
35 – 39	2	37	0.05
	$\sum f = 16$		

Harmonic Mean $= \frac{\Sigma f}{\Sigma\left(\frac{f}{x}\right)} = \frac{16}{0.66} = 24.24$

Median

It is the middle most value of the observations arranged in ascending and descending order of magnitude.

To find the median, first arrange the data in increasing order. If there are an odd number of observations, the median is the middle value in order. If there is an even number of observations, the median is the average of two middle values in order.

Example

- **Odd data:** 1,2,**3**,4,5 here median is 3.
- **Even data:** 1,2,**3**,**4**,5,6 here median is $\frac{3+4}{2} = 3.5$.

Formulae

- **For Ungroup Data**

$M = \left(\frac{n+1}{2}\right)^{th}$ for Odd data

$M = \frac{1}{2}\left[\left(\frac{n}{2}\right)^{th} + \left(\frac{n}{2}+1\right)^{th}\right] = \frac{1}{2}\left[\left(\frac{n}{2}\right)^{th} + \left(\frac{n+2}{2}\right)^{th}\right]$ for Even data

- **For Group Data**

$M = L + \frac{H}{f}\left(\frac{\sum f}{2} - C\right) = L + \frac{H}{f}\left(\frac{n}{2} - C\right)$ we may write $\sum f = n$

$L =$ Lower class boundary of median class

$H =$ Class interval

$f =$ Frequency of median class

$C =$ Cumulative Preceding frequency

22. Find the median of Bradley's summer weekly hours 25, 32, 36, 32, 18, 28, 30, 36, 12, 16, 35, 36.

Solution

First we must put the values in order from lowest to highest: 12, 16, 18, 25, 28, 30, 32, 32, 35, 36, 36, 36. There are two values in the middle, 30 and 32, with five values below and above. The average of the two middle values is 31, which is the median.

23. Find the median of 45,32,21,65,36,53,48,76,27.

Solution

$M = \left(\frac{n+1}{2}\right)^{th} = \left(\frac{9+1}{2}\right)^{th} = \left(\frac{10}{2}\right)^{th} = 5^{th}\text{term} = 45$

24. Find the median of 45,32,21,65,36,53,48,27.

Solution

$$M = \frac{1}{2}\left[\left(\frac{8}{2}\right)^{th} + \left(\frac{8}{2}+1\right)^{th}\right] = \frac{1}{2}\left[4^{th} + 5^{th}\right] = \frac{36+45}{2} = 40.5$$

25. Calculate Median for grouped data;

Solution

Classes	Frequency (f)	Class Boundaries	Cumulative Frequency
10 – 14	5	9.5 – 14.5	5
15 – 19	12	14.5 – 19.5	5 + 12 = **17**
20 – 24	**30**	**19.5 – 24.5**	**17 + 30 = 47**
25 – 29	25	24.5 – 29.5	47 + 25 = 72
30 – 34	6	29.5 – 34.5	72 + 6 = 78
	$\sum f = 78$		

$$M = L + \frac{H}{f}\left(\frac{\sum f}{2} - C\right) = 19.5 + \frac{5}{30}\left(\frac{78}{2} - 17\right) = 385.5$$

26. Find the median for the data in each of the following lists.

a. 4, 8, 1, 14, 9, 21, 12

b. 46, 23, 92, 89, 77, 108

Solution

a. Ranking the numbers from smallest to largest gives 1, 4, 8, 9, 12, 14, 21. The middle number is 9. Thus 9 is the median.

b. Ranking the numbers from smallest to largest gives 23, 46, 77, 89, 92, 108. The two middle numbers are 77 and 89. The mean of 77 and 89 is 83. Thus 83 is the median of the data.

27. The median of the ranked list 3, 4, 7, 11, 17, 29, 37 is 11. If the maximum value 37 is increased to 55, what effect will this have on the median?

Answer

The median will remain the same because 11 will still be the middle number in the ranked list.

Mode
The French word Mode mean ‘Fashion’ has been adopted to convey the idea of most frequent. The Mode is the value that has the most number of observations (frequency), but must occur more than once. Most occurring value of the data set is called mode.

Remember for Ungrouped Data

- A distribution having a single mode called **unimodel** mode. e.g. 2,1,7,2,6,4,2 has a unimodel mode that is 2.
- A distribution having two modes called **bimodel** mode. e.g. 2,5,3,2,4,3,2,3 has a bimodel mode that is 2 and 3.
- A distribution having more than two modes called **multimodel** mode. e.g. 4,1,2,3,6,4,5,2,3,4,2,1,3 has a multimodel mode that is 2,3 and 4.

28. Find the mode of Bradley’s summer weekly hours 25, 32, 36, 32, 18, 28, 30, 36, 12, 16, 35, 36.

Solution
Having the values in order makes this easier: 12, 16, 18, 25, 28, 30, 32, 32, 35, 36, 36, 36. The values that occurs the most often is 36, which is the only mode here.

Remember for Grouped Data

Mode $= L + \frac{(f_m - f_1)}{(f_m - f_1)+(f_m - f_2)} \times H$

$L =$ Lower class boundary of maximum frequency
$H =$ Class interval
$f_m =$ Maximum frequency
$f_1 =$ Preceding frequency of Maximum frequency
$f_2 =$ Following frequency of Maximum frequency

29. Calculate Median for grouped data;

Solution

Classes	Frequency (f)	Class Boundaries
15 – 19	5	14.5 – 19.5
20 – 24	**10**	**19.5 – 24.5**
25 – 29	2	24.5 – 29.5
30 – 34	4	29.5 – 34.5
35 – 39	3	34.5 – 39.5

Mode $= \mathrm{L} + \frac{(\mathrm{f_m} - \mathrm{f_1})}{(\mathrm{f_m} - \mathrm{f_1})+(\mathrm{f_m} - \mathrm{f_2})} \times \mathrm{H} = 19.5 + \frac{(10-5_1)}{(10-5)+(10-2)} \times 5 = 9.42$

30. Find the mode of the data in each of the following lists.

a. 18, 15, 21, 16, 15, 14, 15, 21 b. 2, 5, 8, 9, 11, 4, 7, 23

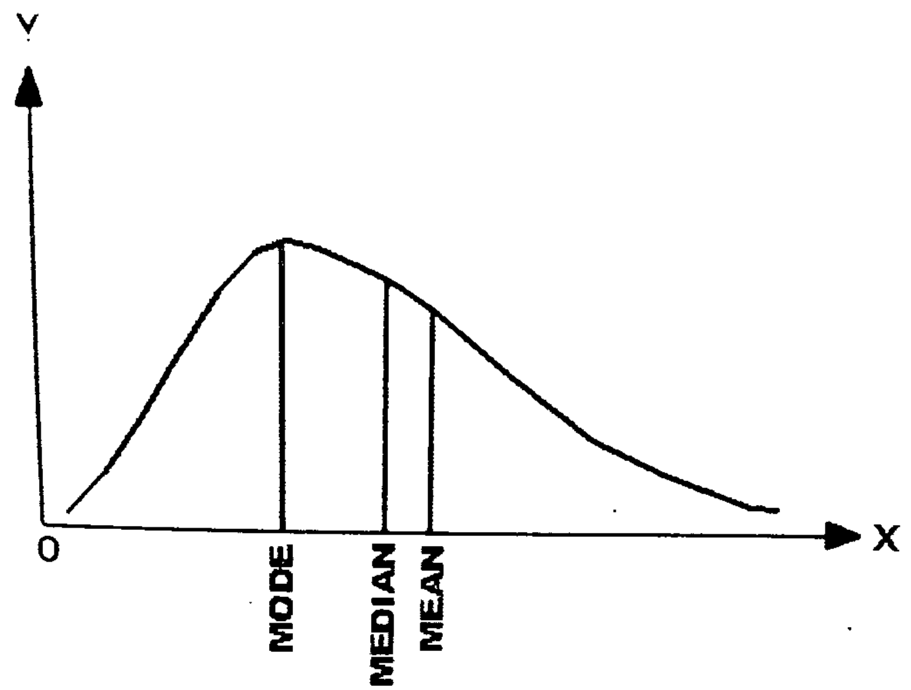
Solution

a. In the list 18, 15, 21, 16, 15, 14, 15, 21, the number 15 occurs more often than the other numbers. Thus 15 is the mode.

b. Each number in the list 2, 5, 8, 9, 11, 4, 7, 23 occurs only once. Because no number occurs more often than the others, there is no mode.

Empirical Relation between Mean, Median and Mode

In a single – peaked frequency distribution, the values of the mean, median and mode coincide if the frequency distribution is absolutely symmetrical. But if these values differ, the frequency distribution is said to be skewed or asymmetrical.



Experience tells us that in a unimodal curve of moderate skewness, the median is usually sandwiched between the mean and the mode and between them the following approximate relation hold good.

$\text{Mean} - \text{Mode} = 3(\text{Mean} - \text{Median})$

Or $\text{Mode} = 3\text{Median} - 2\text{Mean}$

This empirical relation does not hold in case of a J – Shaped or an extremely skewed distribution.

Comparative Properties of the Mean, the Median, and the Mode

The mean of a set of data:

■ is the most sensitive of the averages. A change in any of the numbers changes the mean.
■ can be different from each of the numbers in the set.
■ can be changed drastically by changing an extreme value.

The median of a set of data:

■ is usually not changed by changing an extreme value.
■ is generally easy to compute.

The mode of a set of data:

■ may not exist, and when it does exist it may not be unique.
■ is one of the numbers in the set, provided a mode exists.
■ is generally not changed by changing an extreme value.
■ is generally easy to compute.

Note

In the following example, we compare the mean, the median, and the mode of the salaries of five employees of a small company.

Salaries: \$370,000 \$60,000 \$32,000 \$16,000 \$16,000

The sum of the five salaries is \$494,000. Hence the mean is

The median is the middle number, \$32,000. Because the \$16,000 salary occurs the most, the mode is \$16,000. The data contain one extreme value that is much larger than the other values. This extreme value makes the mean considerably larger than the median. Most of the employees of this company would probably agree that the median of \$32,000 better represents the average of the salaries than does either the mean or the mode.

$$\frac{\$494{,}000}{5} = \$98{,}800$$

Measure of Dispersion/ Deviation

Deviation: A Deviation is defined as ‘a difference of any value of the variable from any constant’. $D_i = x_i - A$

Dispersion: The word dispersion has a technical meaning in statistics. It means the **spread** or **scatterness** of observations in a data set. The measures that are used to determine the degree or extent of variation in a data set are called **measure of dispersion**. If observations are closed to the centre we say that dispersion is small. If observations are spread away from the centre we say that dispersion is large. These include Range, Variance, Standard Deviation, Mean Deviation and Quartile Deviation. These are as follows;

- **Absolute Measure of Dispersion**
  Dispersion expressed in terms of original data.
  These always in same unit.
  We can’t change these units.
  These are restricted measures.
  We can’t compare the values of data.
  These provide actual values.
- **Relative Measure of Dispersion**
  Relative measure of dispersion is used to calculate the comparison of dispersion in two or more than two sets of observations.
  These are unit free.
  These are unrestricted.
  We can compare the values of data.
  These provide data in ratio or percentage form.

Variance and Standard Deviation

Variance: Variance is defined as the mean of the squared deviation of $x_i; (i = 1,2,3, \dots, n)$ observations from their arithmetic mean.

Standard Deviation: Standard Deviation is defined as positive square root of the mean of the squared deviation of $x_i; (i = 1,2,3, \dots, n)$ observations from their arithmetic mean

Standard Deviation is the measure of how far, on average, the data is from the mean.

Another related measure, is the **Variance** which is standard deviation squared.

The standard deviation and variance for a **SAMPLE** are calculated by the following symbols and formulas:

- **Variance** $= s^2 = \frac{\sum(X-\bar{X})^2}{n}$ for ungroup data
- **Variance** $= s^2 = \frac{\sum f(X-\bar{X})^2}{\sum f}$ for group data
- **Standard Deviation** $= s = \sqrt{\frac{\sum(X-\bar{X})^2}{n}}$ for ungroup data
- **Standard Deviation** $= s = \sqrt{\frac{\sum f(X-\bar{X})^2}{\sum f}}$ for group data

The standard deviation and variance for a **POPULATION** are calculated by the following symbols and formulas:

- **Variance** $= \sigma^2 = \frac{\sum(X-\mu)^2}{n}$
- **Standard Deviation** $= \sigma = \sqrt{\frac{\sum(X-\mu)^2}{n}}$

Computational formulae of Variance and Standard Deviation:

- **Variance** $= s^2 = \frac{\sum X^2}{n} - \left(\frac{\sum X}{n}\right)^2$ for ungroup data
- **Variance** $= s^2 = \frac{\sum fX^2}{\sum f} - \left(\frac{\sum fX}{\sum f}\right)^2$ for group data
- **Standard Deviation** $= s = \sqrt{\frac{\sum X^2}{n} - \left(\frac{\sum X}{n}\right)^2}$ for ungroup data
- **Standard Deviation** $= s = \sqrt{\frac{\sum fX^2}{\sum f} - \left(\frac{\sum fX}{\sum f}\right)^2}$ for group data

31. The following numbers were obtained by sampling a population.

2, 4, 7, 12, 15

Find the standard deviation of the sample.

Solution

$$\bar{X} = \frac{\sum X}{n} = \frac{32+4+7+12+15}{5} = \frac{40}{5} = 8$$

x	$x - \bar{x}$	$(x - \bar{x})^2$
2	$2 - 8 = -6$	$(-6)^2 = 36$
4	$4 - 8 = -4$	$(-4)^2 = 16$
7	$7 - 8 = -1$	$(-1)^2 = 1$
12	$12 - 8 = 4$	$4^2 = 16$
15	$15 - 8 = 7$	$7^2 = 49$
		118 ← The sum of the squared deviations

Variance $= s^2 = \frac{\sum(X-\bar{X})^2}{n} = \frac{118}{5} = 23.6$

Standard Deviation $= s = \sqrt{\frac{\sum(X-\bar{X})^2}{n}} = \sqrt{23.6} = 4.858$

32. The marks of six students in Mathematics are as follows. Determine variance and standard deviation.

Student No.	1	2	3	4	5	6
Marks	60	70	30	90	80	42

Solution

Let X = marks of a student.

$\bar{X} = \frac{\sum X}{n} = \frac{372}{6} = 62$

X	X^2	$X - \bar{X}$	$(X - \bar{X})^2$
60	3600	– 2	4
70	4900	8	64
30	900	– 32	1024
90	8100	28	784
80	6400	18	324
42	1764	– 20	400
$\sum X = 372$	$\sum X^2 = 25664$	$\sum(X - \bar{X}) = 0$	$\sum(X - \bar{X})^2 = 2600$

Variance $= s^2 = \frac{\sum(X-\bar{X})^2}{n} \approx 433.3333$

computational Variance $= s^2 = \frac{\sum X^2}{n} - \left(\frac{\sum X}{n}\right)^2 \approx 433.3333$

Standard Deviation $= s = \sqrt{\frac{\sum(X-\bar{X})^2}{n}} \approx 20.81666$

computational Standard Deviation $= s = \sqrt{\frac{\sum X^2}{n} - \left(\frac{\sum X}{n}\right)^2} \approx 20.81666$

33. For the following data showing weights of toffee boxes in gm. Determine variance and standard deviation.

X	4.5	14.5	24.5	34.5	44.5	54.5	64.5
f	2	10	5	9	6	7	1

Solution

Let X = marks of a student.

$\bar{X} = \frac{\sum X}{\sum f} = \frac{241.5}{40} = 6.0376$

X	f	$X - \bar{X}$	$(X - \bar{X})^2$	$f(X - \bar{X})^2$	fX	fX^2
4.5	2	– 28	784	1568	9	40.5
14.5	10	-18	324	3240	145	2102.5
24.5	5	– 8	64	320	122.5	3001.5
34.5	9	2	4	36	310.5	10712.25
44.5	6	12	144	864	267	11881.5
54.5	7	22	484	3388	381.5	20791.75
64.5	1	32	1024	1024	64.5	4160.25

$\textbf{Variance} = s^2 = \frac{\sum f(X-\bar{X})^2}{\sum f} = \frac{10600}{40} = 265$

$\textbf{computational Variance} = s^2 = \frac{\sum fX^2}{\sum f} - \left(\frac{\sum fX}{\sum f}\right)^2 = \frac{52690}{40} - \left(\frac{1300}{40}\right)^2 = 261$

$\textbf{Standard Deviation} = s = \sqrt{261} = 16.155$

$\textbf{computational Standard Deviation} = s = \sqrt{261} = 16.155$

34. A consumer group has tested a sample of eight size D batteries from each of three companies. The results of the tests are shown in the following table. According to these tests, which company produces batteries for which the values representing hours of constant use have the smallest standard deviation?

Company	Hours of Constant Use per Battery
EverSoBright	6.2, 6.4, 7.1, 5.9, 8.3, 5.3, 7.5, 9.3
Dependable	6.8, 6.2, 7.2, 5.9, 7.0, 7.4, 7.3, 8.2
Beacon	6.1, 6.6, 7.3, 5.7, 7.1, 7.6, 7.1, 8.5

Solution

The mean for each sample of batteries is 7 hours.

The batteries from Ever So Bright have a standard deviation of

$$\textbf{Standard Deviation} = s = \sqrt{\frac{\sum(X-\bar{X})^2}{n}} = \sqrt{\frac{12.34}{7}} \approx 1.328 \text{ hours}$$

The batteries from Dependable have a standard deviation of

$$\textbf{Standard Deviation} = s = \sqrt{\frac{\sum(X-\bar{X})^2}{n}} = \sqrt{\frac{3.62}{7}} \approx 0.719 \text{ hours}$$

The batteries from Beacon have a standard deviation of

$$\textbf{Standard Deviation} = s = \sqrt{\frac{\sum(X-\bar{X})^2}{n}} = \sqrt{\frac{5.38}{7}} \approx 0.877 \text{ hours}$$

The batteries from Dependable have the smallest standard deviation. According to these results, the Dependable company produces the most consistent batteries with regard to life expectancy under constant use.

35. Can the variance of a data set be smaller than the standard deviation of the data set?

Answer

Yes. The variance is smaller than the standard deviation whenever the standard deviation is less than 1.

Range

Measures of Variation (or measures of spread) are descriptive measures that indicate how much variation is in the data or how spreads out the data values are from each other.

- The Minimum (Min) is the lowest value in the data set.
- The Maximum (Max) is the highest value in the data set.

The **Range** is the difference between Minimum and Maximum values. It is the difference between largest and smallest values of a data set.

Range $= X_{max} - X_{min}$

Coefficient of Range $= \frac{X_{max}-X_{min}}{X_{max}+X_{min}}$

36. Find the min, max, and range of Bradley’s summer weekly hours.

Solution

Having the values in order makes this easier as well: 12, 16, 18, 25, 28, 30, 32, 32, 35, 36, 36, 36. Here min = 12, max = 36, and Range = 36 − 12 = 24 hours. We could also say the range is from 12 to 36 hours.

37. Calculate range for 2,12,15,4,7,10,13.

Solution

Range $= X_{max} - X_{min} = 15 - 2 = 13$

Coefficient of Range $= \frac{X_{max}-X_{min}}{X_{max}+X_{min}} = \frac{15-2}{15+2} = \frac{13}{17} = 0.76$

38. Calculate Range for grouped data;

Solution

Classes	Frequency (f)	Class Boundaries
15 – 19	5	**14.5** – 19.5
20 – 24	3	19.5 – 24.5
25 – 29	2	24.5 – 29.5
30 – 34	4	29.5 – 34.5
35 – 39	2	34.5 – **39.5**

Range $= X_{max} - X_{min} = 39.5 - 14.5 = 25$

Coefficient of Range $= \frac{39.5-14.5}{39.5+14.5} = \frac{15-2}{15+2} = \frac{25}{54} = 0.46$

Mean Deviation/ the Average Deviation

A defect of range is that it is based on only two extreme observations. The mean deviation does take into consideration based on all values. The mean of absolute deviation of observations from mean is called mean deviation.

Formulae

- **For Ungroup Data**

  $M.D = \frac{\sum|X-\bar{X}|}{n}$

Coefficient of M.D = $\frac{MD}{\bar{X}=mean}$ or Coefficient of M.D = $\frac{MD}{median}$

- **For group Data**

  $M.D = \frac{\sum f|X-\bar{X}|}{\sum f}$

39. Find MD of 2,5,6,6,8,9,12,13,10,23.

Solution

$\bar{X} = \frac{\sum X}{n} = \frac{100}{10} = 10$

X	$X-\bar{X}$	$\|X-\bar{X}\|$
2	-8	8
5	-5	5
6	-4	4
6	-4	4
8	-2	2
9	-1	1
12	2	2
13	3	3
16	6	6
23	13	13

$M.D = \frac{\sum|X-\bar{X}|}{n} = \frac{48}{10} = 4.8$

Coefficient of M.D = $\frac{MD}{\bar{X}} = \frac{4.8}{10} = 0.48$

40. Find MD of the following frequency distribution table;

Weight	65-84	85-104	105-124	125-144	145-164	165-184	185-204
Frequency	9	10	17	10	5	4	5

Solution

Weight	f	X	fX	$X-\bar{X}$	$f\lvert X-\bar{X}\rvert$
65 - 84	9	74.5	670.5	-48.0	432.0
85-104	10	94.5	945.0	-28.0	280.0
105-124	17	114.5	1946.5	-8.0	136.0
125-144	10	134.5	1345.0	12.0	120.0
145-164	5	154.5	772.5	32.0	160.0
165-184	4	174.5	698.0	52.0	208.0
185-204	5	194.5	972.5	72.0	360.0
Total	60	…	7350.0	…	1696.0

$$\bar{X} = \frac{\sum fX}{\sum f} = \frac{7350}{60} = 122.5$$

$$M.D = \frac{\sum f|X-\bar{X}|}{\sum f} = \frac{1696.0}{60} = 28.27$$

Quartiles

Quartiles are the values of the variate that divide a set of data into four equal parts after arranging the observations in ascending order of magnitude.

Deciles

Deciles are the values of the variate that divide a set of data into ten equal parts after arranging the observations in ascending order of magnitude.

Percentiles

Percentiles are the values of the variate that divide a set of data into one hundred equal parts after arranging the observations in ascending order of magnitude.

Quartile Deviation/ Semi Inter – Quartile Range

It is also called semi inter – quartile range. The SIQR is the measure of dispersion defined by the difference between third quartile and the first quartile and half of the range is called quartile deviation. The QD is also an absolute measure of dispersion. Its relative measure called coefficient of quartile deviation.

Formulae

$$Q.D = \frac{Q_3 - Q_1}{2}$$

Where $Q_3 = \left(\frac{3(n+1)}{4}\right)^{th}$ and $Q_1 = \left(\frac{n+1}{4}\right)^{th}$

Coefficient of Q.D $= \frac{Q_3 - Q_1}{Q_3 + Q_1}$

41. Find QD of 45,32,21,65,36,53,48,76,27.

Solution

Arrange: 21,27,32,36,45,48,53,65,76

$$Q_1 = \left(\frac{n+1}{4}\right)^{th} = \left(\frac{9+1}{4}\right)^{th} = \left(\frac{10}{4}\right)^{th} = (2.5)^{th} = 27.5$$

$$Q_3 = \left(\frac{3(n+1)}{4}\right)^{th} = \left(\frac{3(9+1)}{4}\right)^{th} \left(\frac{3(10)}{4}\right)^{th} = \left(\frac{30}{4}\right)^{th} = (7.5)^{th} = 53.5$$

$$Q.D = \frac{Q_3 - Q_1}{2} = \frac{53.5 - 27.5}{2} = \frac{26}{2} = 13$$

Coefficient of Q.D $= \frac{Q_3 - Q_1}{Q_3 + Q_1} = \frac{53.5 - 27.5}{53.5 + 27.5} = \frac{26}{81} = 0.32$

42. Use Medians to Find the Quartiles of a Data Set

The following table lists the calories per 100 milliliters of 25 popular beers. Find the quartiles of the data.

Calories, per 100 Milliliters, of Selected Beer

43	37	42	40	53	62	36	32	50	49
26	53	73	48	45	39	45	48	40	56
41	36	58	42	39					

Solution

Step 1: Rank the data, as shown in the following table.

1) 26	**2)** 32	**3)** 36	**4)** 36	**5)** 37	**6)** 39	**7)** 39	**8)** 40	**9)** 40
10) 41	**11)** 42	**12)** 42	**13)** 43	**14)** 45	**15)** 45	**16)** 48	**17)** 48	**18)** 49
19) 50	**20)** 53	**21)** 53	**22)** 56	**23)** 58	**24)** 62	**25)** 73		

Step 2: The median of these 25 data values has a rank of 13. Thus the median is 43. The second quartile Q_2 is the median of the data, so $Q_2 = 43$

Step 3: There are 12 data values less than the median and 12 data values greater than the median. The first quartile is the median of the data values less than the median. Thus Q_1 is the mean of the data values with ranks of 6 and 7.

$Q_1 = \frac{39+39}{2} = 39$

The third quartile is the median of the data values greater than the median. Thus Q_3 is the mean of the data values with ranks of 19 and 20.

$Q_3 = \frac{50+53}{2} = 51.5$

Coefficient of Variation

For comparing two very different data sets we will use a special measure called the **Coefficient of Variation** (or **CV**). It is equal to the standard deviation divided by the mean, converted into a percent. It has no units, it is only a ratio as percent. The formulas are slightly different, depending upon the data set being from a sample or a population. The CV states how big the standard deviation is, relative to the average size of the data.

For a sample: CV = $\frac{s}{\bar{X}} \times 100\%$ where s is standard deviation.

For a population: CV = $\frac{\sigma}{\mu} \times 100\%$ where σ is standard deviation.

43. Which data set is more spread out, the weight of elephants in a herd: s = 1,175 pounds and $\bar{X} = 12{,}342$ pounds, or the price of regular unleaded gasoline in a US: s = \$0.26 and $\bar{X} = \$3.73$?

Solution

For the elephant weights, CV = $\frac{s}{\bar{X}} \times 100\% = \frac{1{,}175}{12{,}342} \times 100\% = 9.5\%$

For the gas prices, CV = $\frac{\sigma}{\mu} \times 100\% = \frac{0.26}{3.73} \times 100\% = 7.0\%$

The weights of elephants are a more spread out set of data than US gas prices. This is NOT because the values are larger. Another set of large values could have a lower CV than gas prices.

44. Without graph judge shape of following data set. Also find Coefficient of Variation for the following data.

daily sales of laptops: 5,3,7,8,9,8,7,6,1,0,2,10,9,8,8,8,9,9,10,12,7

Solution

Based on the data, without graphing, we can observe the following:

- The data is not uniformly distributed, with a few low values (0, 1, 2) and a few high values (10, 12).
- The data appears to be skewed to the right, with more values concentrated on the higher side.
- There is a possible outlier (0) which is significantly lower than the other values.

To calculate the Coefficient of Variation (CV), we need to calculate the mean and standard deviation of the data:

Mean = (sum of all values) / (total number of values)

Mean = (5 + 3 + ... + 12 + 7) / 21 = 146 / 21 = 6.95

Standard Deviation $= s = \sqrt{\frac{\sum(X-\bar{X})^2}{n}} = \sqrt{\frac{194.95}{21}} = 3.047$

CV = $\frac{s}{\bar{X}} \times 100\% = \frac{3.047}{6.95} \times 100\% = 43.8\%$

RULES OF COUNTING

(MULTIPLICATION RULE, FACTORIAL, PERMUTATION, COMBINATION)

Combinatorics

The study of counting the different results of a task is a branch of mathematics called **combinatorics**.

Experiment

In combinatorics, an activity with an observable outcome is called an **experiment**. For example, if a coin is flipped, the side facing upward will be a head or a tail. The two possible outcomes can be listed as {H,T}. If a regular six-sided die is rolled, the possible outcomes can be listed as {1, 2, 3, 4, 5, 6}.

1. Suppose we roll two standard six-sided dice, first one colored green followed by a second one colored red. How many different results are possible?
   Solution
   There are 36 different possibilities.
   Remember
   Rolling two dice, as shown in Example 1, is an example of a multistage experiment. (This experiment involved two stages; first one die was rolled, then a second die was rolled.)

Arrangements with repetition
The type of counting problems in which we repeatedly select from the same group of choices is called **Arrangements with repetition**.
Arrangements with repetition, also known as permutations with repetition, are arrangements of objects where some items are identical. The presence of identical items in permutations reduces the number of unique arrangements.
If we make r selections from a group of n choices, a total of $\frac{n \times n \times n \times .. \times n = n^r}{r}$ different arrangements are possible.
Here are some examples of arrangements with repetition:

- Ordering boys and girls without distinguishing between students of the same grade
- Ordering cars of certain colors without distinguishing between cars of the same color
- Selecting ice cream flavors where one flavor can be selected multiple times

2. **How many seven symbol license plates are possible if both numerals and uppercase letters can be used in any order?**
   $n^r = 36^7 = 78364164096$

3. **How many six character passwords can be made by combining lower case letters, upper case letters, numerals and the characters $\propto$, $ and &?**
   Each password character can be selected from the set of 26 lower case letters, 26 upper case letters, 10 numerals and 3 symbols, making a total of 65 characters to choose from.
   $n^r = 65^6 = 75418890625$

4. **Suppose you coach a team of 4 swimmers. How many different ways can you put together a four – persons relay team?**
   You can choose any of the four swimmers for the first leg. Once you’ve chosen the first swimmer, you have three swimmers left to choose from for the second leg. Therefore the number of possible choices for the first two legs combined is $4 \times 3 = 12$. You then have two swimmers to choose from for the third leg, and only one choice for the last leg. The total number of possible arrangements for the relay is $4 \times 3 \times 2 \times 1 = 24$.

Fundamental Principle of Counting
Let E be a multistage experiment. If $n_1, n_2, n_3, \dots, n_r$ are the numbers of possible outcomes of each of the r stages of E, then there are $n_1, n_2, n_3, \dots, n_r$ possible outcomes for E.

5. In horse racing, a trifecta consists of choosing the exact order of the first three horses across the finish line. If there are eight horses in the race, how many trifectas are possible, assuming there are no ties?
Solution
Any one of the eight horses can be first, so $n_1 = 8$. Since a horse cannot finish both first and second, there are seven horses that can finish second; thus $n_2 = 7$. Similarly, there are six horses that can finish third; $n_3 = 6$. By the counting principle, there are $8.7.6 = 336$ possible trifectas.

n factorial
n factorial is the product of the natural numbers 1 through n and is symbolized by $n!$.
i.e. $n! = n(n-1)(n-2) \ldots 3.2.1$

6. Evaluate Factorial Expressions
- $\mathbf{5}! - \mathbf{3}! = (5.4.3.2.1) - (3.2.1) = 120. - 6 = 114$
- $\frac{\mathbf{9}!}{\mathbf{6}!} = \frac{9.8.7.6!}{6!} = 9.8.7 = 504$

7. How many arrangements of the letter PAKPATTAN can be made?
Total letters $= n = 9$
P repeated time $= 2$
A repeated time $= 3$
K repeated time $= 1$
T repeated time $= 2$
N repeated time $= 1$
Total arrangements $= \frac{9!}{2!.3!.1!.2!.1!} = \frac{9.8.7.6.5.4.3!}{2.1.3!.1.2.1} = 15120$

Circular Permutations
The permutation of things which can be represented by the points on a circle.
Circular Permutation $= \frac{(n-1)!}{2}$

8. In how many ways can 4 keys be arranged on a circular key ring?
$n = 4$
Possible arrangement $= \frac{(n-1)!}{2} = \frac{(4-1)!}{2} = \frac{3!}{2} = \frac{3.2.1}{2} = 3$

9. How many necklaces can be made from 6 beads of different colours?
$n = 6$
Possible arrangement $= \frac{(n-1)!}{2} = \frac{(6-1)!}{2} = \frac{5!}{2} = \frac{120}{2} = 60$

Permutations
An ordering arrangement of n objects is called Permutation.
Or A permutation is an arrangement of distinct objects in a definite order. The counting principle can be used to determine the total number of possible permutations of a group of objects.
Remember

- Permutation formula $= P(n,r) = {}^{n}P_{r} = \frac{n!}{(n-r)!}$
- $\frac{n!}{(n-r)!} = n(n-1)(n-2)\ldots(n-k+1)$

10. A university tennis team consists of six players who are ranked from 1 through 6. If a tennis coach has 10 players from which to choose, how many different tennis teams can the coach select?
Solution
Because the players on the tennis team are ranked from 1 through 6, a team with player A in position 1 would be different from a team with player A in position 2. Therefore, the number of different teams is the number of permutations of 10 players selected six at a time.
$P(10,6) = \frac{10!}{(10-6)!} = \frac{10!}{4!} = \frac{10.9.8.7.6.5.4!}{4!} = 10.9.8.7.6.5 = 151200$

11. Class Schedule: A middle school principal needs to schedule 6 different classes. Algebra, English, history, Spanish, Science and gym in 6 different time periods. How many different class schedules are possible?
Solution
Number of possible schedules $= 6! = 6.5.4.3.2.1 = 720$
The principal can schedule the 6 classes in 720 different ways.

12. Leadership Election: A city has 12 candidates running for three leadership positions. The top vote getter will become the mayor, the second vote getter will become the deputy mayor and the third vote getter will become the treasurer. How many outcomes are possible for the three leadership positions?
Solution
${}^{12}P_3 = \frac{12!}{(12-3)!} = \frac{12!}{9!} = \frac{12.11.10.9!}{9!} = 12.11.10 = 1320$

13. Batting Orders: A little league manager has 15 children on her team. How many ways can she form a 9 player batting order?
Solution
${}^{15}P_9 = \frac{15!}{(15-9)!} = \frac{15!}{6!} = \frac{15.14.13.12.11.10.9.8.7.6!}{6!} = 15.14.13.12.11.10.9.8.7$
${}^{15}P_9 = 1816214400$

Combinations

An arrangement of n objects without caring of order. A combination is a collection of objects for which the order is not important. The three-letter sequences acb and bca are different permutations but the same combination.

$\text{Combination formula} = C(n,r) = \frac{P(n,r)}{r!} = {}^{n}C_{r} = \frac{n!}{r!.(n-r)!}$

14. A basketball team consists of 11 players.

a. How many different ways can a coach choose the starting five players, assuming the position of a player is not considered?

b. How many different ways can a coach choose the starting five players if the positions of the players are considered?

Solution

a) $C(11,5) = \frac{11!}{5!.(11-5)!} = \frac{11!}{5!.6!} = \frac{11.10.9.8.7.6!}{5!.6!} = \frac{11.10.9.8.7}{5.4.3.2.1} = 462$

There are 462 possible five-player starting teams.

b) $P(11,5) = \frac{11!}{(11-5)!} = \frac{11!}{6!} = \frac{11.10.9.8.7.6!}{6!} = 11.10.9.8.7 = 55440$

There are 55,440 permutations.

Remark

Some counting problems require the use of more than one counting technique to determine the total number of possible outcomes.

15. A committee of five professors is to be chosen from five mathematicians and six economists. How many different committees are possible if the committee must include two mathematicians and three economists?

Solution

By using counting principle,

$C(5,2).C(6,3) = \frac{5!}{2!.3!}.\frac{6!}{3!.3!} = 10.20 = 200$

There are 200 possible committees consisting of two mathematicians and three economists.

16. How many diagonal can be formed by 6 sided polygon?

Solution

Number of diagonals $= {}^{6}C_{2} - 6 = \frac{6!}{2!.(6-2)!} - 6 = \frac{6.5.4}{2.1.4} - 6 = 15 - 6 = 9$

17. From a standard deck of playing cards, a hand of five cards is chosen.

a. How many different five-card hands are possible?
b. How many different five-card hands consist of two kings and three queens? (This hand is an example of a "full house" in the game of poker.)
c. How many different hands consist of five cards of the same suit? (This is a "flush" in poker.)

Solution

a) $C(52,5) = \frac{52!}{5!.47!} = \frac{52.51.50.49.48.47!}{5!.47!} = \frac{52.51.5.49.48.}{5!} = 2{,}598{,}960$

b) $C(4,2).(4,3) = \frac{4!}{2!.2!}.\frac{4!}{3!.1!} = 6.4 = 24$

c) $4(13\,,5) = 4.\frac{13!}{5!.8!} = 4.1287 = 5148$

18. Ice Cream Combinations: Suppose that you select 3 different flavors of ice cream in a shop that carries 12 flavors. How many flavor combinations are possible?

Solution

$^{12}C_3 = \frac{12!}{3!.(12-3)!} = \frac{12!}{3!.9!} = 220$

19. Poker Hands: How many different five card poker hands can be dealt from a standard deck of 52 cards? What is the probability of one particular hand, such a royal flush of hearts (a hand consisting of a king, queen, Jack and 10 of hearts)?

Solution

$^{52}C_5 = \frac{52!}{5!.(52-5)!} = \frac{52!}{5!.47!} = 2598960$

20. (Not) Winning the Lottery: Suppose you play a lottery in which the winner is chosen by drawing 6 balls at random from a drum containing 52 numbered balls (number 1 through 52). What is the probability that your 6 numbers will match the 6 winning numbers?

Solution

$^{52}C_5 = \frac{52!}{5!.(52-5)!} = \frac{52!}{5!.47!} = 2598960$

21. Birthday Coincidence: Suppose there are 25 students in your class and find each of the following probabilities. Assume that there are 365 days in a year.

Solution

$^{52}C_6 = \frac{52!}{6!.(52-6)!} = \frac{52!}{6!.4!} = 20358520$

22. Prove that ${}^{n}C_{r} \times r! = {}^{n}P_{r}$

Proof

$${}^{n}C_{r} \times r! = \frac{n!}{r!.(n-r)!} \times r! = \frac{n!}{(n-r)!} = {}^{n}P_{r}$$

23. Prove that ${}^{n}C_{r} = {}^{n}C_{n-r}$

Proof

$${}^{n}C_{n-r} = \frac{n!}{(n-r)!.(n-n+r)!} = \frac{n!}{r!.(n-r)!} = {}^{n}C_{r}$$

24. Prove that ${}^{n-1}C_{r} + {}^{n-1}C_{r-1} = {}^{n}C_{r}$

Proof

$${}^{n-1}C_{r} + {}^{n-1}C_{r-1} = \frac{(n-1)!}{r!.(n-1-r)!} + \frac{(n-1)!}{(r-1)!.(n-1-r+1)!}$$

$${}^{n-1}C_{r} + {}^{n-1}C_{r-1} = \frac{(n-1)!}{r!.(n-1-r)!} + \frac{(n-1)!}{(r-1)!.(n-r)!}$$

$${}^{n-1}C_{r} + {}^{n-1}C_{r-1} = \frac{(n-1)!}{r(r-1)!.(n-r-1)!} + \frac{(n-1)!}{(r-1)!.(n-r)(n-r-1)!}$$

$${}^{n-1}C_{r} + {}^{n-1}C_{r-1} = \frac{(n-1)!}{(r-1)!.(n-r-1)!}\left[\frac{1}{r} + \frac{1}{(n-r)}\right]$$

$${}^{n-1}C_{r} + {}^{n-1}C_{r-1} = \frac{(n-1)!}{(r-1)!.(n-r-1)!}\left[\frac{n-r+r}{r(n-r)}\right]$$

$${}^{n-1}C_{r} + {}^{n-1}C_{r-1} = \frac{(n-1)!}{(r-1)!.(n-r-1)!}\left[\frac{n}{r(n-r)}\right]$$

$${}^{n-1}C_{r} + {}^{n-1}C_{r-1} = \frac{n(n-1)!}{r(r-1)!.(n-r)(n-r-1)!}$$

$${}^{n-1}C_{r} + {}^{n-1}C_{r-1} = \frac{n!}{r!.(n-r)!} = {}^{n}C_{r}$$

$${}^{n-1}C_{r} + {}^{n-1}C_{r-1} = {}^{n}C_{r}$$

PROBABILITY

Probability

Numerical evaluation of a chance that a particular event would occur. **Or** measurement of uncertainty is called probability.

Sample Space

The set S consisting of all possible outcome of a given experiment is called a sample space.

Event

An event is a subset of a sample space.

Experiments

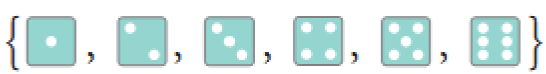
Experiments are activities with observable outcomes. Here are some examples of experiments:

- Flip a coin and observe the outcome as heads or tails.
- Select a company and observe its annual profit.
- Record the time a person spends at the checkout line in a supermarket.

1. A single die is rolled once. What is the sample space for this experiment?

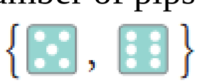
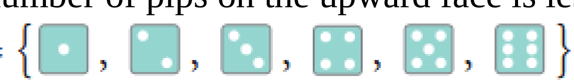
Solution

The sample space is all possible outcomes of the experiment.

$S =$ 

Note

An **event** is a subset of a sample space. Using the sample space of above Example, here are some possible events:

- There are an even number of pips (dots) on the upward face. The event is
  $E_1 =$ 
- The number of pips on the upward face is greater than 4. The event is
  $E_2 =$ 
- The number of pips on the upward face is less than 20. The event is
  $E_3 =$ 
  Because the number of pips on the upward face is always less than 20, this event will always occur. The event and the sample space are the same.
- The number of pips on the upward face is greater than 15. The event is $E_4 = \varphi$ the empty set. This is an impossible event, because it is not possible for the number facing up to be greater than 15.

Basic Probability Formula

For an experiment with sample space S of equally likely outcomes, the probability P(E) of an event E is given by

$$\mathbf{P(E)} = \frac{\mathbf{n(E)}}{\mathbf{n(S)}}$$

Where n(E) is the number of elements in the event and n(S) is the number of elements in the sample space.

2. A fair coin, one for which it is equally likely that heads or tails will result from a single toss of the coin, is tossed three times. What is the probability that two heads and one tail are tossed?

Solution

$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$

$E = \{HHT, HTH, THH\}$

$P(E) = \frac{n(E)}{n(S)} = \frac{3}{8}$

3. Is it possible that the probability of some event could be 1.23?

Answer

No. All probabilities must be between 0 and 1, inclusive.

4. Two fair dice are tossed once. What is the probability that the sum of the pips on the upward faces of the two dice equals 8?

Solution

$P(E) = \frac{n(E)}{n(S)} = \frac{5}{36}$

5. A five-card hand is dealt from a standard 52-card deck of playing cards. What is the probability that the hand has five cards of the same suit?

Solution

Let E be the event of drawing five cards of the same suit. The total number of outcomes for E is $4.C(13,5) = 5148$, so $n(E) = 5148$. The sample space is the total number of different five-card hands, which is given by $(52,5) = 2{,}598{,}960$ .Thus $n(S) = 2{,}598{,}960$. The probability of is E

$P(E) = \frac{n(E)}{n(S)} = \frac{5148}{2{,}598{,}960} \approx 0.00198$

There is only about a 0.2% chance of being dealt five cards of the same suit.

6. Two dice are rolled once. Find the probability of getting same number on upper face when two dice are rolled once.

Solution

There are 6 possible outcomes for each die, and since there are two dice, there are a total of $6 \times 6 = 36$ possible outcomes.

The outcomes where the two dice show the same number are:

(1,1), (2,2), (3,3), (4,4), (5,5), (6,6)

There are 6 outcomes where the two dice show the same number.

So, the probability of getting the same number on both dice is:

6 (favorable outcomes) / 36 (total outcomes) = 1/6

Therefore, the probability of getting the same number on both dice is 1/6 or approximately 0.17 (rounded to two decimal places).

7. In a refrigerator 5 Sprite, 4 Fanta and 3 Dew. 5 bottles are selected at random. Find the probability that 3 are Fanta and 2 are Dew.

Solution

Let's use the combination formula to find the total number of ways to select 5 bottles from the total:

$C(16, 5) = 16! / (5! \times 11!) = 4368$

Now, let's find the number of ways to select 3 Fanta and 2 Dew:

$C(4, 3) = 4! / (3! \times 1!) = 4$ (ways to select 3 Fanta)

$C(7, 2) = 7! / (2! \times 5!) = 21$ (ways to select 2 Dew)

Multiply the number of ways to select 3 Fanta and 2 Dew:

$4 \times 21 = 84$

Now, divide the favorable outcomes (84) by the total outcomes (4368):

84 / 4368 = 1/52

So, the probability of selecting 3 Fanta and 2 Dew is 1/52 or approximately 0.019.

Experimental/Empirical Probability

When a probability is based on data gathered from an experiment, it is called an experimental probability or an empirical probability.

Example

For instance, if we tossed a thumbtack 100 times and recorded the number of times it landed "point up," the results might be as shown in the table at the left. From this, the empirical probability of "point up" is

Point up	15
Side	85
Total	100

$P(\text{point up}) = \frac{15}{100} = 0.15$

8. A survey of the registrar of voters office in a city showed the following information on the ages and party affiliations of registered voters. If one voter is chosen from this survey, what is the probability that the voter is a Republican? Round to the nearest hundredth.

Age	Republican	Democrat	Independent	Other	Total
18–28	205	432	98	112	847
29–38	311	301	109	83	804
39–49	250	251	150	122	773
50+	272	283	142	107	804
Total	1038	1267	499	424	3228

Solution

Let R be the event that a Republican is selected. Then

$P(R) = \frac{\text{Number of Republicans in the survey}}{\text{Total number of people surveyed}} = \frac{1038}{3228} \approx 0.32$

The probability that the selected person is a Republican is 0.32.

Odds and odds Ratio

- Odds are simply the ratio of something happening to something not happening.
- odds are also defined as the probability of an event occurring divided by the probability of the event not occurring.
- An odds ratio is the odds of the event in one group (e.g. exposed group), divided by the odds in another group (not exposed).

Odds in Favor

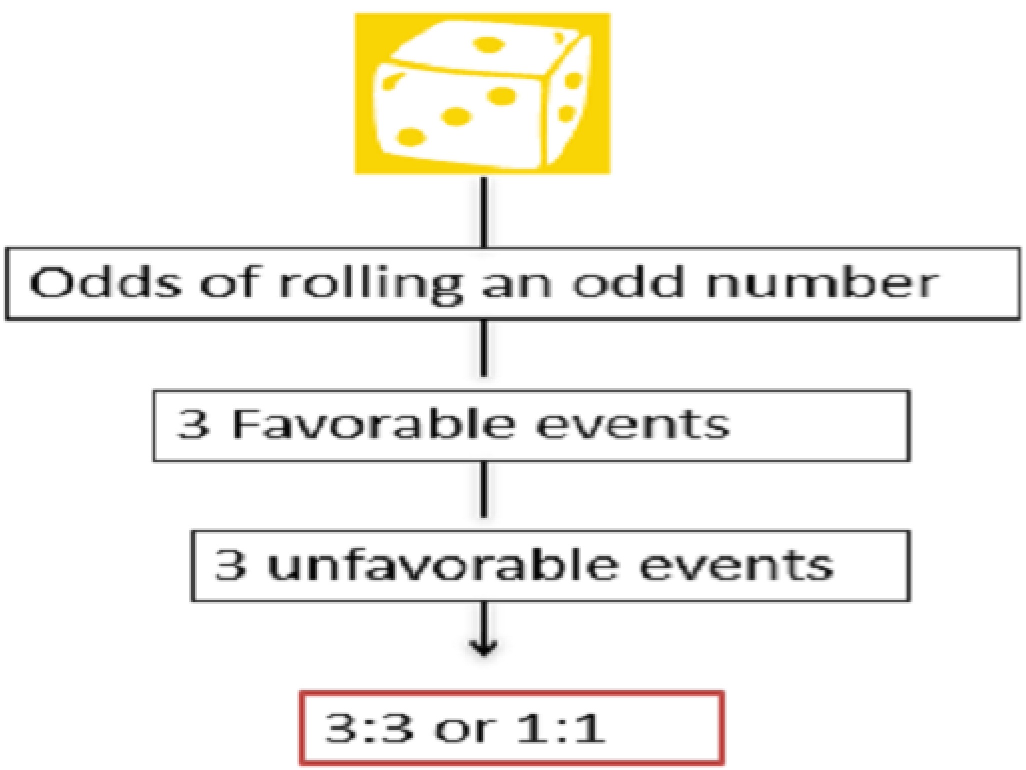
The odds in favor of an event is the ratio of the number of favorable outcomes of an experiment to the number of unfavorable outcomes.

Odds vs Probability

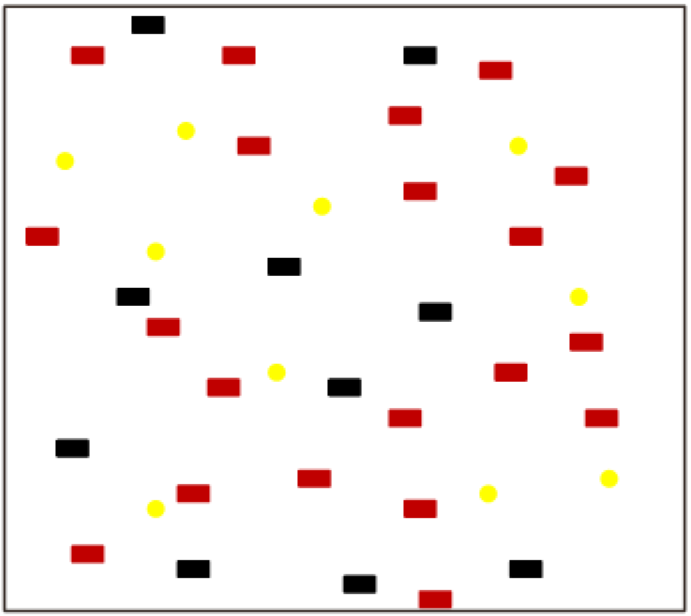
$$\text{Odds} = \frac{\text{number of favorable outcomes}}{\text{number of unfavorable outcomes}}$$

$$\text{Probability} = \frac{\text{number of favorable outcomes}}{\text{total number of outcomes}}$$

9. What are the odds that you will roll an odd number on the die?



10. You have a sack of 10 tootsie rolls, 20 Jolly Ranchers, and 10 gumballs. What are the odds that you will stick your hand in the bag and pull out a tootsie roll?



10 favorable events

30 unfavorable events (not tootsie)

Odds = 10:30 or 1:3

11. A fair coin, one for which it is equally likely that heads or tails will result from a single toss of the coin, is tossed three times. Find the odds in favor of two heads and one tail.

Solution

$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$

$\text{favorable events} = E = \{HHT, HTH, THH\}$

$\text{unfavorable events} = U = \{HHH, HTT, THT, TTH, TTT\}$

$\text{Odds in favor of two heads and one tail} = \frac{n(E)}{n(U)} = \frac{3}{5}$

Odds Against

The odds against an event are the ratio of the number of unfavorable outcomes of an experiment to the number of favorable outcomes.

Example

For instance, in horse racing, the odds against a horse winning a race are posted. Odds posted as 7 to 2 mean that $9(7+2)$ in races, the horse is estimated to lose 7 times and win 2 times.

12. A charity sells 100 raffle tickets. There is one grand prize and four smaller prizes. What are the odds in favor of winning a prize?

Solution

There are five favorable outcomes and 95 unfavorable outcomes. The odds in favor of winning a prize are $\frac{5}{95} = \frac{1}{19}$. The odds in favor of winning a prize are 1 to 19.

Mutually Exclusive Events

Two events A and B are mutually exclusive if they cannot occur at the same time.
That is, A and B are mutually exclusive when $A \cap B = \varphi$.

Example

Suppose we draw a single card from a standard deck of playing cards. The sample space S is the 52 cards of the deck. Therefore $n(S) = 52$, Now consider the events

E_1 = A 4 is drawn = $\{♠4, ♥4, ♦4, ♣4\}$

E_2 = A spade is drawn = $\{♠A, ♠2, ♠3, ♠4, ♠5, ♠6, ♠7, ♠8, ♠9, ♠10, ♠J, ♠Q, ♠K\}$

It is possible, on one draw, to satisfy the conditions of both events: The ♠4 could be drawn. This card is an element of both E_1 and E_2.

Now compare the events

E_3 = A 5 is drawn = $\{♠5, ♥5, ♦5, ♣5\}$

E_4 = A king is drawn = $\{♠K, ♥K, ♦K, ♣K\}$

In this case, it is not possible to draw one card that satisfies the conditions of both events. Two events that cannot both occur at the same time are called **mutually exclusive events**. Thus, the events E_3 and E_4 are mutually exclusive events, whereas E_1 and E_2 are not.

Probability of Mutually Exclusive Events

If A and B are two mutually exclusive events, then the probability of A or B written $P(A \cup B)$, occurring is given by $P(A \cup B) = P(A) + P(B)$.

13. Suppose a single card is drawn from a standard deck of playing cards. Find the probability of drawing a 5 or a king.

Solution

Let A = {♣5, ♠5, ♥5, ♦5} and B = {♣K, ♠K, ♥K, ♦K}. There are 52 cards in a standard deck of playing cards; thus $n(S) = 52$. Because the events are mutually exclusive, we can use the formula for the probability of mutually exclusive events.

$P(A \cup B) = P(A) + P(B) = \frac{1}{13} + \frac{1}{13} = \frac{2}{13}$

The Addition Rule for Probabilities

Let A and B be two events in a sample space S. Then

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

14.Single Roll of two Dice

If $P(A) = \frac{5}{36}$; $P(B) = \frac{6}{36}$; $P(A \cap B) = \frac{1}{36}$ then

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$P(A \cup B) = \frac{5}{36} + \frac{6}{36} - \frac{1}{36} = \frac{10}{36} = \frac{5}{18}$

On a single roll of two dice, the probability of rolling a sum of 8 or a double is $\frac{5}{18}$.

Conditional Probability

The probability of an event B occurring based on knowing that event A has already occurred is called a conditional probability, and is denoted by $P(A|B)$.

Conditional Probability Formula

Let A and B be two events in a sample space S. Then the conditional probability of B given that A has occurred is

$P(A|B) = \frac{P(A \cap B)}{P(B)}$

The symbol $P(A|B)$ is read "the probability of B given A."

15. Flu Data

S = {all people participating in the test}

F = {people contracting the flu}

V = {people who were vaccinated}

$P(F|V) = \frac{P(F \cap V)}{P(V)}$

$P(F|V) = \frac{21/490}{219/490} = \frac{21}{219} \approx 0.096$

	F	No *F*	Total
V	21	198	219
No *V*	76	195	271
Total	97	393	490

V: Vaccinated
F: Contracted the flu

16. The data in the table below show the results of a survey for determining the numbers of adults who have had financial help from their parents for certain purchases.

Age	College Tuition	Buy a Car	Buy a House	Total
18–28	405	253	261	919
29–39	389	219	392	1000
40–49	291	146	245	682
50–59	150	71	112	333
60+	62	15	98	175
Total	1297	704	1108	3109

If one person is selected from this survey, what is the probability that the person received financial help for purchasing a home given that the person is between the ages of 29 and 39?

Solution

A = {adults between 29 and 39}

B = {adults receiving financial help for a home purchase}

$P(B|A) = \frac{P(A \cap B)}{P(A)}$

$P(B|A) = \frac{392/3109}{1000/3109} = \frac{392}{1000} = 0.392$

The conditional probability of B given A is 0.392.

Product Rule for Probabilities

If A and B are two events from the sample space S, then

$P(A \cap B) = P(A).P(B|A)$

17. Here is the possible outcomes of drawing two cards from a deck. Find the probability of drawing an ace on both the first and second draws.

Solution

On the first draw, there are four aces in the deck of 52 cards. Therefore $P(A) = \frac{4}{52} = \frac{1}{13}$, On the second draw, there are only 51 card remaining and only three aces (an ace was drawn on the first draw). Therefore $(B|A) = \frac{3}{51} = \frac{1}{17}$. Putting these calculations together, we have

$P(A \cap B) = P(A).P(B|A) = \frac{1}{13}.\frac{1}{17} = \frac{1}{221}$

Probability of Successive Events
The probability of two or more events occurring in succession is the product of the conditional probabilities of the individual events.

18. A box contains four red, three white, and five green balls. Suppose three balls are randomly selected from the box in succession without replacement.

a. What is the probability that first a red, then a white, and then a green ball are selected?
b. What is the probability that two white balls followed by one green ball are selected?

Solution

a) Let
A = {a red ball is selected first}
B = {a white ball is selected second}
C = {a green ball is selected third}
$P(A \cap B \cap C) = P(A).P(B|A).P(C| \cap B) = \frac{4}{12}.\frac{3}{11}.\frac{5}{10} = \frac{1}{22}$

b) Let
A = {a white ball is selected first}
B = {a white ball is selected second}
C = {a green ball is selected third}
$P(A \cap B \cap C) = P(A).P(B|A).P(C| \cap B) = \frac{3}{12}.\frac{2}{11}.\frac{5}{10} = \frac{1}{44}$

Independent Events
If A and B are two events in a sample space and $P(B|A) = P(B)$ then A and B are called independent events.

19. Consider tossing a coin twice. Compute the probability that the second toss comes up heads, given that the first coin toss came up heads.

Solution
A = event of a head on the first toss = {HH, HT}
B = {= event of a head on the second toss = {HH, TH}
S = sample space = {HH, H, TH, TT}
$P(B|A) =$ the probability of a head on the second toss given a head on the first toss
$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}$
However $P(B) = \frac{n(B)}{n(S)} = \frac{2}{4} = \frac{1}{2}$
Therefore $P(B|A) = P(B)$, and the events are independent.

In general, this allows us to simplify the product rule when two events are independent; the probability of two independent events occurring in succession is simply the product of the probabilities of the individual events.

Product Rule for Independent Events

If A and B are two independent events from the sample space S, then

$P(A \cap B) = P(A).P(B)$

20. A pair of dice are tossed twice. What is the probability that the first roll is a sum of 7 and the second roll is a sum of 11?

Solution

The rolls of a pair of dice are independent; the probability of a sum of 11 on the second roll does not depend on the outcome of the first roll. Let A = {sum of 7 on the first roll} and let B = {sum of 11 on the second roll}. Then

$P(A \cap B) = P(A).P(B) = \frac{6}{36}.\frac{2}{36} = \frac{1}{108}$

21. Suppose that a company claims that it has a test that is 95% effective in determining whether an athlete is using a steroid. That is, if an athlete is using a steroid, the test will be positive 95% of the time. In the case of a negative result, the company says its test is 97% accurate. That is, even if an athlete is not using steroids, it is possible that the test will be positive in 3% of the cases. Suppose this test is given to a group of athletes in which 10% of the athletes are using steroids. What is the probability that a randomly chosen athlete actually uses steroids given that the athlete’s test is positive?

Solution

Let S be the event that an athlete uses steroids and let T be the event that the test is positive. Then

$P(S \mid T) = \frac{P(S \cap T)}{P(T)} = \frac{P(S).P(T)}{P(T)} = \frac{0.1 \times 0.95}{0.1 \times 0.95 + 0.9 \times 0.03} \approx 0.779$

Given that an athlete tests positive, the probability that the athlete actually uses steroids is approximately 77.9%.

22. A child will have cystic fibrosis if the child inherits the recessive gene from both parents. Using F for the normal allele and f for the mutant allele, suppose a parent who is Ff (said to be a carrier) and a parent who is FF (does not have the mutant allele) decide to have a child.

a. What is the probability that the child will have cystic fibrosis? To have the disease, the child must be ff.

b. What is the probability that the child will be a carrier?

Solution

Make a Punnett square.

Parents	F	F
F	FF	FF
f	Ff	Ff

a. To have the disease, the child must be ff. From the table, there is no combination of the alleles that will produce ff. Therefore, the child cannot have the disease, and the probability is 0.

b. To be a carrier, one allele must be f. From the table, there are two cases out of four in which the child will have one f. The probability that the child will be a carrier is $\frac{2}{4} = \frac{1}{2}$.

SETS & EXPRESSIONS

(VENN DIAGRAM, ALGEBRAIC EXPRESSIONS)

Sets

A set is a well-defined collection of distinct objects. It is a collection of unique objects, known as elements or members, that can be anything (numbers, letters, people, etc.).

There are three different ways of describing a set

i. **The Descriptive Method:** A set may be described in words. For instance, the set of all vowels of the English alphabets.
ii. **The Tabular Method:** A set may be described by listing its elements within brackets. If A is the set mentioned above, then we may write: A = {a,e,i,o,u}.
iii. **Set-builder method:** It is sometimes more convenient or useful to employ the method of set-builder notation in specifying sets. This is done by using a symbol or letter for an arbitrary member of the set and stating the property common to all the members. Thus the above set may be written as:

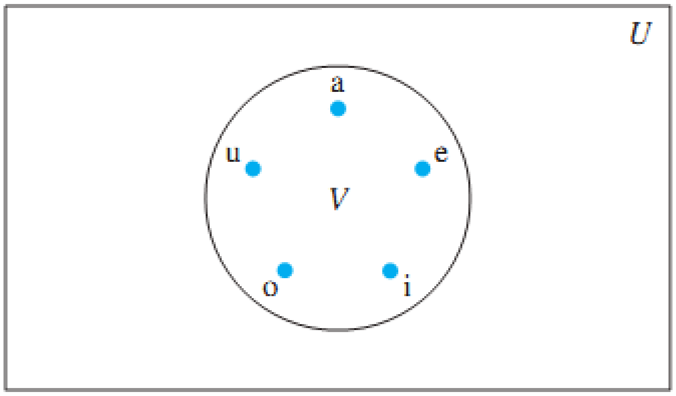
A = {x | x is a vowel of the English alphabet}

This is read as A is the set of all x such that x is a vowel of the English alphabet.

Venn Diagrams

Venn diagrams are very useful in depicting visually the basic concepts of sets and relationships between sets. They were first used by an English logician and mathematician John Venn (1834 to 1883 A.D).
In a Venn diagram, a rectangular region represents the universal set and regions bounded by simple closed curves represent other sets, which are subsets of the universal set. For the sake of beauty these regions are generally shown as circular regions.

1. Draw a Venn diagram that represents V, the set of vowels in the English alphabet.
   Solution
   We draw a rectangle to indicate the universal set U, which is the set of the 26 letters of the English alphabet. Inside this rectangle we draw a circle to represent V. Inside this circle we indicate the elements of V with points (see Figure).

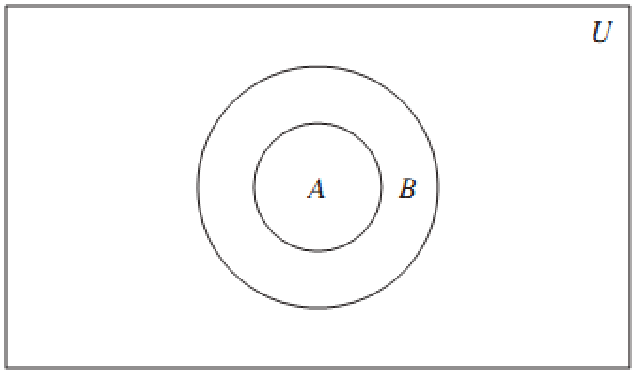


Subset

The set A is a subset of B, and B is a superset of A, if and only if every element of A is also an element of B. We use the notation $A \subseteq B$ to indicate that A is a subset of the set B. If, instead, we want to stress that B is a superset of A, we use the equivalent notation $B \supseteq A$.(So, $A \subseteq B$ and $B \supseteq A$ are equivalent statements.)

For example; The set of all odd positive integers less than 10 is a subset of the set of all positive integers less than 10, the set of rational numbers is a subset of the set of real numbers, the set of all computer science majors at your school is a subset of the set of all students at your school, and the set of all people in China is a subset of the set of all people in China (that is, it is a subset of itself). Each of these facts follows immediately by noting that an element that belongs to the first set in each pair of sets also belongs to the second set in that pair.

2. Draw a Venn diagram showing that A is a subset of B.
   Solution



Equal Sets

To show that two sets A and B are equal, show that A ⊆ B and B ⊆ A.

Sets may have other sets as members. For instance, we have the sets

A ={∅, {a}, {b}, {a, b}} and B ={x | x is a subset of the set {a, b}}.

Note that these two sets are equal, that is, A = B. Also note that {a}∈ A, but a ∉ A.

The Size of a Set

Let S be a set. If there are exactly n distinct elements in S where n is a nonnegative integer, we say that S is a finite set and that n is the cardinality of S. The cardinality of S is denoted by |S|.

Remark

- The term cardinality comes from the common usage of the term cardinal number as the size of a finite set.
- A set is said to be infinite if it is not finite.

Examples

- Let A be the set of odd positive integers less than 10. Then |A| = 5.
- Let S be the set of letters in the English alphabet. Then |S| = 26.
- Because the null set has no elements, it follows that |∅| = 0.
- The set of positive integers is infinite.

Power Set

Given a set S, the power set of S is the set of all subsets of the set S. The power set of S is denoted by P(S).

If a set has n elements, then its power set has 2^n elements.

3. What is the power set of the set {0, 1, 2}?

Solution

The power set P({0, 1, 2}) is the set of all subsets of {0, 1, 2}. Hence,

P ({0, 1, 2}) ={∅, {0}, {1}, {2}, {0, 1}, {0, 2}, {1, 2}, {0, 1, 2}}.

Note that the empty set and the set itself are members of this set of subsets.

4. What is the power set of the empty set? What is the power set of the set {∅}?

Solution

The empty set has exactly one subset, namely, itself. Consequently,

P (∅) = {∅}.

The set {∅} has exactly two subsets, namely, ∅ and the set {∅} itself. Therefore,

P ({∅}) ={∅, {∅}}.

Union of Sets

Let A and B be sets. The union of the sets A and B, denoted by A ∪ B, is the set that contains those elements that are either in A or in B, or in both.

That is $A \cup B = \{x \mid x \in A \vee x \in B\}$

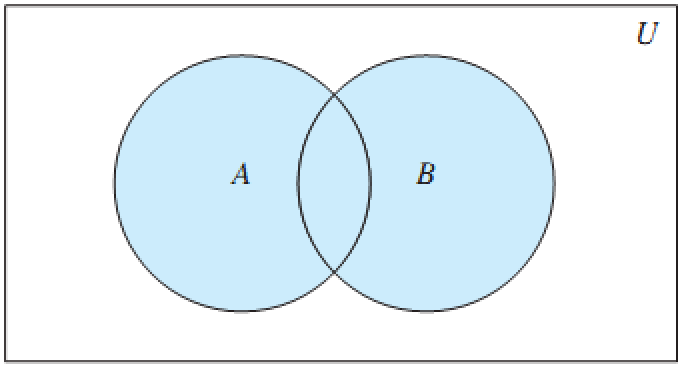
5. Find the union of the sets {1, 3, 5} and {1, 2, 3}

Solution

$\{1, 3, 5\} \cup \{1, 2, 3\} = \{1, 2, 3, 5\}$.

6. Draw a Venn diagram of union of two sets.

Solution



Intersection of Sets

Let A and B be sets. The intersection of the sets A and B, denoted by A ∩ B, is the set containing those elements in both A and B.

That is $A \cap B = \{x \mid x \in A \wedge x \in B\}$

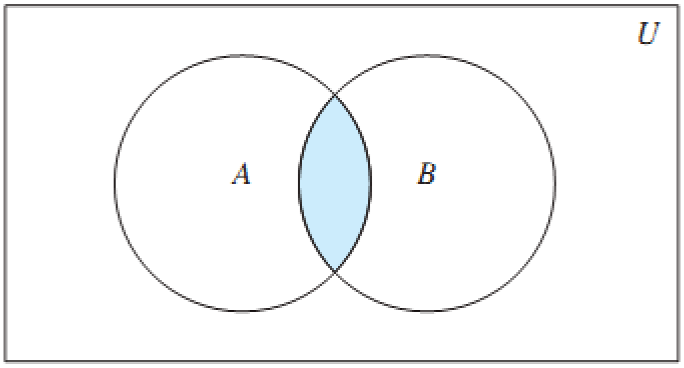
7. Find the intersection of the sets {1, 3, 5} and {1, 2, 3}

Solution

$\{1, 3, 5\} \cap \{1, 2, 3\} = \{1, 3\}$

8. Draw a Venn diagram of intersection of two sets.

Solution



Disjoint Sets

Two sets are called disjoint if their intersection is the empty set.

9. Give an example of disjoint sets.

Solution

Let A ={1, 3, 5, 7, 9} and B ={2, 4, 6, 8, 10}.

Because $A \cap B = \emptyset$,

Therefore A and B are disjoint.

Difference of Sets

Let A and B be sets. The difference of A and B, denoted by A − B, is the set containing those elements that are in A but not in B. The difference of A and B is also called the complement of B with respect to A. i.e. $A - B = \{x \mid x \in A \wedge x \notin B\}$.

For example;

The difference of {1, 3, 5} and {1, 2, 3} is the set {5};

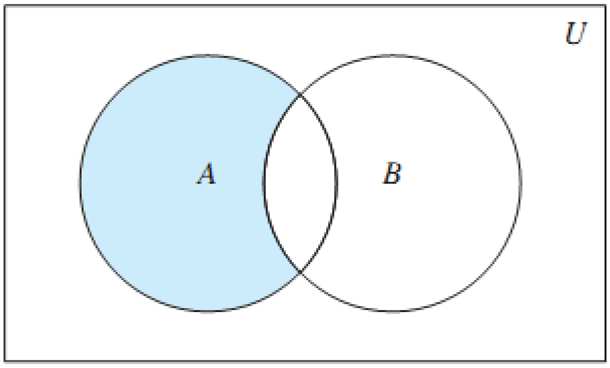
that is, {1, 3, 5}−{1, 2, 3}={5}.

This is different from the difference of {1, 2, 3} and {1, 3, 5},

which is the set {2}.

10. Draw a Venn diagram of difference of two sets.

Solution



$A - B$ is shaded.

Complement of a Set

Let U be the universal set. The complement of the set A, denoted by $\overline{A}$, is the complement of A with respect to U. Therefore, the complement of the set A is U − A.

That is $\overline{A} = \{x \in U \mid x \notin A\}$.

Remember that $A - B = A \cap \overline{B}$.

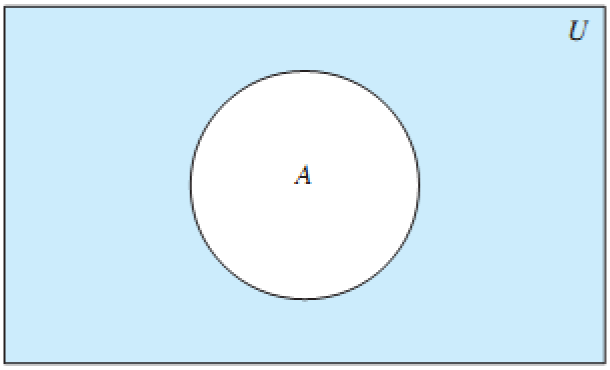
11. Give two examples of complement of sets.

Solution

- Let A ={a, e, i, o, u} (where the universal set is the set of letters of the English alphabet). Then $\overline{A}$ = {b, c, d, f, g, h, j, k, l, m, n, p, q, r, s, t, v, w, x, y, z}.
- Let A be the set of positive integers greater than 10 (with universal set the set of all positive integers). Then $\overline{A}$ = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}.

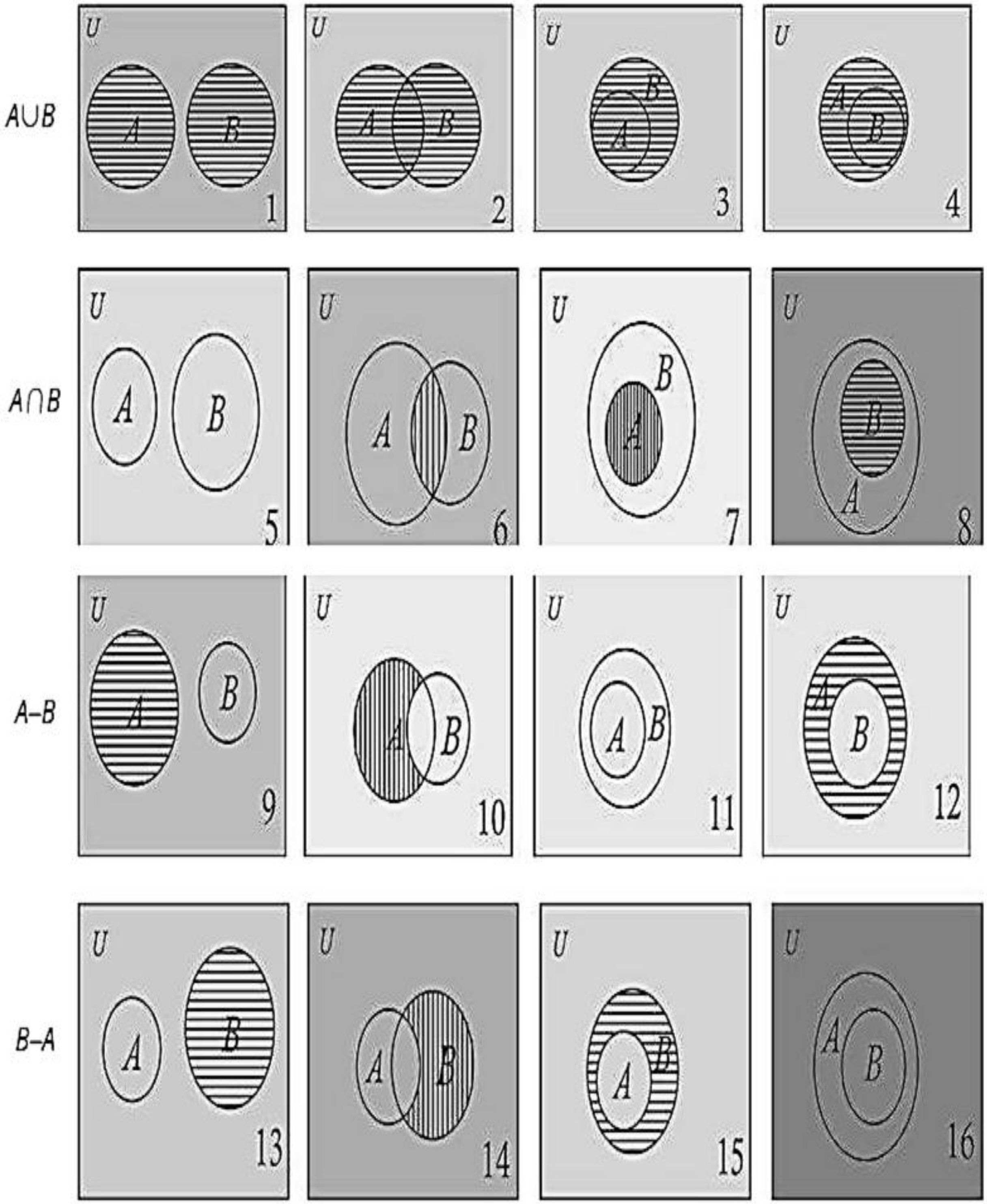
12. Draw a Venn diagram of complement of a set A.

Solution



$\overline{A}$ is shaded.

Summary



Practical scenarios involving algebraic expressions: linear and quadratic

Evaluate each exponential expression in Exercises 1–22.

1) $5^2 . 2$ **Solution:** $5^2 . 2 = 25 \times 2 = 50$

2) $6^2 . 2$ **Solution:** $6^2 . 2 = 36 \times 2 = 72$

3) $(-2)^6$ **Solution:** $(-2)^6 = 64$

4) $(-2)^4$ **Solution:** $(-2)^4 = 16$

5) -2^6 **Solution:** $-2^6 = -64$

6) -2^4 **Solution:** $-2^4 = -16$

7) $(-3)^0$ **Solution:** $(-3)^0 = +1$

8) $(-9)^0$ **Solution:** $(-9)^0 = +1$

9) -3^0 **Solution:** $-3^0 = -1$

10) -9^0 **Solution:** $-9^0 = -1$

11) 4^{-3} **Solution:** $4^{-3} = \frac{1}{4^3} = \frac{1}{64}$

12) 2^{-6} **Solution:** $2^{-6} = \frac{1}{2^6} = \frac{1}{64}$

13) $2^2 . 2^3$ **Solution:** $2^2 . 2^3 = 4 \times 8 = 32$ OR $2^{2+3} = 2^5 = 32$

14) $3^3 . 3^2$ **Solution:** $3^3 . 3^2 = 27 \times 9 = 243$ OR $3^{3+2} = 3^5 = 243$

15) $(2^2)^3$ **Solution:** $(2^2)^3 = (4)^3 = 4 \times 4 \times 4 = 64$

16) $(3^3)^2$ **Solution:** $(3^3)^2 = (27)^2 = 27 \times 27 = 729$

17) $\frac{2^8}{2^4}$ **Solution:** $\frac{2^8}{2^4} = \frac{2\times2\times2\times2\times2\times2\times2\times2}{2\times2\times2\times2} = \frac{256}{16} = 16$

18) $\frac{3^8}{3^4}$ **Solution:** $\frac{3^8}{3^4} = \frac{3\times3\times3\times3\times3\times3\times3\times3}{2\times2\times2\times2} = \frac{6561}{81} = 81$

19) $3^{-3} . 3$ **Solution:** $3^{-3} . 3 = \frac{1}{3^3} . 3 = \frac{1}{27} \times 3 = \frac{1}{9}$

20) $2^{-3} . 2$ **Solution:** $2^{-3} . 2 = \frac{1}{2^3} . 2 = \frac{1}{8} \times 2 = \frac{1}{4}$

21) $\frac{2^3}{2^7}$ **Solution:** $\frac{2^3}{2^7} = \frac{8}{128} = \frac{1}{16}$

22) $\frac{3^4}{3^7}$ **Solution:** $\frac{3^4}{3^7} = \frac{81}{2187} = \frac{1}{27}$

Simplify each exponential expression in Exercises 23–64.

23) $x^{-2}y$ **Solution:** $x^{-2}y = \frac{y}{x^2}$

24) xy^{-3} **Solution:** $xy^{-3} = \frac{x}{y^3}$

25) x^0y^5 **Solution:** $x^0y^5 = 1 \times y^5 = y^5$

26) x^7y^0 **Solution:** $x^7y^0 = x^7 \times 1 = x^7$

27) $x^3 . x^7$ **Solution:** $x^3 . x^7 = x^{3+7} = x^{10}$

28) $x^{11} . x^5$ **Solution:** $x^{11} . x^5 = x^{11+5} = x^{16}$

29) $x^{-5} . x^{10}$ **Solution:** $x^{-5} . x^{10} = x^{-5+10} = x^5$

30) $x^{-6} . x^{12}$ **Solution:** $x^{-6} . x^{12} = x^{-6+12} = x^6$

31) $(x^3)^7$ **Solution:** $(x^3)^7 = x^{21}$

32) $(x^{11})^5$ **Solution:** $(x^{11})^5 = x^{55}$

33) $(x^{-5})^3$ **Solution:** $(x^{-5})^3 = x^{-15} = 1/x^{15}$

34) $(x^{-6})^4$ **Solution:** $(x^{-6})^4 = x^{-24} = 1/x^{24}$

35) $\frac{x^{14}}{x^7}$ **Solution:** $\frac{x^{14}}{x^7} = x^{14-7} = x^7$

36) $\frac{x^{30}}{x^{10}}$ **Solution:** $\frac{x^{30}}{x^{10}} = x^{30-10} = x^{20}$

37) $\frac{x^{14}}{x^{-7}}$ **Solution:** $\frac{x^{14}}{x^{-7}} = x^{14+7} = x^{21}$

38) $\frac{x^{30}}{x^{-10}}$ **Solution:** $\frac{x^{30}}{x^{-10}} = x^{30}x^{10} = x^{30+10} = x^{40}$

39) $(8x^3)^2$ **Solution:** $(8x^3)^2 = (8)^2.(x^3)^2 = 64x^6$

40) $(6x^4)^2$ **Solution:** $(6x^4)^2 = (6)^2.(x^4)^2 = 36x^8$

41) $\left(-\frac{4}{x}\right)^3$ **Solution:** $\left(-\frac{4}{x}\right)^3 = \frac{-(4)^3}{x^3} = \frac{-64}{x^3}$

42) $\left(-\frac{6}{y}\right)^3$ **Solution:** $\left(-\frac{6}{y}\right)^3 = \frac{-(6)^3}{y^3} = \frac{-216}{y^3}$

43) $(-3x^2y^5)^2$ **Solution:** $(-3x^2y^5)^2 = (-3)^2.(x^2)^2.(y^5)^2 = 9x^4y^{10}$

44) $(-3x^4y^6)^3$ **Solution:** $(-3x^4y^6)^3 = (-3)^3.(x^4)^3.(y^6)^3 = -27x^{12}y^{18}$

45) $(3x^4)(2x^7)$ **Solution:** $(3x^4)(2x^7) = 3x^4 \times 2x^7 = 6x^{11}$

46) $(11x^5)(9x^{12})$ **Solution:** $(11x^5)(9x^{12}) = 11x^5 \times 9x^{12} = 99x^{17}$

47) $(-9x^3y)(-2x^6y^4)$ **Solution:** $(-9x^3y)(-2x^6y^4) = 18x^9y^5$

48) $(-5x^4y)(-6x^7y^{11})$ **Solution:** $(-5x^4y)(-6x^7y^{11}) = 30x^{11}y^{12}$

49) $\frac{8x^{20}}{2x^4}$ **Solution:** $\frac{8x^{20}}{2x^4} = \frac{4x^{20}}{x^4} = 4x^{20}x^{-4} = 4x^{20-4} = 4x^{16}$

50) $\frac{20x^{24}}{10x^6}$ **Solution:** $\frac{20x^{24}}{10x^6} = 2x^{24}x^{-6} = 2x^{24-6} = 2x^{18}$

51) $\frac{25a^{13}b^4}{-5a^2b^3}$ **Solution:** $\frac{25a^{13}b^4}{-5a^2b^3} = \frac{-5a^{13}b^4}{a^2b^3} = -5a^{11}b^1$

52) $\frac{35a^{14}b^6}{-7a^7b^3}$ **Solution:** $\frac{35a^{14}b^6}{-7a^7b^3} = -5a^{14}a^{-7}b^6b^{-3} = -5a^7b^3$

53) $\frac{14b^7}{7b^{14}}$ **Solution:** $\frac{14b^7}{7b^{14}} = 2b^7.b^{-14} = 2b^{-7} = \frac{2}{b^{+7}}$

54) $\frac{20b^{10}}{10b^{20}}$ **Solution:** $\frac{20b^{10}}{10b^{20}} = 2b^{10}b^{-20} = 2b^{10-20} = 2b^{-10} = \frac{2}{b^{10}}$

55) $(4x^3)^{-2}$ **Solution:** $(4x^3)^{-2} = \frac{1}{(4x^3)^2} = \frac{1}{16x^6}$

56) $(10x^2)^{-3}$ **Solution:** $(10x^2)^{-3} = \frac{1}{(10x^2)^{+3}} = \frac{1}{1000x^6}$

57) $\frac{24x^3y^5}{32x^7y^{-9}}$ **Solution:** $\frac{24x^3y^5}{32x^7y^{-9}} = \frac{3y^{14}}{4x^4}$

58) $\frac{10x^4y^9}{30x^{12}y^{-3}}$ **Solution:** $\frac{10x^4y^9}{30x^{12}y^{-3}} = \frac{y^{12}}{3x^8}$

59) $\left(\frac{5x^3}{y}\right)^{-2}$ **Solution:** $\left(\frac{5x^3}{y}\right)^{-2} = (5x^3y^{-1})^{-2} = 5^{-2}x^{-6}y^2 = \frac{y^2}{25x^6}$

60) $\left(\frac{4x^4}{y}\right)^{-3}$ **Solution:** $\left(\frac{4x^4}{y}\right)^{-3} = \left(\frac{y}{4x^4}\right)^3 = \frac{y^3}{64x^{12}}$

61) $\left(\frac{-15a^4b^2}{5a^{10}b^{-3}}\right)^3$ **Solution:** $\left(\frac{-15a^4b^2}{5a^{10}b^{-3}}\right)^3 = \left(\frac{-3a^4b^2}{a^{10}b^{-3}}\right)^3 = \left(\frac{-27a^{12}b^6}{a^{30}b^{-9}}\right) = \frac{-27a^{12}b^6}{a^{30}b^{-9}} = \left(\frac{-27b^{15}}{a^{18}}\right)$

62) $\left(\frac{-30a^{14}b^8}{10a^{17}b^{-2}}\right)^3$ **Solution:** $\left(\frac{-30a^{14}b^8}{10a^{17}b^{-2}}\right)^3 = \left(\frac{-3a^{14}b^8}{a^{17}b^{-2}}\right)^3 = \frac{-27a^{42}b^{24}}{a^{51}b^{-6}} = \frac{-27b^{30}}{a^9}$

63) $\left(\frac{3a^{-5}b^2}{12a^3b^{-4}}\right)^0$ **Solution:** $\left(\frac{3a^{-5}b^2}{12a^3b^{-4}}\right)^0 = 1$

64) $\left(\frac{4a^{-5}b^3}{12a^3b^{-5}}\right)^0$ **Solution:** $\left(\frac{4a^{-5}b^3}{12a^3b^{-5}}\right)^0 = 1$

In Exercises 65–76, write each number in decimal notation without the use of exponents.

65) 3.8×10^2 **Solution:** $3.8 \times 10^2 = 380$

66) 9.2×10^2 **Solution:** $9.2 \times 10^2 = 920$

67) 6×10^{-4} **Solution:** $6 \times 10^{-4} = 6 \times \frac{1}{10^4} = 0.0006$

68) 7×10^{-5} **Solution:** $7 \times 10^{-5} = 7 \times \frac{1}{10^5} = 0.00007$

69) -7.16×10^6 **Solution:** $-7.16 \times 10^6 = -7160000$

70) -8.17×10^6 **Solution:** $-8.17 \times 10^6 = -8,170,000$

71) 7.9×10^{-1} **Solution:** $7.9 \times 10^{-1} = 7.9 \times \frac{1}{10} = 0.79$

72) 6.8×10^{-1} **Solution:** $6.8 \times 10^{-1} = 6.8 \times \frac{1}{10} = 0.68$

73) -4.15×10^{-3} **Solution:** $-4.15 \times 10^{-3} = -4.15 \times \frac{1}{10^3} = -0.00415$

74) -3.14×10^{-3} **Solution:** $-3.14 \times 10^{-3} = -3.14 \times \frac{1}{10^3} = -0.00314$

75) -6.00001×10^{10} **Solution:** $-6.00001 \times 10^{10} = -60,000,100,000$

Solution: $-7.00001 \times 10^{10} = -70,000,100,000$

Practice Plus

In Exercises 107–114, simplify each exponential expression. Assume that variables represent nonzero real numbers

107) $\frac{(x^2y)^{-3}}{(x^2y)^3}$ **Solution:** $\frac{(x^2y)^{-3}}{(x^2y)^3} = \frac{x^6y^{-3}}{x^6y^{-3}} = x^{6-6}y^{-3-(-3)} = x^0y^0 = 1$

108) $\frac{(xy^{-2})^{-2}}{(x^{-2}y)^{-3}}$ **Solution:** $\frac{(xy^{-2})^{-2}}{(x^{-2}y)^{-3}} = \frac{x^{-2}y^4}{x^6y^{-3}} = x^{-2-6}y^{4-(-3)} = x^{-8}y^7 = \frac{y^7}{x^8}$

109) $(2x^{-3}yz^{-6})(2x)^{-5}$ **Solution:** $(2x^{-3}yz^{-6})(2x)^{-5} = 2x^{-3}yz^{-6}.2^{-5}x.^{-5} = 2^{-4}x^{-8}yz^{-6} = \frac{y}{2^4x^8z^6} = \frac{y}{16x^8z^6}$

110) $(3x^{-4}yx^{-7})(3x)^{-3}$ **Solution:** $(3x^{-4}yx^{-7})(3x)^{-3} = 3x^{-4}yz^{-7}.3^{-3}x.^{-3} = 3^{-2}x^{-7}yz^{-7} = \frac{y}{3^2x^7z^7} = \frac{y}{9x^7z^7}$

111) $\left(\frac{x^3y^4z^5}{(x^{-3}y^{-4}z^{-5})}\right)^{-2}$ **Solution:** $\left(\frac{x^3y^4z^5}{(x^{-3}y^{-4}z^{-5})}\right)^{-2} = (x^6y^8z^{10})^{-2} = x^{-12}y^{-16}z^{-20} = \frac{1}{x^{12}y^{16}z^{20}}$

112) $\left(\frac{x^4y^5z^6}{(x^{-4}y^{-5}z^{-6})}\right)^{-4}$ **Solution:** $\left(\frac{x^4y^5z^6}{(x^{-4}y^{-5}z^{-6})}\right)^{-4} = (x^8y^{10}z^{12})^{-4} = x^{-32}y^{-40}z^{-48} = \frac{1}{x^{32}y^{40}z^{48}}$

In Exercises 1–16, evaluate each algebraic expression for the given value or values of the variable(s).

1) $7 + 5x$, for $x = 10$ **Solution:** $7 + 5x$, for $x = 10$ => $7 + 5(10) = 7+50 = 57$

2) $8 + 6x$, for $x = 5$ **Solution:** $8 + 6x$, for $x = 5$ => $8 + 6(5) = 8 + 30 = 38$

3) $6x - y$, for $x = 3$ and $y = 8$ **Solution:** $6x - y$, for $x = 3$ and $y = 8$ => $6(3) - 8 = 18 - 8 = 10$

4) $8x - y$, for $x = 3$ and $y = 4$ **Solution:** $8x - y$, for $x = 3$ and $y = 4$ => $8(3) - 4 = 24 - 4 = 20$

5) $x^2 + 3x$, for $x = 8$ **Solution:** $x^2 + 3x$, for $x = 8$ => $(8)^2 + 3(8) = 64 + 24 = 88$

6) $x^2 + 5x$, for $x = 6$ **Solution:** $x^2 + 5x$, for $x = 6$ => $(6)^2 + 5(6) = 36 + 30 = 66$

7) $x^2 - 6x + 3$, for $x = 7$ **Solution:** $x^2 - 6x + 3$, for $x = 7$ => $(7)^2 - 6(7) + 3 = 49 - 42 + 3 = 10$

8) $x^2 - 7x + 4$, for $x = 8$ **Solution:** $x^2 - 7x + 4$, for $x = 8$ => $(8)^2 - 7(8) + 4 = 64 - 56 + 4 = 12$

9) $4 + 5(x-7)^3$, for $x = 9$ **Solution:** $4 + 5(x-7)^3$, for $x = 9$ => $4 + 5(9-7)^3 = 4 + 5(8) = 4 + 40 = 44$

10) $6 + 5(x-6)^3$, for $x = 8$ **Solution:** $6 + 5(x-6)^3$, for $x = 8$ => $6 + 5(8-6)^3 = 6 + 5(8) = 6 + 40 = 46$

11) $x^2 - 3(x - y)$, for $x = 8$ and $y = 2$

Solution: $x^2 - 3(x - y)$, for $x = 8$ and $y = 2$ => $(8)^2 - 3(8-2) = 64 - 3(6) = 64 - 18 = 46$

12) $x^2 - 4(x - y)$, for $x = 8$ and $y = 3$

Solution: $x^2 - 4(x - y)$, for $x = 8$ and $y = 3$ => $(8)^2 - 4(8-3) = 64 - 4(5) = 64 - 20 = 44$

13) $\frac{5(x+2)}{2x-14}$, for $x = 10$ **Solution:** $\frac{5(x+2)}{2x-14}$, for $x = 10$ => $\frac{5(10+2)}{2(10)-14} = \frac{5(12)}{6} = \frac{60}{6} = 10$

14) $\frac{7(x-3)}{2x-16}$, for $x = 9$ **Solution:** $\frac{7(x-3)}{2x-16}$, for $x = 9$ => $\frac{7(9-3)}{2(9)-16} = \frac{7(6)}{2} = 7(3) = 21$

15) $\frac{2x+3y}{x+1}$, for $x = -2$ and $y = 4$

Solution: $\frac{2x+3y}{x+1}$, for $x = -2$ and $y = 4$ => $\frac{2(-2)+3(4)}{-2+1} = \frac{-4+12}{-1} = \frac{8}{-1} = -8$

16) $\frac{2x+y}{xy-2x}$, for $x = -2$ and $y = 4$

Solution: $\frac{2x+y}{xy-2x}$, for $x = -2$ and $y = 4$ => $\frac{2(-2)+4)}{(-2)(4)-2(-2)} = \frac{-4+4}{-8+4} = \frac{0}{4} = 0$

Solution:

In Exercises 21–32, simplify each algebraic expression.

21) $5(3x+4)-4$ **Solution:** $5(3x+4)-4 = 15x+20-4 = 15x+16$

22) $2(5x+4)-3$ **Solution:** $2(5x+4)-3 = 10x+8-3 = 10x+5$

23) $5(3x-2)+12x$ **Solution:** $5(3x-2)+12x = 5 = 15x-10+12x = 27x-10$

24) $2(5x-1)+14x$ **Solution:** $2(5x-1)+14x = 10x-2+14 = 24x-2$

25) $7(3y-5)+2(4y+3)$ **Solution:** $7(3y-5)+2(4y+3) = 21y-35+8y+6 = 29y-29$

26) $4(2y-6)+3(5y+10)$ **Solution:** $4(2y-6)+3(5y+10) = 8y-24+15y+30 = 23y+6$

27) $5(3y-2)-(7y+2)$ **Solution:** $5(3y-2)-(7y+2) = 15y-10+7y-2 = 8y-12$

28) $4(5y-3)-(6y+3)$ **Solution:** $4(5y-3)-(6y+3) = 20y-12+6y-3 = 14y-15$

29) $7-4[3-(4y-5)]$

Solution: $7-4[3-(4y-5)] = 7-4[3-4y+5] = 7-4[8-4y] = 7-32+16y = 16y-25$

30) $6-5[8-(2y-4)]$

Solution: $6-5[8-(2y-4)] = 6-5[8-2y+4] = 6-5[-2y+12] = 6+10y-60 = 10y-54$

31) $18x^2+4-[6(x^2-2)+5]$

Solution: $18x^2+4-[6(x^2-2)+5] = 18x^2+4-[6x^2-12+5] = 18x^2+4-[6x^2-7] = 18x^2+4-6x^2+7 = 12x^2+11$

32) $14x^2+5-[7(x^2-2)+4]$

Solution: $14x^2+5-[7(x^2-2)+4] = 14x^2+5-[7x^2-14+4] = 14x^2+5-[7x^2-10] = 14x^2+5-7x^2+10 = 7x^2+15$

In Exercises 33–38, write each algebraic expression without parentheses.

33) $-(-14x)$ **Solution:** $-(-14x) = 14x$

34) $-(-17y)$ **Solution:** $-(-17y) = 17y$

35) $-(2x-3y-6)$ **Solution:** $-(2x-3y-6) = -2x+3y+6$

36) $-(5x-13y-1)$ **Solution:** $-(5x-13y-1) = -5x+13y+1$

37) $\frac{1}{3}(3x)+[(4y)+(-4y)]$ **Solution:** $\frac{1}{3}(3x)+[(4y)+(-4y)] = x+[4y-4y] = x+0 = x$

38) $\frac{1}{2}(2y)+[(-7x)+7x)]$ **Solution:** $\frac{1}{2}(2y)+[(-7x)+7x)] = y+[-7x+7x] = y+0 = y$

In Exercises 39–48, use the order of operations to simplify each expression.

39) $8^2-16\div 2^2\cdot 4-3$ **Solution:** $8^2-16\div 2^2\cdot 4-3 = 64-16\div 4.4-3 = 64-4.4-3 = 64-16-3 = 48-3 = 45$

40) $10^2-100\div 5^2\cdot 2-3$ **Solution** $10^2-100\div 5^2\cdot 2-3 = 100-100\div 25.2-3 = 100-4.2-3 = 100-8-3 = 92-3 = 89$

41) $\frac{5.2-3^2}{[3^2-(-2)]^2}$ **Solution:** $\frac{5.2-3^2}{[3^2-(-2)]^2} = \frac{5.2-9}{[9-(-2)]^2} = \frac{10-9}{[9+2]^2} = \frac{10-9}{11^2} = \frac{1}{121}$

42) $\frac{10+2+3.4}{(12-3.2)^2}$ **Solution:** $\frac{10+2+3.4}{(12-3.2)^2} = \frac{5+12}{(12-6)^2} = \frac{17}{6^2} = \frac{17}{36}$

43) $8-3[-2(2-5)-4(8-6)]$

Solution: $8-3[-2(2-5)-4(8-6)] = 8-3[-2(-3)-4(2)] = 8-3[6-8] = 8-3[-2] = 8+6 = 14$

44) $8-3[-2(5-7)-5(4-2)]$

Solution: $8-3[-2(5-7)-5(4-2)] = 8-3[-2(-2)-5(2)] = 8-3[4-10] = 8-3[-6] = 8+18 = 26$

45) $\frac{2(-2)-4(-3)}{5-8}$ **Solution:** $\frac{2(-2)-4(-3)}{5-8} = \frac{-4+12}{-3} = \frac{8}{-3} = -\frac{8}{3}$

46) $\frac{6(-4)-5(-3)}{9-10}$ **Solution:** $\frac{6(-4)-5(-3)}{9-10} = \frac{-24+15}{-1} = \frac{-9}{-1} = 9$

47) $\frac{(5-6)^2-2|3-7|}{89-3.5^2}$ **Solution:** $\frac{(5-6)^2-2|3-7|}{89-3.5^2} = \frac{(-1)^2-2|-4|}{89-3.25} = \frac{1-2(4)}{89-75} = \frac{1-8}{14} = \frac{-7}{14} = -\frac{1}{2}$

48) $\frac{12\div 3.5|2^2+3^2|}{7+3-6^2}$ **Solution:** $\frac{12\div 3.5|2^2+3^2|}{7+3-6^2} = \frac{12\div 3.5|4+9|}{7+3-36} = \frac{4.5|13|}{10-36} = \frac{20(13)}{-26} = \frac{260}{-26} = -10$

In Exercises 1–16, solve each linear equation.

1) $7x - 5 = 72$ **Solution:** $7x - 5 = 72 \Rightarrow 7x = 72 + 5 \Rightarrow \frac{7x}{7} = \frac{77}{7} \Rightarrow x = 11$

2) $6x - 3 = 63$ **Solution:** $6x - 3 = 63 \Rightarrow 6x = 63 + 3 \Rightarrow \frac{6x}{6} = \frac{66}{6} \Rightarrow x = 11$

3) $11x - (6x - 5) = 40$ **Solution:** $11x - 6x + 5 = 40 \Rightarrow 5x = 40 - 5 \Rightarrow \frac{5x}{5} = \frac{35}{5} \Rightarrow x = 7$

4) $5x - (2x - 10) = 35$

Solution: $5x - 2x + 10 = 35 \Rightarrow 3x = 35 - 10 \Rightarrow \frac{3x}{3} = \frac{25}{3} \Rightarrow x = \frac{25}{3}$

5) $2x - 7 = 6 + x$ **Solution:** $2x - x = 6 + 7 \Rightarrow x = 13$

6) $3x + 5 = 2x + 13$ **Solution:** $3x - 2x = 13 - 5 \Rightarrow x = 8$

7) $7x + 4 = x + 16$ **Solution:** $7x - x = 16 - 4 \Rightarrow \frac{6x}{6} = \frac{12}{6} \Rightarrow x = 2$

8) $13x + 14 = 12x - 5$ **Solution:** $13x - 12x = -5 - 14 \Rightarrow x = -19$

9) $3(x - 2) + 7 = 2(x + 5)$ **Solution:** $3x - 6 + 7 = 2x + 10 \Rightarrow 3x - 2x = 10 - 1 \Rightarrow x = 9$

10) $2(x - 1) + 3 = x - 3(x + 1)$

Solution: $2x - 2 + 3 = x - 3x - 3 \Rightarrow 2x + 1 = -2x - 3 \Rightarrow 2x + 2x = -3 - 1$

$\frac{4x}{4} = \frac{-4}{4} \Rightarrow x = -1$

11) $\frac{x+3}{6} = \frac{3}{8} + \frac{x-5}{4}$

Solution: $\frac{x+3}{6} = \frac{3+2x-10}{8} \Rightarrow \frac{x+3}{6} = \frac{2x-7}{8}$ By cross multiply $8(x+3) = 6(2x-7)$

$8x + 24 = 12x - 42 \Rightarrow 8x - 12x = -42 - 24 = \frac{-14x}{+4} = \frac{+66}{+4} \Rightarrow x = \frac{33}{2}$

12) $\frac{x+1}{4}=\frac{1}{6}+\frac{2-x}{3}$

Solution: $\frac{x+1}{4}=\frac{1+4-2x}{6}$ => $\frac{x+1}{4}=\frac{5-2x}{6}$. By cross multiply

$6(x+1)=4(5-2x)$

$6x+6=20-8x$ => $6x+8x=20-6$ => $\frac{14x}{14}=\frac{14}{14}$ => $x=1$

13) $\frac{x}{4}=2+\frac{x-3}{3}$

Solution: $\frac{x}{4}=\frac{6+x-3}{3}$ => $\frac{x}{4}=\frac{3+x}{3}$ By cross multiply $3(x)=4(3+x)$

$3x=12+4x$ => $3x-4x=12$ => $-x=12$ => $x=-12$

14) $5+\frac{x-2}{3}=\frac{x+3}{8}$

Solution: $\frac{15+x-2}{3}=\frac{x+3}{8}$ => $\frac{13+x}{3}=\frac{x+3}{8}$ By cross multiply $8(13+x)=3(x+3)$

$104+8x=3x+9$ => $104-9=3x-8x$ => $\frac{95}{5}=\frac{-5x}{5}$ => $-x=19$ => $x=-19$

15) $\frac{x+1}{3}=5-\frac{x+2}{7}$

Solution: $\frac{x+1}{3}=\frac{35-x+2}{7}$ => $21\left[\frac{x+1}{3}=5\frac{x+2}{7}\right]$ => $7x+7=105-3x-6$

$7x+3x=99-7$ => $10x=92$ => $x=\frac{92}{10}=\frac{46}{5}$

The solution set is $\left\{\frac{46}{5}\right\}$

16) $\frac{3x}{5}-\frac{x-3}{2}=\frac{x+2}{3}$

Solution: $\frac{3x}{5}-\frac{x-3}{2}=\frac{x+2}{3}$ => $30\left[\frac{3x}{5}-\frac{x-3}{2}=\frac{x+2}{3}\right]$

$18x-15x+45=10x+20$ => $3x-10x=20-45$ => $-7x=-25$

$x=\frac{25}{7}$ => The solution set is $\left\{\frac{25}{7}\right\}$

Exercises 17–26 contain rational equations with variables in denominators. For each equation, a. Write the value or values of the variable that make a denominator zero. These are the restrictions on the variable. b. Keeping the restrictions in mind, solve the equation.

17) $\frac{1}{x-1}+5=\frac{11}{x-1}$ **Solution:** $\frac{1}{x-1}+5=\frac{11}{x-1}\ (x\neq 1)$

$\frac{1}{x-1}+5=\frac{11}{x-1}$ => $1+5(x-1)=11$ => $1+5x-5=11$ => $5x=15$ => $x=3$

The solution set is $\{3\}$.

18) $\frac{3}{x+4}-7=\frac{-4}{x+4}$ **Solution:** $\frac{3}{x+4}-7=\frac{-4}{x+4}\ (x\neq -4)$

$\frac{3}{x+4}-7=\frac{-4}{x+4}$ => $3-7(x+4)=-4$ => $3-7x-28=-4$ => $-7x=21$ => $x=-3$

The solution set is $\{-3\}$

19) $\frac{8x}{x+1} = 4 - \frac{8}{x+1}$ **Solution:** $\frac{8x}{x+1} = 4 - \frac{8}{x+1} (x \neq -1)$

$\frac{8x}{x+1} = 4 - \frac{8}{x+1}$ => $8x = 4(x+1) - 8$ => $8x = 4x + 4 - 8$ => $4x = -4$ => $x = -1$ => no solution

The solution set is the empty set, $\varnothing$.

20) $\frac{2}{x-2} = \frac{x}{x-2} - 2$ **Solution:** $\frac{2}{x-2} = \frac{x}{x-2} - 2 (x \neq 2)$

$\frac{2}{x-2} = \frac{x}{x-2} - 2$ => $2 = x - 2(x-2)$ => $2 = x - 2x + 4$ => $x = 2$ => no solution

The solution set is the empty set $\varnothing$.

21) $\frac{3}{2x-2} + \frac{1}{2} = \frac{2}{x-1}$ **Solution:** $\frac{3}{2x-2} + \frac{1}{2} = \frac{2}{x-1} (x \neq 1)$

$\frac{3}{2x-2} + \frac{1}{2} = \frac{2}{x-1}$ => $\frac{3}{2(x-1)} + \frac{1}{2} = \frac{2}{x-1}$ => $3 + 1(x-1) = 4$ => $3 + x - 1 = 4$ => $x = 2$

The solution set is {2}.

22) $\frac{3}{2x-2} = \frac{5}{2x+6} + \frac{1}{x-2}$ **Solution:** $\frac{3}{2x-2} = \frac{5}{2x+6} + \frac{1}{x-2} (x \neq -3, x \neq 2)$

$\frac{3}{2x-2} = \frac{5}{2x+6} + \frac{1}{x-2}$ => $6(x-2) = 5(x-2) + 2(x+3)$ => $6x - 12 = 5x - 10 + 2x + 6$

$-x = 8$ => $x = -8$ The solution set is {−8}.

23) $\frac{2}{x+1} - \frac{1}{x-1} = \frac{2x}{x^2-1}$ **Solution:** $\frac{2}{x+1} - \frac{1}{x-1} = \frac{2x}{x^2-1} (x \neq 1, x \neq -1)$

$\frac{2}{x+1} - \frac{1}{x-1} = \frac{2x}{x^2-1}$ => $\frac{2}{x+1} - \frac{1}{x-1} = \frac{2x}{(x+1(x-1)}$

$2(x-1) - 1(x+1) = 2x$ => $2x - 2 - x - 1 = 2x$ => $-x = 3$ => $x = -3$ The solution set is {−3}

24) $\frac{4}{x+5} + \frac{2}{x-5} = \frac{32}{x^2 - 25}$ **Solution:** $\frac{4}{x+5} + \frac{2}{x-5} = \frac{32}{x^2-25}; x \neq 5, -5$

$\frac{4}{x+5} + \frac{2}{x-5} = \frac{32}{x^2-25} (x \neq 5, x \neq -5)$ => $4(x-5) + 2(x+5) = 32$ => $4x - 20 + 2x + 10 = 32$

$6x = 42$ => $x = 7$ the solution set is {7}

25) $\frac{1}{x-4} - \frac{5}{x+2} = \frac{6}{x^2-2x-8}$ **Solution:** $\frac{1}{x-4} - \frac{5}{x+2} = \frac{6}{x^2-2x-8} (x \neq -2, 4)$

$\frac{1}{x-4} - \frac{5}{x+2} = \frac{6}{x^2-2x-8}$ => $\frac{1}{x-4} - \frac{5}{x+2} = \frac{6}{(x-4)(x+2)} (x \neq 4, x \neq -2)$

$1(x+2) - 5(x-4) = 6$ => $x + 2 - 5x + 20 = 6$ => $-4x = -16$ => $x = 4$ The solution set is the empty set, $\varnothing$

26) $\frac{1}{x-3} - \frac{2}{x+1} = \frac{8}{(x-3(x+1)}$ **Solution:** $\frac{1}{x-3} - \frac{2}{x+1} = \frac{8}{(x-3(x+1)}; x \neq -1, 3$

$\frac{1}{x-3} - \frac{2}{x+1} = \frac{8}{(x-3(x+1)}$ => $1(x+1) - 2(x-3) = 8$ => $x + 1 - 2x + 6 = 8$

$-x + 7 = 8$ => $-x = 1$ => $x = -1$ The solution set is the empty set, $\varnothing$

In Exercises 43–54, solve each absolute value equation or indicate the equation has no solution.

43) $|x-2|=7$

Solution $|x-2|=7$

$x-2=7$ or $x-2=-7$

$x=9$, or $x=-5$

The solution set is $\{9,-5\}$

44) $|x+1|=5$

Solution

$|x+1|=5$

$x+1=5$ or $x+1=-5$

$x=4$ or $x=-6$

The solution set is $\{-6, 4\}$

45) $|2x-1|=5$

Solution

$|2x-1|=5$

$2x-1=5$ or $2x-1=-5$

$2x=6$ or $2x=-4$

$x=3$ or $x=-2$

The solution set is $\{3,-2\}$

46) $|2x-3|=11$

Solution

$|2x-3|=11$

$2x-3=11$ or $2x-3=-11$

$2x=14$ or $2x=-8$

$x=7$, or $x=-4$

The solution set is $\{-4, 7\}$

47) $2|3x-2|=14$

Solution

$2|3x-2|=14$

$|3x-2|=7$

$3x-2=7$ or $3x-2=-7$

$3x=9$ or $3x=-5$

$x=3$ or $x=-5/3$

The solution set is $\{3,-5/3\}$

48) $3|2x-1|=21$

Solution

$3|2x-1|=21$

$|2x-1|=7$

$2x-1=7$ or $2x-1=-7$

$2x=8$ or $2x=-6$

$x=4$ or $x=-3$

The solution set is $\{4,-3\}$

49) $2|4-\frac{5}{2}x|+6=18$

Solution

$$2\left|4-\frac{5}{2}x\right|+6=18$$

$$2\left|4-\frac{5}{2}x\right|=12$$

$$\left|4-\frac{5}{2}x\right|=6$$

$4-\frac{5}{2}x=6$ or $4-\frac{5}{2}x=-6$

$-\frac{5}{2}x=2$ or $-\frac{5}{2}x=-10$

$x=-\frac{4}{5}$ or $x=4$

The solution set is $\left\{-\frac{4}{5},4\right\}$

50) $4|1-\frac{3}{4}x|+7=10$

Solution

$$4\left|1-\frac{3}{4}x\right|+7=10$$

$$4\left|1-\frac{3}{4}x\right|=3$$

$$\left|1-\frac{3}{4}x\right|=\frac{3}{4}$$

$1-\frac{3}{4}x=\frac{3}{4}$ or $1-\frac{3}{4}x=-\frac{3}{4}$

$-\frac{3}{4}x=-\frac{1}{4}$ or $-\frac{3}{4}x=-\frac{7}{4}$

$x=\frac{1}{3}$ or $x=\frac{7}{3}$

The solution set is $\left\{\frac{1}{3},\frac{7}{3}\right\}$

51) $|x+1|+5=3$

Solution

$|x+1|+5=3$

$|x+1|=-2$

No solution

The solution set is $\{\ \}$

52) $|x+1|+6=2$

Solution

$|x+1|+6=2$

$|x+1|=-4$

The solution set is $\{\ \}$.

53) $|2x-1|+3=3$

Solution

$|2x-1|+3=3$

$|2x-1|=0$

$2x-1=0 \Rightarrow 2x=1 \Rightarrow x=\frac{1}{2}$

The solution set is $\left\{\frac{1}{2}\right\}$

54) $|3x-2|+4=4$

Solution

$|3x-2|+4=4$

$|3x-2|=0$

$3x-2=0 \Rightarrow 3x=2 \Rightarrow x=\frac{2}{3}$

The solution set is $\left\{\frac{2}{3}\right\}$

In Exercises 55–60, solve each quadratic equation by factoring.

55) $x^2-3x-10=0$

Solution: $x^2-5x+2x-10=0$

$x(x-5)+2(x-5)=0$

$(x-5)(x+2)=0$

$x-5=0 \Rightarrow x=5$ and

$x+2=0 \Rightarrow x=-2$

56) $x^2-13x+36=0$

Solution: $x^2-9x-4x+36=0$

$x(x-9)-4(x-9)=0$

$(x-9)(x-4)=0$

$x-9=0,\quad x=9$

and $x-4=0 \Rightarrow x=4$

57) $x^2=8x-15$

Solution: $x^2-8x+15=0$

$x^2-5x-3x+15=0$

$x(x-5)-3(x-5)=0$

$(x-5)(x-3)=0$

$x=5=0 \Rightarrow x=5$

and $x-3=0 \Rightarrow x=3$

58) $x^2=-11x-10$

Solution: $x^2+11x+10=0$

$x^2+10x+x+10=0$

$x(x+10)+1(x+10)=0$

$(x+10)(x+1)=0$

$x+10=0,$ or $x+1=0$

$x=-10,$ or $x=-1$

59) $5x^2=20x$

Solution: $5x^2-20x=0 \Rightarrow 5x(x-4)=0$

$5x=0$ and $x-4=0$

$x=0/5 \Rightarrow x=0,$ $x=4$

60) $3x^2=12x$

Solution: $3x^2-12x=0 \Rightarrow 3x(x-4)=0$

$3x=0$ and $x-4=0$

$x=0$ and $x=4$

In Exercises 61–66, solve each quadratic equation by the square root property.

61) $3x^2=27$

Solution: $3x^2=27 \Rightarrow x^2=9$

$\sqrt{x^2}=\pm\sqrt{9} \Rightarrow x=\pm 3$

The solution set is $\{+3\}$

62) $5x^2=45$

Solution: $5x^2=45 \Rightarrow x^2=9$

$\sqrt{x^2}=\pm\sqrt{9} \Rightarrow x=\pm 3$

The solution set is $\{\pm 3\}$

63) $5x^2+1=51$

Solution: $5x^2=51 \Rightarrow 5x^2=50$

$x^2=10\ \sqrt{x^2}=\pm\sqrt{10}$

$x=\pm\sqrt{10}$

The solution set is $\{\pm\sqrt{10}\}$

64) $3x^2-1=47$

Solution: $3x^2-1=47 \Rightarrow 3x^2=48$

$x^2=16 \Rightarrow \sqrt{x^2}=\pm\sqrt{16}$

$x=\pm 4$

The solution set is $\{\pm 4\}$

65) $3(x-4)^2=15$

Solution: $(x-4)^2=5$

$\sqrt{(x-4)^2}=\pm\sqrt{5}$

$x-4=\pm\sqrt{5} \Rightarrow x=4\pm\sqrt{5}$

The solution set is $\{4\pm\sqrt{5}\}$

66) $3(x+4)^2=21$

Solution: $(x+4)^2=7$

$\sqrt{(x-4)^2}=\pm\sqrt{7}$

$x+4=\pm\sqrt{7} \Rightarrow x=-4\pm\sqrt{7}$

The solution set is $\{-4\pm\sqrt{7}\}$

In Exercises 67–74, solve each quadratic equation by completing the square.

67) $x^2+6x=7$

$x^2+6x+9=7+9$

$(x+3)^2=16$

$x+3=\pm 4$

$x=-3\pm 4$

The solution set is $\{-7,1\}$

68) $x^2+6x=-8$

Solution: $x^2=+6x-8$

$x^2+6x+9=-8+9$

$(x+3)^2=1 \Rightarrow x+3=\pm 1$

$x=-3\pm 1$

The solution set is $\{-4,-2\}$

69) $x^2 - 2x = 2$

Solution: $x^2 - 2x = 2 \Rightarrow (x^2) - 2(x)(1) = 2$

By completing square adding $(1)^2$ on both side

$(x)^2 - 2(1) + (1)^2 = (1)^2 + 2 \Rightarrow (x-1)^2 = 3$

Taking square roots on both sides

$\sqrt{(x-1)^2} = \sqrt{3} \Rightarrow (x-1) = \pm\sqrt{3}$

$(x = 1 \pm \sqrt{3})$

70) $x^2 + 4x = 12$

Solution: $x^2 + 4x = 12 \Rightarrow (x)^2 + 2(x)(2) = 12$

By completing square adding $(2)^2$ on both side

$(x)^2 + 2(x)(2) + (2)^2 = 12 + (2)^2$

$(x+2)^2 = 12 + 4 = 16$

Taking square roots on both sides

$\sqrt{(x-2)^2} = \sqrt{16} \Rightarrow x + 2 = \pm 4$

$x + 2 = 4$ and $x + 2 = -4$

$x = 4 - 2 = 2$ and $x = -4 - 2 = -6$

71) $x^2 - 6x - 11 = 0$

Solution: $x^2 - 6x = 11 = 0 \Rightarrow x^2 - 6x = 11$

$(x)^2 - 2(x)(3) = 11$

By completing square adding $(3)^2$ on both side

$(x)^2 - 2(x)(3) + (3)^2 = (3)^2 + 11$

$(x-3)^2 = 9 + 11 = 20$

Taking square roots on both sides

$\sqrt{(x-3)^2} = \sqrt{20} \Rightarrow x - 3 = \pm\sqrt{20}$

$x = 3 \pm 2\sqrt{5}$

72) $x^2 - 2x - 5 = 0$

Solution: $x^2 - 2x - 5 = 0 \Rightarrow x^2 - 2x = 5$

$(x)^2 - 2(x)(1) = 5$

By completing square adding $(1)^2$ on both side

$(x)^2 - 2(x)(1) + (1)^2 = 5 + (1)^2$

$(x-1)^2 = 5 + 1 \Rightarrow (x-1)^2 = 6$

Taking square roots on both sides

$\sqrt{(x-1)^2} = \sqrt{6} \Rightarrow x - 1 = \pm\sqrt{6}$

$(x = 1 \pm \sqrt{6})$

73) $x^2 + 4x + 1 = 0$

Solution: $x^2 + 4x = -1 \Rightarrow (x)^2 + 2(x)(2) = -1$

By completing square adding $(2)^2$ on both side

$(x)^2 + 2(x)(2) + (2)^2 = -1 + (2)^2$

$(x+2)^2 = -1 + 4$

$(x+2)^2 = 3$ Taking square roots on both sides

$\sqrt{(x-2)^2} = \sqrt{3} \Rightarrow x + 2 = \pm\sqrt{3}$

$(x = -2 \pm \sqrt{3})$

74) $x^2 + 6x - 5 = 0$

Solution: $x^2 + 6x = 5 \Rightarrow (x)^2 + 2(x)(2) = 5$

By completing square adding $(3)^2$ on both side

$(x)^2 + 2(x)(3) + (3)^2 = 5 + (3)^2$

$(x+2)^2 = 5 + 9 = 14$

Taking square roots on both sides

$\sqrt{(x-3)^2} = \sqrt{14} \Rightarrow x + 3 = \pm\sqrt{14}$

$(x = -3 \pm \sqrt{14})$

In Exercises 75–82, solve each quadratic equation using the quadratic formula.

75) $x^2 + 8x + 15 = 0$

Solution: $a = 1,\ b = 8,\ c = 15$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-8 \pm \sqrt{(8)^2 - 4(1)(6)}}{2(1)}$$

$$x = \frac{-8 \pm \sqrt{64 - 60}}{2} = \frac{-b \pm \sqrt{4}}{2} = \frac{-8 \pm 2}{2}$$

$$x = \frac{-8+2}{2} = \frac{-8-2}{2} = -3$$

$$x = \frac{-8-2}{2} = \frac{10}{2} = -5$$

76) $x^2 + 8x + 12 = 0$

Solution: $a = 1,\ b = 8,\ c = 12$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-8 \pm \sqrt{(8)^2 - 4(1)(12)}}{2(1)}$$

$$x = \frac{-8 \pm \sqrt{64 - 48}}{2} = \frac{-b \pm \sqrt{16}}{2} = \frac{-8 \pm 4}{2}$$

$$x = \frac{-8+4}{2} = \frac{-4}{2} = -2$$

$$x = \frac{-8-4}{2} = \frac{12}{2} = -6$$

77) $x^2 + 5x + 3 = 0$

Solution: $a = 1,\ b = 5,\ c = 3$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-5 \pm \sqrt{(5)^2 - 4(1)(3)}}{2(1)}$$

$$x = \frac{-5 \pm \sqrt{25 - 12}}{2} = \frac{-5 \pm \sqrt{13}}{2}$$

78) $x^2 + 5x + 2 = 0$

Solution: $a = 1,\ b = 5,\ c = 2$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-5 \pm \sqrt{(5)^2 - 4(1)(2)}}{2(1)}$$

$$x = \frac{-5 \pm \sqrt{25 - 8}}{2} = \frac{-5 \pm \sqrt{17}}{2}$$

79) $3x^2 - 3x - 4 = 0$

Solution: $a = 3,\ b = -3,\ c = -4$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(3)(-4)}}{2(3)}$$

$$x = \frac{(3) \pm \sqrt{9 - 48}}{6} = \frac{3 \pm \sqrt{57}}{6}$$

80) $5x^2 + x - 2 = 0$

Solution: $a = 5,\ b = 1,\ c = -2$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(1) \pm \sqrt{(1)^2 - 4(5)(-2)}}{2(5)}$$

$$x = \frac{-1 \pm \sqrt{1 - 40}}{10} = \frac{-1 \pm \sqrt{41}}{10}$$

81) $4x^2 = 2x + 7$

Solution: $4x^2 - 2x - 7 = 0$

$a = 4,\ b = -2,\ c = -7$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(4)(-7)}}{2(4)}$$

$$x = \frac{2 \pm \sqrt{4 + 112}}{8} = \frac{2 \pm \sqrt{116}}{8}$$

$$x = \frac{2 \pm 2\sqrt{29}}{8} = \frac{1 \pm \sqrt{29}}{4}$$

The solution set is $\left\{\frac{1 + \sqrt{29}}{4}, \frac{1 - \sqrt{29}}{4}\right\}$

82) $3x^2 = 6x - 1$ **(ADS.24)**

Solution: $3x^2 - 6x + 1 = 0$

$a = 3,\ b = -6,\ c = 1$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(3)(1)}}{2(3)}$$

$$x = \frac{6 \pm \sqrt{36 - 12}}{6} = \frac{6 \pm \sqrt{24}}{6}$$

$$x = \frac{6 \pm 2\sqrt{6}}{6} = \frac{3 \pm \sqrt{6}}{3}$$

The solution set is $\left\{\frac{3 + \sqrt{6}}{3}, \frac{3 - \sqrt{6}}{3}\right\}$

Compute the discriminant of each equation in Exercises 83–90. What does the discriminant indicate about the number and type of solutions?

83) $x^2 - 4x - 5 = 0$

Solution: $x^2 - 4x - 5 = 0,\ a = 1, b = -4,\ c = -4$

Discriminant $= b^2 - 4ac$

$= (-4)^2 - 4(1)(-5) = 16 + 20$

$= 36;$ 2 unequal real solutions

84) $4x^2 - 2x + 3 = 0$

Solution: $4x^2 - 2x + 3 = 0$

Discriminant $= b^2 - 4ac$

Discriminant $= (-2)^2 - 4(4)(3) = 4 - 48$

$= -44;$ 2 complex imaginary solutions

85) $2x^2 - 11x + 3 = 0$

Solution: $2x^2 - 11x + 3 = 0$

Discriminant $= b^2 - 4ac$

$= (-11)^2 - 4(2)(3) = 121 - 24$

$= 97;$ 2 unequal real solutions

86) $2x^2 + 11x - 6 = 0$

Solution: $2x^2 + 11x - 6 = 0$

Discriminant $= b^2 - 4ac$

$= 11^2 - 4(2)(-6) = 121 + 48$

$= 169;$ 2 unequal real solutions

87) $x^2 = 2x - 1$

Solution: $x^2 = 2x - 1$

$x^2 - 2x + 1 = 0$

Discriminant $= b^2 - 4ac$

$= (-2)2 - 4(1)(1) = 4 - 4$

$= 0;$ 1 real solutions

88) $3x^2 = 2x - 1$

Solution: $3x^2 = 2x - 1$

$3x^2 - 2x + 1 = 0$

Discriminant $= b^2 - 4ac$

$= (-2)^2 - 4(3)(1) = 4 + 12$

$= -8;$ 2 complex imaginary solutions

89) $x^2 - 3x - 7 = 0$

Solution: $x^2 - 3x - 7 = 0$

Discriminant $= b^2 - 4ac$

$= (-3)^2 - 4(1)(-7) = 9 + 28$

$= 37;$ 2 unequal real solutions

90) $3x^2 + 4x - 2 = 0$

Solution: $3x^2 + 4x - 2 = 0$

Discriminant $= b^2 - 4ac$

$= 4^2 - 4(3)(-2) = 16 + 24$

$= 40.$ 2 unequal real solutions

Recommended Texts

1. Akar, G. K., Zembat, İ. Ö., Arslan, S., & Thompson, P. W. (2023). Quantitative Reasoning in Mathematics and Science Education. 1^{st} Ed., Springer, USA.
2. Peck, R., Olsen, C., & Devore, J. L. (2015). Introduction to statistics and data analysis. 5^{th} Ed Brooks Cole, USA.
3. Devlin, K. J. (2012). Introduction to mathematical thinking. Palo Alto, CA: Keith Devlin.

Suggested Readings

1. Triola, M. F., Goodman, W. M., Law, R., & Labute, G. (2006). Elementary statistics. Reading, MA Pearson/Addison-Wesley.
2. Blitzer, R., & White, J. (2005). Thinking mathematically. Pearson Prentice Hall.

حرفِ آخر (10-09-2024)

خوش رہیں خوشیاں بانٹیں اور جہاں تک ہوسکے دوسروں کے لیے آسانیاں پیدا کریں۔

اللہ تعالٰی آپ کو زندگی کے ہر موڑ پر کامیابیوں اور خوشیوں سے نوازے۔(اٰمین)

محمد عثمان حامد

چک نمبر 105 شمالی (گودھے والا) سرگودھا

UNIVERSITY OF SARGODHA

PUNJAB, PAKISTAN