SET THEORY

Equivalent SETs

Relation b/w $A \sim B$ is an "Equivalence Relation"

Two sets A And B are said to be Equivalent sets if $\exists$ $\boxed{f: A \to B}$ which is $\boxed{\text{BIJECTIVE}}$

- $[0,1] \sim [a,b]$ by function $\boxed{f(x) = a + (b-a)x}$
- $[0,1] \sim (0,1)$
- $[0,1] \sim (0,1]$
- $[0,1] \sim [0,1)$
- $(0,1] \sim (0,1)$

by function Jo $\boxed{\text{Piecewise}}$ $\boxed{\text{Hoga}}$...!

PYQ 2022

Q: Prove that $\mathbb{R} \sim \mathbb{R}^+$

$f: \mathbb{R} \to \mathbb{R}^+$ by

$f(x) = e^x \quad \forall\, x \in \mathbb{R}$

One to One: for any $x, y \in \mathbb{R}$

Let $f(x) = f(y)$

$\Rightarrow e^x = e^y$

$\Rightarrow x = y$

$\Rightarrow f$ is one to one!

Onto: for any $y \in \mathbb{R}^+$

$\exists\ \ln x \in \mathbb{R}$ s.t $f(\ln x) = e^{\ln x} = x$

$\Rightarrow f$ is onto

$\Rightarrow f$ is Bijective

$\mathbb{R} \sim \mathbb{R}^+$

PYQ 2022

Q: Prove that $(0,1) \sim (-\infty, \infty)$

$(0,1) \sim (-\infty, \infty)$ But $(-\infty, \infty) \sim \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$

- $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \sim (-\infty, \infty)$
- $(0,1) \sim (-\infty, \infty)$
- $[a,b] \sim (a,b)$

Transitive Property

By Transitive property

$(0,1) \sim \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$f(x) = \pi x - \frac{\pi}{2}$

One to one: for $x, y \in (0,1)$

Let $f(x) = f(y)$

$\Rightarrow \cancel{\pi} x - \cancel{\frac{\pi}{2}} = \cancel{\pi} y - \cancel{\frac{\pi}{2}} \Rightarrow x = y \Rightarrow f$ is one to one

Onto: for $y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ is the image of $\dfrac{y + \frac{\pi}{2}}{\pi} \in (0,1)$

since $y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$-\frac{\pi}{2} < y < \frac{\pi}{2}$

$\Rightarrow -\frac{\pi}{2} + \frac{\pi}{2} < y + \frac{\pi}{2} < \frac{\pi}{2} + \frac{\pi}{2}$

$\Rightarrow 0 < y + \frac{\pi}{2} < \pi$

$\Rightarrow \frac{0}{\pi} < \dfrac{y + \frac{\pi}{2}}{\pi} < \frac{\cancel{\pi}}{\cancel{\pi}}$ 0, 1

Denumerable Set → **Infinite Set** → **Countable Set** → **Non-Denumable OR Uncountable**

Denumerable Set

- A set which is Equivalent to $\boxed{\mathbb{N}}$ (natural numbers)
- $\mathbb{Z}$, E and O.

Infinite Set

A set A is called Infinite set if it is equivalent to one of its proper subset.

Countable Set

A set is called countable if it is denumerable OR finite.

Non-Denumable OR Uncountable

A set which is infinite OR does not have SAME cardinality as $\mathbb{N}$.

PYQ 2019

Q: Prove that $\mathbb{N} \times \mathbb{N}$ is Denumerable set.

→

PYQ 2018, 2021

Prove that $[0,1]$ is not denumerable OR uncountable

Suppose on contrary $[0,1]$ is denumerable. The we write as

→

PYQ 2020

Prove OR Disprove that $[a,b]$ is uncountable

We know that $[0,1] \sim [a,b]$

And $[0,1]$ is non

$\Rightarrow [a,b]$ is also uncountable.

Suppose that $[a,b]$ is countable

$\Rightarrow [a,b] \sim \mathbb{N}$

But $[a,b] \sim [0,1]$

By Transitive property $[0,1] \nsim \mathbb{N}$ which is contradiction

$\Rightarrow [a,b]$ is non-denumbel.

$[0,1] = \{x_1, x_2, x_3, \dots\}$
Divide the interval $[0,1]$ into three sub-intervals $\left[0, \frac{1}{3}\right], \left[\frac{1}{3}, \frac{2}{3}\right]$ $\left[\frac{2}{3}, \frac{3}{3}\right]$
$x_1 \in [0,1]$, Let $[a_1, b_1]$ be such interval that $x_1 \notin [a_1, b_1] \in I_1$

-denumerable set
o to subintervals $x_2 \notin [a_2, b_2] \in I_2$
Continuing,

By Cantor intersection Thm:
$\bigcap^{\infty} I_n \neq \phi$

PYQ 2014, 2018, 2021

Q: Prove that a subset of Denumerable set is either finite or Denumerable.

Let $A = \{a_1, a_2, \dots\}$ be denumerable set. Take $B \subseteq A$. If B is finite then we are done!
If B is not finite then
Let a_{n_1} be smallest element in B,
But $a_{n_2} > a_{n_1}$ and $a_{n_2} \in B$.
continuing in this way, we get
$B = \{a_{n_1}, a_{n_2}, \dots\}$
This that $a_{n_i} \in B$ which sequence of distinct element.
$\Rightarrow B$ is a Denumerables set.

→

PYQ 2020

Prove that Q is countable But $\mathbb{R}$ is uncountable.

$\mathbb{R} = Q \cup Q'$ Here
$\mathbb{R}$ is uncountable
To prove that $\mathbb{R}$ is uncountable we will show Q' are uncountable
Suppose on contrary that Q' is denumerable.
$\therefore Q$ is denumerable
and union of Denumerable sets is also Denumerable set So, $Q \cup Q' = \mathbb{R}$ is denumerable But

→

PYQ 2023

Prove that Q is Denumerable →

We want to show that $Q \sim \mathbb{N}$
Since $Q = Q^+ \cup Q^- \cup \{0\}$
Define a mapping
$f: Q^+ \to \mathbb{N} \times \mathbb{N}$
by $f\left(\frac{p}{q}\right) = (p, q)\ \forall\ p/q \in Q^+$
One to one: for any $p/q, m/n \in Q^+$
Let $f\left(\frac{p}{q}\right) = f\left(\frac{m}{n}\right)$
$\Rightarrow (p, q) = (m, n)$
$\Rightarrow p = m$ and $q = n$
$\Rightarrow \frac{p}{q} = \frac{m}{n}$
$\Rightarrow f$ is one to one.
Onto: for any $(m, n) \in \mathbb{N} \times \mathbb{N}$
$\exists\ m/n \in Q^+$ such that

this contradiction to the fact that $\mathbb{R}$ is not denumerable

$\Rightarrow Q'$ is not denumerable.

CARDINAL NUMBER

- Two sets have $\boxed{\text{SAME}}$ cardinality if they are $\boxed{\text{Equivalent}}$!
- Finite sets have finite cardinality
- cardinality of Infinite sets is Transfinite cardinality"
- Cardinality of $\mathbb{N}$ is $\boxed{\aleph_0}$ → alpha-naught
- cardinality of $\boxed{I=[0,1]}$ is c → power of continuum

Jo Bi sets I Aur $\mathbb{N}$ Kay Equivalent hain, they also have SAME

PYQ 2018–2022

CANTOR'S THEOREM:-

For any set A, we have $\boxed{\#(A)} < \boxed{\#(P(A))}$

$A=\{1,2,3\} \rightarrow 3$

$P(A)=2^3=8$

$\#(A) < \# P(A) \rightarrow$ cantor

$\boxed{3<8}$

Proof: Let A be any set, $P(A)$ be power set of A.

Define $\boxed{g: A \rightarrow P(A)}$ by

$\boxed{g(a)=\{a\}}$ $\forall a \in A$

one to one:- for any $a,b \in A$

Let $g(a)=g(b)$

$\Rightarrow \{x\}=\{y\}$

$\Rightarrow x=y$

$\Rightarrow g$ is one to one!

Hence.

$\Rightarrow \#(A) \leq \# P(A)$

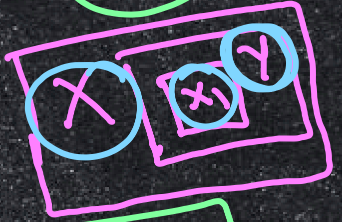
If we prove that

$\#(A) \neq \# P(A)$ then

cardinality as $\mathbb{N}$ and I !

PYQ 2015, 2017, 2018

SCHROEDER-BERSTEIN'S THEOREM: $X \sim Y$



Let X, Y and X_1 be sets such that $X \supseteq Y \supseteq X_1$ and $\boxed{X \sim X_1}$ then $\boxed{X \sim Y}$

$X \sim X_1$

PYQ 2018, 2020, 2021

Q: Prove that $\aleph_0 + \aleph_0 = \aleph_0$, $n + \aleph_0 = \aleph_0$, $c + c = c$

$\aleph_0 + \aleph_0 = \aleph_0$

Let $E = \{2, 4, 6 \ldots\}$

$O = \{1, 3, 5 \ldots\}$

Then $E \cap O = \phi$

$E \cup O = \mathbb{N} \longrightarrow (1)$

$\because E \sim \mathbb{N} \Rightarrow \#(E) = \#(\mathbb{N})$

$\because O \sim \mathbb{N} \Rightarrow \#(O) = \#(\mathbb{N})$

$\Rightarrow \#(E) = \aleph_0, \#(O) = \aleph_0$

$\#(\mathbb{N}) = \aleph_0$

using (c) $\#(E) + \#(O) = \aleph_0$

$\boxed{\aleph_0 + \aleph_0 = \aleph_0}$

$c + c = c$

$\#(I) = [0,1] = c$

Take $[0,2]$

can write as

$\boxed{[0,2] = [0,1) \cup [1,2]} \longrightarrow (2)$

Also

$[0,1) \sim [0,1] \Rightarrow$

And $\#([0,1]) = \#[0,1]$

$[1,2] \sim [0,1)$

$\#[1,2] = \#[0,1)$

using (c)

$\Rightarrow \#[0,2] = \#([0,1)) + \#[1,2]$

$\boxed{n + \aleph_0 = \aleph_0}$

we are done.

Suppose on contrary that $\#A \boxed{=} \#P(A)$.

$\exists$ a mapping $f : A \to P(A)$ which is $\boxed{\text{bijective}}$.

Let $\boxed{a \in A}$ called $\boxed{\text{bad}}$ element if $a \notin f(a)$

Let B be set of Bad element of A

$B = \{x \mid x \in A \text{ and } x \notin f(x)\}$.

Now $B \subseteq A$ i.e $\boxed{B \in P(A)}$

since $f : A \to P(A)$ is onto

So, $\exists$ $\boxed{b \in A}$ such that $f(b) = B$ But be Def. of B which is impossible

So, $f : A \to P(A)$ is not onto.

Hence

$$\boxed{\#(A) \neq \#P(A)}$$

we have

$$\boxed{\#(A) < \#P(A)}$$

BAAT BAN GYI !

$f : A \to P(A)$

$\boxed{c = c + c}$

2021

PYQ

Q: Prove that $\aleph_0 \cdot c = c$

proof: $\#(\mathbb{N}) = \aleph_0$ and $\#(\mathbb{R}) = c$

Also $\mathbb{Z} \sim \mathbb{N}$ and $[0,1) \sim [0,1] \sim \mathbb{R}$

$\Rightarrow \#(\mathbb{Z}) = \aleph_0 \quad \Rightarrow [0,1) = c$

$f : \mathbb{Z} \times [0,1) \longrightarrow \mathbb{R}$ to prove $\aleph_0 \cdot c = c$.

$f(n, a) = n + a \quad \forall (n, a) \in \mathbb{Z} \times [0,1)$

One to One:- for any $x, y \in \mathbb{Z} \times [0,1)$

Let $f(x) = f(y)$

$\Rightarrow n + x = n + y$

$\Rightarrow x = y \Rightarrow f$ is one to one!

Onto: for any $x \in \mathbb{R}$ written as $x = [x] + a$

$\exists \; ([x], a) \in \mathbb{Z} \times [0,1)$ such that

$$\boxed{f([x] + a) = [x] + a = x}$$

$\Rightarrow f$ is bijective

$\Rightarrow \#(\mathbb{Z} \times [0,1)) = \#(\mathbb{R})$

$\Rightarrow \boxed{\aleph_0 \cdot c = c}$

CHARACTERISTIC Function $\longrightarrow$ $\boxed{X \neq \phi}$ set. $\boxed{A \subseteq X}$ then

$\chi_A : X \rightarrow \boxed{\{0, 1\}}$

defined by

$$\chi_A = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

$$\boxed{C(X) = \{ \chi_A : A \subseteq X \}}$$

PYQ 2022

Q: Prove that $\# C(A) = 2^{\alpha}$ where $\# A = \alpha$

Define a Mapping $f: 2^A \to C(A)$

by $f(E) = \chi_E \quad \forall E \in 2^A$

One to One: for any $E, F \in 2^A$

Let $f(E) = f(F)$

$\Rightarrow \chi_E = \chi_F$

$\Rightarrow E = F \Rightarrow$ one to one)

Onto: for any $\chi_E \in C(A)$

$\exists\ E \in 2^A$ such that

$f(E) = \chi_E$

$\boxed{\text{PRECEDE}} \rightarrow$ Two sets A and B. Then A is said to be precedes B iff $\exists$ a one to one Mapping $\boxed{f: A \to B}$

$\boxed{A \prec B}$

$B \nprec A$

$\# (A) < \# (B)$

$\alpha < \beta$

Continuum Hypothesis

Supposed by "CANTOR"

$\#(\mathbb{N}) = \aleph_0$
and
$\#(\mathbb{R}) = c$

As $\mathbb{N} \subset \mathbb{R}$

So, $\aleph_0 < c$ But

By Cantor's Thm:

$\aleph_0 < 2^{\aleph_0}$

$\therefore 2^{\aleph_0} = c$

Natural to Ask to that is there exist β such that $\aleph_0 < \beta < c$ - The Answer is No.

Partial Ordered Set

$\boxed{X \neq \phi}$ (set)

R, Relation defined on X then R is called partial ordering relation if satisfies

TOSET

↓

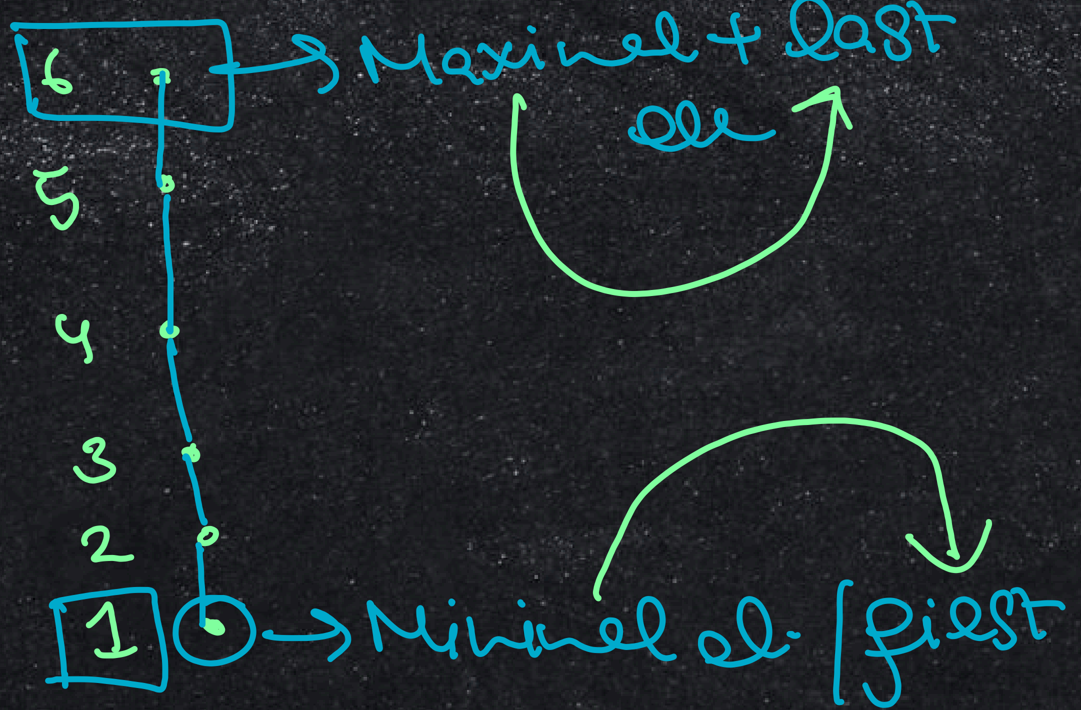
POSET + Compalability

$a, b \in S$

$a < b,\ b < a$

R is
- Reflexive ✓
- Anti-Symmetric ✓
- Transitive ✓

Then (X, R) called POSET OR ORDERED SET



Disjoint

$\{A_i : i \in I\}$ → family of set

$\cup A_i \Longrightarrow B$

$B \cap A_i = \{\text{sigeti}\}$