RING THEORY ON ONE PAGE

Ring Theory

1. Construction of New Rings

- **Direct Sum**: $R \oplus S = \{(r, s) \mid r \in R, s \in S\}$.
  - *Example*: $\mathbb{Z} \oplus \mathbb{Z}$: pairs of integers with component-wise addition/multiplication
- **Polynomial Ring**: $R[x]$, polynomials with coefficients in R.
  - *Example*: $\mathbb{Z}[x]$: polynomials like $2x^3 - x + 5$.
- **Matrix Ring**: $M_n(R)$, $n \times n$ matrices over R.
  - *Example*: $M_2(\mathbb{R})$: 2×2 real matrices.

2. Divisors, Units, and Associates

- **Unit**: Element with a multiplicative inverse.
  - *Example*: In $\mathbb{Z}$, units are ± 1. In $\mathbb{Z}[i]$ (Gaussian integers), units are $\pm 1, \pm i$.
- **Associates**: Elements differing by a unit multiplier.
  - *Example*: In $\mathbb{Z}$, 2 and -2 are associates.

3. Special Domains

- **Unique Factorization Domain (UFD)**:
  - Every non-unit factors uniquely into irreducibles.
  - *Example*: $\mathbb{Z}[x]$ (polynomials with integer coefficients).
  - *Non-example*: $\mathbb{Z}[\sqrt{-5}]$: $6 = 2 \cdot 3 = (1 + \sqrt{-5})(1 - \sqrt{-5})$.
- **Principal Ideal Domain (PID)**:
  - Every ideal is principal (generated by one element).
  - *Example*: $\mathbb{Z}$.
  - *Non-example*: $\mathbb{Z}[x]$ (ideal $(2, x)$ is not principal).
- **Euclidean Domain**:
  - Has a division algorithm.
  - *Example*: $\mathbb{Z}$, $F[x]$ (polynomials over a field F).

Hierarchy:

$$\text{Euclidean Domain} \subsetneq \text{PID} \subsetneq \text{UFD}$$

3. Algebraic Extensions

- **Definition**: Every element is algebraic.
  - *Example*: $\mathbb{Q}(\sqrt{2}, \sqrt{3})$ over $\mathbb{Q}$.

4. Polynomials

- **Reducible**: Factors into lower-degree polynomials.
  - *Example*: $x^2 - 4 = (x - 2)(x + 2)$ over $\mathbb{Q}$.
- **Irreducible**: Cannot be factored further.
  - *Example*: $x^2 + 1$ over $\mathbb{R}$.
- **Minimal Polynomial**: Smallest monic irreducible polynomial with root α.
  - *Example*: Minimal polynomial of $\sqrt[3]{2}$ over $\mathbb{Q}$ is $x^3 - 2$.

Field Extensions

1. Algebraic vs. Transcendental Elements

- **Algebraic**: Satisfies a polynomial equation over the base field.
  - *Example*: $\sqrt{2}$ over $\mathbb{Q}$ (root of $x^2 - 2$).
- **Transcendental**: No such polynomial exists.
  - *Example*: π over $\mathbb{Q}$.

2. Degree of Extension

- **Notation**: $[L : K] = \dim_K L$.
  - *Example*: $[\mathbb{Q}(\sqrt{2}) : \mathbb{Q}] = 2$.
- **Tower Law**: $[L : K] = [L : F][F : K]$.
  - *Example*: $[\mathbb{Q}(\sqrt{2}, \sqrt{3}) : \mathbb{Q}] = 4$.