GENERAL TENSOR

Einstein's Summation Convention:-

$$a_1 x^1, a_2 x^2, \cdots \quad a_n x^n$$

$$\boxed{\sum_{i=1}^{n} a_i x^i}$$

GENERAL TENSOR

Ex: $\frac{\partial \phi}{\partial x^1} dx^1 + \frac{\partial \phi}{\partial x^2} dx^2 + \frac{\partial \phi}{\partial x^3} dx^3 + \cdots \frac{\partial \phi}{\partial x^n} dx^n$

$$\sum_{i=1}^{n} \frac{\partial \phi}{\partial x^i} dx^i = \frac{\partial \phi}{\partial x^i} dx^i$$

GENERAL TENSOR

Ex: $(x^1)^2+(x^2)^2+(x^3)^2+\cdots(x^n)^2$ write in summation convention form.

$$\sum_{i=1}^{n}(x^i)^2 = \boxed{(x^i)^2}$$

$(x^i)^2$

Superscript
↓
general Tensor.

GENERAL TENSOR

Dummy Index :-

$\boxed{a_i x^i}$ → $i \to$ Dummy

indices

Free Index :-

$a_{ij} x^i$

indices

free index = i

Dumy = j

GENERAL TENSOR

DOUBLE SUM:-

$a_{ij}\, x^i x^j$

$= a_{1j} x^1 x^j + a_{2j} x^2 x^j + \cdots$

$+ a_{nj} x^n x^j$ (summed over ij)

$= a_{11} x^1 x^1 + a_{12} x^1 x^2 + \cdots a_{1n} x^1 x^n$

$+ a_{21} x^1 x^1 + a_{22} x^1 x^2 + \cdots a_{2n} x^1 x^n$

$+ \cdots \cdots$

$+ a_{n1} x^n x^1 + a_{n2} x^n x^2 + \cdots a_{nn} x^n x^n$

$a_i x^i \rightarrow$ index (1)

$\sum_{i=1}^{n} a_i x^i = n$ Term

$a_{ij} x^i x^j \Rightarrow$ indices (2)

$\sum \sum = n^2$ Terms

convention

GENERAL TENSOR

Ex:- If $\bar{x}^i = a_{ij} x^j$ express from $Q = g_{ij} \bar{x}^i \bar{x}^j$ in terms of x-variables

$\bar{x}^i = a_{ir} x^r$

$\bar{x}^j = a_{js} x^s$

$\bar{x}^i = a_{ij} x^j = a_{ir} x^r$

$\bar{x}^j = a_{js} x^s$

$$Q = g_{ij} a_{ir} x^r a_{js} x^s = \boxed{g_{ij} a_{ir} a_{js} x^r x^s}$$

GENERAL TENSOR

Kronecker delta $\delta^i_j \rightarrow$ substitution operation

$$\delta^i_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

$\delta^2_1 = 0$

$\delta^3_1 = 0$

$\delta^1_1 = 1$ $\delta^1_2 = 0$

$\delta^1_3 = 0$

$\delta^2_2 = 1$ $\delta^3_3 = 1$

GENERAL TENSOR

Theorem:- Show that $x^1, x^2 - - - x^n$ are n independent variable then $\frac{\partial x^i}{\partial x^j} = \delta^i_j$

$$\frac{\partial x^1}{\partial x^1} = 1$$

If $i = j$

$$\frac{\partial x^i}{\partial x^i} = 1$$

$$\delta^i_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

$$\frac{\partial x^2}{\partial x^1} = 0$$

If $i \neq j$

$$\frac{\partial x^i}{\partial x^j} = 0$$

$$\boxed{\frac{\partial x^i}{\partial x^j} = \delta^i_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}}$$

$$\frac{\partial x^3}{\partial x^1} = 0$$

GENERAL TENSOR

Theorem: $$\frac{\partial x^i}{\partial \bar{x}^k}\,\frac{\partial \bar{x}^k}{\partial x^j} = \delta^i_j$$

$x^i \to$ function of coordinate $\to x^k$

$\bar{x}^k$

$$\boxed{\frac{\partial x^i}{\cancel{\partial \bar{x}^k}}\,\frac{\cancel{\partial \bar{x}^k}}{\partial x^j} = \frac{\partial x^i}{\partial x^j} = \delta^i_j}$$

GENERAL TENSOR

Theorem :

$$\frac{\partial x^i}{\cancel{\partial \bar{x}^j}} \frac{\cancel{\partial \bar{x}^j}}{\partial x^k} = \delta^i_k$$

GENERAL TENSOR

Theorem: Prove that $\boxed{\delta^i_k \delta^k_j = \delta^i_j}$

$$\delta^i_k \delta^k_j = \delta^i_j \rightarrow \text{dummy remove}$$

$\downarrow$

free index

GENERAL TENSOR

Ex: Prove that $\delta^i_i = n$

LHS

$$\delta^i_i = \delta^1_1 + \delta^2_2 + \delta^3_3 + \cdots \delta^n_n$$

using

$$\delta^1_1 = 1$$

$$= 1 + 1 + 1 + \cdots 1 \text{ (n terms)}$$

$$\boxed{= n}$$

GENERAL TENSOR

$$\delta^i_j \, \delta^i_j = n \qquad \delta^i_j = 1 \text{ if } i=j$$

LHS:

$$\delta^1_j \delta^1_j + \delta^2_j \delta^2_j + \cdots \; \delta^n_j \delta^n_j$$

$$= \delta^1_1 \delta^1_1 + \cancelto{0}{\delta^1_2 \delta^1_2} + \cancelto{0}{\delta^1_3 \delta^1_3} + \cdots \; \cancelto{0}{\delta^1_n \delta^1_n}$$

$$+ \cancelto{0}{\delta^2_1 \delta^2_1} + \delta^2_2 \delta^2_2 + \cdots \; + \cancelto{0}{\delta^2_n \delta^2_n}$$

$$+ \cdots\cdots$$

$$+ \cancelto{0}{\delta^n_1 \delta^n_n} + \cancelto{0}{\delta^n_2 \delta^n_2} + \cdots \; \delta^n_n \delta^n_n$$

GENERAL TENSOR

$$\delta^1_1\delta^1_1 + \delta^2_2\delta^2_2 + \delta^3_3\delta^3_3 + \dots + \delta^n_n\delta^n_n$$

$$= 1 + 1 + 1 + \dots + 1 \quad (n \text{ terms})$$

$$\boxed{= n}$$

GENERAL TENSOR

Ex:- Prove that $\delta^i_j \, \delta^j_k \, \delta^k_i = n$

$$(\delta^i_j \, \delta^j_k) \, \delta^k_j = n$$

$$\delta^i_k \, \delta^k_j = n$$

$$\boxed{\delta^j_j = n}$$

$$\delta^1_1 = 1 \; --- \; 1 + (n \text{ times}) = n$$

GENERAL TENSOR

Ex: Prove that

$$\delta^i_k A_{ik}$$

$$= A^i_i$$

$$\boxed{\delta^i_j A_{jk} = A^i_k}$$

$$\delta^i_j (\delta^k_l A_{ik}) = A_{jl}$$

$$\delta^i_j (A_{il}) = A_{jl}$$

$$\boxed{A_{jl} = A_{jl}}$$

GENERAL TENSOR

TRANSFORMATION:

Transform

Cartesian Coordinate $\longrightarrow$ Polar coordi

(x, y) $\qquad$ (r, θ)

$x \to \bar{x}$

$$x^1 = \bar{x}^1 \cos(\bar{x}^2)$$

$$x^2 = \bar{x}^2 \sin(\bar{x}^2)$$

$$\boxed{\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}}$$

$\bar{x}^1$

$$r = \sqrt{x^2 + y^2}$$

$$\underset{\bar{x}^2}{\theta} = \tan^{-1}(y/x)$$

$$\boxed{(r, \theta) = (\bar{x}^1, \bar{x}^2)}$$

$$(x, y) = (x^1, x^2)$$

$$\bar{x}^1 = \sqrt{(x^1)^2 + (x^2)^2}$$

$$\bar{x}^2 = \tan^{-1}\frac{x^2}{x^1}$$

GENERAL TENSOR

Cartesian $\longrightarrow$ Cylindrical

$x = r\cos\theta$ $\quad$ $r = \sqrt{x^2 + y^2}$

$y = r\sin\theta$ and $\theta = \tan^{-1}(y/x)$

$z = z$ $\quad$ $z = z$

$(x, y, z) \to (x^1, x^2, x^3)$ $\quad$ $(r, \theta, z) = (\bar{x}^1, \bar{x}^2, \bar{x}^3)$

$x \to \bar{x}$

$x^1 = \bar{x}^1 \cos\bar{x}^2$ $\quad$ $\bar{x}^1 = \sqrt{(x^1)^2 + (x^2)^2}$

$x^2 = \bar{x}^1 \sin\bar{x}^2$ $\quad$ $\bar{x}^2 = \tan^{-1}(x^2/x^1)$

$x^3 = \bar{x}^3$ $\quad$ $\bar{x}^3 = x^3$

GENERAL TENSOR

Cartesian Sperical Coordinates

$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \end{cases}$$

$$\begin{cases} r = \sqrt{x^2+y^2+z^2} \\ \theta = \tan^{-1}\dfrac{\sqrt{x^2+y^2}}{z} \\ \phi = \tan^{-1}(y/z) \end{cases}$$

$$(x, y, z) = (x^1, x^2, x^3)$$

$$(r, \theta, \phi) = (\bar{x}^1, \bar{x}^2, \bar{x}^3)$$

$$x^1 = \bar{x}^1 \sin\bar{x}^2 \cos\bar{x}^3$$

$$x^2 = \bar{x}^1 \sin\bar{x}^2 \sin\bar{x}^3$$

$$x^3 = \bar{x}^1 \cos\bar{x}^2$$

GENERAL TENSOR

Tensor Notation for Matrices:

(a) Lower - index matrix notation - a_{11}, a_{22}, a_{12}

$$[a_{ij}]_{mn} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & & a_{mn} \end{bmatrix}$$

GENERAL TENSOR

(b) upper index

$a^{11}, a^{12}, a^{22}, \ldots,$

$= [a^{ij}]_{mn}$

(c) Mixed Inde

$a^1_1, a^1_2 + \ldots a^1_n$

$= [a_{ij}]_{mn}$

GENERAL TENSOR

Scalar → quantity → number → constant

Vector → quantity →

Tensor → Zero order → Scalar

$\underbrace{\text{ORDER / RANK}}$ → Tensor → Same Baat

↳ suffices → used.

GENERAL TENSOR

ZEROTH-ORDER Tensor:-

↳ Scalar / Invariant

$$(x^1, x^2, \dots x^n) \xrightarrow{\phi} (\bar{x}^1, \bar{x}^2, \dots \bar{x}^n)$$

$$\boxed{\phi = \bar{\phi}}$$ invariant

GENERAL TENSOR

first order Tensor:- → 1 suffix.

A^i → quantity → Tensor

A^i → first order

contravariant ← A^i → $(x^1, x^2 - - - x^n)$ → x-coordinate system

↓

$\bar{x}$ - coordinate

$$\bar{A}^p = \frac{\partial \bar{x}^p}{\partial x^i} A^i$$

Transformation Bridges

GENERAL TENSOR

$A_j \rightarrow$ covariant $\rightarrow$ x - coordinates

$\downarrow$

$\bar{x}$-coordinate

$\bar{A}_q$

$$\boxed{\bar{A}_q = \frac{\partial x^j}{\partial \bar{x}^q} A_j}$$

GENERAL TENSOR

Second Order Tensor:-

(a) Contravariant $\boxed{A^{ij}}$ → x-coordinate

↓

$\bar{A}^{pq}$ $\qquad$ $\bar{x}$ coordinate

$$\bar{A}^{pq} = \frac{\partial \bar{x}^p}{\partial x^i} \frac{\partial \bar{x}^q}{\partial x^j} A^{ij}$$

GENERAL TENSOR

(b) Covariant

$A_{si} \rightarrow x\text{-coordinate}$

$\bar{A}_{pj} \rightarrow \bar{x}$

$$\bar{A}_{pj} = \frac{\partial x^{s}}{\partial \bar{x}^{p}} \frac{\partial x^{i}}{\partial \bar{x}^{j}} A_{si}$$

GENERAL TENSOR

(C) Mixed Tensor of order Two:

$$A^i_j \Rightarrow \bar{A}^p_q$$

$$x \nearrow \qquad \nwarrow \bar{x}$$

$$\bar{A}^p_q = \frac{\partial \bar{x}^p}{\partial x^i} \frac{\partial x^j}{\partial \bar{x}^q} A^i_j$$

i contravariant

j covariant

GENERAL TENSOR

Ex: If A^i and B^j are two contravariant Tensors then $\boxed{A^iB^j}$ is a contravariant of order 2.

$$\bar{A}^p = \frac{\partial \bar{x}^p}{\partial x^i} A^i$$

$$\bar{B}^q = \frac{\partial \bar{x}^q}{\partial x^j} B^j$$

$$\bar{A}^p \bar{B}^q = \frac{\partial \bar{x}^p}{\partial x^i} \frac{\partial \bar{x}^q}{\partial x^j} A^i B^j$$

$$\boxed{\bar{C}^{pq} = \frac{\partial \bar{x}^p}{\partial x^i} \frac{\partial \bar{x}^q}{\partial x^i} C^{ij}}$$

GENERAL TENSOR

Ex: If A^i and B_j, A^iB_j Tensor of order 2.

$$\bar{A}^p = \frac{\partial \bar{x}^p}{\partial x^i} A^i$$

Contravariant

$$\bar{B}_q = \frac{\partial x^j}{\partial \bar{x}^q} B_j$$

covariant

$$\bar{A}^p \bar{B}_q = \frac{\partial \bar{x}^p}{\partial x^i} \frac{\partial x^j}{\partial \bar{x}^q} A^i B_j$$

$$\boxed{\bar{C}^p_q = \frac{\partial \bar{x}^p}{\partial x^i} \frac{\partial x^j}{\partial \bar{x}^q} C^i_j}$$

GENERAL TENSOR

Theorem: Prove that δ^i_j mixed Tensor of order 2.

PPP

$\delta^i_j \downarrow$ x-coordinate

$\bar{\delta}^p_q$ $\bar{x}$-coordinate

$$\boxed{\bar{\delta}^p_q} = \frac{\partial \bar{x}^p}{\partial x^i}\frac{\partial x^j}{\partial \bar{x}^q}\delta^i_j$$

$$= \frac{\partial \bar{x}^p}{\partial x^i}\left(\frac{\partial x^j}{\partial \bar{x}^q}\delta^i_j\right) = \frac{\partial \bar{x}^p}{\cancel{\partial x^i}}\frac{\partial \cancel{x^i}}{\partial \bar{x}^q} = \frac{\partial \bar{x}^p}{\partial \bar{x}^q} = \boxed{\delta^p_q}$$

$\bar{\delta}^p_q = \delta^p_q = 1$ $\boxed{p=q}$ and 0 if $p \neq q$

GENERAL TENSOR

Transformation law :-

$$A^i_{jk} \rightarrow x$$

$$\bar{A}^p_{qr} \rightarrow \bar{x}$$

$$\boxed{\bar{A}^p_{qr} = \frac{\partial \bar{x}^p}{\partial x^i}\frac{\partial x^j}{\partial \bar{x}^q}\frac{\partial x^k}{\partial \bar{x}^r} A^i_{jk}}$$

$\frac{\partial x^j}{\partial x^i}$ ← Transformation Bridge.

GENERAL TENSOR

Inverse Transformation law:

Transformation

If $$\boxed{\bar{A}^i = \frac{\partial \bar{x}^i}{\partial x^j} A^j}$$ then prove that

Inverse

$$\boxed{A^j = \frac{\partial x^j}{\partial \bar{x}^i} \bar{A}^i}$$

$$\frac{\partial x^k}{\partial \bar{x}^i} \bar{A}^i = \frac{\partial x^k}{\partial \bar{x}^i} \frac{\partial \bar{x}^i}{\partial x^j} A^j$$

$$= \frac{\partial x^k}{\cancel{\partial \bar{x}^i}} \frac{\cancel{\partial \bar{x}^i}}{\partial x^j} A^j = \frac{\partial x^k}{\partial x^j} A^j = \delta^k_j A^j = \boxed{A^k}$$

GENERAL TENSOR

$$\frac{\partial x^{k}}{\partial \bar{x}^{i}}\bar{A}^{i} = A^{k}$$

K Replace by j

$$\boxed{A^{j} = \frac{\partial x^{j}}{\partial \bar{x}^{i}}\bar{A}^{i}}$$

GENERAL TENSOR

Ex: $\bar{A}^{pq} = \frac{\partial \bar{x}^p}{\partial x^i} \frac{\partial \bar{x}^q}{\partial x^j} A^{ij} \Rightarrow \boxed{A^{ij} = \frac{\partial x^i}{\partial \bar{x}^p} \frac{\partial x^j}{\partial \bar{x}^q} \bar{A}^{pq}}$

$$\frac{\partial x^k}{\partial \bar{x}^p} \frac{\partial x^m}{\partial \bar{x}^q} \bar{A}^{pq} = \frac{\partial x^k}{\cancel{\partial \bar{x}^p}} \frac{\partial x^m}{\cancel{\partial \bar{x}^q}} \frac{\cancel{\partial \bar{x}^p}}{\partial x^i} \frac{\cancel{\partial \bar{x}^q}}{\partial x^j} A^{ij} = A^{km}$$

$$= \frac{\partial x^k}{\partial x^i} \frac{\partial x^m}{\partial x^j} A^{ij}$$

$$= \delta^k_i \left(\delta^m_j A^{ij}\right) = \delta^k_i A^{im} = A^{km}$$

GENERAL TENSOR

$$\frac{\partial x^i}{\partial \bar{x}^p} \frac{\partial x^j}{\partial \bar{x}^q} \bar{A}^{pq} = A^{ij}$$

$$\boxed{A^{ij} = \frac{\partial x^i}{\partial \bar{x}^p} \frac{\partial x^j}{\partial \bar{x}^q} \bar{A}^{pq}}$$

GENERAL TENSOR

RANK OF TENSOR:-

→ Number of indices attached to it!

GENERAL TENSOR

Ex: How many components does a Tensor have of Rank 5 in a space of 4 Dimension?

$$\text{No. of components} = (\text{Dimensions})^{\text{Rank}}$$

$$= (4)^5 = \boxed{1024}$$

GENERAL TENSOR

Algebra operations on Tensors:-

$$A^i_j \rightarrow C^i_j$$

$$\bar{C}^p_q = \frac{\partial \bar{x}^p}{\partial x^i} \frac{\partial x^j}{\partial \bar{x}^q} C^i_j$$

$$\bar{A}^p_q = \frac{\partial \bar{x}^p}{\partial x^i} \frac{\partial x^j}{\partial \bar{x}^q} A^i_j$$

$$\bar{A}^p_q + \bar{C}^p_q = \frac{\partial \bar{x}^p}{\partial x^i} \frac{\partial x^j}{\partial \bar{x}^q}$$

$$A^i_j + C^i_j$$

GENERAL TENSOR

Inner product of Tensors:

- Outer product
- Contraction

Ex: Prove contraction of Tensor

A^i_j is a scalar / Invariant

mixed Tensor $\leftarrow$ $\bar{A}^p_q = \frac{\partial \bar{x}^p}{\partial x^i} \frac{\partial x^j}{\partial \bar{x}^q} A^i_j$

GENERAL TENSOR

$$\boxed{p = q}$$

$$\bar{A}^p_p = \frac{\cancel{\partial \bar{x}^p}}{\partial x^i} \frac{\partial x^j}{\cancel{\partial \bar{x}^p}} A^i_j$$

$$= \frac{\partial x^j}{\partial x^i} A^i_j$$

$$= \delta^j_i A^i_j$$

$$= A^i_i = \sum_{i=1}^{n} A^i_i = n \rightarrow \text{Scalar invariant}$$

GENERAL TENSOR

Ex: Show that any inner product of Tensors A^i_j and B^{kl}_m is a tensor of Rank 3:

$$\bar{A}^p_q = \frac{\partial \bar{x}^p}{\partial x^i}\frac{\partial x^j}{\partial \bar{x}^q} A^i_j$$

$$\bar{B}^{rs}_t = \frac{\partial \bar{x}^r}{\partial x^k}\frac{\partial \bar{x}^s}{\partial x^l}\frac{\partial x^m}{\partial \bar{x}^t} B^{kl}_m$$

outer product:

$$\bar{A}^p_q \bar{B}^{rs}_t = \frac{\partial \bar{x}^p}{\partial x^i}\frac{\partial x^j}{\partial \bar{x}^q}\frac{\partial \bar{x}^r}{\partial x^k}\frac{\partial \bar{x}^s}{\partial x^l}\frac{\partial x^m}{\partial \bar{x}^t} A^i_j B^{kl}_m$$

For contraction $(p=q)$

$$\bar{A}^p_q \bar{B}^{rs}_p = \frac{\cancel{\partial \bar{x}^p}}{\partial x^i}\frac{\partial x^j}{\partial \bar{x}^q}\frac{\partial \bar{x}^r}{\partial x^k}\frac{\partial \bar{x}^s}{\partial x^l}\frac{\partial x^m}{\cancel{\partial \bar{x}^p}} A^i_j B^{kl}_m$$

$$\frac{\partial x^m}{\partial x^i} = \delta^m_i$$

GENERAL TENSOR

$$\bar{A}^{p}_{q}\,\bar{B}^{rs}_{p} = \frac{\partial x^{j}}{\partial \bar{x}^{q}}\,\frac{\partial \bar{x}^{r}}{\partial x^{k}}\,\frac{\partial \bar{x}^{s}}{\partial x^{l}}\,\delta^{m}_{i}\,A^{i}_{j}\,B^{kl}_{m}$$

$$= \frac{\partial x^{i}}{\partial \bar{x}^{q}}\,\frac{\partial \bar{x}^{r}}{\partial x^{k}}\,\frac{\partial \bar{x}^{s}}{\partial x^{l}}\,\delta^{m}_{i}\,A^{i}_{j}\,B^{kl}_{m}$$

$$\boxed{= \frac{\partial x^{i}}{\partial \bar{x}^{q}}\,\frac{\partial \bar{x}^{r}}{\partial x^{k}}\,\frac{\partial \bar{x}^{s}}{\partial x^{l}}\,A^{m}_{j}\,B^{kl}_{m}}$$

Rank / order 3.

GENERAL TENSOR

QUOTIENT Theorem:-

inner product

(X) ? $(T) = $ Tensor

any quantity

$\Rightarrow X$ is tensor.

GENERAL TENSOR

Ex: $A_{ij} \rightarrow$ quantity in x-coordinate,

$A_{ij} B^i = C_j$, $B^i \rightarrow$ contravariant tensor

$C_j \Rightarrow$ Convariant. Prove that A_{ij} is covariant Tensor of order 2.

$A_{ij} \rightarrow \bar{A}_{pq}$ $\bar{B}^p = B^i$, $\bar{C}_q = \frac{\partial x^j}{\partial \bar{x}^q} C_i$

$\bar{A}_{pq} \bar{B}^p = \bar{C}_q$

$\bar{A}_{pq} \bar{B}^p = \frac{\partial x^j}{\partial \bar{x}^q} C_i$

GENERAL TENSOR

$$\bar{A}_{pq}\bar{B}^p = \frac{\partial x^j}{\partial \bar{x}^q} C_j$$

using $\bar{B}^p = \dfrac{\partial \bar{x}^p}{\partial x^i} B^i$

$$= \frac{\partial x^j}{\partial \bar{x}^q} A_{ij} B^i$$

$$\frac{\partial x^i}{\partial \bar{x}^p}\bar{B}^p = \frac{\partial x^i}{\cancel{\partial \bar{x}^p}}\frac{\cancel{\partial \bar{x}^p}}{\partial x^i} B^i$$

$$= \frac{\partial x^j}{\partial \bar{x}^q} A_{ij}\left(\frac{\partial x^i}{\partial \bar{x}^p}\bar{B}^p\right)$$

$$= \delta^i_i B^i$$

$$\boxed{\frac{\partial x^i}{\partial \bar{x}^p}\bar{B}^p = B^i}$$

$$\bar{A}'_{pq}\bar{B}^p = \frac{\partial x^j}{\partial \bar{x}^q}\frac{\partial x^i}{\partial \bar{x}^p} A^{ij}\bar{B}^p$$

GENERAL TENSOR

$$\bar{A}_{pq}\bar{B}^{p} - \frac{\partial x^{j}}{\partial \bar{x}^{q}}\frac{\partial x^{i}}{\partial \bar{x}^{p}}A_{ij}\bar{B}^{p} = 0$$

$$\Rightarrow \bar{B}^{p}\left(\bar{A}_{pq} - \frac{\partial x^{i}}{\partial x^{p}}\frac{\partial x^{j}}{\partial \bar{x}^{q}}A_{ij}\right) = 0$$

$$\Rightarrow \bar{B}^{p} \neq 0$$

Covariant.

$$\boxed{\bar{A}_{pq} = \frac{\partial x^{i}}{\partial x^{p}}\frac{\partial x^{j}}{\partial \bar{x}^{q}}A_{ij}}$$

order 2

GENERAL TENSOR

Symmetric And Skew-Symmetric Tensors:-

Theorem: Every Tensor can be expressed as sum of two tensors, one is symmetric and other is skew-symmetric.

$$A^{ij} = \frac{1}{2}\left(A^{ij} + A^{ji}\right) + \frac{1}{2}\left(A^{ij} - A^{ji}\right)$$

$$= B^{ij} + C^{ij}$$

Skew-Symmetric Tensor

Symmetric

GENERAL TENSOR

$$B^{ij} = \frac{1}{2}\left(A^{ij} + A^{ji}\right) = \frac{1}{2}\left(A^{ji} + A^{ij}\right) = B^{ji}$$

$\Rightarrow B^{ij}$ is symmetric

$$C^{ij} = \frac{1}{2}\left(A^{ij} - A^{ji}\right) = \frac{1}{2}\left(A^{ji} - A^{ij}\right)$$

$$= -C^{ji}$$

$\Rightarrow C^{ij}$ is skew symetric

GENERAL TENSOR

Line element And Metric Tensor:-

$(x, y, z) \rightarrow$ line element $\rightarrow (ds^2)$

$$ds^2 = dx^2 + dy^2 + dz^2$$

$$dr = \frac{\partial r}{\partial u^1} du^1 + \frac{\partial r}{\partial u^2} du^2 + \frac{\partial r}{\partial u^3} du^3$$

$$ds^2 = g_{ij}\, dx^i dx^j \quad (i, j = 1, 2 \cdots n)$$

Reimannian metric $\leftarrow$ $\boxed{ds^2 = g_{ij}\, dx^i dx^j}$

$g_{ij} \rightarrow$ metric Tensor

GENERAL TENSOR

Metric Tensor $= g_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$

2 indice → Covariant Tensor of Rank 2

order 2

Ex: If $ds^2 = g_{ij}\, dx^i dx^j$ is an invariant, show that g_{ij} is a symmetric covariant tensor of order 2.

$\boxed{ds^2}$ → invariant.

GENERAL TENSOR

$$ds^2 = g_{ij}\, dx^i dx^j$$

$$\bar{g}_{pq}\, d\bar{x}^p d\bar{x}^p = g_{ij}\, dx^i dx^j$$

$$= \frac{\partial x^i}{\partial \bar{x}^p} \frac{\partial x^j}{\partial \bar{x}^q} g_{ij}\, dx^i dx^j$$

Covariant tensor of order 2

$$= \frac{\partial x^i}{\partial \bar{x}^p} d\bar{x}^p \frac{\partial x^j}{\partial \bar{x}^q} d\bar{x}^q\, g_{ij}$$

$$= g_{ij} \frac{\partial x^i}{\partial \bar{x}^p} \frac{\partial x^j}{\partial \bar{x}^q}\, d\bar{x}^p d\bar{x}^p$$

GENERAL TENSOR

Metric Tensor in Cylindrical Coordinates

$$(r, \theta, z) \rightarrow (x^1, x^2, x^3)$$

$$g_{11} = 1 \qquad g_{22} = r^2 \qquad g_{33} = 1$$

$$g_{12} = g_{21} = g_{23} = g_{32} = g_{13} = g_{31} = 0$$

matrix form $= \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

GENERAL TENSOR

Metric Tensor in Spherical Coordinate:

$$(r, \theta, \phi) \longrightarrow (x^1, x^2, x^3)$$

$$\boxed{g_{11} = 1} \qquad g_{22} = r^2 \qquad g_{33} = r^2 \sin^2\theta$$

$$g_{12} = g_{21} = g_{13} = g_{31} = g_{23} = g_{31} = 0$$

metric form $= \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2\theta \end{pmatrix}$

GENERAL TENSOR

Christofel Symbols: → 1^{st} kind

$$[ij,k] = \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial x^{j}} + \frac{\partial g_{jk}}{\partial x^{i}} - \frac{\partial g_{ij}}{\partial x^{k}}\right)$$

(a) $i=j=k$

Spherical Coordinate:

$$[ii,i] = \frac{1}{2}\frac{\partial g_{ii}}{\partial x^{i}}$$

$$[11,1] = \frac{1}{2}\frac{\partial}{\partial x^{1}} g_{11} = \frac{1}{2}\frac{\partial}{\partial r}(1) = 0$$

$$[22,2) = \frac{1}{2}\frac{\partial}{\partial x^{2}} g_{22} = \frac{1}{2}\frac{\partial}{\partial \theta} r^{2} = 0$$

GENERAL TENSOR

(b) $i = j \neq k$

$$[ii,k] = -\frac{1}{2}\frac{\partial}{\partial x^k} g_{ii}$$

$$[11,2] = -\frac{1}{2}\frac{\partial}{\partial x^2} g_{11} = -\frac{1}{2}\frac{\partial}{\partial \theta}(1) = 0$$

(c) $i = k \neq j$

$$[ij,i] = \frac{1}{2}\frac{\partial}{\partial x^j} g_{ii}$$

(d) $i \neq j \neq k$

$$\boxed{[ij,k] = 0}$$

GENERAL TENSOR

Christofell of 2nd kind:

$$\left\{ {m \atop ij} \right\} = g^{mk} [i j , k] \quad \leftarrow 1^{st} \text{ kind}$$

(a) $i = j = m$

$$\left\{ {i \atop ii} \right\} = \frac{1}{2} \frac{\partial}{\partial x^i} \log g_{ii}$$

$$\left\{ {1 \atop 2\,2} \right\} = -\frac{1}{2 g_{11}} \frac{\partial g_{22}}{\partial x^1}$$

$$= -\frac{1}{2} \frac{1}{1} \frac{\partial (\cancel{x^2})}{\partial \cancel{x}}$$

$$= -\frac{1}{\cancel{2}} \cancel{2} x = -x$$

(b) $i = j \neq m$

$$\left\{ {m \atop ii} \right\} = -\frac{1}{2 g_{ii}} \frac{\partial}{\partial x^i} g_{jj}$$

GENERAL TENSOR

(c) $i = m \neq j$

$$\left\{ {i \atop i\ j} \right\} = \frac{1}{2} \frac{\partial}{\partial x^j} \log g_{ii}$$

(d) $i \neq m \neq j$

$$\left\{ {i \atop i\ j} \right\} = 0$$