1. Number Systems and Errors

Round-Off Errors and Computer Arithmetic

- **Definition:** Round-off error occurs when a number is approximated due to limitations in computer representation.
- **Explanation:** Computers store numbers in finite precision, leading to small discrepancies.
- **Example:** Storing π as **3.1416** instead of its infinite decimal expansion.

Error Estimation

- **Definition:** The process of determining the error in numerical computations.
- **Explanation:** Errors can be absolute ($|x_{true} - x_{approx}|$) or relative ($\frac{x_{true} - x_{approx}}{|x_{true}|}$).
- **Example:** If true value = 2.345 and approximation = 2.34, then absolute error = **0.005**.

Floating-Point Arithmetic

- **Definition:** Representation of real numbers in a finite number of bits using scientific notation.
- **Explanation:** A number is stored as $\pm m \times 10^e$, where m is the mantissa and e is the exponent.
- **Example:** 0.00123 in floating-point format could be stored as 1.23×10^{-3}.

2. Solution of Non-Linear Equations

Iterative Methods and Convergence

- **Definition:** Iterative methods approximate solutions using successive refinements.
- **Explanation:** Convergence occurs when iterations produce values closer to the true solution.

1. Bisection Method

- **Explanation:** A root-finding method that repeatedly bisects an interval.
- **Example:** Find root of $f(x) = x^3 - x - 2$ in $[1, 2]$ by halving the interval.

2. Fixed-Point Iteration Method

- **Explanation:** Transforms an equation into $x = g(x)$ and iterates using $x_{n+1} = g(x_n)$.
- **Example:** Solve $x^3 - x - 1 = 0$ using $x = \sqrt[3]{x+1}$.

2. Stirling's Formula

- **Explanation:** A symmetric interpolation formula for equal intervals.

3. Laplace Everett's Formula

- **Explanation:** Improves accuracy by averaging differences.

4. Bessel's Formula

- **Explanation:** Used when interpolation points are equidistant.

3. Regula Falsi (False Position) Method

- **Explanation:** Uses a weighted average of function values to find the root.
- **Example:** Find the root of $f(x) = x^2 - 3$ in $[1, 2]$ using the formula:

$$x_{new} = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

4. Secant Method

- **Explanation:** Similar to Regula Falsi but does not require sign change.
- **Example:** Solve $f(x) = e^x - 3x$ using two initial approximations.

5. Newton's Method

- **Explanation:** Uses the formula $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$.
- **Example:** Solve $f(x) = x^2 - 2$ using $x_0 = 1.5$.

3. Systems of Linear Equations

Direct Methods

- **Definition:** Solve equations in a finite number of steps.

1. Gaussian Elimination Method

- **Explanation:** Converts the system into an upper triangular matrix and solves by back-substitution.
- **Example:** Solve

$$\begin{cases} 2x + 3y = 8 \\ 4x - y = 3 \end{cases}$$

2. Gauss-Jordan Method

- **Explanation:** Converts a system to reduced row-echelon form.

3. Matrix Inversion Method

- **Explanation:** If $AX = B$, solve using $X = A^{-1}B$.

4. Factorization (Doolittle, Crout, and Cholesky) Method

- **Explanation:** Decomposes a matrix into lower and upper triangular matrices.

Iterative Methods and Convergence

- **Definition:** Approximate solutions iteratively.

1. Gauss-Jacobi Method

- **Explanation:** Solves each equation for one variable at a time.

2. Gauss-Seidel Method

- **Explanation:** Uses updated values within each iteration for faster convergence.

Ill-Conditioned System and Condition Number

- **Definition:** A system where small changes in input cause large changes in the output.

Eigenvalues and Eigenvectors

- **Definition:** If $Ax = \lambda x$, then λ is an eigenvalue and x is an eigenvector.

Power and Rayleigh Quotient Method

- **Explanation:** Power method finds the dominant eigenvalue, while Rayleigh quotient refines approximations.

4. Interpolation and Polynomial Approximation

Difference Operators

- **Definition:** Operators like forward (Δ), backward (∇), and central differences help in numerical differentiation.

Interpolation with Unequal Intervals

- **Definition:** Estimating values between data points when intervals are not uniform.

1. Lagrange's Interpolation Formula

- **Explanation:** Uses Lagrange polynomials to estimate unknown values.
- **Example:** Given $(1, 2), (3, 10), (4, 20)$, estimate $f(2)$.

2. Newton's Divided Difference Formula

- **Explanation:** Uses divided differences to construct interpolation polynomials.

Interpolation with Equal Intervals

- **Definition:** Estimating values when intervals are uniform.

1. Gregory Newton Forward/Backward Interpolation

- **Explanation:** Uses forward or backward differences to estimate values.

2. Error in Polynomial Interpolation

- **Explanation:** Given by

$$f(x) - P_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} \prod_{i=0}^{n} (x - x_i)$$

Central Difference Interpolation Formulae

- **Definition:** Methods for interpolation using central differences.

1. Gauss's Forward/Backward Interpolation

- **Explanation:** Uses symmetric differences around a central value.