TOPOLOGY

CHAPTER FIVE

A^d | A'

limit pt may or may not belongs to set.

ACCUMALUTION POINT

limit point / cluster / derived set.

$p \in G$ (open)

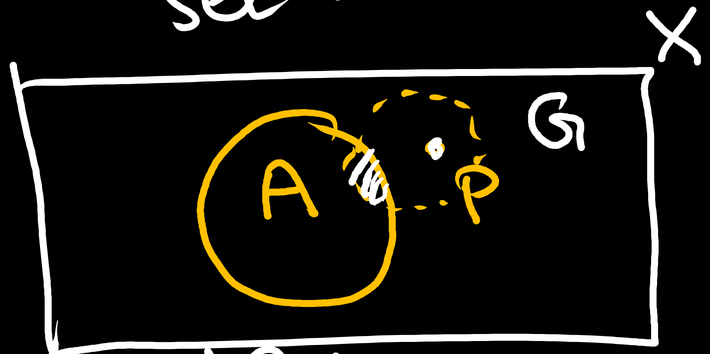
$(G - \{p\}) \cap A \neq \phi$

(X, τ)

$A \subseteq X$

then $p \in X$ limit pt. of A iff for every open set G containing p s.t

The set of limit pts of a set A called derived set.



TOPOLOGY

CHAPTER FIVE

$c \in \{b,c,d,e\} - \{c\} \cap \{a,b,c\} \neq \phi$

$c \in \{c,d\} - \{c\} \cap \{a,b,c\} = \phi$

c is not limit pt.

$\boxed{A^d = \{b,d,e\}}$

Example 2.1: The class $\mathcal{T} = \{X, \emptyset, \{a\}, \{c,d\}, \{a,c,d\}, \{b,c,d\,e\}\}$ → open sets defines a topology on $X = \{a,b,c,d,e\}$. Consider the subset $\boxed{A = \{a,b,c\}}$ of X.

$a \in \{a\} - \{a\} \cap \{a,b,c\} = \phi$

$a \in \{a,c,d\} - \{a\} \cap \{a,b,c\} \neq \phi$

$a \in X - \{a\} \cap \{a,b,c\} \neq \phi$

a is not limit point of A.

$b \in \{b,c,d,e\} - \{b\} \cap \{a,b,c\} \neq \phi$

$b \in X - \{b\} \cap \{a,b,c\} \neq \phi$ b is a limit pt. of A.

$\neq \phi$

$d \in \{a,c,d\} - \{d\}$

$d \in \{b,c,d,e\} - \{d\}$

$\cap A \neq \phi$

d is a limit pt. of A.

$e \in \{b,c,d,e\} - \{e\}$

$\cap A \neq \phi$

TOPOLOGY

CHAPTER FIVE

Example 2.1: The class $\mathcal{T} = \{X, \emptyset, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d\ e\}\}$ defines a topology on $X = \{a, b, c, d, e\}$. Consider the subset $A = \{a, b, c\}$ of X. Observe that $b \in X$ is a limit point of A since the open sets containing b are $\{b, c, d, e\}$ and X, and each contains a point of A different from b, i.e. c. On the other hand, the point $a \in X$ is not a limit point of A since the open set $\{a\}$, which contains a, does not contain a point of A different from a. Similarly, the points d and e are limit points of A and the point c is not a limit point of A. So $A' = \{b, d, e\}$ is the derived set of A.

TOPOLOGY

CHAPTER FIVE

(X, τ) indiscrete

$\tau = \{\phi, X\}$ $p \in X$

Example 2.2: Let X be an indiscrete topological space, i.e. X and $\emptyset$ are the only open subsets of X. Then X is the only open set containing any point $p \in X$. Hence p is an accumulation point of every subset of X except the empty set $\emptyset$ and the set consisting of p alone, i.e. the singleton set $\{p\}$. Accordingly, the derived set A' of any subset A of X is as follows:

$$A' = \begin{cases} \emptyset & \text{if } A = \emptyset \\ \{p\}^c = X \setminus \{p\} & \text{if } A = \{p\} \\ X & \text{if } A \text{ contains two or more points} \end{cases}$$

$\Rightarrow$ p is limit pt. of every subset of X. except, ϕ, $\{p\}$

$$A^d = \begin{cases} \phi & \text{if } A = \phi \\ \{p\}^c & \text{if } A = \{p\} \\ X & \text{if } A \text{ contain 2 or more points} \end{cases}$$

TOPOLOGY

CHAPTER FIVE

$\rightarrow G \cap A = \phi$ equivalently

$\rightarrow G \cap A = \{p\}$

12. Let A be a subset of a topological space $(X, \mathcal{T})$. When will a point $p \in X$ not be a limit point of A?

(X, τ)

$A \subseteq X$

$p \in X$

is limit pt of A if for every open set G

$p \in G$, G is open

$\Rightarrow G - \{p\} \cap A \neq \phi$

So, p is not limit of A if there exist $\boxed{\text{an open set}}$ s.t.

$p \in G$

but

$\boxed{G - \{p\} \cap A = \phi}$

TOPOLOGY

CHAPTER FIVE

12. Let A be a subset of a topological space $(X, \mathcal{T})$. When will a point $p \in X$ not be a limit point of A?

Solution:

The point $p \in X$ is a limit point of A iff every open neighborhood of p contains a point of A other than p, i.e.,

$$p \in G \text{ and } G \in \mathcal{T} \quad \text{implies} \quad (G \setminus \{p\}) \cap A \neq \emptyset$$

So p is not a limit point of A if there exists an open set G such that

$$p \in G \quad \text{and} \quad (G \setminus \{p\}) \cap A = \emptyset$$

or, equivalently, $$p \in G \quad \text{and} \quad G \cap A = \emptyset \text{ or } G \cap A = \{p\}$$

or, equivalently, $$p \in G \quad \text{and} \quad G \cap A \subset \{p\}$$

or, equivalently, $p \in G$ and $G \cap A \subset \{p\}$

TOPOLOGY

CHAPTER FIVE

$(X, \mathcal{T})$, $\mathcal{T} = P(X)$

13. Let A be any subset of a discrete topological space X. Show that the derived set A' of A is empty.

every subset of a discrete space is open.

Let $p \in X$

then $G = \{p\}$

$p \in G$

$\boxed{G - \{p\} \cap A = \phi}$ $p \in X$, $p \notin A'$

$A' = \phi$

13. Let A be any subset of a discrete topological space X. Show that the derived set A' of A is empty.

Solution:

Let p be any point in X. Recall that every subset of a discrete space is open. Hence, in particular, the singleton set $G = \{p\}$ is an open subset of X. But

$$p \in G \quad \text{and} \quad G \cap A = (\{p\} \cap A) \subset \{p\}$$

Hence, by the above problem, $p \notin A'$ for every $p \in X$, i.e. $A' = \emptyset$.

Hence, by the above problem, $p \notin A'$ for every $p \in X$, i.e. $A' = \emptyset$.

TOPOLOGY

CHAPTER FIVE

14. Consider the topology

$$\mathcal{T} = \{X, \emptyset, \{a\}, \{a,b\}, \{a,c,d\}, \{a,b,c,d\}, \{a,b,e\}\}$$

on $X = \{a,b,c,d,e\}$. Determine the derived sets of (i) $A = \{c,d,e\}$ and (ii) $B = \{b\}$.

$a \in \{a\} - \{a\} \cap \{c,d,e\} = \phi$

$a \in \{a,b\} - \{a\} \cap \{c,d,e\} = \phi$

$a \in \{a,c,d\} - \{a\} \cap \{c,d,e\} = \phi$

$a \in \{a,b,c,d\} - \{a\} \cap \{c,d,e\} \neq \phi$

$a \in \{a,b,e\} - \{a\} \cap \{c,d,e\} \neq \phi$

$a \in X - \{a\} \cap \{c,d,e\} \neq \phi$

a is not limit pt. of A.

$e \in \{a,b,e\} - \{e\} \cap A$

$e \in X - \{e\} \cap A \neq \phi \quad = \phi$

e is also not limit point.

TOPOLOGY

CHAPTER FIVE

$a \in \{a,b\} - \{a\} \cap B \neq \emptyset$
$a \in \{a\} - \{a\} \cap \{b\} = \emptyset$

a is not limit pt. of B.

14. Consider the topology

$$\mathcal{T} = \{X, \emptyset, \{a\}, \{a,b\}, \{a,c,d\}, \{a,b,c,d\}, \{a,b,e\}\}$$

on $X = \{a,b,c,d,e\}$. Determine the derived sets of (i) $A = \{c,d,e\}$ and (ii) $B = \{b\}$.

Solution:

(i) Note that $\{a,b\}$ and $\{a,b,e\}$ are open subsets of X and that

$$a, b \in \{a,b\} \quad \text{and} \quad \{a,b\} \cap A = \emptyset$$

$$e \in \{a,b,e\} \quad \text{and} \quad \{a,b,e\} \cap A = \{e\}$$

Hence a, b and e are not limit points of A. On the other hand, every other point in X is a limit point of A since every open set containing it also contains a point of A different from it. Accordingly, $A' = \{c,d\}$.

(ii) Note that $\{a\}$, $\{a,b\}$ and $\{a,c,d\}$ are open subsets of X and that

$$a \in \{a\} \quad \text{and} \quad \{a\} \cap B = \emptyset$$

$$b \in \{a,b\} \quad \text{and} \quad \{a,b\} \cap B = \{b\}$$

$$c, d \in \{a,c,d\} \quad \text{and} \quad \{a,c,d\} \cap B = \emptyset$$

Hence a, b, c and d are not limit points of $B = \{b\}$. But e is a limit point of B since the open sets containing e are $\{a,b,e\}$ and X and each contains the point $b \in B$ different from e. Thus $B' = \{e\}$.

TOPOLOGY

CHAPTER FIVE

15. Prove: If A is a subset of B, then every limit point of A is also a limit point of B, i.e., $A \subset B$ implies $A' \subset B'$.

Let $p \in X$ is a limit pt. of A.

then

$\Rightarrow (G - \{p\}) \cap A \neq \phi$

$\Rightarrow (G - \{p\}) \cap B \neq \phi$ $\quad \boxed{\because A \subset B}$

$\Rightarrow p \in B^d$

$\Rightarrow p$ is also a limit pt. of B.

TOPOLOGY

CHAPTER FIVE

15. Prove: If A is a subset of B, then every limit point of A is also a limit point of B, i.e., $A \subset B$ implies $A' \subset B'$.

Solution:

Recall that $p \in A'$ iff $(G \setminus \{p\}) \cap A \neq \emptyset$ for every open set G containing p. But $B \supset A$; hence

$$(G \setminus \{p\}) \cap B \supset (G \setminus \{p\}) \cap A \neq \emptyset$$

So $p \in A'$ implies $p \in B'$, i.e. $A' \subset B'$.

TOPOLOGY

CHAPTER FIVE

16. Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be topologies on X such that $\boxed{\mathcal{T}_1 \subset \mathcal{T}_2,}$ i.e. every $\mathcal{T}_1$-open subset of X is also a $\mathcal{T}_2$-open subset of X. Furthermore, let A be any subset of X.
   (i) Show that every $\mathcal{T}_2$-limit point of A is also a $\mathcal{T}_1$-limit point of A.
   (ii) Construct a space in which a $\mathcal{T}_1$-limit point is not a $\mathcal{T}_2$-limit point.

Let $p \in \mathcal{T}_2$ limit pt. of A.

s.t

$G - \{p\} \cap A \neq \phi$

for every $G \in \mathcal{T}_2$

Such that $p \in G$

But $\boxed{\mathcal{T}_1 \subset \mathcal{T}_2 \text{ in}}$ particular

$G - \{p\} \cap A \neq \phi$ for $G \in \mathcal{T}_1$

such that $G \in p$

p is limit pt. of $\mathcal{T}_2$.

TOPOLOGY

CHAPTER FIVE

16. Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be topologies on X such that $\mathcal{T}_1 \subset \mathcal{T}_2$, i.e. every $\mathcal{T}_1$-open subset of X is also a $\mathcal{T}_2$-open subset of X. Furthermore, let A be any subset of X.

    (i) Show that every $\mathcal{T}_2$-limit point of A is also a $\mathcal{T}_1$-limit point of A.

    (ii) Construct a space in which a $\mathcal{T}_1$-limit point is not a $\mathcal{T}_2$-limit point.

(ii) Construct a space in which a $\mathcal{T}_1$-limit point is not a $\mathcal{T}_2$-limit point.

TOPOLOGY

CHAPTER FIVE

16. Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be topologies on X such that $\mathcal{T}_1 \subset \mathcal{T}_2$, i.e. every $\mathcal{T}_1$-open subset of X is also a $\mathcal{T}_2$-open subset of X. Furthermore, let A be any subset of X.

(i) Show that every $\mathcal{T}_2$-limit point of A is also a $\mathcal{T}_1$-limit point of A.

(ii) Construct a space in which a $\mathcal{T}_1$-limit point is not a $\mathcal{T}_2$-limit point.

Solution:

(i) Let p be a $\mathcal{T}_2$-limit point of A; i.e. $(G \setminus \{p\}) \cap A \neq \emptyset$ for every $G \in \mathcal{T}_2$ such that $p \in G$. But $\mathcal{T}_1 \subset \mathcal{T}_2$; so, in particular, $(G \setminus \{p\}) \cap A \neq \emptyset$ for every $G \in \mathcal{T}_1$ such that $p \in G$, i.e. p is a $\mathcal{T}_1$-limit point of A.

(ii) Consider the usual topology $\mathcal{U}$ and the discrete topology $\mathcal{D}$ on $\mathbf{R}$. Note that $\mathcal{U} \subset \mathcal{D}$ since $\mathcal{D}$ contains every subset of $\mathbf{R}$. By Problem 13, 0 is not a $\mathcal{D}$-limit point of the set $A = \{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$ since A' is empty. But 0 is a limit point of A with respect to the usual topology on $\mathbf{R}$.

since A' is empty. But 0 is a limit point of A with respect to the usual topology on $\mathbf{R}$.

TOPOLOGY

CHAPTER FIVE

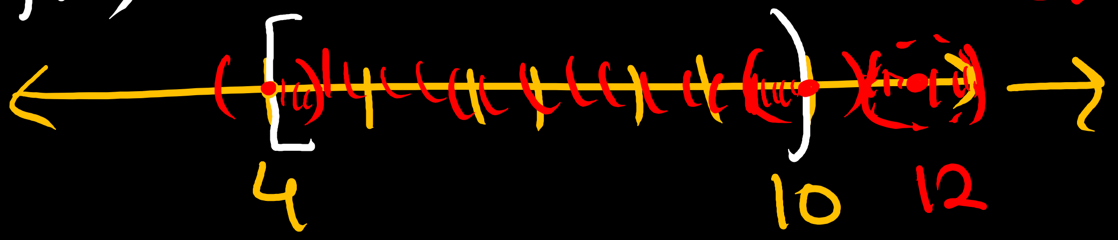
A^d of $\mathbb{N}$ and $\mathbb{Z}$ ale empty!

64. Consider the topological space $(\mathbf{R}, \mathcal{T})$ where $\mathcal{T}$ consists of $\mathbf{R}$, $\emptyset$ and all open infinite intervals $E_a = (a, \infty)$, $a \in \mathbf{R}$. Find the derived set of: (i) the interval $[4, 10)$; (ii) $\mathbf{Z}$, the set of integers.

$[4,10)$

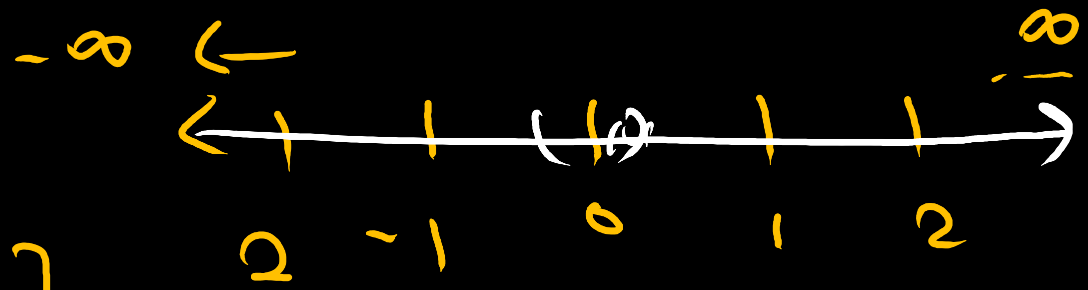
A

$G-\{12\} \cap A \neq \phi$



$A^d = [4,10]$

$4 \in G-\{4\} \cap A \neq \phi$



(a,b)
$[a,b)$
$(a,b]$
$[a,b]$

$A' = [a,b]$

$0 \in G-\{0\} \cap Z = \phi$

TOPOLOGY

CHAPTER FIVE

64. Consider the topological space $(\mathbf{R}, \mathcal{T})$ where $\mathcal{T}$ consists of $\mathbf{R}$, $\emptyset$ and all open infinite intervals $E_a = (a, \infty)$, $a \in \mathbf{R}$. Find the derived set of: (i) the interval $[4, 10)$; (ii) $\mathbf{Z}$, the set of integers.