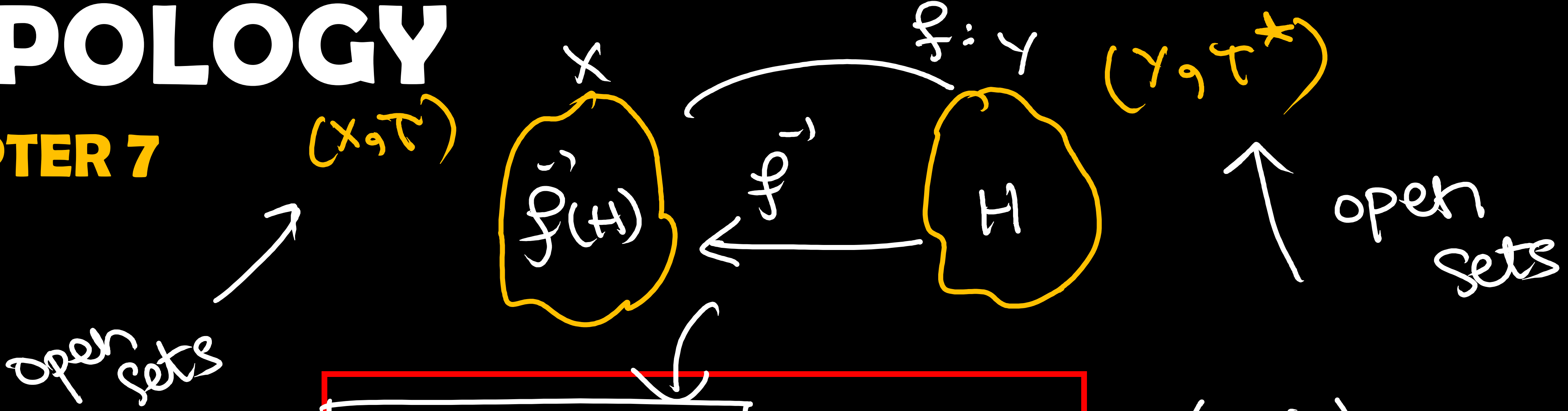
TOPOLOGY

CHAPTER 7



Continuous functions

(X, T) and (Y, T^*) be two topological space. A func. f from X to Y is continuous iff inverse image $f^{-1}(H)$ of every open set H in T^* is a subset of X.

$H \in T^*$

$\Rightarrow \boxed{f^{-1}(H) \in T}$

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$f^{-1}(\phi)=\phi \in T$ $\quad f^{-1}(Y)=X \in T$, $\quad f^{-1}\{x\}=a,b \in T$

$f^{-1}\{y\}=\phi \in T$

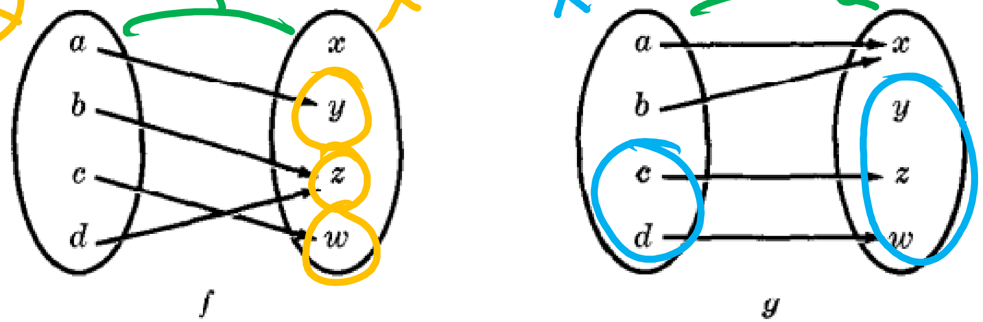
$f^{-1}(x,y)=a,b \in T$

$f^{-1}(y,z,w)=c,d \notin T$

Example 1.1: Consider the following topologies on $X = \{a, b, c, d\}$ and $Y = \{x, y, z, w\}$ respectively:

$$\mathcal{T} = \{X, \emptyset, \{a\}, \{a, b\}, \{a, b, c\}\}, \qquad \mathcal{T}^* = \{Y, \emptyset, \{x\}, \{y\}, \{x, y\}, \{y, z, w\}\}$$

Also consider the functions $f : X \to Y$ and $g : X \to Y$ defined by the diagrams below:



$f \qquad\qquad g$

The function f is continuous since the inverse of each member of the topology $\mathcal{T}^*$ on Y is a member of the topology $\mathcal{T}$ on X. The function g is not continuous since $\{y, z, w\} \in \mathcal{T}^*$, i.e. is an open subset of Y, but its inverse image $g^{-1}[\{y, z, w\}] = \{c, d\}$ is not an open subset of X, i.e. does not belong to $\mathcal{T}$.

$f^{-1}(\phi)=\phi \in T$ $\quad f^{-1}(\{x\})=\phi \in T$ $\quad f^{-1}(\{x,y\})=a \in T$

$f^{-1}(Y)=X \in T$ $\quad f^{-1}(\{y\})=a \in T$ $\quad f^{-1}(z,y,w)=X \in T$

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Example 1.24 Consider any discrete space $(X, \mathscr{D})$ and any topological space $(Y, \mathcal{T})$. Then every function $f: X \to Y$ is $\mathscr{D}$-$\mathcal{T}$ continuous. For if H is any open subset of Y, its inverse $f^{-1}[H]$ is an open subset of X since every subset of a discrete space is open.

$T \to$ Discrete T.S $\longrightarrow$ any T.S

$f^{-1}(H) \longleftarrow H$

every subset

is $\boxed{\text{open}}$

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Example 1.3: Let $f: X \to Y$ where X and Y are topological spaces, and let $\mathcal{B}$ be a base for the topology on Y. Suppose for each member $\boxed{B \in \mathcal{B},\ f^{-1}[B]}$ is an open subset of X; then f is a continuous function. For let H be an open subset of Y; then $H = \cup_i B_i$, a union of members of $\mathcal{B}$. But

$$\boxed{f^{-1}[H] = f^{-1}[\cup_i B_i] = \cup_i f^{-1}[B_i]}$$

and each $f^{-1}[B_i]$ is open by hypothesis; hence $f^{-1}[H]$ is the union of open sets and is therefore open. Accordingly, f is continuous.

$$(X, \tau) \xrightarrow{f} (Y, \tau^*) \to \mathcal{B}$$

(open sets)

open sets

$H \in \mathcal{B}$

$$f^{-1}(H) = f^{-1}(\cup_i B_i)$$
$$= \cup_i f^{-1}[B_i)$$

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Proposition 7.1: A function $f : X \to Y$ is continuous iff the inverse of each member of a base $\mathcal{B}$ for Y is an open subset of X.

generates τ^*

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subbase

Theorem 7.2: Let $\mathcal{S}$ be a subbase for a topological space Y. Then a function $f: X \to Y$ is continuous iff the inverse of each member of $\mathcal{S}$ is an open subset of X.

X

ET

Y

$\leftarrow t^* \longrightarrow$ Base $\longrightarrow$ Subbase

Continous func.

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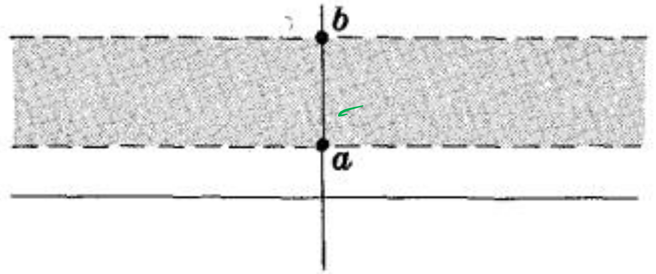
$\pi : \mathbb{R}^2 \to \mathbb{R}$

$\pi(x, y) = x$

Example 1.4: The projection mappings from the plane $\mathbf{R}^2$ into the line $\mathbf{R}$ are both continuous relative to the usual topologies. Consider, for example, the projection $\boxed{\pi : \mathbf{R}^2 \to \mathbf{R}}$ defined by $\boxed{\pi(\langle x, y\rangle) = y}$. Then the inverse of any open interval (a, b) is an infinite open strip as illustrated below:

f^{-1}

$\mathbb{R} \to \mathbb{R}^2$



$\pi^{-1}[(a, b)]$ is shaded

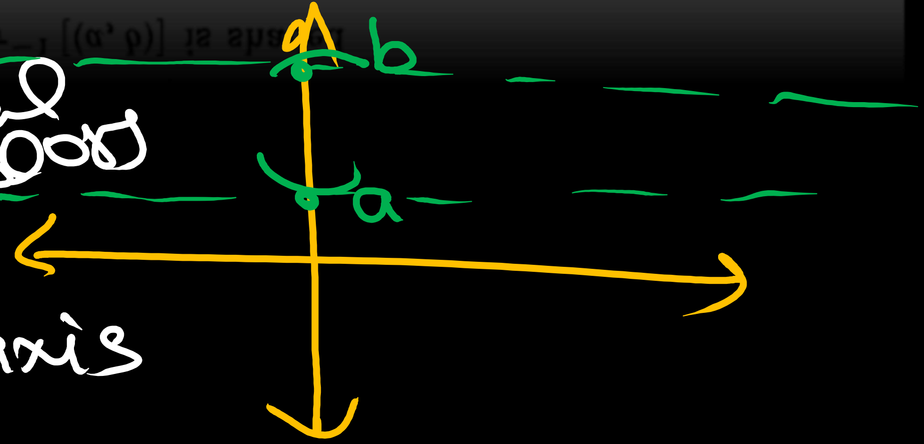
Hence by Proposition 7.1, the inverse of every open subset of $\mathbf{R}$ is open in $\mathbf{R}^2$, i.e. π is continuous.

$\boxed{\pi(x, y) = y}$

$\mathbb{R}^2 \to \mathbb{R} \to$ usual topology

$\mathbb{R}$ ↓ y-axis

infinite open strip ← (a, b) ← y-axis



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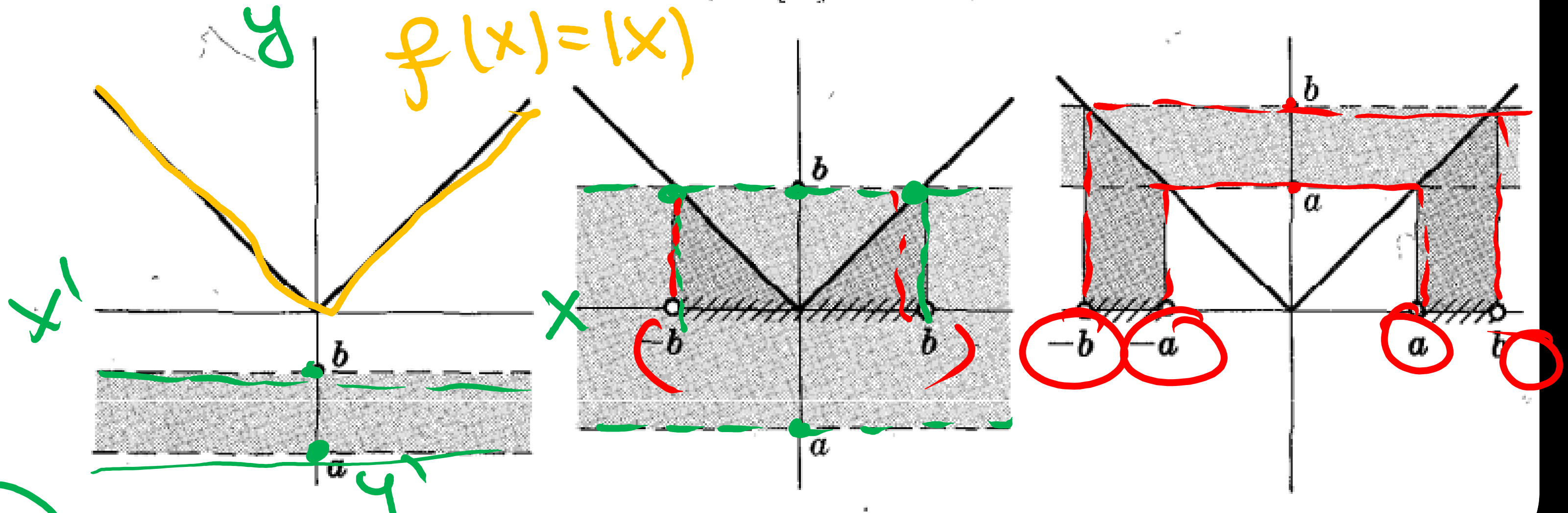
$f(x) = |x|$

Example 1.5: The absolute value function f on $\mathbf{R}$, i.e. $\boxed{f(x) = |x|}$ for $x \in \mathbf{R}$, is continuous. For if $A = (a, b)$ is an open interval in $\mathbf{R}$, then

$$f^{-1}[A] = \begin{cases} \boxed{\emptyset} & \text{if } a < b \leq 0 \\ \boxed{(-b, b)} & \text{if } a < 0 < b \\ (-b, -a) \cup (a, b) & \text{if } 0 \leq a < b \end{cases}$$

as illustrated below. In each case $f^{-1}[A]$ is open; hence f is continuous.



$f^{-1}[A] = \emptyset$ $\quad$ $f^{-1}[A] = (-b, b)$ $\quad$ $f^{-1}[A] = (-b, -a) \cup (a, b)$

$\boxed{a < 0 < b}$

$(a, b) \Rightarrow f^{-1}(A) = \phi \; , \; a, b < 0$

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Theorem 7.3: A function $f: X \to Y$ is continuous if and only if the inverse image of every closed subset of Y is a closed subset of X.