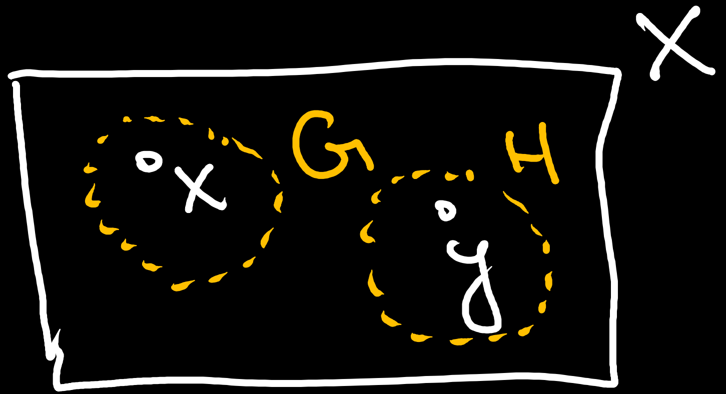
TOPOLOGY

CHAPTER 10

HAUSDROFF SPACES

T_2-Space



$G \cap H = \phi$

Every haudroff space is T_1-space.

(X, τ) be T.S then it T_2 | Haudroff iff

$x, y \in X \quad x \neq y$

$\exists$ G and H open set $G \cap H = \phi$

s.t

$x \in G$

$y \notin G$

$y \in H$

$x \notin H$

TOPOLOGY

CHAPTER 10

HAUSDORFF SPACES

A topological space X is a *Hausdorff space* or *T_2-space* iff it satisfies the following axiom:

[$\mathbf{T_2}$] Each pair of distinct points $a, b \in X$ belong respectively to disjoint open sets.

$a \neq b \in X$

$G \cap H = \phi$

TOPOLOGY

CHAPTER 10

In other words, there exist open sets G and H such that

$$a \in G, \; b \in H \quad \text{and} \quad G \cap H = \emptyset$$

Observe that a Hausdorff space is always a T_1-space.

TOPOLOGY

CHAPTER 10

$p \in G \cap H$

$d(x,p) < \frac{1}{3}\varepsilon$

$d(y,p) < \frac{1}{3}\varepsilon$

Triangular Inequality

$d(x,y) \leq d(x,p) + d(y,p)$

$\frac{1}{3}\varepsilon + \frac{1}{3}\varepsilon$

$\frac{2}{3}\varepsilon$

$d(x,y) \leq \frac{2}{3}\varepsilon$

Example 2.1: We show that every metric space X is Hausdorff.

Let $a, b \in X$ be distinct points; hence by $[M_4]$ $d(a,b) = \epsilon > 0$. Consider the open spheres $G = S(a, \frac{1}{3}\epsilon)$ and $H = S(b, \frac{1}{3}\epsilon)$, centered at a and b respectively. We claim that G and H are disjoint. For if $p \in G \cap H$, then $d(a,p) < \frac{1}{3}\epsilon$ and $d(p,b) < \frac{1}{3}\epsilon$; hence by the Triangle Inequality,

$$d(a,b) \leq d(a,p) + d(p,b) < \tfrac{1}{3}\epsilon + \tfrac{1}{3}\epsilon = \tfrac{2}{3}\epsilon$$

But this contradicts the fact that $d(a,b) = \epsilon$. Hence G and H are disjoint, i.e. a and b belong respectively to the disjoint open spheres G and H. Accordingly, X is Hausdorff.

Let $x, y \in X$

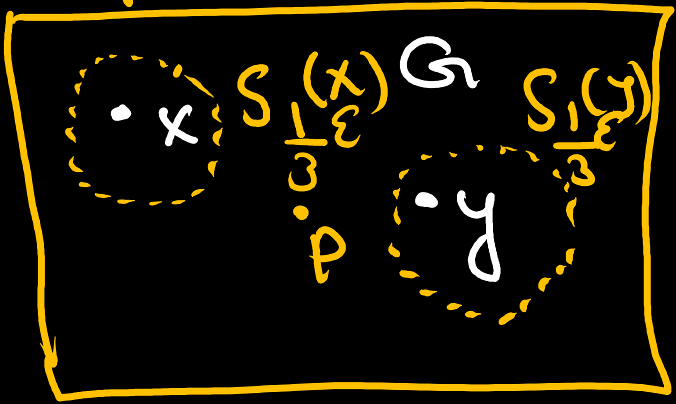
$\rightarrow \boxed{d(x,y) = \varepsilon > 0}$

Every open sphere open set.

Suppose on contrary

$G \cap H = \phi$

(X, d)



$\boxed{G \cap H \neq \phi}$

TOPOLOGY

CHAPTER 10

Theorem 10.3: Every metric space is a Hausdorff space.

TOPOLOGY

CHAPTER 10

Example 2.2: Let $\mathcal{T}$ be the cofinite topology, i.e. $\mathcal{T}_1$-topology, on the real line **R**. We show that $(\mathbf{R}, \mathcal{T})$ is not Hausdorff. Let G and H be any non-empty $\mathcal{T}$-open sets. Now G and H are infinite since they are complements of finite sets. If $G \cap H = \emptyset$, then G, an infinite set, would be contained in the finite complement of H; hence G and H are not disjoint. Accordingly, no pair of distinct points in **R** belongs, respectively, to disjoint $\mathcal{T}$-open sets. Thus $\mathcal{T}_1$-spaces need not be Hausdorff.

$(\mathbb{R}, T_c) \rightarrow$ not T_2-Space

Suppose on contrary

$(\mathbb{R}, T_c)$ is T_2-Space

$\exists$ open set G and H s.t $G \cap H = \phi$

$\Rightarrow G \subseteq H^c$

infinite

finite

which impossible

So, $(\mathbb{R}, T_c)$ is no T_2-Space.

TOPOLOGY

CHAPTER 10

Convergent seq: $\rightarrow \lim_{n \to \infty} a_n = l$ (finite) $\rightarrow$ unique

As noted previously, a sequence $\langle a_1, a_2, \ldots \rangle$ of points in a topological space X could, in general, converge to more than one point in X. This cannot happen if X is Hausdorff:

Theorem 10.4: If X is a Hausdorff space, then every convergent sequence in X has a unique limit.

The converse of the above theorem is not true unless we add additional conditions.

$(X, T) \rightarrow$ T.S & convergent seq $\rightarrow$ converges to one point

$\rightarrow$ X is Hausdorff.

only one.

TOPOLOGY

CHAPTER 10

net / filter → converges to one point
↓
X
↓
T_2-space

Remark: The notion of a sequence has been generalized to that of a *net* (Moore-Smith sequence) and to that of a *filter* with the following results:

Theorem 10.4A: X is a Hausdorff space if and only if every convergent net in X has a unique limit.

Theorem 10.4B: X is a Hausdorff space if and only if every convergent filter in X has a unique limit.

The definitions of net and filter and the proofs of the above theorems lie beyond the scope of this text.